Disjoinable Lagrangian tori and semisimple symplectic cohomology

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We derive constraints on Lagrangian embeddings in completions of certain stable symplectic fillings whose symplectic cohomologies are semisimple. Manifolds with these properties can be constructed by generalizing the boundary connected sum operation to our setting, and are related to birational surgeries like blow-downs and flips. As a consequence, there are many nontoric (noncompact) monotone symplectic manifolds whose wrapped Fukaya categories are proper.

1 Introduction

1.1 Motivations and related works

The number of disjoinable (nondisplaceable) Lagrangian submanifolds with certain topology in a symplectic manifold M is an invariant which measures the "size" or "complexity" of M.

Among considerations along these lines, three cases are of particular interest, namely when $L \subset M$ is diffeomorphic to S^n, $\mathbb{CP}^{n/2}$ and T^n, where $n = \frac{1}{2} \dim_{\mathbb{R}}(M)$, as they correspond to interesting surgeries in the symplectic or algebraic category.

In the case of Lagrangian spheres, this viewpoint is addressed in the construction of Smith, Thomas and Yau [48], where the surgery replaces conifold singularities with

Lagrangian S^3's. On the other hand, when M is a Liouville manifold which carries a dilation $b \in SH^1(M)$ (in the sense of Seidel and Solomon [45]) located in the degree 1 symplectic cohomology, Seidel proved in [43] that there is an integer N depending on the twisted Floer cohomology $\widetilde{H}^*$ such that if $(L_1, \ldots, L_r)$ is a collection of disjoinable Lagrangian spheres, then $r \leq N$.

It's an easy observation that with slight modifications, Seidel's arguments in [43] can be adapted to the case when $(L_1, \ldots, L_r)$ is a collection of Lagrangian submanifolds diffeomorphic to $\mathbb{CP}^{n/2}$ in Liouville manifolds with dilations (when n is a multiple of 4, one works over a field $\mathbb{K}$ with char$(\mathbb{K}) = 2$). This is the case, for example, when $M = T^*\mathbb{CP}^{n/2}$. From the Weinstein neighborhood theorem, it is easy to see that a collection of Lagrangian $\mathbb{CP}^{n/2}$'s provides the starting point of the Mukai flop; see Kawamata [21].

For the above two cases, the method of [43] applies mainly because Lagrangian submanifolds diffeomorphic to S^n or $\mathbb{CP}^{n/2}$ can be equipped with a b–equivariant structure when M admits a dilation b; see Section 4 of [45]. From a more algebraic point of view established in Seidel [42], spherical and projective objects in the Fukaya category $\mathcal{F}(M)$ are $\mathbb{C}^*$–equivariant as A_∞–modules over the endomorphism algebra of split-generators in $\mathcal{F}(M)$ since they are rigid and simple, or, more concretely,

$$(1) \qquad\qquad HF^1(L_i, L_i) = 0, \quad HF^0(L_i, L_i) \cong \mathbb{C}$$

for $1 \leq i \leq r$. However, taking a mirror-symmetric point of view, Lagrangian tori are less likely to be equivariant because they correspond to skyscraper sheaves of points in $D^b\mathrm{Coh}(M^\vee)$, the derived category of coherent sheaves of the mirror.

When M is closed and monotone, its mirror is expected to be a Landau–Ginzburg model $(M^\vee, W)$ — see Auroux, Katzarkov and Orlov [6] — where $W: M^\vee \to \mathbb{K}$ is a superpotential taking values in an algebraically closed field $\mathbb{K}$. Under the additional assumption that W is Morse, the triangulated category of matrix factorizations $H^0(MF(M^\vee, W))$ is then split-generated by skyscraper sheaves supported at the critical points of W, or, equivalently, a set of idempotents after passing to its split-closure. From this point of view, the semisimplicity assumption imposed on the *small* quantum cohomology ring $QH^*(M)$ in Entov and Polterovich [16], which is expected to be mirror to the semisimplicity of $H^0(MF(M^\vee, W))$, seems to be natural. More precisely, let $(L_1, \ldots, L_r)$ be a collection of monotone Lagrangian tori in M such that they are disjoinable by Hamiltonian isotopies and $HF^*(L_i, L_i) \neq 0$ for all i; then

it is essentially a consequence of Theorem 1.25 of [16] that

$$r \leq \dim_{\mathbb{K}} QH^{\mathrm{even}}(M).$$ (2)

This is a beautiful example where the closed string invariant $QH^*(M)$ is used to give global constraints on Lagrangian embeddings, which belongs to the open string sector. See also the work of Biran and Cornea [8] for studies of the same flavor.

The assumption that $QH^*(M)$ is semisimple imposes strong restrictions on M, and many known examples of symplectic manifolds M with semisimple $QH^*(M)$ can be obtained by starting from known ones and applying birational surgeries, such as blow-ups and reverse flips to M. The open string counterpart of this observation has been investigated recently by Charest and Woodward:

Theorem 1.1 (Charest and Woodward [10]) *Let (M_+, ω_{M_+}) be a compact symplectic manifold with $[\omega_{M_+}] \in H^2(M_+; \mathbb{Q})$. Suppose M_+ is obtained from a compact symplectic manifold M_- with $[\omega_{M_-}] \in H^2(M_-; \mathbb{Q})$ by a reverse simple flip or blow-up with trivial center with multiplicity r; then in a contractible neighborhood of the exceptional locus there exist a Lagrangian torus $L \subset M_+$, r distinct local systems $\xi_L^1, \ldots, \xi_L^r$ and weak bounding cochains $b_L^1, \ldots, b_L^r$ such that $HF^*((L, \xi_L^i, b_L^i), (L, \xi_L^i, b_L^i)) \neq 0$ for $1 \leq i \leq r$.*

This paper continues the exploration of the above picture in the case when M is a connected monotone symplectic manifold obtained by completing a *stable symplectic filling* of a $(2n-1)$–dimensional contact manifold (V, ξ), where ξ is a cooriented contact structure. Briefly, this means that V is the boundary of a codimension 0 submanifold $M^{\mathrm{in}} \subset M$, and there exists a contact form θ_V which, together with the restriction of ω_M, form a stable Hamiltonian structure on V; see Section 2.1 for details. This assumption is made here so that the symplectic cohomology $SH^*(M)$ can be shown to be well defined as a ring by standard arguments based on maximum principles; see Section 2.1. For a discussion of Hamiltonian Floer theory on more general open manifolds, we refer to Groman [20].

In the special case when M is the total space of a negative line bundle $\mathcal{L} \to B$ over a closed monotone symplectic manifold (B, ω_B), its Floer theory has been studied extensively in Oancea [32], Ritter [36; 37] and Ritter and Smith [38]. These are examples of *convex* symplectic manifolds, namely $d\theta_V = \omega_V$, so the symplectic filling

$(M^{\mathrm{in}}, \omega_M)$ of (V, ξ) is *strong* in the sense of Massot, Niederkrüger and Wendl [28]. Roughly speaking, the upshot is that the symplectic topology of M has features similar to closed monotone symplectic manifolds, with quantum cohomology replaced by symplectic cohomology.

More precisely, the split-generation of certain summands of the monotone Fukaya category $\mathcal{F}_\lambda(M)$ by Lagrangian tori (equipped with local systems) is similar to the closed case, from which one can deduce the following closed string counterpart of the homological mirror symmetry conjecture:

$$(3) \qquad\qquad SH^*(M) \cong \mathrm{Jac}(W);$$

see [37; 38] for a proof in the case of negative line bundles, and refer to Section 4.3 of this paper for a more general statement. This suggests that $SH^*(M)$ should be the appropriate replacement of $QH^*(M)$ for the purpose of studying Lagrangian embeddings in the current setting. Here $\mathrm{Jac}(W)$ is the Jacobi ring of the superpotential $W \colon (\mathbb{K}^*)^n \to \mathbb{K}$ on the mirror Landau–Ginzburg model.

Motivated by this we will prove an analogue of Entov and Polterovich's theorem in the noncompact setting, which gives an upper bound for the number of certain disjoinable nondisplaceable Lagrangian tori in terms of $SH^*(M)$. See Theorem 1.4.

On the other hand, it is a viewpoint established by Seidel and Smith in [40] that the existence of nondisplaceable Lagrangian tori should result in the nonvanishing of the symplectic cohomology. Combined with Theorem 1.1, this suggests that for stable symplectic fillings M, reverse MMP transitions (there is no minimal model program for noncompact varieties, but it still makes sense to talk about MMP transitions) will contribute nontrivially to $SH^*(M)$. In particular, an analogue of Conjecture 1.1 in [10] adapted to the noncompact case should imply the following:

Conjecture 1.2 *Let*

$$(4) \qquad\qquad M_+ \dashrightarrow \cdots \dashrightarrow M_i \dashrightarrow \cdots \dashrightarrow M_-$$

be a sequence of symplectic MMP transitions among completions M_i of stable fillings such that the quantum cohomology $QH^*(Z_i)$ of the center $Z_i \subset M_i$ (which is a compact symplectic submanifold) in each step has semisimple nonzero eigensummands. Suppose $SH^*(M_-) = 0$ or is semisimple; then $SH^*(M_+)$ is semisimple.

A piece of this conjecture will be proved in Corollaries 1.7 and 1.8.

1.2 New results

We now turn to the main contents of this paper. From now on, the coefficient field $\mathbb{K}$ will mean the Novikov field

$$(5) \qquad \mathbb{K} := \left\{ \sum_{i=0}^{\infty} a_i q^{n_i} \;\middle|\; a_i \in \mathbb{F}, \, n_i \in \mathbb{R}, \, \lim_{i \to \infty} n_i = \infty \right\},$$

where $\mathbb{F}$ is an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$. This particular choice is picked here simply for convenience, but the use of a Novikov field is necessary due to the nonexistence of an a priori energy estimate for Floer solutions.

We follow the convention of McDuff and Salamon [29, Section 11.1] for the discussion of quantum cohomology $QH^*(M)$. For any $a \in QH^*(M)$, denote by $a^{\star j}$ its jth power under quantum cup product. Let N_M be the minimal Chern number of M; recall that $QH^*(M)$ can be equipped with a $\mathbb{Z}/2N_M$–grading and as $\mathbb{K}$–vector spaces we have the decomposition

$$(6) \qquad QH^{2i}(M) = \bigoplus_{2j \equiv 2i \bmod 2N_M} H^{2j}(M; \mathbb{K}).$$

To ensure that the symplectic cohomology $SH^*(M)$ is well defined, it suffices to impose the assumption that M is *semipositive*, ie for any class $[u] \in \pi_2(M)$ we have

$$(7) \qquad 3 - n \le c_1(M)([u]) < 0 \implies \omega_M([u]) \le 0.$$

The following definition is motivated by the work of Ritter [36; 37], where the symplectic cohomologies of negative line bundles $\mathcal{O}(-m) \to \mathbb{CP}^{n-1}$ with $1 \le m \le n-1$ are studied. His main result, which says that $SH^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-m))$ is the quotient of $QH^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-m))$ by the zero eigensummand of the quantum multiplication by the first Chern class, can be regarded as a symplectic analogue of the quantum Lefschetz hyperplane theorem; see Lee [24].

Definition 1.3 Let M be the completion of a stable filling M^{in} of the contact boundary (V, ξ). Suppose M is semipositive. We say that M^{in} is a Lefschetz domain at level j if

(i)   there exists an integer j with $1 \le j \le n-1$ such that $H^{2j}(V; \mathbb{Q}) = 0$ and $c_1(M)^{\star j} \in H^{2j}(M; \mathbb{K}) \subset QH^{\mathrm{even}}(M)$;

(ii) as a ring, $SH^*(M)$ is semisimple and is the localization of $QH^*(M)$ at $c_1(M)$.

The topological restriction on V imposed by condition (i) above is satisfied for a large class of good contact toric manifolds by the computations in Luo [27]. This will be clarified in the proof of Proposition 2.12 and in Section 2.5. By the aforementioned works of Ritter, (ii) is satisfied for the negative line bundles $\mathcal{O}(-m) \to \mathbb{C}\mathbb{P}^{n-1}$ with $1 \le m \le n-1$. Condition (i) is not needed for the proof of Theorem 1.4 if M^{in} is a strong symplectic filling, although it actually holds for $\mathcal{O}(-m) \to \mathbb{C}\mathbb{P}^{n-1}$, and the unit disc bundle $\mathcal{O}(-m)_{\le 1}$ is a Lefschetz domain at level j for any $1 \le j \le n-1$. In the simplest case, when $m = 1$, $c_1(M)^{\star j} = c_1(M)^j \in H^{2j}(M; \mathbb{K})$.

For our first theorem, only the $\mathbb{Z}/2$–grading on $SH^*(M)$ will be relevant.

Theorem 1.4 *Assume M^{in} is a monotone Lefschetz domain (at any level j) in the sense of Definition 1.3. Let $(L_1, \ldots, L_r)$ be a collection of closed monotone Lagrangian submanifolds in M which are oriented and Spin. Assume $(L_1, \ldots, L_r)$ are pairwise disjoinable by Hamiltonian isotopies and $m_0(L_i) \ne 0$ for all i. Suppose that $HF^*(L_i, L_i) \ne 0$ and $\mathrm{pt} \in C_*(L_i)$ defines a cocycle in each $HF^*(L_i, L_i)$; then*

$$\text{(8)} \qquad r \le \dim_{\mathbb{K}} SH^0(M).$$

In the above, the value $m_0(L_i) \in \mathbb{K}$ is determined by the Maslov index 2 holomorphic discs bounded by L_i; see Section 2.2.

Following [8], we say that an unobstructed Lagrangian submanifold $L \subset M$ is *wide* if $HF^*(L, L) \cong H^*(L; \mathbb{K})$, and is *narrow* if $HF^*(L, L) = 0$. Every known monotone Lagrangian submanifold is either wide or narrow. It is proved in [8] that any two closed non-narrow monotone Lagrangian submanifolds in $\mathbb{C}\mathbb{P}^n$ intersect. We have the following partial analogue in the noncompact case:

Corollary 1.5 *Let M be the total space of $\mathcal{O}(-m)^{\oplus n/(m+1)} \to \mathbb{C}\mathbb{P}^{mn/(m+1)}$ with $1 \le m \le n-1$; then any two closed wide monotone Lagrangian submanifolds $L_1, L_2 \subset M$ with $m_0(L_i) \ne 0$ have nontrivial intersection.*

Proof When $m = 1$, it can be checked that M^{in} is a Lefschetz domain; see Lemma 2.16, Proposition 3.10 and Corollary 3.11. By Theorem 1.4, it suffices to show that $\dim_{\mathbb{K}} SH^0(M) = 1$, but this follows from the computations in Section 3.4. The remaining cases when $2 \le m \le n-1$ are completely parallel to the $m = 1$ case; see Section 3.4 for details. $\square$

Let us move onto the construction of Lefschetz domains. Our method here is based on a generalization of the boundary connected sum operation. Such an operation was originally introduced by Weinstein [50] for convex symplectic manifolds and has been generalized to weak symplectic fillings in Geiges and Zehmisch [18]; see also Ghiggini, Niederkrüger and Wendl [19] for a further extension, which allows one to add a subcritical handle to any weak symplectic filling. We actually need a slight modification of their construction so that it can be adapted to the category of stable fillings; see Section 3.1 for details. It then follows that if both of M^{in} and $(M')^{\mathrm{in}}$ are stable fillings, our construction will yield another stable filling $M^{\mathrm{in}} \#_{\partial} (M')^{\mathrm{in}}$. Meanwhile, it follows from our construction that if M and M' are semipositive, then so is $M \#_{\partial} M'$, the completion of $M^{\mathrm{in}} \#_{\partial} (M')^{\mathrm{in}}$. In particular, $SH^*(M \#_{\partial} M')$ is therefore well defined.

Theorem 1.6 *Let M^{in} and $(M')^{\mathrm{in}}$ be stable symplectic fillings of the contact manifolds (V, ξ) and (V', ξ'), respectively, whose completions M and M' are semipositive. Then*

$$(9) \qquad SH^*(M \#_{\partial} M') \cong SH^*(M) \oplus SH^*(M')$$

as rings.

Theorem 1.6 can be adapted to verify some special cases of Conjecture 1.2. The following are two examples; their proofs will appear in Section 3.6.

Corollary 1.7 *Let M_- be a semipositive symplectic manifold, which is the completion of a stable filling, such that $SH^*(M_-) = 0$ or $SH^*(M_-)$ is semisimple. Suppose $M_+ = \mathrm{Bl}_x(M_-)$ is the blow-up of M_- at a point x on the cylindrical end $M \setminus M^{\mathrm{in}}$; then $SH^*(M_+)$ is semisimple.*

Denote by E^{in} the stable filling of the total space of the unit sphere bundle ∂E^{in} of $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$. Since ∂E^{in} can be realized as a circle bundle over $\mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$, there is a standard contact structure ξ_{BW} constructed in Boothby and Wang [9]. When $1 \leq n_1 \leq n_2$, we have $c_1(E) > 0$; to indicate this we shall denote it by E_+. Under the local model of the standard flip

$$(10) \qquad \mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2} \dashrightarrow \mathcal{O}(-1)^{\oplus n_2+1} \to \mathbb{CP}^{n_1-1},$$

E_+ is mapped to the negative vector bundle E_-, which is not semipositive. Suppose that there is some regularization scheme under which $SH^*(E_-)$ is well defined; then one can argue similarly as in Section 3.4 to deduce that $SH^*(E_-) = 0$.

On the other hand, it follows from the computations in Section 3.4 that $SH^*(E_+)$ is semisimple. As a consequence we have the following:

Corollary 1.8 *Let U^{in} be a stable filling and M_- be the symplectic manifold obtained by attaching a cylindrical end to the boundary connected sum $U^{\text{in}} \#_\partial E_-^{\text{in}}$, with $SH^*(U) = 0$ or semisimple. Under the reverse simple flip $M_- \dashrightarrow M_+$ which replaces $\mathbb{CP}^{n_1-1} \subset E_-^{\text{in}}$ by $\mathbb{CP}^{n_2} \subset E_+^{\text{in}}$, $SH^*(M_+)$ is semisimple.*

Another implication of Theorem 1.6 is that it can be used to produce more examples of Lefschetz domains which satisfy the constraints on Lagrangian embeddings established in Theorem 1.4. See Section 4.1. This enables us to apply Theorem 1.4 to some nontoric monotone symplectic manifolds; see Section 4.3. In particular, for the blow-ups of points on $\mathbb{C}^2$ studied in Smith [47] and Leung and Li [26], the bound obtained in Theorem 1.4 is sharp.

If M^{in} is a monotone Lefschetz domain and the equality in the bound (8) can be achieved by a certain collection of monotone Lagrangian submanifolds $(L_1, \ldots, L_r)$ or, more generally, a collection of monotone Lagrangian branes $((L_1, \xi_{L_1}), \ldots, (L_r, \xi_{L_r}))$, where

$$(11) \qquad\qquad \xi_{L_i} \colon \pi_1(L_i) \to U_{\mathbb{K}}$$

is a group homomorphism, with $U_{\mathbb{K}} \subset \mathbb{K}$ the group of units, then a byproduct of Theorem 1.4 is the following:

Theorem 1.9 *M^{in} is a monotone Lefschetz domain. Let $((L_1, \xi_{L_1}), \ldots, (L_r, \xi_{L_r}))$ be a collection of monotone Lagrangian branes with $m_0(L_i, \xi_{L_i}) \neq 0$ and $r = \dim_{\mathbb{K}} SH^0(M)$ such that $m_0(L_i, \xi_{L_i}) \neq m_0(L_j, \xi_{L_j})$ whenever $L_i \cap L_j \neq \varnothing$. Suppose*

$$(12) \qquad\qquad HF^*((L_i, \xi_{L_i}), (L_i, \xi_{L_i})) \neq 0, \quad 1 \leq i \leq r,$$

and every $\text{pt} \in C_*(L_i)$ defines a Floer cocycle. Then the nonzero eigensummands of the derived Fukaya category $\bigsqcup_{\lambda \neq 0} D^\pi \mathcal{F}_\lambda(M)$ and the derived wrapped Fukaya category $\bigsqcup_{\lambda \neq 0} D^\pi \mathcal{W}_\lambda(M)$ are split-generated by $(L_1, \xi_{L_1}), \ldots, (L_r, \xi_{L_r})$.

We remark that $L_i = L_j$ for $i \neq j$ is allowed in the above. Recall that the monotone wrapped Fukaya category $\mathcal{W}(M)$ for M^{in} a strong filling is defined in Abouzaid and Seidel [3] and Ritter and Smith [38]. The definition can be easily extended to the case of stable fillings by allowing noncompact monotone Lagrangian submanifolds $L \subset M$ which are *stable Lagrangian fillings* of their Legendrian boundaries, ie those

of the form $\partial L^{\text{in}} \times [1, \infty)$ on the cylindrical end, where $\partial L^{\text{in}} \subset V$ is a Legendrian submanifold and $\theta_V|_{L^{\text{in}}}$ vanishes near ∂L^{in}. See Section 2.2.

In the special case when M is the total space of $\mathcal{O}(-m) \to \mathbb{CP}^{n-1}$, where $1 \le m \le n-1$, the above theorem has been proved by Ritter [37], since in this case the eigensummands $QH^*(M)_\lambda$ and $SH^*(M)_\lambda$ for $\lambda \ne 0$ are 1–dimensional. However, the symplectic manifold obtained by blowing up $\mathbb{C}^2$ at more than one point (with equal amounts) mentioned above already gives an example where $QH^*(M)_{\lambda \ne 0}$ and $SH^*(M)_{\lambda \ne 0}$ are semisimple but not 1–dimensional.

To apply Theorem 1.9 to concrete examples, it remains to find a collection of Lagrangian branes satisfying the conditions above. For toric negative line bundles, this can be done using standard toric techniques; see [38; 37]. With our tools, it is easy to generalize their results to toric negative vector bundles which split as a direct sum of line bundles; see Section 4.3. We also find the generators for the Fukaya categories of $\text{Bl}_S(\mathbb{C}^n)$, where S is a finite set of distinct points, using essentially elementary methods. This gives a nontoric example for which the nonzero eigensummands of the wrapped Fukaya category $\bigsqcup_{\lambda \ne 0} \mathcal{W}_\lambda(M)$ are cohomologically finite.

Contents

The structure of this paper is as follows. In Section 2 we collect some basic algebraic preliminaries which are needed in the proof of Theorem 1.4. Both of the closed and the open string invariants can be generalized to the setting of stable symplectic fillings, at least when their completions are monotone, as do the open–closed string maps relating these two flavors of Floer theory. These generalizations are mainly based on the relevant maximum principles, which can be extended to the current setup with the help of a convenient class of tame almost complex structures. Theorem 1.4 will be proved in Section 2.4. The proof is a combination of an argument outlined in [43] and a modification of the proof of the nonvanishing of OC^0 appeared in [38].

Section 3 is devoted to another important issue of this paper, namely the semisimplicity of symplectic cohomologies. We translate the insight provided by Woodard's Theorem 1.1 to construct manifolds with semisimple symplectic cohomologies using reverse MMP transitions. This step is far from complete and has only been carried out for very restrictive cases, eg Corollaries 1.7 and 1.8. Our approach here is to replace certain birational surgeries with a surgery which connects the original manifold to the exceptional pieces (created by blow-ups or reverse flips) by symplectic 1–handles. Such a symplectic handle attachment is by no means new, and has been studied in great

detail by Cieliebak [12] and McLean [31]. Our contribution here is solely to observe that, based on the works of Cieliebak and Volkov [14] and Massot, Niederkrüger and Wendl [28], such a surgery can be carried out within the category of stable fillings (Section 3.1). Theorem 1.6 can then be proved by mimicking the arguments presented in [14; 31]; see Section 3.2. To get back to the original surgeries via birational maps, one applies a deformation argument which depends on the invariance of $SH^*(M)$ under certain symplectomorphisms (Section 3.6). The computation of the symplectic cohomologies of the exceptional pieces is possible by generalizing Ritter's work on Seidel representations [36; 37]; see Sections 3.3 and 3.4.

The last section contains some interesting examples and implications of the main results. In particular, there are monotone symplectic manifolds with proper wrapped Fukaya categories which are neither convex nor toric.

Acknowledgements

2   Basic structures

This section collects some standard materials on the Hamiltonian and Lagrangian Floer theories, together with the open–closed string maps which relate these two flavors of Floer theory. Standard references concerning these topics are [1; 3; 38; 43; 42; 41; 46]; see also [8; 7; 16] for some precursors. Once the relevant maximum principles are established, the constructions of the A_∞ structures and open–closed string maps in our case are essentially the same with the fundamental work of Ritter and Smith [38], whether the symplectic filling M^{in} is strong plays no role. For this reason, our account here will be quite brief; detailed constructions can be found in [38].

2.1   Hamiltonian Floer theory

In order to clarify our geometric setup, it should be suitable here to recall some standard notions about symplectic fillings. Throughout this paper, (V, ξ) will be used to denote a

$(2n-1)$–dimensional cooriented closed contact manifold. Let $(M^{\text{in}}, \omega_M)$ be a compact symplectic manifold whose boundary $\partial M^{\text{in}} \cong V$ as oriented manifolds. Denote by ω_ξ the restriction of ω_M on ξ. The following notion is first introduced in [28] in order to study the flexibility of tight contact structures on manifolds with dimension larger than 3.

Definition 2.1 (Massot, Niederkrüger and Wendl [28]) We say that $(M^{\text{in}}, \omega_M)$ is a *weak symplectic filling* of (V, ξ) if, for every choice of the contact form θ_V defining ξ,

$$(13) \qquad \theta_V \wedge (d\theta_V + \omega_\xi)^{n-1} > 0, \quad \theta_V \wedge \omega_\xi^{n-1} > 0.$$

For the purpose of having well-defined Floer theories on M^{in}, we need to impose further restrictions on the symplectic filling M. Denote by ω_V the restriction of the symplectic form ω_M on V; we say that (ω_V, θ_V) form a *stable Hamiltonian structure* on V if

$$(14) \qquad \theta_V \wedge \omega_V^{n-1} > 0, \quad \ker \omega_V \subset \ker d\theta_V.$$

Note that in our case, since $\ker d\theta_V$ is 1–dimensional, we actually have $\ker \omega_V = \ker d\theta_V$. Because of this, we will not distinguish between the Reeb vector field associated to the stable Hamiltonian structure (namely the one generating $\ker \omega_V$ and normalized to 1 by θ_V) and the usual Reeb vector field on the contact manifold (V, θ_V), and will use R to denote both of them.

Definition 2.2 (Latschev and Wendel [23]) A weak filling $(M^{\text{in}}, \omega_M)$ of (V, ξ) is said to be *stable* if there exists a contact form θ_V on (V, ξ) such that (ω_V, θ_V) is a stable Hamiltonian structure on V.

Let $V \subset M$ be an oriented hypersurface in a $2n$–dimensional symplectic manifold (M, ω_M), and $\xi \subset TV$ be the cooriented hyperplane distribution induced by a nowhere-vanishing 1–form θ_V on V such that ω_ξ is symplectic and induces positive orientation. By Lemma 2.6 of [28], a neighborhood of V in M is symplectomorphic to

$$(15) \qquad \left((1-\varepsilon, 1+\varepsilon) \times V, \omega_V + d((r-1)\theta_V)\right)$$

for some $\varepsilon > 0$, where $V \subset M$ is identified naturally as $\{1\} \times V$, and the direction of $\frac{\partial}{\partial r}$ is such that $\iota_{\partial/\partial r} \omega_M^n = \theta_V \wedge \omega_M^{n-1}$.

In particular, when $(M^{\text{in}}, \omega_M)$ is a weak filling of (V, ξ), there exists a tubular neighborhood of $V \subset M^{\text{in}}$ symplectomorphic to

$$(16) \qquad \left((1-\varepsilon, 1] \times V, \omega_V + d((r-1)\theta_V)\right),$$

where $\varepsilon > 0$ is taken to be sufficiently small. This is usually called a *collar neighborhood* of V. Note that when M^{in} is a stable filling of (V, ξ), every hypersurface $\{r\} \times V$ where $1 - \varepsilon < r \leq 1$ in the collar neighborhood is stably filled.

Using the vector field $\frac{\partial}{\partial r}$, we can extend the weak symplectic filling M^{in} to a noncompact symplectic manifold M by attaching to M^{in} a *cylindrical end*, namely

$$(17) \qquad\qquad M = M^{\mathrm{in}} \cup_{\partial M^{\mathrm{in}}} [1, \infty) \times V,$$

where the symplectic form on the cylindrical end is given by

$$(18) \qquad\qquad \omega_M|_{M \setminus M^{\mathrm{in}}} = \omega_V + d((r-1)\theta_V), \quad r \in [1, \infty).$$

We call M the *completion* of M^{in}.

From now on, M^{in} will be a stable symplectic filling of (V, ξ), whose completion M satisfies the semipositivity condition (7). Pick a Hamiltonian function $H \colon M \to \mathbb{R}$ which outside a compact subset of M has the form $h(r)$, where $h(r)$ is linear in r with positive slope $h'(r) > 0$ not equal to the Reeb period. Hamiltonians which possess this shape will be called *admissible*. Using the stability condition (14), it is easy to see that the corresponding Hamiltonian vector field is given by $X_H = h'(r)R$ for $r \gg 0$, where R is the Reeb vector field for (V, θ_V). Just as in the strong filling case, we can choose the contact form θ_V generically subject to the restriction that (ω_V, θ_V) is still a stable Hamiltonian structure on V, while the Reeb periods of R form a discrete period spectrum $\mathcal{P}_M \subset \mathbb{R}$. We shall assume from now on that such a choice of θ_V has been fixed.

Throughout this paper, we shall define the Floer complex $CF^*(H)$ using only contractible periodic orbits of X_H. Since we are mainly interested in the cases when M is simply connected, this convention should be appropriate here and it also simplifies some of our arguments below.

Let $\mathcal{L}_0 M \subset \mathcal{L}M$ be the connected component of the free loop space of M consisting of contractible loops. We can define a covering $\widetilde{\mathcal{L}}_0 M \to \mathcal{L}_0 M$ by considering the pairings (u, x) where $u \colon \mathbb{D} \to M$ is a disc with boundary $x \in \mathcal{L}_0 M$. Then $\widetilde{\mathcal{L}}_0 M$ is defined by modding out the equivalence relation which identifies two pairs (u, x) and (u', x') if $u \# \overline{u'} \in \pi_2(M)_0$, namely both ω_M and $c_1(M)$ vanish on the sphere $u \# \overline{u'}$, where by $\overline{u'}$ we mean the disc u' with its orientation reversed. Recall that with this covering, an action functional

$$(19) \qquad\qquad A_H \colon \widetilde{\mathcal{L}}_0 M \to \mathbb{R}$$

can be defined and its critical points correspond to the generators of the Floer complex $CF^*(H)$. On the other hand, there is also a well-defined Conley–Zehnder index $\mathrm{ind}_{CZ}(u, x) \in \mathbb{Z}$ for every element $(u, x) \in \widetilde{\mathcal{L}}_0 M$, which can be used to equip the vector space $CF^*(H)$ over $\mathbb{K}$ with a $\mathbb{Z}$–grading.

To define the Floer differential on $CF^*(H)$, we need to write down the Floer equation. To this end, we need to specify a set of almost complex structures to work with.

Definition 2.3 An almost complex structure J on M tamed by ω_M in the interior of the domain M^{in} is said to be admissible if

   (i)   it is of contact type on the cylindrical end, ie $dr \circ J = -\theta_V$;

   (ii) J preserves the contact structure ξ and $J|_\xi$ is tamed by ω_ξ and $d\theta_V$.

We need the following important result proved in [28]:

Theorem 2.4 [28, Theorem D and Proposition 2.1] *M^{in} is a weak symplectic filling if and only if it admits an admissible almost complex structure J in the sense of Definition 2.3. Moreover, once there exists an admissible almost complex structure on M, then the space of such almost complex structures, which we denote by $\mathcal{J}(M)$, is contractible.*

Lemma 2.5 *Let $J \in \mathcal{J}(M)$; then it is also tamed by ω_M on the cylindrical end.*

Proof To show this, one can make use of the decomposition $TM = \xi \oplus \langle R \rangle \oplus \langle \partial_r \rangle$ of the tangent bundle on the cylindrical end of M, and consider separately the cases when the vector field X lies in ξ, $\langle R \rangle$ or $\langle \partial_r \rangle$.

In the case when X lies in ξ, the fact that $\omega_M(X, JX) > 0$ is a direct consequence of condition (ii) in Definition 2.3.

When X coincides with a nonzero multiple of the Reeb vector field R, $\omega_M(X, JX) > 0$ follows from the contact type condition (i) in Definition 2.3.

In the remaining case when X is a nonzero multiple of $\frac{\partial}{\partial r}$, use again condition (i) of Definition 2.3, and the simple fact that $\omega_M(\partial_r, R) = 1$.   □

Let (S, j) be a Riemann surface which is topologically a punctured sphere equipped with a subclosed 1–form γ. Fix a set of cylindrical ends $\varepsilon_k \colon \mathbb{R}_\pm \times S^1 \to S$ for the punctures, with coordinates (s, t) such that $j\partial_s = \partial_t$. With such a parametrization,

$\varepsilon_k^* \gamma = w_k \, dt$ on every cylindrical end of S, where $w_k > 0$ is the weight associated to ε_k. Suppose $u \colon S \to M$ is a smooth map which solves the equation $(du - X_H \otimes \gamma)^{0,1} = 0$ (where the $(0, 1)$–part is taken with respect to a $J \in \mathcal{J}(M)$) and converges to the 1–periodic orbits $x_1, \ldots, x_d$ and y of X_H when $s \to \pm\infty$. For the purpose of defining the Floer differential on $CF^*(H)$ and the product structure on $SH^*(M)$, we need to show that for (H, J) admissible, the image of any such solution u lies in a compact subset of M determined by $r(x_1), \ldots, r(x_d)$ and $r(y)$, the radial coordinates of the periodic orbits. The proof of the following lemma is a slight modification of that of Lemma D.1 of [35]:

Lemma 2.6 *If the function $\rho = r \circ u$ has a local maximum, then it is constant.*

Proof Choose local holomorphic coordinates (s, t) on S so that it is compatible with our parametrizations on the cylindrical ends fixed above; then $\gamma = \gamma_s \, ds + \gamma_t \, dt$. Using this we can rewrite the Floer equation locally as

$$(20) \qquad \partial_s u + J \partial_t u = X_H \gamma_s + J X_H \gamma_t.$$

On the cylindrical end, we have the decomposition $TM = \xi \oplus \langle R \rangle \oplus \langle \partial_r \rangle$. The trick is to eliminate the effect of ω_V so that the proof goes along the same lines as in the case of strong fillings [35]. To do this, write correspondingly the Floer solution u as (v, ρ), where v takes its value in V. Projecting to the $\langle R \rangle$ and $\langle \partial_r \rangle$ directions, one gets

$$(21) \qquad \begin{cases} \theta_V(\partial_s v) + \partial_t \rho - \gamma_s h'(\rho) = 0, \\ \partial_s \rho - \theta_V(\partial_t v) + h'(\rho)\gamma_t = 0. \end{cases}$$

From this and the fact that $d\theta_V(\partial_s v, \partial_t v) \geq 0$ (since $d\theta_V$ tames $J|_\xi$), we deduce

$$(22) \qquad \Delta\rho + \frac{h''(\rho) \, d\rho \wedge \gamma}{ds \wedge dt} \geq -\frac{h'(\rho) \, d\gamma}{ds \wedge dt}.$$

Observe that the right-hand side is nonnegative, so the statement follows from the maximum principle for elliptic operators. □

Recall that in general we need to perform a t–dependent perturbation of H on the cylindrical end to ensure that all the Hamiltonian orbits are nondegenerate, which creates an additional term $-(\partial_t h_t') \, dt \wedge \gamma$ in the above computations. However, such a term actually vanishes by our requirement that $\gamma = w_k \, dt$ on the cylindrical ends, so the argument above still holds.

With Lemma 2.6, one can build a moduli space $\mathcal{M}(x; y)$ which consists of solutions of the Floer equation asymptotic to the Hamiltonian orbits x and y, modding out the

reparametrizations $\mathbb{R}$. Then $\mathcal{M}(x; y)$ is a smooth manifold with expected dimension by choosing a regular $J \in \mathcal{J}_{\mathrm{reg}}(M) \subset \mathcal{J}(M)$, whose existence is governed by the Sard–Smale theorem. One can further separate the moduli space $\mathcal{M}(x; y)$ according to the lifts of x and y in $\widetilde{\mathcal{L}}_0 M$, which gives us another moduli space $\mathcal{M}(\widetilde{x}; \widetilde{y})$, where $\widetilde{x}, \widetilde{y} \in \widetilde{\mathcal{L}}_0 M$ are connected by a lift of u. Let

$$(23) \qquad E(u) = \frac{1}{2} \int_S \|du - X_H \otimes \gamma\|^2 \, d\mathrm{vol}_S$$

be the energy of $u \colon S \to M$. Since we have assumed that M is semipositive, the maximum principle above and the a priori energy estimate

$$(24) \qquad E(u) = A_H(\widetilde{x}) - A_H(\widetilde{y}), \quad u \in \mathcal{M}(x; y),$$

ensure that $\mathcal{M}(\widetilde{x}; \widetilde{y})$ can be compactified by adding broken trajectories, by Gromov compactness for tame almost complex structures. The Floer differential d on $CF^*(H)$ is the defined by counting rigid elements of the compactified moduli space $\overline{\mathcal{M}}(\widetilde{x}; \widetilde{y})$. This defines the Hamiltonian Floer cohomology group $HF^*(H)$. It is a standard continuation argument to show that $HF^*(H)$ is independent of the choice of the admissible almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(M)$.

To define the symplectic cohomology of M (or M^{in}), one must establish the required continuation maps. To do this, we need to specify the particular class of homotopies of the Floer data that are allowed. Let (H_s) be a *monotone homotopy* of admissible Hamiltonians, ie $\partial_s h'_s \leq 0$; then a similar argument as in the proof of Lemma 2.6 can be used to prove a maximum principle for the solutions of $(du - X_{H_s} \otimes \gamma)^{0,1} = 0$. This in particular shows that the continuation maps

$$(25) \qquad \kappa \colon HF^*(wH) \to HF^*((w+1)H)$$

are well defined provided that $wh'(r), (w+1)h'(r) \notin \mathcal{P}_M$. It's easy to see these Hamiltonian Floer cohomologies form a directed system indexed by all the possible slopes at infinity. Taking its direct limit, we get the symplectic cohomology

$$(26) \qquad SH^*(M) \cong \varinjlim_w HF^*(wH).$$

Lemma 2.6 together with an a priori energy estimate similar to (24) guarantees that the pair-of-pants product

$$(27) \qquad HF^*(w_1 H) \otimes HF^*(w_\infty H) \to HF^*(w_0 H)$$

can be defined, where the weights are chosen so that $w_1 + w_\infty = w_0$ and

(28) $$w_0 h'(r), w_1 h'(r), w_\infty h'(r) \notin \mathcal{P}_M.$$

Passing to direct limits, we get a product on $SH^*(M)$ which makes it an algebra over $\mathbb{K}$.

When M is monotone, for the purpose of defining open–closed string maps, we remark that using the telescope construction [3, Section 2], the direct limit can be taken on the chain level, which yields a complex computing $SH^*(M)$,

(29) $$SC^*(M) := \bigoplus_{w=1}^{\infty} CF^*(wH)[\boldsymbol{q}],$$

where $\boldsymbol{q}$ is a formal variable and the differential is given by

(30) $$\nu^1(x + \boldsymbol{q}\, y) = (-1)^{\deg(x)}\, dx + (-1)^{\deg(y)}(\boldsymbol{q}\, dy + \kappa y - y).$$

The definitions of the Floer differential $d\colon CF^*(wH) \to CF^*(wH)$, the continuation map $\kappa\colon CF^*(wH) \to CF^*((w+1)H)$ and the pair-of-pants product on the chain level in this setup make use of the moduli spaces of weighted popsicles; see Section 2.2. By Lemma 2.6 and the monotonicity assumption on M, the same construction as in Section 4.6 of [38] carries over with no modification to the stable filling case.

2.2 Fukaya categories

When passing to open string invariants, we further restrict ourselves to the case when M^{in} is a stable symplectic filling whose completion M is monotone, in order to avoid possible technical complexities. The Lagrangian submanifolds $L \subset M$ we shall consider are assumed to be oriented and monotone. Since we are working over a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$, we also require that L is Spin, and actually fix a choice of Spin structure, so that various moduli spaces that appear below will be oriented. When L is noncompact, we also require that on the cylindrical end, L is modeled on a Legendrian cone. More precisely,

(i) L intersects V transversely along ∂L^{in}, where $L^{\mathrm{in}} = L \cap M^{\mathrm{in}}$;

(ii) $\theta_V|_L$ vanishes on $\partial L^{\mathrm{in}} \times (1 - \varepsilon, \infty)$.

Lagrangian submanifolds $L \subset M$ satisfying these constraints will be called *admissible*.

Let $S = \mathbb{D} \setminus \{z_0, \dots, z_d\}$ be a disc with $d + 1$ boundary punctures, and equip it with strip-like ends $\varepsilon_k\colon \mathbb{R}_{\pm} \times [0,1] \to S$ for $i = 1, \dots, d$. Denote by $\partial_i S \subset \partial S$ the ith boundary component between z_i and z_{i+1}. Another important auxiliary datum on S

is the choice of a subclosed 1–form γ such that $\varepsilon_k^* \gamma = dt$, $d\gamma = 0$ near ∂S and $\gamma|_{\partial S} = 0$.

We still work with the class of tame almost complex structures $\mathcal{J}(M)$ specified by Definition 2.3. Consider a solution $u: S \to M$ of

(31)
$$\begin{cases} (du - X_H \otimes \gamma)^{0,1} = 0, \\ u(\partial_i S) \subset L_i, \\ \lim_{s \to \pm\infty} u(\varepsilon_k(s, \cdot)) = x_k, \end{cases}$$

where H is an admissible Hamiltonian as defined in Section 2.1 and x_k is a time-1 chord of the Hamiltonian flow of X_H with ends on two adjacent Lagrangian submanifolds corresponding to the boundary labels of ∂S. The proof of the following lemma is similar to that of Lemma D.2 of [35]:

Lemma 2.7 *The function $\rho = r \circ u$ with $u: S \to M$ a solution of (31) can't have a local maximum unless it is constant.*

Proof One can argue similarly as in the proof of Lemma 2.6, except that in this case ρ may achieve its maximum on ∂S. To exclude this possibility, note that, with our parametrization, ∂_t is in the outward or inward normal direction of ∂S. When ∂_t is outward-pointing, by Hopf's lemma $\partial_t \rho > 0$ at any maxima on ∂S. But, by (31), we have

(32)
$$\partial_t \rho = \gamma_s h'(\rho) - \theta_V(\partial_s v) = \theta_V(\partial_s v),$$

using the fact that $\gamma|_{\partial S} = 0$. On the other hand, since $\partial_s v \in TL$ and θ_V vanishes outside a compact subset of L, $\partial_t \rho = 0$: a contradiction. One can argue similarly when ∂_t is inward-pointing, and get a contradiction as well. $\square$

Using this lemma one can associate various algebraic structures to admissible Lagrangian submanifolds in M provided that the Lagrangian submanifolds under consideration are tautologically unobstructed. Since M is monotone, it remains to consider the Floer theory of admissible Lagrangian submanifolds with minimal Maslov number 2. For this we need the following lemma:

Lemma 2.8 *For $J \in \mathcal{J}(M)$, any J–holomorphic disc $u: \mathbb{D} \to M$ bounded by an admissible Lagrangian submanifold $L \subset M$ satisfies*

(33)
$$u(\partial \mathbb{D}) \subset M^{\mathrm{in}} \cup \{r_0\} \times V$$

for some fixed $r_0 > 1$.

Proof We can argue similarly as in Lemma 2.6. By projecting the Cauchy–Riemann equation to the directions $\langle R \rangle$ and $\langle \partial_r \rangle$, we get in this case $\Delta \rho \geq 0$. On the other hand, by the argument in Lemma 2.7, a local maximum can't appear on $\partial \mathbb{D}$. From these one concludes that if $u(\partial \mathbb{D}) \not\subseteq M^{\mathrm{in}}$, then ρ must be constant on $u(\partial \mathbb{D}) \cap (M \setminus M^{\mathrm{in}})$, which proves the lemma. $\square$

By the above lemma, the construction of the moduli spaces $\mathcal{M}_1(L, \beta)$ of Maslov index 2 J–holomorphic discs with 1 boundary marked point in the class $\beta \in \pi_2(M, L)$ reduces to the closed monotone case. Using this we can define the obstruction (modulo usual transversality and invariance issues)

$$(34) \qquad \mathfrak{m}_0(L) = \sum_{\beta \in \pi_2(M,L)} q^{\omega_M(\beta)} \, \mathrm{ev}_* [\mathcal{M}_1(L, \beta)] = m_0(L)[L]$$

for any admissible Lagrangian $L \subset M$, where $\mathrm{ev} \colon \mathcal{M}_1(L, \beta) \to L$ is the evaluation map at the boundary marked point.

From now on all the Lagrangian submanifolds involved are assumed to be admissible and have the same m_0–value. Denote by $CF^*(L_0, L_1; wH)$ the free $\mathbb{K}$–module generated by time-1 chords of the flow of wX_H, where w is an integer. Note that in order for the Floer complex $CF^*(L_0, L_1; wH)$ to be finitely generated, one needs to choose the contact form θ_V generically (subject to the condition that it defines a stable Hamiltonian structure on V together with the original ω_V) so that there is no Reeb chord of integer period. By counting the solutions of (31) in the case when S is a strip with one input and one output, the Floer differential

$$(35) \qquad \delta \colon CF^*(L_0, L_1; wH) \to CF^{*+1}(L_0, L_1; wH)$$

can be defined. By our assumption that $m_0(L_0) = m_0(L_1)$, $\delta^2 = 0$, and the associated cohomology group will be denoted by $HF^*(L_0, L_1; wH)$. As in the case of closed strings, one can take the direct limit with respect to w by building the continuation maps

$$(36) \qquad \kappa \colon HF^*(L_0, L_1; wH) \to HF^*(L_0, L_1; (w+1)H);$$

these maps can be shown to be well defined by combining the parametrized version of Lemmas 2.6 and 2.7 above. This defines the wrapped Floer cohomology $HW^*(L_0, L_1)$.

However, for the construction of A_∞ structures, we need to work on the chain level and run the telescope construction [3]. Analogous to $SC^*(M)$, the wrapped Floer

complex is defined to be

$$(37) \qquad CW^*(L_0, L_1) = \bigoplus_{w=1}^{\infty} CF^*(L_0, L_1; wH)[\boldsymbol{q}],$$

with the differential

$$(38) \qquad \mu^1(x + \boldsymbol{q}\, y) = (-1)^{\deg(x)} \delta x + (-1)^{\deg(y)} (\boldsymbol{q}\delta y + \kappa y - y).$$

The complex $(CW^*(L_0, L_1), \mu^1)$ so defined computes the wrapped Floer cohomology $HW^*(L_0, L_1)$.

The admissible Lagrangian submanifolds $L \subset M$ form the objects of the wrapped Fukaya category $\mathcal{W}(M)$, and the morphisms between two admissible Lagrangian submanifolds L_0 and L_1 are defined to be the wrapped Floer complex $CW^*(L_0, L_1)$. Note that this is only well defined when the Lagrangian submanifolds involved have the same m_0. Because of this, $\mathcal{W}(M)$ is understood as a disjoint union of the full subcategories $\mathcal{W}_\lambda(M)$ consisting of admissible Lagrangian submanifolds with $m_0(L) = \lambda \in \mathbb{K}$.

The construction of the A_∞ structures on $\mathcal{W}_\lambda(M)$ is rather involved compared to the exact case [1], due to the fact that there is no obvious way to bypass the telescope construction. Instead of working with Riemann surfaces $S = \mathbb{D} \setminus \{z_0, \ldots, z_d\}$ equipped with strip-like ends and subclosed 1–forms, we need to endow S with an additional structure. To describe this, fix a finite collection of labels $p_f \in \{1, \ldots, d\}$ indexed by the set F. This determines a map

$$(39) \qquad \boldsymbol{p} \colon F \to \{1, \ldots, d\}.$$

Associated to $\boldsymbol{p}$ there is a collection of holomorphic maps $\boldsymbol{\phi} = (\phi_f)_{f \in F}$ such that each $\phi_f \colon S \to \mathbb{R} \times [0, 1]$ extends to an isomorphism on the compactification $\bar{S} \to \mathbb{D}$ such that

$$(40) \qquad \phi_f(z_0) = -\infty, \quad \phi_f(z_{p_f}) = +\infty.$$

The quadruple $(S, \boldsymbol{\varepsilon}, \gamma, \boldsymbol{\phi})$ is called a *weighted popsicle*, where $\boldsymbol{\varepsilon}$ is a set of strip-like ends and γ is a carefully chosen subclosed 1–form whose definition involves the specifications of the weights $\boldsymbol{w} = \{w_0, \ldots, w_d\}$, where $w_0 = \sum_{k=1}^{d} w_k + |F|$.

Fix a set $\boldsymbol{x} = \{x_0, \ldots, x_d\}$ of Hamiltonian chords of X_H with weights $\boldsymbol{w}$; denote by $\mathcal{M}^{d+1, \boldsymbol{p}, \boldsymbol{w}}(\boldsymbol{x})$ the moduli space of solutions $u \colon S \to M$ of (31). In order to define the A_∞ structure on $\mathcal{W}_\lambda(M)$, one needs to choose a family $\mathcal{I}_{S, \boldsymbol{\phi}, \boldsymbol{w}}(M) \subset \mathcal{J}(M)$ of domain-dependent admissible almost complex structures which are compatible with

the strip-like ends; see [3]. Namely $\mathcal{I}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M)$ varies smoothly over the moduli space of stable weighted popsicles $\mathcal{M}^{d+1,\boldsymbol{p},\boldsymbol{w}}$, and compatible with the gluing. However, to achieve transversality of the moduli spaces $\mathcal{M}^{d+1,\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x})$, a set of infinitesimal deformations

$$(41) \qquad \mathcal{K}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M) \subset T\mathcal{J}(M)$$

with superexponential decay along the strip-like ends $\boldsymbol{\varepsilon}$ of S must be introduced, where $T\mathcal{J}(M)$ denotes the tangent bundle over the infinite-dimensional manifold $\mathcal{J}(M)$. Note that although we are using tame almost complex structures instead of compatible ones, it is easy to see the fact that $\omega_M(\,\cdot\,,K\cdot)$ is no longer symmetric with $K \in \mathcal{K}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M)$ does not affect the arguments in [3]. Exponentiating the elements in $\mathcal{K}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M)$ to get actual deformations of $\mathcal{I}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M)$, we get a family of admissible almost complex structures $\mathcal{J}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M)$ such that

$$(42) \qquad \mathcal{I}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M) \subset \mathcal{J}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M) \subset \mathcal{J}(M).$$

The upshot is: with a generic choice of almost complex structures in $\mathcal{J}_{S,\boldsymbol{\phi},\boldsymbol{w}}(M)$, the moduli space $\mathcal{M}^{d+1,\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x})$ is a smooth manifold with expected dimension.

Since we are working over a Novikov field $\mathbb{K}$, the corresponding a priori energy estimate is tautological, and Gromov compactness holds as in the usual case. Counting isolated solutions in $\mathcal{M}^{d+1,\boldsymbol{p},\boldsymbol{w}}(\boldsymbol{x})$ defines a map

$$(43) \quad \mu^{d,\boldsymbol{p},\boldsymbol{w}}\colon CF^*(L_{d-1},L_d;w_d H)[\boldsymbol{q}] \otimes \cdots \otimes CF^*(L_0,L_1;w_1 H)[\boldsymbol{q}]$$
$$\to CF^*(L_0,L_d;w_0 H)[\boldsymbol{q}]$$

for $d \geq 2$. For details see [38].

Taking the weighted sum over all the possible $\boldsymbol{p}$ and $\boldsymbol{w}$, we obtain the A_∞ structure maps

$$(44) \qquad \mu^d_{\mathcal{W}}\colon CW^*(L_{d-1},L_d) \otimes \cdots \otimes CW^*(L_0,L_1) \to CW^*(L_0,L_d)[2-d]$$

of $\mathcal{W}_\lambda(M)$.

On the object level, the Fukaya category of compact Lagrangians $\mathcal{F}(M)$ consists of all the closed admissible Lagrangian submanifolds in M. As before, $\mathcal{F}(M)$ is a disjoint union of the full subcategories $\mathcal{F}_\lambda(M)$ whose objects are closed Lagrangian submanifolds $L \subset M$ with $m_0(L) = \lambda$. The morphism between two objects L_0 and L_1 of $\mathcal{F}_\lambda(M)$ is given by the usual Floer complex $CF^*(L_0,L_1)$ generated by chords of X_H with H a compactly supported Hamiltonian.

Note that the A_∞ operations $\mu_{\mathcal{F}}^d$ on $\mathcal{F}_\lambda(M)$ are well defined by Lemma 2.7, without referring to popsicles and weights. However, it is an easy consequence of Lemma 2.8 that

$$(45) \qquad\qquad HW^*(L_0, L_1) \cong HF^*(L_0, L_1)$$

for L_0 and L_1 any two objects of $\mathcal{F}_\lambda(M)$. In fact, one can even construct an A_∞ functor

$$(46) \qquad\qquad \mathcal{A} \colon \mathcal{F}(M) \to \mathcal{W}(M)$$

which is cohomologically full and faithful, by allowing $w = 0$ in the definition of the wrapped Floer complex (37). When M^{in} is a strong filling, this is the *acceleration functor* defined in [38].

2.3 Open–closed maps

As in the last subsection, our standing assumption is that M^{in} is a stable symplectic filling whose completion M is monotone. One way to relate the Hamiltonian and Lagrangian flavors of Floer theory is to use the open–closed or closed–open string maps. Let S be a Riemann surface with both boundary and interior punctures (resp. interior marked points), the definitions of these maps involve the study of the moduli spaces of J–holomorphic maps $u \colon S \to M$ with Lagrangian boundary conditions and asymptotic to Hamiltonian chords and orbits (resp. hitting locally finite cycles). In particular, a combination of Lemmas 2.6 and 2.7 ensures that the required maximum principle holds for defining these maps.

We first consider the case of compact Lagrangian submanifolds. Fix a disc S with $d + 1$ boundary punctures and an interior marked point $*$, which is an output and can be fixed to be the origin. Denote by $CC_*(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M))$ the Hochschild chain complex of $\mathcal{F}_\lambda(M)$. On the chain level, the degree d open–closed map

$$(47) \qquad\qquad \mathrm{OC}^d \colon CC_d(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M)) \to QC^{d+n}(M)$$

is defined by counting the solutions $u \colon S \to M$ of (31) which pass through some fixed choice of locally finite cycle $c \in QC_*^{\mathrm{BM}}(M)$ at the interior marked point $*$, where $QC_*^{\mathrm{BM}}(M)$ denotes the Borel–Moore model of the quantum chain complex of the noncompact manifold M. We remark that the transversality of the moduli spaces involved in this definition relies on the fact that M is monotone; see Section 5.4 of [38]

for details. Summing over d gives us a chain map, which induces on the cohomology level the open–closed string map

(48) $$\text{OC: } HH_*(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M)) \to QH^{*+n}(M).$$

For the closed–open map, consider again $S = \mathbb{D} \setminus \{z_0, \ldots, z_d\}$, but now the interior marked point $* \in S$ is an input and the puncture between the boundary components $\partial_0 S$ and $\partial_n S$ is an output. By counting the rigid solutions $u: S \to M$ which satisfy (31) and an additional intersection condition at $*$, we get a chain map

(49) $$\text{CO}^d: QC^d(M) \to CC^d(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M)),$$

where the right-hand side is the degree d Hochschild cochain complex. Summing over d and passing to cohomologies yields the closed–open string map

(50) $$\text{CO: } QH^*(M) \to HH^*(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M)).$$

The general case of (possibly noncompact) admissible Lagrangians is more complicated. Since we need to work on the chain level, the construction involves popsicles with additional interior punctures. As in the case of A_∞ operations, counting the solutions of (31) which are asymptotic to the Hamiltonian 1–orbits of $w_0 X_H$ at $* \in S$, with specified weights $\boldsymbol{w} = \{w_0, \ldots, w_{d+1}\}$ and the subclosed 1–form γ, defines a map

(51) $$\text{OC}^{d, \boldsymbol{p}, \boldsymbol{w}}: CF^*(L_d, L_0; w_{d+1}H)[\boldsymbol{q}] \otimes \cdots \otimes CF^*(L_0, L_1; w_1 H)[\boldsymbol{q}]$$
$$\to CF^*(w_0 H)[\boldsymbol{q}],$$

where the Lagrangian submanifolds involved are assumed to satisfy $m_0(L_i) = \lambda$ for some fixed $\lambda \in \mathbb{K}$. Refer to [38] for details of this construction. Summing up the $\text{OC}^{d, \boldsymbol{p}, \boldsymbol{w}}$'s as $\boldsymbol{p}$ and $\boldsymbol{w}$ vary yields the map

(52) $$\text{OC}^d: CC_d(\mathcal{W}_\lambda(M), \mathcal{W}_\lambda(M)) \to SC^{d+n}(M).$$

This is a chain map, so we get from this the open–closed string map

(53) $$\text{OC: } HH_*(\mathcal{W}_\lambda(M), \mathcal{W}_\lambda(M)) \to SH^{*+n}(M).$$

The construction of the closed–open map is similar, except that $*$ will be an input, while the puncture that separates $\partial_0 S$ and $\partial_n S$ is considered to be an output.

Denote by $\text{Spec}(\star c_1(M)) \subset \mathbb{K}$ the set of eigenvalues of the quantum multiplication by $c_1(M)$. Recall that we have a decomposition

(54) $$QH^*(M) = \bigoplus_{\lambda \in \text{Spec}(\star c_1(M))} QH^*(M)_\lambda.$$

Via the ring homomorphism (PSS map)

$$(55) \qquad\qquad c^*\colon QH^*(M) \to SH^*(M)$$

obtained by composing a sequence of continuation maps, $SH^*(M)$ can be realized as a $QH^*(M)$–module. This defines subalgebras $SH^*(M)_\lambda \subset SH^*(M)$, namely the λ– generalized eigensummand of the multiplication by $c^*(c_1(M))$. Note that by definition, for monotone Lefschetz domains (Definition 1.3), we actually have

$$(56) \qquad SH^*(M) = \bigoplus_{\lambda \in \mathrm{Spec}(\star c_1(M))} SH^*(M)_\lambda, \quad SH^*(M)_0 = 0$$

and the homomorphism c^* above is simply the localization of $QH^*(M)$ at $c_1(M)$.

The following result relates the generalized eigenvalues λ and the m_0–values of the Lagrangians via the open–closed maps. Since its proof has nothing to do with the fact that the symplectic filling M^{in} may not be strong, the argument of [38] extends to our case without any modification.

Proposition 2.9 (Ritter and Smith [38]) *The images of the open–closed string maps*

$$(57) \qquad \begin{aligned} \mathrm{OC}\colon HH_*(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M)) &\to QH^{*+n}(M), \\ \mathrm{OC}\colon HH_*(\mathcal{W}_\lambda(M), \mathcal{W}_\lambda(M)) &\to SH^{*+n}(M) \end{aligned}$$

lie in the generalized eigensummands $QH^*(M)_\lambda$ and $SH^*(M)_\lambda$, respectively. A similar statement holds for the closed–open string maps.

Using this fact and the construction of the acceleration functor $\mathcal{A}$, we have the commutative diagram

$$(58) \qquad \begin{array}{ccc} HH_*(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M)) & \xrightarrow{\ HH_*(\mathcal{A})\ } & HH_*(\mathcal{W}_\lambda(M), \mathcal{W}_\lambda(M)) \\ {\scriptstyle \mathrm{OC}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{OC}} \\ QH^{*+n}(M)_\lambda & \xrightarrow{\ \ c^*\ \ } & SH^{*+n}(M)_\lambda \end{array}$$

which appears in [38] as the *acceleration diagram* when M^{in} is a strong filling.

We note that (58) is a commutative diagram of $QH^*(M)$–modules, with the $QH^*(M)$– module structures for the top line given as follows. Given a locally finite cycle $c \in QC_*^{\mathrm{BM}}(M)$, we can define an endomorphism ϕ_c of the diagonal $(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M))$– bimodule $(\mathcal{D}_\lambda, \mu_{\mathcal{D}}^{p|1|q})$ as follows. Let S be a disc $\mathbb{D}$ with $d_1 + 1 + d_2$ boundary punctures removed and an interior marked point $*$. Among these $d_1 + 1 + d_2$ boundary

punctures, we fix a module input at $1 \in \partial \mathbb{D}$ and a module output at $-1 \in \partial \mathbb{D}$. Now the components of ∂S in the upper-half plane $\mathbb{H}$ are labeled by Lagrangian submanifolds $L_0, \ldots, L_{d_1}$, while the boundary components $\partial S \setminus \mathbb{H}$ are associated with Lagrangian labels $L'_0, \ldots, L'_{d_2}$. All these Lagrangian submanifolds involved are objects of $\mathcal{F}_\lambda(M)$. The definition of

$$(59) \quad \phi_c^{d_1|1|d_2} \colon CF^*(L_{d_1}, L_{d_1-1}) \otimes \cdots \otimes CF^*(L_1, L_0) \otimes CF^*(L_0, L'_0)$$
$$\otimes CF^*(L'_0, L'_1) \otimes \cdots \otimes CF^*(L'_{d_2-1}, L_{d_2})$$
$$\to CF^*(L_{d_1}, L'_{d_2})$$

is similar to the A_∞ bimodule structure maps $\mu_\mathcal{D}^{d_1|1|d_2} = \pm \mu_\mathcal{F}^{d_1+d_2+1}$ of $\mathcal{D}_\lambda$, namely the Hamiltonian chords with ends on $L_0, \ldots, L_{d_1}$ act on the left, and the Hamiltonian chords with ends on $L'_0, \ldots, L'_{d_2}$ act on the right. However, in this case the solutions $u \colon S \to M$ of (31) are required to satisfy an additional intersection condition at $*$ specified by the locally finite cycle c. Summing up the $\phi_c^{d_1|1|d_2}$'s we get the endomorphism $\phi_c \in \mathrm{End}(\mathcal{D}_\lambda)$.

By Theorem 8.1 of [38], the unital $\mathbb{K}$–algebra homomorphism

$$(60) \qquad QH^*(M) \to H^*(\mathrm{End}(\mathcal{D}_\lambda)) \cong \mathrm{End}(HH_*(\mathcal{D}_\lambda, \mathcal{D}_\lambda))$$

defined by $c \mapsto \phi_c$ endows $HH_*(\mathcal{F}_\lambda(M), \mathcal{F}_\lambda(M))$ with a $QH^*(M)$–module structure.

Similarly, $HH_*(\mathcal{W}_\lambda(M), \mathcal{W}_\lambda(M))$ also admits the structure of a $QH^*(M)$–module.

Proposition 2.10 (Ritter and Smith [38]) *The maps in the acceleration diagram* (58) *are* $QH^*(M)$*–module homomorphisms.*

For most of our applications, the most relevant are the degree 0 parts of the open–closed and closed–open maps

$$(61) \qquad \mathrm{OC}^0 \colon HF^*(L, L) \to QH^{*+n}(M)_\lambda, \quad \mathrm{CO}^0 \colon QH^*(M)_\lambda \to HF^*(L, L),$$

where $L \subset M$ is a closed admissible Lagrangian submanifold with $m_0(L) = \lambda$. Notice that, by definition, on the right-hand side of CO^0 we get $\mathrm{Hom}(\mathbb{K}, CF^*(L, L))$. Since the Hochschild differential in this case can be identified with the Floer differential μ^1 on $CF^*(L, L)$, by evaluating at $1 \in \mathbb{K}$ one gets a map CO^0 with image in $HF^*(L, L)$. Similarly, we have the degree 0 part of the open–closed maps for possibly noncompact admissible Lagrangians: just replace the $QH^*(M)_\lambda$ in (61) with $SH^*(M)_\lambda$.

Finally we need the following important lemma, which computes explicitly the image of $c_1(M) \in QH^2(M)$ under the closed–open map CO^0. Its proof is simply an adaptation of the argument in Lemma 9.1 of [38] to our setting.

Lemma 2.11 (Auroux, Kontsevich and Seidel) *Let CO^0 be the degree 0 open–closed string map defined in* (61); *then*

$$(62) \qquad\qquad CO^0(c_1(M)) = m_0(L)[L].$$

Our proof of Theorem 1.4 relies on the following nontriviality result of the open–closed string map, which is a mild generalization of Theorem 12.19 in [38]:

Proposition 2.12 *Suppose the monotone symplectic manifold M satisfies in addition item* (i) *of Definition 1.3, and let $L \subset M$ be a closed admissible Lagrangian submanifold with $m_0(L) \neq 0$. If $pt \in C_*(L)$ defines a cocycle in $HF^*(L, L)$, then*

$$(63) \qquad\qquad OC^0([pt]) \neq 0 \in QH^*(M).$$

Proof If the symplectic filling M^{in} of (V, ξ) is strong, the proposition follows from Theorem 12.19 in [38]. Otherwise $c_1(M)$ itself may not be exact at infinity, but we can mimic their argument with $c_1(M)$ replaced by its quantum power

$$(64) \qquad\qquad c_1(M)^{\star j} = c_1(M) \star \cdots \star c_1(M) \in QH^{2j}(M).$$

We want to show that the class $c_1(M)^{\star j}$ can be represented by a compactly supported differential form. To see this, use the assumption that $c_1(M)^{\star j} \in H^{2j}(M; \mathbb{K})$. Since we care only about the behavior of $c_1(M)^{\star j}$ on the cylindrical end, we can restrict it to $M \setminus M^{in}$. By our assumption, any representative of $c_1(M)^{\star j}$ defines a cohomology class in $H^{2j}(M \setminus M^{in}; \mathbb{K})$. From the obvious identification between $H^{2j}(M \setminus M^{in}; \mathbb{K})$ and $H^{2j}(V; \mathbb{K})$ and the assumption that $H^{2j}(V; \mathbb{Q}) = 0$, we see that this cohomology class must be trivial.

This allows us to choose a compact locally finite cycle $c \in H_*^{BM}(M)$ which represents $c_1(M)^{\star j}$, where by $H_*^{BM}(M)$ we mean the Borel–Moore homology. By Lemma 2.11 and the fact that $CO^0 : QH^*(M) \to HF^*(L, L)$ is a unital algebra homomorphism, we get

$$(65) \qquad\qquad CO^0(c_1(M)^{\star j}) = m_0(L)^j [L],$$

which shows that c is not a boundary in $H_*^{BM}(M)$. By definition, $m_0(L)^j$ is then the coefficient before $PD(c^\vee)$ in the disc counting $OC^0([pt])$, where by $c^\vee \in H_*(M; \mathbb{K})$

we mean the dual of c under intersection pairing and PD is the Poincaré dual. More precisely,

(66) $$\mathrm{OC}^0([\mathrm{pt}]) = m_0(L)^j \mathrm{PD}(c^\vee) + \text{other terms}.$$

Since the other terms on the right-hand side counting Maslov index 2 discs correspond to cycles in $H_*(M; \mathbb{K})$ which are linear independent with $c^\vee$, they cannot cancel the term $m_0(L)^j \mathrm{PD}(c^\vee)$. The disc countings with Maslov index not equal to 2 will have different powers in the Novikov parameter q, so they will not cancel the first term either. Now the statement follows from our assumption that $m_0(L) \neq 0$. □

2.4   Proof of Theorem 1.4

We have now introduced all the algebraic tools needed in the proof of Theorem 1.4. Our argument follows the proof of Theorem 1.3 sketched in Section (1d) of [43] closely, for which the following version of Cardy relation plays a key role:

Proposition 2.13   (Cardy relation)   *Let $L_1, L_2 \subset M$ be two objects which belong to the same eigensummand of the monotone Fukaya category $\mathcal{F}(M)$, and $[a_i] \in HF^*(L_i, L_i)$ for $i = 1, 2$. Then*

(67) $$\mathrm{OC}^0([a_2]) \star \mathrm{OC}^0([a_1]) = (-1)^{n(n+1)/2} \mathrm{Str}([a] \mapsto (-1)^{|a| \cdot |a_2|}[a_2] \cdot [a] \cdot [a_1]),$$

where the left-hand side is the quantum intersection product on $QH^*(M)$, while the right-hand side is the supertrace of the endomorphism $HF^*(L_1, L_2)$ given by composition with $[a_1]$ and $[a_2]$.

The above Cardy relation is stated in its more general form in Section 5.2 of [42], but the special case above is enough for our applications here. Note also that the quantum product on the left-hand side of (67) is taken to be the ordinary intersection product on $H^*(M; \mathbb{K})$ in [42] since the work is done in the exact category.

The proof of the above result relies on a degeneration and gluing analysis of the moduli space of annuli with two boundary marked points, lying respectively on the two boundary components. See for example Section 11.4 of [38] for a detailed analysis. The same argument extends to the current setting by Lemma 2.7.

Assume that M^{in} is a monotone Lefschetz domain. By assumption, $SH^*(M)$ is semisimple, so we have the decomposition

(68) $$SH^0(M) = \bigoplus_{i \in I} \mathbb{K} v_i,$$

where I is some finite index set and $(v_i)_{i \in I}$ is a collection of idempotents in $SH^*(M)$ which are orthogonal to each other. Here we are using the $\mathbb{Z}/2$–grading on $SH^*(M)$, so $SH^0(M)$ is a commutative ring. By Lemma 2.6, there is a well-defined pair-of-pants product on $SH^*(M)$, which restricts to one on the even degree part $SH^0(M)$. By assumption, the pair-of-pants product is nontrivial on each summand $\mathbb{K}v_i$.

For any closed admissible Lagrangian submanifold $L \subset M$, consider the open–closed string map OC^0 composed with the unital algebra homomorphism c^* induced by the localization of $QH^*(M)$ at $c_1(M)$,

$$
(69) \qquad HF^*(L, L) \xrightarrow{\mathrm{OC}^0} QH^{*+n}(M) \xrightarrow{c^*} SH^{*+n}(M).
$$

We can decompose the even part of the quantum cohomology $QH^0(M)$ and hence $SH^0(M)$ into different generalized eigenspaces with respect to $\star c_1(M)$ and work in a fixed eigensummand of the Fukaya category $\mathcal{F}_\lambda(M)$. By Proposition 2.9, the image of the map $c^* \circ \mathrm{OC}^0$ then lies in the summand $SH^{*+n}(M)_\lambda$.

Let L be a closed admissible Lagrangian submanifold with $m_0(L) = \lambda \neq 0$; consider the restriction of the map (69) to the degree n part of Floer cohomology,

$$
(70) \qquad c^* \circ \mathrm{OC}^0 : HF^n(L, L) \to SH^0(M)_\lambda \subset \bigoplus_{i \in I} \mathbb{K}v_i.
$$

By Proposition 2.10, both of the maps OC^0 and c^* respect the $QH^*(M)$–module structure; it follows that the image of $c^* \circ \mathrm{OC}^0$ in $SH^0(M)_\lambda$ consists of a subset of the summands in (68). Now Proposition 2.12 together with the assumption that $SH^*(M)_\lambda \cong QH^*(M)_\lambda$ for any $\lambda \neq 0$ (condition (ii) of Definition 1.3) shows that this subset is nonempty if $[\mathrm{pt}] \in HF^n(L, L)$ is nontrivial. Now let $L_i, L_j \subset M$ be two closed monotone Lagrangian submanifolds with $m_0(L_i) = m_0(L_j) = \lambda$ which are disjoinable by Hamiltonian isotopy; then $HF^*(L_i, L_j)$ is well defined and vanishes. By Proposition 2.13, the images of $HF^n(L_i, L_i)$ and $HF^n(L_j, L_j)$ under OC^0 must be mutually orthogonal with respect to the quantum product on $QH^0(M)_\lambda$. This is also true for the images of $HF^n(L_i, L_i)$ and $HF^n(L_j, L_j)$ in $SH^0(M)_\lambda$ under the composition $c^* \circ \mathrm{OC}^0$ since c^* is a $\mathbb{K}$–algebra homomorphism. By the nonvanishing of the pair-of-pants product on each summand $\mathbb{K}v_i \subset SH^0(M)_\lambda$, we see that the images of $HF^n(L_i, L_i)$ and $HF^n(L_j, L_j)$ under $c^* \circ \mathrm{OC}^0$ must lie in different summands of $SH^0(M)_\lambda$; therefore, the number of disjoinable Lagrangians in $\mathcal{F}(M)_\lambda$ is bounded by $\dim_\mathbb{K} SH^0(M)_\lambda$. Collecting all the eigensummands together finishes the proof.

2.5 Contact toric manifolds

We digress a little bit and discuss briefly the vanishing condition $H^{2j}(V;\mathbb{Q}) = 0$, where $1 \le j \le n-1$ is a fixed integer. This plays a role when applying Theorems 1.4 and 1.9 to concrete examples. See Section 4.1.

The concept of a *contact toric manifold* is introduced by Lerman [25]. For a $(2n-1)$–dimensional closed contact manifold (V, ξ), this means that V carries an effective T^n–action which preserves the contact structure ξ.

Consider the positive half of the symplectization of (V, ξ). Concretely this is defined by

$$(71) \qquad S\xi := (\mathbb{R}_+^* \times V, dr \wedge \theta_V + r\, d\theta_V),$$

where $r \in \mathbb{R}_+^*$ and θ_V is a T^n–invariant contact form. One can lift the T^n–action on V to T^*V, since $S\xi \subset T^*V$ is preserved under this action, we see that $S\xi$ is a symplectic toric manifold. Denote by

$$(72) \qquad \mu_{S\xi}\colon S\xi \to (\mathfrak{t}^\vee)^n \cong \mathbb{R}^n$$

the toric moment map. The *moment cone* associated to (V, ξ) is defined to be the set

$$(73) \qquad C_V := \mu_{S\xi}(S\xi) \cup \{0\}.$$

Definition 2.14 [25] A rational polyhedral cone

$$(74) \qquad C = \{x \in \mathbb{R}^n \mid \langle x, v_i \rangle \ge 0 \text{ for all } i \in I\},$$

where I is a finite index set and $v_i \in \mathbb{Z}^n$, is said to be good if

- every codimension m face is the intersection of m facets of C;
- for every subset $J \subset I$, $\mathbb{Z}\langle (v_j)_{j \in J} \rangle$ is a direct summand of $\mathbb{Z}^n$ with rank $|J|$.

A contact toric manifold (V, ξ) is called *good* if $\dim_{\mathbb{R}}(V) > 3$ and C_V is a strictly convex good cone. Note that when $\dim_{\mathbb{R}}(V) = 3$ and the T^2–action on V is not free, then it is known that V is diffeomorphic to a lens space [25].

Using a linear transformation in $\mathrm{SL}(n, \mathbb{Z})$, we can place $C_V \setminus \{0\}$ in the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_+^*$. To ensure that V is a circle bundle over a smooth toric manifold, one needs to impose the additional requirement that the intersection between C_V and the hyperplane $\boldsymbol{H} := \mathbb{R}^{n-1} \times \{1\}$ is a *Delzant polytope*. With these assumptions, the following result follows essentially by applying the Gysin sequence:

Proposition 2.15 (Luo [27]) *For a good contact toric manifold* (V, ξ) *such that* $C_V \cap H$ *is Delzant,*

$$(75) \qquad H^{2j}(V; \mathbb{Q}) = 0, \quad \left\lceil \tfrac{1}{2}n \right\rceil \leq j \leq n - 1.$$

This contains the following example as a special case:

Lemma 2.16 *Let* (V, ξ_{BW}) *be the total space of the unit sphere bundle associated to* $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$ *equipped with its standard contact structure. Then* (V, ξ_{BW}) *is a good contact toric manifold such that* $C_V \cap H$ *is a Delzant polytope.*

Proof Since the total space of $\mathcal{O}(-1, -1) \to \mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$ can be regarded as the blow-up of $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$ along the subvariety $\mathbb{CP}^{n_1-1} \subset \mathbb{CP}^{n_2}$ in the zero section, we have an identification between the total spaces of $\mathcal{O}(-1, -1) \to \mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$ and $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$ away from their zero sections. It follows that the unit disc bundle of $\mathcal{O}(-1, -1) \to \mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$ gives a strong symplectic filling of the unit sphere bundle associated to $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, from which it is easy to see that $C_V \cap H$ is the moment polytope associated to $\mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$. $\quad\square$

Note that the same is true for the more general case of negative vector bundles $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$. Since their ideal contact boundaries (V, ξ_{BW}) are circle bundles and can be strongly filled by the unit disc bundle of $\mathcal{O}(-1, -m) \to \mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$, so $C_V \cap H$ is still the moment polytope of $\mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$.

3 Semisimplicity

This section studies another major issue of this paper, namely symplectic manifolds with semisimple symplectic cohomologies. We introduce the surgery of boundary connected sums between stable symplectic fillings, and then prove that symplectic cohomology is well behaved under such a surgery. Based on this we further investigate some special cases of reverse MMP transitions and show that they can be used to construct new examples of symplectic manifolds whose symplectic cohomologies are semisimple.

3.1 Handle attachment

The idea of attaching handles to Weinstein domains dates back to [50]. In [19], the authors observed that such a surgery can be generalized to the case of weak symplectic fillings. See also [18]. For our purposes, we recall here only the construction for 1–handles.

Start with the more general case when $(M^{\mathrm{in}}, \omega_M)$ is a weak symplectic filling of a $(2n-1)$–dimensional contact manifold (V, ξ). For any pair of distinct points $v_+, v_- \in V$, there exist neighborhoods $U_\pm \subset M^{\mathrm{in}}$ containing $v_\pm$ such that one can deform ω_M in a collar neighborhood of M^{in} to ω'_M so that the new symplectic filling $(M^{\mathrm{in}}, \omega'_M)$ becomes locally strong in $U_\pm$. More precisely, there are Liouville vector fields $Z_\pm$ on $U_\pm$ such that the restrictions of their duals are the contact form

$$\theta_V|_{U_\pm \cap V} = \iota_{Z_\pm} \omega'_M|_{U_\pm \cap V}. \tag{76}$$

The existence of such a deformation is ensured by Lemma 2.10 of [28].

To construct a symplectic structure on the 1–handle H_1, equip $\mathbb{R}^{2n}$ with coordinates

$$(x^-, y^-, x_1^+, y_1^+, \ldots, x_{n-1}^+, y_{n-1}^+). \tag{77}$$

Take $H_1(\delta) = [-1, 1] \times \mathbb{B}(\delta)^{2n-1} \subset \mathbb{R}^{2n}$, where $\mathbb{B}(\delta)$ denotes a ball of radius δ. We use y^- to represent the coordinate on the interval $[-1, 1]$. Denote respectively by $\partial_- H_1(\delta)$ and $\partial_+ H_1(\delta)$ the boundary components of $H_1(\delta)$

$$\partial_- H_1(\delta) = (\{-1\} \cup \{1\}) \times \mathbb{B}(\delta)^{2n-1}, \quad \partial_+ H_1(\delta) = [-1, 1] \times \partial \mathbb{B}(\delta)^{2n-1}. \tag{78}$$

The two-point set $\{\pm 1\} \times \{0\}$ is called the *core* of $\partial_- H_1(\delta)$, where $0 \in \mathbb{B}(\delta)^{2n-1}$ is the center of the ball, while the core of $\partial_+ H_1(\delta)$ is the $(2n-2)$–sphere $\{0\} \times \partial \mathbb{B}(\delta)^{2n-1}$.

Following Section 3 of [50], we equip $\mathbb{R}^{2n}$ with the symplectic form

$$\omega_{\mathbb{R}^{2n}} = 2\, dx^- \wedge dy^- + 4 \sum_{i=1}^{n-1} dx_i^+ \wedge dy_i^+. \tag{79}$$

Note that the coefficient before $2\, dx^- \wedge dy^-$ above is reduced, which follows the convention of [19, Section 3.1]. The form $\omega_{\mathbb{R}^{2n}}$ has a primitive

$$\theta_{\mathbb{R}^{2n}} = 4x^-\, dy^- + 2y^-\, dx^- + 2 \sum_{i=1}^{n-1} (x_i^+\, dy_i^+ - y_i^+\, dx_i^+), \tag{80}$$

and the corresponding Liouville vector field is given by

$$Z_{\mathbb{R}^{2n}} = 2x^- \frac{\partial}{\partial x^-} - y^- \frac{\partial}{\partial y^-} + \frac{1}{2} \sum_{i=1}^{n-1} \left(x_i^+ \frac{\partial}{\partial x_i^+} + y_i^+ \frac{\partial}{\partial y_i^+} \right). \tag{81}$$

The vector field $Z_{\mathbb{R}^{2n}}$ points outwards along $\partial_+ H_1(\delta)$, so that it will serve as part of the contact boundary for the new weak filling M_1^{in} after attaching the handle. It is

inward-pointing along $\partial_- H_1(\delta)$, so we can glue this concave boundary component to part of the pseudoconvex boundary of the original weak filling M^{in}.

For $\delta > 0$ small enough, $(\partial_- H_1(\delta), \theta_{\mathbb{R}^{2n}})$ is contact isomorphic to small neighborhoods $U_\pm(\delta) \subset U_\pm$ of $v_\pm$. It follows that there is a contact form θ_V for ξ such that we can glue $\partial_- H_1(\delta)$ to $U_\pm(\delta)$ to obtain a symplectic manifold with corners

$$(82) \qquad (M^{\mathrm{in}}, \omega_M) \cup_{\partial_- H_1(\delta)} (H_1(\delta), \omega_{\mathbb{R}^{2n}}),$$

which weakly fills

$$(83) \qquad (V \setminus U_\pm(\delta), \xi) \cup (\partial_+ H_1(\delta), \xi_\partial),$$

where ξ_∂ is the contact structure induced by $\theta_{\mathbb{R}^{2n}}$. After rounding off the corners we get the desired weak symplectic filling M_1^{in} of the contact boundary (V_1, ξ_1). This completes the construction.

Since we are interested in the behaviors of symplectic cohomologies under handle attachment, we need to show that the above surgery can be done within the category of stable fillings, rather than just weak ones, so that $SH^*(M^{\mathrm{in}}, \omega_M')$ is well defined for semipositive M. The following is a slight modification of Corollary 2.12 in [28]:

Proposition 3.1 *There exists a deformation ω_M' of ω_M such that $\omega_V' = d\theta_V$ when restricted to $V \cap U_\pm$ and (ω_V', θ_V) form a stable Hamiltonian structure on V.*

Proof Without loss of generality, we can assume that $[\omega_V] \in H^2(V; \mathbb{Q})$ represents a rational cohomology class. By Proposition 2.18 of [14], we can find a closed 2–form ω_V' in $[\omega_V]$ such that (ω_V', θ_V) forms a stable Hamiltonian structure on V. Fix a large constant $C \gg 0$; recall that

$$(84) \qquad \omega_V' = d\theta_V + \frac{1}{C} t_F,$$

where t_F is a Thom form associated to some tubular neighborhood D_S of a codimension 2 contact submanifold $S \subset V$ representing a multiple of the class $\mathrm{PD}([\omega_V])$ and F is a function compactly supported in the fiber of $D_S \to S$ such that

$$(85) \qquad \mathrm{supp}(t_F) \subset \mathrm{supp}(F).$$

If v_+ or v_- lies on S, we can perturb S a little bit to another contact submanifold $S' \subset V$ such that $S' \cap \{v_+, v_-\} = \varnothing$ and $[S] = [S']$ in $H_{2n-3}(V; \mathbb{Z})$. Since we can shrink $\mathrm{supp}(F)$ to make it sufficiently small, it follows that there are small neighborhoods $U_\pm$ of $v_\pm$ such that $U_\pm \cap \mathrm{supp}(t_F) = \varnothing$. Now ω_V' is the 2–form we want,

since it is cohomologous to ω_V; Lemma 2.10 of [28] implies the existence of such a deformation. $\square$

Corollary 3.2 *Let* $(M^{\mathrm{in}}, \omega_M)$ *be a stable filling of* (V, ξ); *then* M_1^{in} *is a stable filling of* (V_1, ξ_1).

Proof By Proposition 3.1, (ω_V', θ_V) is a stable Hamiltonian structure on V. On the other hand, the symplectic structure on the handle $H_1(\delta)$ strongly fills the boundary component $\partial_+ H_1(\delta)$. The result now follows from (83). $\square$

From now on we shall always assume that the deformation of ω_M we made in the construction of M_1^{in} satisfies the requirement of Proposition 3.1. This ensures that $SH^*(M_1^{\mathrm{in}})$ is well defined.

3.2 Proof of Theorem 1.6

From now on assume $(M^{\mathrm{in}}, \omega_M)$ is a semipositive stable filling and $(M^{\mathrm{in}}, \omega_M')$ is the deformation provided by Proposition 3.1. Before attaching the handle, we need the following lemma, which shows that the deformation of the symplectic structure we did on the collar of M^{in} does not affect its symplectic cohomology.

Lemma 3.3 *There is an isomorphism of $\mathbb{K}$–algebras*

$$SH^*(M, \omega_M) \cong SH^*(M, \omega_M'). \tag{86}$$

Proof Note first that since $\omega_M' = \omega_M + \beta$ is a deformation of ω_M in some collar neighborhood of M^{in}, M equipped with ω_M' is also semipositive; in particular, $SH^*(M, \omega_M')$ is well defined. The argument is similar to Remark 35 of [34]. More precisely, one can shrink slightly the domain M^{in} using the radial vector field $\frac{\partial}{\partial r}$ on the collar to a subdomain $U^{\mathrm{in}} \subset M^{\mathrm{in}}$ so that $\beta = 0$ on U^{in}. Moreover, we can replace every Hamiltonian $H \colon M \to \mathbb{R}$ involved in the definition of $SH^*(M)$ with another Hamiltonian H_U so that H_U has the same slope as H at infinity and reaches such a slope in the collar of U^{in}. Using these Hamiltonians, the Floer solutions involved in the definition of $HF^*(H_U)$ are all contained in U^{in}, where $\beta = 0$. In view of Lemma 2.6, the same is true for the Floer trajectories defining the pair-of-pants product on $SH^*(M)$, so the isomorphism also preserves the ring structure. $\square$

Attaching the handle $H_1(\delta)$ to M^{in} gives us the new stable filling M_1^{in}. It's easy to see the completion M_1 of M_1^{in} is also semipositive, so $SH^*(M_1)$ is well defined. We now imitate the argument of [12] to prove Theorem 1.6.[1]

One can regard $(M^{\text{in}}, \omega'_M)$ as a subdomain of M_1^{in} by enlarging the latter one a little bit using the vector field $\frac{\partial}{\partial r}$. Consider a cofinal family of Hamiltonians $K_m \colon M^{\text{in}} \to \mathbb{R}$ such that for any fixed m, $K_m < 0$ and is C^2–small in the interior of M^{in} and is linear with slope $a_m > 0$ on the collar neighborhood $(1, 1 + \varepsilon] \times V$, with $a_m \notin \mathcal{P}_M$ for any m. For convenience, we take K_m to be a small perturbation of the piecewise smooth function

$$(87) \qquad K_m = \begin{cases} a_m r & \text{on } \{r \geq 1\} \times V, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have tacitly extended the domain M^{in} to $M^{\text{in}} \cup (1, 1 + \varepsilon] \times V$ with $\varepsilon > 0$ sufficiently small such that it is still a subdomain of M_1^{in}. For convenience, we do not change the notation after such an extension.

Assuming $\delta > 0$ is taken to be sufficiently small, one can extend the Hamiltonian K_m over the handle $H_1(\delta)$ to obtain a Hamiltonian $H_m \colon M_1^{\text{in}} \to \mathbb{R}$ such that H_m has slope $b_m > a_m$ in a collar neighborhood of ∂M_1^{in}, and the only 1–periodic orbit of X_{H_i} on $H_1(\delta)$ is a critical point x_m in its interior. With this definition, (H_m) forms a cofinal family of Hamiltonians on M_1^{in}. Note that we always assume $\{b_m\} \cap \mathcal{P}_{M_1} = \varnothing$. One can arrange that when taking $\delta > 0$ to be very small, $\partial_+ H_1(\delta)$ is always transverse to the Liouville vector fields $Z_\pm$ existing locally, so the parameter δ does not affect the completion M_1 of M_1^{in}. For more details of this step, see [12] or [31].

Recall that the action functional in the nonexact case is given by

$$(88) \qquad A_H(u, x) = \int_{\mathbb{D}^2} u^* \omega_M - \int_{S^1} H(t, x(t)) \, dt,$$

where $(u, x) \in \widetilde{\mathcal{L}}_0 M$ is defined in Section 3.3, and $H \colon S^1 \times M \to \mathbb{R}$ is a time-dependent Hamiltonian function.

For any critical point x of K_m in the interior of M^{in}, we have

$$(89) \qquad \int_{S^1} K_m(t, x(t)) \, dt = K_m(x) < 0,$$

which implies that $A_{K_m}(u_x, x) > 0$. Similarly, for the case of a nonconstant 1–periodic orbit x of X_{K_m} in the boundary $V = \partial M^{\text{in}}$, we also have $A_{K_m}(u, x) > 0$ for any

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[1]The proof of [12, Theorem 1.11] contains an error, which can be fixed. See [17].

choice of $u\colon \mathbb{D}^2 \to M^{\mathrm{in}}$ since $\int_{\mathbb{D}^2} u^* \omega'_M > 0$ and the value of K_m can be made sufficiently small near ∂M^{in}.

This proves that the only generator of the Floer complex $CF^*(H_m)$ with negative action is the critical point $x_m \in H_1(\delta)$ (with the constant disc u_{x_m} in the lift $\widetilde{\mathcal{L}}_0 M$ of $\mathcal{L}_0 M$). Since the Floer differential decreases the symplectic action, x_m generates a subcomplex $CF^*(M_1, M; H_m) \subset CF^*(H_m)$ of the Floer complex. One can take the direct limit of its cohomology,

$$(90) \qquad SH^*(M_1, M) := \varinjlim_m HF^*(M_1, M; H_m).$$

This is called the *relative symplectic cohomology* in [12], and is identical to the symplectic cohomology associated to the *stable symplectic cobordism* $M_1^{\mathrm{in}} \setminus M^{\mathrm{in}}$ defined in [13].

To show that $SH^*(M) \cong SH^*(M_1)$, we first prove that $SH^*(M_1, M) = 0$. Note that although in general neither of $c_1(M)$ and $c_1(M_1)$ will vanish, the difference $M_1^{\mathrm{in}} \setminus M^{\mathrm{in}}$ is a stable cobordism whose completion $M_1 \setminus M$ satisfies $c_1(M_1 \setminus M) = 0$ and is in fact exact, so one can use the Conley–Zehnder index $\mathrm{ind}_{\mathrm{CZ}}(x_m)$ to equip $SH^*(M_1, M)$ with a $\mathbb{Z}$–grading. When $m \to \infty$, the slope of H_m increases and so does $\mathrm{ind}_{\mathrm{CZ}}(x_m)$, from which one sees that the unique generator x_m in the complex $CF^*(M_1, M; H_m)$ does not live in the direct limit, namely $SH^*(M_1, M) = 0$.

It remains to show that any Floer trajectory $u\colon S^1 \times \mathbb{R} \to M_1^{\mathrm{in}}$ defining the Floer differential on Hamiltonian orbits of X_{H_m} with positive action must be contained in $M^{\mathrm{in}} \subsetneq M_1^{\mathrm{in}}$. In the exact case, this follows from Lemma 7.2 of [3]. In our case, one can prove this by applying Lemma 2.6 to the stable filling $(M^{\mathrm{in}}, \omega'_M)$. Since all the Hamiltonian orbits of X_{H_m} with $A_{H_m}(u, x) > 0$ are contained in M^{in}, Lemma 2.6 shows that the value of the function $r \circ u\colon M_1 \to \mathbb{R}_+$ cannot be larger than $1 + \varepsilon$. A similar application of Lemma 2.6 to J–holomorphic maps from pair-of-pants surfaces $u\colon P \to M_1^{\mathrm{in}}$ shows that the product structure on Hamiltonian orbits with positive actions is unaffected under handle attachment. Combined with the vanishing of $SH^*(M_1, M)$, this shows that there is a $\mathbb{K}$–algebra isomorphism $SH^*(M) \cong SH^*(M_1)$.

Note that although one needs to perturb the Hamiltonian H so that it becomes t–dependent in order for the orbits of X_H to be nondegenerate, without loss of generality one can assume that H stays the same near $\partial H_1(\delta)$, so it does not affect the validity of Lemma 2.6.

Now let M^{in} and $(M')^{\mathrm{in}}$ be stable fillings of (V, ξ) and (V', ξ'), respectively, and whose completions M and M' are semipositive. Choose points $v \in V$ and $v' \in V'$; by applying the modifications of the symplectic structures on the collar neighborhoods of M^{in} and $(M')^{\mathrm{in}}$ as described in Section 3.1, the boundary connected sum $M^{\mathrm{in}} \#_\partial (M')^{\mathrm{in}}$ can be defined by connecting the neighborhoods $U(\delta) \subset M^{\mathrm{in}}$ and $U'(\delta) \subset (M')^{\mathrm{in}}$ containing v and v', respectively, using a 1–handle $H_1(\delta) \subset \mathbb{R}^{2n}$. Denote by $M \#_\partial M'$ the completion of $M^{\mathrm{in}} \#_\partial (M')^{\mathrm{in}}$, the above argument together with Lemma 3.3 shows that

$$\text{(91)} \qquad SH^*(M \#_\partial M') \cong SH^*(M) \oplus SH^*(M')$$

as rings.

3.3 Seidel representation

We establish here a mild extension of Ritter's generalization of Seidel representation [39] on convex symplectic manifolds. This will be used in the next subsection to do computations for the quantum and symplectic cohomologies in certain nonconvex cases.

In this subsection, M will be the completion of a stable filling M^{in} of the contact boundary (V, ξ) which carries an additional structure, namely a Hamiltonian circle action g compatible with the Reeb flow at infinity. For the purpose of discussing Seidel representation, we shall further assume that M is *strongly semipositive*, by which we will mean

$$\text{(92)} \qquad 2 - n \leq c_1(M)([u]) < 0 \implies \omega_M([u]) \leq 0.$$

This stronger assumption (called *weak+ monotonicity* by Ritter) is imposed so that the transversality argument in Section 5.5 of [36] holds.

Denote by $\mathrm{Ham}_\ell(M, \omega_M)$ the space of Hamiltonians $H \colon M \to \mathbb{R}$ of the form

$$\text{(93)} \qquad H(v, r) = f(v) r$$

for $r \gg 0$, where $v \in V$ and $f \colon V \to \mathbb{R}$ is invariant under the Reeb flow. Similarly one can define $\mathrm{Ham}_{\ell \geq 0}(M, \omega_M)$ and $\mathrm{Ham}_{\ell > 0}(M, \omega_M)$ by requiring that $f(v) \geq 0$ and $f(v) > 0$. These Hamiltonians are referred to as *weakly admissible* in [31]. The motivation of this enlargement (rather than requiring that $f(v)$ is a constant, as in the usual definition of $SH^*(M)$) is that the Hamiltonians induced by rotations about toric divisors considered below (see Section 3.4) are in general not linear at infinity.

However, by an extension of the maximum principle proved by Ritter in Appendix C of [37], these Hamiltonians can be used to compute $SH^*(M)$ when M^{in} is a strong filling. Unfortunately, his argument does not work in the general case of stable fillings as the circle action g may not preserve the contact form θ_V due to the nonexactness of ω_V. Because of this, we present an alternative argument below, which works also for stable fillings.

Lemma 3.4 Let $(H_m) \subset \mathrm{Ham}_{\ell \geq 0}(M, \omega_M)$ be a cofinal sequence of weakly admissible Hamiltonians of the form $f_m(v)r$ at infinity such that the $f_m \colon V \to \mathbb{R}$ involved in the definitions are invariant under the Reeb flow, and $\lim_{m \to \infty} \min_{v \in V} f_m(v) = \infty$. Under the additional assumption that

$$(94) \qquad \max_{v \in V} f_m(v) \leq \min_{v \in V} f_{m+1}(v),$$

the direct limit $\varinjlim_m HF^*(H_m)$ can be defined and is isomorphic to the symplectic cohomology group $SH^*(M)$.

Proof Fix a weakly admissible Hamiltonian $H \in \mathrm{Ham}_{\ell \geq 0}(M, \omega_M)$; since f is invariant under the Reeb flow, the projections of the equation $(du - X_H \otimes \gamma)^{0,1} = 0$ to the directions $\langle R \rangle$ and $\langle \partial_r \rangle$ are well defined, in the sense that the splitting $TM = \xi \oplus \langle R \rangle \oplus \langle \partial_r \rangle$ is preserved by the flow of X_H. Thus one can argue similarly as in Lemma 2.6 to show that maximum principle holds for the solutions $u \colon S \to M$. Note that one needs to require that $f(v) \geq 0$ if γ is not closed, since the right-hand side of (22) now takes the form $-f(v) \, d\gamma/(ds \wedge dt)$.

Choose a homotopy (f_s) which interpolates $f(v)$ and $\min_{v \in V} f(v)$ such that each $f_s(v)$ is invariant under the Reeb flow. By projecting to $\langle R \rangle$ and $\langle \partial_r \rangle$ we get the equation

$$(95) \qquad \Delta \rho + h_s'' \partial_s \rho + \partial_s f_s \geq 0.$$

Therefore, in order to ensure that the continuation map $HF^*(K_{\min}) \to HF^*(H(v,r))$ is well defined, where $K_{\min}$ is the Hamiltonian which has the form $(\min_{v \in V} f(v))r$ for $r \gg 0$, it suffices to require $\partial_s f_s \leq 0$.

Similarly, under the condition that $\partial_s f_s \leq 0$, one can find a homotopy (f_s) invariant under the Reeb flow which interpolates $\max_{v \in V} f(v)$ and $f(v)$, so there is a continuation map $HF^*(H(v,r)) \to HF^*(K_{\max})$, where $K_{\max}$ is of the form $(\max_{v \in V} f(v))r$ at infinity. From this it is easy to see that under the condition (94), there exists a

sequence of Hamiltonians (K_m) which are linear for $r \gg 0$ with increasing slopes such that all the continuation maps $HF^*(K_m) \to HF^*(H_m)$ and $HF^*(H_m) \to HF^*(K_{m+1})$ are well defined. In particular, the continuation maps $HF^*(H_m) \to HF^*(H_{m+1})$ are well defined. Passing to direct limits, we get $\varinjlim_m HF^*(H_m) \cong SH^*(M)$. $\qquad\square$

Moreover, we need the following, whose proof is an adaptation of the argument of Theorem A.15 in the appendix of [35].

Proposition 3.5 *The above isomorphism*

$$(96) \qquad\qquad \varinjlim_m HF^*(H_m) \cong SH^*(M)$$

preserves product structures on both sides.

Proof Let K be a Hamiltonian on M linear at infinity and $H_s \colon M \to \mathbb{R}$ be an $s-$ dependent Hamiltonian of the form $h_s = f_s(v)r$ when $r \gg 0$, just as in the proof of Lemma 3.4. To prove the statement, it is enough to show that the following diagram commutes, where $w_1 + w_\infty = w_0$:

$$(97) \qquad
\begin{array}{ccc}
HF^*(w_1 K) \otimes HF^*(w_\infty K) & \xrightarrow{\;\psi_P\;} & HF^*(w_0 K) \\[2pt]
{\scriptstyle \varphi_1 \otimes \varphi_\infty} \downarrow & & \uparrow {\scriptstyle \kappa_0} \\[2pt]
HF^*(w_1' H_s) \otimes HF^*(w_\infty' H_s) & \xrightarrow{\;\psi_P'\;} & HF^*(w_0' H_s)
\end{array}$$

In the above, the vertical arrows are continuation maps, while the map ψ_P' is defined in the same way as ψ_P using the nonlinear Hamiltonian H_s. By Lemma 3.4, the continuation maps φ_1, φ_∞ and κ_0 are well defined under the conditions $w_1 K' \ll w_1' h_s'$, $w_\infty K' \ll w_\infty' h_s'$ and $w_0' h_s' \ll w_0 K'$ when $r \gg 0$. Suppose the weights and homotopies have been arranged in this way; then, passing to direct limits, one sees that the resulting commutative diagram gives precisely what we want.

From now on the argument is completely identical to Theorem A.15 of [35]. Namely consider the glued surface defining the operation $\kappa_0 \circ \psi_P' \circ (\varphi_1 \otimes \varphi_\infty)$ by attaching certain continuation cylinders corresponding to κ_0, φ_1 and φ_∞ to the surface P defining the pair-of-pants product ψ_P', one can choose a 1–parameter interpolation

$$(98) \qquad\qquad (P_\lambda, \gamma_\lambda, K_\lambda, J_\lambda)_{0 \le \lambda \le 1}$$

which connects this glued surface together with the corresponding auxiliary data to that of (P, γ, K, J) defining ψ_P. The fact that any solution $u \colon P_\lambda \to M$ of the equation

$$(99) \qquad\qquad (du - X_{K_\lambda} \otimes \gamma_\lambda)^{0,1} = 0$$

satisfies the maximum principle can be proved by combining Lemmas 2.6 and 3.4. This gives us a 1–parameter family of moduli spaces $\mathcal{M}_\lambda(x_1, x_\infty; x_0)$, from which one can extract a chain homotopy from $\kappa_0 \circ \psi'_P \circ (\varphi_1 \otimes \varphi_\infty)$ to ψ_P on the chain level, which yields the commutative diagram (97) after passing to cohomologies. □

With the above facts established, the whole machinery of [36; 37] can be generalized to our case. It seems appropriate to collect here some important facts which will be useful later.

Given $g \in \pi_1(\mathrm{Ham}_\ell(M, \omega_M))$, we can lift the action of g on $\mathcal{L}_0 M$ to $\widetilde{\mathcal{L}}_0 M$ to obtain a map $\widetilde{g} \colon \widetilde{\mathcal{L}}_0 M \to \widetilde{\mathcal{L}}_0 M$. Any two lifts of g differ by an element in the deck transformation group $\pi_2(M)/\pi_2(M)_0$. This defines an extension of $\pi_1(\mathrm{Ham}_\ell(M, \omega_M))$,

$$(100) \quad 1 \to \pi_2(M)/\pi_2(M)_0 \to \widetilde{\pi}_1(\mathrm{Ham}_\ell(M, \omega_M)) \to \pi_1(\mathrm{Ham}_\ell(M, \omega_M)) \to 1.$$

Any lift $\widetilde{g}$ of g acts on the generators of the Floer complex $CF^*(H)$ by $x \mapsto \widetilde{g}^{-1} \cdot x$, which results in an isomorphism

$$(101) \qquad\qquad \mathcal{R}_{\widetilde{g}} \colon HF^*(H) \to HF^{*+2I(\widetilde{g})}(g^* H),$$

where the index $I(\widetilde{g})$ is defined in the usual way by trivializing $u^* TM$ along the boundary of $\mathbb{D}^2$ and computing the degree of the loop induced by $\widetilde{g}$ inside $\mathrm{Sp}(2n, \mathbb{R})$. Taking direct limit with respect to the slope of H at infinity, we obtain a $\mathbb{K}$–algebra automorphism of $SH^*(M)$, whose inverse is induced from $\widetilde{g}^{-1}$.

To adapt Seidel's original approach [39] to the current setting, assume further that the time-dependent Hamiltonian $H_g \colon S^1 \times M \to \mathbb{R}$ associated to g belongs to $\mathrm{Ham}_{\ell \geq 0}(M, \omega_M)$. This assumption ensures that the action of g on the original Hamiltonian H has the effect of slowing down the flow of X_H at infinity, which enables us to construct a continuation map

$$(102) \qquad\qquad \kappa_H \colon HF^*(g^* H) \to HF^*(H).$$

Denote by H_ε a Hamiltonian with very small slope $\varepsilon > 0$ at infinity. Composing with $\mathcal{R}_{\widetilde{g}}$ and taking $H = H_\varepsilon$ in (102), we get a $\mathbb{K}$–algebra homomorphism

$$(103) \qquad\qquad \mathcal{S}_{\widetilde{g}} \colon QH^*(M) \to QH^{*+2I(\widetilde{g})}(M),$$

where we used the identification $HF^*(H_\varepsilon) \cong QH^*(M)$. Since $H_{g^{-1}}$ does not necessarily lie in $\mathrm{Ham}_{\ell \geq 0}(M, \omega_M)$, $\mathcal{S}_{\widetilde{g}}(1)$ may not be invertible.

The following result is due to Ritter when M^{in} is a strong filling; see [36; 37]:

Theorem 3.6 (Seidel representation) *There is a group homomorphism*

(104)
$$\mathcal{R}: \tilde{\pi}_1(\mathrm{Ham}_\ell(M, \omega_M)) \to SH^*(M)^\times$$

given by $\tilde{g} \mapsto \mathcal{R}_{\tilde{g}}(1)$, *where* $SH^*(M)^\times$ *denotes the invertible elements with respect to the product structure on* $SH^*(M)$.

On the other hand, the homomorphism

(105)
$$\mathcal{S}: \tilde{\pi}_1(\mathrm{Ham}_{\ell \geq 0}(M, \omega_M)) \to QH^*(M)$$

is only well defined for Hamiltonians with nonnegative slopes when $r \gg 0$.

Theorem 3.6 enables us to identify the continuation maps $HF^*(H_{m-1}) \to HF^*(H_m)$ for a sequence of weakly admissible Hamiltonians (H_m) compatible with the Reeb flow with the isomorphisms $\mathcal{R}_{\tilde{g}}^{-m} \circ \kappa_{H_\varepsilon} \circ \mathcal{R}_{\tilde{g}}^m$; such an explicit realization of continuation maps makes the computation of the symplectic cohomologies possible. The following is an easy consequence whose proof reduces essentially to linear algebra:

Corollary 3.7 (Ritter [36; 37]) *For any loop* $g \in \pi_1(\mathrm{Ham}_{\ell > 0}(M, \omega_M))$,

(106)
$$SH^*(M) \cong QH^*(M)/\ker \mathcal{S}_{\tilde{g}}^d$$

as algebras over $\mathbb{K}$, *where* d *is a positive integer taken to be sufficiently large.*

It remains to determine the image $\mathcal{S}_{\tilde{g}}(1)$ in $QH^*(M)$. Let $F(g)$ be the fixed locus of the Hamiltonian S^1–action g; then dg acts on the tangent space $T_x M$ with $x \in F(g)$ as a unitary matrix, which preserves the quotient space $T_x M / T_x F(g) \cong \mathbb{C}^{n-f}$, where $f = \dim_{\mathbb{C}} F(g)$.

Lemma 3.8 (Ritter [36; 37]) *Let* $\tilde{g}$ *be the lift of* g *which maps the pair* (u_x, x) *to itself, where* $x \in F(g)$ *and* u_x *is the constant disc associated to* x. *If* $dg \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n-f}, \mathbb{C}^{n-f})$ *has degree* 1, *then*

(107)
$$\mathcal{S}_{\tilde{g}}(1) = \mathrm{PD}[F(g)] + \text{higher-order } q \text{ terms} \in QH^{2n-2f}(M).$$

Moreover, if $f = 1$, *then the higher-order* q *terms in* (107) *vanish.*

3.4 Local computations

This subsection is mainly devoted to the computation of $QH^*(E_+)$ and $SH^*(E_+)$ when E_+ is the total space of the vector bundle $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, where $1 \leq n_1 \leq n_2$.

Note that these symplectic manifolds are just the exceptional pieces arising from blow-ups of points and standard reverse flips. For $n_1 = 1$, this has been done in [36]. Our computations will recover this as a special case.

For a monotone negative vector bundle $\mathcal{V} \to B$ over a closed monotone symplectic manifold, Ritter [36] considers the Seidel representation associated to the Hamiltonian S^1–action h given by rotation along the fibers, and proves that $\mathcal{S}_{\tilde{h}}(1) \in QH^*(\mathcal{V})$ is given by the pullback of the top Chern class of $\mathcal{V}$. Since in our case the bundle splits and the total space E_+ is toric, we can consider at the same time the Hamiltonian S^1–actions given by rotations about the toric divisors of E_+, and obtain a toric description of the element $\mathcal{S}_{\tilde{h}}(1) \in QH^{2n_1}(E_+)$, which makes the explicit computation of $SH^*(E_+)$ possible.

Following Section 11 of [36], we equip the total space of $p: E_+ \to \mathbb{CP}^{n_2}$ with the symplectic form

(108) $$\omega_{E_+} = p^*\omega_{\mathrm{FS}} + \pi \cdot \omega_{\mathbb{C}^{n_1}},$$

where ω_{FS} is the Fubini–Study form on $\mathbb{CP}^{n_2}$ and $\omega_{\mathbb{C}^{n_1}}$ means here a 2–form on E_+ which restricts to the standard area form on fibers and vanishes on the base. For cohomological reasons one sees that (E_+, ω_{E_+}) is not conical at infinity when $n_1 \geq 2$. In particular, it is not admissible in the sense of [37].

We remark that $QH^*(E_+)$ and $SH^*(E_+)$ are well defined. This is proved for monotone negative vector bundles in [36]. Alternatively, one can proceed by noticing that ω_{E_+} restricted to the unit sphere bundle $\partial E_+^{\mathrm{in}}$, together with a suitable choice of the contact form θ_{BW} for the contact structure ξ_{BW}, form a stable Hamiltonian structure on $\partial E_+^{\mathrm{in}}$. So the general argument for stable symplectic fillings presented in Section 2.1 solves the problem.

The standard reference for the geometry of toric vector bundles is Section 7.3 of [15]. Let $D_1, \ldots, D_{n_2+1}$ be the toric divisors of $\mathbb{CP}^{n_2}$. Denote by p_i for $1 \leq i \leq n_1$ the bundle projection

(109) $$\mathcal{O}(-1) \oplus \cdots \oplus \widehat{\mathcal{O}(-1)} \oplus \cdots \oplus \mathcal{O}(-1) \to \mathbb{CP}^{n_2}$$

which misses the ith summand. Now the toric divisors of E_+ are given by

(110) $$p^{-1}(D_1), \ldots, p^{-1}(D_{n_2+1}), p_1^{-1}(\mathbb{CP}^{n_2}), \ldots, p_{n_1}^{-1}(\mathbb{CP}^{n_2}).$$

Recall that the toric divisors are in one-to-one correspondence with homogenous coordinates on E_+, and we denote these coordinates by

$$(111) \qquad x_1, \ldots, x_{n_2+1}; y_1, \ldots, y_{n_1}.$$

For each tuple $(\tau_1, \ldots, \tau_{n_2+1}; \zeta_1, \ldots, \zeta_{n_1}) \in \mathbb{Z}^{n_2+1} \times \mathbb{Z}^{n_1}$, we can associate to it an Hamiltonian S^1–action on E_+ given by

$$(112) \qquad \begin{aligned} x_1 &\mapsto e^{2\pi i \tau_1 t} x_1, \ldots, x_{n_2+1} \mapsto e^{2\pi i \tau_{n_2+1} t} x_{n_2+1}, \\ y_1 &\mapsto e^{2\pi i \zeta_1 t} y_1, \ldots, y_{n_1} \mapsto e^{2\pi i \zeta_{n_1} t} y_{n_1}. \end{aligned}$$

Picking the standard basis of $\mathbb{Z}^{n_2+1} \times \mathbb{Z}^{n_1}$, we get rotations

$$(113) \qquad g_1, \ldots, g_{n_2+1}, h_1, \ldots, h_{n_1} \in \pi_1(\mathrm{Ham}(E_+, \omega_{E_+}))$$

about the toric divisors of E_+. Note that $\prod_{i=1}^{n_2+1} g_i = 1$. Define $h := \prod_{j=1}^{n_1} h_j$.

The next result serves as a replacement for the last two properties in the definition of an admissible noncompact toric manifold introduced in [37]. Its proof is similar to Theorem 4.5 of [37].

Lemma 3.9 *For any fixed $t \in S^1$, denote by $H_1, \ldots, H_{n_2+1}, K_1, \ldots, K_{n_1}$ the Hamiltonian functions which define the Hamiltonian S^1–actions in* (113) *and by $K := \sum_{j=1}^{n_1} K_j$ the Hamiltonian function which define the S^1–action h. Then $H_1, \ldots, H_{n_2+1} \in \mathrm{Ham}_{\ell \geq 0}(E_+, \omega_{E_+})$ and $K \in \mathrm{Ham}_{\ell > 0}(E_+, \omega_{E_+})$.*

Proof It is easy to see that the last n_1 components of the toric moment map

$$\mu_{E_+} \colon E_+ \to \mathbb{R}^n$$

are given in homogenous coordinates by

$$(114) \qquad (x_1, \ldots, x_{n_2+1}, y_1, \ldots, y_{n_1}) \to \left(\tfrac{1}{2} |y_1|^2, \ldots, \tfrac{1}{2} |y_{n_1}|^2 \right).$$

Denote by $\mathbb{CP}^{n_2}$ and $\mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$, respectively, the zero sections of E_+ and $\mathcal{O}_{\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2}}(-1, -1)$. It follows from Section 11.2 of [36] that there is an identification between $E_+ \setminus \mathbb{CP}^{n_2}$ and $\mathcal{O}(-1, -1) \setminus (\mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2})$ which preserves the radial coordinate. Denote by $|w|$ the radial coordinate on the negative line bundle $\mathcal{O}(-1, -1) \to \mathbb{CP}^{n_1-1} \times \mathbb{CP}^{n_2}$; it then follows that

$$(115) \qquad \sum_{j=1}^{n_1} |y_j|^2 = |w|^2.$$

On the other hand, it is proved in Lemma 45 of [36] that $r = \frac{1}{1+\pi}\left(1 + \frac{1}{2}|w|^2\right)$. Combined with (115), we have

$$(116) \qquad r = \frac{1}{1+\pi}\left(1 + \sum_{j=1}^{n_1} \tfrac{1}{2}|y|_j^2\right).$$

But, for points outside the locus $\mu_{E_+}^{-1}(0)$, we have $K_j = \frac{1}{2}|y_j|^2$. Using (116) and the fact that $K := \sum_{j=1}^{n_1} K_j$, we see that $K = (1+\pi)r - 1$, which has strictly positive slope at infinity.

All the Hamiltonians H_i arise from the base $\mathbb{CP}^{n_2}$, and the fact that they have nonnegative slope at infinity essentially follows from the nonnegativity of the corresponding Hamiltonians on the base. More precisely, we can lift the Hamiltonian vector fields X_{H_i} to $\mathbb{C}^{n+1}$, so that they become the standard angular vector fields $\partial/\partial\theta_i$. Since the bundle map $p: E_+ \to \mathbb{CP}^{n_2}$ corresponds precisely to the projection to the x–coordinates, one deduces easily that $dp \cdot \partial/\partial\theta_i = X_{B_i}$, where $B_i: \mathbb{CP}^{n_2} \to \mathbb{R}_+$ are the corresponding Hamiltonians on the base. Combining with the expression of the symplectic form ω_{E_+}, we deduce

$$(117) \qquad dH_i = \omega_{E_+}\left(\cdot, \frac{\partial}{\partial\theta_i}\right)$$
$$= (1 + \pi r^2)p^*\omega_{\mathbb{C}^n}\left(\cdot, \frac{\partial}{\partial\theta_i}\right) + \pi(d(r^2)\wedge\theta)\left(\cdot, \frac{\partial}{\partial\theta_i}\right)$$
$$= C_1 r d(p^*B_i) + C_2\omega_{\mathbb{C}^{n_1}}\left(\frac{\partial}{\partial\theta_i}\right),$$

where $C_1, C_2 > 0$ are some constants and (r, θ) is the polar coordinate on the fiber of the line bundle $\mathcal{O}(-1,-1) \to \mathbb{CP}^{n_1-1}\times\mathbb{CP}^{n_2}$. Integrating on both sides, we get $H_i = C_1 r p^*B_i$. Notice that p^*B_i is a function on $(\partial E_+^{\text{in}}, \xi_{\text{BW}})$ which is invariant under the Reeb flow and $B_i \geq 0$; the claim follows. $\qquad\square$

It is clear from our proof that K_j does not lie in $\text{Ham}_{\ell > 0}(E_+, \omega_{E_+})$ for any j as long as $n_1 > 1$. It is therefore necessary to consider the Hamiltonian S^1–action h instead of any individual h_j.

Now our computation reduces essentially to standard toric geometry and the general theory of Seidel representation recalled in Section 3.3. By Theorem 3.6, we then have well-defined elements

$$(118) \qquad \mathcal{S}_{\widetilde{g}_i}(1), \mathcal{S}_{\widetilde{h}}(1) \in QH^*(E_+), \quad \mathcal{R}_{\widetilde{g}_i}(1), \mathcal{R}_{\widetilde{h}}(1) \in SH^*(E_+)^{\times}.$$

One can normalize the choices of lifts $\widetilde{g}_i$ and $\widetilde{h}$ of g_i and h so that the constant pairing $(u_x, x) \in \widetilde{\mathcal{L}}_0 E_+$ maps to itself under the actions of $\widetilde{g}_i$ and $\widetilde{h}$. By Lemma 3.8,

$$(119) \qquad \mathcal{S}_{\widetilde{g}_i}(1) = \mathrm{PD}[p^{-1}(D_i)], \quad \mathcal{S}_{\widetilde{h}}(1) = \mathrm{PD}[\mathbb{C}\mathbb{P}^{n_2}] + \text{higher-order } q \text{ terms.}$$

For our particular choice of $h \in \pi_1(\mathrm{Ham}_{\ell>0}(E_+, \omega_{E_+}))$, it follows from Theorem 72 of [36] that the higher-order q terms in the expression of $\mathcal{S}_{\widetilde{h}}(1)$ actually vanish, therefore $\mathcal{S}_{\widetilde{h}}(1) \in QH^{2n_1}(E_+)$ represents the pullback of top Chern class of E_+ via the bundle map $p \colon E_+ \to \mathbb{C}\mathbb{P}^{n_2}$.

The rest of the computation can be completed by applying standard toric techniques; see for example [30].

Proposition 3.10 *Let E_+ be the total space of $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{C}\mathbb{P}^{n_2}$; then*

$$(120) \qquad QH^*(E_+) \cong \mathbb{K}[x]/(x^{n_2+1} - (-1)^{n_1} q^{n_2+1-n_1} x^{n_1})$$

and

$$(121) \qquad SH^*(E_+) \cong \mathbb{K}[x]/(x^{n_2+1-n_1} - (-1)^{n_1} q^{n_2+1-n_1}).$$

Proof It follows from Section 7.3 of [15] that the rays of the fan of E_+ are given by

$$e_1 = (b_1, 1, \ldots, 0), \ldots, e_{n_1} = (b_{n_1}, 0 \ldots, 1),$$

$$(122) \qquad e_{n_1+1} = (b_{n_1+1}, 0, \ldots, 0), \ldots, e_{n_2+1} = (b_{n_2+1}, 0, \ldots, 0),$$

$$f_1 = (0, 1, \ldots, 0), \ldots, f_{n_1} = (0, 0, \ldots, 1) \in \mathbb{Z}^{n_1+n_2+1},$$

where $b_1, \ldots, b_{n_2+1}$ are rays of the fan of $\mathbb{C}\mathbb{P}^{n_2}$. So the new linear relations are given by

$$(123) \qquad x_i + y_1 = 0, \ldots, x_i + y_{n_1} = 0, \quad i = 1, \ldots, n_2 + 1.$$

It's easy to see that in our case the primitive set of E_+ is precisely the primitive set of the base $\mathbb{C}\mathbb{P}^{n_2}$, so we have the relation

$$(124) \qquad \sum_{i=1}^{n_2+1} e_i = \sum_{j=1}^{n_1} f_j$$

among the rays of the fan. This gives rise to the quantum Stanley–Reisner relation

$$(125) \qquad \prod_{i=1}^{n_2+1} x_i = q^{n_2+1-n_1} \prod_{j=1}^{n_1} y_j.$$

Using the original linear relations $x_i = x$ for all $1 \leq i \leq n_2 + 1$ for the base $\mathbb{CP}^{n_2}$ to simplify the above relations, we get the presentation of $QH^*(E_+)$.

By Corollary 3.7, $SH^*(E_+)$ is the localization of $QH^*(E_+)$ at $c_{n_1}(E_+)$, ie at x^{n_1} by our computations of $QH^*(E_+)$, from which the result follows. $\square$

Corollary 3.11 *Let $j = \lceil \frac{n}{2} \rceil$; then $c_1(E_+)^{\star j} \in H^{2j}(E_+; \mathbb{K})$. In particular, E_+^{in} is a Lefschetz domain at level $\lceil \frac{n}{2} \rceil$ in the sense of Definition 1.3.*

Proof The class $c_1(E_+)$ is represented by a nonzero multiple of x in the above presentation of $QH^*(E_+)$, so our computation in Proposition 3.10 shows that $SH^*(E_+)$ is in fact a localization of $QH^*(E_+)$ at $c_1(E_+)$. The class $c_1(E_+)^{\star j}$, as well as x^j, generates $H^{2j}(E_+; \mathbb{K})$. Note that $H^{2j}(E_+; \mathbb{K}) \neq 0$ follows from the assumptions that $j = \lceil \frac{n}{2} \rceil$ and $n_1 \leq n_2$. The fact that E_+^{in} is a Lefschetz domain at level $\lceil \frac{n}{2} \rceil$ then follows from Lemma 2.16. $\square$

Exactly the same method works for the negative vector bundles $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, where $m \geq 1$, $mn_1 \leq n_2$ and $n_1 + n_2 = n$. The result is

$$(126) \qquad QH^*(\mathcal{O}_{\mathbb{P}^{n_2}}(-m)^{\oplus n_1}) \cong \mathbb{K}[x]/(x^{n_2+1} - q^{n_2+1-mn_1}(-m)^{mn_1} x^{mn_1})$$

and

$$(127) \qquad SH^*(\mathcal{O}_{\mathbb{P}^{n_2}}(-m)^{\oplus n_1}) \cong \mathbb{K}[x]/(x^{n_2+1-mn_1} - q^{n_2+1-mn_1}(-m)^{mn_1}).$$

These computations imply that one can pick any j with $\lceil \frac{n}{2} \rceil \leq j \leq \lceil \frac{mn}{m+1} \rceil$ to turn the total space of the unit ball bundle $\mathcal{O}(-m)_{\leq 1}^{\oplus n_1} \to \mathbb{CP}^{n_2}$ into a Lefschetz domain.

In particular, for the negative vector bundles $\mathcal{O}(-m)^{\oplus n/(m+1)} \to \mathbb{CP}^{mn/(m+1)}$ that appeared in Corollary 1.5, their symplectic cohomologies are 1–dimensional and are located in the even degree.

3.5 Symplectomorphisms of contact type

If one thinks on the topological level, taking M' to be E_+ in Theorem 1.6 should yield some interesting special cases of Conjecture 1.2. However, to do this geometrically, it still remains to modify the symplectic structures on the completion of $M^{\mathrm{in}} \#_\partial E_+^{\mathrm{in}}$, so that they coincide with the ones obtained by blow-ups or reverse flips. To study the behavior of $SH^*(M)$ under certain deformations of symplectic forms, it is useful to extend the invariance property of $SH^*(M)$ under a particular class of symplectomorphisms to the stable filling case.

The class of symplectomorphisms we are interested in is called *symplectomorphisms of contact type at infinity*. Roughly speaking, it consists of the symplectomorphisms which preserve the contact structures on the boundaries. It is defined in [40] for Liouville manifolds and can be generalized to our case as follows:

Definition 3.12 Let M^{in} and $(M')^{\mathrm{in}}$ be stable symplectic fillings of the contact manifolds (V, ξ) and (V', ξ'). A symplectomorphism $\phi\colon M \to M'$ is of contact type at infinity if, on a collar neighborhood, ϕ is of the form

$$(128) \qquad \phi(r, y) = (r - f(v), \eta(v)),$$

where $f\colon V \to \mathbb{R}$ is a smooth function and $\eta\colon V \to V'$ is a contactomorphism such that $\eta^* \theta_{V'} = e^{f(v)} \theta_V$.

With this concept, Theorem 8 of [34] can be generalized as follows:

Proposition 3.13 *Let $\phi\colon M \to M'$ be a symplectomorphism of contact type at infinity, and assume M and M' are semipositive; then $SH^*(M) \cong SH^*(M')$ as $\mathbb{K}$–algebras.*

Proof The key point is to have a version of the maximum principle similar to Lemma 7 of [34] so that it guarantees that the required continuation maps are well defined. The argument is similar to the parametrized version of Lemma 2.6, with the additional complexity that f is s–dependent. Choose an interpolation f_s from $f\colon V \to \mathbb{R}$ to 0 which is supported on a closed interval $I \subset \mathbb{R}$. Write again $u = (v, \rho)$ for a solution to the parametrized Floer equation $(du - X_{H_s} \otimes dt)^{0,1} = 0$, where X_{H_s} is the Hamiltonian vector field associated to H_s, which has the form $h(r - f_s(v))$, where h is linear for $r \gg 0$. Note that

$$(129) \qquad \rho = (r - f_s(v)) \circ u$$

in our case. What is different from the argument in [34] is that we shall use an admissible almost complex structure $J \in \mathcal{J}(M)$, so in particular $dr \circ J = -\theta_V$ for $r \gg 0$. To indicate the dependence of h on s, we denote it as h_s in the computations below. As in the proof of Lemma 2.6, we can project the Floer equation to the $\langle R \rangle$ and $\langle \partial_r \rangle$ directions to get

$$(130) \qquad \begin{cases} \theta_V(\partial_s v(s, t)) + \partial_t \rho = 0, \\ \partial_s(\rho + f_s \circ v(s, t)) - \theta_V(\partial_t v(s, t)) + h'_s(\rho) = 0, \end{cases}$$

where we have used the fact that f_s is independent of t. Since J restricted to ξ is tamed by $d\theta_V$, we have

$$(131) \qquad \partial_s \big(\theta_V(\partial_t v(s,t))\big) - \partial_t \big(\theta_V(\partial_s v(s,t))\big) \geq 0.$$

Substitute the expressions of $\theta_V(\partial_s v(s,t))$ and $\theta_V(\partial_t v(s,t))$ in the above inequality; we get an inequality satisfied by $\Delta\rho$,

$$(132) \qquad \Delta\rho + \partial_s^2 f_s + h_s'' \partial_s \rho + \partial_s h_s' \geq 0.$$

Since the term $h_s'' \partial_s \rho$ above only involves the first-order derivative in ρ, to ensure that the maximum principle for elliptic operators applies, we only require $(f_s)_{s \in I}$ to satisfy

$$(133) \qquad \partial_s h_s' \leq -\partial_s^2 f_s.$$

One can assume that the homotopy satisfies $\partial_s^2 f_s \leq C$ for some large constant $C \gg 0$, so that the above condition becomes $\partial_s h_s' \leq -C$. Denote by h_- the value of h_s for $s \ll 0$, and by h_+ the value of h_s for $s \gg 0$. After integrating on I one sees that it suffices to require $h_-' \gg h_+'$. The homotopy (f_s) from 0 to f can be built similarly provided that $h_-' \ll h_+'$.

The above argument enables us to build continuation maps $HF^*(H_+) \to HF^*(H_-)$ when $h_-' \gg h_+'$ for $r \gg 0$. Now the argument is identical to that of Theorem 8 of [34]; namely one picks a sequence of Hamiltonians (K_m), linear at infinity, which alternates the sequence of nonlinear Hamiltonians (H_m) of the form $h_m(r - f_{s_m}(v))$ at infinity, so that the continuation maps $HF^*(K_m) \to HF^*(H_m)$ and $HF^*(H_m) \to HF^*(K_{m+1})$ can be defined. Suppose that the slopes of (H_m) and (K_m) tend to infinity when $m \to \infty$; passing to direct limits, one sees that $\varinjlim_m HF^*(H_m) \cong SH^*(M)$. This finishes the proof by realizing the obvious isomorphism $\varinjlim_m HF^*(H_m) \cong SH^*(M')$.

For the product structures, one can argue similarly as in Proposition 3.5, with H_s replaced by a Hamiltonian of the form $h_s = h(r - f_s(v))$ when $r \gg 0$, and Lemma 3.4 replaced by Proposition 3.13. $\square$

A byproduct of the proposition above is that the *growth rate*

$$(134) \qquad \Gamma(M) \in \{-\infty\} \cup \mathbb{R}_+ \cup \{\infty\}$$

of $SH^*(M)$ introduced in [40] is well defined under the assumption that M^{in} is a stable filling whose completion M is semipositive. It follows from our definition that for every Lefschetz domain M^{in}, $\Gamma(M) = 0$.

3.6 Blow-ups and flips

We now collect all the tools and results established in the last five subsections to prove Corollaries 1.7 and 1.8.

We start by recalling the concept of a symplectic flip, which is considered in [10] for closed symplectic manifolds. The definition in [10] applies straightforwardly to noncompact symplectic manifolds; in particular, it makes sense to consider symplectic flips in our setup.

Definition 3.14 [10, Definition 2.7] We say that two symplectic manifolds W_- and W_+ are related by a simple symplectic flip if there exists a symplectic vector space $\widetilde{W} \cong \mathbb{C}^{n+1}$ with a Hamiltonian S^1–action whose moment map is $\mu_{\widetilde{W}} \colon \widetilde{W} \to \mathbb{R}$ such that $W_\pm = \mu_{\widetilde{W}}(\pm\varepsilon)/S^1$ are symplectic reductions at values $\pm\varepsilon$ for some small $\varepsilon > 0$, and the following are satisfied:

(i) $\sum_{i=1}^{n+1} \lambda_i > 0$, where $\lambda_1, \dots, \lambda_{n+1}$ are the weights of the Hamiltonian S^1–action;

(ii) $\lambda_i \neq 0$ for every i;

(iii)   at least two weights of $\{\lambda_i\}$ are positive, and at least two weights are negative.

Write $\widetilde{W}_\pm$ for the sum of the positive and negative weight spaces in $\widetilde{W}$, so that $\widetilde{W} = \widetilde{W}_- \oplus \widetilde{W}_+$. The semistable loci are

$$(135) \qquad \widetilde{W}^{\mathrm{ss},-} = (\widetilde{W}_- \setminus \{0\}) \times \widetilde{W}_+, \quad \widetilde{W}^{\mathrm{ss},+} = \widetilde{W}_+ \times (\widetilde{W}_+ \setminus \{0\}).$$

In particular, W_+ is obtained from W_- by replacing a weighted projective space of dimension $n_- - 1$ with a weighted projective space of dimension $n_+ - 1$, where $n_\pm$ are the number of positive and negative weights.

More generally, let M_- and M_+ be nonempty symplectic manifolds. We say that M_+ is obtained from M_- by a symplectic flip if there exist open covers

$$(136) \qquad\qquad\qquad M_\pm = U_\pm \cup W_\pm$$

such that the following hold:

(i) U_+ is diffeomorphic to U_- and there is a family of symplectic forms $\omega_{U,t} \in \Omega^2(U_\pm)$, where $t \in [-\varepsilon, \varepsilon]$, together with symplectic embeddings $i_\pm \colon U_\pm \to M_\pm$ such that $i_\pm^* \omega_{M_\pm} = \omega_{U,\pm\varepsilon}$.

(ii) $W_\pm$ are related by a simple flip.

(iii)   Under the canonical identification $H^2(M_-; \mathbb{R}) \cong H^2(M_+; \mathbb{R})$ which maps $c_1(M_-)$ to $c_1(M_+)$, we have

(137) $$[\omega_{M_-}] - [\omega_{M_+}] = 2\varepsilon' c_1(M_-)$$

for some $\varepsilon' > 0$.

In the above, $\widetilde{W}$ is the local model used to perform the variation of GIT construction, which results in the (generally nonisomorphic) symplectic manifolds W_+ and W_-. The neighborhood $U_\pm \subset M_\pm$ is the complement of $W_\pm$, where the symplectic structure does not change up to isotopy.

A particular simple case of interest is when all the weights of the Hamiltonian S^1–action on $\widetilde{W}$ are equal to ± 1. In this case the center of the flip is necessarily trivial, meaning that there is a symplectic $\mathbb{CP}^{n_2}$ embedded in M_+. Since the weights are ± 1, a tubular neighborhood of $\mathbb{CP}^{n_2} \subset M_+$ is symplectomorphic to $\mathcal{O}(-1)^{\oplus n_1} \to \mathbb{CP}^{n_2}$. Under the simple flip $M_+ \dashrightarrow M_-$, the $\mathbb{CP}^{n_2}$ is replaced by a $\mathbb{CP}^{n_1-1} \subset M_-$ whose tubular neighborhood is identified with $\mathcal{O}(-1)^{\oplus n_2+1} \to \mathbb{CP}^{n_1-1}$. Note that here n_2 is the number of positive weights of our S^1–action.

We now proceed to the proofs of Corollaries 1.7 and 1.8. By allowing $n_1 = 1$ in E_+, we can treat the cases of blow-ups and reverse simple flips simultaneously. We first show that in both cases the manifold M_+ admits the structure of the completion of a stable symplectic filling, so in particular $SH^*(M_+)$ is well defined.

Lemma 3.15   *There exists a compact submanifold with boundary $M_+^{\mathrm{in}} \subset M_+$ whose boundary $V_+ := \partial M_+^{\mathrm{in}}$ carries a stable Hamiltonian structure $(\omega_{V_+}, \theta_{V_+})$, so that M_+ is the completion of M_+^{in}, where $\omega_{V_+} := \omega_{M_+}|_{V_+}$ is the restriction of the symplectic form and θ_{V_+} is a contact form on V_+.*

Proof   Let $M_+ = \mathrm{Bl}_x(M_-)$, with $x \in M_- \setminus M_-^{\mathrm{in}}$. Outside of a compact subset of M_+, the symplectic form satisfies $\omega_{M_+} = \pi^* \omega_{M_-}$, where $\pi \colon M_+ \dashrightarrow M_-$ is the blow-down map. Recall that M_- is the completion of a stable filling of the contact boundary (V_-, ξ_-); without loss of generality, one can assume that $\omega_{M_+} = \pi^* \omega_{M_-}$ holds on $\pi^{-1}(V_-)$. Define $V_+ := \pi^{-1}(V_-)$ and $\theta_{V_+} := \pi^* \theta_{V_-}$; then it is clear that θ_{V_+} is a contact form on V_+. Since $(\omega_{V_-}, \theta_{V_-})$ is a stable Hamiltonian structure, it is easy to see the same is true for $(\omega_{V_+}, \theta_{V_+})$. Such a geometric setup clearly makes M_+ the completion of M_+^{in}.

In the case of a reverse simple flip, a similar argument holds since the flip $M_+ \dashrightarrow M_-$ can be regarded as first blowing up the symplectic submanifold $\mathbb{CP}^{n_2} \subset M_+$, and then blowing down along the $\mathbb{CP}^{n_1-1}$ direction. $\qquad\square$

We shall use the convenient notation $(M_\#, \omega_{M_\#})$ for the completions of the boundary connected sums $M_-^{\text{in}} \#_\partial E_+^{\text{in}}$ and $U^{\text{in}} \#_\partial E_+^{\text{in}}$, respectively, in the blow-up and reverse flip cases. Denote by $(V_\#, \xi_\#)$ the boundary of $(M_\#, \omega_{M_\#})$.

Lemma 3.16 *There is a contactomorphism* $\eta\colon (V_+, \xi_+) \xRightarrow{\cong} (V_\#, \xi_\#)$.

Proof By our proof of Lemma 3.15, there is a contactomorphism between (V_+, ξ_+) and (V_-, ξ_-). In particular, $(\partial E_+^{\text{in}}, \xi_{\text{BW}})$ and $(\partial E_-^{\text{in}}, \xi_{\text{BW}})$ are contactomorphic.

In the blow-up case, the contact structure on $(V_\#, \xi_\#)$ comes from the contact connected sum $(V_-, \xi_-) \# (S^{2n-1}, \xi_{\text{std}})$, where ξ_{std} is the standard contact structure on S^{2n-1}, which is contactomorphic to (V_-, ξ_-).

In the reverse flip case, (V_-, ξ_-) is the contact connected sum $(V, \xi) \# (\partial E_-^{\text{in}}, \xi_{\text{BW}})$, while $(V_\#, \xi_\#)$ is the contact connected sum $(V, \xi) \# (\partial E_+^{\text{in}}, \xi_{\text{BW}})$. Since $(\partial E_+^{\text{in}}, \xi_{\text{BW}})$ and $(\partial E_-^{\text{in}}, \xi_{\text{BW}})$ are contactomorphic, so are (V_-, ξ_-) and $(V_\#, \xi_\#)$. $\qquad\square$

Proof of Corollaries 1.7 and 1.8 By the above lemma, we can change the contact hypersurface from (V_+, ξ_+) to $(V_\#, \xi_\#)$ in (M_+, ω_{M_+}) when deforming the symplectic form from ω_{M_+} to $\omega_{M_\#}$. Under this change of contact hypersurface, $(V_\#, \xi_\#) \subset (M_+, \omega_{M_+})$ is still stably filled and the cylindrical end $[1, \infty) \times \partial M_+$ can be identified with $[1, \infty) \times (V_\#, \xi_\#)$. There is a symplectomorphism $\phi\colon (M_+, \omega_{M_+}) \to (M_+, \omega'_{M_+})$ which realizes the above process, and ω_{M_+} is cohomologous to ω'_{M_+}. By Proposition 3.13, we have $SH^*(M_+, \omega_{M_+}) \cong SH^*(M_+, \omega'_{M_+})$ as $\mathbb{K}$–algebras.

On the other hand, the symplectic manifold $(M_\#, \omega_{M_\#})$ can be regarded as (M_+, ω'_{M_+}) with the symplectic structure ω'_{M_+} deformed in M_+^{in} to $\omega_{M_\#}$. Since the symplectic form ω_{M_+} is cohomologous to $\omega_{M_\#}$ by our choice, and ω_{M_+} is cohomologous to ω'_{M_+}, the difference $\omega'_{M_+} - \omega_{M_\#}$ is a compactly supported exact form. Moser's lemma then produces a symplectomorphism

$$\text{(138)} \qquad\qquad \varphi\colon (M_\#, \omega_{M_\#}) \to (M_+, \omega'_{M_+}),$$

and therefore an identification of the Floer complexes $CF^*(H)$ and $CF^*(\varphi^* H)$. Since φ is an identity outside a compact subset of M_+, it follows that after taking direct

limits we get an isomorphism

$$
SH^*(M_\#, \omega_{M_\#}) \cong SH^*(M_+, \omega'_{M_+})
\tag{139}
$$

which preserves the $\mathbb{K}$–algebra structures.

By Theorem 1.6, proved in Section 3.2,

$$
SH^*(M_\#) \cong SH^*(Y) \oplus SH^*(E_+),
\tag{140}
$$

where by Y we mean M_- in the case of blow-ups, and U in the case of reverse flips. By the computations in Section 3.4, $SH^*(E_+) \cong \mathbb{K}^{n_2+1-n_1}$ is semisimple. Combined with our assumption that $SH^*(Y) = 0$ or $SH^*(Y)$ is semisimple, the same is true for $SH^*(M_\#)$. Since we have proved

$$
SH^*(M_+, \omega_{M_+}) \cong SH^*(M_+, \omega'_{M_+}) \cong SH^*(M_\#, \omega_{M_\#})
\tag{141}
$$

as $\mathbb{K}$–algebras, $SH^*(M_+)$ is semisimple. □

3.7 Remark on the general case

Although Corollaries 1.7 and 1.8 are only stated for blow-ups and flips with trivial centers, it is not hard to see the method presented here works when the center Z of the blow-up or flip is a smooth toric Fano variety (with the algebra $QH^*(Z)/\ker c_1(Z)$ being semisimple).

As a simple example which can be extracted essentially from known computations, consider the case when M_- is the total space of the negative vector bundle

$$
\mathcal{O}(-1)^{\oplus(n+1)/2} \to \mathbb{CP}^{(n-1)/2}.
\tag{142}
$$

Since $c_1(M_-) = 0$, it follows from [20] or [36] that $SH^*(M_-) = 0$. By blowing up along the zero section $\mathbb{CP}^{(n-1)/2} \subset M_-$ we obtain another symplectic manifold M_+, which can be identified with the total space of the line bundle $\mathcal{O}(-1,-1) \to \mathbb{CP}^{(n-1)/2} \times \mathbb{CP}^{(n-1)/2}$. Note that $QH^*(\mathbb{CP}^{(n-1)/2})$, the quantum cohomology of the blowing-up center, is semisimple [22]. Using the method of [37], it can be shown that $SH^*(M_+) \cong \mathbb{K}^{(n^2+2n-7)/4}$ is semisimple (the simplest case $n = 3$ is covered by Corollary 4.16 of [37], and the general case follows similarly).

A possible approach to prove Conjecture 1.2 for more general blow-ups and flips (which are not necessarily performed on the cylindrical end) would be to try to generalize the Mayer–Vietoris sequence for symplectic cohomologies established in [13] to the nonexact case. More precisely, let M^{in} be a Liouville domain and $U_1^{\mathrm{in}}, U_2^{\mathrm{in}} \subset M^{\mathrm{in}}$ Li-

ouville cobordisms such that $M^{\mathrm{in}} = U_1^{\mathrm{in}} \cup U_2^{\mathrm{in}}$. Denote by A^{in} the Liouville cobordism $U_1^{\mathrm{in}} \cap U_2^{\mathrm{in}}$; it follows from [13] that there is an exact triangle

(143)

$$
\begin{array}{ccc}
SH^*(M) & \longrightarrow & SH^*(U_1) \oplus SH^*(U_2) \\
& \nwarrow \qquad \swarrow & \\
{[-1]} & SH^*(A) &
\end{array}
$$

Assuming the above long exact sequence holds for stable symplectic fillings, we illustrate here briefly how to adapt it to study the symplectic cohomology in the special case when $M_- \dashrightarrow M_+$ is a blow-up or reverse flip with trivial center $Z = \{\mathrm{pt}\}$. Take U_1^{in} above to be the Lefschetz domain E_+^{in}, and U_2^{in} to be the stable symplectic cobordism $M_+^{\mathrm{in}} \setminus \mathcal{O}(-1)_{\leq 1+\varepsilon}^{\oplus n_1}$, where $\mathcal{O}(-1)_{\leq 1+\varepsilon}^{\oplus n_1}$ is a slight enlargement of the unit ball bundle E_+^{in}. It follows that $A^{\mathrm{in}} = [0, \varepsilon] \times \partial E_+^{\mathrm{in}}$ is a trivial cobordism, therefore one would have $SH^*(A) = 0$ in view of Albers and Kang [4].

In order to compute $SH^*(U_2)$, it would be convenient to change to an alternative symplectic filling of the boundary component $\partial E_+^{\mathrm{in}} \subsetneq \partial U_2^{\mathrm{in}}$ of the symplectic cobordism U_2^{in}. For this we have an obvious choice, which is E_-^{in}. As mentioned in the introduction, one should then expect an isomorphism

(144) $$SH^*(M_-) \cong SH^*(U_2)$$

of symplectic cohomology groups. Note that although in general the symplectic cohomology for symplectic cobordisms depends on the choice of a filling, certain invariance results can be proved when the contact boundary is dynamically convex; see Section 9.5 of [13].

The above procedure should be enough to determine $SH^*(M_+)$ additively. Recovering its ring structure requires more detailed analysis. Note that if the center $Z \subset M_-$ is not toric, then the computation of $SH^*(U_1)$ would be substantially more difficult.

4 Applications

We collect in this section some interesting consequences of Theorems 1.4 and 1.6. In particular, Theorem 1.9 will be proved in Section 4.2.

4.1 Lefschetz manifolds

The Lefschetz condition (Definition 1.3) imposed on completions of stable symplectic fillings is very strong. The condition (i) has been discussed briefly in Section 2.5, and

(ii) is particularly hard to check without explicit computations of $SH^*(M)$. However, Theorem 1.6 and its corollaries still enable us to get some new examples in terms of known ones.

The following is a simple consequence of Theorem 1.6:

Proposition 4.1 *Suppose M^{in} and $(M')^{\mathrm{in}}$ are two Lefschetz domains at the same level j; then their boundary connected sum $M^{\mathrm{in}} \#_\partial (M')^{\mathrm{in}}$ is still a Lefschetz domain at the same level.*

Proof It's easy to see from the Mayer–Vietoris sequence that additively we have

$$(145) \qquad H^*(M \#_\partial M'; \mathbb{K}) \cong \frac{H^*(M; \mathbb{K}) \oplus H^*(M'; \mathbb{K})}{H^*(\mathrm{pt}; \mathbb{K})}.$$

For the product structure, suppose $u\colon S^2 \to M \#_\partial M'$ is any J–holomorphic sphere contributing to the quantum product on $H^*(M \#_\partial M'; \mathbb{K})$. It is clear that its image $\mathrm{im}(u)$ must intersect the handle $H_1(\delta)$ nontrivially. More precisely, we actually have $\mathrm{im}(u) \cap \partial_- H_1(\delta) \neq \varnothing$. Since any such smooth map u can be regarded as a Floer trajectory which is asymptotic to orbits of a Hamiltonian function with very small slope on the cylindrical end of $M \#_\partial M'$, we can apply Lemma 7.2 of [3] to the restriction of u to $u^{-1}(\mathrm{im}(u) \cap M^{\mathrm{in}})$ to exclude its existence. This is possible because the symplectic structure on $M \#_\partial M'$ restricted to a small neighborhood of $H_1(\delta)$ is exact, and the boundary components $\partial_- H_1(\delta)$ are concave with respect to the local Liouville vector fields $Z_\pm$. Note that using an almost complex structure $J \in \mathcal{J}(M)$ which is not necessarily compatible with ω_M does not affect the proof there. From this we see that the quantum product structures on both summands in (145) are unchanged under the boundary connected sum, and, for $a_1 \in H^*(M; \mathbb{K})$ and $a_2 \in H^*(M'; \mathbb{K})$, we have $a_1 \star a_2 = 0$. This completely determines the ring structure of $QH^*(M \#_\partial M')$.

With this realization, $c_1(M \#_\partial M') \in QH^*(M \#_\partial M')$ is given by

$$(146) \qquad (c_1(M), c_1(M')) \in H^*(M; \mathbb{K}) \oplus H^*(M'; \mathbb{K}),$$

so it follows from Theorem 1.6 that $SH^*(M \#_\partial M')$ is the localization of $QH^*(M \#_\partial M')$ at $c_1(M \#_\partial M')$.

On the other hand, it is obvious that $H^{2j}(V \# V'; \mathbb{Q})$ vanishes because of the vanishing of $H^{2j}(V; \mathbb{Q})$ and $H^{2j}(V'; \mathbb{Q})$. Since $c_1(M)^{\star j} \in H^{2j}(M; \mathbb{K})$ and $c_1(M')^{\star j} \in H^{2j}(M'; \mathbb{K})$, it follows that $c_1(M \#_\partial M')^{\star j} \in H^{2j}(M \#_\partial M'; \mathbb{K})$. $\qquad \square$

By our proof of Corollaries 1.7 and 1.8 in Section 3.6, we can translate the above result in terms of blow-ups of points and reverse simple flips.

Corollary 4.2 *If M is the completion of a Lefschetz domain, then so is its blow-up $\mathrm{Bl}_x(M)$ at any point $x \in M \setminus M^{\mathrm{in}}$.*

Proof As we have already noted in the introduction, for the total space of $\mathcal{O}(-1)_{\leq 1} \to \mathbb{CP}^{n-1}$, we can make it into a Lefschetz domain by taking j to be any integer with $1 \leq j \leq n-1$. Replace $QH^*(\mathrm{Bl}_x(M))$ by the Hamiltonian Floer cohomology $HF^*(H_\varepsilon)$ for a Hamiltonian function $H_\varepsilon \colon \mathrm{Bl}_x(M) \to \mathbb{R}$ with sufficiently small slope $\varepsilon > 0$ at infinity. Note that we can choose ε to be so small that under a symplectomorphism ϕ of contact type at infinity, the pullback $\phi^* H_\varepsilon$ still defines the quantum cohomology, namely

$$(147) \qquad HF^*(\phi^* H_\varepsilon) \cong QH^*(\mathrm{Bl}_x(M)).$$

In fact, $\phi^* H_\varepsilon$ has the form $\varepsilon(r - f(v))$ for $r \gg 0$. Since $f(v)$ is bounded, we may assume $f(v) < \frac{r}{2}$, and replace ε by $\frac{\varepsilon}{2}$ whenever necessary. Finally, apply Moser's lemma, we deduce that

$$(148) \qquad QH^*(\mathrm{Bl}_x(M)) \cong QH^*(M \#_\partial \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$$

as algebras over $\mathbb{K}$. The corollary then follows from Proposition 4.1. $\square$

This in particular shows that the manifold $\mathrm{Bl}_S(\mathbb{C}^n)$, obtained by blowing up $\mathbb{C}^n$ at a finite set of distinct points $S \subset \mathbb{C}^n$, is the completion of a Lefschetz domain. One can also take any split vector bundle $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, with $mn_1 \leq n_2 + 1$ and do the blow-ups away from the zero section.

Analogously, we have the following:

Corollary 4.3 *Let $(U^{\mathrm{in}}, \omega_U)$ be a Lefschetz domain at level $j = \lceil \frac{n}{2} \rceil$, and denote by $U \#_\partial E_-$ the completion of the boundary connected sum $U^{\mathrm{in}} \#_\partial E_-^{\mathrm{in}}$. Under the reverse simple flip $U \#_\partial E_- \dashrightarrow M$ along $\mathbb{CP}^{n_1-1} \subset E_-^{\mathrm{in}}$, M^{in} is also a Lefschetz domain.*

Proof This is similar to the proof of Corollary 4.2, except that in this case condition (i) in Definition 1.3 may not be satisfied for E_+ for any choice of j with $1 \leq j \leq n-1$. However, it follows from Corollary 3.11 that E_+^{in} is a Lefschetz domain at level $\lceil \frac{n}{2} \rceil$. $\square$

Note that the Lefschetz domain U^{in} in the above can be taken to be the total space of $\mathcal{O}(-m)_{\leq 1}^{\oplus n_1} \to \mathbb{CP}^{n_2}$, where $mn_1 \leq n_2 + 1$, by the computations in Section 3.4. We

can actually go beyond this case a little bit. For instance, take the blow-up of E_- at any point x away from the zero section $\mathbb{CP}^{n_1-1}$, and then perform a reverse simple flip $\mathrm{Bl}_x(E_-) \dashrightarrow M$ along $\mathbb{CP}^{n_1-1} \subset E_-$. The resulting manifold M is also Lefschetz.

Let U_1^{in} and U_2^{in} be Lefschetz domains. An embedding $\iota: U_1^{\mathrm{in}} \hookrightarrow U_2^{\mathrm{in}}$ is a *Lefschetz embedding* if it is a symplectomorphism of contact type onto its image. Following [40], one can introduce the following notion:

Definition 4.4 Let M be a noncompact symplectic manifold without boundary. M is a Lefschetz manifold if there exists a sequence of Lefschetz domains $\{U_k^{\mathrm{in}}\}$ and embeddings $\iota_k: U_k^{\mathrm{in}} \hookrightarrow M$, which are symplectomorphisms of contact type onto its image, such that the images $\{\iota_k(U_k^{\mathrm{in}})\}$ exhaust M.

Clearly, every completion of a Lefschetz domain is a Lefschetz manifold. On the other hand, there are certainly Lefschetz manifolds which do not come from completion. For example, one can take $M = \mathrm{Bl}_{\mathbb{Z}^{2n}}(\mathbb{C}^n)$ to be the blow-up of $\mathbb{C}^n$ at every point in $\mathbb{Z}^{2n} \subset \mathbb{C}^n$.

With some effort, the Viterbo functoriality [49] can be generalized to the current case. Namely, for every Lefschetz embedding $\iota: U_1^{\mathrm{in}} \hookrightarrow U_2^{\mathrm{in}}$ between Lefschetz domains, there is a pullback map

$$\tag{149} \mathcal{T}_\iota: SH^*(U_2) \to SH^*(U_1).$$

Using this one can associate to every Lefschetz manifold M its symplectic cohomology, by taking the inverse limit

$$\tag{150} SH^*(M) := \varprojlim_k SH^*(U_k).$$

As an example, it follows from Proposition 4.9 below that

$$\tag{151} SH^*(\mathrm{Bl}_{\mathbb{Z}^{2n}}(\mathbb{C}^n)) \cong \prod_{\alpha \in \mathbb{Z}^{2n}} \mathbb{K}[x_\alpha]/(x_\alpha^n + q^n),$$

which shows that the symplectic cohomology of a Lefschetz manifold need not be finite-dimensional.

4.2 Split-generation

Let M be a monotone symplectic manifold obtained by completing the stable filling of the contact manifold (V, ξ). Using the algebraic tools introduced in Section 2, one can prove the following:

Theorem 4.5 (Ritter and Smith [38]) *Let $\mathcal{B} \subset \mathcal{F}_\lambda(M)$ be an full A_∞ subcategory. If the open–closed map*

$$(152) \qquad\qquad OC\colon HH_*(\mathcal{B}, \mathcal{B}) \to QH^*(M)_\lambda$$

hits an invertible element, then $\mathcal{B}$ split-generates $D^\pi \mathcal{F}_\lambda(M)$. Similarly, a full A_∞ subcategory $\mathcal{B} \subset \mathcal{W}_\lambda(M)$ split-generates $D^\pi \mathcal{W}_\lambda(M)$ if

$$(153) \qquad\qquad OC\colon HH_*(\mathcal{B}, \mathcal{B}) \to SH^*(M)_\lambda$$

hits an invertible element of $SH^*(M)_\lambda$.

This is proved in [38] when M^{in} is a strong filling, and extends easily to the form stated above once the relevant maximum principles have been established. See Sections 2.1 and 2.2. The key ingredient of the proof is a breaking analysis of the moduli space of marked annuli, namely the one involved in the Cardy relation (Proposition 2.13).

For the purpose of proving Theorem 1.9, we need to extend the Fukaya categories $\mathcal{W}(M)$ and $\mathcal{F}(M)$ by allowing as their objects the Lagrangian branes (L, ξ_L), where $L \subset M$ is an admissible or closed admissible Lagrangian submanifold, and $\xi_L \in \mathrm{Hom}(\pi_1(L), U_{\mathbb{K}})$ determines a unitary rank 1 local system with the fiber at the point $l \in L$ given by $\mathbb{K}_l^L \cong \mathbb{K}$. Under this extension, we treat the underlying Lagrangian submanifold L as a Lagrangian brane equipped with a trivial local system.

One can generalize the definition (34) of $\mathfrak{m}_0$ to the admissible Lagrangian brane (L, ξ_L) by

$$(154) \qquad \mathfrak{m}_0(L, \xi_L) = \sum_{\beta \in \pi_2(M,L)} q^{\omega_M(\beta)} \xi_L(\partial\beta) \, \mathrm{ev}_*[\mathcal{M}_1(L, \beta)] = m_0(L, \xi_L)[L],$$

where we have identified ξ_L with a cohomology class of $H^1(L; U_{\mathbb{K}})$, so $\xi_L(\partial\beta) \in U_{\mathbb{K}}$.

Fix two Lagrangian branes (L_0, ξ_{L_0}) and (L_1, ξ_{L_1}) with $m_0(L_0, \xi_{L_0}) = m_0(L_1, \xi_{L_1})$; the morphism spaces between them are given by the Floer complex

$$(155) \quad CF^*((L_0, \xi_{L_0}), (L_1, \xi_{L_1}); wH) := \bigoplus_{x \in CF^*(L_0, L_1; wH)} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}_{x(0)}^{L_0}, \mathbb{K}_{x(1)}^{L_1}),$$

where the direct sum ranges over all Hamiltonian chords of X_H with weight $w \in \mathbb{Z}$. The term $q^{\omega_M([u])}x$ in the original Floer differential ∂y is now deformed to be

$$(156) \qquad\qquad q^{\omega_M([u])} h_{\ell_1}^{L_1} \circ o_y \circ h_{\ell_0}^{L_0} \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}_{x(0)}^{L_0}, \mathbb{K}_{x(1)}^{L_1})$$

in the Floer differential of $o_y \in \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}_{y(0)}^{L_0}, \mathbb{K}_{y(1)}^{L_1})$, where $\ell_0 \colon [0, 1] \to L_0$ and $\ell_1 \colon [0, 1] \to L_1$ are paths swept out by $u(\partial \mathbb{D})$ along the oriented arcs connecting the boundary punctures -1 to 1, and 1 to -1, respectively. The $h_{\ell_i}^{L_i} \colon \mathbb{K}_{\ell_i(0)}^{L_i} \to \mathbb{K}_{\ell_i(1)}^{L_i}$ are the corresponding parallel transport maps. With this definition, it is easy to see the original Fukaya categories embed fully faithfully into the corresponding extensions. For convenience we shall denote the extended Fukaya categories still by their original notations $\mathcal{W}(M)$ and $\mathcal{F}(M)$.

The definitions of open–closed string maps can also be generalized to the current case so that the generation criterion stated above still holds for the extended version of Fukaya categories. In particular, the zeroth-order open–closed maps

$$\text{(157)} \qquad\qquad \mathrm{OC}^0 \colon HF^*((L, \xi_L), (L, \xi_L)) \to QH^{*+n}(M)$$

and

$$\text{(158)} \qquad\qquad \mathrm{OC}^0 \colon HW^*((L, \xi_L), (L, \xi_L)) \to SH^{*+n}(M)$$

are $\mathbb{K}$–algebra homomorphisms.

Proof of Theorem 1.9   From now on we restrict ourselves to the special case when M^{in} is a monotone Lefschetz domain. With all the preliminaries at hand, Theorem 1.9 is a simple corollary of the generation criterion. Notice that the argument presented in Section 2.4 for Theorem 1.4 still holds after replacing the collection of monotone Lagrangian submanifolds $(L_1, \ldots, L_r)$ with a collection of Lagrangian branes

$$((L_1, \xi_{L_1}), \ldots, (L_r, \xi_{L_r})).$$

The same method as in Proposition 2.12 shows that $\mathrm{OC}^0([\mathrm{pt}]) \neq 0$ in $QH^0(M)$ for $\mathrm{pt} \in L$. By assumption, (L_i, ξ_{L_i}) and (L_j, ξ_{L_j}) lie in different eigensummands of $\mathcal{F}(M)$ whenever $L_i \cap L_j \neq \varnothing$, so by fixing a particular eigensummand $\mathcal{F}_\lambda(M)$ with $\lambda \neq 0$, we only need to deal with disjoinable Lagrangians L_i and L_j with $HF^*((L_i, \xi_{L_i}), (L_j, \xi_{L_j})) = 0$. This reduces the problem to the original setting of Theorem 1.4.

Denote by $\mathcal{B}$ the full A_∞ subcategory of $\mathcal{F}(M)$ formed by the Lagrangian branes $(L_1, \xi_{L_1}), \ldots, (L_r, \xi_{L_r})$ with $m_0(L_i, \xi_{L_i}) \neq 0$. By our assumption, the even degree part of the nonzero eigensummand $QH^0(M)_{\neq 0}$ is commutative and semisimple, therefore isomorphic to $\bigoplus_{i \in I} \mathbb{K} v_i$, with I a finite index set and $(v_i)_{i \in I} \subset QH^*(M)$ a

collection of idempotents. Since $r = \dim_{\mathbb{K}} SH^0(M)$ and M is Lefschetz, the image of $HH_*(\mathcal{B}, \mathcal{B})$ under the open–closed map OC contains an element of the form

$$(159) \qquad (a_1, \ldots, a_{|I|}) \in QH^0(M)_{\neq 0} \cong \bigoplus_{i \in I} \mathbb{K} v_i,$$

where $a_i \neq 0$ for every i. This defines an invertible element when restricted to every eigensummand $QH^*(M)_\lambda$ with $\lambda \neq 0$. Denote by $\mathcal{B}_\lambda \subset \mathcal{B}$ the full A_∞ subcategory which belongs to the A_∞ category $\mathcal{F}_\lambda(M)$; it follows from Theorem 4.5 that $\mathcal{B}_\lambda$ split-generates $D^\pi \mathcal{F}_\lambda(M)$.

By the acceleration diagram (58), we further have $\mathrm{OC}(HH_*(\mathcal{B}_\lambda, \mathcal{B}_\lambda))$ hits an invertible element of $SH^*(M)_\lambda$, which proves the split-generation of $D^\pi \mathcal{W}_\lambda(M)$ by $\mathcal{B}_\lambda$. □

4.3 Lagrangian tori

Theorem 1.9 proved in the last subsection is useful in finding explicit generators of Fukaya categories for monotone Lefschetz domains. As an application, we discuss three examples here, and show that certain eigensummands of their derived wrapped Fukaya categories are split-generated by Lagrangian tori (equipped with local systems).

Negative vector bundles Let M be the total space of the negative vector bundle $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, where $m \geq 1$, $mn_1 \leq n_2$ and $n_1 + n_2 = n$. Since M is toric, we may apply standard toric techniques to find generators of the nonzero eigensummands of the Fukaya categories $\mathcal{F}(M)$ and $\mathcal{W}(M)$. Since the situation here is very similar to that of Section 12 of [38], our discussions below will be quite sketchy.

We first remark that the monotone symplectic form ω_M is taken so that its restriction to the base is $mn_1 \cdot \omega_{\mathrm{FS}}$, where ω_{FS} is the Fubini–Study metric on $\mathbb{CP}^{n_2}$. From this we deduce the monotonicity constant for M,

$$(160) \qquad \lambda_M = \frac{1 + n_2 - mn_1}{mn_1}.$$

Let $\mu_M \colon M \to \Delta_M \subset \mathbb{R}^n$ be the moment map with respect to the standard T^n–action on M. The moment polytope Δ_M is given by

$$(161) \quad \Delta_M =$$
$$\left\{ y \in \mathbb{R}^n \;\middle|\; y_1 \geq 0, \ldots, y_n \geq 0, \; \sum_{j=1}^{n_2} y_j - my_{n_2+1} \leq m, \ldots, \sum_{j=1}^{n_2} y_j - my_n \leq m \right\}.$$

From this one can write down the superpotential $W\colon (\mathbb{K}^*)^n \to \mathbb{K}$,

$$(162) \qquad W(z) = \sum_{j=1}^{n} z_j + q^{n_2+1-mn_1} z_1^{-1} \cdots z_{n_2}^{-1} z_{n_2+1}^{m} \cdots z_n^{m}.$$

One can check that for every solution $x = q(-m)^{1/\lambda_M}$ of the equation

$$(163) \qquad x^{n_2+1-mn_1} = q^{n_2+1-mn_1}(-m)^{mn_1},$$

there is a critical point

$$(164) \qquad z_c = (x, \ldots, x, -mx, \ldots, -mx),$$

whose critical value is

$$(165) \qquad W(z_c) = (n_2 - mn + 1)x.$$

A consequence of these computations is the following closed string homological mirror symmetry statement, which generalizes Corollary 12.11 of [38]:

Proposition 4.6 *For M the total space of $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, with $mn_1 \leq n_2$, we have the ring isomorphism*

$$(166) \qquad SH^*(M) \cong \mathrm{Jac}(W).$$

Proof Recall that the Jacobi ring $\mathrm{Jac}(W)$ is defined by

$$(167) \qquad \mathrm{Jac}(W) \cong \mathbb{K}[z_1^{\pm}, \ldots, z_n^{\pm}]/(\partial_{z_1}W, \ldots, \partial_{z_n}W).$$

This can be explicitly computed using the expression (162), and the result follows by comparing $\mathrm{Jac}(W)$ with our computations of $SH^*(M)$ in Section 3.4.   □

We now turn our attention to Lagrangian submanifolds of M. Analogous to the closed toric Fano case, we show that there is a nondisplaceable Lagrangian torus in M, which is a fiber of μ_M.

Proposition 4.7 *There is a monotone Lagrangian torus $L \subset M$, which is a T^{n_1}–bundle over the Clifford torus $T_{\mathrm{Cl}}^{n_2} \subset \mathbb{CP}^{n_2}$ such that $HF^*(L, L) \neq 0$ and $\mathrm{pt} \in C_*(L)$ defines a Floer cocycle. Moreover, it can be equipped with $n_2 + 1 - mn_1$ different local systems $\xi_L^1, \ldots, \xi_L^{n_2+1-mn_1} \in \mathrm{Hom}(\pi_1(L), U_{\mathbb{K}})$, such that*

$$(168) \qquad HF^*((L, \xi_L^i), (L, \xi_L^i)) \neq 0, \quad i = 1, \ldots, n_2 + 1 - mn_1.$$

Proof Making the change of variable $y = q^{mn_1 \lambda_M} z_{n_2+1}^m \cdots z_n^m$ in the expression of the superpotential (162), we get

$$(169) \qquad W(z) = W_{\mathbb{P}}(y) + z_{n_2+1} + \cdots + z_n,$$

where $W_{\mathbb{P}}(y)$ is the superpotential for the Landau–Ginzburg model of $\mathbb{CP}^{n_2}$, which has the form

$$(170) \qquad W_{\mathbb{P}}(y) = z_1 + \cdots + z_{n_2} + \frac{y}{z_1 \cdots z_{n_2}}.$$

Recall that $W_{\mathbb{P}}(y)$ has $n_2 + 1$ critical points with nonzero critical values. Given any critical point $z(y)$ of $W_{\mathbb{P}}(y)$, a detailed computation shows that there is a critical point $z_c = (z(y), z_{n_2+1}, \ldots, z_n)$ of W. Under the valuation map $\mathrm{val}_q \colon (\mathbb{K}^*)^n \to \mathbb{R}^n$, the critical point z_c is mapped to a point $\mathrm{val}_q(z_c) \in \Delta_M$, and the fiber of μ_M over this point is exactly our monotone Lagrangian torus L.

Geometrically, consider the Hamiltonian T^{n_1}–action given by rotations along the fibers of the vector bundle $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, which gives rise to a moment map $\nu_M \colon M \to \mathbb{R}^{n_1}$. The monotone Lagrangian torus $L \subset M$ is then given by the image of the Clifford torus $T_{\mathrm{Cl}}^{n_2} \subset \mathbb{CP}^{n_2}$ under the monotone Lagrangian correspondence

$$(171) \qquad \nu_M^{-1}(\lambda_L) \subset (\mathbb{CP}^{n_2}, -\omega_{\mathrm{FS}}) \times (M, \omega_M)$$

for some $\lambda_L \in \mathbb{R}^{n_1}$ corresponding to the last n_1 coordinates of the critical point z_c. Note that $\nu_M^{-1}(\lambda_L)$ is topologically a T^{n_1}–bundle over $\mathbb{CP}^{n_2}$.

It follows from the general result of Cho [11] on the Floer cohomologies of toric fibers that $HF^*(L, L) \neq 0$ and $\mathrm{pt} \in C_*(L)$ defines a cocycle in $HF^*(L, L)$. The choices of local systems correspond to the ambiguity in choosing a solution of (163). In fact, $m_0(L, \xi_L^i)$ equals the critical value $W(z_c)$ of the corresponding critical point z_c of W. $\qquad \square$

Since $\dim_{\mathbb{K}} SH^*(M) = n_2 + 1 - mn_1$ by the computations in Section 3.4, as a corollary of Theorem 1.9 we have:

Theorem 4.8 Let M be the total space of $\mathcal{O}(-m)^{\oplus n_1} \to \mathbb{CP}^{n_2}$, where $mn_1 \leq n_2$. The nonzero eigensummands $\bigsqcup_{\lambda \neq 0} D^\pi \mathcal{F}_\lambda(M)$ of the split-closed derived Fukaya category and those of the split-closed derived wrapped Fukaya category $\bigsqcup_{\lambda \neq 0} D^\pi \mathcal{W}_\lambda(M)$ of M are split-generated by the Lagrangian branes (L, ξ_L^i) for $i = 1, \ldots, n_2 + 1 - mn_1$.

Blow-ups of symplectic vector spaces The material here is a generalization of [26, Section 5.1]. Consider the symplectic blow-up of $\mathbb{C}^n$ at a finite set of points $S = \{q_1, \ldots, q_m\}$; as we have noticed in Section 4.1, $M = \mathrm{Bl}_S(\mathbb{C}^n)$ is a Lefschetz manifold. Its symplectic cohomology can be computed using Theorem 1.6 and known results.

Proposition 4.9 *Let* $M = \mathrm{Bl}_S(\mathbb{C}^n)$; *then*

$$(172) \qquad SH^*(M) \cong \bigoplus_{i=1}^m \mathbb{K}[x_i]/(x_i^n + q^n \cdot n).$$

Proof Let $M_\#$ be the boundary connected sum of m copies of the unit ball bundle associated to $\mathcal{O}(-1) \to \mathbb{C}\mathbb{P}^{n-1}$. By Theorem 1.6 and the computation of $SH^*(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ in [36], we have that $SH^*(M_\#) \cong \bigoplus_{i=1}^m \mathbb{K}[x_i]/(x_i^n + q^n)$ is semisimple. On the other hand, by our proof of Corollaries 1.7 and 1.8 in Section 3.6, $SH^*(M) \cong SH^*(M_\#)$. $\square$

For the purpose of dealing with Fukaya categories, we shall blow up with equal amounts at every point of S, so that M is monotone. To simplify our exposition, we shall impose the following two simplifying assumptions:

 (i) $S \subset \mathbb{C}$ lies on a common complex plane.
 (ii) $\mathrm{dist}_{\mathbb{C}}(q_i, q_j) \gg 0$ for any $i \neq j$.

Note that (i) in the above gives us a Morse–Bott fibration

$$(173) \qquad\qquad \pi \colon M \to \mathbb{C}.$$

This is obtained by starting with the trivial projection $\mathbb{C}^n \to \mathbb{C}$ to the complex plane containing S, and then attaching an exceptional $\mathbb{C}\mathbb{P}^{n-1}$ to the fiber over the points in S. As a consequence, the critical loci of π form a disjoint union of m copies of $\mathbb{C}\mathbb{P}^{n-2}$. We remark that once there is such a Morse–Bott fibration π on M, a generalization of the method developed in [5; 31] for studying symplectic cohomologies of Lefschetz fibrations is possible, which reduces the computation of $SH^*(M)$ on the additive level to the Morse complexes of the critical loci of π.

(ii) is needed to ensure that M can be equipped with a symplectic form ω_M so that every exceptional divisor $\mathbb{C}\mathbb{P}^{n-1} \subset M$ has area π.

Since (M, ω_M) is the completion of a monotone Lefschetz domain, Theorem 1.4 applies to M. Namely, for $(L_1, \ldots, L_r)$ a set of wide monotone Lagrangian submanifolds of M which are disjoinable by Hamiltonian isotopies, we have

$$(174) \qquad\qquad r \leq m(n-1).$$

We shall find a set of monotone Lagrangian branes in M which satisfies the assumptions in Theorem 1.9 and realizes the upper bound of (174). To do this, take the unit ball bundles $E_+^{\text{in}} \subset M$ of every exceptional divisor; their boundaries ∂E_+^{in} are circle bundles over $\mathbb{CP}^{n-1}$. Taking the circle bundle over the Clifford torus $T_{\text{Cl}}^{n-1} \subset \mathbb{CP}^{n-1}$ gives us a Lagrangian torus $L_i \subset M$, so altogether we get m disjoint Lagrangian tori $L_1, \ldots, L_m \subset M$.

Under the Morse–Bott fibration $\pi\colon M \to \mathbb{C}$, these Lagrangian tori project to circles $\gamma_1, \ldots, \gamma_m \subset \mathbb{C}$ with the same radii. Taking the Clifford torus $T_{\text{Cl}}^{n-2} \subset \mathbb{CP}^{n-2}$ in every connected component of the critical loci of π, the corresponding *relative vanishing cycles* (see [44]) of π are given by the standard product tori $T_{\text{Cl}}^{n-2} \times S^1 \subset \mathbb{C}^{n-1}$, so L_i can be regarded as a matching torus of π.

Lemma 4.10 *The Lagrangian tori $L_1, \ldots, L_m$ are monotone, and $HF^*(L_i, L_i) \neq 0$ with $\mathrm{pt} \in C_*(L_i)$ defining a Floer cocycle for every i. Moreover, one can equip each L_i with $n-1$ different local systems $\xi_{L_i}^1, \ldots, \xi_{L_i}^{n-1}$ such that*

$$(175) \qquad HF^*((L_i, \xi_{L_i}^j), (L_i, \xi_{L_i}^j)) \neq 0, \quad 1 \leq j \leq n-1.$$

Proof The fact that L_i is monotone follows conceptually from the fact that L_i is the image of $T_{\text{Cl}}^{n-1} \subset \mathbb{CP}^{n-1}$ under the monotone Lagrangian correspondence given by the boundary of the unit sphere bundle $\partial E_+^{\text{in}} \subset M$. More concretely, let $u\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, L_i)$ be a J–holomorphic disc bounded by L_i, where we take J to be the standard complex structure on M. Applying the maximum principle to $\pi \circ u$, one sees that the image of u lies inside $\pi^{-1}(D_i)$, where $D_i \subset \mathbb{C}$ is the disc bounded by γ_i. This enables us to identify the moduli spaces $\mathcal{M}_1^M(L_i, \beta_i)$ with the moduli space $\mathcal{M}_1^{\mathcal{O}(-1)}(L, \beta)$, where the superscripts indicate the symplectic manifolds that contain the relevant Lagrangian submanifolds, $\beta_i \in \pi_2(M, L_i)$ and $\beta \in \pi_2(\mathcal{O}_{\mathbb{P}^{n-1}}(-1), L)$, and $L \subset \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ is the nondisplaceable monotone Lagrangian torus corresponding to the critical points of the mirror superpotential; see Proposition 4.7. Note that the moduli space $\mathcal{M}_1^{\mathcal{O}(-1)}(L, \beta)$ is regular with respect to the standard complex structure J, since $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ is toric. This identification makes use of the Morse–Bott fibration $\pi\colon \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathbb{C}$, which is obtained as a specialization of (173). Because of this, the monotonicity of L_i follows from that of L. This also enables us to identify the pearl complexes of L_i and L computing the Floer cohomologies [7], so the rest of the lemma follows from Proposition 4.7. □

When $n = 2$, this in particular shows that the bound obtained in Theorem 1.4 for the number of disjoinable monotone wide Lagrangian submanifolds is sharp for $\mathrm{Bl}_S(\mathbb{C}^2)$. Combining Lemma 4.10 with Proposition 4.9 and Theorem 1.9, we get the following:

Theorem 4.11 *Let M be the monotone Lefschetz manifold $\mathrm{Bl}_S(\mathbb{C}^n)$. The nonzero eigensummands of the derived Fukaya categories*

$$\bigsqcup_{\lambda \neq 0} D^\pi \mathcal{F}_\lambda(M) \quad and \quad \bigsqcup_{\lambda \neq 0} D^\pi \mathcal{W}_\lambda(M)$$

are split-generated by the Lagrangian branes

$$(L_1, \xi_{L_1}^1), \ldots, (L_1, \xi_{L_1}^{n-1}), \ldots, (L_m, \xi_{L_m}^1), \ldots, (L_m, \xi_{L_m}^{n-1}).$$

A reverse flip   As in Section 4.1, denote by M the $2n$–dimensional Lefschetz manifold obtained by the reverse simple flip

$$(176) \qquad\qquad\qquad\qquad \mathrm{Bl}_S(E_-) \dashrightarrow M,$$

where $S \subset E_- \setminus E_-^{\mathrm{in}}$ is a finite set of points with $|S| = m$ on the cylindrical end of E_- such that for every point $p \in S$ there is a symplectic embedding $\mathbb{B}_p(\sqrt{2\pi}) \hookrightarrow E_-$, where $\mathbb{B}_p(\sqrt{2\pi})$ is a ball with radius $\sqrt{2\pi}$ centered at p and $\mathbb{B}_p(\sqrt{2\pi}) \cap \mathbb{B}_q(\sqrt{2\pi}) = \varnothing$ for any two different points $p, q \in S$. It's then clear that M can be equipped with a monotone symplectic form ω_M so that the symplectic area of any rational curve coming from blow-up is π. Note that this then forces that ω_M restricted to $\mathbb{CP}^{n_2} \subset M$ equals $((n-1)/(n_2 - n_1 + 1))\omega_{\mathrm{FS}}$. In particular, as the completion of a stable filling, the monotone Fukaya category $\mathcal{F}(M)$ and the monotone wrapped Fukaya category $\mathcal{W}(M)$ of M are well defined. By Theorem 1.6, we know that $SH^*(M)$ is semisimple and

$$(177) \qquad\qquad\qquad \dim_{\mathbb{K}} SH^*(M) = n_2 + 1 - n_1 + m(n-1).$$

From the above discussions, it is easy to deduce the following:

Theorem 4.12   *There are $m + 1$ monotone Lagrangian tori $L_0; L_1, \ldots, L_m \subset M$ such that there exist unitary local systems*

$$(178) \qquad \begin{aligned} \xi_{L_0}^1, \ldots, \xi_{L_0}^{n_2+1-n_1} &: \pi_1(L_0) \to U_{\mathbb{K}}, \\ \xi_{L_i}^1, \ldots, \xi_{L_i}^{n-1} &: \pi_1(L_i) \to U_{\mathbb{K}}, \quad i = 1, \ldots, m, \end{aligned}$$

such that the Lagrangian branes

$$(L_0, \xi_{L_0}^1), \ldots, (L_0, \xi_{L_0}^{n_2+1-n_1}),$$
$$(L_1, \xi_{L_1}^1), \ldots, (L_1, \xi_{L_1}^{n-1}), \ldots, (L_m, \xi_{L_m}^1), \ldots, (L_m, \xi_{L_m}^{n-1})$$

split-generate the nonzero eigensummands of $D^\pi \mathcal{F}(M)$ *and* $D^\pi \mathcal{W}(M)$.

Proof Instead of relying on the Morse–Bott fibration, we can use Lemma 2.8 to localize the analysis of Maslov 2–holomorphic discs to domains isomorphic to $\mathcal{O}(-1)_{\leq 1}$ or $\mathcal{O}(-1)_{\leq 1}^{\oplus n_1}$. This ensures that the Lagrangian tori $L_1, \ldots, L_m$ created by blowing up at S are monotone and nondisplaceable. On the other hand, although we have changed the monotonicity constant to $n - 1$, it is not hard to check the argument above for negative vector bundles still holds, and can be applied to the Lagrangian torus L_0 created by the reverse flip. The result then follows from Theorems 4.8 and 4.11. $\square$

4.4 Mirror symmetry

The monotone Lefschetz manifold $M = \mathrm{Bl}_S(\mathbb{C}^2)$ that appears above is of particular interest since the mirror construction due to Abouzaid, Auroux and Katzarkov [2] can be applied to it to get its mirror Landau–Ginzburg model $(M^\vee, W)$.

We briefly recall here the geometry of the mirror $(M^\vee, W)$; the details can be found in Section 9 of [2]. Denote by $\overline{M}^\vee$ the resolution of the (A_{m-1}) singularity $\{xy = z^m\} \subset \mathbb{K}^3$. Note that as a toric Calabi–Yau surface, $\overline{M}^\vee$ can be covered by m affine coordinate charts $\overline{U}_i \cong \mathbb{K}^2$ with coordinates (x_i, y_i) for $1 \leq i \leq m$, and z defines a regular function on $\overline{M}^\vee$ which restricts to $q^\varepsilon x_i y_i - q^\varepsilon$ on each $\overline{U}_i$, where $q^\varepsilon \in \mathbb{K}$ is some constant. $M^\vee$ is the complement of the surface $\{z = 1\} \subset \overline{M}^\vee$.

Denote by x the regular function on $\overline{M}^\vee$ defined by x_1, or its restriction on $M^\vee$; then the superpotential $W \colon M^\vee \to \mathbb{K}$ is given by $x + z$. Denote by $D_{\mathrm{sing}}^\pi(W^{-1}(-q^\varepsilon))$ the split-closure of the triangulated category of singularities defined in [33], where $-q^\varepsilon$ is the unique critical value of W. The following statement concerning homological mirror symmetry is easy to prove:

Proposition 4.13 *Let* $\mathcal{W}_\lambda(M)$ *denote the unique nonzero summand of the monotone wrapped Fukaya category of M. There is an equivalence between triangulated categories*

$$(179) \qquad\qquad D^\pi \mathcal{W}_\lambda(M) \cong D_{\mathrm{sing}}^\pi(W^{-1}(-q^\varepsilon)).$$

Proof The wrapped Fukaya category $\mathcal{W}_\lambda(M)$ has been computed in the last subsection. In this case, $D^\pi \mathcal{W}_\lambda(M)$ is split-generated by the Lagrangian tori $L_1, \ldots, L_m$ in Theorem 4.11, and

(180) $$HW^*(L_i, L_i) \cong HF^*(L_i, L_i) \cong \mathrm{Cl}_2$$

are isomorphic to the Clifford algebra.

On the other hand, $M^\vee$ is covered by m affine charts $U_i \cong \mathbb{K}^2 \setminus \{x_i y_i = 1\}$, and the superpotential W on each chart U_i has the form

(181) $$W_i = x_i^i y_i^{i-1} + q^\varepsilon x_i y_i - q^\varepsilon,$$

where $q^\varepsilon \in \mathbb{K}^*$. Elementary calculations show that $W \colon M^\vee \to \mathbb{K}$ has m nondegenerate critical points $p_1, \ldots, p_m$. (Note that the critical point $(-q^\varepsilon, 0)$ on U_2 can be identified with the critical point $(0, -q^{-\varepsilon})$ on U_1; all the other critical points are origins of the charts $U_2, \ldots, U_m$.) It follows that the triangulated category $D^\pi_{\mathrm{sing}}(W^{-1}(-q^\varepsilon))$ is split-generated by skyscraper sheaves $\mathcal{O}_{p_i}$ for $1 \le i \le m$. It's not hard to compute $\mathrm{Hom}(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$ in $D^\pi_{\mathrm{sing}}(W^{-1}(-q^\varepsilon))$, from which one sees that it is isomorphic to $HW^*(L_i, L_i)$. Note that the computation of $\mathrm{Hom}(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$ is the simplest instance of Knörrer periodicity, which holds over any field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. $\square$

More generally, the monotone wrapped Fukaya category should play a role in understanding the mirror symmetry for blow-ups of toric varieties in higher dimensions [2], which will then involve blowing up along noncompact submanifolds. For example, let $\mathbb{S} \subset \mathbb{C}^3$ be the disjoint union of two complex planes,

(182) $$\mathbb{C}_x \sqcup \mathbb{C}_y \subset \mathbb{C}^2_{x,y} \times \{z_1\} \sqcup \mathbb{C}^2_{x,y} \times \{z_2\} \subset \mathbb{C}^3_{x,y,z},$$

with $z_1 \ne z_2$. Then $M = \mathrm{Bl}_{\mathbb{S}}(\mathbb{C}^3)$ can be equipped with a monotone symplectic structure. It has a mirror Landau–Ginzburg model $(M^\vee, W)$ described in Section 11 of [2], where $M^\vee$ is an open dense subset of the resolved conifold $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{CP}^1$. In fact, algebraically, M can be realized as a partial compactification of T^*S^3. The symplectic topology of these manifolds will be studied elsewhere.

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