

The dihedral genus of a knot

PATRICIA CAHN

ALEXANDRA KJUCHUKOVA

Let $K \subset S^3$ be a Fox p -colored knot and assume K bounds a locally flat surface $S \subset B^4$ over which the given p -coloring extends. This coloring of S induces a dihedral branched cover $X \rightarrow S^4$. Its branching set is a closed surface embedded in S^4 locally flatly away from one singularity whose link is K . When S is homotopy ribbon and X a definite four-manifold, a condition relating the signature of X and the Murasugi signature of K guarantees that S in fact realizes the four-genus of K . We exhibit an infinite family of knots K_m with this property, each with a Fox 3-colored surface of minimal genus m . As a consequence, we classify the signatures of manifolds X which arise as dihedral covers of S^4 in the above sense.

57M12, 57M25, 57Q60

1 Introduction

The slice-ribbon conjecture of Fox [7] asks whether every smoothly slice knot in S^3 bounds a ribbon disk in the four-ball. The analogous question can be asked in the topological category, namely: does every topologically slice knot bound a locally flat homotopy-ribbon disk in B^4 ? Recall that a properly embedded surface with boundary $F' \subset B^4$ is *homotopy ribbon* if the fundamental group of its complement is generated by meridians of $\partial F'$ in S^3 . Ribbon disks are easily seen to be homotopy ribbon whereas homotopy-ribbon disks need not be smooth.

For knots of higher genus, the generalized topological slice-ribbon conjecture asks whether the topological four-genus of a knot is always realized by a homotopy-ribbon surface in B^4 . When a knot K admits Fox p -colorings, we approach this problem by studying locally flat, oriented surfaces $F' \subset B^4$ with $\partial F' = K$ over which some p -coloring of K extends, in the sense defined in Section 2.1. The minimal genus of such a surface, when one exists, we call the p -dihedral genus of K .

When K is slice and p square-free, it is classically known that the colored surface F' for K can always be chosen to be a disk. This is essentially a consequence of a result

of Casson and Gordon [6, Lemma 3]; a detailed explanation can be found in work of Geske, Kjuchukova and Shaneson [9, Lemma 9]. Put differently, p -dihedral genus and classical four-genus coincide for slice knots. Furthermore, the topological slice-ribbon conjecture is true for p -colorable slice knots if and only if the minimal p -dihedral genus for these knots can always be realized by homotopy-ribbon surfaces. With this in mind, given a square-free integer p and a p -colorable knot K , we ask:

Question 1 Is the (topological) four-genus of K equal to its (topological) p -dihedral genus?

Question 2 Is the p -dihedral genus of K realized by a homotopy-ribbon surface?

When both of these questions are answered in the affirmative for a knot K with respect to some integer p , it follows that the topological four-genus and homotopy-ribbon genus of K are equal; that is, the generalized topological slice-ribbon conjecture holds for K . If K is not slice, requiring that it satisfy Questions 1 and 2 is a priori a stronger condition than satisfying the generalized slice-ribbon conjecture; however, the advantage of this point of view is that dihedral genus can be studied using dihedral branched covers.

Specifically, our approach is the following. Start with a branched cover of $f': X' \rightarrow B^4$ branched along a locally flat properly embedded surface F' with $\partial F' = K$; that is, F' is a properly embedded topological submanifold of B^4 . We now construct a new cover $f: X \rightarrow S^4$ by taking the cones of $\partial X'$, S^3 and the map f . The branching set of f is a surface F embedded in S^4 locally flatly except for one singular point whose link is K . Depending on the knot K and the map f , it may be the case that this construction yields a total space X that is again a manifold. In general, X has one singular (nonmanifold) point z , the preimage of the singularity on F . The link of z is the cover $f|_1$ of S^3 branched along K . We will consider the signature of X whether it is a manifold or has an isolated singularity. In the latter case, by the signature of X we mean the Novikov signature of the manifold with boundary obtained by deleting an open neighborhood of z in X .

When $f: X \rightarrow S^4$ in the above construction is a p -fold *irregular dihedral cover* (see the definition on page 1945), an invariant of (p -colored) knots, Ξ_p , is extracted from this construction. This invariant is our main tool. In a general setting, Ξ_p can be thought of as a defect term in the formula for the signature of a branched cover, resulting from the fact that the branching set is not locally flat. Put differently, the

presence of a cone singularity K on the branching set causes the signature of the cover to deviate from the smooth case by a term denoted by $\Xi_p(K, \rho)$. This term depends only on the isotopy class of the knot K and its Fox p -coloring ρ , but not on the locally flat part of the branching set.

Given a dihedral cover $f: X \rightarrow S^4$ whose branching set is orientable with one singularity, we in fact have

$$(1) \quad \Xi_p(K, \rho) = -\sigma(X),$$

by Kjuchukova [13, Theorem 1.4], when X is a manifold, and

$$(2) \quad \Xi_p(K, \rho) = -\sigma(X', \partial X')$$

when X has a singularity, by Geske, Kjuchukova and Shaneson [9, Theorem 7]. In the latter formula, $\partial X'$ is the dihedral cover of K induced by $f|$, and $\sigma(\cdot, \cdot)$ denotes the Novikov signature of a manifold with boundary. Of course, the first formula for $\Xi_p(K, \rho)$ in terms of X is a special case of the second, since the signature of a manifold is unchanged by deleting an open neighborhood of a point.

Unless explicitly stated otherwise, we will only consider orientable branching sets. Thus, we take the above signature equation to be the definition of $\Xi_p(K, \rho)$. In (5) we recall an explicit formula [13] for Ξ_p which does not rely on constructing the cover X . We also note that $\Xi_p(K, \rho)$ can be computed algorithmically from a colored diagram of K ; see Cahn and Kjuchukova [4]. We often suppress notation and write $\Xi_p(K)$ when the choice of coloring is clear, or when a knot admits a unique p -coloring (up to permuting the colors). Thus, for a two-bridge knot K , we will write simply $\Xi_p(K)$. The main result of this paper, Theorem 1, obtains a certain genus bound for K from $\Xi_p(K)$.

As implied by the above, the signature defect $\Xi_p(K)$ is defined for a knot K which arises as the only singularity on the branching set (not necessarily orientable) of an irregular dihedral cover [9]. A knot K is called p -admissible over S^4 , or simply p -admissible, if there exists a p -fold dihedral cover $f: X \rightarrow S^4$ whose branching set is embedded and locally flat except for one singularity whose link is K . If, in addition, the covering space X is a topological manifold, K is called *strongly* p -admissible. The distinguishing property of *strongly* p -admissible knots¹ is that their dihedral covers are S^3 . Admissibility of knots is studied by Kjuchukova and Orr in [14].

¹Like the invariant Ξ_p , the notion of (strong) p -admissibility of a knot may depend on the choice of coloring. We do not dwell on this presently since all examples in this paper are two-bridge knots and their colorings are unique.

In [Section 2](#), we put side by side the relevant notions of knot four-genus, recall several definitions, and state our main results, [Theorems 1, 2 and 3](#). In [Theorem 1](#), we give a lower bound on the homotopy-ribbon p -dihedral genus of a colored knot K in terms of the invariant $\Xi_p(K)$. We also give a sufficient condition for when this bound is sharp.

In [Theorems 2 and 3](#), we construct, for any integer $m \geq 0$, infinite families of knots for which the 3-dihedral genus and the topological four-genus are both equal to m . The basis of this construction are the knots K_m pictured in [Figure 1](#). The various four-genera of these knots are computed with the help of [Theorem 1](#). In particular, for these knots, the lower bound on genus obtained via branched covers is exact and the generalized topological slice-ribbon conjecture is seen to hold. The proofs of [Theorems 1, 2 and 3](#) are given in [Section 4](#).

The technique we apply is the following. Given a strongly p -admissible knot K , one can evaluate $\Xi_p(K)$ by realizing K as the only singularity on the branch surface of a dihedral cover of S^4 . Each of the knots K_m arises as the only singularity on the branching set of a 3-fold dihedral cover

$$f_m: \#^{2m+1} \overline{\mathbb{CP}^2} \rightarrow S^4.$$

The branching set of f_m is the boundary union of the cone on K_m with the surface F'_m realizing the four-genus of K_m . We construct these covering maps explicitly using singular triplane diagrams, a technique introduced by Cahn and Kjuchukova in [\[3\]](#). Equivalently, we construct a family of covers $\#^{2m+1} \mathbb{CP}^2 \rightarrow S^4$, again with oriented, connected branching sets, with the mirror images of the knots K_m as singularities. This construction appears in [Section 3](#). As a corollary of this construction, we realize all odd integers as values of Ξ_3 . In [Theorem 5](#), we prove that the range of values of Ξ_3 on strongly admissible knots is precisely the set of odd integers.

We work in the topological category, except where explicitly stated otherwise. Throughout, F denotes a closed, connected, oriented surface, and F' a connected, oriented surface with boundary. D_p denotes the dihedral group of order $2p$, and p is always assumed odd.

Acknowledgements The idea of dihedral genus first appeared in discussion with Kent Orr [\[14\]](#). The examples used in [Theorem 2](#) were inspired by joint work with Ryan Blair [\[1\]](#). This work was partially supported by NSF-DMS grant 1821212 to P Cahn and 1821257 to A Kjuchukova. Kjuchukova thanks the MPIM for its support and hospitality.

2 Dihedral four-genus and the main theorems

2.1 Some old and new notions of knot genus

We study the interplay between the following notions of four-genus for a Fox p -colorable knot $K \subset S^3$. Classically, the *smooth (resp. topological) four-genus* is the minimum genus of a smooth (resp. locally flat) embedded orientable surface in B^4 with boundary K . The *smooth (resp. topological) p -dihedral genus* of a p -colored knot K is, informally, the minimum genus of such a surface F' in B^4 over which the p -coloring of K extends. Precisely, a given p -coloring ρ of K extends over F' if there exists a homomorphism $\bar{\rho}$ which makes the following diagram commute (where i_* is the map induced by inclusion):

$$\begin{array}{ccc} \pi_1(S^3 - K) & \xrightarrow{i_*} & \pi_1(B^4 - F') \\ \downarrow \rho & \nwarrow \bar{\rho} & \\ D_p & & \end{array}$$

The p -dihedral genus above is defined for a knot K with a fixed coloring ρ , and hence we denote it by $g_p(K, \rho)$ in the topological case. We define the p -dihedral genus of a p -colorable knot K to be the minimum p -dihedral genus of K over all p -colorings ρ of K , and denote this by $g_p(K)$ in the topological case. Note that not every p -colored knot K admits a surface F' as above. In [14], we determine a necessary and sufficient condition for the existence of a connected oriented surface that fits into this diagram. When there is no surface over which a given coloring ρ of K extends, we define $g_p(K, \rho)$ to be infinite, and similarly for the refined notions of dihedral genus defined below.

The *ribbon genus* of K is the minimum genus of a smooth embedded orientable surface F' in B^4 with boundary K such that F' has only local minima and saddles with respect to the radial height function on B^4 . The *smooth (topological) homotopy-ribbon genus* of a knot K is the minimum genus of a smooth (locally flat) embedded orientable surface F' in B^4 with boundary K such that $i_*: \pi_1(S^3 - K) \twoheadrightarrow \pi_1(B^4 - F')$, that is, inclusion of the boundary into the surface complement induces a surjection on fundamental groups. Finally, given a p -colorable or p -colored knot, its *ribbon p -dihedral genus* or *smooth (topological) homotopy-ribbon p -dihedral genus* are defined in the obvious way. Observe that all notions of dihedral genus refer to surfaces embedded in the four-ball, even though “four” is not among the multitude of qualifiers we inevitably use.

As a straightforward consequence of the definitions, the following inequalities hold among the *smooth* four-genera of a knot:

four-genus

$\sqcup \wedge$

$\leq$

hom. ribbon genus

$\sqcup \wedge$

$\leq$

ribbon genus

$\sqcup \wedge$

p -dihedral genus $\leq$ p -dihedral hom. ribbon genus $\leq$ p -dihedral ribbon genus

Excluding the last column, the inequalities make sense and hold in the topological category too.

2.2 The main theorems

Denote by $g_4(K)$ the topological 4-genus of a knot K , and by $g_p(K, \rho)$ the topological homotopy-ribbon p -dihedral genus of a knot K with coloring ρ . Again, the minimum such genus over all colorings ρ of K is $g_p(K)$. Let $\sigma(K)$ be the (Murasugi) signature of the knot K . We relate $g_p(K, \rho)$, $\Xi_p(K, \rho)$ and $\sigma(K)$. Here, $\Xi_p(K, \rho)$ denotes the invariant discussed in Section 1; it is reviewed in more detail in this section and, in particular, we recall that it can be computed using (5).

Theorem 1 (A) *Let K be a p -admissible knot with p -coloring ρ and denote by M the irregular dihedral cover of K determined by ρ . Then*

(3)

$$g_p(K, \rho) \geq \frac{|\Xi_p(K, \rho)| - \operatorname{rk} H_1(M; \mathbb{Z})}{p - 1} - \frac{1}{2}.$$

(B) *Let K be a p -admissible knot and $F' \subset B^4$ a locally flat homotopy-ribbon oriented surface for K over which a given p -coloring ρ of K extends. Denote by $c(K)$ the cone on K , viewed as embedded in $D^4 = c(S^3)$. If the associated singular dihedral cover of S^4 branched along $F' \cup_K c(K)$ is a definite manifold, then the inequality (3) is sharp. In particular, F' realizes the dihedral genus $g_p(K, \rho)$ of K . If, in addition, the equality*

$$|\sigma(K)| = \frac{2|\Xi_p(K, \rho)|}{p - 1} - 1$$

holds, then the topological four-genus and the topological homotopy-ribbon p -dihedral genus of K coincide and equal $\frac{1}{2}|\sigma(K)|$, so the generalized topological slice-ribbon conjecture holds for K .

Remark If K has multiple p -colorings, denote by $\min_p(K)$ the minimum value of

$$|\Xi_p(K, \rho)| - \operatorname{rk} H_1(M; \mathbb{Z})$$

over all such colorings of K . [Theorem 1](#) implies

(4)

$$\mathfrak{g}_p(K) \geq \frac{\min_p(K)}{p-1} - \frac{1}{2}.$$

Theorem 2 For every integer $m \geq 0$, there exists a knot K_m and corresponding 3-coloring ρ_m such that

$$g_4(K_m) = \mathfrak{g}_3(K_m) = \frac{1}{2}|\Xi_3(K_m, \rho_m)| - \frac{1}{2} = m.$$

That is, the inequality (4) is sharp for these knots and computes their 3-dihedral genus as well as their topological four-genus. The generalized slice-ribbon conjecture holds for these knots.

Theorem 3 For any integer $m \geq 0$, there exist infinite families of knots whose 3-dihedral genus and topological four-genus are both equal to m .

2.3 Singular dihedral covers of S^4 and the invariant Ξ_p

In this section, we revisit the definition of a singular branched cover, and dihedral covers in particular. We also review the context in which the invariant Ξ_p arises, as well as a couple of techniques for its calculation.

Definition Let Y be a manifold and $B \subset Y$ a codimension-two submanifold with the property that there exists a surjection $\varphi: \pi_1(Y - B) \twoheadrightarrow D_p$. Denote by $\overset{\circ}{X}$ the covering space of $Y - B$ corresponding to the conjugacy class of subgroups $\varphi^{-1}(\mathbb{Z}/2\mathbb{Z})$ in $\pi_1(Y - B)$, where $\mathbb{Z}/2\mathbb{Z} \subset D_p$ is any reflection subgroup. The completion of $\overset{\circ}{X}$ to a branched cover $f: X \rightarrow Y$ is called the *irregular dihedral p -fold cover* of Y branched along B .

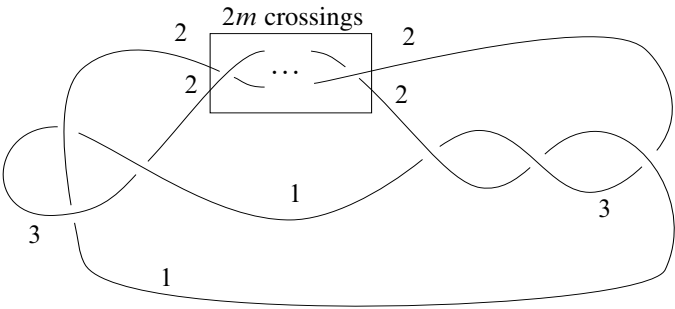


Figure 1: The knot K_m , where $m \geq 0$, and its 3-coloring. We have $K_0 = 6_1$, $K_1 = 8_{11}$, $K_2 = 10_{21}$ and $K_3 = 12a723$.

The manifolds whose irregular dihedral covers we will consider are S^3 , B^4 and S^4 . The Ξ_p invariant was originally defined in the more general context of a dihedral cover of an arbitrary four-manifold Y with a singularly embedded branching set [13].

Recall the following construction from Section 1. Let F' be a surface with connected boundary K , properly embedded in B^4 and locally flat. Given a branched cover of manifolds with boundary $f': X' \rightarrow B^4$, one constructs a *singular* branched cover of S^4 by coning off $\partial X'$, ∂B^4 and the map f' . The resulting covering map, $f: X \rightarrow S^4$, has total space $X := X' \cup_{\partial X'} c(\partial X')$, where $c(\partial X')$ denotes the cone on $\partial X'$. The branching set is a closed surface $F := F' \cup_K c(K)$ embedded in S^4 with a singularity (the cone point) whose link is K . The space X obtained in this way is a manifold if and only if $\partial X' \cong S^3$.

Denote by $\sigma(X', \partial X')$ the Novikov signature of the manifold with boundary given as a cover of B^4 branched along F' . When $\partial X' = S^3$, denote by $\sigma(X)$ the signature of the manifold X . In this case, we have $\sigma(X) = \sigma(X', \partial X')$.

Given $f': X' \rightarrow B^4$ as before with f' an (irregular) dihedral covering map, we always assume that the associated homomorphism $\rho: \pi_1(S^3 - K) \rightarrow D_p$ is surjective or, equivalently, that $\partial X'$ is connected. In this case, assuming that F' is orientable, $\Xi_p(K, \rho) = -\sigma(X', \partial X')$ by [9, Theorem 7]. In particular, when X is a manifold, this equation reduces to the earlier result $\Xi_p(K, \rho) = -\sigma(X)$ [13, Theorem 1.4].

Below, we recall two formulas for $\Xi_p(K, \rho)$ from [13]. Equation (5) allows $\Xi_p(K, \rho)$ to be computed in terms of K and its coloring using [4] and [2]. Equation (6) expresses $\Xi_p(K, \rho)$ in terms of a singular branched cover of S^4 in the more general case where the branching set is a possibly nonorientable surface.

Refocusing for a moment on the case where the dihedral branched cover X of S^4 is a manifold, we note that there exist many infinite families of knots $K \subset S^3$ whose irregular dihedral covers are homeomorphic to S^3 . For example, this is a property shared by all p -colorable two-bridge knots (a well-known fact recalled in the proof of [13, Lemma 3.3]). By definition, if a p -admissible knot K has S^3 as its dihedral cover, then it is in fact strongly p -admissible. We are then able study invariants of K using four-dimensional techniques such as trisections. Criteria for admissibility of singularities are discussed in more detail in [3], where we also use the invariant $\Xi_p(K)$ to give a homotopy-ribbon obstruction for strongly p -admissible knots K . A generalization of this ribbon obstruction to all p -admissible knots appears in [9].

We conclude this section by reviewing the formula for computing the invariant Ξ_p given in [13]. Let p be an odd integer and K a p -admissible knot. Let V be a Seifert surface for K and V° the interior of V . Denote by $\beta \subset V^\circ$ a mod p characteristic knot² for K , as defined in [5]. Also denote by L_V the symmetrized linking form for V and by σ_{ζ^i} the Tristram–Levine ζ^i -signature, where ζ is a primitive p^{th} root of unity. Finally, let $W(K, \beta)$ be the cobordism constructed in [5] between the p -fold cyclic cover of S^3 branched along β and the p -fold dihedral cover of S^3 branched along K and determined by ρ . We briefly describe the manifold $W(K, \beta)$. Let Σ be the p -fold cyclic branched cover of β and let $\Sigma_p(\beta) \times [0, 1] \rightarrow S^3 \times [0, 1]$ be the induced cyclic cover branched along $\beta \times [0, 1]$. Letting $\mathbb{Z}/2\mathbb{Z}$ act on an appropriate subset of $\Sigma_p(\beta) \times \{0\}$, one obtains $W(K, \beta)$ as a quotient of $\Sigma_p(\beta) \times [0, 1]$ by this action. One boundary of this quotient, namely $\Sigma_p(\beta) \times \{1\}$, is clearly the p -fold cyclic cover of β . The other boundary component, that is, the image of $\Sigma_p(\beta) \times \{0\}$ under the $\mathbb{Z}/2\mathbb{Z}$ action, is the dihedral cover of α as shown in [5, Proposition 1.1]. By [13, Theorem 1.4],

$$(5) \quad \Xi_p(K, \rho) = \frac{p^2 - 1}{6p} L_V(\beta, \beta) + \sigma(W(K, \beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta).$$

The Novikov signature $\sigma(W(K, \beta))$ can be computed in terms of linking numbers in the dihedral cover of K [13, Proposition 2.5]. Thus, the above formula allows $\Xi_p(K)$ to be evaluated directly from a p -colored diagram of K , without direct reference to a four-dimensional construction. An explicit algorithm for performing this computation is outlined in [4]. Note also that when a knot K is realized as the only singularity on an embedded surface $F \subset S^4$ and moreover this surface is presented by a Fox p -colored singular triplane diagram, [3] gives a method for computing $\Xi_p(K)$ from this data, via the signature of the associated cover of S^4 . This technique is reviewed and applied in Section 3 below.

We also review the context in which (1) and (2) arise, allowing us to relate $\Xi_p(K, \rho)$ to the signature of a singular branched cover X of S^4 . Consider an irregular dihedral cover $f: X \rightarrow S^4$ whose branching set F is an embedded surface, *not necessarily orientable*, locally flat away from one singularity $z \in F$ of type K . The induced coloring of F is, as always, an extension of ρ . Once again we denote by X' the

²Precisely, if K admits multiple p -colorings, one must work with a characteristic knot corresponding to the coloring in question. The sense in which a characteristic knot determines a coloring is laid out in [5, Proposition 1.1]. The examples we construct always admit a unique p -coloring, up to permuting the colors, and therefore a unique equivalence class of mod p characteristic knots.

dihedral cover of B^4 branched along the complement in F of a neighborhood of the singular point z . Note also that X' is obtained by deleting from X a small open neighborhood of $f^{-1}(z)$. We have

(6)
$$\Xi_p(K) = -\frac{1}{4}(p-1)e(F) - \sigma(X', \partial X'),$$

where $e(F)$ denotes the self-intersection number of F . This is a special case of the signature formula for dihedral branched covers over an arbitrary base³ given in [9, Theorem 7]. Note that, when F is orientable and X a manifold, (6) reduces to (1), that is, $\Xi_p(K) = -\sigma(X)$. In this case, the Ξ_p invariant of a singularity can be understood entirely in terms of the signature of the branched cover and, in particular, can be computed using four-manifold techniques. We further note that it is possible to realize *all* connected sums $\#^n \mathbb{CP}^2$ as 3-fold dihedral covers of S^4 with one knot singularity on a connected, embedded branching set, if one allows the branching set to be nonorientable [1]. By contrast, we see in Theorem 5 that orientability of the branching set, together with a single singular point, implies that the signature of such a cover is odd.

3 Knots with equal topological and dihedral genera

In this section we construct families of knots for which the topological, ribbon and 3-dihedral genus are equal. We use trisections of four-manifolds [8], triplane diagrams [15], and singular triplane diagrams [3], all of which we review informally for the reader’s convenience.

Given a smooth, oriented, 4-manifold X , a $(g; k_1, k_2, k_3)$ -trisection of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ into three 4-handlebodies with boundary such that

- $X_i \cong \natural^{k_i}(B^3 \times S^1)$,
- $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ is a closed, oriented surface of genus g ,
- $Y_{ij} = \partial(X_i \cup X_j) \cong \#^{k_l}(S^2 \times S^1)$, where $i, j, l \in \{1, 2, 3\}$ are distinct,
- $\Sigma_g \subset Y_{ij}$ is a Heegaard surface for Y_{ij} .

Every embedded surface $F \subset S^4$ can be described combinatorially by a $(b; c_1, c_2, c_3)$ -triplane diagram [15]. This is a set of three b -strand trivial tangles (A, B, C) such that

³The reference [9] is written in the language of intersection homology. In the case of a singular branched cover $f: X \rightarrow S^4$, this is equivalent to the Novikov signature $\sigma(X', \partial X')$ since X has only an isolated singularity.

each boundary union of tangles $A \cup \bar{B}$, $B \cup \bar{C}$ and $C \cup \bar{A}$ is a c_i -component unlink, for $i = 1, 2, 3$ respectively. Here $\bar{T}$ denotes the mirror image of T . To obtain F from (A, B, C) , one views each of $A \cup \bar{B}$, $B \cup \bar{C}$ and $C \cup \bar{A}$ as unlinks in bridge position in the spokes Y_{12} , Y_{23} and Y_{31} of the standard genus-0 trisection of S^4 , glues c_i disks to the components of each of these unlinks, and pushes these disks into the X_i to obtain an embedded surface.

We introduce *singular triplane diagrams* and their colorings in [3]. A $(b; 1, c_2, c_3)$ singular triplane diagram is a triple of b -strand trivial tangles (A, B, C) . As above, $B \cup \bar{C}$ and $C \cup \bar{A}$ are c_2 - and c_3 -component unlinks. $A \cup \bar{B}$ is a knot K . To build a surface with one singularity of type K , one again views each of $A \cup \bar{B}$, $B \cup \bar{C}$ and $C \cup \bar{A}$ in bridge position in the three spokes Y_{12} , Y_{23} and Y_{31} of the standard genus-0 trisection of S^4 and glues c_2 and c_3 disks to the components of each of the two unlinks. Rather than glue disks to $A \cup \bar{B}$, one attaches the cone on K . Note that by interchanging the order of the tangles A and B , one obtains a surface with singularity $\bar{K}$, the mirror of K .

A p -colored singular triplane diagram is a singular triplane diagram together with an assignment of values in $\{1, 2, \dots, p\}$ to the arcs of the diagram such that on each tangle, the assignment is a Fox p -coloring and such that the colors along the endpoints of each tangle agree. Such a coloring induces a coloring on the corresponding singular surface.

We use 3-colored singular triplane diagrams to construct a family of 3-fold dihedral covers of S^4 which realize the knots K_m given in Figure 1 as singularities on the branching sets. This construction allows us to compute the values of $\Xi_3(K_m)$ using the induced trisections of the corresponding branched cover. As a corollary, we obtain Theorem 5, which establishes the range of the invariant Ξ_3 .

Proposition 4 *Each knot K_m in Figure 1 arises as the only singularity on a 3-fold dihedral branched cover $f_m: \#^{2m+1} \overline{\mathbb{CP}^2} \rightarrow S^4$ whose branching set F_m is an oriented surface of genus m , embedded smoothly in S^4 away from the one singular point. Equivalently, each knot $\bar{K}_m$ arises as the only singularity on a 3-fold dihedral branched cover $\bar{f}_m: \#^{2m+1} \mathbb{CP}^2 \rightarrow S^4$, also with an embedded oriented branching set of genus m .*

Remark By deleting a small neighborhood of the singularity on the branching set in S^4 , one obtains an oriented, 3-colored surface in $F'_m \subset B^4$ with $\partial F'_m = K_m$. In Section 4, we prove that the genus of F'_m is minimal, that is, equal to $g_4(K_m)$. Moreover, by construction, each surface F'_m is ribbon.

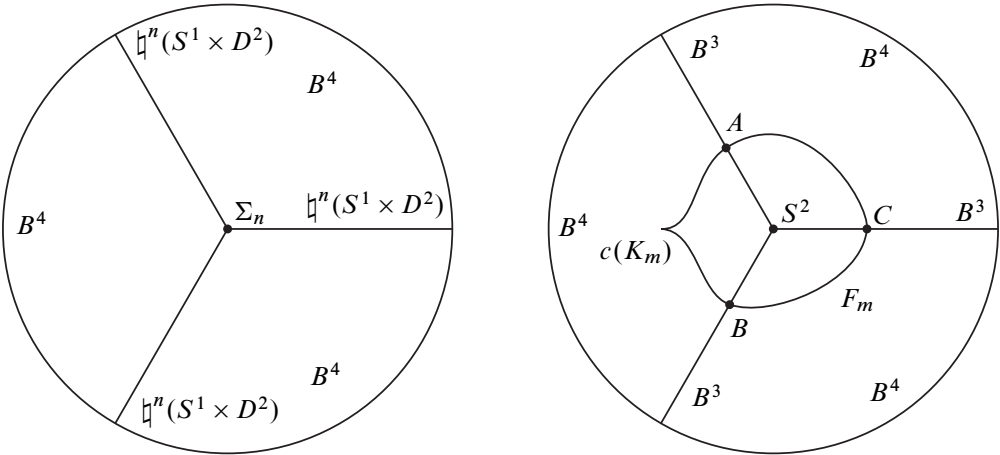


Figure 2: An $(n; 0, 0, 0)$ –trisection of $\#^n \overline{\mathbb{CP}}^2$, obtained as a branched cover of S^4 over a trisected surface F_m with one singularity K_m .

Proof of Proposition 4 We will construct the surface F_m and will give its Fox coloring using a colored (singular) triplane diagram. From this information, we will produce a tripartition of the dihedral cover of S^4 determined by this coloring. We will identify this cover as $\#^n \overline{\mathbb{CP}}^2$, where $n = 2m + 1$.

The colored triplane diagram (A_n, B_n, C_n) for F_m , where $m = \frac{1}{2}(n - 1)$, is shown in Figure 3. We write the value $i \in \{1, 2, 3\}$ next to an arc of a tangle or knot if the homotopy class of the meridian of that arc is mapped to the reflection in D_3 fixing i .

The union $A_n \cup \overline{B}_n$ is the knot $\overline{K}_m$, while $B_n \cup \overline{C}_n$ and $C_n \cup \overline{A}_n$ are each 2–component unlinks; see Figure 4 for a verification when $n = 3$. A triplane diagram with b bridges and c_i components in each link diagram has Euler characteristic $c_1 + c_2 + c_3 - b$; hence, the surface F_m with singularity $\overline{K}_m$ has Euler characteristic $3 - n$ and genus $m = \frac{1}{2}(n - 1)$ since F_m is connected and orientable, and since the tangles A_n , B_n and C_n have $b = n + 2$ bridges.

The fact that F_m is orientable requires a careful check. Consider the cell structure on F_m corresponding to its triplane structure. To show that F_m is orientable, we show that it is possible to coherently orient the faces of this cell structure so that each edge (a bridge in one of the three tangles A_n , B_n or C_n) inherits two different orientations from the two faces adjacent to it. This is shown in Figure 4 in the case $m = 1$ (or $n = 3$).

An Euler characteristic computation shows that the 3–fold dihedral branched cover of the bridge sphere S^2 , branched along the $2(n + 2)$ endpoints of the bridges, is a

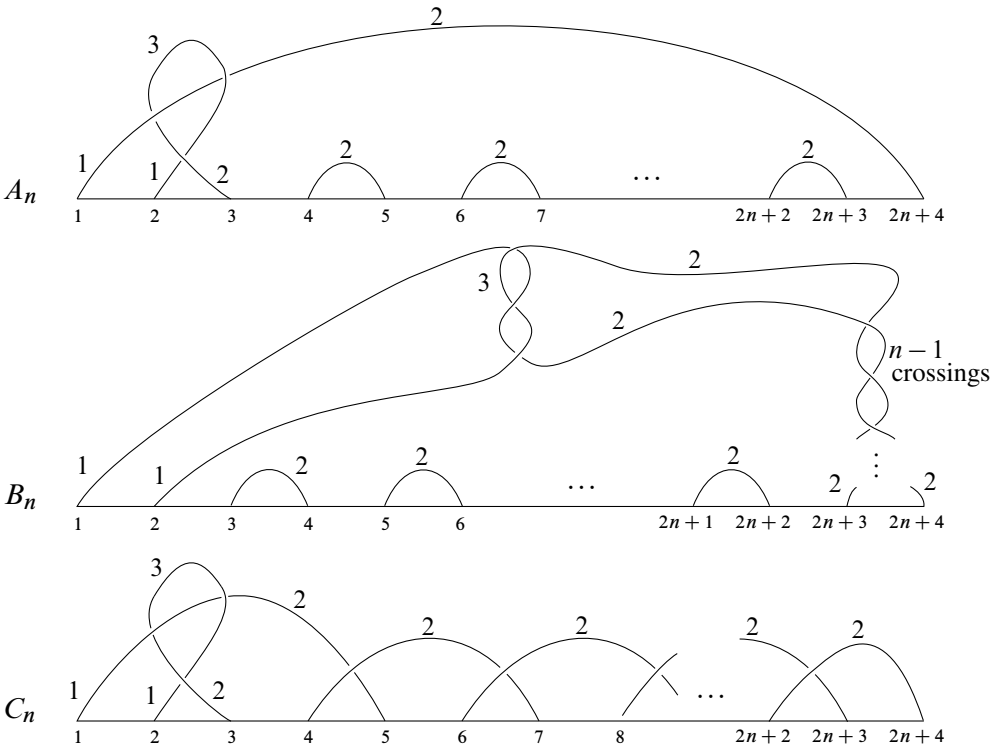


Figure 3: A colored triplane diagram corresponding to a branched covering $\#^n \overline{\mathbb{CP}}^2 \rightarrow S^4$, in the case where n is odd. The numbers $\{1, 2, 3\}$ along the arcs describe the coloring. There is one singularity K_m on the branching set, where $m = \frac{1}{2}(n - 1)$. By reversing the roles of A_n and B_n , one obtains a branched covering $\#^n \overline{\mathbb{CP}}^2 \rightarrow S^4$ with singularity $\bar{K}_m$.

surface Σ_n of genus n . We now show the 3-colored triplane diagram (A_n, B_n, C_n) gives rise to a genus- n trisection of $\#^n \overline{\mathbb{CP}}^2$ with central surface Σ_n , following a method explained in [3]. The branching set F_m is orientable and has one singularity of type K_m , so it will follow from (6) that $\Xi_3(K_m) = -\sigma(\#^n \overline{\mathbb{CP}}^2) = n$.

If a properly embedded b -strand tangle $(T, \partial T) \subset (B^3, S^2)$ with arcs $t_1, t_2, \dots, t_b$ is trivial, then by definition there exists a collection of disjoint arcs $d_1, d_2, \dots, d_b$ in S^2 such that the boundary unions $t_i \cup d_i$ bound a collection of disjoint disks in B_3 . We refer to the d_i as *disk bottoms*. The existence of such a collection of disks is equivalent to the arcs of T being simultaneously isotopic to a collection of disjoint arcs (the d_i) in S^2 .

To determine the trisection diagram, we must first find the disk bottoms for the three tangles A_n , B_n and C_n , then lift them from the bridge sphere S^2 to its irregular

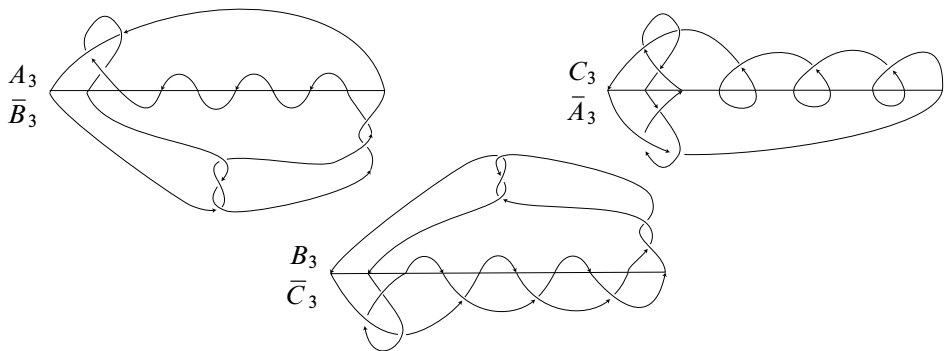


Figure 4: The links $A_3 \cup \bar{B}_3$, $B_3 \cup \bar{C}_3$ and $C_3 \cup \bar{A}_3$. Note that $A_3 \cup \bar{B}_3$ is the knot $\bar{K}_1$.

dihedral cover Σ_n . The curves in the trisection diagram are formed by certain lifts of these disk bottoms; we identify these lifts later.

The disk bottoms for each tangle A_n , B_n and C_n are depicted in Figure 5, in the case $n = 3$. In Figure 7, we draw just three of the disk bottoms for each of A_n (blue), B_n (red) and C_n (green) on the same copy of S^2 .

In the next step of the proof, we use a construction of the irregular 3–fold dihedral cover $\Sigma_n \rightarrow S^2$, branched along $2(n + 2)$ points in S^2 , due to Hilden [11]. We review this construction now; the reader should refer to Figure 6 for an example in the case $n = 3$. In this construction, the meridians of two branch points map to the transposition $(2\ 3)$ (equivalently, are colored “1”), and the meridians of the remaining $2n + 2$ branch points map to the transposition $(1\ 3)$ (equivalently, are colored “2”). One first constructs the 6–fold regular dihedral cover $R_n \rightarrow S^2$ branched along $2(n + 2)$ points determined by this coloring. The resulting surface has genus $3n + 1$. The 3–fold irregular dihedral cover Σ_n is obtained from this regular one by an involution, namely 180° rotation about the vertical axis.

Next, we lift the disk bottoms from the bridge sphere to Σ_n , where Σ_n is constructed as above. Each disk bottom has three lifts to Σ_n , two of which fit together to form a closed curve. Not all of these closed curves are necessarily essential curves on Σ_n ; see [3] for further examples. However, we may choose $n - 2$ disk bottoms for each tangle (A_n, B_n, C_n) whose lifts are essential. These lifts are shown in Figure 8, again in the case $n = 3$.

The resulting curves form a trisection diagram for $\#^n \overline{\mathbb{CP}}^2$. Moreover, the standard trisection of S^4 , branched along F_m , lifts to an $(n; 0, 0, 0)$ –trisection of $\#^n \overline{\mathbb{CP}}^2$. This

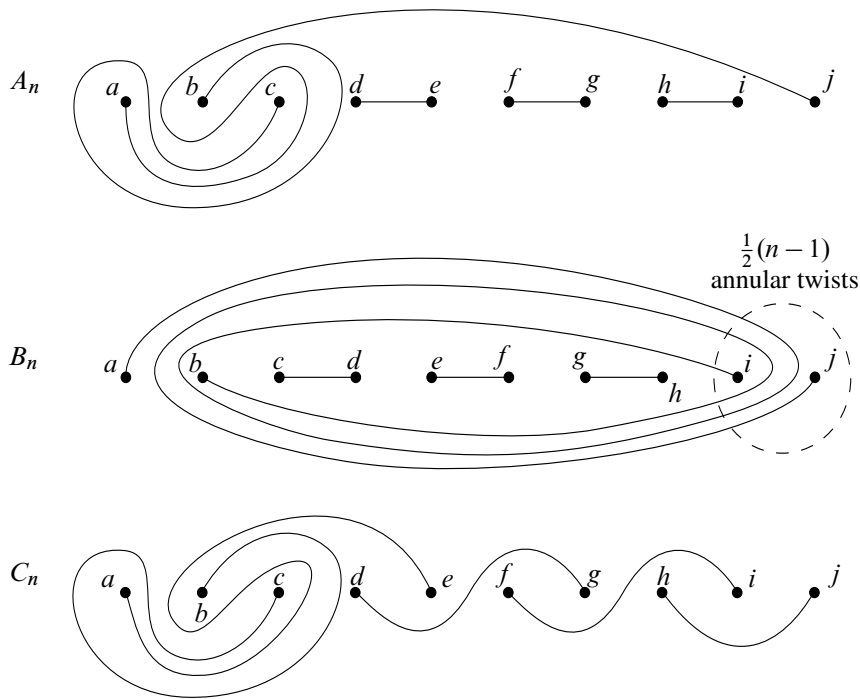


Figure 5: Disk bottoms for the triplane diagram (A_n, B_n, C_n) when $n = 3$.

can be found by analyzing the lifts of the three pieces of the trisection of (S^4, F_m) ; for details see [3, Theorem 8]. $\square$

We use the above construction to establish the range of the invariant Ξ_3 .

Theorem 5 *Let n be an integer. There exists a **strongly** 3–admissible singularity K_n and a 3–coloring ρ_n of K_n such that $\Xi_3(K_n, \rho_n) = n$ if and only if $n \in 2\mathbb{Z} + 1$.*

Remark The proof of Theorem 5 is slightly more general than what the theorem statement requires. That is, we establish that $\Xi_p(K, \rho)$ is odd whenever $p \equiv 3 \pmod 4$. Realizability of all odd integers by Ξ_p is open for $p \neq 3$.

Proof of Theorem 5 We have given a construction realizing each of the knots K_m as the only singularity on a branched cover $\#^{2m+1} \overline{\mathbb{CP}}^2 \rightarrow S^4$ whose branching set is oriented. By (6), it follows that $\Xi_3(K_m) = -\sigma(\#^{2m+1} \overline{\mathbb{CP}}^2) = 2m + 1$, where $m \geq 0$. Note also that $\Xi_p(\bar{K}_m) = -\Xi_p(K_m)$, as proved in [3], where $\bar{K}$ denotes the mirror image of K . Of course, K is (strongly) p –admissible if and only if $\bar{K}$ is. This

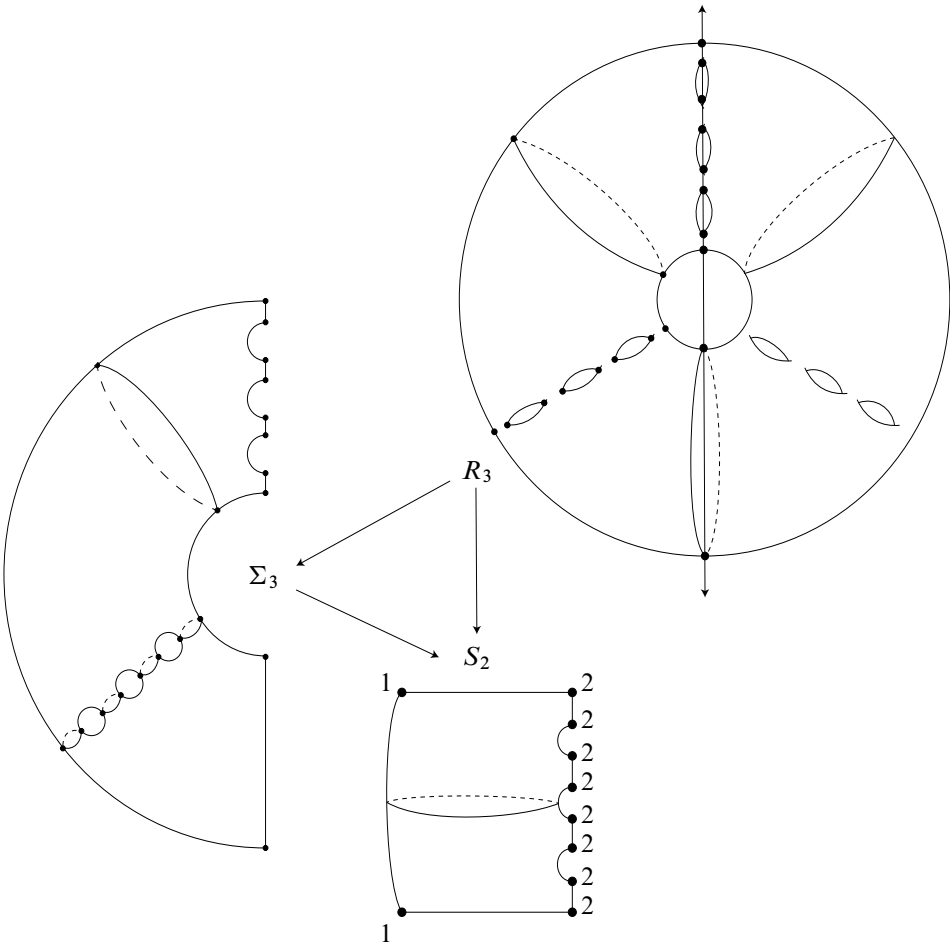


Figure 6: A 6-fold regular dihedral cover R_n of S^2 branched along $2(n+2)$ points, with $n=3$; the irregular cover Σ_n is the quotient of R_n by 180° rotation about the vertical axis.

proves that all odd integers are contained in the range of the invariant Ξ_3 on strongly 3-admissible knots.

Conversely, we will verify that for any p -coloring ρ of any strongly p -admissible singularity K , the integer $\Xi_p(K, \rho)$ is odd. It suffices to assume that $p \equiv 3 \pmod 4$. We use (5). Since p is odd, $p^2 \equiv 1 \pmod 4$, so $(p^2 - 1)/(6p) L_V(\beta, \beta)$ is even. It follows from [13, Equation 2.20] that, if $p \equiv 3 \pmod 4$, the rank of $H_2(W(K, \beta); \mathbb{Z})$ is odd, and hence so is the signature. Lastly, each σ_{ζ^i} is an even integer. It follows that $\Xi_p(K)$ is odd. $\square$

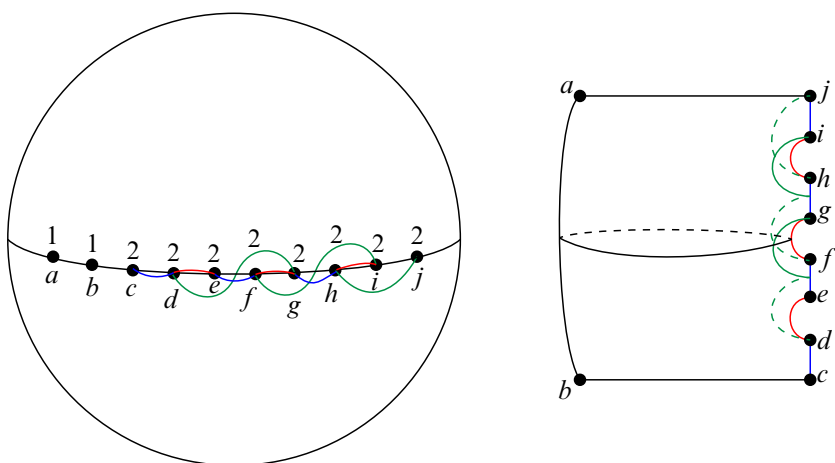


Figure 7: Disk bottoms for the triplane diagram (A_n, B_n, C_n) when $n = 3$, drawn on the bridge sphere.

Remark The knot K_m has bridge number 2, showing that two-bridge knots realize the full range of Ξ_p when $p = 3$. This answers a question posed in [12]. It is not known whether the full range of Ξ_p is realized by two-bridge knots when $p \neq 3$. It would be of interest to establish that it is “sufficient” to consider two-bridge knots when constructing singular dihedral covers of four-manifolds since p -admissibility is particularly easy to detect for two-bridge singularities [14].

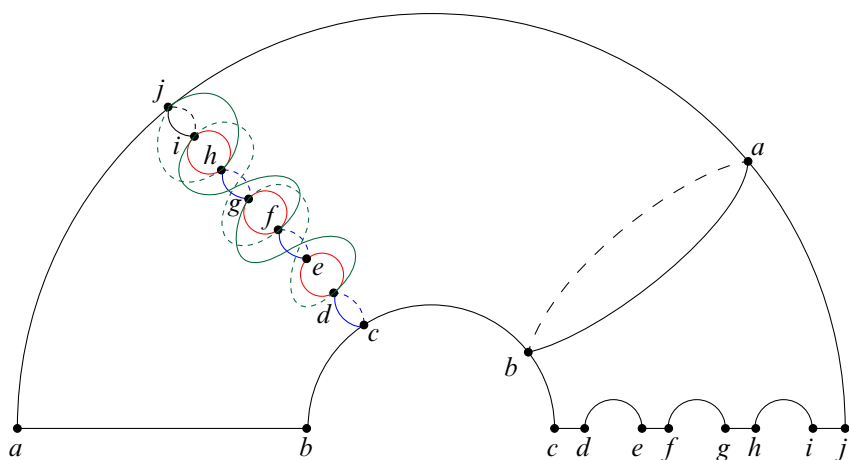


Figure 8: Lifts of disk bottoms to the 3-fold irregular dihedral cover of S^2 , for the triplane diagram (A_n, B_n, C_n) , when $n = 3$.

4 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1 (A) Given a p -admissible knot K with p -coloring ρ , we wish to prove the inequality

$$g_p(K, \rho) \geq \frac{|\Xi_p(K, \rho)| - \text{rk } H_1(M)}{p - 1} - \frac{1}{2},$$

where M denotes the dihedral cover of S^3 branched along K corresponding to ρ . Throughout, all homology groups are with $\mathbb{Z}$ coefficients.

For K a p -admissible knot, by the definition of homotopy-ribbon dihedral genus, there exists a topologically locally flat orientable homotopy-ribbon surface F' for K such that the genus of F' equals $g_p(K, \rho)$. (If ρ does not extend over any locally flat, orientable, homotopy-ribbon surface, $g_p(K, \rho) = \infty$ and the inequality is trivial.) Recall that, since F' is orientable, $|\Xi_p(K, \rho)| = \sigma(X', \partial X')$; the right-hand side denotes the Novikov signature of X' as a manifold with boundary. We will find an upper bound for $|\sigma(X', \partial X')|$ in terms of the Euler characteristic of $X = X' \cup c(\partial X')$.

Let

$$\bar{\rho}: \pi_1(B^4 - F') \rightarrow D_p$$

be the homomorphism which extends the coloring $\rho: \pi_1(S^3 - K) \rightarrow D_p$ and induces the cover $X' \rightarrow B^4$ branched over F' . Let $\widehat{M}$ be the unbranched irregular dihedral cover of $S^3 - K$ corresponding to ρ , and M the induced branched cover. Denote by $F \subset S^4$ the singular surface which is the boundary union of F' and the cone on K , so that X is the dihedral cover of S^4 with branching set F .

We will show that X is simply connected. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{i_*} & \pi_1(X') \\ i_{M*} \uparrow & & \uparrow i_{X*} \\ \pi_1(\widehat{M}) & \xrightarrow{j_*} & \pi_1(\widehat{X}') \\ \downarrow p_* & & \downarrow q_* \\ \pi_1(S^3 - K) & \xrightarrow{i_*} & \pi_1(B^4 - F') \\ \downarrow \rho & \swarrow \bar{\rho} & \\ D_p & & \end{array}$$

All maps in the diagram are either induced by inclusions or by covering maps. Clearly p_* and q_* are injective, as they are induced by covering maps, and i_{M*} and i_{X*}

are surjective, as they are induced by inclusions of unbranched covering spaces into their branched counterparts. The homomorphisms ρ and $\bar{\rho}$ are surjective by definition. Finally, since F' is a homotopy-ribbon surface for K , the homomorphism i_* is surjective.

We now show that j_* is surjective as well. Consider an element $\gamma \in \pi_1(\widehat{X}')$. Since i_* is surjective, there exists an element $\delta \in \pi_1(S^3 - K)$ such that $i_*(\delta) = q_*(\gamma)$. We have that $\bar{\rho} \circ q_*(\gamma) \in \mathbb{Z}/2\mathbb{Z} \subset D_p$, the reflection subgroup which determines the cover $\widehat{X}'$ of $B^4 - F'$. By commutativity of the lower triangle, $\rho(\delta) = \bar{\rho} \circ q_*(\gamma) \in \mathbb{Z}/2$, so $\delta \in \text{im } p_*$. Take $\tilde{\delta} \in \pi_1(\widehat{M})$ such that $p_*(\tilde{\delta}) = \delta$. Consider $q_* \circ j_*(\tilde{\delta})$, which by commutativity is equal to $i_* \circ p_*(\tilde{\delta})$. Now $q_* \circ j_*(\tilde{\delta}) = i_*(\delta) = q_*(\gamma)$. By injectivity of q_* , we have $j_*(\tilde{\delta}) = \gamma$, so j_* is indeed surjective.

Observe that, since j_* and i_{X*} are both surjective, $i_*: \pi_1(M) \rightarrow \pi_1(X')$ is surjective as well.

By the Seifert–van Kampen theorem, we have $\pi_1(X) \simeq \pi_1(c(M)) *_{\pi_1(M)} \pi_1(X')$. The cone $c(M)$ is contractible, so $\pi_1(c(M))$ is trivial. Hence

$$\pi_1(c(M)) *_{\pi_1(M)} \pi_1(X') \simeq \pi_1(X') / \text{im } i_*(\pi_1(M)).$$

The quotient $\pi_1(X') / \text{im } i_*$ is trivial since i_* is surjective. Hence X is simply connected and, in particular, $H_1(X) = 0$. We also know that $H_0(X) = 1$ since X is path-connected.

Next, consider

$$\chi(X) = \text{rk } H_4(X) - \text{rk } H_3(X) + \text{rk } H_2(X) + 1.$$

Since $X = X' \cup c(\partial X')$,

$$H_n(X) = H_n(X' / \partial X') = \tilde{H}_n(X', \partial X').$$

By Lefschetz duality and universal coefficients, we see that

$$\text{rk } H_3(X) = \text{rk } H_3(X', \partial X') = \text{rk } H^1(X') = \text{rk } H_1(X'),$$

and

$$\text{rk } H_4(X) = \text{rk } H_4(X', \partial X') = \text{rk } H^0(X') = \text{rk } H_0(X') = 1,$$

and hence

$$\chi(X) = 2 + \text{rk } H_2(X) - \text{rk } H_1(X').$$

By [13, Equation 1.1] we have

$$(7) \quad \chi(X) = 2p - \frac{1}{2}(p-1)\chi(F) - \frac{1}{2}(p-1).$$

Recall that the Novikov signature $\sigma(X, \partial X')$ is the signature of the intersection form defined on the image of the map $i_*: H_2(X') \rightarrow H_2(X', \partial X')$ induced by inclusion. Thus,

(8)
$$|\Xi_p(K, \rho)| = \sigma(X, \partial X') \leq \text{rk } \text{im } i_* \leq \text{rk } H_2(X', \partial X') = \text{rk } H_2(X).$$

The result follows by combining this inequality with the two formulas for $\chi(X)$ above. We substitute

$$\chi(F) = 2 - 2g(F)$$

into (7), and

$$\text{rk } H_2(X) = \chi(X) - 2 + \text{rk } H_1(X')$$

into (8), which gives

$$|\Xi_p(K, \rho)| \leq 2p - \frac{1}{2}(p-1)(2-2g(F)) - \frac{1}{2}(p-1) - 2 + \text{rk } H_1(X').$$

Simplifying, we obtain

$$g(F) \geq \frac{|\Xi_p(K, \rho)| - \text{rk } H_1(X')}{p-1} - \frac{1}{2}.$$

Finally, since the inclusion $i: M \rightarrow X'$ induces a surjection on fundamental groups, we also know that $\text{rk } H_1(M) \geq \text{rk } H_1(X')$. Hence

$$g(F) \geq \frac{|\Xi_p(K, \rho)| - \text{rk } H_1(M)}{p-1} - \frac{1}{2}.$$

(B) Let K be a p -admissible knot with respect to a coloring ρ and let $F' \subset B^4$ be a homotopy-ribbon, locally flat oriented surface with boundary K such that ρ extends over F' . Denote by $F \subset S^4$ the surface with singularity K obtained as a boundary union of F' and the cone on K , and denote by X the dihedral cover of S^4 determined by the induced coloring of F . We assume that X is a definite manifold. In particular, K is in fact strongly admissible with respect to the coloring ρ , and the corresponding branched cover M of S^3 along K is again S^3 . We then wish to show that the inequality (3) is sharp:

$$g_p(K, \rho) = \frac{|\Xi_p(K, \rho)|}{p-1} - \frac{1}{2}.$$

Precisely, we will show that the right-hand side of this equation equals the genus of F' . That is, F' will be seen to realize the lower bound from (A) on the dihedral genus $g_p(K, \rho)$ of K .

Since X is a definite manifold, $\text{rk } H_2(X) = |\sigma(X)|$. By the proof of (A), X is simply connected; by Poincaré duality we have $\chi(X) = 2 + \text{rk } H_2(X)$ and hence

$$|\Xi_p(K, \rho)| = |\sigma(X)| = \chi(X) - 2.$$

On the other hand, denoting by $g(F)$ the genus of F , by (7) we have

$$\chi(X) = 2p - \frac{1}{2}(p-1)(2 - 2g(F)) - \frac{1}{2}(p-1).$$

Putting these two equations together, we conclude that

$$g(F) = \frac{|\Xi_p(K, \rho)|}{p-1} - \frac{1}{2}.$$

By assumption, the coloring ρ extends over F' , so

$$g(F') \geq \mathfrak{g}_p(K, \rho) \geq \frac{|\Xi_p(K, \rho)|}{p-1} - \frac{1}{2}.$$

Thus, F' realizes the p -dihedral genus of K .

In the second part of the theorem, we assume in addition that

$$|\sigma(K)| = \frac{2|\Xi_p(K, \rho)|}{p-1} - 1,$$

where $\sigma(K)$ is the signature of the knot K . We wish to show that the topological four-genus and the topological homotopy-ribbon p -dihedral genus of K are both equal to $\frac{1}{2}|\sigma(K)|$.

The additional assumption here can be rewritten as $|\sigma(K)| = 2\mathfrak{g}_p(K, \rho)$. Murasugi's signature bound [16, Theorem 9.1] states that $g_4(K) \geq \frac{1}{2}|\sigma(K)|$. Thus, we have $g_4(K) \geq \mathfrak{g}_p(K, \rho)$. But $g_4(K) \leq \mathfrak{g}_p(K) \leq \mathfrak{g}_p(K, \rho)$ in general, so $g_4(K) = \mathfrak{g}_p(K)$. $\square$

Proof of Theorem 2 Our aim is to show that the equalities

$$g_4(K_m) = \mathfrak{g}_3(K_m) = \frac{1}{2}|\Xi_3(K_m, \rho_m)| - \frac{1}{2} = m$$

hold for the 3-colored knots K_m introduced in the previous section. In particular, it will follow that the generalized topological slice-ribbon conjecture holds for these knots.

By Theorem 1(B), it suffices to show that

- (1) each K_m is the boundary of a homotopy-ribbon surface F'_m such that $\mathfrak{g}_3(K) = g(F'_m)$, and

(2) the signature $\sigma(K_m)$ satisfies the equality

$$|\sigma(K)| = \frac{2|\Xi_p(K, \rho)|}{p-1} - 1$$

for $p = 3$.

We first address (1). Surfaces F'_m realizing the lower bound on dihedral homotopy-ribbon genus for the knots K_m are constructed in the proof of Proposition 4: we have shown $g(F'_m) = m$ and $|\Xi_3(K_m)| = 2m + 1$, so

$$\frac{|\Xi_p(K_m)|}{p-1} - \frac{1}{2} = m.$$

We note that, since the knots K_m are two-bridge, each of them has a unique 3–coloring (up to permuting the colors), so there is no distinction between $\mathfrak{g}_p(K_m, \rho_m)$ and $\mathfrak{g}_p(K_m)$. By construction, the surface $F'_m \subset B^4$ obtained by deleting a small neighborhood of the singularity K_m is ribbon since $A_m \cup \bar{B}_m$ only bounds the cone on K_m , while the unlinks $B_m \cup \bar{C}_m$ and $C_m \cup \bar{A}_m$ bound standard unknotted disks in B^4 .

We now address (2). We will compute the signature $\sigma(K_m)$, and show it is equal to $2m = 2|\Xi_p(K)|/(p-1) - 1$.

The signature of K can be computed using the Goeritz matrix $G(K)$, the matrix of a quadratic form associated to a knot diagram via a checkerboard coloring, and hence a (not necessarily orientable) spanning surface; this technique was introduced by Gordon and Litherland [10]. The advantage of this technique is that the dimension of the Goeritz matrix associated to a projection of a knot may be much smaller than the dimension of the corresponding Seifert matrix; indeed, the dimension of $G(K_m)$ is 4 for all m .

Gordon and Litherland proved that the signature of a knot is equal the signature of the Goeritz matrix of a diagram of the knot plus a certain correction term: $\sigma(K) = \sigma(G(K)) - \mu$. We start by computing the Goeritz matrix $G(K_m)$ and its signature.

One first computes the *unreduced* Goeritz matrix. To do this, one chooses a checkerboard coloring of the knot diagram, and labels the “white” regions $X_1, \dots, X_k$. Such a labeling for the K_m is shown in Figure 9. The entries g_{ij} of the unreduced Goeritz matrix are computed as follows:

$$g_{ij} = \begin{cases} -\sum \eta(c) & \text{for } i \neq j \text{ and } c \text{ a double point incident to } X_i \text{ and } X_j, \\ -\sum_{s \in \{1, \dots, k\} \setminus \{i\}} g_{is} & \text{for } i = j. \end{cases}$$

The signs $\eta(c)$ are computed as in Figure 10; shaded areas correspond to “black” regions of the checkerboard coloring.

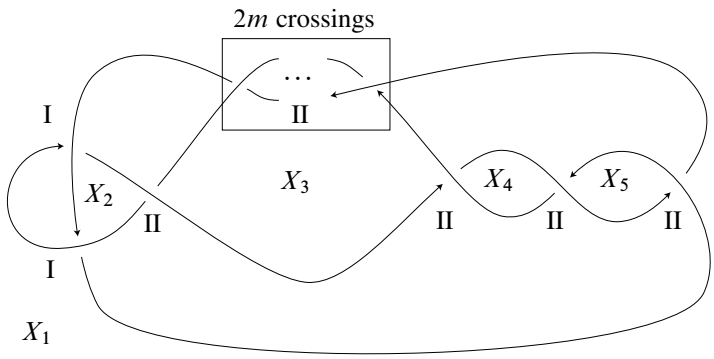


Figure 9: The “white” regions of a checkerboard coloring of K_m , labeled $X_1, X_2, \dots, X_5$.

The unreduced Goeritz matrix of K_m is

$$G'(K_m) = \begin{pmatrix} -2m-3 & -2 & -2m & 0 & -1 \\ -2 & -3 & -1 & 0 & 0 \\ -2m & -1 & -2m-2 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 \\ -1 & 0 & 0 & -1 & -2 \end{pmatrix}.$$

The Goeritz matrix $G(K_m)$ is obtained by deleting the first row and column of $G'(K_m)$. The characteristic polynomial of this matrix is

$$p_{G(K_m)}(\lambda) = (\lambda + 3)(\lambda(\lambda + 3)^2 + 2(\lambda + 1)(\lambda + 3)m + 3).$$

Hence $\lambda = -3$ is an eigenvalue. In addition, since $m \geq 0$, it is straightforward to verify that any root of the cubic factor must be negative (if λ is nonnegative, the cubic, as written above, is a sum of three nonnegative terms). Hence, $\sigma(G(K_m)) = -4$.

The correction term $\mu(K)$ in Gordon and Litherland’s formula for $\sigma(K)$ is computed as follows. Each crossing c of K can be classified as type I or type II, as shown in Figure 10. Let $\mu(K) = \sum_c \eta(c)$, where the sum is taken over all type II crossings.

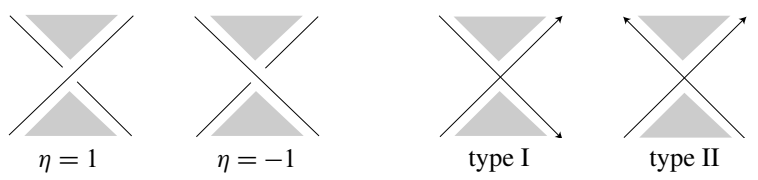


Figure 10: Incidence numbers η and type I and II crossings.

The knot K_m has $4 + 2m$ type II crossings, each of negative sign; see [Figure 9](#). Hence $\sigma(K_m) = -4 + (4 + 2m) = 2m$. $\square$

Proof of Theorem 3 Given any nonnegative integer m , our goal is to construct an infinite family of knots whose 3–dihedral and topological 4–genus are both equal to m . Let K_m denote the knot given in [Theorem 2](#) whose 3–dihedral and topological 4–genus equal m . We will prove that, given a nontrivial ribbon knot γ , the knot $K_m \# \gamma$ has the desired property. The theorem follows by taking repeated connect sums of K_m with γ .

Let γ denote any ribbon knot and let $D \subset B^4$ be a ribbon disk with $\partial D = \gamma$. The knot $K_m \# \gamma$ has 3–dihedral genus and topological four-genus equal to m , as we now show. It is clear that the smooth and topological four-genera of $K_m \# \gamma$ are both equal to m since the knot is smoothly concordant to K_m . Next, note that the given 3–coloring ρ_m of K_m induces a 3–coloring ρ_γ of $K_m \# \gamma$ which restricts trivially to γ . Moreover, since ρ_m extends over F'_m , ρ_γ extends over the ribbon surface $F'_m \natural D$, where $\natural$ denotes boundary connected sum. Therefore, the ribbon 3–dihedral genus of $K_m \# \gamma$ is at most m . Since g_4 is a lower bound for the topological 3–dihedral genus, which in turn is a lower bound for the ribbon 3–dihedral genus, it follows that these genera are equal, as claimed. $\square$

References

- [1] **R Blair, P Cahn, A Kjuchukova**, *Blowing up dihedral covers*, in preparation
- [2] **P Cahn, A Kjuchukova**, *Linking numbers in three-manifolds*, preprint (2016) [arXiv](#)
- [3] **P Cahn, A Kjuchukova**, *Singular branched covers of four-manifolds*, preprint (2017) [arXiv](#)
- [4] **P Cahn, A Kjuchukova**, *Computing ribbon obstructions for colored knots*, preprint (2018) [arXiv](#)
- [5] **S E Cappell, J L Shaneson**, *Linking numbers in branched covers*, from “Four-manifold theory” (C Gordon, R Kirby, editors), Contemp. Math. 35, Amer. Math. Soc., Providence, RI (1984) 165–179 [MR](#)
- [6] **A J Casson, C M Gordon**, *Cobordism of classical knots*, from “À la recherche de la topologie perdue” (L Guillou, A Marin, editors), Progr. Math. 62, Birkhäuser, Boston (1986) 181–199 [MR](#)
- [7] **R H Fox**, *Some problems in knot theory*, from “Topology of 3–manifolds and related topics” (M K Fort, Jr, editor), Prentice-Hall, Englewood Cliffs, NJ (1962) 168–176 [MR](#)
- [8] **D Gay, R Kirby**, *Trisecting 4–manifolds*, Geom. Topol. 20 (2016) 3097–3132 [MR](#)

- [9] **C Geske, A Kjuchukova, J L Shaneson**, *Signatures of topological branched covers*, preprint (2019) [arXiv](#) To appear in Int. Math. Res. Not.
- [10] **CM Gordon, RA Litherland**, *On the signature of a link*, Invent. Math. 47 (1978) 53–69 [MR](#)
- [11] **HM Hilden**, *Every closed orientable 3–manifold is a 3–fold branched covering space of S^3* , Bull. Amer. Math. Soc. 80 (1974) 1243–1244 [MR](#)
- [12] **A Kjuchukova**, *On the classification of irregular dihedral branched covers of four-manifolds*, PhD thesis, University of Pennsylvania (2015) [MR](#) Available at <https://search.proquest.com/docview/1728869559>
- [13] **A Kjuchukova**, *Dihedral branched covers of four-manifolds*, Adv. Math. 332 (2018) 1–33 [MR](#)
- [14] **A Kjuchukova, K Orr**, *Knots arising as singularities on branched covers between four-manifolds*, in preparation
- [15] **J Meier, A Zupan**, *Bridge trisections of knotted surfaces in S^4* , Trans. Amer. Math. Soc. 369 (2017) 7343–7386 [MR](#)
- [16] **K Murasugi**, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965) 387–422 [MR](#)

Department of Mathematics and Statistics, Smith College
Northampton, MA, United States

Max Planck Institute for Mathematics
Bonn, Germany

pcahn@smith.edu, sashka@mpim-bonn.mpg.de

Received: 29 December 2018 Revised: 24 July 2019

