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Algebraic & Geometric Topology 20 (2020) 1235–1277

Towards the $K(2)$–local homotopy groups of Z

PRASIT BHATTACHARYA
PHILIP EGGER

Previously (Adv. Math. 360 (2020) art. id. 106895), we introduced a class $\widetilde{\mathcal{Z}}$ of 2–local finite spectra and showed that all spectra $Z \in \widetilde{\mathcal{Z}}$ admit a v_2–self-map of periodicity 1. The aim here is to compute the $K(2)$–local homotopy groups $\pi_* L_{K(2)} Z$ of all spectra $Z \in \widetilde{\mathcal{Z}}$ using a homotopy fixed point spectral sequence, and we give an almost complete answer. The incompleteness lies in the fact that we are unable to eliminate one family of d_3–differentials and a few potential hidden 2–extensions, though we conjecture that all these differentials and hidden extensions are trivial.

55N20, 55Q10, 55Q51

1 Introduction

We recently (see [6]) introduced the class of all finite 2–local type 2 spectra Z such that there is an isomorphism

$$H_* Z \cong A(2) /\!/ E(Q_2)$$

of $A(2)$–modules, where $A(2)$ is the subalgebra of the Steenrod algebra generated by Sq^1, Sq^2 and Sq^4. We denote this class by $\widetilde{\mathcal{Z}}$. Let $K(n)$ denote the height n Morava K–theory and $k(n)$ its connective cover. Let tmf denote the connective spectrum of topological modular forms. The two key features of $\widetilde{\mathcal{Z}}$ (see [6] for details) are as follows:

- Every $Z \in \widetilde{\mathcal{Z}}$ admits a self-map $v\colon \Sigma^6 Z \to Z$ which induces multiplication by v_2^1 on $K(2)_*$–homology of Z, ie Z admits a v_2^1–self-map.
- Every $Z \in \widetilde{\mathcal{Z}}$ satisfies $\mathrm{tmf} \wedge Z \simeq k(2)$.

The purpose of this paper is to compute the $K(2)$–local homotopy groups of any $Z \in \widetilde{\mathcal{Z}}$.

It is difficult to overestimate the importance of $K(n)$–local computations in stable homotopy theory. At every prime p, the homotopy groups of $L_{K(1)} S^0$ have been known to capture the patterns in chromatic layer 1 of the stable homotopy groups

Published: 27 May 2020 DOI: 10.2140/agt.2020.20.1235

of spheres (also known as the image of J) since work of Adams [1]. Likewise, the chromatic fracture square, the chromatic convergence theorem of Ravenel [22, Theorem 7.5.7], as well as the nilpotence and periodicity theorems of Hopkins and Smith [13, Theorems 3 and 9], suggest that the $K(n)$–local homotopy groups of S^0 or other finite spectra encapsulate information about the patterns in the n^{th} chromatic layer of the stable homotopy groups of spheres.

However, our motivation to compute the $K(2)$–local homotopy groups of Z comes from its relevance to the telescope conjecture due to Ravenel [20, Section 10]. One of the various formulations of the telescope conjecture is as follows. Let X be a p–local type n spectrum. By [13, Theorem 9], X admits a v_n–self-map $v\colon \Sigma^t X \to X$, ie a self-map such that $K(n)_* v$ is an isomorphism. Then the homotopy groups of the telescope of X,

$$T(X) := \underrightarrow{\text{hocolim}}(X \xrightarrow{v} \Sigma^{-t} X \xrightarrow{v} \Sigma^{-2t} X \xrightarrow{v} \cdots),$$

are the v_n–inverted homotopy groups of X, ie $\pi_* T(X) = v_n^{-1}\pi_* X$. Since $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, the localization of a spectrum with respect to $K(n)$ can be thought of as, roughly speaking, another way of "inverting v_n" in the homotopy groups of X. Moreover, there is always a natural map

$$\iota\colon T(X) \to L_{K(n)} X.$$

Telescope conjecture (Ravenel) *For every type n spectrum X, the map ι is a weak equivalence.*

It follows from the thick subcategory theorem [13, Theorem 7] that if the telescope conjecture is true for one p–local type n finite spectrum then it is true for all p–local type n finite spectra (see [22]). For chromatic height $n=1$, the telescope conjecture was proved by Miller [17, Theorem 4.11] using the mod p Moore spectrum $M_p(1)$ when $p > 2$, and by Mahowald [16, Theorem 1.0] using the bo–resolution [15, Theorem 2.4] of the finite spectrum $Y := M_2(1) \wedge C\eta$ when $p = 2$. While the telescope conjecture is true for $n \le 1$ at every prime, it remains an open question for all other pairs (n, p).

We claim that in the case $n = 2$ and $p = 2$, the 2–local type 2 spectra $Z \in \widetilde{\mathcal{Z}}$ are the most appropriate ones to consider in our study of the telescope conjecture. Firstly, they all admit a v_2^1–self-map, whereas other type 2 spectra with known v_2–periodicity, such as $M(1,4)$ and the A_1 spectra, only admit v_2^{32}–self-maps, as shown, respectively, by Behrens, Hill, Hopkins and Mahowald [5, Theorem 1.1] and by Bhattacharya, Egger and Mahowald [7, Main Theorem]. Lower periodicity is desirable for computational reasons. Moreover, the fact that $\mathrm{tmf}\wedge Z \simeq k(2)$ makes the E_1–page of the tmf-based Adams spectral sequence readily computable. Also, the $Z \in \widetilde{\mathcal{Z}}$ are in many ways the "correct" height 2 analogue of Y (the spectrum used in the proof of the telescope conjecture at chromatic height 1 at the prime 2). This is because Y is a type 1 spectrum which satisfies properties analogous to Z; ie it admits a v_1^1–self-map, as shown by Davis and Mahowald [10, Theorem 1.2], and satisfies $bo\wedge Y \simeq k(1)$. We will further strengthen our claim by giving an almost complete computation of the $K(2)$–local homotopy groups of any $Z \in \widetilde{\mathcal{Z}}$, which is the "easier side" of the telescope conjecture because of its computational accessibility.

We will use a homotopy fixed point spectral sequence

$$E_2^{s,t} := H^s(\mathbb{G}_2; (E_2)_t(-)) \Rightarrow \pi_{t-s} L_{K(2)}(-),$$

of which (2-8) and (4-1) are consequences. This spectral sequence can be derived from the work of Morava [18] and Devinatz and Hopkins [12]. We will give further details in Section 2.

To compute the homotopy fixed point spectral sequence, we need to understand the action of the big Morava stabilizer group $\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on $(E_2)_*Z$, where $\mathbb{S}_2$ is the small Morava stabilizer group (see Section 2 for details). This action can be understood by explicitly analyzing the BP_*BP–comodule structure on BP_*Z via the map

$$\phi\colon BP_*BP \to \mathrm{Hom}^c(\mathbb{S}_2, (E_2)_*Z)$$

due to Devinatz and Hopkins [11]. The real hard work in this paper is to compute the BP_*BP–comodule structure on BP_*Z and obtain the action of $\mathbb{S}_2$ on $(E_2)_*Z$ via the map ϕ. The group $\mathbb{S}_2$ has a finite quaternion subgroup Q_8 (to be described in Section 4) and the pivotal result of this paper is Theorem 4.5, where we prove that there is an isomorphism

$$(E_2)_0 Z \cong \mathbb{F}_4[Q_8]$$

of modules over the group ring $\mathbb{F}_4[Q_8]$. Part of the proof of Theorem 4.5 is a nontrivial exercise in representation theory, which we have banished to Lemma A.1 in the appendix in order to avoid distracting from the main mathematical issues at hand. Theorem 4.5 provides another point of comparison between Y and Z; note that $\mathbb{G}_1 = \mathbb{Z}_2^\times \cong \mathbb{Z}/2\times\mathbb{Z}_2$, and it can easily be seen that

$$(E_1)_0 Y \cong \mathbb{F}_2[\mathbb{Z}/2].$$

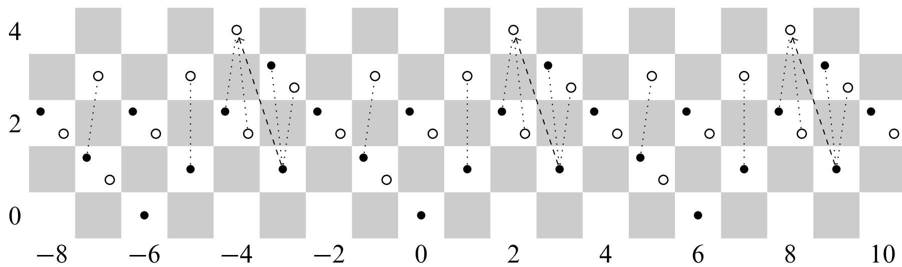


Figure 1: The E_2–page of (4-1) with possible differentials and hidden extensions.

In Section 5, we run the algebraic duality resolution spectral sequence, a convenient tool to compute the group cohomology with coefficients in $(E_2)_*Z$. Finally, in Section 6 we compute the E_2–page of (4-1). We locate two possible families of v_2–linear d_3–differentials and several possible hidden extensions. Using the inclusion $S^0 \hookrightarrow Z$ of the bottom cell, we are able to eliminate one of the two v_2–linear d_3–differentials and some of the possible hidden extensions.

Summary of results

In Figure 1, we summarize all possibilities for $\pi_* L_{K(2)} Z$ from the work in this paper. Figure 1 is a part of the homotopy fixed point spectral sequence (4-1), where we represent possible d_3–differentials using dashed arrows and hidden extensions by dotted lines. Any generator which is a multiple of a specific element ζ in the E_2–page (to be discussed in Section 6) is displayed using a $\circ$, otherwise using a $\bullet$. Since the homotopy groups of $L_{K(2)}Z$ are periodic with respect to multiplication by v_2^1, which has bidegree $(s, t-s) = (0, 6)$, the different possible answers can be read off from the portion $0 \le t-s \le 5$.

In Beaudry, Behrens, Bhattacharya, Culver and Xu [4], the tmf–resolution for one particular model of $Z \in \widetilde{\mathcal{Z}}$ is studied to compute its unlocalized homotopy groups. This computation shows that the potential d_3–differentials and hidden extensions as indicated in Figure 1 are trivial, giving us a complete computation of the $K(2)$–local homotopy groups of that particular spectrum Z. We expect the same thing to happen for every spectrum $Z \in \widetilde{\mathcal{Z}}$.

Conjecture 1 *For every $Z \in \widetilde{\mathcal{Z}}$, the $K(2)$–local homotopy groups of Z are given by*

$$\pi_* L_{K(2)} Z \cong \mathbb{F}_2[v_2^{\pm 1}] \otimes E(a_1, a_3, a_5, \zeta),$$

where $|a_i| = i$, $|\zeta| = -1$ and $|v_2| = 6$.

The spectrum Z in the unpublished work mentioned above would be the first finite 2–local spectrum for which we have complete knowledge of its $K(2)$–local homotopy groups. It can be built using iterated cofiber sequences of five different self-maps (see [6]) starting from S^0. Thus, one could work backwards from $\pi_* L_{K(2)} Z$, using Bockstein spectral sequences iteratively to get information about $\pi_* L_{K(2)} S^0$.

Organization of the paper

The results in this paper are independent of the choice of $Z \in \widetilde{\mathcal{Z}}$, and hence Z will refer to an arbitrary spectrum $Z \in \widetilde{\mathcal{Z}}$ for the rest of the paper.

We devote Section 2 to recalling some fundamental results which connect the theory of formal group laws to homotopy theory.

In Section 3 we compute the BP_*BP–comodule structure of BP_*Z.

In Section 4, we briefly recall some of the details of the height 2 Morava stabilizer group $\mathbb{S}_2$ and compute the action of $\mathbb{S}_2$ on the generators of $(E_2)_*Z$.

In Section 5, we compute the group cohomology with coefficients in $(E_2)_*Z$ using the duality spectral sequence as well as a result of Henn, reported by Beaudry [3].

In Section 6, we analyze the homotopy fixed point spectral sequence for Z and eliminate one of the two possible $\mathbb{F}_2[v_2^{\pm 1}]$–linear families of d_3–differentials and some of the possible hidden extensions.

In the appendix, we include the representation theory exercise omitted from the proof of Theorem 4.5.

Acknowledgements

The authors would like to thank Agnès Beaudry and Mark Behrens for their helpful comments and suggestions. We also thank Paul Goerss for his tireless work advising Egger's PhD thesis, which was the starting point of this project. The idea of understanding eight-dimensional Q_8–representations, as expressed in the appendix, stemmed from a discussion with Noah Snyder. We are also thankful to the referee for identifying some issues in earlier drafts and providing many helpful details and suggestions for improvement.

2 Formal group laws and homotopy theory

The theory of formal group laws was developed by number theorists and eventually found by Lazard and Quillen to have deep relations with homotopy theory. We will review these relations, primarily following Lubin and Tate [14] and Ravenel [21]. We will conclude with a formula relating the action of the Morava stabilizer group on a Morava module to the structure of a corresponding BP_*BP–comodule.

Definition 2.1 Let R be a $\mathbb{Z}_{(p)}$–algebra. A *formal group law* over R is a power series $F(x,y) \in R[\![x,y]\!]$ satisfying

- $F(x,y) = F(y,x)$,
- $x = F(x,0)$,
- $F(F(x,y),z) = F(x,F(y,z))$.

When R is a graded $\mathbb{Z}_{(p)}$–algebra we set $|x| = |y| = -2$ and we require that $F(x,y)$ be a homogeneous expression in degree -2.

Definition 2.2 Given formal group laws F and G over R, a *homomorphism* from F to G is a power series $f \in R[\![x]\!]$ such that $f(0) = 0$ and

$$f(F(x,y)) = G(f(x), f(y)).$$

A homomorphism f is an *isomorphism* if $f'(0)$ is a unit in R, and an isomorphism f is said to be *strict* if $f'(0) = 1$. A strict isomorphism from F to the additive formal group law is called a *logarithm* of F.

Notation 2.3 We will often use the notation $x +_F y$ to denote $F(x,y)$ and $[n]_F(x)$ to denote the n–fold sum $x +_F \cdots +_F x$. We will denote the set of formal group laws over R by $\mathrm{FGL}(R)$, and the groupoid of formal group laws over R with strict isomorphisms by $(\mathrm{FGL}(R), \mathrm{SI}(R))$. When R is torsion-free, then the image of F in $(R \otimes \mathbb{Q})[\![x,y]\!]$ has a logarithm, which we will denote by $\log_F \in (R \otimes \mathbb{Q})[\![x]\!]$.

Definition 2.4 Let R be a torsion-free $\mathbb{Z}_{(p)}$–algebra and let F be a formal group law over R. Then F is called *p–typical* if its logarithm is

$$\log_F(x) = \sum_{i \geq 0} l_i x^{p^i}$$

with $l_0 = 1$.

Now we recall the p–local analogue of the famous theorem of Lazard and Quillen [19, Theorem 2]. All formal groups discussed will be assumed to be p–typical unless otherwise stated.

The assignment of a $\mathbb{Z}_{(p)}$–algebra R to the set $\mathrm{FGL}(R)$ is functorial, and we denote this functor by

$$\mathrm{FGL}(-)\colon \mathbb{Z}_{(p)}\textbf{–algebra} \to \textbf{Set}.$$

Similarly, the functor which assigns a graded $\mathbb{Z}_{(p)}$–algebra R_* to the set of homogeneous formal group laws over R_* of degree -2 is denoted by

$$\overline{\mathrm{FGL}}(-)\colon \textbf{Graded } \mathbb{Z}_{(p)}\textbf{–algebra} \to \textbf{Set}.$$

Theorem 2.5 (Cartier, Lazard and Quillen) *The covariant functor* $\mathrm{FGL}(-)$ *defined on the category of* $\mathbb{Z}_{(p)}$*–algebras is represented by the* $\mathbb{Z}_{(p)}$*–algebra*

$$\widetilde{V} = \mathbb{Z}_{(p)}[\tilde{v}_1, \tilde{v}_2, \dots],$$

ie $\mathrm{FGL}(R) \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\widetilde{V}, R)$. *The covariant functor* $\overline{\mathrm{FGL}}(-)$ *defined on the category of graded* $\mathbb{Z}_{(p)}$*–algebras is represented by the graded* $\mathbb{Z}_{(p)}$*–algebra*

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

with $|v_i| = 2(p^i - 1)$, *ie* $\overline{\mathrm{FGL}}(R_*) \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(BP_*, R_*)$, *where* $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(BP_*, R_*)$ *is the set of graded* $\mathbb{Z}_{(p)}$*–algebra maps from* BP_* *to* R_*.

Example 2.6 (Honda formal group law) Defining the ring homomorphism

$$\phi_n\colon \widetilde{V} \to \mathbb{F}_{p^n}, \qquad \tilde{v}_i \mapsto \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{if } i \neq n, \end{cases}$$

gives the *Honda formal group law* Γ_n over $\mathbb{F}_{p^n}$. This formal group law satisfies

$$[p]_{\Gamma_n}(x) = x^{p^n}.$$

A theorem of Lazard says that Γ_n is unique in that every formal group law of height n over a separably closed field of characteristic p is isomorphic to Γ_n, though this isomorphism might not be strict.

Remark 2.7 The generators $\tilde{v}_i \in \widetilde{V}$ are defined by the property that

$$[p]_{\mathcal{F}_{\widetilde{V}}}(x) = px +_{\mathcal{F}_{\widetilde{V}}} \sum_{i>1}^{\mathcal{F}_{\widetilde{V}}} \tilde{v}_i x^{p^i},$$

where $\mathcal{F}_{\widetilde{V}}$ is the universal p–typical formal group law over $\widetilde{V}$. Similarly, the $v_i \in BP_*$ are defined by the property that

$$[p]_{\mathcal{F}_{BP_*}}(\bar{x}) = p\bar{x} +_{\mathcal{F}_{BP_*}} \sum_{i>0}^{\mathcal{F}_{BP_*}} v_i \bar{x}^{p^i},$$

where $\mathcal{F}_{BP_*}$ is the universal p–typical formal group law over BP_* and $|\bar{x}| = -2$. The generators $\{\tilde{v}_i : i > 0\}$ and $\{v_i : i > 0\}$ are often called the *Araki generators* in the literature.

Consider the functor

$$\rho\colon \mathbb{Z}_{(p)}\textbf{–algebra} \to \textbf{Graded } \mathbb{Z}_{(p)}\textbf{–algebra}$$

which sends $R \mapsto R[u^{\pm 1}]$, where u is a formal variable in degree -2. If F is a formal group law over R, then

$$\bar{F}(\bar{x}, \bar{y}) := uF(u^{-1}\bar{x}, u^{-1}\bar{y}),$$

where $|\bar{x}| = |\bar{y}| = -2$, is a formal group law over $R[u^{\pm 1}]$. Mapping $F \mapsto \bar{F}$ defines a natural transformation between the functors $\mathrm{FGL}(-)$ and $\overline{\mathrm{FGL}}(-) \circ \rho$. Since $\bar{\mathcal{F}}_{\widetilde{V}}$ is a formal group law over the graded ring $\widetilde{V}[u^{\pm 1}]$, we obtain a map

$$\theta\colon BP_* \to \widetilde{V}[u^{\pm 1}] \tag{2-1}$$

and it follows from comparing the p–series (see Remark 2.7) that $\theta(v_i) = u^{1-p^i}\tilde{v}_i$.

We can also ask about how to represent *groupoids* of formal group laws. We can do this in two ways, either by considering the groupoid of formal group laws with isomorphisms, or the smaller groupoid of formal group laws with strict isomorphisms.

Lemma 2.8 *Let F be a p–typical formal group law and let G be an arbitrary formal group law over a $\mathbb{Z}_{(p)}$–algebra R, and let f be an isomorphism from F to G. Then G is p–typical if and only if*

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i},$$

where $t_i \in R$ for every i and $t_0 \in R^\times$.

If we want f to be a strict isomorphism, then we must have $t_0 = 1$. In the context of graded $\mathbb{Z}_{(p)}$–algebras, t_i is forced to be in degree $2(p^i - 1)$. Thus we can define a Hopf algebroid (BP_*, BP_*BP) with

$$BP_*BP = BP_*[t_1, t_2, \dots : |t_i| = 2(p^i - 1)],$$

which represents the functor

$$(\overline{\mathrm{FGL}}(-), \overline{\mathrm{SI}}(-))\colon \textbf{Graded } \mathbb{Z}_{(p)}\textbf{–algebra} \to \textbf{Groupoid}$$

which assigns a graded $\mathbb{Z}_{(p)}$–algebra R_* to the groupoid of p–typical formal group laws over R_* with strict isomorphisms. Let $\eta_L, \eta_R\colon BP_* \to BP_*BP$ denote the left and the right units of the Hopf algebroid BP_*BP. Note that the universal isomorphism $f\colon \eta_L^*\mathcal{F}_{BP_*} = \mathcal{F}_{BP_*} \to \eta_R^*\mathcal{F}_{BP_*}$ satisfies the formula

$$f^{-1}(\overline{x}) = \overline{x} +_{\mathcal{F}_{BP_*}} \sum_{i\geq 1}^{\mathcal{F}_{BP_*}} t_i \overline{x}^{p^i},$$

where $|\overline{x}| = -2$.

Similarly, one can consider the case where R is ungraded and f is an isomorphism that need not be strict. Thus we define

$$\widetilde{VT} = \widetilde{V}[\tilde{t}_0^{\pm 1}, \tilde{t}_1, \tilde{t}_2, \dots : |\tilde{t}_i| = 0],$$

getting a Hopf algebroid $(\widetilde{V}, \widetilde{VT})$, which represents the functor

$$(\mathrm{FGL}(-), I(-))\colon \mathbb{Z}_{(p)}\textbf{–algebra} \to \textbf{Groupoid}$$

which assigns a $\mathbb{Z}_{(p)}$–algebra R to the groupoid of p–typical formal group laws over R with isomorphisms. In this case the universal isomorphism $\tilde{f}\colon \eta_L^*\mathcal{F}_{\widetilde{V}} = \mathcal{F}_{\widetilde{V}} \to \eta_R^*\mathcal{F}_{\widetilde{V}}$ satisfies the formula

$$\tilde{f}^{-1}(x) = \sum_{i\geq 0}^{\mathcal{F}_{\widetilde{V}}} \tilde{t}_i x^{p^i}.$$

Let us define

$$\begin{aligned} \overline{\mathcal{F}}_{\widetilde{V}}(\overline{x}, \overline{y}) &= u\mathcal{F}_{\widetilde{V}}(u^{-1}\overline{x}, u^{-1}\overline{y}), \\ \widehat{G}(\overline{x}, \overline{y}) &= \tilde{t}_0 u \eta_R^* \mathcal{F}_{\widetilde{V}}(\tilde{t}_0^{-1}u^{-1}\overline{x}, \tilde{t}_0^{-1}u^{-1}\overline{y}), \\ \hat{f}(\overline{x}) &= \tilde{t}_0 u \tilde{f}(u^{-1}\overline{x}), \end{aligned}$$

where $|\overline{x}| = |\overline{y}| = -2$. It is easy to see that the triple $(\overline{\mathcal{F}}_{\widetilde{V}}, \hat{f}, \widehat{G})$ is an element of the groupoid $(\mathrm{FGL}(\widetilde{VT}[u^{\pm 1}]), \mathrm{SI}(\widetilde{VT}[u^{\pm 1}]))$. Hence the map θ of (2-1) can be extended to a left BP_*–linear map

$$\theta\colon BP_*BP \to \widetilde{VT}[u^{\pm 1}]. \tag{2-2}$$

Since

$$\hat{f}^{-1}(\overline{x}) = u\tilde{f}^{-1}(\tilde{t}_0^{-1}u^{-1}\overline{x}) = u\biggl(\sum_{i\geq 0}^{\mathcal{F}_{\widetilde{V}}} \tilde{t}_i \tilde{t}_0^{-p^i} u^{-p^i} \overline{x}^{p^i}\biggr) = \sum_{i\geq 0}^{\overline{\mathcal{F}}_{\widetilde{V}}} \tilde{t}_i \tilde{t}_0^{-p^i} u^{1-p^i} \overline{x}^{p^i}$$

and

$$\hat{f}^{-1}(\bar{x}) = \theta(\tilde{f}^{-1}(x)),$$

we get that the map θ in (2-2) satisfies

$$\theta(t_i) = \tilde{t}_i \tilde{t}_0^{-p^i} u^{1-p^i}. \tag{2-3}$$

Now we briefly recall the notion of deformation, which arose in number theory and has important implications for homotopy theory.

Definition 2.9 Let k be a field of characteristic $p > 0$ and Γ a formal group law over k. A *deformation* of (k, Γ) to a complete local ring B with projection

$$\pi\colon B \to B/\mathfrak{m}$$

is a pair (G, i) where G is a formal group law over B and

$$i\colon k \to B/\mathfrak{m}$$

is a homomorphism satisfying $i\Gamma = \pi G$.

A morphism from $(G_1, i_1) \to (G_2, i_2)$ is defined only when $i_1 = i_2$, in which case it consists of an isomorphism

$$f\colon G_1 \to G_2$$

of formal group laws over B such that

$$f(x) \equiv x \bmod \mathfrak{m}.$$

Such morphisms are also called $\star$–isomorphisms. Note that the set $\mathrm{Def}_\Gamma(B)$ of deformations of (k, Γ) to B with $\star$–isomorphisms forms a groupoid. The work of Lubin and Tate [14, Theorem 3.1] guarantees the existence of a universal deformation. More precisely:

Theorem 2.10 (Lubin and Tate) *Let Γ be a formal group law of finite height over a field k of characteristic $p > 0$. Then there exists a complete local ring $E(k, \Gamma)$ with residue field k and a deformation $(F_\Gamma, \mathrm{id}) \in \mathrm{Def}_\Gamma(E(k, \Gamma))$ such that for every $(G, i) \in \mathrm{Def}_\Gamma(B)$, there is a unique continuous[1] ring homomorphism $\theta\colon E(k, \Gamma) \to B$ and a unique $\star$–isomorphism from (G, i) to $(\theta F_\Gamma, i)$.*

[1] A ring homomorphism of local ring is continuous if the image of the maximal ideal of the domain is contained in the maximal ideal of the codomain.

Remark 2.11 It is well known (see [14]) that if k is a perfect field and Γ has height n, then a choice of F_Γ determines an isomorphism

$$E(k,\Gamma) \cong W(k)[\![u_1,\dots,u_{n-1}]\!]$$

of complete local rings, where $W(k)$ is the ring of Witt vectors on k.

The automorphism group $\mathrm{Aut}(\Gamma/k)$ of Γ acts on $E(k,\Gamma)$ as follows (also see [11, Section 1]). Let $\gamma \in k[\![x]\!]$ be an invertible power series. Choose an invertible power series $\tilde{\gamma} \in E(k,\Gamma)[\![x]\!]$ as a lift of γ and define $\widetilde{F}_\gamma$ over $E(k,\Gamma)$ by

$$\widetilde{F}_\gamma(x,y) := \tilde{\gamma}^{-1}\big(F_\Gamma(\tilde{\gamma}(x),\tilde{\gamma}(y))\big).$$

Note that the lift $\widetilde{F}_\gamma$ depends on the choice of lift $\tilde{\gamma}$. Since $(\widetilde{F}_\gamma,\mathrm{id}) \in \mathrm{Def}_\Gamma(E(k,\Gamma))$, the Lubin–Tate theorem gives us a unique homomorphism

$$\tilde{\phi}_\gamma\colon E(k,\Gamma) \to E(k,\Gamma)$$

and a unique $\star$–isomorphism

$$\hat{\gamma}\colon \widetilde{F}_\gamma \to \tilde{\phi}_\gamma F_\Gamma.$$

The composite

$$f_\gamma\colon F_\Gamma \xrightarrow{\tilde{\gamma}^{-1}} \widetilde{F}_\gamma \xrightarrow{\hat{\gamma}} \tilde{\phi}_\gamma F_\Gamma$$

does not depend on the choice of $\tilde{\gamma}$ and is an element of the groupoid

$$\big(\mathrm{FGL}(E(k,\Gamma)), I(E(k,\Gamma))\big).$$

Therefore the classifying map for F_Γ,

$$\tilde{\theta}_\Gamma\colon \widetilde{V} \to E(k,\Gamma),$$

can be extended to a left $\widetilde{V}$–linear map

$$\tilde{\theta}_\Gamma\colon \widetilde{VT} \to \mathrm{Map}^c(\mathrm{Aut}(\Gamma/k), E(k,\Gamma)).$$

Let us simply denote $\theta_\Gamma(\tilde{t}_i)(\gamma)$ by $\tilde{t}_i(\gamma)$ for $\gamma \in \mathrm{Aut}(\Gamma/k)$. The elements $\tilde{t}_i(\gamma)$ satisfy the equation

$$f_\gamma^{-1}(x) = \sum_{i\ge 0}^{F_\Gamma} \tilde{t}_i(\gamma) x^{p^i}.$$

One can also consider the graded formal group law $\overline{\Gamma}$ over $k[u^{\pm 1}]$. Note that $\mathrm{Aut}(\Gamma/k) \cong \mathrm{Aut}(\overline{\Gamma}/k[u^{\pm 1}])$ via the invertible map $\gamma(-) \mapsto u\gamma(u^{-1}-)$. One can similarly define the graded universal deformation formal group law $\overline{F}_\Gamma$ over the graded ring $E(k,\Gamma)[u^{\pm 1}]$. Let $\gamma \in \mathrm{Aut}(\Gamma/k)$ act on $E(k,\Gamma)[u^{\pm 1}]$ via the ring homomorphism $\phi_\gamma\colon E(k,\Gamma)[u^{\pm 1}] \to E(k,\Gamma)[u^{\pm 1}]$ such that

$$\phi_\gamma(x) = \begin{cases} \tilde{\phi}_\gamma(x) & \text{if } x \in E(k,\Gamma), \\ \tilde{t}_0(\gamma)x & \text{if } x = u. \end{cases}$$

Notice that

$$\phi_\gamma \bar{F}_\Gamma(\bar{x},\bar{y}) = \tilde{t}_0(\gamma)u\tilde{\phi}_\gamma F_\Gamma(\tilde{t}_0(\gamma)^{-1}u^{-1}\bar{x}, \tilde{t}_0(\gamma)^{-1}u^{-1}\bar{y})$$

and

$$\hat{f}_\gamma(\bar{x}) = \tilde{t}_0(\gamma)uf_\gamma(u^{-1}\bar{x})$$

is a strict isomorphism between $\bar{F}_\Gamma$ and $\phi_\gamma \bar{F}_\Gamma$. Thus we have a left BP_*–linear map

(2-4) $$\theta_{\bar{\Gamma}}\colon BP_*BP \to \mathrm{Map}^c(\mathrm{Aut}(\Gamma/k), E(k,\Gamma)[u^{\pm 1}]).$$

It can be easily checked that $\theta_{\bar{\Gamma}}$ is identical to the composite map

$$BP_*BP \xrightarrow{\theta} \widetilde{VT}[u^{\pm 1}] \xrightarrow{\tilde{\theta}_\Gamma[u^{\pm 1}]} \mathrm{Map}^c(\mathrm{Aut}(\Gamma/k), E(k,\Gamma)[u^{\pm 1}]).$$

Let us denote the map $\theta_{\bar{\Gamma}}(t_i)(-)$ simply by $t_i(-)$. It follows from (2-3) that

(2-5) $$t_i(\gamma) = \tilde{t}_i(\gamma)\tilde{t}_0(\gamma)^{-p^i}u^{1-p^i}$$

for $\gamma \in \mathrm{Aut}(\Gamma/k)$. Also keep in mind that f_γ fits into the commutative diagram

$$\begin{array}{ccc} F_\Gamma & \xrightarrow{f_\gamma} & \tilde{\phi}_\gamma F_\Gamma \\ \wr\downarrow & & \wr\downarrow \\ \Gamma & \xrightarrow[\gamma]{} & \Gamma \end{array}$$

where the vertical squiggly arrows are reduction modulo $m = (p, u_1, \dots, u_{n-1})$. Thus, for $\gamma^{-1} = a_0x +_\Gamma a_1x^p +_\Gamma +a_2x^{p^2} +_\Gamma \cdots \in k[\![x]\!]$, we have

(2-6) $$\tilde{t}_i(\gamma) \equiv a_i \quad \text{and} \quad t_i(\gamma) \equiv a_ia_0^{-p^i}u^{1-p^i} \quad \mod m.$$

It follows from [21, Corollary 4.3.15] that, when Γ has height n and $k \le n$,

(2-7) $$\tilde{t}_k(\gamma_1\gamma_2) \equiv \sum_{i=0}^{k} \tilde{t}_i(\gamma_1)\tilde{t}_{k-i}(\gamma_2)^{p^i} \mod m.$$

Now let's focus on the Honda formal group law Γ_n over $\mathbb{F}_{p^n}$ and let F_n denote its universal deformation. By Remark 2.11, we have

$$E(\mathbb{F}_{p^n}, \Gamma_n) = W(\mathbb{F}_{p^n})[\![u_1, \dots, u_{n-1}]\!],$$

where $W(\mathbb{F}_{p^n})$ are the Witt vectors of $\mathbb{F}_{p^n}$, which has an action of the *small Morava stabilizer group*

$$\mathbb{S}_n := \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}).$$

Note that the map ϕ_n of Example 2.6 which defines the Honda formal group law factors through $\mathbb{F}_p$. Therefore Γ_n has coefficients in $\mathbb{F}_p$. Consequently, the *big Morava stabilizer group*

$$\mathbb{G}_n := \mathrm{Aut}(\Gamma_n/\mathbb{F}_p) = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

acts on $E(\mathbb{F}_{p^n}, \Gamma_n)$. The Lubin–Tate universal formal group law F_n over $E(\mathbb{F}_{p^n}, \Gamma_n)$ is given by the ring homomorphism

$$\theta\colon \widetilde{V} \to E(\mathbb{F}_{p^n}, \Gamma_n), \qquad \widetilde{v}_i \mapsto \begin{cases} u_i & \text{if } i < n, \\ 1 & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases}$$

which means that

$$[p]_{\Gamma_n}(x) = px +_{F_n} u_1 x^p +_{F_n} \cdots +_{F_n} u_{n-1} x^{p^{n-1}} +_{F_n} x^{p^n}.$$

We also have a graded formal group law $\overline{F}_n$ over the graded ring $(E_n)_*$, defined as $(E_n)_* := E(\mathbb{F}_{p^n}, \Gamma_n)[u^{\pm 1}]$, which is given by the ring homomorphism

$$\theta\colon BP_* \to (E_n)_*, \qquad v_i \mapsto \begin{cases} u_i u^{1-p^i} & \text{if } i < n, \\ u^{1-p^n} & \text{if } i = n, \\ 0 & \text{if } i > n. \end{cases}$$

By the Landweber exact functor theorem,

$$(E_n)_*(-) := (E_n)_* \otimes_{BP_*} BP_*(-)$$

is a homology theory, thus it is represented by a spectrum E_n, known as *Morava E–theory*. By a theorem of Hopkins and Miller, reported by Rezk [23], the action of $\mathbb{G}_n$ on $(E_n)_*$ lifts to one on E_n itself whose homotopy fixed point spectrum is

$$(E_n)^{h\mathbb{G}_n} \simeq L_{K(n)} S^0,$$

which gives us a homotopy fixed point spectral sequence

$$E_2^{s,t} := H^s(\mathbb{G}_n; (E_n)_t) \Rightarrow \pi_{t-s} L_{K(n)} S^0.$$

The E_2–page of this spectral sequence can be found using a Lyndon–Hochschild–Serre spectral sequence

$$H^{s_1}\big(\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p); H^{s_2}(\mathbb{S}_n; (E_n)_*)\big) \Rightarrow H^{s_1+s_2}(\mathbb{G}_n; (E_n)_*),$$

which as a consequence of Hilbert's Theorem 90 reduces to

$$H^s(\mathbb{G}_n;(E_n)_*) = H^s(\mathbb{S}_n;(E_n)_*)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}.$$

Thus, the spectral sequence of interest to us is

$$E_2^{s,t} := H^s(\mathbb{S}_n;(E_n)_t)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)} \Rightarrow \pi_{t-s} L_{K(n)} S^0. \tag{2-8}$$

3 The BP_*BP–comodule BP_*Z

For every $Z \in \widetilde{\mathcal{Z}}$, there is, by definition, an isomorphism

$$H_*Z \cong (A(2)/\!/E(Q_2))_* = \mathbb{F}_2[\xi_1,\xi_2]/(\xi_1^8,\xi_2^4)$$

of $A(2)_*$–comodules [6]. We will use this fact to determine the BP_*BP–comodule structure of BP_*Z. One can use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*BP \otimes H^*Z, \mathbb{F}_2) \Rightarrow BP_{t-s}Z$$

to compute BP_*Z as a BP_*–module. Note that

$$H^*BP = A/\!/E(Q_0,Q_1,Q_2,\dots),$$

where Q_i are the Milnor primitives. By a change of rings, the E_2–page of the above Adams spectral sequence is isomorphic to

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*BP \otimes H^*Z, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{E(Q_0,Q_1,\dots)}(H^*Z, \mathbb{F}_2). \tag{3-1}$$

Let g denote the generator of H_*Z in degree 0. As an $E(Q_0,Q_1,Q_2)$–module, $A(2)/\!/E(Q_2)$ is a direct sum of 8 copies of $E(Q_0,Q_1)$, generated by the elements in the set

$$\mathcal{G} = \{g^*, (\xi_1^2 g)^*, (\xi_1^4 g)^*, (\xi_1^6 g)^*, (\xi_2^2 g)^*, (\xi_1^2\xi_2^2 g)^*, (\xi_1^4\xi_2^2 g)^*, (\xi_1^6\xi_2^2 g)^*\}.$$

Since $H^*Z \cong_{A(2)} A(2) \otimes_{E(Q_2)} \mathbb{F}_2$ and Q_2 is in the center of $A(2)$, Q_2 acts trivially on H^*Z. Using the iterative formula

$$Q_i = \mathrm{Sq}^{2^i} Q_{i-1} + Q_{i-1} \mathrm{Sq}^{2^i},$$

one can inductively argue that Q_i for $i \geq 2$ acts trivially on H^*Z. Thus, we have completely determined H^*Z as a module over $E(Q_0,Q_1,\dots)$ from its $A(2)$–module structure. Thus, as an $E(Q_0,Q_1,\dots)$–module,

$$H^*Z \cong E(Q_0,Q_1,\dots) \otimes_{E(Q_2,Q_3,\dots)} \mathcal{G},$$

and therefore the E_2–page of (3-1) is isomorphic to

$$E_2^{*,*} \cong \mathbb{F}_2[v_2, v_3, \dots] \otimes \mathcal{G}^*,$$

where v_i has bidegree $(s,t) = (1, |Q_i|) = (1, 2^{i+1}-1)$. Due to sparseness, the Adams spectral sequence (3-1) collapses at the E_2–page. Hence, as a BP_*–module,

$$BP_*Z \cong BP_*/(2, v_1)\langle x_0, x_2, x_4, x_6, y_6, y_8, y_{10}, y_{12}\rangle, \tag{3-2}$$

where x_i and y_i are generators in degree i chosen in such a way that the map $BP_*Z \to H_*Z$ sends

$$\begin{aligned} &x_0 \mapsto g, && x_2 \mapsto \xi_1^2 g, && x_4 \mapsto \xi_1^4 g, && x_6 \mapsto \xi_1^6 g, \\ &y_6 \mapsto \xi_2^2 g, && y_8 \mapsto \xi_1^2\xi_2^2 g, && y_{10} \mapsto \xi_1^4\xi_2^2 g, && y_{12} \mapsto \xi_1^6\xi_2^2 g. \end{aligned}$$

This identification allows us to infer the BP_*BP–comodule structure of BP_*Z from the $A(2)_*$–comodule structure of H_*Z via the diagram

$$\begin{array}{ccc} BP_*Z & \xrightarrow{\psi} & BP_*BP \otimes_{BP_*} BP_*Z \\ \downarrow & & \downarrow \\ H_*Z & \xrightarrow[\psi_2]{} & A(2)_* \otimes H_*Z \end{array}$$

First notice that the coaction map

$$\psi_2\colon H_*Z \to A(2)_* \otimes H_*Z$$

sends

$$\begin{aligned} g &\mapsto 1|g, \\ \xi_1^2 g &\mapsto \xi_1^2|g + 1|\xi_1^2 g, \\ \xi_1^4 g &\mapsto \xi_1^4|g + 1|\xi_1^4 g, \\ \xi_1^6 g &\mapsto \xi_1^6|g + \xi_1^4|\xi_1^2 g + \xi_1^2|\xi_1^4 g + 1|\xi_1^6 g, \\ \xi_2^2 g &\mapsto \xi_2^2|g + \xi_1^4|\xi_1^2 g + 1|\xi_2^2 g, \\ \xi_1^2\xi_2^2 g &\mapsto \xi_1^2\xi_2^2|g + (\xi_1^6 + \xi_2^2)|\xi_1^2 g + \xi_1^2|\xi_2^2 g + \xi_1^4|\xi_1^4 g + 1|\xi_1^2\xi_2^2 g, \\ \xi_1^4\xi_2^2 g &\mapsto \xi_1^4\xi_2^2|g + \xi_1^8|\xi_1^2 g + \xi_1^4|\xi_2^2 g + \xi_2^2|\xi_1^4 g + \xi_1^4|\xi_1^6 g + 1|\xi_1^4\xi_2^2 g, \\ \xi_1^6\xi_2^2 g &\mapsto \xi_1^6\xi_2^2|g + (\xi_1^4\xi_2^2 + \xi_1^{10})|\xi_1^2 g + (\xi_1^2\xi_2^2 + \xi_1^8)|\xi_1^4 g + \xi_1^6|\xi_2^2 g \\ &\qquad + (\xi_2^2 + \xi_1^6)|\xi_1^6 g + \xi_1^4|\xi_1^2\xi_2^2 g + \xi_1^2|\xi_1^4\xi_2^2 g + 1|\xi_1^6\xi_2^2 g. \end{aligned} \tag{3-3}$$

The map

$$BP_*BP \to A_*$$

sends $v_i \mapsto 0$ and $t_i \mapsto \zeta_i^2$, where ζ_i is the image of ξ_i under the canonical anti-automorphism of A_*. Moreover $A(2)_* \cong A_*/(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots)$. Therefore ψ_2, along with the fact that $(2, v_1) \subset BP_*$ acts trivially on BP_*Z, completely determines the composite map

$$BP_*Z \xrightarrow{\psi} BP_*BP \otimes_{BP_*} BP_*Z \to BP_*BP/\mathcal{I}_2 \otimes_{BP_*} BP_*Z,$$

where

$$\mathcal{I}_2 = (v_2, v_3, \dots, t_1^4, t_2^2, t_3, t_4, \dots) \subset BP_*BP.$$

Note that all elements in the generating set $\{x_0, x_2, x_4, x_6, y_6, y_8, y_{10}, y_{12}\}$ of BP_*Z have internal degrees between 0 and 12, whereas $|t_j| > 12$ and $|v_j| > 12$ when $j \geq 3$. Therefore, for $j \geq 3$, t_j and v_j do not appear in the expression for $\psi(x_i)$ and $\psi(y_i)$, though v_2 may be present. Using (3-3) and the fact that $\zeta_1^2 = \xi_1^2$ and $\zeta_2^2 = \xi_2^2 + \xi_1^6$, we easily derive the coaction map ψ on the generators of BP_*Z modulo $(v_2, t_1^4, t_2^2) \in BP_*BP$. We get

$$\begin{aligned}
\psi(x_0) &= 1|x_0,\\
\psi(x_2) &= t_1|x_0 + 1|x_2,\\
\psi(x_4) &= t_1^2|x_0 + 1|x_4,\\
\psi(x_6) &\equiv t_1^3|x_0 + t_1^2|x_2 + t_1|x_4 + 1|x_6,\\
\psi(y_6) &\equiv (t_2 + t_1^3)|x_0 + t_1^2|x_2 + 1|y_6,\\
\psi(y_8) &\equiv t_1t_2|x_0 + t_2|x_2 + t_1^2|x_4 + t_1|y_6 + 1|y_8,\\
\psi(y_{10}) &\equiv t_1^2t_2|x_0 + (t_1^3 + t_2)|x_4 + t_1^2|x_6 + t_1^2|y_6 + 1|y_{10},\\
\psi(y_{12}) &\equiv t_1^3t_2|x_0 + t_1^2t_2|x_2 + t_1t_2|x_4 + t_2|x_6 + t_1^3|y_6 + t_1^2|y_8 + t_1|y_{10} + 1|y_{12}.
\end{aligned} \tag{3-4}$$

Lemma 3.1 *For any $Z \in \widetilde{\mathcal{Z}}$, BP_*Z has one of the four different BP_*BP–comodule structures given below:*

$$\begin{aligned}
\psi(x_0) &= 1|x_0,\\
\psi(x_2) &= t_1|x_0 + 1|x_2,\\
\psi(x_4) &= t_1^2|x_0 + 1|x_4,
\end{aligned}$$

$$
\begin{aligned}
\psi(x_6) &= t_1^3|x_0 + t_1^2|x_2 + t_1|x_4 + 1|x_6,\\
\psi(y_6) &= (t_2 + t_1^3)|x_0 + t_1^2|x_2 + 1|y_6,\\
\psi(y_8) &= (at_1^4 + t_1t_2)|x_0 + t_2|x_2 + t_1^2|x_4 + t_1|y_6 + 1|y_8,\\
\psi(y_{10}) &= (t_1^5 + t_1^2t_2)|x_0 + t_1^4|x_2 + (t_1^3 + t_2)|x_4 + t_1^2|x_6 + t_1^2|y_6 + 1|y_{10},\\
\psi(y_{12}) &= ((b+1)t_1^6 + t_1^3t_2 + (a+b)t_2^2)|x_0 + t_1^2t_2|x_2 + (bt_1^4 + t_1t_2)|x_4 + t_2|x_6\\
&\qquad + t_1^3|y_6 + t_1^2|y_8 + t_1|y_{10} + 1|y_{12},
\end{aligned}
$$

where $a, b \in \mathbb{F}_2$.

Proof For degree reasons, the congruences of (3-4) imply that there are coefficients

$$
\lambda_6^0, \kappa_6^0, \mu_8^0, \lambda_8^0, \lambda_8^2, \mu_{10}^0, \lambda_{10}^0, \mu_{10}^2, \lambda_{10}^2, \lambda_{10}^4,\\
\mu_{12}^0, \nu_{12}^0, \lambda_{12}^0, \kappa_{12}^0, \sigma, \mu_{12}^2, \lambda_{12}^2, \mu_{12}^4, \lambda_{12}^4, \lambda_{12}^6, \kappa_{12}^6
$$

in $\mathbb{F}_2$ such that one has

$$
\begin{aligned}
\psi(x_0) &= 1|x_0,\\
\psi(x_2) &= t_1|x_0 + 1|x_2,\\
\psi(x_4) &= t_1^2|x_0 + 1|x_4,\\
\psi(x_6) &= (t_1^3 + \lambda_6^0 v_2)|x_0 + t_1^2|x_2 + t_1|x_4 + 1|x_6,\\
\psi(y_6) &= (t_1^3 + t_2 + \kappa_6^0 v_2)|x_0 + t_1^2|x_2 + 1|y_6,\\
\psi(y_8) &= (\mu_8^0 t_1^4 + t_1t_2 + \lambda_8^0 v_2 t_1)|x_0 + (t_2 + \lambda_8^2 v_2)|x_2 + t_1^2|x_4 + t_1|y_6 + 1|y_8,\\
\psi(y_{10}) &= (\mu_{10}^0 t_1^5 + t_1^2 t_2 + \lambda_{10}^0 v_2 t_1^2)|x_0 + (\mu_{10}^2 t_1^4 + \lambda_{10}^2 v_2 t_1)|x_2\\
&\qquad + (t_1^3 + t_2 + \lambda_{10}^4 v_2)|x_4 + t_1^2|x_6 + t_1^2|y_6 + 1|y_{10},\\
\psi(y_{12}) &= (\mu_{12}^0 t_1^6 + t_1^3 t_2 + \nu_{12}^0 t_2^2 + \lambda_{12}^0 v_2 t_1^3 + \kappa_{12}^0 v_2 t_2 + \sigma v_2^2)|x_0\\
&\qquad + (\mu_{12}^2 t_1^5 + t_1^2 t_2 + \lambda_{12}^2 v_2 t_1^2)|x_2 + (\mu_{12}^4 t_1^4 + t_1 t_2 + \lambda_{12}^4 v_2 t_1)|x_4\\
&\qquad + (t_2 + \lambda_{12}^6 v_2)|x_6 + (t_1^3 + \kappa_{12}^6 v_2)|y_6 + t_1^2|y_8 + t_1|y_{10} + 1|y_{12}.
\end{aligned}
$$

The counitality condition of ψ

$$
\begin{array}{ccc}
BP_*Z & \xrightarrow{\ \cong\ } & \\
\downarrow{\scriptstyle\psi} & \searrow & \\
BP_*BP \otimes_{BP_*} BP_*Z & \xrightarrow[\epsilon \otimes BP_*Z]{} & BP_* \otimes_{BP_*} BP_*Z
\end{array}
$$

forces $\lambda_6^0=\kappa_6^0=\lambda_8^2=\lambda_{10}^4=\sigma=\lambda_{12}^6=\kappa_{12}^6=0$. After the change of basis

$$\begin{aligned} y_8 &\rightsquigarrow y_8+\lambda_8^0 v_2 x_2,\,, \\ y_{10} &\rightsquigarrow y_{10}+\lambda_{10}^0 v_2 x_4,\,, \\ y_{12} &\rightsquigarrow y_{12}+\kappa_{12}^0 v_2 y_6+(\lambda_{12}^0+\kappa_{12}^0)v_2 x_6, \end{aligned}$$

we have

$$\begin{aligned}
\psi(x_0)&=1|x_0,\\
\psi(x_2)&=t_1|x_0+1|x_2,\\
\psi(x_4)&=t_1^2|x_0+1|x_4,\\
\psi(x_6)&=t_1^3|x_0+t_1^2|x_2+t_1|x_4+1|x_6,\\
\psi(y_6)&=(t_1^3+t_2)|x_0+t_1^2|x_2+1|y_6,\\
\psi(y_8)&=(\mu_8^0t_1^4+t_1t_2)|x_0+t_2|x_2+t_1^2|x_4+t_1|y_6+1|y_8,\\
\psi(y_{10})&=(\mu_{10}^0t_1^5+t_1^2t_2)|x_0+(\mu_{10}^2t_1^4+\lambda_{10}^2v_2t_1)|x_2+(t_1^3+t_2)|x_4+t_1^2|x_6+t_1^2|y_6\\
&\qquad +1|y_{10},\\
\psi(y_{12})&=(\mu_{12}^0t_1^6+t_1^3t_2+\nu_{12}^0t_2^2)|x_0+(\mu_{12}^2t_1^5+t_1^2t_2+(\lambda_8^0+\lambda_{12}^0+\lambda_{12}^2)v_2t_1^2)|x_2\\
&\qquad +(\mu_{12}^4t_1^4+t_1t_2+(\lambda_{10}^0+\lambda_{12}^4+\lambda_{12}^0+\kappa_{12}^0)v_2t_1)|x_4+t_2|x_6\\
&\qquad +t_1^3|y_6+t_1^2|y_8+t_1|y_{10}+1|y_{12}.
\end{aligned}$$

Now we exploit the coassociativity condition

$$\begin{array}{ccc}
BP_*Z & \xrightarrow{\quad\psi\quad} & BP_*BP\otimes_{BP_*}BP_*Z \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\Delta\otimes BP_*Z} \\
BP_*BP\otimes_{BP_*}BP_*Z & \xrightarrow[BP_*BP\otimes\psi]{} & BP_*BP\otimes_{BP_*}BP_*BP\otimes_{BP_*}BP_*Z
\end{array}$$

Applying the coassociativity condition on y_8 tells us nothing, while applying it on y_{10} tells us that

$$\lambda_{10}^2=0,\quad \mu_{10}^0=\mu_{10}^2=1.$$

Applying it on y_{12}, we get

$$\begin{aligned} \lambda_8^0+\lambda_{12}^0+\lambda_{12}^2&=0, \\ \lambda_{10}^0+\lambda_{12}^4+\lambda_{12}^0+\kappa_{12}^0&=0, \end{aligned}\qquad \begin{aligned} \mu_{12}^2&=0,\\ \mu_{12}^4+\mu_{12}^0+1&=0,\\ \mu_8^0+\mu_{12}^0+\nu_{12}^0+1&=0. \end{aligned}$$

Setting $a=\mu_8^0$ and $b=\mu_{12}^0+1$ completes the proof. □

Remark 3.2 By sending $v_i \mapsto 0$ and $t_i \mapsto \zeta_i^2$, we obtain a functor

$$Q\colon (BP_*, BP_*BP)\text{–comodules} \to (\mathbb{F}_2, \Phi A_*)\text{–comodules},$$

where ΦA_* is the double of the dual Steenrod algebra. This functor sends BP_*Z to $\Phi A(1)_*$. Since $A(1)_*$ has four different A_*–comodule structures, it follows that $\Phi A(1)_*$ has four different ΦA_*–comodule structures. The four different BP_*BP–comodule structures on BP_*Z are essentially lifts of the four different ΦA_*–comodule structures on $\Phi A(1)_*$.

Remark 3.3 Let $M_* = BP_*/(2, v_1)\langle g_0, g_2, g_4, g_6\rangle$ be the BP_*BP–comodule with four generators with cooperations

$$\psi(g_0) = 1|g_0, \qquad \psi(g_4) = t_1^2|g_0 + 1|g_4,$$
$$\psi(g_2) = t_1|g_0 + 1|g_2, \qquad \psi(g_6) = t_1^3|g_0 + t_1^2|g_2 + t_1|g_4 + 1|g_6.$$

Then, if $W = A_1 \wedge C\nu$, where A_1 is any of the four 8–cell complexes whose cohomology is isomorphic to $A(1)$, the BP_*BP–comodule BP_*W is isomorphic to M_*.

A straightforward calculation tells us:

Lemma 3.4 *There is an exact sequence of BP_*BP–comodules*

$$0 \to M_* \xrightarrow{\iota} BP_*Z \xrightarrow{\tau} \Sigma^6 M_* \to 0$$

such that $\iota(g_i) = x_i$, $\tau(x_i) = 0$ and $\tau(y_i) = \Sigma^6 g_{i-6}$.

4 The action of the small Morava stabilizer group on $(E_2)_*Z$

To compute the E_2–page of the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\mathbb{S}_2; (E_2)_t Z)^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \Rightarrow \pi_{t-s} L_{K(2)} Z, \tag{4-1}$$

we first need to understand the action of $\mathbb{S}_2 = \mathrm{Aut}(\Gamma_2/\mathbb{F}_4)$ on $(E_2)_*Z$, where Γ_2 is the height 2 Honda formal group law over $\mathbb{F}_4$. Recall from (2-4) the left BP_*–linear map

$$\theta_{\overline{\Gamma}_n}\colon BP_*BP \to \mathrm{Map}^c(\mathbb{S}_2, (E_2)_*).$$

For X a finite spectrum, we deduce the action of $\mathbb{S}_2$ on $(E_2)_*X$ from the knowledge of the BP_*BP–coaction map ψ_X^{BP} on BP_*X via the diagram

(4-2)
$$\begin{array}{ccc} BP_*X & \xrightarrow{\psi_X^{BP}} & BP_*BP \otimes_{BP_*} BP_*X \\ & & \Big\downarrow {\scriptstyle \theta_{\overline{\Gamma}_n} \otimes BP_*X} \\ & & \mathrm{Map}(\mathbb{S}_2, (E_2)_*) \otimes_{BP_*} BP_*X \end{array}$$

The main purpose of this section is to understand the action of $\mathbb{S}_2$ on $(E_2)_*Z$, for all $Z \in \widetilde{\mathcal{Z}}$.

We begin by briefly recalling some key facts about $\mathbb{S}_2$ that we need for the calculations to follow. Let T be a formal variable that need not commute with $W(\mathbb{F}_4)$ and let

$$\mathcal{O}_2 := W(\mathbb{F}_4)\langle T\rangle/(T^2-2, T\omega-\omega^\sigma T),$$

where ω is a root of $x^2+x+1 \in \mathbb{F}_2[x]$, and σ is the Frobenius map on $W(\mathbb{F}_4)$. Note that any element $\gamma \in \mathcal{O}_2$ can be written as a power series

$$\gamma = \sum_{n=0}^{\infty} a_n T^n,$$

where the a_n are Teichmüller lifts of $\mathbb{F}_4^\times$ or are zero. Then γ corresponds to the power series

$$\overline{a}_0 x +_{\Gamma_2} \overline{a}_1 x^2 +_{\Gamma_2} \cdots +_{\Gamma_2} \overline{a}_n x^{2^n} +_{\Gamma_2} \cdots \in \mathbb{F}_4[\![x]\!],$$

where $\overline{a}_n$ is the image of a_n under the quotient map $W(\mathbb{F}_4) \to \mathbb{F}_4$. In fact, this defines an isomorphism from $\mathcal{O}_2$ to $\mathrm{End}(\Gamma_2/\mathbb{F}_4) \subset \mathbb{F}_4[\![x]\!]$ and, consequently, $\mathbb{S}_2$ is isomorphic to the group of units of $\mathcal{O}_2$ (see [21, Lemma A2.2.20]). Recall from Section 2 the map

$$\widetilde{\theta}_{\Gamma_n} \colon \widetilde{VT} \to \mathrm{Map}^c(\mathbb{S}_2, (E_2)_0),$$

where Γ_n is the height n Honda formal group law. To avoid cumbersome notation, let us continue (from Section 2) to denote $\widetilde{\theta}_{\Gamma_n}(\tilde{t}_n)$ simply by $\tilde{t}_n$. From (2-6), we learn that if

$$\gamma^{-1} = \sum_{n\geq 0}^{\Gamma_n} \overline{a}_n x^{2^n} \in \mathbb{S}_2,$$

then

(4-3)
$$\tilde{t}_n(\gamma) \equiv a_n \bmod (2, u_1).$$

Also keep in mind that the Teichmüller lifts a_n satisfy the equation $a_n^4 = a_n$. Therefore we have

$$\tilde{t}_n(-)^4 \equiv \tilde{t}_n(-) \bmod (2, u_1). \tag{4-4}$$

We can also write every element of $\mathcal{O}_2$ as $a + bT$ with $a, b \in W(\mathbb{F}_4)$. Using the isomorphism $\mathbb{S}_2 \cong \mathcal{O}_2^\times$, one defines a determinant map (see [2, Section 2.3])

$$\det\colon \mathbb{S}_2 \to \mathbb{Z}_2^\times,$$

which sends $a + bT \mapsto aa^\sigma - 2bb^\sigma$. Composing this with the quotient map $\mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2$ gives the *norm map*

$$N\colon \mathbb{S}_2 \xrightarrow{\det} \mathbb{Z}_2^\times \twoheadrightarrow \mathbb{Z}_2. \tag{4-5}$$

The kernel of the norm map $N\colon \mathbb{S}_2 \to \mathbb{Z}_2$ is called the *norm one* subgroup and is denoted by $\mathbb{S}_2^1$. In [2, Section 2.3], Beaudry described two elements $\alpha, \pi \in \mathbb{S}_2$ such that $\det(\alpha) = -1$ and $\det(\pi) = 3$, two elements which generate $\mathbb{Z}_2^\times$ topologically. In particular, π defines an isomorphism $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$.

As we will see in this section, the most crucial part of the action of $\mathbb{S}_2$ on $(E_2)_*Z$ is the action of its finite subgroups, which we describe here, following [2; 9].

Proposition 4.1 *Every maximal finite nonabelian subgroup of $\mathbb{S}_2$ is conjugate to a group*

$$G_{24} = Q_8 \rtimes C_3,$$

where Q_8 is the quaternion group

$$Q_8 = \langle \mathsf{i}, \mathsf{j} : \mathsf{i}^4 = \mathbb{1}, \mathsf{i}^2 = \mathsf{j}^2, \mathsf{i}^3\mathsf{j} = \mathsf{j}\mathsf{i} \rangle$$

and the generator $\overline{\omega}$ of C_3 acts by conjugation and permutes i, j and $\mathsf{k} = \mathsf{ij}$. More precisely, we have the relations

(1) $\overline{\omega}\mathsf{i} = \mathsf{j}\overline{\omega}$,

(2) $\overline{\omega}\mathsf{j} = \mathsf{k}\overline{\omega}$.

Notation 4.2 We denote the identity element of Q_8 by $\mathbb{1}$. The order 2 element of Q_8 is often denoted by $-\mathbb{1}$; however, to avoid confusing it with an element of a ring, we will denote it by $\widehat{\mathbb{1}} \in Q_8$. Similarly, we denote $\widehat{\mathsf{i}} = \widehat{\mathbb{1}}\mathsf{i}, \widehat{\mathsf{j}} = \widehat{\mathbb{1}}\mathsf{j}, \widehat{\mathsf{k}} = \widehat{\mathbb{1}}\mathsf{k}$. The center of Q_8, which is an order 2 group generated by $\widehat{\mathbb{1}}$, will be denoted by $C_{\widehat{\mathbb{1}}}$.

The maximal finite subgroup G_{24} is unique up to conjugation as a subgroup of $\mathbb{S}_2$, while as a subgroup of $\mathbb{S}_2^1$, there are two conjugacy classes, G_{24} and $G'_{24} = \pi G_{24} \pi^{-1}$. The group $\mathbb{S}_2^1$ also has a cyclic subgroup

$$C_6 = C_{\widehat{\mathbb{1}}} \times C_3$$

generated by $\widehat{\mathbb{1}}$ and $\overline{\omega}$.

The identification of $\mathbb{S}_2$ with $\mathcal{O}_2^\times$ endows $\mathbb{S}_2$ with a decreasing filtration

$$F_{0/2}\mathbb{S}_2 = \mathbb{S}_2, \, F_{n/2}\mathbb{S}_2 = \{\gamma \in \mathbb{S}_2 : \gamma \equiv 1 \bmod T^n\}. \tag{4-6}$$

One should note that

$$F_{n/2}\mathbb{S}_2/F_{(n+1)/2}\mathbb{S}_2 \cong \begin{cases} C_3 & \text{if } n = 0, \\ \mathbb{F}_4 & \text{if } n \geq 0. \end{cases}$$

Notice from (4-3) that t_n coacts trivially on BP_*Z for $n > 2$. Therefore, following (4-3), we conclude that $F_{n/2}\mathbb{S}_2$ for $n > 2$ acts trivially on $(E_2)_*Z$. We list the generators of the various filtration quotients of $\mathbb{S}_2$ in the following table:

associated graded	generators
$F_{0/2}\mathbb{S}_2/F_{1/2}\mathbb{S}_2 \cong C_3$	$\overline{\omega}$
$F_{1/2}\mathbb{S}_2/F_{2/2}\mathbb{S}_2 \cong Q_8/C_{\widehat{\mathbb{1}}} \cong C_2 \times C_2$	i, j
$F_{2/2}\mathbb{S}_2/F_{3/2}\mathbb{S}_2 \cong C_{\widehat{\mathbb{1}}} \times C_\alpha$	$\widehat{\mathbb{1}}, \alpha$

The subgroup

$$K = \overline{\langle \alpha, F_{3/2}\mathbb{S}_2 \rangle}$$

is known as the Poincaré duality subgroup of $\mathbb{S}_2$, and it is known that

$$\mathbb{S}_2 \cong K \rtimes G_{24}.$$

The subgroup $\mathbb{S}_2^1$ inherits the filtration of (4-6) via $F_{n/2}\mathbb{S}_2^1 := \mathbb{S}_2^1 \cap F_{n/2}\mathbb{S}_2$. In particular, we have

$$F_{n/2}\mathbb{S}_2^1/F_{(n+1)/2}\mathbb{S}_2^1 \cong \begin{cases} C_3 & \text{if } n = 0, \\ \mathbb{F}_4 & \text{if } n \geq 0 \text{ and } n \text{ odd, or } n = 2, \\ \mathbb{F}_2 & \text{otherwise.} \end{cases}$$

There is a corresponding Poincaré duality subgroup K^1 which satisfies

$$\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}.$$

Recall from (3-2) that

$$(E_2)_*Z \cong (E_2)_* \otimes_{BP_*} BP_*Z \cong (E_2)_*/(2, u_1)\langle x_0, x_2, x_4, x_6, y_6, y_8, y_{10}, y_{12} \rangle.$$

u	$\tilde{t}_0 u$
$\overline{x}_0$	$\overline{x}_0$
$\overline{x}_2$	$\tilde{t}_0^2\tilde{t}_1\overline{x}_0+\tilde{t}_0\overline{x}_2$
$\overline{x}_4$	$\tilde{t}_0^1\tilde{t}_1^2\overline{x}_0+\tilde{t}_0^2\overline{x}_4$
$\overline{x}_6$	$\tilde{t}_1^3\overline{x}_0+\tilde{t}_0^2\tilde{t}_1^2\overline{x}_2+\tilde{t}_0\tilde{t}_1\overline{x}_4+\overline{x}_6$
$\overline{y}_6$	$(\tilde{t}_1^3+\tilde{t}_0^2\tilde{t}_2)\overline{x}_0+\tilde{t}_0^2\tilde{t}_1^2\overline{x}_2+\overline{y}_6$
$\overline{y}_8$	$(a\tilde{t}_0^2\tilde{t}_1+\tilde{t}_0\tilde{t}_1\tilde{t}_2)\overline{x}_0+\tilde{t}_2\overline{x}_2+\tilde{t}_1^2\overline{x}_4+\tilde{t}_0^2\tilde{t}_1\overline{y}_6+\tilde{t}_0\overline{y}_8$
$\overline{y}_{10}$	$(\tilde{t}_0\tilde{t}_1^2+\tilde{t}_1^2\tilde{t}_2)\overline{x}_0+\tilde{t}_1\overline{x}_2+(\tilde{t}_0^2\tilde{t}_1^3+\tilde{t}_0\tilde{t}_2)\overline{x}_4+\tilde{t}_0\tilde{t}_1^2\overline{x}_6+\tilde{t}_0\tilde{t}_1\overline{y}_6+\tilde{t}_0^2\overline{y}_{10}$
$\overline{y}_{12}$	$((b+1)\tilde{t}_1^3+\tilde{t}_0^2\tilde{t}_1^3\tilde{t}_2+(a+b)\tilde{t}_0\tilde{t}_2^2)\overline{x}_0+\tilde{t}_0\tilde{t}_1^2\tilde{t}_2\overline{x}_2$
	$+(b\tilde{t}_0\tilde{t}_1+\tilde{t}_1\tilde{t}_2)\overline{x}_4+\tilde{t}_0^2\tilde{t}_2\overline{x}_6+\tilde{t}_1^3\overline{y}_6+\tilde{t}_0^2\tilde{t}_1^2\overline{y}_8+\tilde{t}_0\tilde{t}_1\overline{y}_{10}+\overline{y}_{12}$

Table 1: The action of $\mathbb{S}_2$ on $(E_2)_*Z$.

For our purposes it is convenient to have all the generators in degree 0, so we define

$$\overline{x}_i=u^{i/2}x_i,\quad \overline{y}_i=u^{i/2}y_i$$

in order to have

$$(E_2)_*Z\cong(E_2)_*/(2,u_1)\langle\overline{x}_0,\overline{x}_2,\overline{x}_4,\overline{x}_6,\overline{y}_6,\overline{y}_8,\overline{y}_{10},\overline{y}_{12}\rangle.$$

By (4-2), the action of $\mathbb{S}_2$ can be expressed in terms of maps $\tilde{t}_i\colon \mathbb{S}_2\to E(\mathbb{F}_4,\Gamma_2)$, which are related to the maps $t_i\colon \mathbb{S}_2\to E(\mathbb{F}_4,\Gamma_2)[u^{\pm1}]$ by (2-5). For instance, we have

$$\psi_Z^{BP}(x_2)=1|x_2+t_1|x_0=1|x_2+u^{-1}\tilde{t}_0^{-2}\tilde{t}_1|x_0,$$

so for $\gamma\in\mathbb{S}_2$ we have

$$\gamma(\overline{x}_2)=\gamma(ux_2)=\tilde{t}_0(\gamma)u(x_2+u^{-1}\tilde{t}_0(\gamma)^{-2}\tilde{t}_1(\gamma)x_0)=\tilde{t}_0(\gamma)\overline{x}_2+\tilde{t}_0(\gamma)^{-1}\tilde{t}_1(\gamma)\overline{x}_0.$$

By Lemma 3.1, if we suppress "γ", the action of $\mathbb{S}_2$ can be described as in Table 1 (thanks to (4-4), we can adopt the simplifications $\tilde{t}_i^4=\tilde{t}_i$ for $i=1,2$ and $\tilde{t}_0^3=1$).

Let $\overline{M}_*=(E_2)_*\otimes_{BP_*}M_*$, where M_* is the BP_*BP–comodule introduced in Remark 3.3. Define

$$\overline{g}_i=u^{i/2}g_i$$

to form a set of generators $\{\overline{g}_0,\overline{g}_2,\overline{g}_4,\overline{g}_6\}$ of $\overline{M}_0$. A consequence of Lemma 3.4 and Table 1 is:

Lemma 4.3 *There is an exact sequence*

$$0 \to \overline{M}_0 \xrightarrow{\bar{\iota}} (E_2)_0 Z \xrightarrow{\bar{\tau}} \overline{M}_0 \to 0 \tag{4-7}$$

of Q_8–modules, where $\bar{\iota}(\bar{g}_i) = \bar{x}_i$ and $\bar{\tau}(\bar{y}_i) = \bar{g}_{i-6}$.

We will use the exact sequence of (4-7) (compare with (A-2)) to understand the action of Q_8 on $(E_2)_0 Z$. For this purpose we need the data of $\tilde{t}_i(\gamma)$ modulo $(2, u_1)$ for $\gamma \in Q_8$. By definition of the Honda formal group law Γ_2, we have $[2]_{\Gamma_2}(x) = x^4$ and it follows that

$$[-1]_{\Gamma_2}(x) = \sum_{n \geq 0}^{\Gamma_2} x^{4^n}.$$

Indeed, one has

$$x +_{\Gamma_2} \sum_{n \geq 0}^{\Gamma_2} x^{4^n} = x +_{\Gamma_2} x +_{\Gamma_2} \sum_{n \geq 1}^{\Gamma_2} x^{4^n} = x^4 +_{\Gamma_2} \sum_{n \geq 1}^{\Gamma_2} x^{4^n} = \cdots = 0.$$

Following (2-6) and using the fact that $[-1]_{\Gamma_2}(x)$ is its own inverse, we have

$$\tilde{t}_n(\widehat{\mathbb{1}}) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

modulo $(2, u_1)$. Further, i and j can be chosen so that, modulo $(2, u_1)$, one has

- $\tilde{t}_0(\gamma) = 1$ for all $\gamma \in Q_8$,
- $\tilde{t}_1(\mathsf{i}) = \tilde{t}_1(\hat{\mathsf{i}}) = 1$, $\tilde{t}_1(\mathsf{j}) = \tilde{t}_1(\hat{\mathsf{j}}) = \omega$ and $\tilde{t}_1(\mathsf{k}) = \tilde{t}_1(\hat{\mathsf{k}}) = \omega^2$.

Lemma 4.4 *There is an isomorphism $\overline{M}_0 \cong \mathbb{F}_4[Q_8/C_{\widehat{\mathbb{1}}}]$ of left $\mathbb{F}_4[Q_8]$–modules.*

Proof Since $\bar{\iota}(\bar{g}_i) = \bar{x}_i$, one can read the action of Q_8 off Table 1. With respect to the ordered basis $\mathcal{B} = \{\bar{g}_0, \bar{g}_2, \bar{g}_4, \bar{g}_6\}$ of $\overline{M}_0$, we have

$$(\mathsf{i})_{\mathcal{B}} = (\hat{\mathsf{i}})_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\mathsf{j})_{\mathcal{B}} = (\hat{\mathsf{j}})_{\mathcal{B}} = \begin{bmatrix} 1 & \omega & \omega^2 & 1 \\ 0 & 1 & 0 & \omega^2 \\ 0 & 0 & 1 & \omega \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\mathsf{k})_{\mathcal{B}} = (\hat{\mathsf{k}})_{\mathcal{B}} = \begin{bmatrix} 1 & \omega^2 & \omega & 1 \\ 0 & 1 & 0 & \omega \\ 0 & 0 & 1 & \omega^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the basis $\mathcal{C} = \{v_0, v_1, v_2, v_3\}$ of $\overline{M}_0$, where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & \omega & \omega^2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the change of basis matrix from $\mathcal{B} \to \mathcal{C}$. It can be readily checked that

$$(\mathsf{i})_{\mathcal{C}} = Q^{-1}(\mathsf{i})_{\mathcal{B}} Q = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\mathsf{j})_{\mathcal{C}} = Q^{-1}(\mathsf{j})_{\mathcal{B}} Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\mathsf{k})_{\mathcal{C}} = Q^{-1}(\mathsf{k})_{\mathcal{B}} Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last three matrices are equal to those representing the same transformations in the basis $\mathcal{B}_4$ of $\mathbb{F}_4[Q_8/C_{\hat{\mathbb{1}}}]$ given in (A-3), and thus we conclude that $\overline{M}_0$ and $\mathbb{F}_4[Q_8/C_{\hat{\mathbb{1}}}]$ are isomorphic as $\mathbb{F}_4[Q_8]$–modules. □

Theorem 4.5 *There is an isomorphism*

$$(E_2)_0 Z \cong \mathbb{F}_4[Q_8]$$

of $\mathbb{F}_4[Q_8]$–modules.

Proof Let $\mathcal{B}$, $\mathcal{C}$ and Q be as in the proof of Lemma 4.4. Let

$$\mathcal{B}_Z = \{\bar{x}_0, \bar{x}_2, \bar{x}_4, \bar{x}_6, \bar{y}_6, \bar{y}_8, \bar{y}_{10}, \bar{y}_{12}\}$$

be the usual ordered basis of $(E_2)_0 Z$ and let $\mathcal{C}_Z = \{c_0, c_1, c_2, c_3, c_0', c_1', c_2', c_3'\}$ be another basis of $(E_2)_0 Z$, where $\widetilde{Q} = \left(\begin{smallmatrix} Q & * \\ 0 & Q \end{smallmatrix}\right)$ is a change of basis matrix from $\mathcal{B}_Z \to \mathcal{C}_Z$.

By Lemma 4.3, in the exact sequence (4-7), we have $\bar{\iota}(v_i) = c_i$ and $\bar{\tau}(c_i') = v_i$. From Table 1, we know that

$$(\hat{\mathbb{1}})_{\mathcal{B}_Z} = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix},$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & a+b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$(\widehat{\mathbb{1}})_{\mathcal{C}_Z} = \widetilde{Q}^{-1}(\widehat{\mathbb{1}})_{\mathcal{B}_8}\widetilde{Q} = \begin{bmatrix} I & Q^{-1}MQ \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}.$$

The remainder of the proof that $(E_2)_0 Z \cong V_8(\mathbb{F}_4)$ will continue in Lemma A.1. □

Because Q_8 acts trivially on u we get the following corollary:

Corollary 4.6 *There is an isomorphism*

$$(E_2)_* Z \cong \mathbb{F}_4[u^{\pm 1}][Q_8]$$

of graded Q_8–modules.

Theorem 4.7 *There is an isomorphism*

$$(E_2)_0 Z \cong \mathbb{F}_4[G_{24}] \otimes_{\mathbb{F}_4[C_3]} \mathbb{F}_4$$

of $\mathbb{F}_4[G_{24}]$–modules.

Proof Note that $\overline{\omega} \in G_{24} \subset \mathbb{S}_2$ corresponds to the power series $\omega x \in \mathrm{Aut}(\Gamma_n/\mathbb{F}_4)$, where $\omega \in \mathbb{F}_4^{\times}$. Keep in mind that $\tilde{t}_0(\overline{\omega}) = \omega^{-1} = \omega^2$ by (4-3). Note that in $\mathbb{F}_4[G_{24}] \otimes_{\mathbb{F}_4[C_3]} \mathbb{F}_4$, we have the relations

$$\overline{\omega}\,\mathsf{i} \otimes 1 = \mathsf{j}\overline{\omega} \otimes 1 = \mathsf{j} \otimes 1, \quad \overline{\omega}\,\mathsf{j} \otimes 1 = \mathsf{j}\overline{\omega} \otimes 1 = \mathsf{k} \otimes 1.$$

To establish the result, we must check that the same relations hold with the $\mathbb{F}_4[\mathbb{Q}_8]$–module generator of $(E_2)_* Z$, which is $\overline{y}_{12}$. From Table 1, we observe that this is indeed the case, ie

$$(\overline{\omega}\,\mathsf{i}) \cdot \overline{y}_{12} = \mathsf{j} \cdot \overline{y}_{12}, \quad (\overline{\omega}\,\mathsf{j}) \cdot \overline{y}_{12} = \mathsf{k} \cdot \overline{y}_{12}.$$

□

Corollary 4.8 *There is an isomorphism*

$$(E_2)_* Z \cong \mathbb{F}_4[G_{24}] \otimes_{\mathbb{F}_4[C_3]} \mathbb{F}_4[u^{\pm 1}]$$

of $\mathbb{F}_4[G_{24}]$–modules.

Lemma 4.9 *The element $\alpha^{-1}\pi \in \mathbb{S}_2$ acts trivially on the generators of $(E_2)_* Z$.*

Proof By definition of α and π [2, Section 2.3], $\pi \equiv \alpha \mod F_{3/2}\mathbb{S}_2$. Therefore, $\alpha^{-1}\pi \in F_{3/2}\mathbb{S}_2$. The result follows from the fact that t_n coacts trivially on $BP_* Z$ (ie the BP_*BP–comodule structure of $BP_* Z$ does not contain any terms involving t_n) for $n \geq 3$. □

Further note that

$$\tilde{t}_0(\alpha) \equiv 1, \quad \tilde{t}_1(\alpha) \equiv 0 \quad \text{and} \quad \tilde{t}_2(\alpha) \equiv \omega \quad \mod (2, u_1).$$

A direct inspection of Table 1 shows that α acts nontrivially on $(E_2)_*Z$, in fact we have:

Corollary 4.10 *The fixed point modules $(E_2)_*Z^{C_{\widehat{\mathbb{1}}}}$ and $(E_2)_*Z^{C_\alpha}$ both equal $\mathbb{F}_4[u^{\pm 1}]\{\bar{x}_0, \bar{x}_2, \bar{x}_4, \bar{x}_6\}$.*

Corollary 4.11 *The subgroup $F_{2/2}\mathbb{S}_2$ acts trivially on $(E_2)_*Z^{C_{\widehat{\mathbb{1}}}}$.*

Proof We know that $F_{3/2}\mathbb{S}_2$ acts trivially as t_n coacts trivially on BP_*Z for $n \geq 3$. Furthermore, $F_{2/2}\mathbb{S}_2/F_{3/2}\mathbb{S}_2$ is generated by the image of $\widehat{\mathbb{1}}$ and α, both of which act trivially on $(E_2)_*Z^{C_{\widehat{\mathbb{1}}}}$. □

5 The duality resolution spectral sequence for Z

Now that we have complete knowledge of the action of $\mathbb{S}_2^1$ on $(E_2)_*Z$, we are all set to calculate the group cohomology $H^*(\mathbb{S}_2^1; (E_2)_*Z)$, which is the key step to finding the E_2–page of the descent spectral sequence (4-1). We will use the duality resolution spectral sequence, a convenient tool to calculate the E_2–page. The duality resolution spectral sequence comes from the duality resolution, which is a finite $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module resolution of $\mathbb{Z}_2$. First we fix some notation.

Notation 5.1 Throughout this section, we will let

- $S_2 := F_{1/2}\mathbb{S}_2$,
- $S_2^1 := F_{1/2}\mathbb{S}_2^1$, and
- IS_2^1 be the augmentation ideal of the group ring $\mathbb{Z}_2[\![S_2^1]\!]$.

Note that every element in IS_2^1 can be written as an infinite sum of elements of the form $a_g(\mathbb{1} - g)$, where $\mathbb{1}$ denotes the neutral element of $\mathbb{S}_2$, $g \in S_2^1$ and $a_g \in \mathbb{Z}_2[\![S_2^1]\!]$.

Theorem 5.2 (Goerss, Henn, Mahowald and Rezk; Beaudry [3]) *Let $\mathbb{Z}_2$ be the trivial $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. There is an exact sequence of complete left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\epsilon} \mathbb{Z}_2 \to 0,$$

where $\mathscr{C}_0 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}]\!]$, $\mathscr{C}_3 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!]$ and $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!]$. Let e be the unit in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ and e_p be the resulting generator in $\mathscr{C}_p$. The map ∂_p can be chosen *to satisfy*

- $\partial_1(e_1) = (e - \alpha) \cdot e_0$,
- $\partial_2(e_2) \equiv (e + \alpha) \cdot e_1 \bmod (2, (IS_2^1)^2)$,
- $\partial_3(e_3) = \pi(e + i + j + k)(e - \alpha^{-1})\pi^{-1} \cdot e_2$.

Let $F_0 = G_{24}$, $F_1 = F_2 = C_6$ *and* $F_3 = G'_{24}$. *For a profinite* $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$*–module* M, *there is a first quadrant spectral sequence*

$$E_1^{p,q} = \mathrm{Ext}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, M) \cong H^q(F_p; M) \Rightarrow H^{p+q}(\mathbb{S}_2^1; M)$$

with differentials $d_r\colon E_1^{p,q} \to E_1^{p+r,q-r+1}$.

Since the map $BP_* \to (E_2)_*$ sends $v_2 \mapsto u^{-3}$, we will denote u^{-3} by v_2. Let us now recall Shapiro's lemma, an important result in group cohomology, which will be used throughout the rest of this section.

Lemma 5.3 (Shapiro's lemma) *Let* G *be a finite group,* $H \subset G$ *be a subgroup and let* M *be a* $\mathbb{Z}[H]$*–module. Then, for every* n, *we have*

$$H^n(G; \mathrm{CoInd}_H^G(M)) = H^n(H; M),$$

where $\mathrm{CoInd}_H^G(M) = \mathrm{Hom}_{\mathbb{Z}[H]}(Z[G], M)$.

Remark 5.4 If $H \subset G$ is a subgroup of finite index, then

$$\mathrm{CoInd}_H^G(M) \cong \mathrm{Ind}_H^G(M) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M.$$

In all instances in which we invoke Shapiro's lemma, G will be a finite group, hence one need not distinguish between ${}_H^G(M)$ and $\mathrm{Ind}_H^G(M)$ for any normal subgroup H of G.

Corollary 4.8 along with Shapiro's lemma implies that

$$H^p(G_{24}; (E_2)_* Z) \cong H^p(C_3; \mathbb{F}_4[u^{\pm 1}]).$$

The generator $\overline{\omega}$ of C_3 acts nontrivially on u (see Table 1), but fixes u^3, so that

$$H^p(G_{24}; (E_2)_* Z) = \begin{cases} \mathbb{F}_4[u^{\pm 3}] & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases} \tag{5-1}$$

Lemma 5.5 *Let* $G'_{24} = \pi G_{24} \pi^{-1}$ *in* $\mathbb{S}_2^1$. *Then we have*

$$H^q(G'_{24}; E_* Z) \cong H^q(G_{24}; E_* Z) \cong \mathbb{F}_4[v_2^{\pm 1}].$$

Proof Notice that $G'_{24} \cap G_{24} \supset C_{\widehat{\mathbb{1}}}$. Also keep in mind that $\pi \equiv \alpha \bmod F_{3/2}\mathbb{S}_2^1$. Therefore, by Corollary 4.10, π acts trivially on $(E_2)_0 Z^{C_{\widehat{\mathbb{1}}}}$. Let $Q'_8 = \pi Q_8 \pi^{-1} \subset G'_{24}$, $\mathrm{i}' = \pi \mathrm{i} \pi^{-1}$ and $\mathrm{j}' = \pi \mathrm{j} \pi^{-1}$. Thus we have

$$G'_{24} \cong Q'_8 \rtimes C_3.$$

Note that

$$\mathrm{i}' \equiv \mathrm{i} \quad \text{and} \quad \mathrm{j}' \equiv \mathrm{j} \quad \bmod F_{2/2}\mathbb{S}_2^1$$

because $\pi \in F_{2/2}\mathbb{S}_2^1$. Therefore, the actions of i' and j' on $(E_2)_0 Z^{C_{\widehat{\mathbb{1}}}}$ are exactly the same as that of i and j, respectively. It follows that

$$(E_2)_0 Z^{C_{\widehat{\mathbb{1}}}} \cong \mathbb{F}_4[Q'_8/C_{\widehat{\mathbb{1}}}]$$

as an $\mathbb{F}_4[Q'_8]$–module. Applying the arguments of Theorem 4.5 to this case, one sees

$$(E_2)_0 Z \cong \mathbb{F}_4[Q'_8]$$

as an $\mathbb{F}_4[Q'_8]$–module, and the result follows. □

Now we shall focus on computing $H^q(C_6; (E_2)_* Z)$. Take $C_{\widehat{\mathbb{1}}}$ to be the center of Q_8 and consider $C_6 = C_{\widehat{\mathbb{1}}} \times C_3$. While $C_{\widehat{\mathbb{1}}}$ fixes all the $\overline{x}_i$ in addition to fixing u, the group C_3 does not fix the $\overline{x}_i$; however, it does fix the x_i. This observation will be crucial for the computation that follows. Because $C_{\widehat{\mathbb{1}}}$ is the 2–Sylow subgroup of C_6, we have

$$H^q(C_6; (E_2)_* Z) \cong H^q(C_{\widehat{\mathbb{1}}}; \mathbb{F}_4[u^{\pm 1}][Q_8])^{C_3}.$$

Because $(E_2)_* Z \cong \mathbb{F}_4[u^{\pm 1}][Q_8]$ is a free $\mathbb{F}_4[C_{\widehat{\mathbb{1}}}]$–module we have

$$\begin{aligned} H^q(C_{\widehat{\mathbb{1}}}; \mathbb{F}_4[u^{\pm 1}][Q_8])^{C_3} &\cong ((E_2)_* Z^{C_{\widehat{\mathbb{1}}}})^{C_3} \\ &\cong \mathbb{F}_4[u^{\pm 1}][\overline{x}_0, \overline{x}_2, \overline{x}_4, \overline{x}_6]^{C_3} \\ &\cong \mathbb{F}_4[u^{\pm 3}][x_0, x_2, x_4, x_6] \end{aligned} \tag{5-2}$$

concentrated at $q = 0$. Essentially deriving from Table 1, we list the actions of i, j and ij on the generators x_0 x_2 x_4 and x_6, which will come in handy later on:

(5-3)

x	x_0	x_2	x_4	x_6
$\mathrm{i}\cdot x$	x_0	$u^{-1}x_0 + x_2$	$u^{-2}x_0 + x_4$	$u^{-3}x_0 + u^{-2}x_2 + u^{-1}x_4 + x_6$
$\mathrm{j}\cdot x$	x_0	$\omega u^{-1}x_0 + x_2$	$\omega^2 u^{-2}x_0 + x_4$	$u^{-3}x_0 + \omega^2 u^{-2}x_2 + \omega u^{-1}x_4 + x_6$
$\mathrm{ij}\cdot x$	x_0	$\omega^2 u^{-1}x_0 + x_2$	$\omega u^{-2}x_0 + x_4$	$u^{-3}x_0 + \omega u^{-2}x_2 + \omega^2 u^{-1}x_4 + x_6$

To summarize (5-1) and (5-2), as well as to establish notation, we rewrite the E_1–page of the duality resolution spectral sequence for Z as

$$E_1^{p,q} = \begin{cases} \mathbb{F}_4[v_2^{\pm 1}]\langle x_{0,0}\rangle & \text{if } p=0 \text{ and } q=0,\\ \mathbb{F}_4[v_2^{\pm 1}]\langle x_{1,0}, x_{1,2}, x_{1,4}, x_{1,6}\rangle & \text{if } p=1 \text{ and } q=0,\\ \mathbb{F}_4[v_2^{\pm 1}]\langle x_{2,0}, x_{2,2}, x_{2,4}, x_{2,6}\rangle & \text{if } p=2 \text{ and } q=0,\\ \mathbb{F}_4[v_2^{\pm 1}]\langle x_{3,0}\rangle & \text{if } p=3 \text{ and } q=0,\\ 0 & \text{otherwise,} \end{cases}$$

where the internal grading of $x_{i,j}$ is j. To compute the differentials in this spectral sequence, we need the following result:

Theorem 5.6 *For every $Z \in \widetilde{\mathcal{Z}}$, $H^*(S_2^1; (E_2)_*Z)$ is isomorphic to*

$$H^*(K^1; \mathbb{F}_2) \otimes \mathbb{F}_4[u^{\pm 1}]$$

as an $\mathbb{F}_4[u^{\pm 1}]$–module.

Proof We begin with the calculation of $H^*(F_{2/2}\mathbb{S}_2^1; (E_2)_*Z)$. Since

$$F_{2/2}\mathbb{S}_2^1 \cong K^1 \times C_{\hat{\mathbb{1}}},$$

we have a Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(K^1; H^q(C_{\hat{\mathbb{1}}}; (E_2)_*Z)) \Rightarrow H^{p+q}(F_{2/2}\mathbb{S}_2^1; (E_2)_*Z). \tag{5-4}$$

Since $H^q(C_{\hat{\mathbb{1}}}; (E_2)_*Z) \cong ((E_2)_*Z)^{C_{\hat{\mathbb{1}}}}$ (concentrated at $q=0$) and K^1 acts trivially on $((E_2)_*Z)^{C_{\hat{\mathbb{1}}}}$ by Corollary 4.11, the spectral sequence (5-4) collapses and we have

$$H^*(F_{2/2}\mathbb{S}_2^1; (E_2)_*Z) \cong H^*(K^1; \mathbb{F}_2) \otimes (E_2)_*Z^{C_{\hat{\mathbb{1}}}}.$$

Note that

$$(E_2)_*Z^{C_{\hat{\mathbb{1}}}} \cong \mathbb{F}_4[u^{\pm 1}]\langle \bar{x}_0, \bar{x}_2, \bar{x}_4, \bar{x}_6\rangle \cong \mathbb{F}_4[u^{\pm 1}][Q_8/C_{\hat{\mathbb{1}}}].$$

Now we run yet another Lyndon–Hochschild–Serre spectral sequence,

$$E_2^{p,q} = H^p(Q_8/C_{\hat{\mathbb{1}}}; H^q(F_{2/2}\mathbb{S}_2^1; (E_2)_*Z)) \Rightarrow H^{p+q}(S_2^1; (E_2)_*Z), \tag{5-5}$$

to compute $H^*(S_2^1; (E_2)_*Z)$. Notice that

$$\begin{aligned} E_2^{p,q} &= H^p(Q_8/C_{\hat{\mathbb{1}}}; H^q(F_{2/2}\mathbb{S}_2^1; (E_2)_*Z)) \\ &= H^p(Q_8/C_{\hat{\mathbb{1}}}; H^q(K^1; \mathbb{F}_2) \otimes (E_2)_*Z^{C_{\hat{\mathbb{1}}}}) \\ &= H^p(Q_8/C_{\hat{\mathbb{1}}}; H^q(K^1; \mathbb{F}_2) \otimes \mathbb{F}_4[u^{\pm 1}][Q_8/C_{\hat{\mathbb{1}}}]) \end{aligned}$$

$$= \begin{cases} H^q(K^1;\mathbb{F}_2)\otimes \mathbb{F}_4[u^{\pm 1}] & \text{if } p=0, \\ 0 & \text{if } p\neq 0, \end{cases}$$

by Shapiro's Lemma 5.3. Thus the spectral sequence (5-5) collapses at the E_2–page and we get

$$H^*(S_2^1;(E_2)_*Z) = H^*(K^1;\mathbb{F}_2)\otimes \mathbb{F}_4[u^{\pm 1}]. \qquad \square$$

We get a complete description of $H^*(S_2^1;(E_2)_*Z)$ from Theorem 5.6 and the following result of Goerss and Henn (see [2, Theorem 2.5.13]):

$$\text{(5-6)} \qquad H^*(K^1;\mathbb{F}_2)\cong \mathbb{F}_2[y_0,y_1,y_2]/(y_0^2,\, y_1^2+y_0y_1,\, y_2^2+y_0y_2).$$

Our next goal is to make use of the formulas in Theorem 5.2 to calculate the d_1–differentials of the duality resolution spectral sequence for Z. Moving forward, there are two things that are handy to keep in mind:

- Z admits a v_2^1–self-map [6]; therefore, differentials in the duality resolution spectral sequence for Z will be v_2^1–linear.
- The d_1–differentials preserve the internal grading.

Lemma 5.7 *The differentials $d_1\colon E_1^{0,0}\to E_1^{1,0}$ and $d_1\colon E_1^{2,0}\to E_1^{3,0}$ are zero, while the differential $d_1\colon E_1^{1,0}\to E_1^{2,0}$ is the v_2–linear map given by*

$$\mathbb{F}_4[v_2^{\pm 1}]\langle x_{1,0},x_{1,2},x_{1,4},x_{1,6}\rangle \to \mathbb{F}_4[v_2^{\pm 1}]\langle x_{2,0},x_{2,2},x_{2,4},x_{2,6}\rangle,$$
$$x_{1,0},x_{1,2},x_{1,4}\mapsto 0,$$
$$x_{1,6}\mapsto \lambda v_2 x_{2,0},$$

where $\lambda\in\mathbb{F}_4^\times$. The duality resolution spectral sequence for Z collapses at the E_2–page.

Proof It follows from Theorem 5.2 that the differential $d_1\colon E_1^{0,0}\to E_1^{1,0}$ is given by

$$d_1(x) = (\mathbb{1}-\alpha)\cdot x,$$

which is zero because α fixes $x_{0,0}$ (this follows from Table 1; also see Corollary 4.10). Likewise, the differential $d_1\colon E_1^{2,0}\to E_1^{3,0}$ is given by

$$d_1(x) = \pi(\mathbb{1}+\mathsf{i}+\mathsf{j}+\mathsf{k})(\mathbb{1}-\alpha^{-1})\pi^{-1}\cdot x,$$

which is zero because by Corollary 4.10 and the fact that $\alpha\equiv\pi \bmod F_{3/2}\mathbb{S}_2^1$, both α and π fix all the $x_{1,i}$. The differential $d_1\colon E_1^{1,0}\to E_1^{2,0}$ is given by

$$d_1(x) = \Theta\cdot x = (\mathbb{1}+\alpha+\mathcal{E})\cdot x,$$

where $\mathcal{E} \in (IS_2^1)^2$. Because α fixes all the $x_{1,j}$, this simplifies to

$$d_1(x) = \mathcal{E} \cdot x.$$

The element $\mathcal{E}$ is a possibly infinite sum of the form

$$\mathcal{E} = \sum a_{g,h}(\mathbb{1} - g)(\mathbb{1} - h)$$

for $a_{g,h} \in \mathbb{Z}_2[\![S_2^1]\!]$ and $g, h \in S_2^1$. In particular, thanks to Table 1 and (2-7), we know that

$$(\mathbb{1} - g)(\mathbb{1} - h) \cdot x_0 = 0, \quad (\mathbb{1} - g)(\mathbb{1} - h) \cdot x_4 = 0,$$
$$(\mathbb{1} - g)(\mathbb{1} - h) \cdot x_2 = 0, \quad (\mathbb{1} - g)(\mathbb{1} - h) \cdot x_6 = (\tilde{t}_1(h)\tilde{t}_1(g)^2 + \tilde{t}_1(h)^2\tilde{t}_1(g))x_0,$$

and it follows that

$$d_1(x_{1,0}) = d_1(x_{1,2}) = d_1(x_{1,4}) = 0,$$

while $d_1(x_{1,6})$ is a multiple of $x_{2,0}$. We know that d_1 is not identically zero, because

$$H^1(\mathbb{S}_2^1; (E_2)_*Z) \cong H^1(S_2^1; (E_2)_*Z)^{C_3}$$

has rank at most 3. Since differentials preserve internal grading,

$$\mathcal{E} \cdot x_{1,6} = \lambda v_2 x_{2,0},$$

where $\lambda \in \mathbb{F}_4^\times$, is forced. Since $E_1^{p,q} = 0$ for $q \neq 0$, the duality resolution spectral sequence for Z collapses at the E_2–page. □

Remark 5.8 Since $H^p(\mathbb{S}_2^1; (E_2)_*Z) \cong H^p(S_2^1; (E_2)_*Z)^{C_3}$ and

$$H^p(S_2^1; (E_2)_*Z) \cong H^*(K^1; \mathbb{F}_2) \otimes \mathbb{F}_4[u^{\pm 1}]$$

by Theorem 5.6, it suffices to understand the action of C_3 on $H^p(K^1; \mathbb{F}_2)$. This action is given by

$$\bar{\omega} \cdot y_0 = y_0, \quad \bar{\omega} \cdot y_1 = y_1 + y_2, \quad \bar{\omega} \cdot y_2 = y_1 \tag{5-7}$$

and can be deduced from [2, Section 2.5]. Despite the fact that the isomorphism of Theorem 5.6 is not an isomorphism of C_3–modules, with careful bookkeeping, the action of C_3 on $H^*(S_2^1; (E_2)_*Z)$ can nonetheless be deduced from the actions of C_3 on $H^*(K^1; \mathbb{F}_2)$ and $\mathbb{F}_4[u^{\pm 1}]$. Therefore, knowledge of (5-6) and (5-7) lets us completely calculate $H^p(\mathbb{S}_2^1; (E_2)_*Z)$ without resorting to the duality resolution.

However, most existing $K(2)$–local computations are done using the duality resolution spectral sequence, which is why we chose this method, providing a better basis for comparison with previous work.

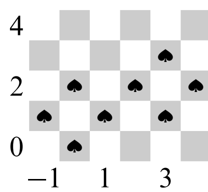
Corollary 5.9 *The homotopy fixed point spectral sequence*

$$E_2^{s,t} = H^s(\mathbb{S}_2^1; (E_2)_t Z) \Rightarrow \pi_{t-s}(E^{h\mathbb{S}_2^1} \wedge Z)$$

with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ *has* E_2*–page*

$$E_2^{s,*} = H^s(\mathbb{S}_2^1; (E_2)_* Z) \cong \begin{cases} \mathbb{F}_4[v_2^{\pm 1}]\langle x_{0,0}\rangle & \text{if } s = 0, \\ \mathbb{F}_4[v_2^{\pm 1}]\langle x_{1,0}, x_{1,2}, x_{1,4}\rangle & \text{if } s = 1, \\ \mathbb{F}_4[v_2^{\pm 1}]\langle x_{2,2}, x_{2,4}, x_{2,6}\rangle & \text{if } s = 2, \\ \mathbb{F}_4[v_2^{\pm 1}]\langle x_{3,0}\rangle & \text{if } s = 3, \end{cases}$$

or in graphical form (in Adams' grading) with each ♠ *denoting a copy of* $\mathbb{F}_4[v_2^{\pm 1}]$:



The spectral sequence collapses at the E_2*–page due to sparseness.*

Remark 5.10 According to recent work of Goerss and Bobkova [8], there is a topological version of the duality resolution, which gives a resolution of the $K(2)$–local sphere. The topological duality resolution can be used to compute $\pi_*(E_2^{h\mathbb{S}_2^1} \wedge Z)$ directly. However, for Z, the algebraic and the topological duality spectral sequences are isomorphic and the computations remain identical as the relevant spectral sequences simply collapse.

6 The $K(2)$–local homotopy groups of Z

The $K(2)$–local homotopy groups of Z can be computed using the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(\mathbb{S}_2; (E_2)_t Z)^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \Rightarrow \pi_{t-s} L_{K(2)} Z$$

given in (4-1), where $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ merely plays the role of "changing the coefficient field from $\mathbb{F}_4$ to $\mathbb{F}_2$".

Recall the norm map (4-5), $N\colon \mathbb{S}_2 \to \mathbb{Z}_2$, whose kernel is $\mathbb{S}_2^1$. By choosing an element $\gamma \in \mathbb{S}_2$ such that $N(\gamma)$ is a topological generator of $\mathbb{Z}_2$, one can produce a map $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{S}_2^1)$ which sends $1 \in \mathbb{Z}_2$ to the conjugation automorphism by γ, which gives an isomorphism

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2.$$

In [2; 3], γ is chosen to be π. However, one can also choose $\gamma = \alpha^{-1}\pi$. We choose $\gamma = \alpha^{-1}\pi$ to get the isomorphism $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$. Note that $\alpha^{-1}\pi \in F_{4/2}\mathbb{S}_2$ and therefore it acts trivially on $(E_2)_*Z$. Consequently, we have:

Lemma 6.1 *The action of γ on $H^*(\mathbb{S}_2^1; (E_2)_*Z)$ is trivial.*

We postpone the proof of Lemma 6.1 until the end of this section, so that we can focus on its immediate consequences. Lemma 6.1 simplifies the calculation of the E_2–page of the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}_2; H^q(\mathbb{S}_2^1; (E_2)_*Z)) \cong E(\zeta) \otimes H^*(\mathbb{S}_2^1; (E_2)_*Z) \Rightarrow H^{p+q}(\mathbb{S}_2; (E_2)_*Z),$$

which collapses due to sparseness. Therefore,

$$H^*(\mathbb{S}_2; (E_2)_*Z)^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \cong [E(\zeta) \otimes H^*(\mathbb{S}_2^1; (E_2)_*Z)]^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)},$$

where ζ has bidegree $(s,t) = (1,0)$. More precisely, as an $\mathbb{F}_2[v_2^{\pm 1}]$–module,

$$H^s(\mathbb{S}_2; (E_2)_*Z)^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \cong \begin{cases} \mathbb{F}_2[v_2^{\pm 1}]\langle x_{0,0}\rangle & \text{if } s = 0, \\ \mathbb{F}_2[v_2^{\pm 1}]\langle \zeta x_{0,0}, x_{1,0}, x_{1,2}, x_{1,4}\rangle & \text{if } s = 1, \\ \mathbb{F}_2[v_2^{\pm 1}]\langle x_{2,2}, x_{2,4}, x_{2,6}, \zeta x_{1,0}, \zeta x_{1,2}, \zeta x_{1,4}\rangle & \text{if } s = 2, \\ \mathbb{F}_2[v_2^{\pm 1}]\langle x_{3,0}, \zeta x_{2,2}, \zeta x_{2,4}, \zeta x_{2,6}\rangle & \text{if } s = 3, \\ \mathbb{F}_2[v_2^{\pm 1}]\langle \zeta x_{3,0}\rangle & \text{if } s = 4, \\ 0 & \text{if } s > 4. \end{cases}$$

In Figure 2, we draw the E_2–page of (4-1). We denote by $\circ$ the generators that are multiples of ζ, and all others by $\bullet$.

It is clear that (4-1) collapses at the E_4–page. The only possibilities are two sets of v_2–linear d_3–differentials

- $d_3(x_{0,0}) = v_2^{-1}\zeta x_{2,6}$, and
- $d_3(x_{1,4}) = v_2 \zeta x_{3,0}$.

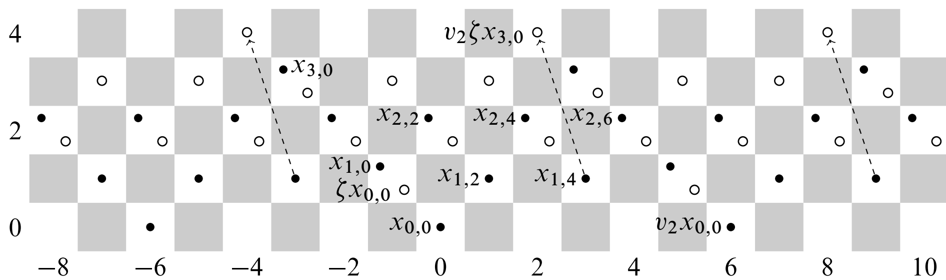


Figure 2: The E_2–page of (4-1).

The v_2–linearity of differentials follows from the fact that Z admits a v_2^1–self-map [6]. However, the generator $x_{0,0}$ cannot support a differential for the following reason:

The inclusion of the bottom ι_0: $S^0 \hookrightarrow Z$ induces a nontrivial map $K(2)$–homology. Therefore, ι_0 induces a nontrivial element in $\bar{\iota} \in \pi_0 L_{K(2)} Z$ which is represented by $x_{0,0}$ in the E_2–page of (4-1). Therefore, $x_{0,0}$ is a permanent cycle.

From the calculation of the classical Adams spectral sequence in [6],

$$\mathrm{Ext}_A^{s,t}(H^*Z, \mathbb{F}_2) \Rightarrow \pi_* Z,$$

we see that $\pi_0 Z \cong \mathbb{Z}/2$. In particular, this means $[\iota_0]$ is the generator of $\pi_0 Z$ and $2[\iota_0] = 0$. Since the map η: $Z \to L_{K(2)} Z$ sends $[\iota_0] \mapsto \bar{\iota}$, it must be the case that $2\bar{\iota} = 0$. Therefore there is no hidden extension supported by $x_{0,0}$.

Moreover it is well known that $\tilde{\zeta}$ is a class in $\pi_{-1} L_{K(2)} S^0$. Let $\hat{\zeta}$ denote the representative of $\tilde{\zeta}$ in the E_2–page of the descent spectral sequence (2-8),

$$E_2^{s,t} = H^s(\mathbb{S}_2; (E_2)_t)^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \Rightarrow \pi_{t-s} L_{K(2)} S^0.$$

A straightforward analysis of the map of descent spectral sequences from (2-8) to (4-1) induced by ι_0 shows that $\hat{\zeta} \cdot x_{0,0} = \zeta x_{0,0}$, which is a nonzero permanent cycle representing $\tilde{\zeta} \cdot \bar{\iota} \in \pi_{-1} L_{K(2)} Z$. Since $2\bar{\iota} = 0$, it follows that

$$2(\tilde{\zeta} \cdot \bar{\iota}) = \tilde{\zeta} \cdot 2\bar{\iota} = 0,$$

ruling out another possible v_2–periodic family of hidden extensions. There are other possibilities of hidden extensions depicted in Figure 1, which we currently cannot rule out, though low-dimensional computations lead us to believe that there exists a particular spectrum Z for which all differentials and possible hidden extensions are zero. Furthermore, as stated in Conjecture 1, we expect that this will be the case for *every* spectrum $Z \in \widetilde{\mathcal{Z}}$.

Proof of Lemma 6.1 Notice that $F_{2/2}\mathbb{S}_2^1 = F_{2/2}\mathbb{S}_2 \cap \mathbb{S}_2^1$ is a normal subgroup of $\mathbb{S}_2$ and

$$\mathbb{S}_2/F_{2/2}\mathbb{S}_2^1 \cong G_{24}/C_{\hat{\mathbb{1}}} \times \mathbb{Z}_2\langle\gamma\rangle.$$

From our work in Theorem 5.6, we see that

$$\text{(6-1)} \qquad \begin{aligned} H^*(\mathbb{S}_2^1;(E_2)_*Z) &\cong \big(H^*(F_{2/2}\mathbb{S}_2^1;(E_2)_*Z)^{\mathbb{Q}_8/C_{\hat{\mathbb{1}}}}\big)^{C_3} \\ &\cong H^*(F_{2/2}\mathbb{S}_2^1;(E_2)_*Z)^{G_{24}/C_{\hat{\mathbb{1}}}}. \end{aligned}$$

Also note that $G_{24}/C_{\hat{\mathbb{1}}}$ is a normal subgroup of $\mathbb{S}_2/F_{2/2}\mathbb{S}_2^1$. Therefore, γ acts on $H^*(F_{2/2}\mathbb{S}_2^1;(E_2)_*Z)^{G_{24}/C_{\hat{\mathbb{1}}}}$ and the isomorphism in (6-1) commutes with the action of γ. Therefore, it suffices to prove that γ acts trivially on $H^*(F_{2/2}\mathbb{S}_2^1;(E_2)_*Z)$.

Since $F_{2/2}\mathbb{S}_2^1 \cong K^1 \times C_{\hat{\mathbb{1}}}$ and γ acts trivially on $C_{\hat{\mathbb{1}}}$, we have a sequence of natural γ–equivariant maps

$$\begin{aligned} H^*(K^1;(E_2)_*Z^{C_{\hat{\mathbb{1}}}}) \cong H^*(F_{2/2}\mathbb{S}_2^1/C_{\hat{\mathbb{1}}};(E_2)_*Z^{C_{\hat{\mathbb{1}}}}) \to H^*(F_{2/2}\mathbb{S}_2^1;(E_2)_*Z^{C_{\hat{\mathbb{1}}}}) \\ \to H^*(F_{2/2}\mathbb{S}_2^1;(E_2)_*Z), \end{aligned}$$

where the first map is induced by the quotient map and the second map is simply inclusion of the coefficients. Note that the composite is an isomorphism. Since K^1 acts trivially on $(E_2)_*Z^{C_{\hat{\mathbb{1}}}}$, the isomorphism

$$H^*(K^1;(E_2)_*Z^{C_{\hat{\mathbb{1}}}}) \cong H^*(K^1;\mathbb{F}_2) \otimes (E_2)_*Z^{C_{\hat{\mathbb{1}}}}$$

is γ–equivariant. Since γ acts trivially on $(E_2)_*Z$ and $H^*(K^1;\mathbb{F}_2)$ (see Lemmas 4.9 and 6.2), the result follows. □

Lemma 6.2 *The action of γ on $H^*(K^1;\mathbb{F}_2)$ is trivial.*

Proof Since $H^*(K^1;\mathbb{F}_2)$ is a ring, it is enough to show that γ acts trivially on the generators (see (5-6)) $y_0, y_1, y_2 \in H^1(K^1;\mathbb{F}_2)$. A fundamental fact of group cohomology says that

$$H^1(K^1;\mathbb{F}_2) \cong \mathrm{Hom}(K^1/\overline{[K^1,K^1](K^1)^2}, \mathbb{F}_2),$$

where $[K^1,K^1]$ denotes the commutator subgroup. It can be deduced from Section 2.5 of [2] that for every element $g \in K^1$, its conjugate $\gamma^{-1}g\gamma$ belongs to the same coset as g, ie $\gamma^{-1}g\gamma \in g\overline{[K^1,K^1](K^1)^2}$. Hence the result holds. □

Appendix A regularity criterion for a representation of Q_8

The quaternionic group Q_8 is an order 8 group which can be presented as

$$Q_8 = \langle \mathsf{i}, \mathsf{j} : \mathsf{i}^4 = \mathbb{1}, \mathsf{i}^2 = \mathsf{j}^2, \mathsf{i}^3\mathsf{j} = \mathsf{j}\mathsf{i}\rangle. \tag{A-1}$$

We will denote the neutral element of Q_8 by $\mathbb{1}$. Often in the literature, ij is denoted by k and i^2 by -1. This is justified as $-1 \in Q_8$ is central and its square is $\mathbb{1}$. However, -1 also denotes the additive inverse of 1 in a ring, and potentially can cause confusion while working with group rings. Therefore we will instead denote -1 by $\widehat{\mathbb{1}} \in Q_8$ and $\widehat{\mathsf{i}} = \widehat{\mathbb{1}}\mathsf{i}$, $\widehat{\mathsf{j}} = \widehat{\mathbb{1}}\mathsf{j}$ and $\widehat{\mathsf{k}} = \widehat{\mathbb{1}}\mathsf{k}$. With this notation, the relations in Q_8 can be rewritten as

- $\mathsf{ij} = \mathsf{k}$, $\mathsf{jk} = \mathsf{i}$ and $\mathsf{ki} = \mathsf{j}$,
- $\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = \widehat{\mathbb{1}}$,
- $(\widehat{\mathbb{1}})^2 = 1$, and
- $\mathsf{ji} = \widehat{\mathsf{k}}$, $\mathsf{kj} = \widehat{\mathsf{i}}$ and $\mathsf{ik} = \widehat{\mathsf{j}}$.

The quotient of the central subgroup of order 2 generated by $\widehat{\mathbb{1}}$ is the *Klein four group* $C_2 \times C_2$. In other words we have an exact sequence of groups

$$\mathbb{1} \to C_2 \xrightarrow{\iota} Q_8 \xrightarrow{q} C_2 \times C_2 \to \mathbb{1}.$$

We will denote the images of $\mathsf{i}, \mathsf{j} \in Q_8$ by $\mathsf{i}, \mathsf{j} \in C_2 \times C_2$.

Let $\mathbb{F}$ be an arbitrary field and let $V_4(\mathbb{F})$ denote the 4–dimensional representation of Q_8 induced by the regular representation of $C_2 \times C_2$ via the quotient map q. Let $V_8(\mathbb{F})$ denote the regular representation of Q_8. When $\operatorname{char} \mathbb{F} = 2$, it is easy to see that there is an exact sequence of $\mathbb{F}[Q_8]$–modules

$$0 \to V_4(\mathbb{F}) \xrightarrow{t} V_8(\mathbb{F}) \xrightarrow{r} V_4(\mathbb{F}) \to 0.$$

More explicitly, let ι_4 and ι_8 be the generators of $V_4(\mathbb{F})$ and $V_8(\mathbb{F})$ as $\mathbb{F}[Q_8]$–modules and define

$$r(g \cdot \iota_8) = q(g) \cdot \iota_4, \quad t(h \cdot \iota_4) = h \cdot \iota_8 + \widehat{\mathbb{1}}h \cdot \iota_8$$

for $h, g \in Q_8$.

The purpose of this appendix is to give a necessary and sufficient condition on an 8–dimensional representation V over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} = 2$ which fits in the exact sequence

$$0 \to V_4(\mathbb{F}) \xrightarrow{\tilde{t}} V \xrightarrow{\tilde{r}} V_4(\mathbb{F}) \to 0, \tag{A-2}$$

under which it is isomorphic to $V_8(\mathbb{F})$. When $\operatorname{char} \mathbb{F} \neq 2$, the problem is straightforward.

Any V which satisfies (A-2) is isomorphic to $V_4(\mathbb{F}) \oplus V_4(\mathbb{F})$, including $V_8(\mathbb{F})$, the regular representation of Q_8. This is because, when $\mathsf{char}\,\mathbb{F} \nmid |Q_8|$ and W is a subrepresentation of V, one can define a complement subrepresentation W' such that $V \cong W \oplus W'$ (Maschke's theorem). In our case, let $W = \operatorname{img} \tilde{\imath}$ and W' be its complement. Since (A-2) is an exact sequence, it follows that

$$W \cong W' \cong V_4(\mathbb{F}).$$

We will soon see that $V_8(\mathbb{F}) \ncong V_4(\mathbb{F}) \oplus V_4(\mathbb{F})$ when $\mathsf{char}\,\mathbb{F} = 2$.

For any $g \in G$, let $e_g \in \mathbb{F}[G]$ denote the element such that

$$g' e_g = e_{g'g}$$

for every $g' \in G$. The collection $\{e_g : g \in G\}$ forms a basis for $\mathbb{F}[G]$. For our convenience, we consider the ordered basis

(A-3) $$\mathcal{B}_4 = \{v_1 = e_{\mathbb{1}} + e_{\mathsf{i}} + e_{\mathsf{j}} + e_{\mathsf{k}},\ v_2 = e_{\mathbb{1}} + e_{\mathsf{j}},\ v_3 = e_{\mathbb{1}} + e_{\mathsf{i}},\ v_4 = e_{\mathbb{1}}\}$$

of $V_4(\mathbb{F})$. Note that

$$(\mathsf{i})_{\mathcal{B}_4} = \begin{bmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}, \quad (\mathsf{j})_{\mathcal{B}_4} = \begin{bmatrix} 1&0&1&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \quad (\mathsf{k})_{\mathcal{B}_4} = \begin{bmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}.$$

Thus any vector space isomorphic to the regular representation of $C_2 \times C_2$ admits a basis $\mathcal{B}$ such that

$$(\mathsf{i})_{\mathcal{B}} = (\mathsf{i})_{\mathcal{B}_4}, (\mathsf{j})_{\mathcal{B}} = (\mathsf{j})_{\mathcal{B}_4}, (\mathsf{k})_{\mathcal{B}} = (\mathsf{k})_{\mathcal{B}_4}.$$

The main result in this appendix is the following:

Lemma A.1 *Let $\mathbb{F}$ be a field with $\mathsf{char}\,\mathbb{F} = 2$. Suppose we have an exact sequence of $\mathbb{F}[Q_8]$–modules*

(A-4) $$0 \to V_4 \xrightarrow{\tilde{\imath}} V_8 \xrightarrow{\tilde{r}} V_4 \to 0,$$

where V_4 is a representation of Q_8 induced from the regular representation of $C_2 \times C_2$. Let $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ be a basis of V_4 such that

$$(\mathsf{i})_{\mathcal{B}} = \begin{bmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}, \quad (\mathsf{j})_{\mathcal{B}} = \begin{bmatrix} 1&0&1&0\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}, \quad (\mathsf{k})_{\mathcal{B}} = \begin{bmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}.$$

Then, for any basis $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_1', c_2', c_3', c_4'\}$ of V_8 with the property that $\tilde{t}(v_i) = c_i$ and $\tilde{r}(c_i') = v_i$, we have

(1) $(\widehat{\mathbb{1}})_{\mathcal{C}} = \left(\begin{smallmatrix} I_4 & M \\ 0 & I_4 \end{smallmatrix}\right)$, *where*

$$M = \begin{bmatrix} c & d & a & b \\ 0 & c & 0 & a \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{bmatrix}$$

for $a, b, c, d \in \mathbb{F}$, and

(2) *if $c \neq 0$ then V_8 is isomorphic to the regular representation of Q_8.*

Proof It follows from (A-4) that

$$(\mathrm{i})_{\mathcal{C}} = \begin{bmatrix} (\mathrm{i})_{\mathcal{B}} & X \\ 0 & (\mathrm{i})_{\mathcal{B}} \end{bmatrix} \quad \text{and} \quad (\mathrm{j})_{\mathcal{C}} = \begin{bmatrix} (\mathrm{j})_{\mathcal{B}} & Y \\ 0 & (\mathrm{j})_{\mathcal{B}} \end{bmatrix}$$

for some 4×4 matrices X and Y. Let x_{ij} and y_{ij} denote the $(i, j)^{\text{th}}$ entry of X and Y, respectively. Since the choice of c_i' is only unique modulo $\operatorname{img} \tilde{t}$, we may apply a change of basis matrix of the form

$$P = \begin{bmatrix} I_4 & \overline{P} \\ 0 & I_4 \end{bmatrix}.$$

In particular, if we choose

$$\overline{P} = \begin{bmatrix} y_{13} & 0 & x_{14} & 0 \\ x_{11} & x_{12} + y_{13} & x_{13} & 0 \\ y_{11} & y_{12} & 0 & y_{14} \\ x_{31} & x_{32} + y_{11} & x_{33} & x_{34} \end{bmatrix},$$

we see that

$$P(\mathrm{i})_{\mathcal{C}} P^{-1} = \begin{bmatrix} (\mathrm{i})_{\mathcal{B}} & \widetilde{X} \\ 0 & (\mathrm{i})_{\mathcal{B}} \end{bmatrix}, \quad P(\mathrm{j})_{\mathcal{C}} P^{-1} = \begin{bmatrix} (\mathrm{j})_{\mathcal{B}} & \widetilde{Y} \\ 0 & (\mathrm{j})_{\mathcal{B}} \end{bmatrix},$$

where

$$\widetilde{X} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_{21} & x_{11} + x_{12} & x_{23} & x_{13} + x_{24} \\ 0 & 0 & 0 & 0 \\ x_{41} & x_{31} + x_{42} & x_{43} & x_{33} + x_{34} \end{bmatrix}$$

and

$$\widetilde{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_{31} + y_{21} & x_{32} + y_{11} + y_{22} & x_{11} + x_{33} + y_{23} & x_{12} + x_{34} + y_{13} + y_{24} \\ y_{31} & y_{32} & y_{11} + y_{33} & y_{12} + y_{34} \\ y_{41} & y_{42} & x_{31} + y_{43} & x_{32} + y_{11} + y_{44} \end{bmatrix}.$$

Thus, without loss of generality we may assume that

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & x_{42} \\ 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_{21} & y_{22} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{33} & y_{43} \\ y_{41} & y_{42} & y_{43} & y_{44} \end{bmatrix}.$$

Now we use the relations (A-1) to get further restrictions on X and Y. While $(\mathsf{i})_C^4 = (\mathsf{j})_C^4 = I_8$ is trivially satisfied, $(\mathsf{i})_C^2 = (\mathsf{j})_C^2$ is true if and only if

$$(\mathsf{i})_C X + X(\mathsf{i})_C = (\mathsf{j})_C Y + Y(\mathsf{j})_C.$$

Thus we get a linear system with free variables y_{23}, y_{24}, y_{32}, y_{33}, y_{34}, y_{42}, y_{43} and y_{44}, and we get

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_{42} & y_{32} & y_{33} & y_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y_{42} & y_{32} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ y_{43} & y_{43}+y_{44} & y_{23} & y_{24} \\ y_{42} & y_{32} & y_{33} & y_{34} \\ 0 & y_{42} & y_{43} & y_{44} \end{bmatrix}.$$

Consequently, $(\widehat{\mathbb{1}})_C = \left(\begin{smallmatrix} I_4 & M \\ 0 & I_4 \end{smallmatrix}\right)$, where

$$M = \begin{bmatrix} y_{42} & y_{32} & y_{33} & y_{34} \\ 0 & y_{42} & 0 & y_{33} \\ 0 & 0 & y_{42} & y_{32} \\ 0 & 0 & 0 & y_{42} \end{bmatrix}.$$

Now, the linear system generated by the relation

$$(\mathsf{i})_C(\mathsf{j})_C = (\widehat{\mathbb{1}})_C(\mathsf{j})_C(\mathsf{i})_C$$

has free variables y_{33}, y_{34}, y_{43} and y_{44}, and basic variables

$$y_{23} = y_{33} + y_{43}, \quad y_{24} = y_{34} + y_{44}, \quad y_{32} = y_{33} + y_{43} + y_{44}, \quad y_{42} = y_{43}.$$

Let $a = y_{33}$, $b = y_{34}$, $c = y_{43}$ and $d = y_{33} + y_{43} + y_{44}$. In terms of a, b, c and d, we have

$$\text{(A-5)} \qquad X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ c & d & a & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & d \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ c & c+d & a+c & a+b+c+d \\ c & d & a & b \\ 0 & c & c & a+c+d \end{bmatrix},$$

(A-6)
$$M = \begin{bmatrix} c & d & a & b \\ 0 & c & 0 & a \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{bmatrix}.$$

Recall that our change of basis matrix was of the form

$$P = \begin{bmatrix} I_4 & \overline{P} \\ 0 & I_4 \end{bmatrix},$$

and thus $P^{-1} = P$ and we have

$$P^{-1}(\widehat{\mathbb{1}})_{\mathcal{C}} P = \begin{bmatrix} I_4 & M \\ 0 & I_4 \end{bmatrix}$$

as $\operatorname{char} \mathbb{F} = 2$. This proves (1).

For (2), we need to find a vector $\overline{v}$ such that

$$\{g\overline{v} : g \in Q_8\}$$

spans V_8. We choose $\overline{v} = c_4' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$ in the basis $\mathcal{C}$. Let

$$A = \begin{bmatrix} \overline{v} & (\widehat{\mathbb{1}})_{\mathcal{C}}\overline{v} & (\mathsf{i})_{\mathcal{C}}\overline{v} & (\widehat{\mathsf{i}})_{\mathcal{C}}\overline{v} & (\mathsf{j})_{\mathcal{C}}\overline{v} & (\widehat{\mathsf{j}})_{\mathcal{C}}\overline{v} & (\mathsf{k})_{\mathcal{C}}\overline{v} & (\widehat{\mathsf{k}})_{\mathcal{C}}\overline{v} \end{bmatrix}.$$

Using (A-5) and (A-6) we see that

$$A = \begin{bmatrix} 0 & b & 0 & a+b & 0 & b+d & a+b+c+d & 0 \\ 0 & a & b & a+b & a+b+c+d & b+d & a+c & 0 \\ 0 & d & 0 & c+d & b & b+d & a+b+c+d & a+b \\ 0 & c & d & c+d & a+c+d & a+d & a+c & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By a tedious but straightforward calculation, we find

$$\det A = c^4,$$

completing the proof of (2). □

Remark A.2 When $\operatorname{char} \mathbb{F} = 2$, the representations $V_4(\mathbb{F}) \oplus V_4(\mathbb{F})$ and $V_8(\mathbb{F})$ are not isomorphic. Without loss of generality we may assume $c = 1$ and $a = b = d = 0$.

Suppose there were an isomorphism between $V_4(\mathbb{F}) \oplus V_4(\mathbb{F})$ and $V_8(\mathbb{F})$. Then there would exist a invertible matrix P such that

$$P \begin{bmatrix} (\widehat{\mathbb{1}})_{\mathcal{B}_4} & 0 \\ 0 & (\widehat{\mathbb{1}})_{\mathcal{B}_4} \end{bmatrix} = (\widehat{\mathbb{1}})_{\mathcal{C}} P.$$

Note that $(\widehat{\mathbb{1}})_{\mathcal{B}_4}$ is simply the identity matrix, while $(\widehat{\mathbb{1}})_{\mathcal{C}}$ is not. It follows easily that any matrix which satisfies the above condition is not invertible, whence a contradiction.

Remark A.3 We are unaware of any classification theorem for 8–dimensional representations of Q_8 over fields of characteristic 2. We suspect that the question of how many isomorphism classes of V satisfy (A-2) can be resolved. A possible guess might be that there are overall four isomorphism classes:

- $c \neq 0$ (when $V \cong V_8(\mathbb{F})$),
- $c = 0$ and $d \neq 0$,
- $c = 0$, $d = 0$ and $a \neq 0$,
- $c = 0$, $a = 0$, $d = 0$ and $b \neq 0$, and
- $a = b = c = d = 0$ (when $V \cong V_4(\mathbb{F}) \oplus V_4(\mathbb{F})$).

Since this is irrelevant to the purpose of the paper, we leave this question to the interested reader to verify.

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Department of Mathematics, University of Virginia
Charlottesville, VA, United States
Center for Neuroprosthetics, Swiss Federal Institute of Technology (EPFL)
Geneva, Switzerland

`pb9wh@virginia.edu, philip.egger@epfl.ch`

Received: 17 April 2018 Revised: 22 August 2019