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Equivariant loops on classifying spaces

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We compute the homology of the space of equivariant loops on the classifying space of a simplicial monoid M with anti-involution, provided $\pi_0(M)$ is central in the homology ring of M. The proof is similar to McDuff and Segal's proof of the group completion theorem. Then we give an analogous computation of the homology of the C_2–fixed points of a Γ–space-type delooping of an additive category with duality with respect to the sign circle. As an application we show that this fixed-point space is sometimes group complete, but in general not.

55P35, 55P48, 55P91

1 Introduction

It is well known that for a grouplike simplicial monoid M the natural map

$$\lambda_M \colon |M| \to \Omega|BM|$$

is a weak homotopy equivalence, where B denotes the bar construction and $|-|$ denotes geometric realization. In the nongrouplike case the classical group completion theorem of McDuff and Segal [8] and Quillen [11, Theorem Q.4] states that for a simplicial monoid M satisfying certain conditions, λ_M induces an isomorphism of $H_*(M)$–modules,

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(\Omega|BM|).$$

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The first goal of this paper is to investigate the corresponding situation when M has an anti-involution, ie an involution $\overline{(-)}\colon M \to M$ such that $\overline{(mn)} = \overline{n}\overline{m}$. This extra structure induces a natural action on $|BM|$ of the cyclic group C_2 of order two.

We let $\mathbb{R}^{1,1}$ denote the sign-representation of C_2 on $\mathbb{R}$ and write $S^{1,1}$ for its one-point compactification. For a pointed C_2–space X we write $\Omega^{1,1}X$ for the space $\mathrm{Map}_*(S^{1,1}, X)$ with the conjugation action of C_2. In his Ph.D. thesis [18], Nisan Stiennon has shown that λ_M is in fact an equivariant map

$$|M| \to \Omega^{1,1}|BM|,$$

and that if M is grouplike, then the induced map on fixed points

$$|M|^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}$$

is a weak equivalence. In view of the group completion theorem it is natural to ask what happens when M is not grouplike. The simplicial monoid M acts from the left on the fixed points M^{C_2} by the formula $(m,n) \mapsto m \cdot n \cdot \overline{m}$, and homotopy coherently on the equivariant loop space $(\Omega^{1,1}|BM|)^{C_2}$ in a similar way. Note that the fixed points M^{C_2} will usually not inherit a monoid structure from M, only a left M–action. Using these actions we provide an analog of the group completion theorem:

Theorem 3.12 *Let M be a simplicial monoid with anti-involution such that $\pi_0 M$ is in the center of $H_*(M)$. Then the map*

$$\lambda_M^{C_2}\colon |M|^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}$$

induces an isomorphism

$$\pi_0(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} \pi_0(\Omega^{1,1}|BM|)^{C_2}$$

of left $\pi_0(M)$–sets and an isomorphism of left $H_*(M)$–modules

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*((\Omega^{1,1}|BM|)^{C_2}).$$

In [16, Section 4], Segal proved a variant of the group completion theorem for Γ–spaces. Shimakawa [17] later considered the G–equivariant situation for G a finite group. He described an equivariant delooping machine in terms of Γ_G–spaces and proved a group completion statement for these deloopings provided one is delooping with respect to a representation sphere S^W such that $W^G \neq 0$. In this paper we consider the case $G = C_2$ and $W = \mathbb{R}^{1,1}$. Since $(\mathbb{R}^{1,1})^{C_2} = 0$, Shimakawa's result does not apply. We describe a Segal-type delooping with respect to the sign circle $S^{1,1}$, whose input is an additive category $\mathscr{C}$ with a subcategory of weak equivalences $w\mathscr{C} \subseteq \mathscr{C}$ and duality T. In such a category there is a notion of symmetric form, which is a pair (X, φ) where X is an object of $\mathscr{C}$ and $\varphi\colon X \to TX$ is a map in $w\mathscr{C}$ with certain symmetry properties. These form a monoidal category $\mathrm{Sym}(w\mathscr{C})$ which behaves like a space of fixed points of the duality T acting on $\mathscr{C}$. In particular, there is a homotopy equivalence $|N\mathrm{Sym}(w\mathscr{C})| \simeq |Nw\mathscr{C}|^{C_2}$; see Section 4 for details. The delooping produces a pointed C_2–space $|Nw\mathscr{C}(S^{1,1})|$ which comes with a natural C_2–equivariant map $|Nw\mathscr{C}| \to \Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|$. On underlying spaces this is equivalent to Segal's Γ–space delooping of the E_∞–monoid $|Nw\mathscr{C}|$, while on C_2–fixed points it is a Γ–space approximation to Schlichting's delooping map $|\mathrm{Sym}(w\mathscr{C})| \to \mathrm{GW}(\mathscr{C}, w)$ [12, Section 4] whose target is the Grothendieck–Witt space of $(\mathscr{C}, w)$.

The analog of Theorem 3.12 in this setting is as follows:

Theorem 5.14 *Let $(\mathscr{C}, w\mathscr{C}, T)$ be an additive category with strict duality and weak equivalences. Then the map $|N\mathrm{Sym}(w\mathscr{C})| \to \big(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\big)^{C_2}$ induces isomorphisms*

$$\pi_0(N\mathrm{Sym}(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} \pi_0\big(\big(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\big)^{C_2}\big)$$

of monoids and

$$H_*(N\mathrm{Sym}(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} H_*((\Omega^{1,1}|Nw\mathscr{C}(S^{1,1}))^{C_2})$$

of left $H_*(Nw\mathscr{C})$–modules.

Since only the action of $\pi_0 Nw\mathscr{C}$ is inverted and not the action of $\pi_0 N\mathrm{Sym}(w\mathscr{C})$, the H–space $\big(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\big)^{C_2}$ is in general not grouplike unless $\mathscr{C}$ is enriched in $\mathbb{Z}\big[\frac{1}{2}\big]$–modules (see Propositions 5.15 and 5.17). Therefore, it is in general not equivalent to the Grothendieck–Witt space $\mathrm{GW}(\mathscr{C}, w)$, which is an infinite loop space. The delooping $\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|$ and the theorem above are used in Hesselholt and Madsen's book [4] to prove that the C_2–fixed-point space of real algebraic K–theory is homotopy equivalent to Schlichting's Grothendieck–Witt space [4, Theorem 9.1].

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2 Homology fibrations

In this section we collect some basic facts about homology fibrations of simplicial and bisimplicial sets. We make no claim to originality; the results here can either be found in [10], [2, Section IV.5] or [5] or are easy consequences of the results there.

A map of spaces or simplicial sets inducing an isomorphism on integral homology will be called a *homology equivalence*.

Definition 2.1 A commuting square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

of simplicial sets is called homology cartesian if for any factorization of $f\colon B \to D$ as a trivial cofibration followed by a fibration

$$B \overset{\simeq}{\rightarrowtail} W \twoheadrightarrow D,$$

the induced map from A to the pullback $C \times_D W$ is a homology equivalence.

Note that a homotopy cartesian square is automatically homology cartesian. Just as for homotopy cartesian squares, it doesn't matter which factorization we use or whether we choose to factor f or g. By analogy with the case of homotopy cartesian squares [2, Lemma II.8.22(2)] we have the following lemma, whose proof we omit:

Lemma 2.2 *Let*

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & A'' \\ \downarrow & \boldsymbol{I} & \downarrow & \boldsymbol{II} & \downarrow \\ B & \longrightarrow & B' & \longrightarrow & B'' \end{array}$$

be a diagram of simplicial sets such that the square $\boldsymbol{II}$ *is homotopy cartesian. Then* $\boldsymbol{I}$ *is homology cartesian if and only if the outer rectangle* $\boldsymbol{I}+\boldsymbol{II}$ *is homology cartesian.*

Let Y be a simplicial set. An m–simplex $\sigma \in Y_m$ of Y corresponds to a unique map $\Delta^m \to Y$, which we will also call σ. The simplices of Y form a category $\mathrm{Simp}(Y)$ where an object is a map $\sigma\colon \Delta^m \to Y$, and where a morphism

$$(\sigma\colon \Delta^m \to Y) \to (\tau\colon \Delta^n \to Y)$$

is a map $\alpha\colon [m] \to [n]$ in Δ such that the diagram

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha_*} & \Delta^n \\ & {}_{\sigma}\searrow \quad \swarrow_{\tau} & \\ & Y & \end{array}$$

commutes. Composition is given by composition of maps in Δ.

Let $f\colon X \to Y$ be a map of simplicial sets. Then, for any simplex $\sigma\colon \Delta^m \to Y$, we define $f^{-1}(\sigma)$ to be the pullback in the square

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^m & \xrightarrow[\sigma]{} & Y \end{array}$$

For a diagram

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha_*} & \Delta^n \\ & {}_{\sigma}\searrow \quad \swarrow_{\tau} & \\ & Y & \end{array}$$

there is an induced map

$$f^{-1}(\alpha_*)\colon f^{-1}(\sigma) \to f^{-1}(\tau).$$

The assignments

$$\sigma \mapsto f^{-1}(\sigma)$$

and

$$\big(\alpha_*\colon (\sigma\colon \to \Delta^m) \to (\tau\colon \to \Delta^n)\big) \mapsto \big(f^{-1}(\alpha_*)\colon f^{-1}(\sigma) \to f^{-1}(\tau)\big)$$

form the object and morphism components, respectively, of a functor

$$f^{-1}\colon \mathrm{Simp}(Y) \to \mathrm{sSet}.$$

If $g\colon Z \to Y$ is another map to Y, then a map $h\colon X \to Z$ of objects over Y induces a natural transformation

$$h_*\colon f^{-1} \to g^{-1}.$$

Note that the natural map $\operatorname{colim} f^{-1} \to X$ over Y is an isomorphism; in particular, $\operatorname{colim} \mathrm{id}_Y^{-1} \cong Y$. The homotopy colimit $\operatorname{hocolim} f^{-1}$ is the diagonal of the bisimplicial set $\coprod_* f^{-1}$ (see [2, Example IV.1.8]) which is given in degree n by $\left(\coprod_* f^{-1}\right)_n = \coprod_{\sigma \in N_n \mathrm{Simp}(Y)} f^{-1}(\sigma(0))$. The canonical maps $f^{-1}(\sigma(0)) \to X$ combine to give a map $\coprod_* f^{-1} \to X$ of bisimplicial sets, where X is constant in the "nerve" simplicial direction. This map induces a weak equivalence on diagonal simplicial sets (see [2, Lemma IV.5.2]).

Lemma 2.3 *Let $f\colon X \to Y$ be a map of simplicial sets. The following are equivalent:*

(1) *For every simplex $\sigma\colon \Delta^m \to Y$, the pullback diagram*

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta^m & \xrightarrow{\sigma} & Y \end{array}$$

is homology cartesian.

(2) *For any pair of simplices $\sigma\colon \Delta^m \to Y$ and $\tau\colon \Delta^n \to Y$ and for any diagram*

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\alpha_*} & \Delta^n \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & Y & \end{array}$$

the induced map on pullbacks along f,

$$f^{-1}(\alpha_*)\colon f^{-1}(\sigma) \to f^{-1}(\tau),$$

is a homology equivalence.

In the proof of this lemma we will use the following result, which is proven in [2, Lemma IV.5.11]; see [10, Theorem 1.4] for a different proof of Lemma 2.3.

Theorem 2.4 *Let $X\colon I \to \mathrm{sSet}$ be a functor such that for any morphism $i \to j$ in I the induced map $X(i) \to X(j)$ is a homology equivalence; then for all objects i of I the pullback diagram*

$$\begin{array}{ccc} X(i) & \longrightarrow & \operatorname{hocolim}_I X \\ \downarrow & & \downarrow \\ * & \xrightarrow{i} & NI \end{array}$$

is homology cartesian.

Proof of Lemma 2.3 (1) $\Longrightarrow$ (2) We begin by factoring f as

$$X \overset{g}{\rightarrowtail} W \overset{\bar{f}}{\twoheadrightarrow} Y,$$

where g is a cofibration and a weak equivalence. Condition (1) says precisely that the natural transformation $g_*\colon f^{-1} \to \bar{f}^{-1}$ has components which are homology equivalences. Since $\bar{f}$ is a fibration, the functor $\bar{f}^{-1}$ sends all maps in $\mathrm{Simp}(Y)$ to weak equivalences. Therefore, a map $\alpha\colon \sigma \to \tau$ in $\mathrm{Simp}(Y)$ gives a naturality square

$$\begin{array}{ccc} f^{-1}(\sigma) & \xrightarrow{f^{-1}(\alpha_*)} & f^{-1}(\tau) \\ {\scriptstyle g_{*,\sigma}}\downarrow & & \downarrow{\scriptstyle g_{*,\tau}} \\ \bar{f}^{-1}(\sigma) & \underset{\bar{f}^{-1}(\alpha_*)}{\xrightarrow{\simeq}} & \bar{f}^{-1}(\tau) \end{array}$$

where the vertical maps are homology equivalences and the lower horizontal map is a weak equivalence. It follows that $f^{-1}(\alpha_*)$ is a homology equivalence.

(2) $\Longrightarrow$ (1) By [2, Remark IV.5.18] we have, for every simplex $\sigma\colon \Delta^m \to Y$, a diagram of bisimplicial sets

$$\begin{array}{ccccc} f^{-1}(\sigma) & \longrightarrow & \coprod_* f^{-1} & \xrightarrow{\simeq} & X \\ \downarrow & \boldsymbol{I} & \downarrow & \boldsymbol{II} & \downarrow{\scriptstyle f} \\ \Delta^m & \longrightarrow & \coprod_* \mathrm{id}_Y^{-1} & \xrightarrow{\simeq} & Y \\ {\scriptstyle\simeq}\downarrow & \boldsymbol{III} & \downarrow{\scriptstyle\simeq} & & \\ * & \longrightarrow & \coprod_{N\mathrm{Simp}(Y)} * & & \end{array}$$

Write $d(\boldsymbol{I})$ for the square obtained by taking diagonals in the square $\boldsymbol{I}$ and similarly for the other subdiagrams. The square $d(\boldsymbol{I}+\boldsymbol{III})$ is

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & \operatorname{hocolim} f^{-1} \\ \downarrow & & \downarrow \\ * & \underset{\sigma}{\longrightarrow} & N\mathrm{Simp}(Y) \end{array}$$

which is homology cartesian by Theorem 2.4. Since the square $d(\boldsymbol{III})$ is homotopy cartesian it follows by Lemma 2.2 that $d(\boldsymbol{I})$ is homology cartesian. The square $d(\boldsymbol{II})$ is also homotopy cartesian, so it follows, again by Lemma 2.2, that $d(\boldsymbol{I}+\boldsymbol{II})$ is homology cartesian. $\square$

Definition 2.5 A map $f\colon X \to Y$ of simplicial sets is called a homology fibration if it satisfies the conditions of Lemma 2.3.

Definition 2.6 A map $p\colon E \to B$ of topological spaces is called a homology fibration if for any point $b \in B$ the natural map from the fiber F_b at b to the homotopy fiber hF_b induces an isomorphism on integral homology.

The relation between the two kinds of homology fibrations is given as follows:

Theorem 2.7 [10, Theorem 4.4] *A map $f\colon X \to Y$ of simplicial sets is a homology fibration if and only if the induced map on geometric realizations $|f|\colon |X| \to |Y|$ is a homology fibration of topological spaces.*

Recall Segal's edgewise subdivision functor $\mathrm{Sd}\colon \mathrm{sSet} \to \mathrm{sSet}$, which has $(\mathrm{Sd}\,X)_n = X_{2n+1}$ (see the appendix here and the appendix of [15] for details). An important property of this construction is that the geometric realization of a simplicial set X is naturally homeomorphic to the geometric realization of its subdivision $\mathrm{Sd}\,X$. Using this, we get the next lemma from Theorem 2.7.

Lemma 2.8 *A map $f\colon X \to Y$ of simplicial sets is a homology fibration if and only if the induced map $\mathrm{Sd}\,f\colon \mathrm{Sd}\,X \to \mathrm{Sd}\,Y$ is a homology fibration.*

The next lemma follows easily from Lemma 2.3(2).

Lemma 2.9 *The pullback of a homology fibration along any map is a homology fibration.*

Lemma 2.10 *Let $f\colon X \to Y$ be a homology fibration and let $g\colon Z \to Y$ be any map. Then the pullback square*

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

is homology cartesian.

Proof Factor the map f as

$$X \overset{i}{\rightarrowtail} W \overset{\bar{f}}{\twoheadrightarrow} Y,$$

where i is a trivial cofibration. There is an induced factorization

$$Z \times_Y X \xrightarrow{j} Z \times_Y W \xrightarrow{\bar{h}} Z$$

of h and we must show that j is a homology equivalence.

Let

$$H_q(h^{-1}, \mathbb{Z})\colon \mathrm{Simp}(Z) \to Ab$$

be the composite functor given by

$$\sigma \mapsto h^{-1}(\sigma) \mapsto H_q(h^{-1}(\sigma), \mathbb{Z}),$$

and similarly for $H_q(\bar{h}^{-1}, \mathbb{Z})$, where Ab is the usual category of abelian groups. The natural transformation $j_*\colon H_q(h^{-1}, \mathbb{Z}) \to H_q(\bar{h}^{-1}, \mathbb{Z})$ is an isomorphism of functors, since f is a homology fibration and the maps from the pullbacks over σ to the pullbacks over $g(\sigma)$ are isomorphisms. Recall that for a functor $F\colon I \to Ab$ the simplicial abelian group EF is given in degree n by

$$EF_n = \bigoplus_{i_0 \to \cdots \to i_n} F(i_0)$$

with structure maps as for $\coprod_* F$ (see [2, Section IV.2.1]). The map

$$E(H_q(h^{-1}, \mathbb{Z})) \to E(H_q(\bar{h}^{-1}, \mathbb{Z}))$$

induced by j_* is an isomorphism. By [2, Section IV.5.1] there is a natural first quadrant spectral sequence

$$E_2^{p,q} = \pi_p E(H_q(h^{-1}, \mathbb{Z})) \Rightarrow H_{p+q}(Z \times_Y X, \mathbb{Z})$$

and a corresponding one for $\bar{h}^{-1}$ converging to $H_{p+q}(Z \times_Y W, \mathbb{Z})$. The map j induces an isomorphism of E_2–pages and is therefore a homology equivalence by the comparison theorem for spectral sequences. □

For a homology fibration $f\colon X \to Y$, the functor $H_q(f^{-1}, \mathbb{Z})$ sends all maps to isomorphism and hence factors through the groupoid $G\mathrm{Simp}(Y)$ obtained from $\mathrm{Simp}(Y)$ by inverting all morphisms (see [2, Section IV.5.1]). This groupoid is naturally equivalent to the fundamental groupoid of the geometric realization $|Y|$ (see [2, Theorem III.1.1]). If for any pair of maps $\xi, \zeta\colon \sigma \to \tau$ in $G\mathrm{Simp}(Y)$ the induced maps

$$\xi_*, \zeta_*\colon H_q(f^{-1}(\sigma), \mathbb{Z}) \to H_q(f^{-1}(\tau), \mathbb{Z})$$

agree, we say that the fundamental groupoid acts trivially on the homology of the fibers of f. If Y is also connected, then, for any simplex ρ in Y, there is a unique isomorphism of functors

$$(\sigma \mapsto H_q(f^{-1}\sigma, \mathbb{Z})) \cong (\sigma \mapsto H_q(f^{-1}\rho, \mathbb{Z}))$$

whose value at ρ is the identity map.

Lemma 2.11 *Let $f\colon X \to Y$ be a homology fibration such that the fundamental groupoid of Y acts trivially on the homology of the fibers of f. Then, for any homology equivalence $g\colon Z \to Y$, the induced map*

$$g'\colon Z \times_Y X \to X$$

is a homology equivalence.

Proof Assume, without loss of generality, that Y is connected and choose a fiber F over some vertex of Y. Then, by [2, Section IV.5.1], there are Serre spectral sequences, for f,

$$E_2^{p,q} = H_p(Y, H_q(F)) \Rightarrow H_{p+q}(X),$$

and, for the pullback of f along g,

$$E_2^{p,q} = H_p(Z, H_q(F)) \Rightarrow H_{p+q}(Z \times_Y X).$$

The map induced by g' on E_2–pages is an isomorphism by the universal coefficient theorem and the fact that g is a homology equivalence. It follows that g' is a homology equivalence. □

We now turn to bisimplicial sets. A map of bisimplicial sets will be called a homology equivalence if the induced map on diagonals is a homology equivalence. Note that any element $\sigma \in Y_{m,n}$, called a bisimplex, is classified by a unique map $\sigma\colon \Delta^{m,n} \to Y$ from a representable bisimplicial set.

Lemma 2.12 *Let $f\colon X \to Y$ be a map of bisimplicial sets. The following are equivalent:*

(1) *The diagonal $df\colon dX \to dY$ is a homology fibration.*

(2) *For any pair of bisimplices* $\sigma\colon \Delta^{m,n} \to Y$ *and* $\tau\colon \Delta^{p,q} \to Y$ *and for any diagram*

$$\begin{array}{ccc} \Delta^{m,n} & \xrightarrow{(\alpha,\beta)_*} & \Delta^{p,q} \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & Y & \end{array}$$

the induced map on pullbacks along f

$$f^{-1}(\alpha,\beta)_*\colon f^{-1}(\sigma) \to f^{-1}(\tau)$$

is a homology equivalence.

Proof (1) $\Longrightarrow$ (2) Given a bisimplex $\sigma\colon \Delta^{m,n} \to Y$ we choose a vertex $v \in Y_{0,0}$ belonging to σ. Since pullbacks commute with diagonals, we get a diagram

$$\begin{array}{ccccc} df^{-1}(v) & \longrightarrow & df^{-1}(\sigma) & \longrightarrow & dX \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle df} \\ \Delta^0 & \longrightarrow & \Delta^m \times \Delta^n & \longrightarrow & dY \end{array}$$

in which the two squares and the outer rectangle are pullback diagrams. The middle vertical map is a homology fibration, by Lemma 2.9, since it is a pullback of df. The lower left map is a weak equivalence, so it follows by Lemma 2.11 that the induced map $df^{-1}(v) \to df^{-1}(\sigma)$ is a homology equivalence. By definition this means that the map $f^{-1}(v) \to f^{-1}(\sigma)$ is a homology equivalence. A map $(\alpha,\beta)\colon \sigma \to \tau$ gives a commuting triangle

$$\begin{array}{ccc} & f^{-1}(v) & \\ \swarrow & & \searrow \\ f^{-1}(\sigma) & \xrightarrow[f^{-1}(\alpha,\beta)]{} & f^{-1}(\tau) \end{array}$$

By the argument above the two downward maps are homology equivalences, so it follows that $f^{-1}(\alpha,\beta)$ is as well.

(2) $\Longrightarrow$ (1) The proof follows roughly the same outline as the corresponding proof for simplicial sets. As for simplicial sets there is a category $\mathrm{Simp}(Y)$ of bisimplices of Y and condition (2) says that the functor $f^{-1}\colon \mathrm{Simp}(Y) \to \mathrm{bisSet}$ takes values in homology equivalences. Composing f^{-1} with the diagonal functor $d\colon \mathrm{bisSet} \to \mathrm{sSet}$ gives a functor $df^{-1}\colon \mathrm{Simp}(Y) \to \mathrm{sSet}$ taking values in homology equivalences. For a simplex $\sigma\colon \Delta^n \to dY$ there is a diagram of bisimplicial sets like the one in the proof of Lemma 2.3 and by the same argument we conclude that df is a homology fibration. $\square$

Definition 2.13 A map $f\colon X \to Y$ of bisimplicial sets is called a homology fibration if it satisfies one (and hence both) of the conditions of Lemma 2.12.

The exposition of Propositions 2.14 and 2.15 below and their proofs follows the unpublished lecture notes [5, Lecture 05] closely. If X is a bisimplicial set, we will write X_n for the vertical simplicial set

$$[q] \mapsto X_{n,q}.$$

Proposition 2.14 *Let $f\colon X \to Y$ be a map of bisimplicial sets such that for each $n \geq 0$ the map $f_n\colon X_n \to Y_n$ is a Kan fibration. Assume that for each $\theta\colon [m] \to [n]$ and each $v \in Y_{n,0}$ the induced map on fibers $f_n^{-1}(v) \to f_m^{-1}(\theta^*(v))$ is a homology equivalence. Then f is a homology fibration.*

Proof We show that f satisfies condition (2) of Lemma 2.12. Given a bisimplex $\tau\colon \Delta^{p,q} \to Y$ choose a vertex v of Δ^q and let $(\mathrm{id}_{[p]}, v)_*\colon \Delta^{p,0} \to \Delta^{p,q}$ be the corresponding map of bisimplicial sets. In level n we can form the iterated pullback

$$\begin{array}{ccccc}
\coprod_{\gamma\in\Delta^p_n} f_n^{-1}(v) & \xrightarrow{v_*} & \coprod_{\gamma\in\Delta^p_n} f_n^{-1}\big(((\gamma,\mathrm{id}_{[q]})^*\tau)_n\big) & \longrightarrow & X_n \\
\downarrow & & \downarrow & & \downarrow f_n \\
\coprod_{\gamma\in\Delta^p_n} \Delta^0 & \underset{\coprod v}{\xrightarrow{\simeq}} & \coprod_{\gamma\in\Delta^p_n} \Delta^q & \xrightarrow[\coprod((\gamma,\mathrm{id}_{[q]})^*\tau)_n]{} & Y_n
\end{array}$$

where the map v_* is a weak equivalence since f_n is a fibration. This says precisely that the map $f^{-1}((\mathrm{id}_{[p]}, v)^*\tau) \to f^{-1}(\tau)$ is a levelwise weak equivalence, and so in particular a homology equivalence. Therefore it suffices to consider diagrams of the form

$$\begin{array}{ccc}
\Delta^{m,0} & \xrightarrow{(\alpha,\mathrm{id})_*} & \Delta^{p,0} \\
& {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle w} & \\
& Y &
\end{array}$$

In level n the induced map $f^{-1}(v) \to f^{-1}(w)$ fits as the top row in the diagram

$$\begin{array}{ccccc}
\coprod_{\gamma\in\Delta^m_n} f_n^{-1}((\gamma,\mathrm{id})^*v) & & \longrightarrow & & \coprod_{\delta\in\Delta^p_n} f_n^{-1}((\delta,\mathrm{id})^*w) \\
\uparrow & & & & \uparrow \\
\coprod_{\gamma\in\Delta^m_n} f_m^{-1}(v) & \longleftarrow & \coprod_{\gamma\in\Delta^m_n} f_p^{-1}(w) & \longrightarrow & \coprod_{\delta\in\Delta^p_n} f_p^{-1}(w)
\end{array}$$

where the vertical maps and the lower left-hand map are homology equivalences by assumption on f. The lower right-hand map becomes the weak equivalence

$$\alpha_* \times \mathrm{id}\colon \Delta^m \times f_p^{-1}(w) \to \Delta^p \times f_p^{-1}(w)$$

after taking diagonals. □

Theorem 2.15 *Let $f\colon X \to Y$ be a map of bisimplicial sets such that for each $n \geq 0$ the map $f_n\colon X_n \to Y_n$ is a homology fibration and for each $v \in Y_{n,0}$ the induced map on fibers $f_n^{-1}(v) \to f_m^{-1}(\theta^*(v))$ is a homology equivalence. Then f is a homology fibration.*

Proof We begin by factoring the map $f\colon X \to Y$ as a levelwise trivial cofibration followed by a levelwise fibration $X \xrightarrow{g} W \xrightarrow{h} Y$. Given a bisimplex $\sigma\colon \Delta^{p,q} \to Y$ we get a diagram of bisimplicial sets

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow g \\ h^{-1}(\sigma) & \longrightarrow & W \\ \downarrow & & \downarrow h \\ \Delta^{p,q} & \longrightarrow & Y \end{array} \qquad (f = h \circ g\colon X \to Y)$$

which in level n looks like

$$\begin{array}{ccc} \coprod_{\theta\in\Delta^p_n} f_n^{-1}((\theta,\mathrm{id}_{[q]})^*\sigma) & \longrightarrow & X_n \\ \downarrow & & \simeq \downarrow g_n \\ \coprod_{\theta\in\Delta^p_n} h_n^{-1}((\theta,\mathrm{id}_{[q]})^*\sigma) & \longrightarrow & W_n \\ \downarrow & & \downarrow h_n \\ \coprod_{\theta\in\Delta^p_n} \Delta^q & \xrightarrow[\coprod(\theta,\mathrm{id}_{[q]})^*\sigma]{} & Y_n \end{array} \qquad (f_n = h_n \circ g_n\colon X_n \to Y_n)$$

Since f_n is a homology fibration, the upper left vertical map induces a homology equivalence on each summand, and is therefore a homology equivalence. This says that the map $f^{-1}(\sigma) \to h^{-1}(\sigma)$ is a levelwise homology equivalence and hence a homology equivalence.

Given a vertex $v \in Y_{n,0}$ and a map $\theta\colon [m] \to [n]$, there is a commuting square of fibers

$$\begin{array}{ccc} f_n^{-1}(v) & \longrightarrow & f_m^{-1}(\theta^* v) \\ \downarrow & & \downarrow \\ h_n^{-1}(v) & \longrightarrow & h_m^{-1}(\theta^* v) \end{array}$$

The vertical maps are homology equivalences since f is a levelwise homology fibration and the upper horizontal map is a homology equivalence by assumption on f. From this we see that the lower horizontal map is a homology equivalence, which, by Proposition 2.14, implies that h is a homology fibration. A map $\sigma \to \tau$ in $\mathrm{Simp}(Y)$ induces a square of pullbacks

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & f^{-1}(\tau) \\ \downarrow & & \downarrow \\ h^{-1}(\sigma) & \longrightarrow & h^{-1}(\tau) \end{array}$$

where the vertical maps are homology equivalences. Moreover, the lower horizontal map is a homology equivalence since h is a homology fibration. Hence the top horizontal map is a homology equivalence and f is a homology fibration. □

For a bisimplicial set X we write $\mathrm{Sd}_h X$ for the Segal edgewise subdivision of X in the first (horizontal) variable and $\mathrm{Sd}_v X$ for the subdivision in the second (vertical) variable. Clearly $\mathrm{Sd}_h \mathrm{Sd}_v X = \mathrm{Sd}_v \mathrm{Sd}_h X$ and $d\mathrm{Sd}_h \mathrm{Sd}_v X = \mathrm{Sd}(dX)$.

Lemma 2.16 *Let $f\colon X \to Y$ be a map of bisimplicial sets satisfying the conditions of Theorem 2.15. Then $\mathrm{Sd}_h f$ and $\mathrm{Sd}_v f$ also satisfy the conditions.*

Proof We treat $\mathrm{Sd}_h f$ first. In level n the map $\mathrm{Sd}_h f$ is just the map $f\colon X_{2n+1} \to Y_{2n+1}$, which is a homology fibration by assumption. Assume given a vertex $v \in (\mathrm{Sd}_h Y)_{n,0} = Y_{2n+1,0}$ and a simplicial structure map $\theta\colon [m] \to [n]$. The induced map $\theta^*\colon (\mathrm{Sd}\, Y)_n \to (\mathrm{Sd}\, Y)_m$ is the simplicial structure map $(\theta \sqcup \theta^{\mathrm{op}})^*\colon Y_{2n+1} \to Y_{2m+1}$, so the map on fibers is a homology equivalence by assumption.

Now for the map $\mathrm{Sd}_v f$. For $n \geq 0$ the map $(\mathrm{Sd}_v f)_n$ is the subdivision $\mathrm{Sd}(f_n)$ of the map $f_n\colon X_n \to Y_n$, so by Lemma 2.8 it is a homology fibration. A vertex $v \in (\mathrm{Sd}\, Y_n)_0 = Y_{n,1}$ need not come from a vertex in $Y_{n,0}$, but it can be connected to such a vertex by an edge. Since $\mathrm{Sd}(f_n)$ is a homology fibration it then follows that the fiber over v is homology equivalent to the fiber over a vertex in Y_n. This implies that for any simplicial structure map $\theta\colon [m] \to [n]$ the fiber over v maps by a homology equivalence to the fiber over $(\mathrm{Sd}\,\theta^*)(v)$. □

3 Simplicial monoids with anti-involution

Definition 3.1 An anti-involution on a monoid M is a function $m \mapsto \overline{m}$ from M to itself such that $\overline{(\overline{m})} = m$ and $\overline{m \cdot n} = \overline{n} \cdot \overline{m}$ for all $m, n \in M$. A simplicial monoid with anti-involution is a simplicial monoid M with a self-map of the underlying simplicial set, which is an anti-involution in each simplicial level.

Given a monoid M we can form the bar construction BM, which is the simplicial set $[n] \mapsto M^{\times n}$ with the usual structure maps. If M has the extra structure of an anti-involution, we get extra structure on the bar construction as well. The system of maps

$$\{w_i\colon B_i M \to B_i M\}$$

given in level p by

$$w_p(m_1, m_2, \dots, m_p) = (\overline{m}_p, \dots, \overline{m}_2, \overline{m}_1)$$

together with the simplicial structure maps of BM form a real simplicial set (see the appendix), which we, by abuse of language, also call BM. Similarly, for a simplicial monoid M with anti-involution we get a functor

$$BM\colon (\Delta R)^{\mathrm{op}} \to \mathrm{sSet}.$$

Write $\Delta^1 \boxtimes M$ for the external product bisimplicial set $[p], [q] \mapsto \Delta^1_p \times M_q$. The natural map $\Delta^1 \boxtimes M \to BM$ extends to a map of $\Delta_R^{\mathrm{op}} \times \Delta^{\mathrm{op}}$–sets

$$\Delta^1_R \boxtimes M \to BM,$$

where the real structure maps in the source are given by $w_p(\zeta, m) = (\zeta^{\mathrm{op}}, \overline{m})$. As a consequence, we get a C_2–equivariant map on geometric realizations,

$$|\Delta^1| \times |M| \to |BM|,$$

which sends the C_2–subspace $|\Delta^1| \times \{e\} \cup \{0,1\} \times |M|$ to the basepoint. The group C_2 acts on $|\Delta^1|$ by reflection through the midpoint (see Example A.2), so there is an induced C_2–map $S^{1,1} \wedge |M| \to |BM|$ whose adjoint is the canonical map

$$\lambda_M\colon |M| \to \Omega^{1,1}|BM|.$$

Nonequivariantly, the topological monoid $|M|$ acts by left multiplication on itself and acts homotopy associatively on the loop space by $m \cdot \gamma = \lambda(m) * \gamma$, where $*$ means concatenation of loops. The map λ is C_2–equivariant and commutes up to homotopy with the action of $|M|$. We are interested in the properties of the map induced by λ on C_2–fixed points,

$$\lambda_M^{C_2}\colon |M|^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}.$$

The topological monoid $|M|$ acts continuously on the fixed points $|M|^{C_2}$ by $m \cdot n = mn\overline{m}$ and up to homotopy on the fixed points of the loop space by $m \cdot \gamma = \lambda(m) * \gamma * \lambda(\overline{m})$. These actions commute with $\lambda_M^{C_2}$ up to homotopy.

Definition 3.2 Let N be a commutative monoid. An element $s \in N$ is called a cofinal generator if for any $x \in N$ there is an $n \geq 0$ and an element $y \in N$ such that $xy = s^n$. A vertex t in a simplicial monoid M with $\pi_0(M)$ commutative is called a homotopy cofinal generator if its class $[t] \in \pi_0(M)$ is a cofinal generator.

Example 3.3 Let M be a simplicial monoid such that the monoid $\pi_0(M)$ is finitely generated and commutative. Pick vertices $t_1, \dots, t_n \in M_0$ whose path components $[t_1], \dots, [t_n]$ generate $\pi_0(M)$. Then the vertex $t = t_1 t_2 \cdots t_n$ is a homotopy cofinal generator of M.

From now on let M denote a simplicial monoid with $\pi_0(M)$ in the center of $H_*(M)$ and let t be a homotopy cofinal generator of M. For a simplicial set X with a left M–action we set

$$X_\infty = \operatorname{hocolim}(X \xrightarrow{t\cdot} X \xrightarrow{t\cdot} X \xrightarrow{t\cdot} \cdots).$$

In particular, we have

$$M_\infty = \operatorname{hocolim}(M \xrightarrow{t\cdot} M \xrightarrow{t\cdot} M \xrightarrow{t\cdot} \cdots).$$

The homology of M is a graded ring with the Pontryagin product. Since $[t]$ is in the center, multiplication by $[t]$ on $H_*(M)$ is $H_*(M)$–linear. Hence there is an isomorphism of left $H_*(M)$–modules

$$H_*(M_\infty) \cong \operatorname{colim}(H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} \cdots).$$

Lemma 3.4 *The map $M \to M_\infty$ including M at the start of the diagram induces an isomorphism of right $H_*(M)$–modules*

$$H_*(M)[\pi_0(M)^{-1}] \to H_*(M_\infty).$$

Proof Since $\pi_0(M)$ is central in $H_*(M)$, there is an isomorphism

$$\operatorname{colim}(H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} H_*(M) \xrightarrow{[t]\cdot} \cdots) \cong H_*(M)[t^{-1}]$$

and $H_*(M) \to H_*(M)[t^{-1}]$ is the localization map. The element $[t]$ is a cofinal generator of $\pi_0(M)$, hence the further localization map

$$H_*(M)[t^{-1}] \to H_*(M)[\pi_0(M)^{-1}]$$

is an isomorphism. □

It follows that the vertices M_0 of M act on M_∞ by homology equivalences. The following result can also be found in eg [2, Theorem IV.5.15]:

Lemma 3.5 *Let X be a simplicial set with a right action of M on which M_0 acts by homology equivalences. Then the canonical map $p\colon B(X, M, *) \to BM$ satisfies the conditions of Theorem 2.15. In particular, it is a homology fibration.*

Proof In each level $n \geq 0$ the map p_n is the canonical projection

$$p_n\colon X \times M^{\times n} \to M^{\times n},$$

which is a homology fibration.

Let $v \in M_0^{\times n}$ be a vertex and let $\theta\colon [m] \to [n]$ be a map in Δ. Note that the fiber over any vertex is isomorphic to X. We must show that the map on fibers

$$p_n^{-1}(v) \to p_m^{-1}(\theta^*(v))$$

is a homology equivalence. Since θ^* can be factored into face and degeneracy maps, we reduce to these cases. If $\theta^* = d_j$ with $j \neq 0$, then the map $p_n^{-1}(v) \to p_{n-1}^{-1}(\theta^*(v))$ is an isomorphism, and similarly for $\theta^* = s_i$. If $\theta^* = d_0$, then the induced map corresponds to acting on X by an element of M_0 and is therefore a homology equivalence. □

Recall that we are using the Bousfield–Kan model for the homotopy colimit throughout this paper. If M is a simplicial monoid, we write sSet_M for the category of simplicial sets with right M–action and M–equivariant maps.

Lemma 3.6 *Let I be a small category and let $G\colon I \to \mathrm{sSet}_M$ be any diagram. If X is a simplicial set with left M–action, there is a natural isomorphism of simplicial sets*

$$dB(\operatorname{hocolim} G, M, X) \cong \operatorname{hocolim} dB(G, M, X).$$

Proof Both simplicial sets are obtained by taking iterated diagonals of the trisimplicial set $B\big(\coprod_* G, M, X\big)$ given by

$$[p],[q],[r] \mapsto \Big(\coprod_{\sigma\in N_r(I)} G(\sigma(0))_q\Big) \times M_q^{\times p} \times X_q. \qquad \square$$

Corollary 3.7 *For any simplicial set X with a left M–action, there is an isomorphism of simplicial sets*

$$dB(M_\infty, M, X) \cong (dB(M,M,X))_\infty.$$

Theorem 3.8 (group completion [8; 11; 10]) *Let M be a simplicial monoid such that $\pi_0 M$ is in the center of $H_*(M)$. Then the map λ induces an isomorphism of left $H_*(M)$–modules*

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(\Omega|BM|).$$

Proof Assume first that M has a homotopy cofinal generator t. By Lemma 3.5, the map $B(M_\infty, M, *) \to BM$ is then a homology fibration with fiber M_∞. Taking $X = *$ in Corollary 3.7 shows that the simplicial set $dB(M_\infty, M, *)$ is a homotopy colimit of contractible spaces and hence is contractible. The contracting homotopy induces a map $|M_\infty| \to \Omega|BM|$ which is a homology equivalence and we conclude by Lemma 3.4.

For general M we let $F(M)$ denote the filtering poset of submonoids of M with finitely generated monoid of path components. Then there is an isomorphism of simplicial monoids $\operatorname{colim}_{M_i\in F(M)} M_i \cong M$ and the colimit is filtering. The functors $|-|$, B, Ω, $H_*(-)$ and inverting π_0 commute with the filtering colimit in question, so the result now follows since each $M_i \in F(M)$ has a homotopy cofinal generator by Example 3.3. $\square$

Inspired by the proof of Theorem 3.8 we will now proceed to analyze the map $\lambda_M^{C_2}$. It becomes easier to work with the anti-involution when we take the Segal subdivision in the horizontal (ie bar construction) direction of BM. The output is the bisimplicial set $\mathrm{Sd}_h\, BM$, which has a simplicial action of C_2 and whose fixed points we will now describe. An element in level (p,q) of $\mathrm{Sd}_h\, BM$ is a tuple

$$(m_1, \dots, m_{2p+1}) \in M_q^{\times 2p+1},$$

and the action of the nontrivial element in C_2 is

$$(m_1, \dots, m_p, m_{p+1}, m_{p+2}, \dots, m_{2p+1}) \mapsto (\overline{m}_{2p+1}, \dots, \overline{m}_{p+2}, \overline{m}_{p+1}, \overline{m}_p, \dots, \overline{m}_1).$$

The fixed points of this action are of the form

$$(m_1, \dots, m_p, m_{p+1}, \overline{m}_p, \dots, \overline{m}_1), \quad \text{where } m_{p+1} = \overline{m}_{p+1}.$$

Here, the last p factors are redundant and projection on the first $p+1$ factors gives a bijection

$$b_{p,q}\colon (M_q^{\times 2p+1})^{C_2} \xrightarrow{\cong} M_q^{\times p} \times M_q^{C_2}.$$

The monoid M_q acts on $M_q^{C_2}$ on the left by $(m,n) \mapsto m \cdot n \cdot \overline{m}$. Both this action and the description of the fixed points are compatible with the simplicial structure maps of M. Combining this with the fact that

$$d_p(m_1, \dots, m_p, m_{p+1}, \overline{m}_p, \dots, \overline{m}_1) = (m_1, \dots, m_p \cdot m_{p+1} \cdot \overline{m}_p, \dots, \overline{m}_1),$$

we get the following:

Lemma 3.9 *Let M be a simplicial monoid with anti-involution. Then the maps $b_{p,q}$ determine a natural isomorphism of bisimplicial sets*

$$b\colon (\mathrm{Sd}_h\, BM)^{C_2} \xrightarrow{\cong} B(*, M, M^{C_2}).$$

The map $p\colon B(M_\infty, M, *) \to BM$ induces a map

$$\mathrm{Sd}_h\, p\colon \mathrm{Sd}_h\, B(M_\infty, M, *) \to \mathrm{Sd}_h\, BM$$

on subdivisions. Since p satisfies the conditions of Theorem 2.15, the map $\mathrm{Sd}_h\, p$ does as well, by Lemma 2.16. Therefore $\mathrm{Sd}_h\, p$ is a homology fibration.

Lemma 3.10 *The pullback of $\mathrm{Sd}_h\, p$ along the inclusion*

$$B(*, M, M^{C_2}) \hookrightarrow \mathrm{Sd}_h\, BM$$

is isomorphic to $B(M_\infty, M, M^{C_2})$.

The proof is straightforward. It now follows from Lemma 2.10 that the square

$$(*) \qquad \begin{array}{ccc} B(M_\infty, M, M^{C_2}) & \longrightarrow & \mathrm{Sd}_h\, B(M_\infty, M, *) \\ \downarrow & & \downarrow \\ B(*, M, M^{C_2}) & \longrightarrow & \mathrm{Sd}_h\, BM \end{array}$$

becomes homology cartesian after taking diagonals. We consider M^{C_2} as a bisimplicial set which is constant in the first variable. Define the map

$$i\colon M^{C_2} \to B(M, M, M^{C_2})$$

levelwise by

$$m \mapsto (e, e, \dots, e, m).$$

This map has a retraction r given by

$$r(m_0, m_1, \dots, m_p, m) = m_0 \cdot m_1 \cdots m_p \cdot m \cdot \overline{m}_p \cdots \overline{m}_1 \cdot \overline{m}_0$$

and there is a standard simplicial homotopy $r \circ i \simeq \mathrm{id}$. The map $M \to M_\infty$ of Lemma 3.4 induces a map

$$j\colon B(M, M, M^{C_2}) \to B(M_\infty, M, M^{C_2}).$$

Lemma 3.11 *The map $j \circ i\colon M^{C_2} \to B(M_\infty, M, M^{C_2})$ induces an isomorphism of left $\pi_0(M)$–sets*

$$\pi_0(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} \pi_0(dB(M_\infty, M, M^{C_2}))$$

and an isomorphism of left $H_*(M)$–modules

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*(dB(M_\infty, M, M^{C_2})).$$

Proof We present the argument for homology, the one for π_0 being similar. By Corollary 3.7 there is an isomorphism $(dB(M, M, M^{C_2}))_\infty \cong dB(M_\infty, M, M^{C_2})$. In the diagram

$$\begin{array}{ccccccc}
dB(M, M, M^{C_2}) & \xrightarrow{t\cdot} & dB(M, M, M^{C_2}) & \xrightarrow{t\cdot} & dB(M, M, M^{C_2}) & \xrightarrow{t\cdot} & \cdots \\
\downarrow{\scriptstyle dr} & & \downarrow{\scriptstyle dr} & & \downarrow{\scriptstyle dr} & & \\
M^{C_2} & \xrightarrow{t\cdot} & M^{C_2} & \xrightarrow{t\cdot} & M^{C_2} & \xrightarrow{t\cdot} & \cdots
\end{array}$$

the vertical maps are weak equivalences and hence induce a weak equivalence of homotopy colimits $dB(M, M, M^{C_2})_\infty \xrightarrow{r_\infty} M_\infty^{C_2}$. In homology we get a sequence of isomorphisms of left $H_*(M)$–modules

$$H_*(B(M_\infty, M, M^{C_2})) \xrightarrow{\cong} H_*(M_\infty^{C_2}) \xrightarrow{\cong} H_*(M^{C_2})[\pi_0(M)^{-1}]. \qquad \square$$

Let (X, x) be a based C_2–space with $\sigma\colon X \to X$ representing the action of the nontrivial element of C_2. The homotopy fiber hF_{ι_X} of the canonical inclusion $\iota_X\colon X^{C_2} \hookrightarrow X$

of the fixed points can be identified with the space of paths $\chi\colon \left[0, \frac{1}{2}\right] \to X$ such that $\chi(0) = x$ and $\chi\left(\frac{1}{2}\right) \in X^{C_2}$. There is a map

$$b_X\colon \mathrm{hF}_{\iota_X} \to (\Omega^{1,1} X)^{C_2}$$

given by $b_X(\chi) = \chi * (\sigma \circ \overline{\chi})$, where $*$ is the concatenation operation and $\overline{\chi}$ is the path $t \mapsto \chi(1-t)$. This map is a homeomorphism with inverse given by restricting loops to $\left[0, \frac{1}{2}\right]$.

Now we apply geometric realization to the square $(*)$ to obtain a homology cartesian square of spaces

$$\begin{array}{ccc} |B(M_\infty, M, M^{C_2})| & \longrightarrow & |B(M_\infty, M, *)| \\ \downarrow & & \downarrow \\ |B(*, M, M^{C_2})| & \longrightarrow & |BM| \end{array}$$

The space $|B(M_\infty, M, *)|$ is contractible and so $|B(M_\infty, M, M^{C_2})|$ is homology equivalent to the homotopy fiber of the composite

$$|B(*, M, M^{C_2})| \cong |BM|^{C_2} \hookrightarrow |BM|.$$

By the discussion above, this space is homeomorphic to $(\Omega^{1,1}|BM|)^{C_2}$ and so we get a homology equivalence

$$g\colon |B(M_\infty, M, M^{C_2})| \to (\Omega^{1,1}|BM|)^{C_2}.$$

Theorem 3.12 *Let M be a simplicial monoid with anti-involution such that $\pi_0 M$ is in the center of $H_*(M)$. Then the map*

$$\lambda_M^{C_2}\colon M^{C_2} \to (\Omega^{1,1}|BM|)^{C_2}$$

induces an isomorphism

$$\pi_0(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} \pi_0(\Omega^{1,1}|BM|)^{C_2}$$

of left $\pi_0(M)$–sets and an isomorphism of left $H_*(M)$–modules

$$H_*(M^{C_2})[\pi_0(M)^{-1}] \xrightarrow{\cong} H_*((\Omega^{1,1}|BM|)^{C_2}).$$

Proof Assume first that M has a homotopy cofinal generator $t \in M_0$. By Lemma 3.11, the map

$$|j \circ i|\colon |M^{C_2}| \to |B(M_\infty, M, M^{C_2})|$$

induces the desired localization map on homology. Since the homology equivalence $g\colon |B(M_\infty, M, M^{C_2})| \to (\Omega^{1,1}BM)^{C_2}$ is induced by the contracting homotopy on $|B(M_\infty, M, *)|$, which is homotopic to a homotopy that induces the map $\lambda_M^{C_2}$, we conclude that $\lambda_M^{C_2}$ also induces the desired map on homology.

If M does not have a cofinal generator, we reduce to the above case by a colimit argument as in the proof of Theorem 3.8. A similar argument proves the π_0–statement of the theorem. □

We end this section with a remark that aims to put Theorem 3.12 in a broader context and suggests a possible universal property for the space $(\Omega^{1,1}|BM|)^{C_2}$.

Remark 3.13 Let M be any simplicial monoid with anti-involution and let $f\colon M \to N$ be a map of ordinary simplicial monoids such that N is grouplike and the map $Bf\colon BM \to BN$ is a weak equivalence. Such N and f can always be found as in [11, Section Q.5], after replacing M up to C_2–equivariant weak equivalence by a simplicial monoid with anti-involution whose underlying simplicial monoid is levelwise free. We may then view N as a model for the homotopy-theoretic group completion of M. The simplicial set N has a canonical right M–action, and there is a homotopy cartesian square of spaces

$$\begin{array}{ccc} |B(N, M, M^{C_2})| & \longrightarrow & |B(N, M, *)| \\ \downarrow & & \downarrow \\ |B(*, M, M^{C_2})| & \longrightarrow & |BM| \end{array}$$

The upper right-hand corner $|B(N, M, *)|$ is contractible, so we get a weak equivalence

$$|B(N, M, M^{C_2})| \simeq (\Omega^{1,1}|BM|)^{C_2}.$$

Since $B(N, M, -)$ is a model for the derived induction functor from left M–simplicial sets to left N–simplicial sets, this suggests that $(\Omega^{1,1}|BM|)^{C_2}$ is a kind of initial $|M|$–space under $|M|^{C_2}$ on which M acts homotopy coherently, via homotopy equivalences. This perspective, which we will not develop further here, fits well with Theorem 3.12 above as well as with Stiennon's result [18] for grouplike monoids.

4 Categories with duality

In this section we summarize some facts we will need later. The reader can consult [1; 13; 4] for further details.

Definition 4.1 A small category with duality is a triple $(\mathscr{C}, T, \eta)$ where $\mathscr{C}$ is a small category, $T\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ is a functor and $\eta\colon \mathrm{id} \to T \circ T^{\mathrm{op}}$ is a natural transformation such that for each object c in $\mathscr{C}$ the composite map $Tc \xrightarrow{\eta_{Tc}} TT^{\mathrm{op}}Tc \xrightarrow{T(\eta_c)} Tc$ is the identity on Tc. If $T \circ T^{\mathrm{op}} = \mathrm{Id}_{\mathscr{C}}$ and $\eta = \mathrm{id}$, the duality is said to be strict.

Example 4.2 A monoid M can be thought of as a category $\mathscr{B}_M$ with one object $*$ and $\mathrm{Hom}_{\mathscr{B}_M}(*, *) = M$ as monoids. Then a duality (T, η) on $\mathscr{B}_M$ is the same as a monoid map $t\colon M^{\mathrm{op}} \to M$, ie such that $t(mn) = t(n)t(m)$ for all $m, n \in M$, and an element $\eta \in M$ such that $\eta t^2(m) = m\eta$ for all $m \in M$ and $t(\eta)\eta = e$. The duality is strict if and only if t is an anti-involution on M.

The main example of interest to us is the following (see eg [21]):

Example 4.3 A Wall antistructure is a triple (R, α, ε) where R is a ring, α is an additive map $R \to R$ such that $\alpha(rs) = \alpha(s)\alpha(r)$, and ε is a unit in R such that $\alpha^2(r) = \varepsilon r \varepsilon^{-1}$ and $\alpha(\varepsilon) = \varepsilon^{-1}$. When R is commutative, one can take $\alpha = \mathrm{id}_R$ and $\varepsilon \in \{\pm 1\}$. An interesting class of noncommutative cases is when R is an integral group ring $\mathbb{Z}[G]$ for a discrete group G, with the involution determined by $\alpha(g) = g^{-1}$ and $\varepsilon \in \{\pm 1\}$.

For an antistructure (R, α, ε) there is a naturally associated category with duality $P(R, \alpha, \varepsilon)$ with underlying category $P(R)$ the category of finitely generated projective (f.g.p.) right R–modules. The duality functor on $P(R, \alpha, \varepsilon)$ is $\mathrm{Hom}_R(-, R)$, where for an f.g.p. module P we give $\mathrm{Hom}_R(P, R)$ the right module structure given by $(fr)(p) = \alpha(r)f(p)$. The map

$$\eta_P\colon P \xrightarrow{\cong} \mathrm{Hom}_R(\mathrm{Hom}_R(P, R), R)$$

is the isomorphism given on elements $p \in P$ by $\eta_P(p)(f) = \alpha(f(p))\varepsilon$. It is straightforward to check that the equation $\eta_{\mathrm{Hom}_R(P,R)} \circ \eta_P^* = \mathrm{id}_{\mathrm{Hom}_R(P,R)}$ holds for all f.g.p. modules P. There are several well-known ways to ensure that the category $P(R)$ is small; we will not go into this here.

Definition 4.4 A duality-preserving functor

$$(F, \xi)\colon (\mathscr{C}, T, \eta) \to (\mathscr{C}', T', \eta')$$

consists of a functor $F\colon \mathscr{C} \to \mathscr{C}'$ and a natural transformation

$$\xi\colon F \circ T \to T' \circ F$$

such that for all c in $\mathscr{C}$ the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta'_{F(c)}} & T'(T')^{\mathrm{op}}F(c) \\ {\scriptstyle F(\eta_c)}\downarrow & & \downarrow{\scriptstyle T'(\xi_c)} \\ FTT^{\mathrm{op}}(c) & \xrightarrow[\xi_{T(c)}]{} & T'F^{\mathrm{op}}T^{\mathrm{op}}(c) \end{array}$$

commutes.

Composition is given by $(G,\zeta)\circ(F,\xi) = (G\circ F, \zeta_F \circ G(\xi))$. An *equivalence of categories with duality* is a duality-preserving functor

$$(F,\xi)\colon (\mathscr{C},T,\eta)\to(\mathscr{C}',T',\eta')$$

such that there is a duality-preserving functor $(F',\xi')\colon (\mathscr{C}',T',\eta')\to(\mathscr{C},T,\eta)$ and natural isomorphisms $u\colon F'\circ F \xrightarrow{\cong} \mathrm{Id}_{\mathscr{C}}$ and $u'\colon F\circ F' \xrightarrow{\cong} \mathrm{Id}_{\mathscr{C}'}$ satisfying $\xi'_{F(c)}\circ F'(\xi_c) = T(u_c)\circ u_{T(c)}$ for c in $\mathscr{C}$ and similarly for u'.

Definition 4.5 Let $(\mathscr{C},T,\eta)$ be a category with duality. The category $\mathrm{Sym}(\mathscr{C},T,\eta)$ of symmetric forms in $(\mathscr{C},T,\eta)$ is given as follows:

- The objects of $\mathrm{Sym}(\mathscr{C},T,\eta)$ are maps $f\colon a\to Ta$ such that $f = Tf\circ\eta_a$.
- A morphism from $f\colon a\to Ta$ to $f'\colon a'\to Ta'$ is a map $r\colon a\to a'$ in $\mathscr{C}$ such that the diagram

  $$\begin{array}{ccc} a & \xrightarrow{f} & Ta \\ {\scriptstyle r}\downarrow & & \uparrow{\scriptstyle Tr} \\ a' & \xrightarrow[f']{} & Ta' \end{array}$$

  commutes.
- Composition is given by ordinary composition of maps in $\mathscr{C}$.

The reason for the name "symmetric form" in the preceding definition is the following. Let (R,α,ε) be a Wall antistructure. The category $\mathrm{Sym}(P(R,\alpha,\varepsilon))$ has as objects maps $\varphi\colon P\to \mathrm{Hom}_R(P,R)$ such that the adjoint map $\widetilde{\varphi}\colon P\otimes_{\mathbb{Z}} P\to R$ is a biadditive form on P satisfying

$$\widetilde{\varphi}(pr,qs) = \alpha(r)\widetilde{\varphi}(p,q)s, \quad \widetilde{\varphi}(q,p) = \alpha(\widetilde{\varphi}(p,q))\varepsilon$$

for $r,s\in R$ and $p,q\in P$. A map

$$h\colon (P\xrightarrow{\varphi}\mathrm{Hom}_R(P,R))\to(P'\xrightarrow{\varphi'}\mathrm{Hom}_R(P',R))$$

is an R–module homomorphism $h\colon P \to P'$ such that $\widetilde{\varphi}'(h(p), h(q)) = \widetilde{\varphi}(p,q)$ for all $p,q \in P$. If the map $\varphi\colon P \to \mathrm{Hom}_R(P,R)$ is an isomorphism, we say that (P,φ) is nondegenerate.

Definition 4.6 [14, Lemma 4] For a category with duality $(\mathscr{C}, T, \eta)$, the strictification $\mathscr{D}(\mathscr{C}, T, \eta)$ has as objects triples (c, c', f) where $f\colon c' \to Tc$ is a map and morphisms from (c, c', f) to (d, d', g) are pairs $(r\colon c \to d, s\colon d' \to c')$ such that the diagram

$$\begin{array}{ccc} c' & \xrightarrow{f} & Tc \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle Tr} \\ d' & \xrightarrow[g]{} & Td \end{array}$$

commutes. Composition is given by composition in each component. The duality on $\mathscr{D}(\mathscr{C}, T, \eta)$ is given by sending an object $f\colon c' \to Tc$ to the composite $c \xrightarrow{\eta_c} TT^{\mathrm{op}}c \xrightarrow{Tf} Tc'$ and $(r\colon c \to d, s\colon d' \to c')$ to $(s\colon d' \to c', r\colon c \to d)$.

It is easy to see that the duality on $\mathscr{D}(\mathscr{C}, T, \eta)$ is strict. There are duality-preserving functors

$$(I, \iota)\colon (\mathscr{C}, T, \eta) \to \mathscr{D}(\mathscr{C}, T, \eta),$$

given by $I(c) = (c, Tc, \mathrm{id}_{T(c)})$, $I(f) = (f, Tf)$ and $\iota_c = (\mathrm{id}_{T(c)}, \eta_c)$, and

$$(K, \kappa)\colon \mathscr{D}(\mathscr{C}, T, \eta) \to (\mathscr{C}, T, \eta),$$

given by $K(c, c', f) = c$, $K(r,s) = r$ and $\kappa_{(c,c',f)} = f$. These induce homotopy inverse weak equivalences $N\mathscr{C} \simeq N\mathscr{D}\mathscr{C}$ and $N\mathrm{Sym}(\mathscr{C}) \simeq N\mathrm{Sym}(\mathscr{D}\mathscr{C})$. Both the construction $\mathscr{D}$ and the functors K and I are functorial in $(\mathscr{C}, T, \eta)$ for duality-preserving functors (see [14, Lemma 4]).

A strict duality T on a category $\mathscr{C}$ gives a map

$$NT\colon (N\mathscr{C})^{\mathrm{op}} = N(\mathscr{C}^{\mathrm{op}}) \xrightarrow{\cong} N\mathscr{C}$$

such that $NT \circ (NT)^{\mathrm{op}} = \mathrm{id}_{N\mathscr{C}}$. We know from Lemma A.1 that this is equivalent to extending the simplicial structure of $N\mathscr{C}$ to a real simplicial structure. It follows that the geometric realization has an induced C_2–action given by

$$[(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n, t_0, \dots, t_n)] \mapsto [(c_n \xrightarrow{Tf_n} \cdots \xrightarrow{Tf_1} Tc_0, t_n, \dots, t_0)]$$

for $(c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n, t_0, \dots, t_n) \in N_n\mathscr{C} \times \Delta^n$. Thus, from the topological perspective, the effect of the $\mathscr{D}$–construction is to replace the geometric realization $|N\mathscr{C}|$, which has an action of C_2 in the homotopy category,[1] by the bigger space $|N\mathscr{D}\mathscr{C}|$ which has a strict action of C_2. The actions are compatible in the sense that the maps $|NI|$ and $|NK|$ are mutually inverse isomorphisms of C_2–objects in the homotopy category.

Definition 4.7 Let $\mathscr{C}$ be a category. Its subdivision $\mathrm{Sd}\,\mathscr{C}$ is a category given as follows: An object of $\mathrm{Sd}\,\mathscr{C}$ is a morphism $f\colon a \to b$ in $\mathscr{C}$ and a map from $f\colon a \to b$ to $g\colon c \to d$ is a pair (h,i) of maps such that the following diagram commutes:

$$\begin{array}{ccc} a & \xrightarrow{\ f\ } & b \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle \mathrm{id}} \\ c & \xrightarrow[\ g\ ]{} & d \end{array}$$

Composition is given by $(h',i')\circ(h,i) = (h'\circ h, i\circ i')$.

There is a natural isomorphism $\mathrm{Sd}\,N\mathscr{C} \cong N\,\mathrm{Sd}\,\mathscr{C}$. If $(\mathscr{C},T,\eta)$ is a category with duality then there is an induced functor

$$\mathrm{Sd}\,T\colon \mathrm{Sd}\,\mathscr{C} \to \mathrm{Sd}\,\mathscr{C}$$

given by $\mathrm{Sd}\,T(a \xrightarrow{f} b) = Tb \xrightarrow{Tf} Ta$ and $\mathrm{Sd}\,T(h,i) = (Ti, Th)$. If T is a strict duality then $\mathrm{Sym}(\mathscr{C})$ is the category fixed under the C_2–action defined by $\mathrm{Sd}\,T$.

5 K–theory of additive categories with duality

Definition 5.1 Let $\mathscr{C}$ be a category and let X_1 and X_2 be objects of $\mathscr{C}$. A biproduct diagram for the pair (X_1, X_2) is a diagram

$$X_1 \underset{i_1}{\overset{p_1}{\leftrightarrows}} Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} X_2 \tag{1}$$

in $\mathscr{C}$ such that $p_j \circ i_j = \mathrm{id}_{X_j}$, the p_j's express Y as the product of X_1 and X_2 and the i_j's express Y as a coproduct of X_1 and X_2.

Recall that if $\mathscr{C}$ is a category with a zero object and each pair of objects has a biproduct diagram in $\mathscr{C}$, then the hom sets of $\mathscr{C}$ naturally inherit the structure of commutative monoids such that composition is bilinear (see [7, Section VIII.2] for details). Such a category $\mathscr{C}$ is called *additive* if the hom sets are abelian groups, not just monoids.

[1] In fact, the action on $|N\mathscr{C}|$ refines to a homotopy coherent C_2–action but we will ignore this additional structure in the present paper.

A functor between additive categories is called additive if it preserves biproducts and zero objects. Additive functors induce group homomorphisms on hom groups.

Let X be a finite pointed set. The category $Q(X)$ is defined as follows (see also [4, Section 4]): The objects in $Q(X)$ are the pointed subsets $U \subseteq X$. A morphism $U \to V$ of pointed subsets is a pointed subset of the intersection $U \cap V$. The composition of two subsets $A \subseteq U \cap V$ and $B \subseteq V \cap W$ is $A \cap B \subseteq U \cap W$. Note that $A \subseteq U \cap V$ can be thought of both as a map from U to V and as a map from V to U. This gives an isomorphism $Q(X) \cong Q(X)^{\mathrm{op}}$.

Definition 5.2 Let $\mathscr{C}$ be an additive category and X a finite pointed set. A sum-diagram in $\mathscr{C}$ indexed by X is a functor

$$A\colon Q(X) \to \mathscr{C}$$

such that for any pointed subset $U \subseteq X$ the maps $A(U) \to A(\{u,*\})$ induced by the pointed subsets $\{u,*\} \subseteq U$ induce an isomorphism

$$A(U) \xrightarrow{\cong} \prod_{u \in U \setminus \{*\}} A(\{u,*\}).$$

We write $\mathscr{C}(X)$ for the full subcategory of sum-diagrams in the functor category $\mathrm{Fun}(Q(X),\mathscr{C})$.

A *pointed* category is a category $\mathscr{C}$ with a chosen object $0_{\mathscr{C}}$. When $\mathscr{C}$ is additive, $0_{\mathscr{C}}$ will always be a zero object, but in general it need not be. We say that a functor between pointed categories is pointed if it preserves the chosen objects. Many of the constructions below rely on having chosen basepoints. Therefore we will usually work with pointed categories.

For a pointed additive category and a finite pointed set X we require that the elements of $\mathscr{C}(X)$ be pointed, ie that they send the subset $\{*\}$ to $0_{\mathscr{C}}$. We write $\mathscr{C}^X$ for the (pointed) category $\mathrm{Fun}_*(X,\mathscr{C})$ of pointed functors from X to $\mathscr{C}$, where we think of X as a discrete category. There is a natural evaluation functor $e_X\colon \mathscr{C}(X) \to \mathscr{C}^X$ given on objects by $e_X(A)(x) = A(\{x,*\})$ and similarly for morphisms. The following lemma is easily verified:

Lemma 5.3 *Let $\mathscr{C}$ be a pointed additive category. For any finite pointed set X, the functor*

$$e_X\colon \mathscr{C}(X) \to \mathscr{C}^X$$

is an equivalence of categories.

A pointed map $f\colon X \to Y$ induces a pushforward functor $f_*\colon \mathscr{C}(X) \to \mathscr{C}(Y)$ given by

$$(f_*(A))(U) = A\big(f^{-1}(U \setminus \{*\}) \cup \{*\}\big).$$

Given two composable maps f and g of finite pointed sets it is not hard to see that $(f \circ g)_* = f_* \circ g_*$, so that we get a functor

$$\mathscr{C}(-)\colon \mathit{FinSet}_* \to \mathit{Cat}_*,$$

where FinSet_* is the category of finite sets and pointed maps and Cat_* is the category of small pointed categories and pointed functors between them. This notion is a variant of Segal's Γ–category construction [16]. If S is a pointed simplicial set which is finite in each simplicial level, we can regard it as a functor $S\colon \Delta^{\mathrm{op}} \to \mathit{FinSet}_*$ and form the composite functor

$$\mathscr{C}(S)\colon \Delta^{\mathrm{op}} \xrightarrow{S} \mathit{FinSet}_* \xrightarrow{\mathscr{C}(-)} \mathit{Cat}_*, \tag{2}$$

which is a simplicial pointed category, ie a simplicial object in Cat_*.

Definition 5.4 An additive category with weak equivalences is a pair $(\mathscr{C}, w\mathscr{C})$ where $\mathscr{C}$ is an additive category and $w\mathscr{C} \subseteq \mathscr{C}$ is a subcategory such that all isomorphisms are in $w\mathscr{C}$ and such that if f and g are in $w\mathscr{C}$ then their coproduct $f \oplus g$ is in $w\mathscr{C}$.

A map $F\colon (\mathscr{C}, w\mathscr{C}) \to (\mathscr{C}', w'\mathscr{C}')$ of additive categories with weak equivalences is an additive functor which preserves weak equivalences. It is an equivalence of additive categories with weak equivalences if it is an equivalence on the underlying categories and on the categories of weak equivalences. If $\mathscr{C}$ is pointed, we take $w\mathscr{C}$ to be pointed with the same chosen object as $\mathscr{C}$.

Let $(\mathscr{C}, w\mathscr{C})$ be a pointed additive category with weak equivalences and X a finite pointed set. The subcategory $w\mathscr{C}(X) \subseteq \mathscr{C}(X)$ which has the same objects as $\mathscr{C}(X)$ and morphisms that are pointwise in $w\mathscr{C}$ is a subcategory of weak equivalences. If $f\colon X \to Y$ is a pointed map, then the functor f_* maps $w\mathscr{C}(X)$ into $w\mathscr{C}(Y)$, so there is an induced functor

$$w\mathscr{C}(-)\colon \mathit{FinSet}_* \to \mathit{Cat}_*.$$

As in Lemma 5.3, the functor $we_X\colon w\mathscr{C}(X) \to w\mathscr{C}^X$ induced by e_X is an equivalence of categories. We write S^1 for the simplicial circle $\Delta^1/\partial\Delta^1$, with basepoint $[\partial\Delta^1]$.

Segal showed in [16] that the space $\Omega|Nw\mathscr{C}(S^1)|$ is a model for the algebraic K–theory of $(\mathscr{C}, w\mathscr{C})$, analogous to the space $\Omega|BM|$ for a simplicial monoid M. Here $w\mathscr{C}(S^1)$ is defined as in the construction (2) above.

The functor $w\mathscr{C} \to w\mathscr{C}(S^1_1)$ sending an object c to the diagram with value c on the nontrivial subset of S^1_1 and $0_{\mathscr{C}}$ on $\{*\}$ is an equivalence of categories. There is an induced map

$$\Delta^1 \boxtimes Nw\mathscr{C} \to Nw\mathscr{C}(S^1)$$

of bisimplicial sets, which in turn induces a map

$$\lambda_{\mathscr{C}}\colon |Nw\mathscr{C}| \to \Omega|Nw\mathscr{C}(S^1)|$$

of spaces. In [16, Section 4] Segal proves a group completion theorem for the map $\lambda_{\mathscr{C}}$ analogous to Theorem 3.8. We will mimic the treatment of the monoid case above to reprove Segal's result and extend it to an equivariant statement analogous to Theorem 3.12 in the case that $\mathscr{C}$ has an additive duality.

Lemma 5.5 (see eg [7, Theorem XI.3.1]) *Let $(\mathscr{C}, w\mathscr{C})$ be an additive category with weak equivalences. Then there is a pointed additive category with weak equivalences $(\mathscr{C}', w'\mathscr{C}')$ and an additive equivalence $F\colon (\mathscr{C}, w\mathscr{C}) \to (\mathscr{C}', w'\mathscr{C}')$ such that $(\mathscr{C}', w'\mathscr{C}')$ has a coproduct functor*

$$\oplus\colon \mathscr{C}' \times \mathscr{C}' \to \mathscr{C}'$$

making $\mathscr{C}'$ a strictly unital, strictly associative symmetric monoidal category.

The construction $w\mathscr{C}(S^1)$ makes sense also for nonpointed $\mathscr{C}$ but one must choose a basepoint for $\Omega|Nw\mathscr{C}(S^1)|$ and $\lambda_{\mathscr{C}}$ to be defined. This can be done in such a way that the induced map $F_{S^1}\colon w\mathscr{C}(S^1) \to w'\mathscr{C}'(S^1)$ gives a homotopy equivalence on geometric realizations and there is a commutative diagram

$$\begin{array}{ccc} |Nw\mathscr{C}| & \xrightarrow{\lambda_{\mathscr{C}}} & \Omega|Nw\mathscr{C}(S^1)| \\ \downarrow F & & \downarrow \Omega|NF_{S^1}| \\ |Nw'\mathscr{C}'| & \xrightarrow[\lambda'_{\mathscr{C}}]{} & \Omega|Nw'\mathscr{C}'(S^1)| \end{array}$$

of spaces, in which the vertical maps are homotopy equivalences and H–maps. From now on we assume, without loss of generality, that $(\mathscr{C}, w\mathscr{C})$ is pointed and has a strictly unital and strictly associative coproduct functor $\oplus$ as in Lemma 5.5.

The path components of the nerve $Nw\mathscr{C}$ will be called weak equivalence classes. The set $\pi_0 Nw\mathscr{C}$ of such classes is a commutative monoid under the operation $[a]+[b]=$

$[a \oplus b]$. We assume that $\pi_0 N w\mathscr{C}$ has a cofinal generator represented by an object t of $\mathscr{C}$. Then there is a functor $t \oplus -\colon \mathscr{C} \to \mathscr{C}$ which restricts to an endofunctor on $w\mathscr{C}$. By analogy with the monoid case above, we form the diagram

$$w\mathscr{C} \xrightarrow{t\oplus -} w\mathscr{C} \xrightarrow{t\oplus -} w\mathscr{C} \xrightarrow{t\oplus -} \cdots$$

of categories. We define $\underline{\mathbb{N}}$ to be the poset category of natural numbers with the usual ordering

$$0 \to 1 \to 2 \to \cdots,$$

so that the above diagram of categories becomes a functor $D\colon \underline{\mathbb{N}} \to \mathit{Cat}$ in the obvious way. Now set $w\mathscr{C}_\infty = \underline{\mathbb{N}} \int D$, where $\int$ denotes the Grothendieck construction (see eg [19]). The objects of the category $w\mathscr{C}_\infty$ are pairs $(m, c) \in \mathbb{N} \times \mathrm{ob}\,\mathscr{C}$ and a map $(n, c) \to (n+k, d)$ is a map $(t\oplus -)^k(c) \to d$ in $w\mathscr{C}$ (see also [3, page 8]). Thomason [19, Theorem 1.2] constructs a natural weak equivalence

$$\mathrm{hocolim}(ND) \to N w\mathscr{C}_\infty.$$

Since the nerve $N w\mathscr{C}$ is a simplicial monoid, its homology $H_*(N w\mathscr{C})$ is a ring under the induced Pontryagin product. The following result is the categorical analog of Lemma 3.4:

Lemma 5.6 *The canonical functor $w\mathscr{C} \to w\mathscr{C}_\infty$ sending an object c to $(0, c)$ induces an isomorphism of left $H_*(N w\mathscr{C})$–modules*

$$H_*(N w\mathscr{C})[\pi_0(N w\mathscr{C})^{-1}] \xrightarrow{\cong} H_*(N w\mathscr{C}_\infty).$$

We now recall the simplicial path construction (see [20, Section 1.5] for details). Define the shift functor $P\colon \Delta \to \Delta$ by $P([n]) = [0] \star [n] = [1+n]$ and $P(\alpha) = \mathrm{id}_{[0]} \star \alpha$, where $\star$ denotes the join operation on Δ. For a simplicial object $X\colon \Delta^{\mathrm{op}} \to \mathscr{A}$ the (simplicial) path object PX on X is defined as $PX = X \circ P^{\mathrm{op}}$. The natural transformation $\delta^0\colon \mathrm{Id}_\Delta \to P$ given on objects by $\delta^0\colon [n] \to [1+n]$ gives a natural map $d_0\colon PX \to X$. For a simplicial set X there is a natural map $PX \to X_0$ onto the vertices of X which is a simplicial homotopy equivalence [20, Lemma 1.5.1]. In the case of the simplicial circle the map $d_0\colon PS^1 \to S^1$ induces a map $w\mathscr{C}(PS^1) \to w\mathscr{C}(S^1)$ of simplicial categories, which we will also call d_0. There is a simplicial homotopy equivalence $PS^1 \xrightarrow{\simeq} *$ which induces a weak equivalence $N w\mathscr{C}(PS^1) \xrightarrow{\simeq} N w\mathscr{C}(*) \simeq *$ of bisimplicial sets. Let $\tilde{t}_{n+1}$ be the object of $\mathscr{C}(S^1_{n+1})$ which maps to $(t, 0, \ldots, 0)$ under the equivalence $e_{S^1_{n+1}}\colon \mathscr{C}(S^1_{n+1}) \to \mathscr{C}^{S^1_{n+1}}$. Then $d_0(\tilde{c}_{n+1})$ is the constant diagram with value $0_{\mathscr{C}}$ and adding $\widetilde{t}_{n+1}$ from the left gives a functor

$$\widetilde{t}_{n+1} \oplus -\colon w\mathscr{C}(S^1_{n+1}) \to w\mathscr{C}(S^1_{n+1}),$$

which commutes with d_0. We define $w\mathscr{C}(S^1_{n+1})_\infty$ to be the Grothendieck construction on the diagram

$$w\mathscr{C}(S^1_{n+1}) \xrightarrow{\widetilde{t}_{n+1}\oplus -} w\mathscr{C}(S^1_{n+1}) \xrightarrow{\widetilde{t}_{n+1}\oplus -} w\mathscr{C}(S^1_{n+1}) \xrightarrow{\widetilde{t}_{n+1}\oplus -} \cdots .$$

The $w\mathscr{C}(S^1_{n+1})_\infty$'s assemble to a simplicial category $w\mathscr{C}(PS^1)_\infty$ with a map

$$d_{0,\infty}\colon w\mathscr{C}(PS^1)_\infty \to w\mathscr{C}(S^1).$$

Including $w\mathscr{C}(S^1_{n+1})$ in the first spot of the diagram gives a map $w\mathscr{C}(PS^1) \to w\mathscr{C}(PS^1)_\infty$ such that the diagram

$$\begin{array}{ccc} w\mathscr{C}(PS^1) & \longrightarrow & w\mathscr{C}(PS^1)_\infty \\ & {\scriptstyle d_0}\searrow \quad \swarrow{\scriptstyle d_{0,\infty}} & \\ & w\mathscr{C}(S^1) & \end{array}$$

commutes.

Proposition 5.7 *The induced map on nerves,*

$$Nd_{0,\infty}\colon Nw\mathscr{C}(PS^1)_\infty \to Nw\mathscr{C}(S^1),$$

is a homology fibration of bisimplicial sets.

Proof We will show that the map satisfies the conditions of Theorem 2.15. First, we verify that it is a levelwise homology fibration. The functor $\widetilde{t}_{n+1} \oplus -$ commutes with $(d_{0,\infty})_n$, so the evaluation functors give a commuting square

$$\begin{array}{ccc} w\mathscr{C}(S^1_{n+1})_\infty & \xrightarrow{e_{n,\infty}} & w\mathscr{C}_\infty \times w\mathscr{C}^{\times n} \\ \downarrow & & \downarrow{\scriptstyle p} \\ w\mathscr{C}(S^1_n)_{e_n} & \longrightarrow & w\mathscr{C}^{\times n} \end{array} \tag{3}$$

where the horizontal arrows are equivalences of categories.

Assume given a simplex $\sigma\colon \Delta^m \to Nw\mathscr{C}(S^1_n)$ and consider the resulting diagram

$$\begin{array}{ccccc} \Delta^m & \xrightarrow{\sigma} & Nw\mathscr{C}(S^1_n) & \xleftarrow{N(d_{0,\infty})_n} & Nw\mathscr{C}(S^1_{n+1})_\infty \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle Ne_n} & & \downarrow{\scriptstyle Ne_{n,\infty}} \\ \Delta^m & \xrightarrow[N(e_n\circ\sigma)]{} & Nw\mathscr{C}^{\times n} & \xleftarrow[p]{} & Nw\mathscr{C}_\infty \times Nw\mathscr{C}^{\times n} \end{array}$$

Since the vertical maps are weak equivalences, the induced map on homotopy pullbacks is a weak equivalence. The map p is obviously a homology fibration, so it suffices to show that the map on actual pullbacks is a weak equivalence. Nerves commute with limits, so this pullback can be taken in Cat_*, where it is straightforward to check that the map on pullbacks is an equivalence of categories.

To see that the second condition of Theorem 2.15 holds, we observe that the fiber over an object c in $w\mathscr{C}(S^1_n)$ is equivalent to $w\mathscr{C}_\infty$. Now we conclude by Lemma 5.6 in the same way as in the proof of Lemma 3.5. □

The proof of the following theorem (cf [16, Section 4; 11, Section Q.9]) is similar to that of Theorem 3.8:

Theorem 5.8 (K–theoretic group completion) *The map $\lambda_{\mathscr{C}}$ induces an isomorphism of left $H_*(Nw\mathscr{C})$–modules*

$$H_*(Nw\mathscr{C})[\pi_0(Nw\mathscr{C})^{-1}] \xrightarrow{\cong} H_*\big(\Omega|Nw\mathscr{C}(S^1)|\big).$$

We now turn to additive categories with duality.

Definition 5.9 An additive category with duality and weak equivalences is a tuple $(\mathscr{C}, T, \eta, w\mathscr{C})$ such that

(1) T is additive and η takes values in weak equivalences,

(2) T and η give a duality on $\mathscr{C}$,

(3) T sends (opposites of) weak equivalences to weak equivalences,

(4) $(\mathscr{C}, w\mathscr{C})$ is an additive category with weak equivalences.

Note that the conditions (1) and (3) imply that T and η restrict to a duality structure on the category $w\mathscr{C}$ and so we may apply the Sym–construction. For ease of notation we will often write $\mathrm{Sym}(w\mathscr{C})$ when T and η are clear from the context.

Example 5.10 Let (R, α, ε) be a Wall antistructure. Then the category $P(R, \alpha, \varepsilon)$ becomes an additive category with duality and weak equivalences if we take the weak equivalences to be the isomorphisms. In applications it is sometimes useful to work with bounded chain complexes in $P(R, \alpha, \varepsilon)$ with quasi-isomorphisms as weak equivalences; see eg Schlichting's paper [14].

To get a strict duality we can apply the functor $\mathscr{D}$; the category $\mathscr{D}\mathscr{C}$ is additive since $\mathscr{C}$ is. Taking the weak equivalences in $\mathscr{D}\mathscr{C}$ to be pairs of maps in $w\mathscr{C}$ gives $\mathscr{D}(\mathscr{C}, T, \eta)$ the structure of an additive category with duality and weak equivalences which is a functorial and better-behaved replacement of $(\mathscr{C}, T, \eta, w\mathscr{C})$. There is a square of H–spaces and H–maps

$$\begin{array}{ccc} |Nw\mathscr{C}| & \xrightarrow{\lambda_{\mathscr{C}}} & \Omega|Nw\mathscr{C}(S^1)| \\ \downarrow & & \downarrow \\ |Nw\mathscr{D}\mathscr{C}| & \xrightarrow{\lambda_{\mathscr{D}\mathscr{C}}} & \Omega|Nw\mathscr{D}\mathscr{C}(S^1)| \end{array}$$

where the vertical maps are weak equivalences. Note that Lemma 5.5 also applies to additive categories with duality and weak equivalences, so that we may assume that our categories have a strict duality T, a duality-preserving direct sum functor $(-\oplus-)$ which is strictly associative and strictly unital and that the unit 0 is fixed under the duality; see eg [12, Remark A.11].

Remark 5.11 Let $\nu\colon T(-)\oplus T(-)\to T(-\oplus-)$ be the canonical natural isomorphism. The category $\mathrm{Sym}(w\mathscr{C})$ has a functorial sum operation $\perp$, called the orthogonal sum, given by

$$(f\colon c\to T(c))\perp(g\colon d\to Td) = c\oplus d \xrightarrow{f\oplus g} T(c)\oplus T(d) \xrightarrow{\nu_{c,d}} T(c\oplus d).$$

Under the induced operation the set $\pi_0(\mathrm{Sym}(w\mathscr{C}))$ becomes a commutative monoid with unit represented by the 0–form $0\to 0$. For any object $c\xrightarrow{f} d$ of $\mathrm{Sd}(w\mathscr{C})$ we can form the *hyperbolic form* $H(f)$ on $c\xrightarrow{f} d$, which is the object

$$c\oplus T(d) \xrightarrow{\begin{pmatrix} 0 & T(f) \\ \eta_d\circ f & 0\end{pmatrix}} T(c)\oplus TT(d) \xrightarrow{\nu_{c,T(d)}} T(c\oplus T(d))$$

of $\mathrm{Sym}(w\mathscr{C})$. This is also compatible with maps in $\mathrm{Sd}\, w\mathscr{C}$. Together, the functors $\perp$ and H give an action of $\mathrm{Sd}\, w\mathscr{C}$ on $\mathrm{Sym}(w\mathscr{C})$, in the sense of [3, page 218], which is analogous to the action of M on M^{C_2} of Section 3.

Let X be a pointed C_2–set with $\sigma\colon X\to X$ representing the action of the nontrivial group element. We define a strict duality σ_X on $Q(X)$ by composing the natural isomorphism $Q(X)^{\mathrm{op}} \xrightarrow{\cong} Q(X)$ with the map defined by $U\mapsto\sigma(U)$ and similarly for morphisms. If $\mathscr{C}$ is an additive category with weak equivalences and strict duality, there is an induced duality T_X on $w\mathscr{C}(X)$ given by taking a diagram

$$A\colon Q(X)\to\mathscr{C}$$

to the composite diagram

$$Q(X) \xrightarrow{\sigma_X^{\mathrm{op}}} Q(X)^{\mathrm{op}} \xrightarrow{A^{\mathrm{op}}} \mathscr{C}^{\mathrm{op}} \xrightarrow{T} \mathscr{C}.$$

Clearly the duality T_X is strict and functorial in both X and $(\mathscr{C}, T)$. Let $n_+ = \{0, 1, \dots, n\}$, based at 0, with the action of C_2 taking an element $k \geq 1$ to $n-k+1$ and fixing 0. If $X = 2_+$ then the action interchanges the two nontrivial elements and the duality on $\mathscr{C}(2_+)$ sends the diagram

$$X \underset{i_1}{\overset{p_1}{\leftrightarrows}} Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} X'$$

to the diagram

$$TX' \underset{Tp_2}{\overset{Ti_2}{\leftrightarrows}} TY \underset{Tp_1}{\overset{Ti_1}{\rightleftarrows}} TX.$$

We give $w\mathscr{C}^{\times n}$ the strict duality given on objects by

$$(X_1, \dots, X_n) \mapsto (TX_n, \dots, TX_1)$$

and similarly for maps. The evaluation map

$$e_n \colon w\mathscr{C}(n_+) \to w\mathscr{C}^{\times n}$$

is compatible with these dualities and is an equivalence of categories with duality.

We will now use the real simplicial set $S^{1,1} = \Delta^1_R / \partial \Delta^1_R$ to describe an action of C_2 on the algebraic K–theory space of an additive category with strict duality and weak equivalences. For each $n \geq 0$ the category $w\mathscr{C}(S^{1,1}_n)$ inherits a duality T_n from the action of w_n and the duality T. There are induced maps

$$w_{m,n} \colon N_n w\mathscr{C}(S^{1,1}_m) \to N_n w\mathscr{C}(S^{1,1}_m)$$

given by

$$w_{m,n}(A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n) = (T_m A_n \xrightarrow{T_m f_n} \cdots \xrightarrow{T_m f_1} T_m A_0),$$

which satisfy the relations $w_{m,n} \circ w_{m,n} = \mathrm{id}$ and $w_{m,n} \circ (\alpha, \beta)^* = (\alpha^{\mathrm{op}}, \beta^{\mathrm{op}})^* \circ w_{p,q}$ for maps $(\alpha, \beta) \colon ([m],[n]) \to ([p],[q])$ in $\Delta \times \Delta$. After subdivision these assemble to a map of bisimplicial sets

$$W \colon \mathrm{Sd}\, N w\mathscr{C}(\mathrm{Sd}\, S^{1,1}) \to \mathrm{Sd}\, N w\mathscr{C}(\mathrm{Sd}\, S^{1,1}),$$

which in level (m,n) is the map $N w_{2m+1,2n+1}$.

The bisimplicial set $\operatorname{Sd} Nw\mathscr{C}(\operatorname{Sd} S^{1,1})$ is naturally isomorphic to $N\operatorname{Sd} w\mathscr{C}(\operatorname{Sd} S^{1,1})$ and under this identification the map W comes from a map of simplicial categories

$$\widetilde{W}\colon \operatorname{Sd} w\mathscr{C}(\operatorname{Sd} S^{1,1}) \to \operatorname{Sd} w\mathscr{C}(\operatorname{Sd} S^{1,1}),$$

which squares to the identity and hence defines an action of C_2 on $\operatorname{Sd} w\mathscr{C}(\operatorname{Sd} S^{1,1})$. The function $f_n\colon S^{1,1}_n \to n_+$ given by sending the basepoint to 0 and $f(\alpha) = \#\alpha^{-1}(\{1\})$ for $\alpha = *$ is a C_2–equivariant bijection. Let

$$e^{1,1}_n\colon w\mathscr{C}(S^{1,1}_n) \to w\mathscr{C}^{\times n}$$

be the evaluation map corresponding to f_n. It is an equivalence of categories with duality and in level $2n+1$ it induces a functor

$$\operatorname{Sd} e^{1,1}_{2n+1}\colon \operatorname{Sd} w\mathscr{C}(S^{1,1}_{2n+1}) \to \operatorname{Sd} w\mathscr{C}^{\times 2n+1}$$

which is C_2–equivariant. The category $\operatorname{Sd} w\mathscr{C}^{\times 2n+1}$ has the action given by

$$(f_1, \dots, f_{2n+1}) \mapsto (Tf_{2n+1}, \dots, Tf_1),$$

so a fixed object is of the form $(f_1, \dots, f_n, f_{n+1}, Tf_n, \dots, Tf_1)$ with $Tf_{n+1} = f_{n+1}$. We see that the last n factors are redundant, so evaluation followed by projection on the first $n+1$ coordinates defines a functor

$$\operatorname{Sym}(w\mathscr{C}(S^{1,1}_{2n+1})) \to \operatorname{Sd} w\mathscr{C}^{\times n} \times \operatorname{Sym}(w\mathscr{C})$$

which is an equivalence of categories.

The map $d_0\colon PS^1 \to S^1$ induces a map $\operatorname{Sd} d_0\colon \operatorname{Sd} PS^1 \to \operatorname{Sd} S^1$ and hence a map of simplicial categories

$$\operatorname{Sd} w\mathscr{C}(\operatorname{Sd} PS^1) \to \operatorname{Sd} w\mathscr{C}(\operatorname{Sd} S^1).$$

Define $\operatorname{Pb}(\mathscr{C}, T, w\mathscr{C})$ to be the pullback in the diagram

$$\begin{array}{ccc} \operatorname{Pb}(\mathscr{C}, T, w\mathscr{C}) & \longrightarrow & \operatorname{Sd} w\mathscr{C}(\operatorname{Sd} PS^1) \\ \downarrow & & \downarrow \\ \operatorname{Sym}(w\mathscr{C}(\operatorname{Sd} S^{1,1})) & \longrightarrow & \operatorname{Sd} w\mathscr{C}(\operatorname{Sd} S^1) \end{array}$$

of simplicial categories where the bottom map is the inclusion functor and the right-hand vertical map is induced by $\operatorname{Sd} d_0$. Note that the evaluation map gives an equivalence of categories

$$\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})_n \simeq \mathrm{Sd}\, w\mathscr{C} \times \mathrm{Sd}\, w\mathscr{C}^{\times^n} \times \mathrm{Sym}(w\mathscr{C}).$$

Thinking of $\mathrm{Sym}(w\mathscr{C})$ as a constant simplicial category, we define a map of simplicial categories

$$i\colon \mathrm{Sym}(w\mathscr{C}) \to \mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})$$

which in level n sends an object $f\colon a \to Ta$ to the sum-diagram with value $f\colon a \to Ta$ on subsets containing the unique nontrivial fixed point of $S^{1,1}_{2n+1}$ and id: $0 \to 0$ on subsets not containing it. The morphisms in $i_n(f)$ are identities or 0 as for $\tilde{t}_n$.

Lemma 5.12 *The map i induces a weak equivalence on nerves.*

Proof Under the equivalences $\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C}) \simeq \mathrm{Sd}\, w\mathscr{C} \times \mathrm{Sd}\, w\mathscr{C}^{\times^n} \times \mathrm{Sym}(w\mathscr{C})$, the functor i_n corresponds to the inclusion of $\mathrm{Sym}(w\mathscr{C})$ by

$$(f\colon a \to Ta) \mapsto (\mathrm{id}\colon 0 \to 0,\ \mathrm{id}\colon 0 \to 0,\ \dots,\ \mathrm{id}\colon 0 \to 0,\ f\colon a \to Ta).$$

Applying the simplicial path construction P to $\mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1}))$ gives a simplicial category with a simplicial deformation retraction to $\mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1})_0) = \mathrm{Sym}(w\mathscr{C})$. There are maps

$$P\mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1})) \to \mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1})),$$
$$P\mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1})) \to \mathrm{Sd}\, w\mathscr{C}(\mathrm{Sd}\, PS^1),$$

analogous to the ones for $B(M, M, M^{C_2})$ in Lemma 3.10 and the resulting map $P\mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1})) \to \mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})$ is a levelwise equivalence of categories. □

Now assume that $\mathscr{C}$ has an object t whose class in $\pi_0 N w\mathscr{C}$ is a cofinal generator. By a colimit argument as in the proof of Theorem 3.8, this can be done without loss of generality. The subdivision of the functor $t \oplus -\colon w\mathscr{C} \to w\mathscr{C}$ is the functor that adds $t \xrightarrow{\mathrm{id}} t$ to objects $a \to b$ of $\mathrm{Sd}\, w\mathscr{C}$. Similarly, the subdivision $\mathrm{Sd}\, \tilde{t}_n \oplus -$ of the functor $\tilde{t} \oplus -$, defined earlier, adds the map of sum-diagrams $\tilde{t}_n \xrightarrow{\mathrm{id}} \tilde{t}_n$ to objects $A \to B$ in $\mathrm{Sd}\, w\mathscr{C}(S^{1,1}_n)$. For each $n \geq 0$ there is a diagram

$$\mathrm{Sd}\, w\mathscr{C}(S^1_n) \xrightarrow{\mathrm{Sd}\, \tilde{t}_n \oplus -} \mathrm{Sd}\, w\mathscr{C}(S^1_n) \xrightarrow{\mathrm{Sd}\, \tilde{t}_n \oplus -} \cdots$$

and we define $\mathrm{Sd}\, w\mathscr{C}_\infty$ and $\mathrm{Sd}\, w\mathscr{C}(S^1_n)_\infty$ to be the Grothendieck constructions on the diagrams as before. The map

$$(\mathrm{Sd}\, d_0)_*\colon \mathrm{Sd}\, w\mathscr{C}(S^1_{2n+2}) \to \mathrm{Sd}\, w\mathscr{C}(S^1_{2n+1})$$

commutes with the maps $\mathrm{Sd}\,\tilde{t}_n \oplus -$ and just as before there is an induced map

$$\mathrm{Sd}\, w\mathscr{C}(\mathrm{Sd}\, PS^1)_\infty \to \mathrm{Sd}\, w\mathscr{C}(\mathrm{Sd}\, S^1)$$

which induces a homology fibration on nerves. The maps $\mathrm{Sd}\,\tilde{t}_n \oplus -$ also induce a map on the pullback $\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})$ which commutes with the projection to $\mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1}))$. There results a pullback square of simplicial categories

$$\begin{array}{ccc} \mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})_\infty & \longrightarrow & \mathrm{Sd}\, w\mathscr{C}(\mathrm{Sd}\, PS^1)_\infty \\ \downarrow & & \downarrow \\ \mathrm{Sym}(w\mathscr{C}(\mathrm{Sd}\, S^{1,1})) & \longrightarrow & \mathrm{Sd}\, w\mathscr{C}(\mathrm{Sd}\, S^1) \end{array}$$

where the vertical maps induce homology fibrations on nerves. The inclusion of $\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})$ into $\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})_\infty$ at the start of the diagram defining the latter will be called j.

Lemma 5.13 *The map*

$$j \circ i\colon \mathrm{Sym}(w\mathscr{C}) \to \mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})_\infty$$

induces an isomorphism

$$H_*(N\,\mathrm{Sym}(w\mathscr{C}))[\pi_0 N w\mathscr{C}^{-1}] \xrightarrow{\cong} H_*(N\,\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})_\infty)$$

of left $H_*(N\,\mathrm{Sd}\, w\mathscr{C})$–modules.

Proof By Lemma 5.12, the map i induces an isomorphism $H_*(N\,\mathrm{Sym}(w\mathscr{C})) \cong H_*(N\,\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C}))$ of left $H_*(N\,\mathrm{Sd}\, w\mathscr{C})$–modules. The map

$$\mathrm{Sd}\,\tilde{t}\colon \mathrm{Pb}(\mathscr{C}, T, w\mathscr{C}) \to \mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})$$

induces left multiplication by $[t]$ on $H_*(N\,\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C}))$, and by Thomason's theorem [19, Theorem 1.2] we get a sequence of isomorphisms

$$\begin{aligned} H_*(N\,\mathrm{Pb}(\mathscr{C}, T, w\mathscr{C})_\infty) &\cong \mathrm{colim}\big(H_*(N\,\mathrm{Sym}(w\mathscr{C})) \xrightarrow{[t]\cdot} H_*(N\,\mathrm{Sym}(w\mathscr{C})) \xrightarrow{[t]\cdot} \cdots\big) \\ &\cong H_*(N\,\mathrm{Sym}(w\mathscr{C}))[t^{-1}] \\ &\cong H_*(N\,\mathrm{Sym}(w\mathscr{C}))[\pi_0 N w\mathscr{C}^{-1}] \end{aligned}$$

of left $H_*(N\,\mathrm{Sd}\, w\mathscr{C})$–modules, as desired. □

The proof of the following statement is similar to that of Theorem 3.12. We use that there is a natural ring isomorphism $H_*(N\,\mathrm{Sd}\, w\mathscr{C}) \cong H_*(N w\mathscr{C})$.

Theorem 5.14 *Let $(\mathscr{C}, w\mathscr{C}, T)$ be an additive category with strict duality and weak equivalences. Then the map $|N\operatorname{Sym}(w\mathscr{C})| \to \big(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\big)^{C_2}$ induces isomorphisms*

$$\pi_0(N\operatorname{Sym}(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} \pi_0\big(\big(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\big)^{C_2}\big)$$

of monoids and

$$H_*(N\operatorname{Sym}(w\mathscr{C}))[\pi_0 Nw\mathscr{C}^{-1}] \xrightarrow{\cong} H_*\big(\big(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\big)^{C_2}\big)$$

of left $H_*(Nw\mathscr{C})$–modules.

For a Wall antistructure (R, α, ε) we set

$$K_0^{1,1}(R,\alpha,\varepsilon) = \pi_0\big(\Omega^{1,1}|N\operatorname{Sym}(i\mathscr{D}P(R,\alpha,\varepsilon))(S^{1,1})|\big)^{C_2}.$$

We will now investigate the two fundamental cases when $R = \mathbb{Z}$ and $\alpha = \mathrm{id}_{\mathbb{Z}}$, namely $\varepsilon = 1$ and $\varepsilon = -1$. In the first case observe that $\operatorname{Sym}(iP(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1))$ is the category of nondegenerate symmetric bilinear form spaces over $\mathbb{Z}$.

Proposition 5.15 *The monoid $K_0^{1,1}(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1)$ is not a group.*

Proof By Theorem 5.14, there is an isomorphism

$$K_0^{1,1}(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1) \cong \pi_0|N\operatorname{Sym}(i\mathscr{D}P(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1))|\big[\pi_0\big(Ni\mathscr{D}(P(\mathbb{Z}))\big)^{-1}\big]$$

and the right-hand side is isomorphic to the monoid

$$M = \pi_0 N\operatorname{Sym}(iP(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1))\big[\pi_0\big(Ni(P(\mathbb{Z}))\big)^{-1}\big].$$

We will show that the latter is not a group by finding an element that does not have an inverse.

The n^{th} hyperbolic space H^n is the symmetric bilinear form space with underlying abelian group $\mathbb{Z}^{2n}$ and the symmetric form given by the matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix. The monoid $\pi_0\big(Ni(P(\mathbb{Z}))\big)$, which is isomorphic to $\mathbb{N}$, acts on $\pi_0 N\operatorname{Sym}(iP(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1))$ by adding hyperbolic spaces H^n via the orthogonal sum. Let $\langle 1\rangle$ denote the group $\mathbb{Z}$ with the symmetric bilinear form given by ordinary multiplication. Assume that $[\langle 1\rangle]$ has an inverse in M. Elements of M can be represented as differences $[a] - [H^m]$ where a is in $\operatorname{Sym}(iP(\mathbb{Z}, \mathrm{id}_{\mathbb{Z}}, 1))$.

An inverse for $[\langle 1\rangle]$ is a difference $[a]-[H^m]$ such that $[\langle 1\rangle]+[a]-[H^m]=0$ in M, or equivalently such that for some n the equation

$$[\langle 1\rangle]+[a]+[H^n]=[H^m]+[H^n]$$

holds in $\pi_0 N\mathrm{Sym}(iP(\mathbb{Z},\mathrm{id}_{\mathbb{Z}},1))$. Since $H^m \perp H^n \cong H^{m+n}$, this means that we have an isomorphism

$$\langle 1\rangle \perp a \perp H^n \cong H^{m+n}.$$

On the left-hand side the element $(1,0,0)$ pairs with itself to $1\in\mathbb{Z}$ under the bilinear form. However, on the right-hand side any element gives an even number when paired with itself. We conclude that no such isomorphism exists and hence that $K_0^{1,1}(\mathbb{Z},\mathrm{id}_{\mathbb{Z}},1)$ is not a group. □

The second case is $\mathrm{Sym}(iP(\mathbb{Z},\mathrm{id}_{\mathbb{Z}},-1))$, the category of nondegenerate symplectic bilinear form spaces over $\mathbb{Z}$. We write ${}_{-1}H^n(\mathbb{Z})$ for the symplectic form module with matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

By [9, Section I.4, Corollary 3.5], any symplectic form module over $\mathbb{Z}$ is isomorphic to ${}_{-1}H^n(\mathbb{Z})$ for a uniquely determined $n\ge 0$. We call this number the rank of the symplectic module. The corresponding rank map

$$\pi_0|N\mathrm{Sym}(iP(\mathbb{Z},\mathrm{id}_{\mathbb{Z}},-1))|\to\mathbb{N}$$

is an isomorphism of monoids.

Proposition 5.16 *The rank map induces an isomorphism*

$$K_0^{1,1}(\mathbb{Z},\mathrm{id}_{\mathbb{Z}},-1)\cong\mathbb{Z}.$$

The results above show that the H–space $\left(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|\right)^{C_2}$ is not always the group completion of the topological monoid $N\mathrm{Sym}(\mathscr{C},T,w\mathscr{C})$ in the sense of [8] or [11]. However, there are many important cases where this holds.

Proposition 5.17 *Let $(\mathscr{C},T)$ be an additive category with strict duality such that the functor $\mathrm{Hom}_{\mathscr{C}}(-,-)$ takes values in the category of $\mathbb{Z}[\frac{1}{2}]$–modules. Then the map $|N\mathrm{Sym}(i\mathscr{C})|\to\left(\Omega^{1,1}|Ni\mathscr{C}(S^{1,1})|\right)^{C_2}$ induces isomorphisms*

$$\pi_0(N\mathrm{Sym}(i\mathscr{C}))[\pi_0 N\mathrm{Sym}(i\mathscr{C})^{-1}]\xrightarrow{\cong}\pi_0\left(\left(\Omega^{1,1}|Ni\mathscr{C}(S^{1,1})|\right)^{C_2}\right)$$

of groups and

$$H_*(N\mathrm{Sym}(i\mathscr{C}))[\pi_0 N\mathrm{Sym}(i\mathscr{C})^{-1}] \xrightarrow{\cong} H_*\big((\Omega^{1,1}|Ni\mathscr{C}(S^{1,1})|)^{C_2}\big)$$

of left $H_*(Nw\mathscr{C})$–modules.

Proof The monoid $\pi_0 Nw\mathscr{C}$ acts through the hyperbolic map

$$H\colon \pi_0 Nw\mathscr{C} \to \pi_0 N\mathrm{Sym}(i\mathscr{C}).$$

Since 2 is invertible in all the Hom groups of the category $\mathscr{C}$, any object (X,φ) of $\mathrm{Sym}(i\mathscr{C})$ is an orthogonal summand in an object of the form $H(Y)$ for some object Y of $\mathscr{C}$ by [6, Theorem 1.4]. It follows that H is a cofinal homomorphism of monoids, so that inverting the action of $\pi_0 Nw\mathscr{C}$ is, up to isomorphism, the same as inverting the action of all of $\pi_0 N\mathrm{Sym}(i\mathscr{C})$. Applying Theorem 5.14 now gives the desired result. □

Remark 5.18 The group $\pi_0(N\mathrm{Sym}(i\mathscr{C}))[\pi_0 N\mathrm{Sym}(i\mathscr{C})^{-1}]$ appearing in Proposition 5.17 is isomorphic to the classical Grothendieck–Witt group $\mathrm{GW}_0(\mathscr{C},T)$; see eg [14] for details. For more on the relation between group completion, Grothendieck–Witt groups and the space $(\Omega^{1,1}|Nw\mathscr{C}(S^{1,1})|)^{C_2}$ see [4].

Appendix The category ΔR

The category ΔR has the same objects as the finite ordinal category Δ but more morphisms. In addition to the maps of Δ, there is, for each $n \geq 0$, a morphism $\omega_n\colon [n] \to [n]$, which should be thought of as reversing the ordering on $[n]$. The maps satisfy the relations

$$\omega_n \circ \omega_n = \mathrm{id}_{[n]}, \tag{4}$$

$$\omega_{n-1} \circ \sigma^j = \sigma^{n-1-j} \circ \omega_n, \tag{5}$$

$$\omega_{n+1} \circ \delta^i = \delta^{n+1-i} \circ \omega_n \tag{6}$$

for $0 \geq i, j \geq n$. Following [4], a functor from $(\Delta R)^{\mathrm{op}}$ to sets is called a real simplicial set and similarly for functors into other categories. The maps induced in a real simplicial object by ω_n are denoted by w_n.

If we restrict a real simplicial set X to Δ^{op} then the geometric realization $|(X|_{\Delta^{\mathrm{op}}})|$ carries an action of C_2 which, for $(x,t_0,\dots,t_n) \in X_n \times \Delta^n$, acts by

$$[(x,t_0,\dots,t_n)] \mapsto [(w_n(x),t_n,\dots,t_0)]$$

(see [4] for details). Recall the functor

$$(-)^{\mathrm{op}}\colon \Delta \to \Delta$$

which is the identity on objects and which sends $\delta^i\colon [n]\to[n+1]$ to $(\delta^i)^{\mathrm{op}}=\delta^{n+1-i}$ and $\sigma^j\colon [n]\to[n-1]$ to $(\sigma^j)^{\mathrm{op}}=\sigma^{n-1-j}$. Clearly $(-)^{\mathrm{op}}\circ(-)^{\mathrm{op}}=\mathrm{Id}_\Delta$. Let $\mathscr{A}$ be any category. Given a simplicial object $X\colon \Delta^{\mathrm{op}}\to\mathscr{A}$, its opposite is defined by

$$X^{\mathrm{op}}=X\circ(-)^{\mathrm{op}}.$$

This defines a functor on the functor category $\mathscr{A}^{\Delta^{\mathrm{op}}}$ which squares to the identity.

Lemma A.1 *Giving an extension of a functor $X\colon \Delta^{\mathrm{op}}\to\mathscr{A}$ to the category ΔR^{op} is equivalent to giving a map $\omega\colon X^{\mathrm{op}}\to X$ such that $\omega\circ\omega^{\mathrm{op}}=\mathrm{id}_X$.*

Now recall the functor $\mathrm{Sd}\colon\Delta\to\Delta$ given by $\mathrm{Sd}[n]=[2n+1]$ and $\mathrm{Sd}(\theta)=\theta\star\theta^{\mathrm{op}}$. The Segal edgewise subdivision of X is defined by $\mathrm{Sd}\,X=X\circ\mathrm{Sd}$. This gives an endofunctor on $\mathscr{A}^{\Delta^{\mathrm{op}}}$ which satisfies $\mathrm{Sd}\circ(-)^{\mathrm{op}}=\mathrm{Sd}$, so that $\mathrm{Sd}\,X^{\mathrm{op}}=\mathrm{Sd}\,X$ for any simplicial object X. Given any real simplicial object $X\colon(\Delta R)^{\mathrm{op}}\to\mathscr{A}$, we can regard it as a simplicial object $X|_{\Delta^{\mathrm{op}}}$ with a map $\omega\colon X^{\mathrm{op}}\to X$ as in Lemma A.1. On the subdivision we get a map

$$\mathrm{Sd}(\omega)\colon \mathrm{Sd}\,X^{\mathrm{op}}=\mathrm{Sd}\,X\to\mathrm{Sd}\,X$$

such that $\mathrm{Sd}(\omega)^2=\mathrm{id}_{\mathrm{Sd}\,X}$. In other words, $\mathrm{Sd}(\omega)$ defines an action of C_2 on $\mathrm{Sd}\,X$. For a real simplicial set X the natural homeomorphism $\bigl|\mathrm{Sd}(X|_{\Delta^{\mathrm{op}}})\bigr|\xrightarrow{\cong}|X|_{\Delta^{\mathrm{op}}}|$ of [15, Appendix 1, Proposition A.1] is C_2–equivariant.

Example A.2 The simplicial set Δ^1 extends to a real simplicial set Δ^1_R with $w_n(\alpha)=\alpha^{\mathrm{op}}$. Its geometric realization is the topological 1–simplex with the action of C_2 given by reflection through the middle point. Its boundary $\partial\Delta^1_R$ realizes to the two endpoints, which are interchanged by the C_2–action. We write $S^{1,1}_R$ for the real simplicial set $\Delta^1_R/\partial\Delta^1_R$. The geometric realization $|S^{1,1}_R|$ is C_2–homeomorphic to the usual $S^{1,1}$.

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