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Immersed Möbius bands in knot complements

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We study the 3–dimensional immersed crosscap number of a knot, which is a nonorientable analogue of the immersed Seifert genus. We study knots with immersed crosscap number 1, and show that a knot has immersed crosscap number 1 if and only if it is a nontrivial $(2p,q)$–torus or $(2p,q)$–cable knot. We show that unlike in the orientable case the immersed crosscap number can differ from the embedded crosscap number by arbitrarily large amounts, and that it is neither bounded below nor above by the 4–dimensional crosscap number.

57M25, 57M27; 57M35

1 Introduction

One method to define a measure of the complexity of a knot $K \subset S^3$ is by describing the minimal topological complexity of a compact surface F whose boundary is equal to K. When F is required to be orientable and embedded in S^3 this gives the classical *Seifert genus* $g_3(K)$ of K; when the requirement that F be embedded in S^3 is weakened, and we allow embedded surfaces in B^4, we obtain the *slice genus* $g_4(K)$ of K. Here we are thinking of S^3 as the boundary of B^4.

Loosening the requirement that F be embedded in S^3 in another direction we can instead consider surfaces F which are merely immersed in S^3. While it is easy to verify that every knot K is the boundary of an immersed disk in S^3, if we consider only immersed surfaces which are embedded along a neighborhood of their boundaries we obtain *immersed Seifert surfaces* for K. The minimal genus of any immersed Seifert surface for K is called the *immersed Seifert genus* of K, and is denoted by $g_I(K)$. While g_I takes nontrivial values on knots in S^3, using foliations Gabai [6] proved that the resulting knot invariant is always equal to the Seifert genus. Both invariants are in turn bounded below by the slice genus.

Dropping the requirement that F be orientable, we can instead consider nonorientable immersed spanning surfaces, defined in a similar way as above. This gives rise to a

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nonorientable analogue of the immersed Seifert genus, called the *immersed crosscap number* $\gamma_I(K)$ of the knot K. Our main result involves knots with immersed crosscap number 1, ie knots which bound immersed Möbius bands that are embedded along their boundaries.

For $|p| \geq 2$, we say that K is a (p,q)*–cable knot* if it can be isotoped to lie on the boundary of a solid torus $V \subset S^3$ whose core is knotted, where K represents the class $q \cdot m + p \cdot l$ in $\pi_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$. Here m and l are the homotopy classes of the meridian and the Seifert-framed longitude of ∂V, respectively.

Theorem 1 *A knot $K \subset S^3$ has $\gamma_I(K) = 1$ if and only if K is a nontrivial $(2p,q)$–torus or $(2p,q)$–cable knot.*

In a similar way, we can also define the 3*–dimensional (embedded) crosscap number* $\gamma_3(K)$ and 4*–dimensional (embedded) crosscap number* $\gamma_4(K)$ of a knot, which are nonorientable analogues of the Seifert and slice genus, respectively. Theorem 1 then generalizes a result of Clark [5], who proved that $\gamma_3(K) = 1$ if and only if K is a $(2,q)$–torus or $(2,q)$–cable knot.

Unlike their orientable counterparts, in general $\gamma_I(K)$ may not equal $\gamma_3(K)$, and is not bounded below by $\gamma_4(K)$. More precisely, we present an infinite family of immersed crosscap number 1 knots with unbounded 3– and 4–dimensional crosscap numbers. Furthermore, we also present examples of knots K with $\gamma_I(K) > \gamma_4(K)$.

2 The immersed crosscap number of a knot

2.1 Crosscap numbers of knots

We begin by defining nonorientable analogues of the Seifert, slice, and immersed Seifert genera of knots. Roughly speaking, these values capture the minimum k needed to span the knot by a punctured connected sum of k copies of $\mathbb{RP}^2$, assuming different embedding and immersion requirements.

Let K be a knot in S^3. We say that a compact, embedded, nonorientable surface $F \subset S^3$ with $\partial F = K$ is a *nonorientable spanning surface* for K, and we define the 3*–dimensional* (*embedded*) *crosscap number* of a nontrivial knot K to be

$$\gamma_3(K) = \min\{b^1(F) \mid F \text{ is a nonorientable spanning surface for } K\}.$$

Here $b^1(F)$ is the first Betti number of F. We define γ_3 of the unknot to be 0. As with orientable spanning surfaces, any nonorientable spanning surface F induces a framing on the knot K, though this framing need not equal the Seifert framing on K. Indeed, by boundary summing an embedded Möbius band with unknotted boundary to F we obtain a nonorientable spanning surfaces whose induced framing differs by ± 2 from the framing induced by F.

If we think of $S^3 = \partial B^4$, then a compact, embedded, nonorientable surface $F \subset B^4$ with $\partial F = K \subset S^3$ is a *nonorientable slice surface* for K, and we define the 4–*dimensional* (*embedded*) *crosscap number* of a nonslice knot K (ie $g_4(K) \geq 1$) to be

$$\gamma_4(K) = \min\{b^1(F) \mid F \text{ is a nonorientable slice surface for } K\}.$$

If K bounds an embedded slice disk in B^4, we define $\gamma_4(K) = 0$.

Lastly, suppose that F is the image of an immersion $h\colon \Sigma \to S^3$, where Σ is a compact, nonorientable surface with boundary. Then $F = h(\Sigma)$ is a *nonorientable immersed spanning surface* for K if $h(\partial\Sigma) = K$, and if there is a collar neighborhood A of the boundary $\partial\Sigma$ such that $h(A)$ is embedded and $h^{-1}(h(A)) = A$. The first Betti number $b^1(F)$ of the nonorientable immersed spanning surface F is defined to be $b^1(\Sigma)$, and if K is a nontrivial knot we define the *nonorientable immersed crosscap number* of K to be

$$\gamma_I(K) = \min\{b^1(F) \mid F \text{ is a nonorientable immersed spanning surface for } K\}.$$

In the case when K is the unknot, we again define $\gamma_I(K) = 0$.

Recall that in the orientable case, the Seifert, slice and immersed Seifert genus of a knot K satisfy

$$g_I(K) = g_3(K) \geq g_4(K).$$

Our goal in this section is to determine which of the results above generalize to the nonorientable case.

Firstly, as any nonorientable spanning surface in S^3 can be pushed into B^4 to become a nonorientable slice surface, we clearly have that $\gamma_3(K) \geq \gamma_4(K)$. Furthermore, we also trivially have that $\gamma_3(K) \geq \gamma_I(K)$.

Note, however, that not every nonorientable immersed spanning surface can be pushed to an embedding in B^4 (see [4] for criteria describing when this is possible). Furthermore, not every nonorientable slice surface can be pushed into S^3 to give a nonorientable immersed spanning surface. Hence, we do not have any a priori relations between the invariants $\gamma_I(K)$ and $\gamma_4(K)$, a fact which can be illustrated with a few simple examples.

In [9] Teragaito gives an algorithm for computing the crosscap number of the (p,q)–torus knot $T(p,q)$ using the partial fraction expansions of rational expressions involving p and q. In the case of $T(2k,2k-1)$, where $k \geq 2$, his result implies the following:

Proposition 2 [9, Theorem 1.1] *Let $k \geq 2$. Then $\gamma_3(T(2k,2k-1)) = k$.*

Proof This is a direct specialization of results in [9], though we include the required computation here for completeness. By Theorem 1.1 in [9], if p and q are coprime positive integers with p even, then $\gamma_3(T(p,q)) = N(p,q)$. Here $N(p,q)$ is an integer value introduced by Bredon and Wood [2], and is the minimal genus of a closed, connected, nonorientable surface contained in the lens space $L(p,q)$.

The value of $N(p,q)$ can be computed by starting with a continued fraction expansion of p/q of the form

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where each $a_j > 0$ is an integer. Then, by [2], the integer $N(p,q)$ is half the value obtained by summing up the a_j successively, except that when the partial sum is even the next a_j value is skipped. More precisely, if we define

$$b_0 = a_0, \qquad b_i = \begin{cases} a_i & \text{if } b_{i-1} \neq a_{i-1} \text{ or } \sum_{j=0}^{i-1} b_j \text{ is odd,} \\ 0 & \text{if } b_{i-1} = a_{i-1} \text{ and } \sum_{j=0}^{i-1} b_j \text{ is even,} \end{cases}$$

then $N(p,q) = \frac{1}{2}\sum_{j=0}^{n} b_j$. Since $2k/(2k-1) = 1 + 1/(2k-1)$, it follows then that $N(2k,2k-1) = \frac{1}{2}(1+2k-1) = k$, as required. □

On the other hand Batson [1] finds a lower bound on $\gamma_4(K)$ involving the signature of K, along with the Heegaard Floer d–invariant of certain integer homology spheres. For $T(2k,2k-1)$, again with $k \geq 2$, his results specialize to give $\gamma_4(T(2k,2k-1)) = k-1$.

Proposition 3 *Let p and q be integers, with $2p$ and q coprime and $|q| \geq 2$. Then $\gamma_I(T(2p,q)) = 1$.*

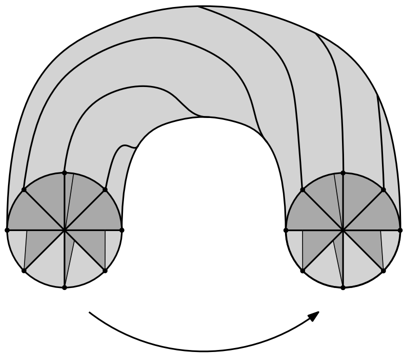

Figure 1: An immersed Möbius band in a solid torus with boundary $T(2p,q)$.

Proof With p and q as above, the knot $K = T(2p,q)$ is nontrivial, and hence $\gamma_I(K) \geq 1$. Furthermore, it is not difficult to construct an immersed Möbius band F whose boundary is K as follows.

Suppose that K is embedded along the boundary of the standard embedding of the solid torus $V \subset S^3$. Suppose that for any disk $D_\theta \subset V$ of the form $\{\theta = \text{constant}\} \cap V$, where θ is the angular polar coordinate, $K \cap D_\theta$ is a collection of $2p$ evenly spaced points $\{x_1, \dots, x_{2p}\}$ around ∂D_θ. In each D_θ, draw p straight lines through the center of D_θ, connecting x_j with x_{p+j} for $1 \leq j \leq p$. As θ ranges from 0 to 2π these lines will sweep out an immersed Möbius band in V, with boundary K (see Figure 1). Furthermore, this Möbius band will be embedded away from the core of V. □

Clearly the above proof of Proposition 3 generalizes to $(2p,q)$–cable knots, a fact which we record here.

Proposition 4 *If K is a $(2p,q)$–cable knot, then $\gamma_I(K) = 1$.*

In particular, for $k \geq 2$, we have $\gamma_I(T(2k, 2k-1)) = 1$. We thus see that both of the quantities $\gamma_3(K) - \gamma_I(K)$ and $\gamma_4(K) - \gamma_I(K)$ can be arbitrarily large.

On the other hand, we can also find knots K for which $\gamma_I(K) > \gamma_4(K)$. Indeed, an immediate corollary to Theorem 1 is that any hyperbolic knot K has $\gamma_I(K) > 1$. Hence, any slice hyperbolic knot K has $\gamma_I(K) > \gamma_4(K)$. The Stevedore knot 6_1 is the simplest example of such a knot. An interesting question would be to ask whether the value of $\gamma_I(K) - \gamma_4(K)$ can be arbitrarily large.

Question *For any $n \in \mathbb{N}$ does there exist a knot K such that $\gamma_I(K) - \gamma_4(K) > n$?*

3 Essential Möbius bands

In what follows we focus on studying nontrivial knots K which bound immersed Möbius bands that are embedded near their boundaries, ie knots with $\gamma_I(K) = 1$. To do so it will be more useful to think of our immersed nonorientable spanning surfaces as lying in the exterior of K, rather than S^3 itself.

More precisely, let $N(K) \subset S^3$ be a small open tubular neighborhood of K, and let $E(K) = S^3 \setminus N(K)$ be the exterior of K. Then $\partial E(K)$ is a torus, with a canonical choice of meridian. An *immersed* (*nonorientable*) *spanning surface* F in $E(K)$ is the image of an immersion $h\colon \Sigma \to E(K)$, where Σ is a (nonorientable) compact surface with boundary, such that $h(\partial\Sigma) = F \cap \partial E(K)$ is a longitude on $\partial E(K)$ and $\partial\Sigma$ has a collared neighborhood A in Σ such that $h(A)$ is embedded and $h^{-1}(h(A)) = A$. Clearly there is a straightforward way to pass between nonorientable immersed spanning surfaces for the knot K in S^3 and nonorientable immersed spanning surfaces in the knot exterior $E(K)$. Furthermore, a nontrivial knot K has $\gamma_I(K) = 1$ if and only if there is a spanning surface in $E(K)$ which is the immersed image of a Möbius band.

3.1 Essential maps

Consider a map $h\colon \Sigma \to E(K)$ with $h(\partial\Sigma) \subset \partial E(K)$. We say that h is π_1*–essential* if $h_*\colon \pi_1(\Sigma) \to \pi_1(E(K))$ is injective. Similarly, we say that h is $\partial\pi_1$*–essential* if $h_*\colon \pi_1(\Sigma, \partial\Sigma) \to \pi_1(E(K), \partial E(K))$ is injective. Finally, we say that h is *essential* if it is both π_1– and $\partial\pi_1$–essential. We will sometimes describe the image of an embedding $h\colon \Sigma \to E(K)$ as being *essential* if the map h is essential.

Now let $K \subset S^3$ be a nontrivial knot and M a Möbius band. To make things more precise, let M be given by the square $[-1, 1] \times [-1, 1]$ in $\mathbb{R}^2$, with vertical edges identified via $(-1, t) \sim (1, -t)$. The *core* of M will be denoted by c, and is the image of $[-1, 1] \times \{0\}$ under the quotient map. Let α be the image of the arc $\{0\} \times [-1, 1]$ under the quotient map. Note that the homotopy class of c generates $\pi_1(M) \cong \mathbb{Z}$, and the homotopy class of α is the only nontrivial class in $\pi_1(M, \partial M)$.

We fix a meridian m and longitude l of $\partial E(K)$, which we think of as generators for $\pi_1(\partial E(K)) \cong \mathbb{Z} \oplus \mathbb{Z}$. (We will write the group operation of $\pi_1(\partial E(K))$ as addition because it is abelian, and suppress explicit reference to a basepoint when there is no danger in doing so.)

Theorem 5 *Let K be a nontrivial knot, and suppose that $h\colon M \to E(K)$ is a proper map where $h(\partial M)$ is homotopic in $\partial E(K)$ to a curve of the form $a \cdot m + b \cdot l$ for some $a, b \in \mathbb{Z}$ with either a or b odd. Then h is essential.*

Proof We begin by showing that h is π_1–essential. Note that since $\pi_1(E(K))$ is torsion-free, it suffices to show that $h_*(c)$ is nontrivial. Suppose then to the contrary that $h_*(c)$ is null-homotopic.

Consider a tubular neighborhood $N(c) \subset M$ of c. Notice that $\partial N(c)$ is a double of the core, and hence $h(\partial N(c))$ is also null-homotopic in $E(K)$. Hence we can take a disk D and glue it to $M \setminus N(c)$, via some identification φ of ∂D with $\partial N(c)$, and then extend the map h across D to get a map $h'\colon M \setminus N(c) \cup_\varphi D \to E(K)$. Note, however, that $M \setminus N(c) \cup_\varphi D$ is homeomorphic to a disk D^2, and hence we obtain a map $h'\colon D^2 \to E(K)$, with $h'(\partial D^2) = h(\partial M)$. As $h(\partial M)$ is nontrivial in $\pi_1(\partial E(K))$, the loop theorem then implies that there is a properly embedded disk in $D' \subset E(K)$, with $\partial D'$ nontrivial in $\pi_1(\partial E(K))$. This contradicts the assumption that K was nontrivial, and hence h must be π_1–essential.

To show that h is $\partial\pi_1$–essential, assume now that there is homotopy taking $h(\alpha)$ to $\partial E(K)$, relative to $\partial h(\alpha)$. Using this homotopy, we can modify h to obtain a new map $h_0\colon M \to E(K)$, which sends $\partial M \cup N(\alpha)$ to $\partial E(K)$, where $N(\alpha)$ is a tubular neighborhood of α in M. Moreover, the restrictions of h and h_0 to ∂M will be homotopic inside $\partial E(K)$.

Consider now the disk $D_0 = M \setminus N(\alpha)$. The map h_0 restricts to give $h_0\colon D_0 \to E(K)$, with $h_0(\partial D_0) \subset E(K)$. Suppose first that $h_0(\partial D_0)$ is a nontrivial loop in $\pi_1(\partial E(K))$. Then, as above, the loop theorem implies that K is the trivial knot, which is a contradiction.

Suppose then that $h_0(\partial D_0)$ is null-homotopic in $\partial E(K)$. This null-homotopy can be viewed as a map $\rho\colon D_0 \to \partial E(K)$, where $\rho|_{\partial D_0} \equiv h_0|_{\partial D_0}$. Then we can define a map $h_1\colon M \to \partial E(K)$ by

$$h_1(x) = \begin{cases} \rho(x) & \text{for } x \in D_0, \\ h_0(x) & \text{for } x \in N(\alpha). \end{cases}$$

Furthermore, we have that $h_1|_{\partial M} \equiv h_0|_{\partial M}$, which is homotopic to $h|_{\partial M}$ in $\partial E(K)$.

Note, however, that ∂M is homotopic to $2{\cdot}c$ in M. Hence in $\pi_1(\partial E(K))$ the homotopy class of $h(\partial M) = a \cdot m + b \cdot l$ will be two times the homotopy class of $h_1(c)$, which contradicts the assumption that at least one of a or b is odd. Thus h must be $\partial\pi_1$–essential, and therefore essential. □

4 Immersed crosscap number one knots

For the remainder of the paper, let K be a knot with $\gamma_I(K)=1$, M a Möbius band, and $h\colon M\to E(K)$ an immersion such that $F=h(M)$ is an immersed nonorientable spanning surface in $E(K)$. We will make use of the following theorem of Cannon and Feustel. Note that we can define the notions of π_1–essential, $\partial\pi_1$–essential and essential as in Section 3.1 for maps into any manifold Y. Let A be an annulus.

Theorem 6 [3, Theorem 4] *Let Y be a compact orientable 3–manifold, and $h\colon A\to Y$ an essential map with $h(\partial A)\subset\partial Y$. Then there exists an essential embedding $f\colon\Sigma\to Y$ with $f(\partial\Sigma)\subset\partial Y$, where Σ is either an annulus or Möbius band. Moreover, if $h|_{\partial A}$ is an embedding, then we may assume that $f(\partial\Sigma)\subset h(\partial A)$.*

We begin the proof of Theorem 1 by first showing that K must be a torus or cable knot, before describing its type.

Lemma 7 *If $\gamma_I(K)=1$, then K is a torus or cable knot.*

Proof By Theorem 5 the map $h\colon M\to E(K)$ is essential. Let $\tau\colon A\to M$ be a double-covering map. As τ_* is injective on both $\pi_1(A)$ and $\pi_1(A,\partial A)$, the map $h\circ\tau\colon A\to E(K)$ will also be essential. Note that $(h\circ\tau)|_{\partial A}$ will not be an embedding, but by pushing the image of the two sheets of the covering map off of $h(\partial M)$ we obtain an essential map which is an embedding along the boundary ∂M. Then by Theorem 6 there is an essential embedding $f\colon\Sigma\to E(K)$, where Σ is either a Möbius band or annulus, and $f(\partial\Sigma)$ is contained in a pair of parallel push-offs of $h(\partial M)$ along $\partial E(K)$.

Suppose first that $\Sigma=M$ is a Möbius band. Then $f\colon M\to E(K)$ is an embedding, with $f(\partial M)$ a longitude of $\partial E(K)$. Hence $\gamma_3(K)=1$, and by [5] it follows that K is a $(2,q)$–torus or $(2,q)$–cable knot.

On the other hand, if $\Sigma=A$ is an annulus, then we obtain an essential annulus embedded in $E(K)$. By [8], the only such annuli are either subsurfaces of decomposing spheres for K or cabling annuli (see also [7]). Since the boundary $\partial f(A)$ is a pair of longitudes of $\partial E(K)$ and not meridians, it follows that $f(A)$ is cannot be extended to a decomposing sphere. Hence K is a torus or cable knot, with $f(A)$ its cabling annulus. □

Notice that, in the above proof, the cabling annulus $f(A)$ we obtain in $E(K)$ has the same boundary slope as the original immersed Möbius band $h(M)$. In other words, $f(\partial A)$ consists of a pair of simple closed curves in $\partial E(K)$ which are parallel to the longitudinal curve $h(\partial M)$.

As torus and cable knots are necessarily prime, we have the following immediate corollary:

Corollary 8 *Any knot K with $\gamma_I(K) = 1$ is prime.*

Now that we've established that any knot K with $\gamma_I(K) = 1$ is a nontrivial (p,q)–torus or (p,q)–cable knot, we proceed to determine what restrictions (if any) this situation places on p and q. We begin by answering the question in the case of torus knots.

Lemma 9 *If K has $\gamma_I(K) = 1$ and is a (p,q)–torus knot, then one of p or q must be even.*

Proof Suppose to the contrary that both p and q are odd. Suppose further that K sits on the standardly embedded torus $T \subset S^3$ obtained by extending the cabling annulus from the proof of Lemma 7, and let V be one of the solid tori bounded by T. Recall that $\pi_1(E(K))$ can be presented as

$$\pi_1(E(K)) = \langle x, y \mid x^p = y^q \rangle,$$

where x and y are the homotopy classes of the cores of the two solid tori in $S^3 \setminus T$. Suppose, without loss of generality, that x is the homotopy class of the core of V.

Note that $T \cap E(K)$ is the cabling annulus obtained in the proof of Lemma 7. Furthermore, if $h(M)$ is an immersed Möbius band bounded by K we can homotope $h(\partial M)$ so that it sits on $\partial E(K) \cap \operatorname{int} V$. Pushing $h(\partial M)$ off of $\partial E(K)$ towards the core of V, we see that it is homotopic to x^p.

Pushing ∂M inside M towards its core c and tracking its image under h, we see that $h(\partial M)$ is also freely homotopic to $h(c)^2$, and hence $h(c)^2$ is conjugate to x^p in $\pi_1(E(K))$. Note, however, that the only relation in the above group presentation does not change the parity of the algebraic length of any of the words in $\pi_1(E(K))$. Hence any representation of $h(c)^2$ as a word in the generators x and y will always have even length, and any word representing a conjugate of x^p will have odd length, a contradiction. Thus either p or q must be even. □

We now state and prove Proposition 10, which will serve to complete the proof of Theorem 1.

Proposition 10 *If K has $\gamma_I(K)=1$ and is a (p,q)–cable knot, then p must be even.*

We will break the proof of Proposition 10 into several lemmas. In what follows, suppose that K is a knot with $\gamma_I(K)=1$ which is a (p,q)–cable knot. Furthermore, suppose that K lies on the boundary T of a solid torus V, this time knotted in S^3, where $T \cap E(K)$ is the cabling annulus A obtained from the proof of Lemma 7. Let W be the closure of $S^3 \setminus V$. Again, $h(\partial M)$ is parallel to the components of ∂A in $\partial E(K)$, but this time we homotope $h(\partial M)$ so that it is embedded and lies on the outside of V, in $\partial E(K) \cap W$.

Lemma 11 *Suppose that $h(M)$ can be homotoped so that it lies entirely in $E(K)\cap W$. Then p must be even.*

Proof Note that $E(K)\cap W$ is homeomorphic to $E(V)=S^3\setminus V$, which can be viewed as the exterior of the knotted core of V. Then $h(\partial M)$ represents the class $q\cdot m+p\cdot l$ in $\pi_1(\partial E(V))$, where as usual m and l are the homotopy classes in $\pi_1(\partial E(V))$ of the meridian and Seifert-framed longitude, respectively. Note that since K is a cable (and hence a satellite knot) we must have $|p|\geq 2$. If p is even, then we are done. Assume, therefore, that p is odd.

By Theorem 5 the map $h\colon M\to E(V)$ is essential, and hence we can find an essential embedding $f\colon \Sigma\to E(V)$, where Σ is either an annulus or Möbius band, and where $f(\partial\Sigma)$ is parallel to $h(\partial M)$ in both cases. Furthermore, $f(\partial\Sigma)$ will represent either $q\cdot m+p\cdot l$ or $2q\cdot m+2p\cdot l$ in $\pi_1(\partial E(V))$, depending on whether Σ is a Möbius band or annulus, respectively.

Suppose first that $\Sigma=M$. Then the boundary of a tubular neighborhood of $N(f(M))$ will be an embedded, essential annulus $A'\subset E(V)$, whose boundary represents $2q\cdot m+2p\cdot l$ in $\pi_1(\partial E(V))$. Then by [8] (see also Theorem 4.13 in [7]) A' must be a cabling annulus for the core of V, which implies that $\partial A'$ represents a class of the form $k\cdot m\pm 2\cdot l\in\pi_1(\partial E(V))$. Thus $p=\pm 1$, a contradiction.

Suppose then that Σ is an annulus. As above we can conclude that $f(\Sigma)$ is a cabling annulus for the core of V, and hence we arrive at the same contradictory conclusion, namely that $p=\pm 1$. □

We now turn our attention to the case when $h(M)$ cannot be arranged to lie entirely in $E(K)\cap W$. Then $h(M)$ must intersect the cabling annulus $A\subset E(K)$ nontrivially. We assume that h is transverse to A, and hence $h^{-1}(A)$ will be a collection of embedded simple closed curve contained in the interior of M. We thus divert our attention momentarily to discuss such curves on the Möbius band M.

Recall that M is given by the square $[-1,1]\times[-1,1]$ in $\mathbb{R}^2$, with vertical edges identified via $(-1,t)\sim(1,-t)$. The core of M will be denoted by c, and is the image of $[-1,1]\times\{0\}$ under the quotient map. Furthermore, let μ be the image under the quotient map of the segments $[-1,1]\times\left\{-\frac{1}{2},\frac{1}{2}\right\}$. While the following result is certainly well known, we know of no reference to it in the literature, and hence we reproduce its proof here.

Lemma 12 *Any simple closed curve in M which does not bound a disk is isotopic to either c or μ.*

Proof Let $\alpha\subset M$ be a simple closed curve, which we fix an orientation on. Let $\delta\subset M$ be the image of the arc $\{-1\}\times[-1,1]$ under the quotient map. Assume that α and δ intersect transversely, and orient δ so that the algebraic intersection $\alpha\cdot\delta$ between α and δ is nonnegative. Suppose that $|\alpha\cap\delta|>\alpha\cdot\delta$. Then we can chose an arc $\tau\subset\alpha$ such that the endpoints of τ consist of both a positive and a negative intersection point of α and δ. Furthermore, we can assume that there are no other intersection points with δ on the interior of τ. Then there is a subarc τ' in δ such that $\tau\cup\tau'$ bounds a disk. After choosing the innermost such disk, we can push τ through δ to the other side, removing one pair of canceling intersection points. We can repeat this until all remaining intersection points between α and δ are positive. If $\alpha\cap\delta=\varnothing$, then α lies in a disk and hence is nullhomotopic. Assume then that $\alpha\cap\delta\neq\varnothing$.

Lift the simple closed curve α to $[-1,1]\times[-1,1]$, where we get a collection of n properly embedded disjoint arcs $\alpha_1,\dots,\alpha_n$, each of which has one endpoint on $\{-1\}\times[-1,1]$ and the other on $\{1\}\times[-1,1]$. Assume that the arcs are labeled in order from top to bottom. The identification of the vertical boundary components then induces an identification of the strands, sending the left endpoint of the j^{th} strand to the right endpoint of the $(n-j+1)^{\text{st}}$ strand. Represent this identification as an element σ of the symmetric group S_n on n letters. Notice that as a permutation $\sigma\circ\sigma=\text{id}$, however the subgroup of S_n generated by σ must act transitively on the set $\{1,\dots,n\}$ as α is connected. Thus $n=1$ or 2, and hence α is isotopic to either c or μ. □

Returning to our map $h\colon M \to E(K)$, we show that all loops in $h^{-1}(A) \subset M$ can be avoided except possibly for curves that are isotopic to μ.

Lemma 13 *The map $h\colon M \to E(K)$ can be modified away from ∂M so that all curves in $h^{-1}(A)$ are isotopic to the curve μ in M.*

Proof We first note that none of the simple closed curves in $h^{-1}(A)$ can be isotopic to the core c of M, since A is orientable and c is an orientation-reversing curve in M.

Next we show that we can modify h away from ∂M so that $h^{-1}(A)$ contains no inessential curves. Let $\alpha \subset h^{-1}(A)$ be a simple closed curve which bounds a disk D in M, and assume that D contains no other such curves. Then $h(D)$ will be an immersed disk which lies in the closure of one of the two components of $E(K) \setminus A$, which we denote by U, and whose boundary $h(\alpha)$ is an immersed loop on A. Since A is essential, the immersed loop $h(\alpha)$ will be null-homotopic on A.

Pick a homotopy which takes $h(\alpha)$ to a small disk D' in A, and extend it to a homotopy of h supported in small neighborhoods of α and A, so that the double-point curve along $h(\alpha)$ now lies in $D' \subset A$.

Let $N(D')$ be the restriction to D' of a small tubular neighborhood of A, which we can parametrize in the usual way as $N(D') = D' \times (-1,1)$. Then $h(D)$ sits entirely on one side of $D' = D' \times \{0\}$, so we can assume without loss of generality that $h(D) \cap (D' \times (-1,0)) = \varnothing$.

Then $h(D)$ can be thought of as a properly immersed disk in the ball

$$B = S^3 \setminus (\operatorname{int} D' \times (-1,0)).$$

Meanwhile, the surface $h(M \setminus \operatorname{int} D)$ will have one boundary component immersed along $h(\alpha)$, which can be pushed slightly off of A into the interior of $E(K) \setminus U$. The disk $h(D) \subset B$ can be reglued to the newly repositioned boundary of $h(M \setminus \operatorname{int} D)$, and by shrinking the ball B down sufficiently we can assume that it is contained entirely in the interior of $E(K) \setminus U$. The resulting immersion will have one less inessential loop intersection with the annulus A. By removing all such inessential loop intersections, we are left with only with loops in $h^{-1}(A)$ that are isotopic to $\mu \subset M$. □

We thus can assume that $h^{-1}(A)$ consists only of a finite collection of parallel curves $\mu_0, \dots, \mu_k$, all of which are isotopic to $\mu \subset M$. Suppose that μ_0 is the innermost of the curves in $h^{-1}(A)$.

Lemma 14 *The curve $h(\mu_0)$ is homotopic to the core κ of the cabling annulus A.*

Proof We first note that since μ_0 is nontrivial in $\pi_1(M)$ and h is essential, $h(\mu_0)$ will be nontrivial in $\pi_1(A)$. Thus $h(\mu_0)$ is homotopic to some nonzero power of κ, say κ^b, with $b \neq 0$.

Notice now that $h(\partial M)$ is homotopic in $\partial E(K)$ to either of the components of $\partial A \subset \partial E(K)$, both of which are in turn homotopic in A to κ. Hence $h(\partial M)$ is homotopic to κ in $\pi_1(E(K))$.

On the other hand, ∂M and μ_0 bound an annulus A_0 in M, and hence $h(\partial M)$ and $h(\mu_0)$ are freely homotopic in $E(K)$ via $h(A_0)$. This implies that $[h(\partial M)] = [h(\mu_0)]$ in $H_1(E(K)) \cong \mathbb{Z}$, and hence that $[\kappa] = b[\kappa]$ in homology. Because the slopes of ∂A and $h(\partial M)$ in $\partial E(K)$ agree, we can compute the boundary slope of $h(\partial M)$ with respect to the Seifert-framed longitude of K, to see that its framing coefficient is $pq \neq 0$. Hence we see that $[h(\partial M)] = [\kappa]$ is nonzero in $H_1(E(K))$. Thus $b = 1$, which completes the proof. □

Proof of Proposition 10 Let M_0 denote the subsurface of M bounded by μ_0, which will also be a Möbius band. Let $h_0\colon M_0 \to E(K)$ denote the restriction of h to M_0. Note that $h_0(\partial M_0)$ will be an immersed curve in the cabling annulus A which is homotopic to the core κ. Furthermore, as μ_0 was the innermost curve in $h^{-1}(A)$, $h_0(M_0)$ will be contained entirely inside either $E(K) \cap V$ or $E(K) \cap W$.

Choose a homotopy of h_0 which is supported in a small neighborhood of ∂M_0, and which first straightens out $h_0(\partial M_0)$ to the embedded core κ, and then pushes it along A towards one of its boundary components, and finally onto $\partial E(K)$. If $h_0(M_0) \subset E(K) \cap V$, then we push $h_0(\partial M_0)$ onto $\partial E(K) \cap \operatorname{int} V$, while if $h_0(M_0) \subset E(K) \cap W$ then we push $h_0(\partial M_0)$ onto $\partial E(K) \cap \operatorname{int} W$.

In the latter case, we obtain a proper map $h_0\colon M_0 \to E(K)$ whose image is contained entirely outside of $E(K) \cap V$. Moreover, h_0 is essential by Theorem 5, and hence by Lemma 11 it follows that p must be even.

Suppose then that $h_0(M_0) \subset E(K) \cap V$. Take the solid torus $E(K) \cap V \cong V$, and perform an inverse satellite operation, embedding it in S^3 as the standardly embedded solid torus V'. In doing so we choose this embedding so that the longitude coming from the Seifert framing on V is identified with the longitude from the Seifert framing on V', though this will not be necessary. Using this choice of identification we obtain a map

$h'\colon M_0 \to V'$, with $h'(\partial M_0)$ embedded on $\partial V'$ as a (p,q)–torus knot K'. Thinking instead of h' as a map to $E(K')$, it follows immediately that h' is essential.

Now, suppose that p is odd. If we cut the torus V' and apply k twists before regluing, we obtain the solid torus V' again, while the knot K' is transformed into a $(p, q+kp)$–torus knot K'' lying on $\partial V'$. As p is odd, we can choose k so that $q+kp$ is also odd. However, since the image of h' is contained entirely in V', we can twist the map h' as well to obtain a new essential map $h''\colon M_0 \to E(K'')$, with $h''(\partial M_0)$ a longitude on $\partial E(K'')$. By Lemma 9 then either p or $q+kp$ must be even, which is a contradiction. Hence p must be even. □

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