

Transverse link invariants from the deformations of Khovanov \mathfrak{sl}_3 -homology

CARLO COLLARI

We make use of the Mackaay–Vaz approach to the universal \mathfrak{sl}_3 -homology to define a family of cycles (called β_3 -invariants) which are transverse braid invariants. This family includes Wu’s ψ_3 -invariant. Furthermore, we analyse the vanishing of the homology classes of the β_3 -invariants and relate it to the vanishing of Plamenevskaya’s and Wu’s invariants. Finally, we use the β_3 -invariants to produce some Bennequin-type inequalities.

57M25, 57R17; 57M27

1 Introduction

Background material and motivations Let (x, y, z) be a system of coordinates in \mathbb{R}^3 . The *symmetric contact structure* on \mathbb{R}^3 is the plane distribution $\xi_{\text{sym}} = \ker(dz + x dy - y dx)$. A *link* is a smooth embedding of a number of copies of \mathbb{S}^1 into \mathbb{R}^3 , and a *knot* is a link with a single component. The presence of ξ_{sym} allows one to distinguish a special class of links (and knots): those which are nowhere tangent to ξ_{sym} (*transverse links*).

Two transverse links are *equivalent* (or the same *transverse type*) if they are ambient isotopic through a one-parameter family of transverse links. In particular, equivalent transverse links represent the same link type (ie the ambient isotopy class of a link).

The study of transverse links has played (and still plays) a key role in the study of low-dimensional topology. The aim of the present paper is to present new invariants for transverse links arising from the deformations of Khovanov \mathfrak{sl}_3 -homology. Each transverse link inherits a natural orientation from ξ_{sym} ; see Etnyre [4]. All transverse invariants defined in this paper are defined with respect to this orientation.

In order to define our invariants we need a particular representation of transverse links. It is a result due to D Bennequin [2] that each transverse link is equivalent to a closed braid. Another result, due to S Orevkov and V Shevchishin [19] and independently N Wrinkle [23], provides a complete set of combinatorial moves relating all braids

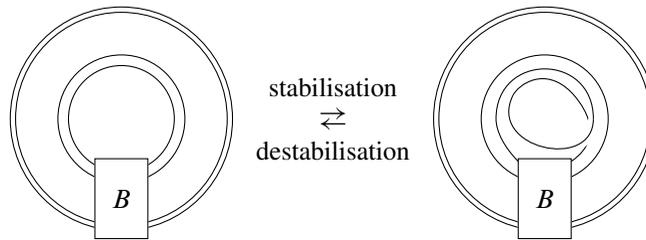


Figure 1: Stabilisation and destabilisation as Reidemeister moves between closures.

whose closure represent the same transverse type. We summarise these results in the following theorem, which shall be referred to as the *transverse Markov theorem* in the rest of the paper.

Theorem 1 (Bennequin [2], Orevkov and Shevchishin [19], and Wrinkle [23]) *Any transverse link is transversely isotopic to the closure of a braid (with axis the z -axis). Moreover, two braids represent the same transverse type if and only if they are related by a finite sequence of braid relations, conjugations, positive stabilisations, and positive destabilisations.¹ These moves are called **transverse Markov moves**.*

Remark Braids are naturally oriented, and their orientation coincides with the orientation of the corresponding transverse link.

Remark By adding the negative stabilisation and destabilisation to the set of transverse Markov moves one recovers the full set of *Markov moves*.

Any sequence of Markov moves between braids naturally translates into a sequence of oriented Reidemeister moves between their closures. In particular, conjugation in the braid group can be seen as a sequence of Reidemeister moves of the second type followed by a planar isotopy, while the braid relations can be seen as either second or third Reidemeister moves. We remark that not all the oriented versions of second and third Reidemeister moves arise in this way; those which can be obtained as a composition of Markov moves and braid relations are called *braid-like* or *coherent*. Finally, a positive (resp. negative) stabilisation translates into a positive (resp. negative) first Reidemeister move, as shown in Figure 1.

¹The *positive* (resp. *negative*) stabilisation of $B \in B_{m-1}$ is the braid $B\sigma_m \in B_m$ (resp. $B\sigma_m^{-1} \in B_m$). The destabilisation is just the inverse process: if one considers a braid of the form $A\sigma_m B$ (resp. $A\sigma_m^{-1} B$), where $A, B \in B_{m-1}$, then its positive (resp. negative) destabilisation is the braid AB .

Transverse links have two *classical invariants*: the link type and the *self-linking number* sl .² The latter is defined as follows. Let B be a braid, and set

$$sl(B) = w(B) - b(B),$$

where w denotes the *writhe*, and b denotes the *braid index* (ie the number of strands). In the light of the transverse Markov theorem, it is immediate that the self-linking number is a transverse invariant. Moreover, from the definition of sl follows easily that a negative stabilisation (resp. destabilisation) does not preserve the equivalence class of the transverse link. Any invariant capable of distinguishing distinct transverse links with the same classical invariants is called *effective*. It is a major problem in the study of transverse links to find effective invariants.

O Plamenevskaya [20], making use of the representation of transverse links as closed braids, introduced a transverse invariant ψ . This invariant is a homology class in Khovanov’s categorification of the Jones polynomial. Since Plamenevskaya’s ground-breaking work, other invariants for transverse links coming from quantum (and also Floer theoretic) link homologies have been introduced. In 2008, H Wu [24] generalised ψ to a family of invariants ψ_N which are homology classes in M Khovanov and L Rozansky’s categorification of the Reshetikhin–Turaev \mathfrak{sl}_N -invariants. Khovanov–Rozansky homologies (and also Khovanov homology, being the case $N = 2$) admit deformations; these “deformed theories” are parametrised by a monic polynomial of degree N called the potential. In 2015, R Lipshitz, L Ng, and S Sarkar [14] extended the definition of the Plamenevskaya invariant to two chains ψ^\pm belonging to the chain complex of (a twisted version of) Lee’s deformation (ie the theory associated to the potential $x^2 - x$). The author in [3] extended the Plamenevskaya invariant to all deformations of Khovanov homology. It is still unknown at the time of writing whether or not all these invariants are effective.

Outline and statement of results The aim of this paper is to extend the definition of Wu’s ψ_3 -invariant to all deformations of Khovanov’s \mathfrak{sl}_3 -homology. As we stated above, these deformations are parametrised by a monic polynomial $\omega \in R[x]$ of degree 3 (the *potential*), where R is the coefficient ring of the homology theory. Furthermore,

²Technically speaking, for links one may record the self-linking number of each component and this defines a slightly stronger transverse invariant, called the self-linking matrix. However, for consistency with the literature in the subject [3; 14; 20; 24] in this paper we shall consider only the self-linking number.

³The \mathfrak{sl}_3 -homology was defined first by Khovanov, and then Khovanov and Rozansky extended the definition to all \mathfrak{sl}_N for $N \geq 2$ with a different technique.

the resulting homology theory is either graded or filtered depending on ω . We make use of the construction of the universal \mathfrak{sl}_3 -homology due to M Mackaay and P Vaz [17] which encodes all the deformations of Khovanov \mathfrak{sl}_3 -homology [18]. We tried to keep the paper as self-contained as possible, so we shall review Mackaay and Vaz's construction in Sections 2 and 3.

In Section 4 we define for each diagram D , each potential ω , and each root x_1 of ω in R , a chain $\beta_{\omega, x_1}(D)$, which turns out to be a cycle (Proposition 11). If $D = \hat{B}$ is the closure of a braid B , we shall denote $\beta_{\omega, x_1}(\hat{B})$ simply by $\beta_{\omega, x_1}(B)$. Our main theorem is the following.

Theorem 2 *Let B be a braid. For each root x_1 of $\omega(x)$, the cycle $\beta_{\omega, x_1}(\overline{B})$ is a transverse invariant, where the overline indicates the mirror braid. More precisely, if B and B' are related by a sequence of transverse Markov moves, and Φ denotes the map associated to the mirror sequence of these moves, then*

$$\Phi(\beta_{\omega, x_1}(\overline{B})) = \beta_{\omega, x_1}(\overline{B'}).$$

Moreover, if the homology theory is filtered (resp. graded), then the filtered degree⁴ (resp. the degree) of $\beta_{\omega, x_1}(\overline{B})$ is $-2 \operatorname{sl}(B)$.

Remark The filtered degree mentioned in Theorem 2 is the filtered degree of *the chain* $\beta_{\omega, x_1}(\overline{B})$, and not the filtered degree of its homology class. The filtered degree of $[\beta_{\omega, x_1}(\overline{B})]$ is not necessarily a meaningful transverse invariant. For instance, if \mathbb{F} is a field and $\omega \in \mathbb{F}[x]$ has three distinct roots, then the filtered degree of $[\beta_{\omega, x_1}(\overline{B})]$ is a concordance invariant (one of the j_i mentioned in Proposition 4). The comparison between the two filtered degrees, under the above hypotheses, gives a weaker version of the first Bennequin-type inequality in Proposition 4.

Motivated by the theorem above, the cycles $\beta_{\omega, x_1}(\overline{B})$ are collectively called β_3 -invariants. We wish to point out that the construction of the β_3 -invariants seems to be generalisable to the \mathfrak{sl}_N case (also with colours). This will be the subject of a forthcoming paper by the author joint with P Wedrich.

Exploring the effectiveness of these invariants is quite a difficult problem. The first reason is that distinguishing nonequivalent transverse knots with the same classical invariants is a subtle problem. This is also confirmed by the fact that there are only a

⁴That is, the index corresponding to the smallest subcomplex in the filtration containing the chain $\beta_{\omega, x_1}(\overline{B})$.

few known families of transverse knots which are not distinguished by their classical invariants. The second reason is the nature of our invariants: these are chains in a diagram-dependent chain complex up to the action of a certain group of chain homotopies. So we leave open the following question.

Question Are the β_3 -invariants effective?

In Section 5 we make use of the homology classes of the β_3 -invariants to define two auxiliary invariants which are easier to analyse. These invariants are the vanishing of the homology class, and the divisibility with respect to a nonunit element of the homology class.

In the case where the base ring is a field we can relate the vanishing of the homology class of the β_3 -invariants to the vanishing of the Plamenevskaya invariant and Wu's ψ_3 -invariant. Our second main result is the following.

Proposition 3 *Let ω be a potential over a field \mathbb{F} , and let $x_1, x_2, x_3 \in \mathbb{F}$ be the roots of ω . Denote by H_ω^\bullet the homology corresponding to the potential ω . Then, given a braid B , we have the following:*

- (1) *If x_1 is a simple root of ω , then $[\beta_{\omega, x_1}(\bar{B})]$ is nontrivial in $H_\omega^\bullet(\bar{B}, \mathbb{F})$.*
- (2) *If x_1 is a double root of ω , then $[\beta_{\omega, x_1}(\bar{B})]$ vanishes if and only if the Plamenevskaya invariant $\psi(B)$ vanishes in $\text{Kh}^\bullet(B, \mathbb{F})$.*
- (3) *If x_1 is a triple root of ω , then $[\beta_{\omega, x_1}(\bar{B})]$ vanishes if and only if Wu's invariant $\psi_3(B)$ vanishes in $H_{x_3}^\bullet(\bar{B}, \mathbb{F})$.*

In particular, the vanishing of $[\beta_{\omega, x_1}(\bar{B})]$ does not depend on the potential ω and on the root x_1 , but only on the multiplicity of x_1 as a root of ω . (However, it may depend on the choice of the base field.)

The previous proposition holds true only if the base ring is a field. This naturally leads to the following question.

Question Assume the base ring R to be a Noetherian domain. Is it true that the vanishing of $[\beta_{\omega, x_1}(\bar{B})]$ depends only on the multiplicity of x_1 and on the vanishing of the ψ - and ψ_3 -invariants?

The previous proposition allows us to relate the effectiveness of the vanishing of the homology classes of the β_3 -invariants with the effectiveness of the vanishing of the ψ - and ψ_3 -invariants. In particular, taking into account the results in [3; 14], we obtain

the noneffectiveness of the vanishing of the homology classes of the β_3 -invariants corresponding to double roots in the following cases: knots with crossing number ≤ 11 , flypes, and two-bridge knots. Most of the examples of distinct transverse knots with the same classical invariants fall into these categories. However, the following question remains open.

Question Is the vanishing of the homology classes of the β_3 -invariants an effective invariant?

Let R be an integral domain and $a \in R \setminus \{0\}$ a nonunit element. Given a potential $\omega \in R[x]$ and one of its roots $x_1 \in R$, the number

$$c_{\omega, x_1}(B; a) = \max\{k \mid \text{there exists } [y] \in H_{\omega}^0(\bar{B}) \text{ such that } a^k [y] = [\beta_{\omega, x_1}(\bar{B})]\} \\ \in \mathbb{N} \cup \{\infty\}$$

is a well-defined transverse invariant, where $c_{\omega, x_1}(B; a) = \infty$ if and only if $[\beta_{\omega, x_1}(\bar{B})]$ is trivial or a -torsion. We analyse these invariants in the case $R = \mathbb{F}[U]$ and $\omega = (x - Ux_1)(x - Ux_2)(x - Ux_3)$, with $x_1, x_2, x_3 \in \mathbb{F}$ distinct, and $a = U$. From our analysis we obtain the following Bennequin-type inequalities.

Proposition 4 *Let B be a braid representing a knot K . Given ω as above, write $\omega_1 = \omega|_{U=1}$. Then*

- (1) $2[\text{sl}(B) + c_{\omega, Ux_i}(B, U)] \leq j_1(K),$
- (2) $2[\text{sl}(B) + c_{\omega, Ux_i}(B, U)] \leq s_{\omega_1}(K),$
- (3) $\text{sl}(B) + c_{\omega, Ux_i}(B, U) - 1 \leq 3\tilde{s}_{\omega_1, x_i}(K)$

for each $i \in \{1, 2, 3\}$, where j_1 (Lobb [16] and Wu [25]; see also Lewark and Lobb [12]), $s_{\omega_1}(K)$, and $\tilde{s}_{\omega_1, x_i}(K)$ [12] are concordance invariants (see also Theorem 18 and subsequent lines for a very quick overview).

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2 Webs and foams

In this section we will briefly review the definition of webs and foams. These objects play the same role played by closed 1-dimensional manifolds and surfaces in Bar-Natan's geometric description of Khovanov homology (and its deformations).

2.1 Webs

Webs were originally introduced by Greg Kuperberg in [9], as a tool to study the representation theory of rank 2 Lie algebras, and used by Khovanov in [7] to define a categorification of the \mathfrak{sl}_3 -Jones polynomial.

Definition A *web* W is a directed trivalent planar graph embedded in \mathbb{R}^2 , possibly with components without vertices (*loops*), satisfying the following properties:

- (a) W has a finite number of vertices and a finite number of loops.
- (b) There are two types of edges in W , the *thin edges* and the *thick edges*, and for each vertex $v \in V(W)$ there is a unique thick edge incident in v .
- (c) Each vertex of W is either a source or a sink.⁵

For technical reasons the empty set is also considered a web (the *empty web*). A trivalent directed (abstract) graph Γ satisfying (a) and (b) shall be called an *abstract web*.

The distinction between thick and thin edges is necessary to keep track of crossings after their resolution (see Section 3.2). Whenever necessary the thick edges shall be drawn thicker and coloured yellow, otherwise no distinction shall be made.

2.2 Foams

Roughly speaking, foams are decorated branched surfaces which are singular along a smooth 1-dimensional manifold of triple points. Let us put aside the decorations and let us start by defining the underlying topological structure of a foam.

A (*topological*) *prefoam* Σ is a compact topological space such that each point has a neighbourhood homeomorphic to one of the four local models in Figure 2. A point $p \in \Sigma$ is called *regular* if it has a neighbourhood which is homeomorphic to either (C) or (D). Nonregular points are called *singular*, and the set of singular points

⁵A vertex v of a directed graph is a *source* if all edges incident to v are directed outwards from v . A vertex v of a directed graph is a *sink* if all edges incident to v are directed towards v .

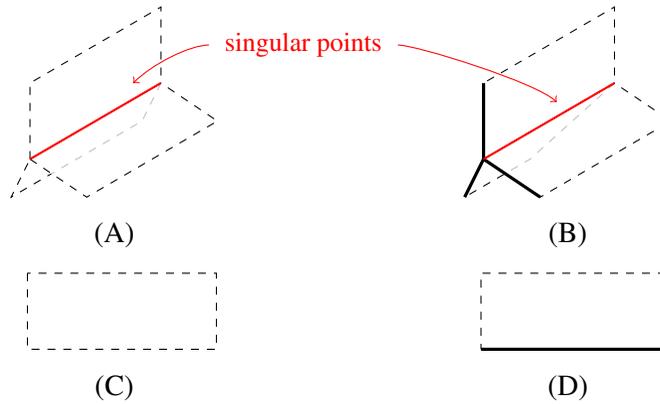


Figure 2: Local models for a prefoam.

is denoted by $\text{Sing}(\Sigma)$. The connected components of $\Sigma \setminus \text{Sing}(\Sigma) \subseteq \Sigma$ are called *regular regions* of Σ . Finally, a *boundary point* for Σ is a point which does not have a neighbourhood homeomorphic to either (A) or (C). A topological prefoam with empty boundary is called *closed*.

Remark The singular locus of prefoam is the disjoint union of circles and arcs, which are called *singular circles* and *singular arcs*. The singular boundary points correspond to the boundary points of the singular arcs.

The choice of an atlas⁶ on a prefoam determines an atlas on each regular region and also on $\text{Sing}(\Sigma)$. A *smooth prefoam* is a topological prefoam with a chosen topological atlas such that the induced atlases on the regular regions and on the singular locus are smooth atlases. Similarly, given a prefoam Σ , an *orientation* on Σ is the choice of an orientation of the closure of each regular region in such a way that the orientation induced in the intersection of two closed regions agrees.

Remark If we choose an orientation on a orientable prefoam, this induces an orientation on the singular locus.

Remark The boundary of a topological prefoam is a (possibly empty) finite trivalent graph whose vertices correspond to singular boundary points. If the prefoam is oriented its boundary is a directed graph. Moreover, each vertex of the boundary graph of

⁶We mean an open cover of Σ together with a homeomorphism of each element of the cover with one of the local models in Figure 2.

an oriented prefoam is either a sink or a source. In other words, the boundary of an oriented prefoam is an abstract web.

Definition A decorated prefoam is an oriented prefoam Σ together with the data of

- (a) a finite number (possibly zero) of marked points, called *dots*, in the interior of each regular region of Σ ;
- (b) a cyclic order on the regular regions incident to a singular arc or circle.

It is possible to define a category **PreFoam** whose objects are abstract webs and whose morphisms are formal R -linear combinations of the triples $(\Sigma, \partial_0 \Sigma, \partial_1 \Sigma)$ satisfying the following properties:

- ▷ Σ is a decorated prefoam.
- ▷ $\partial_0 \Sigma, \partial_1 \Sigma \in \text{Obj}(\mathbf{PreFoam})$, and $\partial_0 \Sigma$ is the source object and $\partial_1 \Sigma$ the target object of the morphism.
- ▷ $\partial \Sigma = \partial_0 \Sigma \sqcup -\partial_1 \Sigma$, where the minus sign denotes the reversal of the orientation.
- ▷ The triple is seen up to boundary-fixing isotopies which do not change the regular regions of the dots and preserve the ordering of the components near each singular arc.

Finally, the composition of two triples $(\Sigma, \partial_0 \Sigma, \partial_1 \Sigma)$ and $(\Sigma', \partial_1 \Sigma, \partial_1 \Sigma')$ is defined as the triple $(\Sigma'', \partial_0 \Sigma, \partial_1 \Sigma')$, where Σ'' is obtained by glueing Σ and Σ' along $\partial_1 \Sigma$.

Definition A foam is a decorated prefoam properly and smoothly⁷ embedded in $\mathbb{R}^2 \times I$. Moreover, we ask the cyclic order of the regular regions at each singular arc or circle to coincide with the cyclic order induced by rotating clockwise⁸ around a singular arc or circle. Foams will be considered up to ambient isotopies of $\mathbb{R}^2 \times I$ which fix the boundary of the foam and do not change the regular regions of the dots.

Given two webs W_0 and W_1 , a foam between W_0 and W_1 is a foam F such that

$$F \cap \mathbb{R}^2 \times \{0\} = W_0 \quad \text{and} \quad F \cap \mathbb{R}^2 \times \{1\} = -W_1.$$

The category **Foam** is the category whose objects are webs, whose morphisms are R -linear combinations of foams between two webs, and whose composition is defined

⁷That is, in such a way that the restriction of the embedding to each regular region and to the singular locus is smooth.

⁸We suppose fixed an orientation of $\mathbb{R}^2 \times I$.

as the glueing of two foams along the shared boundary. For technical reasons the empty foam is also included in **Foam** as an element of $\text{Hom}_{\mathbf{Foam}}(\emptyset, \emptyset)$.

2.3 Local relations

Local relations are equalities among (linear combinations of) foams which are identical except inside a small ball. The relations we are concerned with can be divided into two types:

- ▷ *reduction relations* (see Figure 3),
- ▷ *evaluation relations* (see Figure 4).

The reduction relations depend on the choice of a polynomial $\omega(x) \in R[x]$ (the *potential*) of the form

$$\omega(x) = x^3 + a_2x^2 + a_1x + a_0.$$

Thus, we will hereby suppose ω fixed. In the case R is a graded ring, one may require the coefficients a_2 , a_1 and a_0 and the polynomial ω to be homogeneous in order to obtain a graded theory. Let us get back to the local relations and postpone the matter of the gradings.

The reduction relations allow one to reduce either the number of handles (*genus reduction relation* (GR)) or the number of dots (*dot reduction relation* (DR)) at the expense of trading a single foam for a linear combination of foams.

There are two types of evaluation relations. The first type are called *sphere relations* (S), and concern spheres with fewer than four dots. The second type are called *theta foam relations* (Θ), and concern theta foams (that is, spheres with a disk glued along the equator) with two dots or fewer on each region.

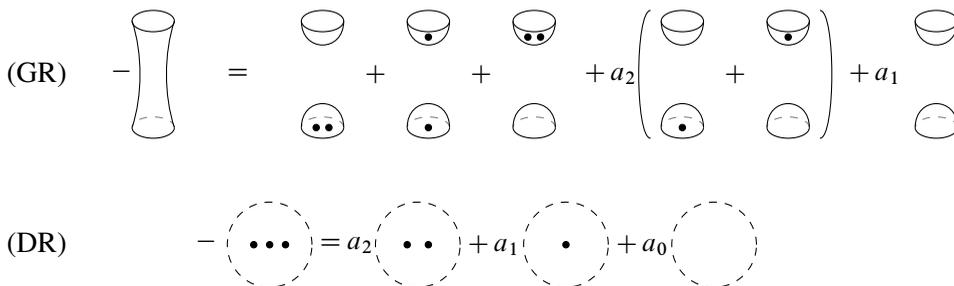


Figure 3: Reduction relations.

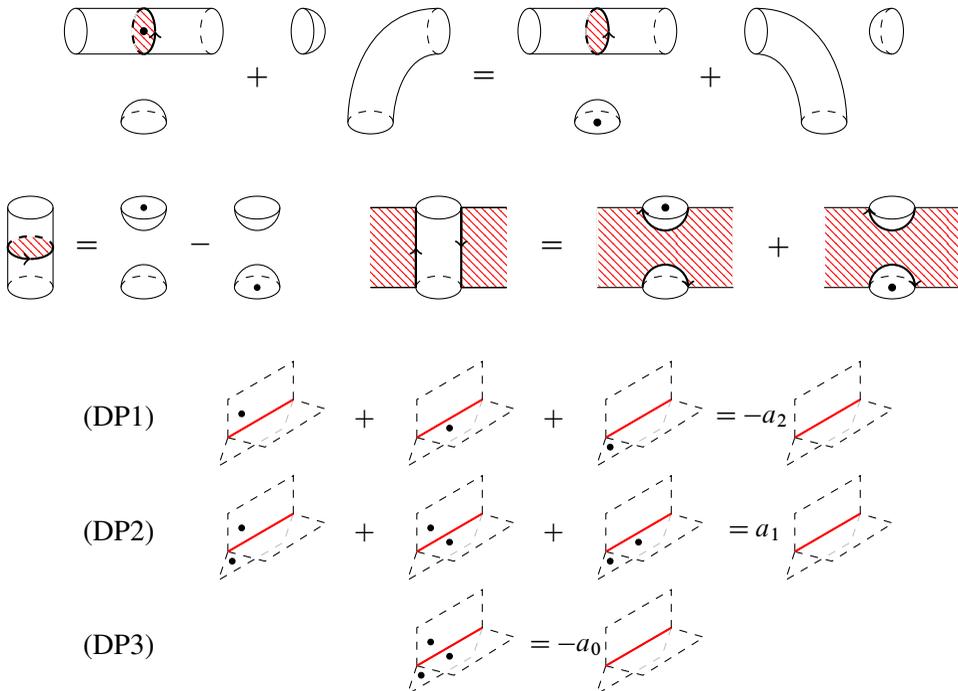
Modulo the local relations, each closed foam is equivalent to a multiple of the empty foam; using the reduction relations one reduces the closed foam to a R -linear combination of (disjoint unions of) spheres and theta foams with fewer than three dots in each regular region. Finally, one uses the evaluation relations to obtain a multiple of the empty foam. For each pair of webs W_0 and W_1 , and each foam $F \in \text{Hom}_{\mathbf{Foam}}(W_0, W_1)$, there is a well-defined R -bilinear pairing

$$\langle \cdot | \cdot \rangle_F: \text{Hom}_{\mathbf{Foam}}(\emptyset, W_0) \otimes \text{Hom}_{\mathbf{Foam}}(W_1, \emptyset) \rightarrow R (= R\langle \emptyset \rangle)$$

given by ‘‘capping off’’ F with an element of $\text{Hom}_{\mathbf{Foam}}(\emptyset, W_0)$ and an element of $\text{Hom}_{\mathbf{Foam}}(W_1, \emptyset)$, and evaluating it.

Now, we can define the category \mathbf{Foam}_ℓ , as the category whose objects are the same as \mathbf{Foam} , but whose morphisms are considered to be equal if the corresponding bilinear forms are equal. Using the local relations it is possible to prove the following result. The reader is referred to [17] for a proof.

Proposition 5 (Mackaay and Vaz [17]) *The following local relations hold in \mathbf{Foam}_ℓ :*



(DP1), (DP2) and (DP3) are also called **dot permutation relations**. □

$$\begin{aligned}
 \text{(S)} \quad & \text{Diagram 1} = \text{Diagram 2} = 0 \quad \text{Diagram 3} = -1 \\
 \text{(}\Theta\text{)} \quad & \text{Diagram 4} = - \text{Diagram 5} = \begin{cases} 1 & (n_1, n_2, n_3) \text{ is } (1, 2, 0) \text{ or a cyclic permutation,} \\ -1 & (n_1, n_2, n_3) \text{ is } (2, 1, 0) \text{ or a cyclic permutation,} \\ 0 & n_1, n_2, n_3 \leq 2 \text{ and } (n_1, n_2, n_3) \text{ not a prior case} \end{cases}
 \end{aligned}$$

Figure 4: Evaluation relations. The numbers n_1, n_2 and n_3 on the theta foam indicate the number of dots in the corresponding region.

3 The \mathfrak{sl}_3 -link homologies via foams

In this section we shall review the construction of a link homology theory via webs and foams. This construction consists of three steps. First we need some machinery coming from category theory, namely cubes and abstract complexes. Then, we shall describe how to build a cube from a link diagram, and apply the machinery developed to get a formal “geometric” complex of foams and webs. Finally, we make use of Bar-Natan’s tautological functors to obtain an honest chain complex and a homology theory.

3.1 Cubes in categories and abstract complexes

Let us review the construction of the complex in the category of webs and foams associated to an oriented link diagram. To define this complex, a bit of abstract nonsense is necessary. Denote by Q_n the standard n -dimensional cube $[0, 1]^n$. Orient each edge of Q_n from the vertex with the lowest number of 1’s to the vertex with highest number of 1’s.

Let R be a ring. An R -linear category is a (small) category \mathcal{C} such that, for each pair of objects A and B , the set of morphisms $\text{Hom}_{\mathcal{C}}(A, B)$ has the structure of a R -module, and the composition is bilinear with respect to this structure. A \mathbb{Z} -linear category is often called a *preadditive category*.

Definition A n -cube in a category \mathcal{C} is the assignment of an object O_v to each vertex v of Q_n and a morphism $F(v, v') \in \text{Hom}_{\mathcal{C}}(O_v, O_{v'})$ to the edge from v to v' , for each (ordered) pair of vertices v and v' . A n -cube in a R -linear category \mathcal{C} is *commutative* (resp. *skew-commutative*) if for each square the composition of the morphism on the edges commutes (resp. anticommutes).

Given a commutative cube, it is easy to prove that it is always possible to change the signs of the morphisms on the edges in such a way that the cube becomes skew-commutative [6]. Now, we wish to assign to a skew-commutative cube a formal chain complex. To do so, one must first give a meaning to a complex over a category.

Definition Let C be a R -linear category. The category $\mathbf{Com}(C)$ of complexes over C is the category defined as follows:

- (1) The objects of $\mathbf{Com}(C)$ are ordered collections of pairs $(C_i, d_i)_{i \in \mathbb{Z}}$, where $C_i \in \text{Obj}(C)$ and $d_i \in \text{Hom}_C(C_i, C_{i+1})$, such that

$$d_{i+1} \circ d_i = 0_{\text{Hom}_C(C_i, C_{i+2})}.$$

- (2) The morphisms between two objects (C_i, d_i) and (C'_j, d'_j) of $\mathbf{Com}(C)$ are collections of maps $(F_i)_{i \in \mathbb{Z}}$ such that

$$F_i \in \text{Hom}_C(C_i, C'_{i+k}) \quad \text{and} \quad F_{i+1} \circ d_i = d'_{i+k} \circ F_i \quad \text{for all } i \in \mathbb{Z}$$

for a fixed $k \in \mathbb{Z}$, called the *degree* of $(F_i)_{i \in \mathbb{Z}}$.

- (3) The composition of two morphisms $(F_i)_{i \in \mathbb{Z}}$ and $(G_j)_{j \in \mathbb{Z}}$ is defined to be $(G_{i+k} \circ F_i)_{i \in \mathbb{Z}}$, where k is the degree of $(F_i)_{i \in \mathbb{Z}}$.

In general, an R -linear category does not have kernels and cokernels. So, even though we can define chain complexes, we cannot define the homology. However, it is possible to define chain homotopy equivalences.

Definition Two morphisms F and G between $C_\bullet = (C_i, d_i)$ and $D_\bullet = (D_i, \partial_i)$, two objects in $\mathbf{Com}(C)$, are *(chain) homotopy equivalent* if there exists a morphism $H \in \text{Hom}_{\mathbf{Com}(C)}(D_\bullet, C_\bullet)$ such that

$$F - G = d \circ H \pm H \circ \partial.$$

Two objects $C_\bullet, D_\bullet \in \text{Obj}(\mathbf{Com}(C))$ are *homotopy equivalent* if there exists two morphisms

$$F \in \text{Hom}_{\mathbf{Com}(C)}(D_\bullet, C_\bullet) \quad \text{and} \quad G \in \text{Hom}_{\mathbf{Com}(C)}(C_\bullet, D_\bullet)$$

such that the compositions $F \circ G$ and $G \circ F$ are homotopy equivalent to the identity morphisms of D_\bullet and C_\bullet , respectively. We will denote by $\mathbf{Com}_h(C)$ the category of complexes over C and morphisms of $\mathbf{Com}(C)$ up to homotopy equivalence.

To conclude the abstract construction of a complex from a skew-commutative cube we need one more definition.

Definition Given an R -linear category \mathcal{C} , the *matrix category over \mathcal{C}* is the category whose objects are formal direct sums of objects in \mathcal{C} and whose morphisms are matrices with entries in the morphism of \mathcal{C} . The composition of two morphisms in the matrix category over \mathcal{C} is given by the usual matrix multiplication rule.

Denote by $\mathbf{Kom}(\mathcal{C})$ the category of complexes over the matrix category over \mathcal{C} , where \mathcal{C} is an arbitrary R -linear category. Given a skew-commutative n -cube Q in \mathcal{C} , define

$$C_i^Q = \bigoplus_{|v|=i} Q_v \quad \text{and} \quad d_i^Q = \sum_{|v|=i} \bigoplus_{v'} F(v, v'),$$

where $F(v, v')$ is defined to be zero if there is no edge from v to v' , and $|v|$ denotes the number of 1's in v . It is an easy verification that (C_i^Q, d_i^Q) is an object in $\mathbf{Kom}(\mathcal{C})$.

3.2 The Khovanov–Kuperberg bracket and the geometric complex

Fix a potential $\omega(x) \in R[x]$. Let D be an oriented link diagram. Fix an order of the crossings of D , say $\{c_1, \dots, c_k\}$. Each crossing has two possible *web resolutions*; see Figure 5. These resolutions come with an integer depending on the crossing and the type of resolution performed.

There is a natural bijection between the vertices of a k -dimensional cube $[0, 1]^k$ and the web resolutions of the diagram⁹ D ; to $v \in \{0, 1\}^k$ is associated the web $W(D, v)$ obtained by replacing, for each i , the crossing c_i with its v_i -web resolution.

To each oriented edge $v \rightarrow v'$ of Q_k is associated a foam $F(v, v')$ between the webs $W(D, v)$ and $W(D, v')$. The foam $F(v, v')$ is everywhere a cylinder, except above a disk where the two webs differ, where the cobordism looks like one of the two elementary web cobordisms depicted in Figure 6. The cube we defined depends on the choice of an ordering of the crossings of D . Moreover, it is easy to see that, by a Morse-theoretic argument, the cube associated to an oriented link diagram is commutative. We can turn this commutative cube into a skew commutative cube $Q(D)$ (but there is the choice of signs for the edges). Finally, using the abstract construction described in Section 3.1 we can associate a complex $\langle D \rangle_\omega$ in $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$ to the skew commutative cube $Q(D)$.

⁹That is, the web obtained by replacing each crossing with a web resolutions.

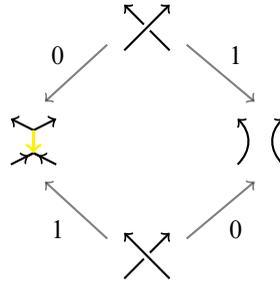


Figure 5: Web resolutions of positive (top) and negative (bottom) crossings.

Remark The complex $\langle D \rangle_\omega$ does not depend on the potential per se; what really depends on ω is the category \mathbf{Foam}/ℓ .

The complex $\langle D \rangle_\omega$ is called the *Khovanov–Kuperberg bracket of D* (with respect to ω). Whenever ω is fixed or clear from the context we will remove it from the notation. With some standard machinery of homological algebra it is easy to show the following proposition.

Proposition 6 [7; 17] *The Khovanov–Kuperberg bracket of an oriented link diagram D does not depend (up to isomorphism in $\mathbf{Kom}(\mathbf{Foam}/\ell)$) on the sign assignment or the order of the crossings used to obtain the cube $Q(D)$.* □

The Khovanov–Kuperberg bracket is the \mathfrak{sl}_3 -analogue of the Khovanov bracket introduced by Bar-Natan in [1]. Exactly as in the case of the Khovanov bracket, to turn the Khovanov–Kuperberg bracket into an invariant of links (and not framed links) we need to shift the homological grading.

Definition Given an R -linear category \mathbf{C} and $C_\bullet = (C_i, d_i)_{i \in \mathbb{Z}} \in \mathbf{Com}(\mathbf{C})$, the *shift of C_\bullet by $k \in \mathbb{Z}$* is the object $C_\bullet(k)$ in $\mathbf{Com}(\mathbf{C})$ defined by

$$C_\bullet(k) = (C_{i+k}, d_{i+k})_{i \in \mathbb{Z}}.$$

Finally, we can define the *geometric \mathfrak{sl}_3 -complex of D* (with respect to ω) by

$$\tilde{C}_\omega^\bullet(D, R) = \langle D \rangle_\omega(-n_+(D)),$$

where $n_+(D)$ (resp. $n_-(D)$) is the number of positive (resp. negative) crossings in D (see Figure 5). This is a link invariant in the sense of the following proposition.

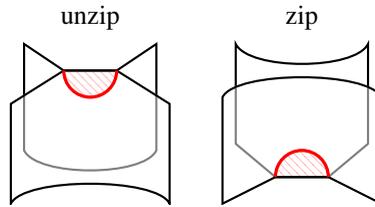


Figure 6: Elementary web cobordisms. The arc in red is a singular arc.

Theorem 7 (Mackaay and Vaz [17]) *Let ω be a potential. If D and D' are oriented link diagrams representing the same link, then $\tilde{C}_\omega^\bullet(D, R)$ and $\tilde{C}_\omega^\bullet(D', R)$ are chain homotopy equivalent. Furthermore, the assignment*

$$\tilde{C}_\omega^\bullet: \mathbf{Link} \rightarrow \mathbf{Kom}_{/\pm h}(\mathbf{Foam}_{/\ell})$$

is a functor from the category **Link** (ie the category of links in \mathbb{R}^3 and properly embedded surfaces in $\mathbb{R}^3 \times [0, 1]$ up to boundary-fixing isotopies) to the category $\mathbf{Kom}_{/\pm h}(\mathbf{Foam}_{/\ell})$ (ie the category obtained from $\mathbf{Kom}_h(\mathbf{Foam}_{/\ell})$ by considering the morphisms up to sign). □

3.3 Tautological functors and the \mathfrak{sl}_3 -homology

The category $\mathbf{Mat}(\mathbf{Foam}_{/\ell})$ is not an abelian category. Thus, it is not possible to define the homology of $\tilde{C}_\omega^\bullet(D, R)$. There are different ways to turn the geometric complex into an honest chain complex. Out of the different possibilities, following [17], we pursue the approach via tautological functors.

Definition The *tautological functor* is the functor

$$T: \mathbf{Foam}_{/\ell} \rightarrow R\text{-Mod}$$

defined on an object $W' \in \text{Obj}(\mathbf{Foam}_{/\ell})$ by

$$T(W') = \text{Hom}_{\mathbf{Foam}_{/\ell}}(\emptyset, W')$$

and on morphisms by composition on the left, that is,

$$T(F): T(W') \rightarrow T(W''), \quad G \mapsto F \circ G,$$

for each $F \in \text{Hom}_{\mathbf{Foam}_{/\ell}}(W', W'')$ and $W', W'' \in \text{Obj}(\mathbf{Foam}_{/\ell})$.

Note that, if we have a disjoint union of the webs W' and W'' , then

$$T(W' \sqcup W'') \simeq T(W') \otimes_R T(W'')$$

as R -modules. Before proceeding further let us show in an example how the functor T works. This example will be useful later on.

Example Let us compute $T(\bigcirc)$ (ie find its isomorphism class as an R -module).

By definition $T(W') = \text{Hom}_{\mathbf{Foam}/\ell}(\emptyset, W')$ is the R -module generated by all foams bounding W' (modulo local relations). All closed components of such a foam evaluate to elements of R . Thus, $T(\bigcirc)$ is generated (as an R -module) by connected foams. Since these foams must bound the circle \bigcirc , which is made of regular boundary points, we can use the genus reduction relation and write any connected foam bounding \bigcirc as an R -linear combination of disks marked with at most two dots. It follows that $T(\bigcirc)$ is generated by dotted disks. Now, consider the epimorphism of R -modules

$$\Phi: R[x] \rightarrow \text{Hom}_{\mathbf{Foam}/\ell}(\emptyset, \bigcirc)$$

mapping x^k to the disk with k dots. The dot reduction relation tells us that $\omega(x)$ is in the kernel. On the other hand, the disk with two dots, the disk with a single dot and the disk with no dots are linearly independent over R (use the pairing to get a linear system). Finally, an easy application of the Euclidean division algorithm (which works in any ring, assuming that the polynomial by which we are dividing has invertible leading term; see [10, Section IV, Theorem 1.1]) shows that $\ker(\Phi)$ is exactly $(\omega(x))$, and thus

$$T(\bigcirc) \simeq \frac{R[x]}{(\omega(x))}.$$

There is a natural way to extend the tautological functor to the category $\mathbf{Kom}(\mathbf{Foam}/\ell)$ (see [1, Section 9]). With an abuse of notation we shall denote the extended functor also by T .

Definition The \mathfrak{sl}_3 -complex (with respect to ω) of an oriented link diagram D is

$$C_\omega^\bullet(D, R) = T(\tilde{C}_\omega^\bullet(D, R)) \in \text{Obj}(\mathbf{Kom}(R - \mathbf{Mod})).$$

The following proposition is an immediate consequence of Theorem 7.

Proposition 8 *The isomorphism class (as an R -module) of $H_\omega^\bullet(D, R)$ is a link invariant, so we may write $H_\omega^\bullet(L, R)$, where L is the oriented link represented by D . Moreover, H_ω^\bullet defines a functor between the category \mathbf{Link} and the category $R\text{-Mod}_{\text{gr}}$ of graded R -modules. \square*

The homology of the \mathfrak{sl}_3 -complex will be called the \mathfrak{sl}_3 -homology of L (with respect to ω).

3.4 Graded and filtered \mathfrak{sl}_3 -homologies

To conclude the background material, we wish to describe how to define a second grading or a filtration on the \mathfrak{sl}_3 -complex. Suppose that R is a graded ring. For the sake of simplicity we shall assume R to be graded over the nonnegative integers. By setting $\deg(x) = 2$, the grading on R induces a grading on $R[x]$. The choice of a homogeneous potential $\omega(x)$ (ie $\deg(a_i) = 2(3 - i)$) induces a graded structure on the category $\mathbf{Foam}_{/\ell}$ (see [1, Definition 6.1; 17, Section 2]); that is, the modules $\text{Hom}_{\mathbf{Foam}_{/\ell}}(W', W)$ (and thus also $T(W)$) become graded R -modules. This structure is defined by setting

$$\deg_{\mathbf{Foam}}(F) = -2\chi(F) + \chi(\partial F) + 2d$$

for each foam F with d dots.

In particular, it follows that the \mathfrak{sl}_3 -complex associated to an oriented diagram D and a potential ω is graded. Furthermore, the differential respects the *quantum degree* qdeg , which is defined by

$$\text{qdeg}(x) = \deg_{\mathbf{Foam}}(x) - |v| + 3n_+(D) - 2n_-(D),$$

where x is a homogeneous element of $T(W(D, v))$ and v a vertex of the cube $Q_{n_+ + n_-}$.

Now, suppose that R is trivially graded (ie supported in degree 0). In this case, the unique homogeneous potential is x^3 (which corresponds to the original theory due to Khovanov). For all the other potentials the quantum grading is not well defined (because the reduction relations are not homogeneous). Nonetheless, one may define a filtered structure on $\text{Hom}_{\mathbf{Foam}_{/\ell}}(W', W)$ by setting

$$\mathcal{F}_i \text{Hom}_{\mathbf{Foam}_{/\ell}}(W', W) = \langle F, \deg_{\mathbf{Foam}}(F) \leq i \rangle_R,$$

where $\deg_{\mathbf{Foam}}$ is defined as above. This filtered structure extends to $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$, and also to the \mathfrak{sl}_3 -complex, and the differential does not increase the filtration level (shifted as in the case of the quantum degree). This (shifted) filtration on the \mathfrak{sl}_3 -complex is called *quantum filtration*.

Remark One can prove that the isomorphism in the example on page 1745 induces the following isomorphism of graded (resp. filtered) R -modules:

$$T(\bigcirc) \simeq \frac{R[x]}{(\omega(x))}(-2),$$

where $\deg(x) = 2$, and in the filtered case the filtration on the left-hand side is induced by the degree.

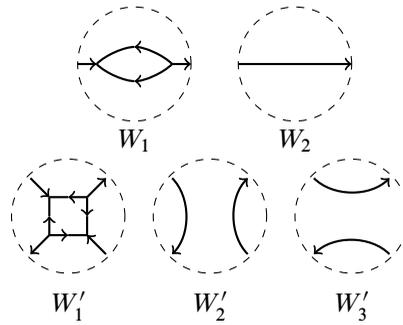


Figure 7: Webs involved in the Khovanov–Kuperberg relations.

The following result is due to Mackaay and Vaz ([17, Lemma 2.9]; compare also [7]).

Proposition 9 (Khovanov–Kuperberg relations) *We have the following isomorphisms of graded (filtered) R -modules:*

Circle removal $T(W' \sqcup \bigcirc) \simeq T(\bigcirc) \otimes T(W')$.

Digon removal $T(W_1) \simeq T(W_2)(-1) \oplus T(W_2)(1)$.

Square removal $T(W'_1) \simeq T(W'_2) \oplus T(W'_3)$.

Here W_1 and W_2 (resp. W'_1 , W'_2 and W'_3) are two (resp. three) webs which are identical but in a small ball where they are as depicted in Figure 7, and (\cdot) indicates the degree (filtration) shift. □

4 Transverse invariants and the universal \mathfrak{sl}_3 -theory

Let R be an integral domain, and fix a potential $\omega(x) \in R[x]$. In this section we define a family of transverse braid invariants in $C_\omega^\bullet(\overline{B}, R)$, where B is a closed braid diagram and the overline denotes the mirror. The elements of this family are in bijection with the (distinct) roots of ω in R . From now on, unless otherwise stated, all tensor products are assumed to be taken over R and all the isomorphisms are assumed to be isomorphisms of R -modules.

4.1 The β -chains

Assume that $\omega(x)$ has a root x_1 in R . It follows that

$$\omega(x) = x^3 + a_2x^2 + a_1x + a_0 = (x - x_1)(x^2 + a'_1x + a'_0)$$

for some $a'_1, a'_0 \in R$ such that

$$(4) \quad a_2 = a'_1 - x_1, \quad a_1 = a'_0 - x_1 a'_1, \quad a_0 = -x_1 a'_0.$$

Let D be an oriented link diagram. Define the *oriented web resolution* \underline{w}_D to be the web resolution where each positive crossing is replaced by its 1–web resolution, and every negative crossing is replaced by its 0–resolution. In other words, the oriented web resolution is the web resolution where both $\nearrow\searrow$ and $\searrow\nearrow$ are replaced by \curvearrowright . It follows that the oriented web resolution is a collection of loops. Moreover, these loops have a natural orientation induced by the orientation of D .

Definition Let D be a oriented link diagram. Consider a family of disjoint unknotted disks $\{\mathbb{D}_\gamma\}_{\gamma \in \underline{w}_D}$ properly embedded in $(\mathbb{R}^2 \times \{0\}) \times [0, 1] \subseteq \mathbb{R}^3 \times [0, 1]$, obtained by pushing the Jordan disks bounding \underline{w}_D in $\mathbb{R}^2 \times \{0\}$. Denote by \mathbb{D}_γ^k the disk \mathbb{D}_γ with k dots on it. The β –chain (with respect to ω) associated to the root x_1 is defined as

$$\beta_{\omega, x_1}(D) = \sum_{S \subseteq \underline{w}_D} \sum_{S' \subseteq \underline{w}_D \setminus S} (a'_1)^{\#S'} (a'_0)^{\#S} \left(\bigsqcup_{\gamma \in \underline{w}_D \setminus (S \cup S')} \mathbb{D}_\gamma^2 \sqcup \bigsqcup_{\gamma \in S'} \mathbb{D}_\gamma^1 \sqcup \bigsqcup_{\gamma \in S} \mathbb{D}_\gamma^0 \right) \in T(\underline{w}_D),$$

where $\#S$ denotes the number of elements in S .

By definition of the \mathfrak{sl}_3 –complex, to each web resolution \underline{w} of D corresponds a direct summand $T(\underline{w})$ in $C_\omega^\bullet(D, R)$ in homological degree $-n_+(D) + |\underline{w}|$ where $n_+(D)$ is the number of positive crossings in D , and $|\underline{w}|$ is the number of 1–web resolutions in the web resolution \underline{w} . In particular, we have

$$T(\underline{w}_D) \subseteq C_\omega^0(D, R).$$

Furthermore, the filtered quantum degree of β_{ω, x_1} can be easily computed. It suffices¹⁰ to look at the maximal quantum degree of the summands in the definition of β_{ω, x_1} . The maximal degree is achieved by the summand where the disks have two dots each.

¹⁰We are using the fact that the disks with dots are a filtered basis for $T(\bigcirc \cdots \bigcirc)$. This follows immediately from the observation that the isomorphism (see (5))

$$T(\underbrace{\bigcirc \cdots \bigcirc}_k) \simeq \frac{R[x_1, \dots, x_k]}{(\omega(x_1), \dots, \omega(x_k))}(-2k),$$

given by sending the disk with d dots bounding the i^{th} circle into x_i^d , is an isomorphism of filtered modules. Here the filtration on $R[x_1, \dots, x_k]/(\omega(x_1), \dots, \omega(x_k))$ is the one induced by the total grading, $\deg(x_i) = 2$ for each i , and $(-2k)$ denotes a shift of $-2k$ in the filtered degree (see the remark on page 1746).

Thus, the filtered quantum degree of $\beta_{\omega, x_1}(D)$ is $2(\#\text{disks in } \underline{w}_D) + 2w(D)$. In the special case $D = \widehat{B}$, we have that

$$2(\#\text{disks in } \underline{w}_{\widehat{B}}) + 2w(B) = 2(b(B) - w(\overline{B})) = -2 \text{sl}(\overline{B}),$$

where the last equality is due to the fact that $b(B) = b(\overline{B})$. The same computation works in the graded case.

From the Khovanov–Kuperberg relations and from the example on page 1745, it follows that, as R -modules,

$$(5) \quad T(\underline{w}_D) \simeq \bigotimes_{\gamma \in \underline{w}_D} \frac{R[x_\gamma]}{(\omega(x_\gamma))} \simeq \frac{R[x_\gamma \mid \gamma \in \underline{w}_D]}{(\omega(x_\gamma))_{\gamma \in \underline{w}_D}},$$

where $\gamma \in \underline{w}_D$ should be read as “ γ is a circle in \underline{w}_D ”. It is easy to see that the isomorphism in (5) maps $\beta_{\omega, x_1}(D)$ to

$$\prod_{\gamma \in \underline{w}_D} [x_\gamma^2 + a'_1 x_\gamma + a'_0] \in T(\underline{w}_D) \subseteq C_\omega^0(D, R).$$

In the rest of the paper we shall freely switch between these two representations of the chains $\beta_{\omega, x_1}(D)$.

Remark The multiplication of $\beta_{\omega, x_1}(D)$ by x_γ , which is an algebraic operation, corresponds “geometrically” to adding a dot to the disk \mathbb{D}_γ in each summand of the “geometric expression” of $\beta_{\omega, x_1}(D)$.

Lemma 10 *Let F be the foam in Figure 8, and consider the morphism of R -modules*

$$T(F): T(\bigcirc \sqcup \bigcirc) \rightarrow T(W_\theta),$$

where W_θ is the theta web (ie the closure of the web W_1 in Figure 7). Defining β as

$$\beta = (x^2 + a'_1 x + a'_0)(y^2 + a'_1 y + a'_0) \in \frac{R[x, y]}{(\omega(y), \omega(x))} \simeq T(\bigcirc \sqcup \bigcirc),$$

we have

$$T(F)(\beta) = 0.$$

Proof Our aim is to prove that $T(F)(\beta)$ can be written as

$$T(F)(\beta) = \sum_{j=1}^k c_j (F_j''' + a_2 F_j'' + a_1 F_j' + a_0 F_j), \quad c_j \in R,$$

where F_j, F_j', F_j'' and F_j''' are of the form shown in Figure 10, and are identical

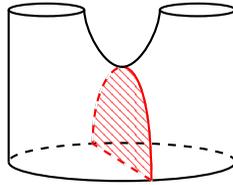


Figure 8: The foam F .

except in a small region where they differ as shown in Figure 9. Since, for each j , $(F_j''' + a_2 F_j'' + a_1 F_j' + a_0 F_j)$ is trivial in \mathbf{Foam}/ℓ by (DR), the claim shall follow. To avoid graphical calculus we make use of polynomials. So, let us denote by the monomial $A^r B^s C^t$ the foam (in $(\mathbb{R}^2 \times \{0\}) \times [0, 1]$) shown in Figure 10, where s , r and t indicate the number of dots in the regions A , B and C , respectively. By definition, $T(F)(\beta)$ can be written as

$$A^2 B^2 + a_1'(A^2 B + AB^2) + (a_1' + 1)^2 AB + a_0'(A^2 + B^2) + a_1' a_0'(A + B) + (a_0')^2.$$

With this notation we can write the dot permutation relations (DP1), (DP2) and (DP3) described in Proposition 5 as follows:

- (DP1) $A + B + C = -a_2,$
- (DP2) $AC + BC + AB = a_1,$
- (DP3) $ABC = -a_0.$

Since all foam relations are local, and since we are allowed to move the dots inside regions, the formal products above satisfy associativity. Using relations (DP1), (DP2) and (DP3) we obtain

$$\begin{aligned} T(F)(\beta) &= B(A^3 + a_2 A^2 + a_1 A + a_0)(A - a_1' + 1) \\ &= (BA^4 + a_2 BA^3 + a_1 BA^2 + a_0 BA) + (-a_1' + 1)(A^3 + a_2 A^2 + a_1 A + a_0), \end{aligned}$$

which is the desired decomposition of $T(F)(\beta)$. □

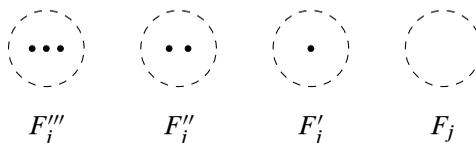


Figure 9: The local difference between the foams F_j , F_j' , F_j'' and F_j''' .

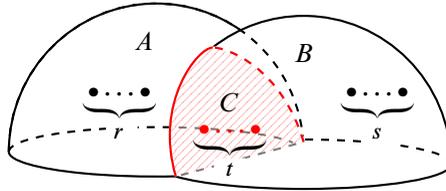


Figure 10: The foam $A^r B^s C^t$.

Remark In the proof of Lemma 10 we identified $T(W_\theta)$ with the R -module

$$M = \frac{R[A, B, C]}{((DP1), (DP2), (DP3))}.$$

(1) The above identification of $T(W_\theta)$ with M completely disregards the quantum grading (resp. filtration). Taking into account the quantum degree (resp. filtration) there is a shift one has to consider. More precisely, we have an isomorphism of graded (resp. filtered) R -modules

$$T(W_\theta) \simeq \frac{R[A, B, C]}{((DP1), (DP2), (DP3))}(-3),$$

where $\deg(A) = \deg(B) = \deg(C) = 2$.

(2) By Proposition 9, the example on page 1745 and the remark on page 1746, we have the following isomorphisms of graded (resp. filtered) R -modules:

$$T(W_\theta) \simeq T(\bigcirc)(-1) \oplus T(\bigcirc)(1) \simeq \frac{R[x]}{(\omega(x))}(-3) \oplus \frac{R[y]}{(\omega(y))}(-1),$$

where $\deg(x) = \deg(y) = 2$. In the follow-up, we shall make use of this representation of $T(W_\theta)$ rather than $M(-3)$. Of course the two representations are isomorphic as graded (resp. filtered) R -modules. An explicit isomorphism is given by

$$M(-3) \rightarrow \frac{R[x]}{(\omega(x))}(-3) \oplus \frac{R[y]}{(\omega(y))}(-1): \begin{cases} A^k \mapsto x^k & \text{for } k = 0, 1, 2, \\ A^k B \mapsto y^k & \text{for } k = 0, 1, 2. \end{cases}$$

Now, we are ready to prove the following result.

Proposition 11 *Let D be an oriented link diagram. Then $\beta_{\omega, x_1}(D)$ is a cycle.*

Proof First, notice that the oriented web resolution is bipartite exactly as the oriented resolution; that is, if two arcs in \underline{w}_D were connected by a crossing in D , then they belong to different circles in \underline{w}_D .

Let \underline{w}' be a web resolution which is obtained from \underline{w}_D by replacing a 0–web resolution with a 1–resolution, and denote by E the set of such resolutions. Notice that each $w' \in E$ is the disjoint union of circles and a theta web. In particular, we have the following isomorphism of R –modules:

$$T(\underline{w}') \simeq \bigotimes_{\gamma \in \underline{w}_D \setminus \{\gamma_1, \gamma_2\}} \frac{R[x_\gamma]}{(\omega(x_\gamma))} \otimes \left(\frac{R[y]}{(\omega(y))} \oplus \frac{R[x]}{(\omega(x))} \right),$$

where $\gamma_1 = \gamma_1(\underline{w}')$ and $\gamma_2 = \gamma_2(\underline{w}')$ are the two circles in \underline{w}_D which are merged into the theta web in \underline{w}' , and the circles in \underline{w}' are identified with the corresponding circles in \underline{w}_D .

By definition, the differential d_{geo} of the geometric complex is of the form

$$d_{\text{geo}}|_{\underline{w}_D} = \sum_{\underline{w}' \in E} \pm F(\underline{w}'),$$

where each $F(\underline{w}')$ is a disjoint union of cylinders and a copy of the foam F in Figure 8, and the boundary of $F(\underline{w}')$ is $\underline{w}' \sqcup \underline{w}_D$. Applying the tautological functor T , we get

$$d(T(\underline{w}_D)) \subseteq \bigoplus_{\underline{w}' \in E} \left(\left(\frac{R[y]}{(\omega(y))} \oplus \frac{R[x]}{(\omega(x))} \right) \otimes \bigotimes_{\gamma \in \underline{w}_D \setminus \{\gamma_1(\underline{w}'), \gamma_2(\underline{w}')\}} \frac{R[x_\gamma]}{(\omega(x_\gamma))} \right),$$

and

$$d|_{T(\underline{w}_D)} = \bigoplus_{\underline{w}' \in E} \pm \left(T(F) \otimes \bigotimes_{\gamma \in \underline{w}_D \setminus \{\gamma_1(\underline{w}'), \gamma_2(\underline{w}')\}} \text{Id}_{R[x_\gamma]/(\omega(x_\gamma))} \right).$$

The statement now follows immediately from Lemma 10. □

4.2 The transverse invariance of the β –chains

Now, let us analyse the behaviour of the β –chains under the maps induced by some Reidemeister moves. These moves include the closures of the mirror images of the transverse Markov moves. During the proofs in this section the homological degree and the quantum degree (or filtration) will be disregarded. We remark that the chain homotopy equivalences associated to the Reidemeister moves described here are of (filtered) degree 0 with respect to the quantum grading (resp. filtration) once the appropriate shifts are taken into account.

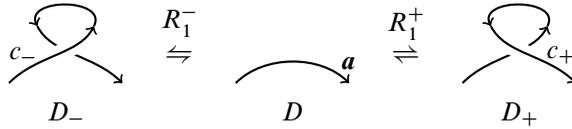


Figure 11: The negative (left) and positive (right) versions of the first Reidemeister move.

Negative first Reidemeister move Let D be an oriented link diagram, and denote by D_- the oriented link diagram obtained from D via a negative first Reidemeister move on a given arc a (Figure 11).

In Figure 12 there is a description of the map associated to a negative Reidemeister move between the geometric complexes (see [17, Section 2.2]). The figure should be read as follows: the foams are all embedded in $(\mathbb{R}^2 \times \{0\}) \times [0, 1]$ and are cylinders except in a small cylinder above the arc a , where they look like the ones depicted in Figure 12.

Remark The maps F and G described in Figure 12 have the opposite sign with respect to the corresponding maps defined in [17, Section 2.2].

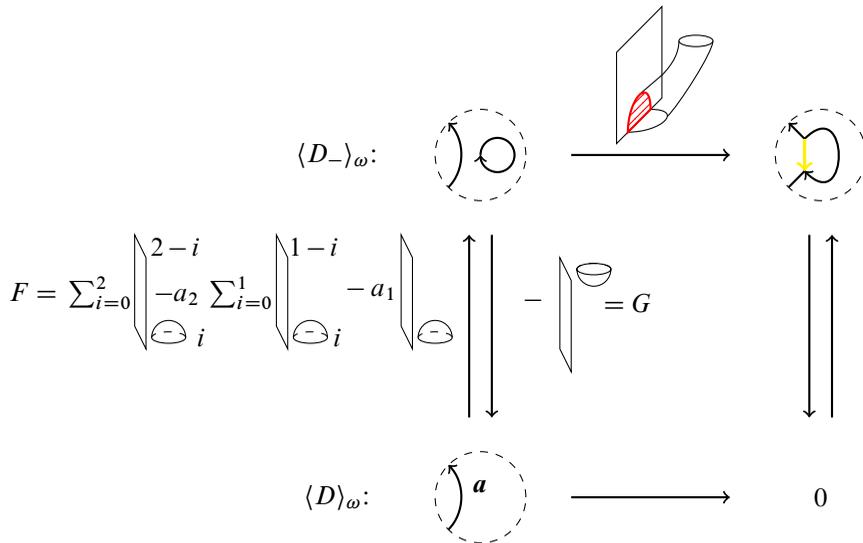


Figure 12: Schematic description of the maps encoding a negative first Reidemeister move. The numbers next to the foams (which are drawn in $(\mathbb{R}^2 \times \{0\}) \times [0, 1]$ and should be read top to bottom) indicate the number of dots. The horizontal maps are the differentials. Notice that there is a difference in the sign of the maps F and G with respect to [17].

Before proceeding we need the following lemma.

Lemma 12 *Let $P(x) \in R[x]$ be the polynomial $x^2 + a'_1x + a'_0$. Then*

$[x^2P(x)] \otimes [1] + [xP(x)] \otimes [y] + a_2[xP(x)] \otimes [1] - x_1[P(x)] \otimes [y] - x_1a'_1[P(x)] \otimes [1]$
is zero in

$$\frac{R[x]}{(\omega(x))} \otimes \frac{R[y]}{(\omega(y))}.$$

Proof First, let us point out that

$$(6) \quad xP(x) = x_1P(x) \pmod{\omega(x)}.$$

Using the equality $a_2 = a'_1 - x_1$, we get

$$\begin{aligned} [x^2P(x)] \otimes [1] + [xP(x)] \otimes [y] + a_2[xP(x)] \otimes [1] \\ = [x^2P(x)] \otimes [1] + [xP(x)] \otimes [y] + a'_1[xP(x)] \otimes [1] - x_1[xP(x)] \otimes [1] \\ = x_1[P(x)] \otimes [y] + a'_1x_1[P(x)] \otimes [1], \end{aligned}$$

where the last equality follows from (6). □

Denote by

$$\Phi_1: C_\omega^\bullet(D) \rightarrow C_\omega^\bullet(D_-)$$

the map associated to the (linear combination of) foam(s) F in Figure 12, and by

$$\Psi_1: C_\omega^\bullet(D_-) \rightarrow C_\omega^\bullet(D)$$

the map associated to the foam denoted by G in the same figure.

Proposition 13 *Let D be an oriented link diagram, and let D_- be the diagram obtained from D via a negative first Reidemeister move. Then*

$$\Psi_1(\beta_{\omega, x_1}(D_-)) = \beta_{\omega, x_1}(D) \quad \text{and} \quad \Phi_1(\beta_{\omega, x_1}(D)) = \beta_{\omega, x_1}(D_-).$$

Proof Notice that \underline{w}_{D_-} is mapped to \underline{w}_D by Ψ_1 and that \underline{w}_{D_-} can be identified with $\underline{w}_D \sqcup \bigcirc$. By the Khovanov–Kuperberg circle removal relation, the example on page 1745 and the sphere relation, we can identify $\Psi_1|_{\underline{w}_D}$ with the map

$$\frac{R[x_{\gamma'}]}{(\omega(x'_{\gamma}))} \otimes \bigotimes_{\gamma \in \underline{w}_D} \frac{R[x_\gamma]}{(\omega(x_\gamma))} \rightarrow \bigotimes_{\gamma \in \underline{w}_D} \frac{R[x_\gamma]}{(\omega(x_\gamma))}$$

given by

$$q_{\gamma'}(x_{\gamma'}) \otimes \bigotimes_{\gamma \in \underline{w}_D} q_\gamma(x_\gamma) \mapsto -\epsilon(q'(x_{\gamma'})) \bigotimes_{\gamma \in \underline{w}_D} q_\gamma(x_\gamma),$$

where γ' indicates the circle in $\underline{w}_{D-} \setminus \underline{w}_D$ and

$$\epsilon: \frac{R[x]}{(\omega(x))} \rightarrow R: [ax^2 + bx + c] \mapsto -a.$$

Since in the case of $\beta_{\omega, x_1}(D)$ we have $q_\gamma(x) = q_{\gamma'}(x) = [P(x)] = [x^2 + a'_1x + a'_0]$, the first part of the statement follows.

Similarly to what has been done with Ψ_1 , we can identify Φ_1 with the map

$$\bigotimes_{\gamma \in \underline{w}_D} \frac{R[x_\gamma]}{(\omega(x_\gamma))} \rightarrow \frac{R[x_{\gamma'}]}{(\omega(x_{\gamma'}))} \otimes \bigotimes_{\gamma \in \underline{w}_D} \frac{R[x_\gamma]}{(\omega(x_\gamma))}$$

mapping $\bigotimes_{\gamma \in \underline{w}_D} q_\gamma(x_\gamma)$ to

$$-\left(\sum_{i=0}^2 (x_{\gamma_a}^{2-i} q_{\gamma_a} \otimes x_{\gamma'}^i) + a_2 \sum_{i=0}^1 (x_{\gamma_a}^{1-i} q_{\gamma_a} \otimes x_{\gamma'}^i) + a_1 (q_{\gamma_a} \otimes 1) \right) \otimes \bigotimes_{\gamma \in \underline{w}_D \setminus \{\gamma_a\}} q_\gamma(x_\gamma),$$

where γ_a is the circle in \underline{w}_D containing the arc \mathbf{a} (see Figure 12) and $q_{\gamma_a} = q_{\gamma_a}(x_{\gamma_a})$. It is now easy to see that

$$\begin{aligned} \Phi_1(\beta_{\omega, x_1}(D)) &= ([x_{\gamma_a}^2 P(x_{\gamma_a})] \otimes [1] + [x_{\gamma_a} P(x_{\gamma_a})] \otimes [x_{\gamma'}] + a_2 [x_{\gamma_a} P(x_{\gamma_a})] \otimes [1] \\ &\quad - x_1 [P(x_{\gamma_a})] \otimes [x_{\gamma'}] - x_1 a_1 [P(x_{\gamma_a})] \otimes [1]) \\ &\quad \otimes \bigotimes_{\gamma \in \underline{w}_D \setminus \{\gamma_a\}} P(x_\gamma) + \beta_{\omega, x_1}(D_-) \\ &= \beta_{\omega, x_1}(D_-), \end{aligned}$$

where the last equality is due to Lemma 12. □

Second Reidemeister move Now, let us turn to the coherent version of the second Reidemeister move. Let D be an oriented link diagram. Let \mathbf{a} and \mathbf{b} be two (unknotted) arcs of D lying in a small ball. Performing a second Reidemeister move on these arcs inserts two adjacent crossings, say c_1 and c_2 , of opposite types.

Recall that a Reidemeister move is coherent if it can be obtained by rotating or taking the mirror image of the one in Figure 13. Denote by D' the link obtained from D by performing a coherent second Reidemeister move. Finally, denote by $\underline{w}'_{D'}$ the web resolution of D' where all crossings but c_1 and c_2 are resolved as in the oriented web resolution.

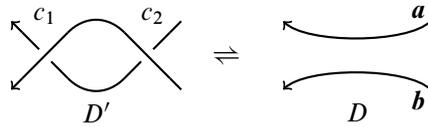


Figure 13: A coherent version of the second Reidemeister move. All other coherent second Reidemeister moves are obtained by rotating or taking the mirror image of the one in figure.

The map associated to the second Reidemeister move at the level of geometric complexes was defined by Mackaay and Vaz as in Figure 14. Denote by Φ_2 and Ψ_2 the two maps

$$\Phi_2: C_\omega^\bullet(D, R) \rightarrow C_\omega^\bullet(D', R) \quad \text{and} \quad \Psi_2: C_\omega^\bullet(D', R) \rightarrow C_\omega^\bullet(D, R)$$

associated to the coherent second Reidemeister move.

Proposition 14 *Let D be an oriented link diagram and let D' be the diagram obtained from D by performing a coherent second Reidemeister move. Then*

$$\Phi_2(\beta_{\omega, x_1}(D)) = \beta_{\omega, x_1}(D') \quad \text{and} \quad \Psi_2(\beta_{\omega, x_1}(D')) = \beta_{\omega, x_1}(D).$$

Proof First notice that w_D can be easily identified with $w_{D'}$. With this identification we have that $\Psi_2|_{T(w_{D'})}$ behaves as the identity map (see Figure 14), and the second

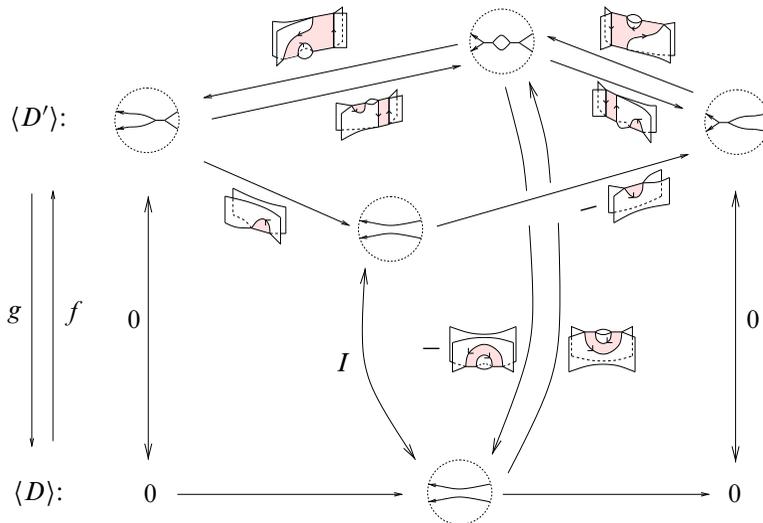


Figure 14: Map associated to the coherent second Reidemeister move in Figure 13.

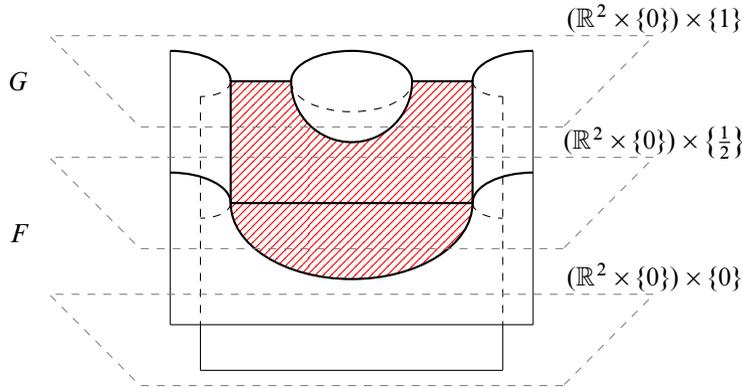


Figure 15: The cobordism F' as a composition of the cobordism F (on the bottom) and the cobordism G (on the top).

part of the statement follows. The map $\Phi_2|_{T(\underline{w}_D)}$ sends $T(\underline{w}_D)$ to $T(\underline{w}_D) \oplus T(\underline{w}'_{D'})$. More precisely, we have

$$\Phi_2|_{\underline{w}_D} = \text{Id}_{\underline{w}_D} \oplus T(F'),$$

where F' is the foam drawn in Figure 15. To conclude it suffices to prove that

$$T(F')(\beta_{\omega, x_1}(D)) = 0.$$

This is immediate from Lemma 10, once one notices that the foam F' is the composition of the foam F in Figure 8 and a foam G (see Figure 15). \square

Third Reidemeister move Finally, we have to prove the invariance of the β -chains under braid-like third Reidemeister moves. Consider the version R_3° of the third Reidemeister move in Figure 16. All the braid-like third Reidemeister moves can be deduced, via a sequence of coherent second Reidemeister moves, from the R_3° move (see [21, Lemma 2.6]). See Figure 17 for an example.

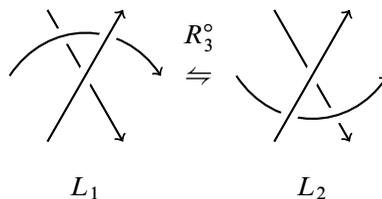


Figure 16: A version of the third Reidemeister move.

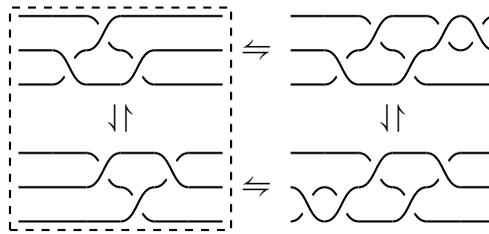


Figure 17: How to recover another braid-like third Reidemeister move (in the dashed box) with an R_3° and two coherent second Reidemeister moves.

Chain maps between the geometric complexes associated to R_3° have been described explicitly by Mackaay and Vaz. Each of these maps is defined as the composition of two maps. First, one defines an element $Q \in \mathbf{Kom}(\mathbf{Foam}/\ell)$ which is *not* the geometric complex associated to a link. Then, one defines chain maps

$$F_i: \langle D_i \rangle_\omega \rightarrow Q \quad \text{and} \quad G_i: Q \rightarrow \langle D_i \rangle_\omega,$$

where $i \in \{1, 2\}$, and D_1 and D_2 are the diagrams on each side of the R_3° move (Figure 16), such that F_i is the up-to-homotopy inverse of G_i . For a description of such maps the reader may refer to Figure 18 (see [17]).

The important thing that the reader should keep in mind is that for each web resolution \underline{w} of D_i such that the crossings involved in R_3° are resolved as in the oriented resolution, there is the same direct summand in Q , and that the restriction of either G_i or F_i to \underline{w} is minus the identity cobordism.

Finally, the maps associated to each direction of R_3° (between the Kuperberg brackets),

$$\tilde{\Psi}_3: \langle D_1 \rangle_\omega \rightarrow \langle D_2 \rangle_\omega \quad \text{and} \quad \tilde{\Phi}_3: \langle D_2 \rangle_\omega \rightarrow \langle D_1 \rangle_\omega,$$

are defined as

$$\tilde{\Psi}_3 = G_2 \circ F_1 \quad \text{and} \quad \tilde{\Phi}_3 = G_1 \circ F_2.$$

It is immediate that these two maps, when restricted to the oriented web resolutions, are cylinders. Denote by Ψ_3 and Φ_3 the maps between \mathfrak{sl}_3 -complexes associated to $\tilde{\Psi}_3$ and $\tilde{\Phi}_3$, respectively. Then the maps Ψ_3 and Φ_3 behave as the identity maps between the summands associated to the oriented web resolutions. So the following proposition is immediate.

Proposition 15 *Let D_1 and D_2 be two oriented link diagrams related by a coherent third Reidemeister move. Then*

$$\Psi_3(\beta_{\omega, x_1}(D_1)) = \beta_{\omega, x_1}(D_2) \quad \text{and} \quad \Phi_3(\beta_{\omega, x_1}(D_2)) = \beta_{\omega, x_1}(D_1). \quad \square$$

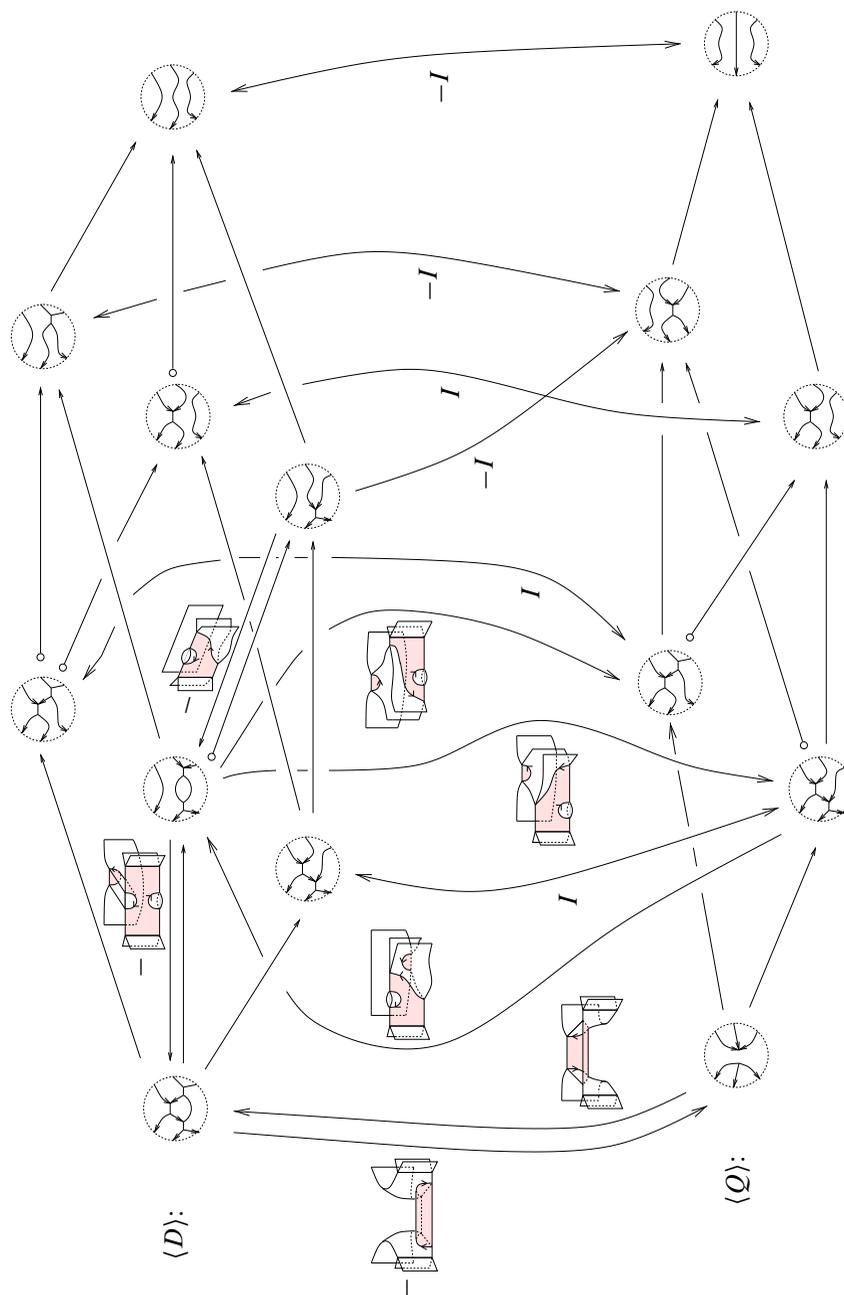


Figure 18: The maps G_1 and F_1 . The maps G_2 and F_2 are obtained by reflecting the local figures along the horizontal axes, and the cobordisms accordingly. A circle at the tail of an arrow indicates that the map gets a minus sign, and an I indicates that the map is the identity web cobordism.

Proof of Theorem 2 Theorem 2 follows immediately by putting together the results concerning the behaviour of β_{ω, x_1} under the maps induced by coherent Reidemeister moves and negative first Reidemeister moves, and the computation of the degrees after the definition of β_{ω, x_i} . \square

We will call $\beta_{\omega, x_1}(\bar{B})$ the β_3 -invariant of B associated to (ω, x_1) .

Remark The ψ_3 -invariant introduced by Wu in [24] is a special case of our construction; more precisely ψ_3 is the homology class of the β_3 -invariant associated to $(x^3, 0)$.

5 Auxiliary invariants

In the previous section, we already proved the existence of transverse invariants in the \mathfrak{sl}_3 -chain complex obtained from a factorisable potential (ie a potential admitting at least a root in the base ring). At this point a natural question arises: How much information do these invariants contain? That is, are these invariants effective? Unfortunately we do not have an explicit answer to this question. This is partly due to the lack of sufficiently simple (nontrivial) examples on which to perform the computations, and also due to the fact that these invariants are, in some sense, difficult to handle. More precisely, they are chains in a diagram-dependent chain complex, and to prove that two braids have distinct invariants one has to prove that there *does not exist* a map (induced by sequences of Reidemeister/Markov moves between the closures of the two braids) sending the β_3 -invariants of one braid to the corresponding β_3 -invariants of the other braid. Proving this is not easy in general. The most natural thing to do in this setting is to prove that the homology classes of the β_3 -invariants associated to the two braids behave differently with respect to a given structure on the \mathfrak{sl}_3 -homology, which is preserved by the maps induced by sequences of Reidemeister/Markov moves. In this section we shall present some ways to extract information from the homology classes of the β_3 -invariants making use of the R -module structure of the \mathfrak{sl}_3 -homology.

5.1 The vanishing of the homology class

The most basic structure on the \mathfrak{sl}_3 -homology is that of an R -module. The simplest way to make use of this structure is to look at the vanishing of the homology class of the β_3 -invariants. That is, if one braid has a β_3 -invariant whose homology class vanishes while the other braid does not, then the two braids represent distinct transverse links.

For the sake of simplicity, let us assume throughout this section the potential to be completely factorisable in R (ie all its roots are in R), and $R = \mathbb{F}$ to be a field. Under these hypotheses we can prove Proposition 3.

Proof of Proposition 3 Before going into the details of the proof, which is quite long, it is worth to schematically describe the idea behind it. We wish to make use of the Mackaay–Vaz classification of the isomorphism types of $H_\omega^\bullet(L, \mathbb{F})$ depending on the multiplicity of the roots of ω (see [17]). This classification, as well as its generalisation due to Rose and Wedrich in [22], is some sort of generalisation of the Chinese remainder theorem (which can be thought of as the case $L = \bigcirc$). In each of the three cases we shall identify the image of the β -cycles under the isomorphism which describes the isomorphism type of H_ω^\bullet . This identification will give us the desired result.

If the potential admits distinct roots in \mathbb{F} , then the homology classes of the corresponding β_3 -invariants are linearly independent by an argument essentially due to Gornik (see [5] and [17, Section 3]). More precisely, in this case the β -cycles are a rescaling of some of the so-called “canonical generators”. Since the independence of the homology classes of the “canonical generators” was proved in [5] and [17], the claim follows in this case.

Remark The proofs in [5; 17] only concern the case $\mathbb{F} = \mathbb{C}$. However, the proof of the independence works over any integral domain R , provided that ω has all roots in R .

Let us shift to the case when potential ω has a double root x_1 , and a simple root x_2 . That is, $x_1 = x_3$. In [17, Theorem 3.18] it was proved that

$$(7) \quad H_\omega^i(L, \mathbb{F}) \cong \bigoplus_{L' \subseteq L} \text{Kh}^{-i-\text{lk}(L', L \setminus L')}(\overline{L'}, \mathbb{F}),$$

where $L' \subseteq L$ means that L' ranges among the sublinks of L , and Kh denotes the original (\mathfrak{sl}_2 -)Khovanov homology.

- Remark** (a) The empty sublink and L are also counted among the sublinks.
 (b) In [17] the isomorphism in (7) is proved only for $\mathbb{F} = \mathbb{C}$. The proof works without change for any field, provided that ω has all roots in \mathbb{F} .
 (c) The isomorphism above is *not* canonical but depends on a number of choices.
 (d) The isomorphism in (7) is a slight rephrasing of [17, Theorem 3.18]. More precisely, the statement of [17, Theorem 3.18] reads

$$U_{a,b,c}^i(L, \mathbb{F}) \cong \bigoplus_{L' \subseteq L} \text{KH}^{i-\text{lk}(L', L \setminus L')}(L', \mathbb{F}),$$

where $U_{a,b,c}^i$ denotes H_ω^i and KH denotes a theory which is equivalent to Khovanov homology. To be precise, KH denotes the Khovanov–Rozansky \mathfrak{sl}_2 –homology with the homological degree reversed, that is, $\text{KH}^{i,j}(K, \mathbb{F}) = \text{Kh}^{-i,j}(\bar{K}, \mathbb{F})$. To see this, the reader can compare (7) and [17, Theorem 3.18] with the analogous (more general) result [22, Theorem 1] and subsequent examples (see the conventions in [7; 8] and the results in [18]).

Items (1) and (2) in the statement shall follow from a careful inspection of the proof of [17, Theorem 3.18]. In order to keep the paper as self-contained as possible, we shall review the crucial steps of the proof.

Let D be an oriented diagram representing an oriented link L . A *colouring of D* is a function ϕ associating to each arc of D (seen as a graph) a root of ω . A colouring ϕ is *compatible* with a web resolution \underline{w} if we can colour all the thick edges of \underline{w} in such a way that at each vertex the set of colours is the set of roots of ω . Denote by $\mathbb{W}_\phi(D)$ the set of resolutions compatible with a colouring ϕ .

Remark Given a web resolution \underline{w} compatible with ϕ , the colour of each thick edge is uniquely determined.

Mackaay and Vaz proved that one can associate a subcomplex $C_\omega^\bullet(\phi, \mathbb{F})$ to each colouring ϕ . Furthermore, the complex $C_\omega^\bullet(D, \mathbb{F})$ decomposes as the direct sum of these complexes. It turns out that the homology of $C_\omega^\bullet(\phi, \mathbb{F})$ is trivial unless ϕ assigns the same colour to all arcs belonging to the same components (*proper colouring*). Finally, one proves that for such colourings

$$H_\omega^\bullet(\phi, \mathbb{F}) \cong \text{Kh}^{-\bullet - \text{lk}(L_\phi, L \setminus L_\phi)}(\bar{L}_\phi, \mathbb{F}),$$

where L_ϕ is the sublink of L corresponding to the subdiagram of D obtained by deleting all arcs whose colour with respect to ϕ is x_1 . Furthermore, to each $\underline{w} \in \mathbb{W}_\phi(D)$ and proper colouring ϕ , it is possible to associate a cycle $\Sigma_\phi(\underline{w}) \in T(\underline{w})$ called the *canonical generator* associated to ϕ and \underline{w} .¹¹ For each proper colouring ϕ the set $\{\Sigma_\phi(\underline{w})\}_{\underline{w} \in \mathbb{W}_\phi(D)}$ generates the \mathbb{F} –vector space $H_\omega^\bullet(\phi, \mathbb{F})$.

Denote by ϕ_i the colouring of D where all arcs are coloured with x_i for $i \in \{1, 2\}$. The sublinks associated to ϕ_1 and ϕ_2 are the whole link L and the empty sublink \emptyset , respectively. The submodules generated by $\beta_{\omega, x_1}(D)$ and $\beta_{\omega, x_2}(D)$ are contained

¹¹Even though they are not canonically defined!

in $C_\omega^\bullet(\phi_1, \mathbb{F})$ and $C_\omega^\bullet(\phi_2, \mathbb{F})$, respectively. Thus, the homology classes of $\beta_{\omega, x_1}(D)$ and $\beta_{\omega, x_2}(D)$ are mapped to $\text{Kh}^0(\bar{L}, \mathbb{F})$ and $\text{Kh}^0(\emptyset, \mathbb{F}) = \mathbb{F}$, respectively, by the isomorphism in (7).

Remark The above reasoning proves, in particular, that the homology classes of β -invariants corresponding to different roots (if nonvanishing) are always linearly independent.

By inspecting the argument in [17], it is not difficult to see that the image of the homology class of $\beta_{\omega, x_2}(D)$ is nontrivial. It is immediate from the definition of the canonical generators that $\beta_{\omega, x_2}(D)$ is $(x_1 - x_2)^r \Sigma_{\phi_2}(\underline{w}_D)$ for a certain r , where \underline{w}_D denotes the oriented web resolution of D . Thus, the homology class of $\beta_{\omega, x_2}(D)$ generates $H_\omega^\bullet(\phi_2, \mathbb{F}) \cong \mathbb{F}$, and this concludes the proof of item (1).

Remark An alternative proof of the nonvanishing of $[\beta_{\omega, x_2}(D)]$ can be found in [13, Proposition 1.3]; the cycle $\psi(D)^{12}$ defined in [13] (for $n = 3$) coincides with $\beta_{x^3-x^2, 1}(D)$. Moreover, the same argument used in [13] to prove that $[\beta_{x^3-x^2, 1}(D)]$ is always nontrivial can be adapted to the case of an arbitrary (degree 3) potential with a double and a single root (see third paragraph in [13, Section 4]).

It takes a bit more care to prove that the image of $[\beta_{\omega, x_1}(D)]$ is (a nonzero multiple of) the Plamenevskaya invariant ψ . First notice that $\beta_{\omega, x_1}(B)$ is $P \cdot \Sigma_{\phi_1}(\underline{w}_D)$, where

$$P = \prod_{\gamma \in \underline{w}_D} (x_2 - x_1)(X_\gamma - x_1) \in R_{\phi_1}(\underline{w}_D) = \frac{\mathbb{F}[X_\gamma \mid \gamma \in \underline{w}_D]}{\langle \omega(X_\gamma) \mid \gamma \in \underline{w}_D \rangle},$$

and X_γ acts on $T(\underline{w}_D)$ by adding a dot on the regular region bounding γ . This is implied by the equality

$$(X - x_1)[(x_2 - x_1)^2 - (X - x_1)^2] \equiv (x_2 - x_1)(X - x_1)(X - x_2) \pmod{\omega(X)},$$

which is easily verified. Then the chain map in [17, (17)] which defines the isomorphism between $H_\omega^\bullet(\phi, \mathbb{F})$ and $\text{Kh}^{-\bullet - \text{lk}(L_\phi, L \setminus L_\phi)}(\bar{L}_\phi, \mathbb{F})$ behaves as follows:

$$\beta_{\omega, x_1}(B) = P \cdot \Sigma_{\phi_1}(\underline{w}_D) \mapsto (x_2 - x_1)^r X \otimes \cdots \otimes X \quad \text{for some } r \in \mathbb{N},$$

where $X \otimes \cdots \otimes X = \tilde{\psi}$ belongs to the summand associated to the oriented resolution in the Khovanov chain complex. Since, $[\tilde{\psi}] = \psi$, item (2) follows.

¹²Not to be confused with Plamenevskaya's ψ -invariant.

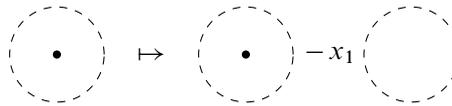


Figure 19

Finally, assume the potential ω has a triple root. Consider the endofunctor Φ of **Foam** described in Figure 19. The endofunctor Φ translates the local relations ℓ_1 associated to the potential $(x - x_1)^3$ into the local relations ℓ_0 associated to the potential x^3 . It follows that Φ induces an equivalence of \mathbb{F} -linear categories between **Foam**/ ℓ_1 and **Foam**/ ℓ_0 . This equivalence of categories induces an isomorphism between the two complexes C_ω^\bullet and $C_{x^3}^\bullet$. Finally, it is immediate to see that this isomorphism sends the β_3 -invariant to the ψ_3 -invariant. \square

Proposition 3, together with the results in [3], implies the following.

Corollary 16 *Let \mathbb{F} be a field of characteristic different from 2. If ω has a double root $x_1 \in \mathbb{F}$, then the vanishing of $[\beta_{\omega, x_1}(\bar{B})]$ is a noneffective invariant for every B that represents a knot with fewer than 12 crossings.* \square

5.2 Divisibility, numerical invariants and Bennequin inequalities

There are also other ways to make use of the R -module structure of the \mathfrak{sl}_3 -homology. Let us recall the definition of the c -invariants. Let R be an integral domain and $a \in R \setminus \{0\}$ a nonunit element. Given a potential ω and a root $x_1 \in R$, the number

$$c_{\omega, x_1}(B; a) = \max\{k \mid \text{there exists } [y] \in H_\omega^0(\bar{B}) \text{ such that } a^k[y] = [\beta_{\omega, x_1}(\bar{B})]\} \\ \in \mathbb{N} \cup \{\infty\}$$

is a well-defined transverse invariant, where $c_{\omega, x_1}(B; a) = \infty$ if and only if $[\beta_{\omega, x_1}(\bar{B})]$ is trivial or a -torsion. These invariants are particularly useful in the case ω has only single roots: in this case the homology classes of the β_3 -invariants are nontrivial and nontorsion (see the paragraph preceding the first remark on page 1761). Now assume

$$(8) \quad R = \mathbb{F}[U] \quad \text{and} \quad \omega(x) = (x - Ux_1)(x - Ux_2)(x - Ux_3),$$

where $x_i \in \mathbb{F}$ for all i , and $x_i \neq x_j$ if $i \neq j$. By setting $\deg(U) = 2$ the theory becomes graded, and we also have the following exact sequences of complexes of \mathbb{F} -vector spaces:

$$(9) \quad 0 \rightarrow C_\omega^{\bullet, q}(\hat{B}, \mathbb{F}[U]) \xrightarrow{U \cdot} C_\omega^{\bullet, q}(\hat{B}, \mathbb{F}[U]) \xrightarrow{\pi_0} C_{x^3}^{\bullet, q}(\hat{B}, \mathbb{F}) \rightarrow 0$$

for each $q \in \mathbb{Z}$, and

$$(10) \quad 0 \rightarrow C_\omega^\bullet(\widehat{B}, \mathbb{F}[U]) \xrightarrow{(U-1)\cdot} C_\omega^\bullet(\widehat{B}, \mathbb{F}[U]) \xrightarrow{\pi_1} C_{\omega|_{U=1}}^\bullet(\widehat{B}, \mathbb{F}) \rightarrow 0.$$

Moreover, it is immediate that

$$\pi_0(\beta_{\omega, x_i}(\overline{B})) = \beta_{x^3, 0}(\overline{B}) \quad \text{and} \quad \pi_1(\beta_{\omega, x_i}(\overline{B})) = \beta_{\omega|_{U=1}, x_i}(\overline{B}).$$

The following proposition follows immediately from (9).

Proposition 17 *Let R , ω and x_i be as above. Then, for each braid B , the following are equivalent:*

- (1) $\psi_3(B) \neq 0$.
- (2) $c_{\omega, Ux_i}(B; U) = 0$ for any choice of ω and x_i .
- (3) $c_{\omega, Ux_i}(B; U) = 0$ for all choices of ω and x_i . □

Now let us recall a few facts about concordance invariants defined from \mathfrak{sl}_3 -link homologies. These invariants were defined in the more general setting of the deformations of \mathfrak{sl}_n Khovanov–Rozansky (KR) homologies. However, the statements here shall be restricted to the case $n = 3$. It is worth noticing that the deformation of \mathfrak{sl}_3 KR homology corresponding to the potential ω coincides with the theory defined in this paper (see [18]).

Theorem 18 [12, Theorem 1.1] *Let K be a knot. Given a potential $\omega \in \mathbb{F}[x]$ with distinct roots $x_1, x_2, x_3 \in \mathbb{F}$, we have the isomorphism of bigraded vector spaces*

$$\text{Gr}^* H_\omega^\bullet(K, \mathbb{F}) \simeq \bigoplus_{i=1}^3 \mathbb{F}(0, j_i) \quad \text{with } j_1 \leq j_2 \leq j_3,$$

where $\text{Gr}^* H_\omega^\bullet$ is the associated graded object of $H_\omega^\bullet(K, \mathbb{F})$ (endowed with the quantum filtration), and $\mathbb{F}(h, q)$ is a copy of \mathbb{F} generated in bidegree (h, q) . Furthermore, the j_i are concordance invariants which provide lower bounds to the value of the slice genus. □

Lewark and Lobb in [12] defined two other concordance invariants. The first one is just a rescaled average of the j_i , that is,

$$s_\omega(K) = \frac{1}{12}(j_1(K) + j_2(K) + j_3(K)) \in \frac{1}{4}\mathbb{Z},$$

which is a concordance quasihomomorphism. The second invariant depends on the choice of a root x_i and is denoted by $\tilde{s}_{\omega, x_i}(K)$. This is a slice-torus knot invariant (see [15; 11]) and, in particular, a concordance homomorphism. Since its definition involves constructions which we do not wish to introduce, we refer the reader to [12] for it. All these invariants are somehow related, as is stated in the following proposition.

Proposition 19 [12, Proposition 2.12] *Let K be a knot. Given a potential $\omega \in \mathbb{F}[x]$ with distinct roots $x_1, x_2, x_3 \in \mathbb{F}$, order the roots in such a way that*

$$\tilde{s}_{\omega, x_1}(K) \leq \tilde{s}_{\omega, x_2}(K) \leq \tilde{s}_{\omega, x_3}(K).$$

Then

$$|j_i - 6\tilde{s}_{\omega, x_i}(K)| \leq 2$$

for each $i \in \{1, 2, 3\}$. □

To conclude this parenthesis we point out that $-j_i(\bar{K}) = j_i(K)$ [12, Proposition 2.13]. It follows that $s_\omega(K) = -s_\omega(\bar{K})$. Now we are ready to prove Proposition 4.

Proof of Proposition 4 Directly from the definition of s_{ω_i} follows that (1) implies (2). Furthermore, from [12, Proposition 2.12] it is immediate that (1) implies (3). So it is sufficient to prove (1). We borrow the notation from the proof of Proposition 3. Denote by $[y]$ a homogeneous element of $H_\omega^0(K, \mathbb{F})$ such that

$$U^{c_{\omega, x_i}(B, U)}[y] = [\beta_{\omega, x_i}(B)].$$

It is immediate that $\pi_1([y]) = \pi([\beta_{\omega, Ux_i}(B)])$, and the latter is a nontrivial multiple of the homology class of the canonical generator $\Sigma_{\phi_i}(\underline{w}_D)$. Since, the quantum filtration is increasing it follows that

$$-2 \text{sl}(B) - 2c_{\omega, Ux_i}(B, U) = \text{qdeg}([y]) \geq \text{Fdeg}([\Sigma_{\phi_i}(\underline{w}_D)]) =: q_i(\bar{K}).$$

Here Fdeg denotes the filtered degree. It is easy to prove that the maximum filtered degree of the elements of a basis of a vector space equipped with an increasing filtration does not depend on the chosen basis. Thus, we obtain that

$$-j_1(K) = j_3(\bar{K}) = \max_i \{q_i(\bar{K})\} \leq -2 \text{sl}(B) - 2c_{\omega, Ux_i}(B, U).$$

Using the isomorphism induced by the endofunctor of **Foam** obtained from the one described in Figure 19 by replacing x_1 with $x_i - x_j$, it follows that $q_i = q_j$; and this concludes the proof. □

Corollary 20 *Let K be a knot and B a braid representing K . If either $j_1(K) = \text{sl}(B)$, $s_{\omega_1}(K) = \text{sl}(B)$ or $3\tilde{s}_{\omega, x_i}(K) = \text{sl}(B) - 1$, then $\psi_3(B) \neq 0$. In particular, if B is a quasipositive braid, $\psi_3(B) \neq 0$.*

Proof The only thing to prove is that quasipositive braids are such that $3\tilde{s}_{\omega, x_i}(K) = \text{sl}(B) - 1$. This is true because $3\tilde{s}_{\omega, x_i}(K)$ is a slice torus invariant, and thus its value on quasipositive braids is exactly $\text{sl}(B) - 1$ (see [11]). \square

References

- [1] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, *Geom. Topol.* 9 (2005) 1443–1499 MR
- [2] **D Bennequin**, *Entrelacements et équations de Pfaff*, from “Third Schnepfenried geometry conference, I”, *Astérisque* 107, Soc. Math. France, Paris (1983) 87–161 MR
- [3] **C Collari**, *Transverse invariants from Khovanov-type homologies*, *J. Knot Theory Ramifications* 28 (2019) art. id. 1950012, 37 pages MR
- [4] **JB Etnyre**, *Legendrian and transversal knots*, from “Handbook of knot theory” (W Menasco, M Thistlethwaite, editors), Elsevier, Amsterdam (2005) 105–185 MR
- [5] **B Gornik**, *Note on Khovanov link cohomology*, preprint (2004) arXiv
- [6] **M Khovanov**, *A categorification of the Jones polynomial*, *Duke Math. J.* 101 (2000) 359–426 MR
- [7] **M Khovanov**, *Patterns in knot cohomology, I*, *Experiment. Math.* 12 (2003) 365–374 MR
- [8] **M Khovanov, L Rozansky**, *Matrix factorizations and link homology*, *Fund. Math.* 199 (2008) 1–91 MR
- [9] **G Kuperberg**, *Spiders for rank 2 Lie algebras*, *Comm. Math. Phys.* 180 (1996) 109–151 MR
- [10] **S Lang**, *Algebra*, 3rd edition, *Graduate Texts in Mathematics* 211, Springer (2002) MR
- [11] **L Lewark**, *Rasmussen's spectral sequences and the \mathfrak{sl}_N -concordance invariants*, *Adv. Math.* 260 (2014) 59–83 MR
- [12] **L Lewark, A Lobb**, *New quantum obstructions to sliceness*, *Proc. Lond. Math. Soc.* 112 (2016) 81–114 MR
- [13] **L Lewark, A Lobb**, *Upsilon-like concordance invariants from \mathfrak{sl}_n knot cohomology*, *Geom. Topol.* 23 (2019) 745–780 MR
- [14] **R Lipshitz, L Ng, S Sarkar**, *On transverse invariants from Khovanov homology*, *Quantum Topol.* 6 (2015) 475–513 MR

- [15] **C Livingston, S Naik**, *Ozsváth–Szabó and Rasmussen invariants of doubled knots*, *Algebr. Geom. Topol.* 6 (2006) 651–657 MR
- [16] **A Lobb**, *A note on Gornik’s perturbation of Khovanov–Rozansky homology*, *Algebr. Geom. Topol.* 12 (2012) 293–305 MR
- [17] **M Mackaay, P Vaz**, *The universal \mathfrak{sl}_3 –link homology*, *Algebr. Geom. Topol.* 7 (2007) 1135–1169 MR
- [18] **M Mackaay, P Vaz**, *The foam and the matrix factorization \mathfrak{sl}_3 link homologies are equivalent*, *Algebr. Geom. Topol.* 8 (2008) 309–342 MR
- [19] **S Y Orevkov, V V Shevchishin**, *Markov theorem for transversal links*, *J. Knot Theory Ramifications* 12 (2003) 905–913 MR
- [20] **O Plamenevskaya**, *Transverse knots and Khovanov homology*, *Math. Res. Lett.* 13 (2006) 571–586 MR
- [21] **M Polyak**, *Minimal generating sets of Reidemeister moves*, *Quantum Topol.* 1 (2010) 399–411 MR
- [22] **DE V Rose, P Wedrich**, *Deformations of colored \mathfrak{sl}_N link homologies via foams*, *Geom. Topol.* 20 (2016) 3431–3517 MR
- [23] **NC Wrinkle**, *The Markov theorem for transverse knots*, PhD thesis, Columbia University (2002) MR Available at <https://search.proquest.com/docview/304800949>
- [24] **H Wu**, *Braids, transversal links and the Khovanov–Rozansky theory*, *Trans. Amer. Math. Soc.* 360 (2008) 3365–3389 MR
- [25] **H Wu**, *On the quantum filtration of the Khovanov–Rozansky cohomology*, *Adv. Math.* 221 (2009) 54–139 MR

Alfréd Rényi Institute of Mathematics
Budapest, Hungary

Current address: *New York University Abu Dhabi*
Saadiyat Island, Abu Dhabi, United Arab Emirates

`carlo.collari.math@gmail.com`

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