The syzygy order of big polygon spaces

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Big polygon spaces are compact orientable manifolds with a torus action whose equivariant cohomology can be torsion-free or reflexive without being free as a module over $H^*(BT)$. We determine the exact syzygy order of the equivariant cohomology of a big polygon space as a function of the length vector defining it. The proof uses a refined characterization of syzygies in terms of certain linearly independent elements in $H^2(BT)$ adapted to the isotropy groups occurring in a given T–space.

1 Introduction

Let $T \cong (S^1)^r$ be a torus, and let X be a T–manifold whose cohomology $H^*(X)$ (with real coefficients) is finite-dimensional. A powerful tool to compute the equivariant cohomology $H_T^*(X)$ is the Chang–Skjelbred sequence

$$(1\text{-}1) \qquad 0 \to H_T^*(X) \to H_T^*(X^T) \to H_T^{*+1}(X_1, X^T),$$

where $X^T \subset X$ is the fixed-point set and the equivariant 1–skeleton X_1 the union of orbits of dimension at most 1. The first map is induced by the inclusion $X^T \hookrightarrow X$ and the second one is the connecting homomorphism in the long exact sequence for the pair (X_1, X^T).

If $H_T^*(X)$ is free as a module over the polynomial ring $R = H^*(BT)$, then the Chang–Skjelbred sequence is exact — see Chang and Skjelbred [6, Proposition 2.4] — which implies that $H_T^*(X)$ can be calculated out of the equivariant 1–skeleton. In many cases of interest, X^T is finite and X_1 a union of 2–spheres glued together at their poles. In such a setting, this approach is called the GKM method after work of Goresky, Kottwitz and MacPherson [12, Theorem 7.2].

It is not hard to find examples of T–manifolds such that (1-1) is exact without $H_T^*(X)$ being free over R; see below. This phenomenon was studied in detail by Allday, Franz and Puppe [1; 2], who characterized those T–manifolds for which the Chang–Skjelbred

sequence is exact; in [9] this is generalized to nonabelian Lie groups. Allday, Franz and Puppe actually proved a more general theorem that involves higher equivariant skeletons. For $-1 \leq i \leq r$, we write $X_i \subset X$ for the union of orbits of dimension at most i, so that $X_0 = X^T$ and $X_r = X$.

Theorem 1.1 (Allday, Franz and Puppe) *The Chang–Skjelbred sequence (1-1) is exact if and only if $H_T^*(X)$ is a reflexive R–module. More generally, for any $1 \leq k \leq r$, the sequence*

$$0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0)$$
$$\to H_T^{*+2}(X_2, X_1) \to \cdots \to H_T^{*+k-1}(X_{k-1}, X_{k-2})$$

is exact if and only if $H_T^*(X)$ is a kth syzygy over R.

See [1, Theorem 1.1]. The additional maps in the sequence above are the connecting homomorphisms for the triples (X_{i+1}, X_i, X_{i-1}).

Recall that an R–module is reflexive if the canonical map to its double-dual is an isomorphism. Syzygies are a notion from commutative algebra that interpolates between torsion-free and free modules; see Section 2 for the precise definition. The first syzygies over R are exactly the torsion-free modules, the second syzygies the reflexive ones and the rth syzygies the free ones.

As a corollary (see the comment following [1, Theorem 5.7]) we get the result of Atiyah and Bredon [5, Theorem on pages 848–849] that the sequence

$$(1\text{-}2) \quad 0 \to H_T^*(X) \to H_T^*(X_0) \to H_T^{*+1}(X_1, X_0)$$
$$\to H_T^{*+2}(X_2, X_1) \to \cdots \to H_T^{*+r}(X_r, X_{r-1}) \to 0$$

is exact if and only if $H_T^*(X)$ is free over R, which strengthens the Chang–Skjelbred theorem.

It is not difficult to construct T–manifolds such that $H_T^*(X)$ is a kth syzygy for $k < r$. For example, the usual rotation action of S^1 on S^2 gives an action of T on $(S^2)^r$ such that $H_T^*(X)$ is free over R. By suitably removing two fixed points, any syzygy order less than r can be realized [1, Section 6.1].

The situation becomes much more intriguing if one looks at compact orientable T–manifolds. For such an X, another result of Allday, Franz and Puppe says that if $H_T^*(X)$ is a syzygy of order $\geq \frac{1}{2}r$, then it is actually free over R [1, Corollary 1.4].

It already appears very difficult to construct compact orientable T–manifolds such that $H_T^*(X)$ is torsion-free, but not free over R. The first such examples were the "mutants of compactified representations" given in 2008 by Franz and Puppe [11, Section 4]. Recently, the first author found a family of T–manifolds, the so-called big polygon spaces, that generalize one of the mutants to arbitrary syzygies [8]. We recall the definition.

Let $\ell \in \mathbb{R}^r$, called a *length vector* in this context. We assume that ℓ is *generic*, meaning

$$(1\text{-}3) \qquad \sum_{j \in J} \ell_j \neq \sum_{j \notin J} \ell_j$$

for any subset $J \subset \{1, \ldots, r\}$. Depending on which side dominates, J is called ℓ–*long* or ℓ–*short*.

Let $p, q \geq 1$. The *big polygon space* $X(\ell) = X_{p,q}(\ell)$ is the real algebraic subvariety of $\mathbb{C}^{(p+q)r}$ defined by the equations

$$(1\text{-}4) \quad \|u_j\|^2 + \|z_j\|^2 = 1 \quad \text{for any } 1 \leq j \leq r \qquad \text{and} \qquad \ell_1 u_1 + \cdots + \ell_r u_r = 0,$$

where $u_1, \ldots, u_r \in \mathbb{C}^p$ and $z_1, \ldots, z_r \in \mathbb{C}^q$. Since ℓ is generic, $X(\ell)$ is a compact orientable manifold with a smooth action of $T = (S^1)^r$ given by scalar multiplication of the z–variables,

$$(1\text{-}5) \qquad g \cdot (u, z) = (u, g_1 z_1, \ldots, g_r z_r).$$

See [8, Lemma 2.1]. The fixed-point set $X(\ell)^T$ is the "space of polygons" $E_{2p}(\ell)$ studied by Farber and Fromm [7].

It turns out that $H_T^*(X(\ell))$ is never free over R. In fact, $H_T^*(X(\ell))$ is not a syzygy of order

$$(1\text{-}6) \qquad \mu(\ell) = \min\{\sigma_\ell(J) \mid J \subset \{1, \ldots, r\} \text{ is } \ell\text{–long and } \sigma_\ell(J) > 0\},$$

where

$$(1\text{-}7) \qquad \sigma_\ell(J) = \#\{j \in J \mid J \setminus j \text{ is } \ell\text{–short}\}.$$

See [8, Proposition 6.3]. Our main result confirms [8, Conjecture 6.6].

Theorem 1.2 *The syzygy order of $H_T^*(X(\ell))$ over R equals $\mu(\ell) - 1$.*

Our proof of Theorem 1.2 is purely algebraic and uses the description of $H^*_T(X(\ell))$ given in [8, Lemma 4.4]. It is inspired by the proof appearing in the second author's PhD thesis [15], which in turn is based on the quotient criterion for syzygies developed by the first author in [10] and on Morse–Bott theory for manifolds with corners.

The largest possible syzygy order for $r = 2m + 1$ and $r = 2m + 2$ is m. It is known that this syzygy order is realized by an essentially unique length vector which for odd r corresponds to the equilateral case $\ell = (1, \ldots, 1)$; see [8, Corollary 6.4] and also Proposition 4.2. From Theorem 1.2 we deduce that syzygies of the next smaller order are also unique or at least almost unique.

Corollary 1.3 *Let $r \geq 3$, and let $\ell \in \mathbb{R}^r$ be a generic length vector with weakly increasing nonnegative components.*

  (i) *Assume that $r = 2m + 1$ is odd. Then $H^*_T(X(\ell))$ is a syzygy of order $m - 1$ if and only if $X(\ell)$ is equivariantly diffeomorphic to $X(0, 0, 1, \ldots, 1)$.*

  (ii) *Assume that $r = 2m + 2$ is even. Then $H^*_T(X(\ell))$ is a syzygy of order $m - 1$ if and only if $X(\ell)$ is equivariantly diffeomorphic to $X(0, 0, 0, 1, \ldots, 1)$ or to $X(1, 1, 1, 2, \ldots, 2)$.*

To relate our algebraic reasoning with equivariant cohomology, we develop a refined criterion for syzygies in equivariant cohomology which is of independent interest. It involves the notion of a *k–localizing subset* $S \subset H^2(BT)$ for a given "nice" T–space X; see again Section 2 for the definitions. For a big polygon space $X(\ell)$, the set $\{t_1, \ldots, t_r\}$ of indeterminates of R is k–localizing for any k.

Theorem 1.4 *Let $S \subset H^2(BT; \mathbb{Z})$ be k–localizing for X for some $k \geq 1$. Then $H^*_T(X)$ is a kth syzygy over R if and only if any linearly independent sequence in S of length at most k is $H^*_T(X)$–regular.*

The proof of Theorem 1.4 appears in Section 2. Theorem 1.2 is proven in Section 3 and Corollary 1.3 in Section 4. In Section 5 we state versions of our results for actions of 2–tori $(\mathbb{Z}_2)^r$ and certain "real" analogues of big polygon spaces which have recently been studied by Puppe [16].

2 A refined characterization of syzygies

From now on, all cohomology is taken with coefficients in a field $\Bbbk$ of characteristic 0 unless stated otherwise. Let $T \cong (S^1)^r$ be a torus; we write $R = H^*(BT)$.

Recall that any element $a \in H^2(BT; \mathbb{Z}) \cong H^1(T; \mathbb{Z})$ can be interpreted as a character $\chi_a \colon T \to S^1$. We write $t_i \in H^2(BT; \mathbb{Z})$ for the element corresponding to the ith coordinate $T \to S^1$, so that $R = \Bbbk[t_1, \ldots, t_r]$. For any linearly independent sequence $\boldsymbol{a} = (a_1, \ldots, a_m)$ in $H^2(BT; \mathbb{Z})$ we write $T(\boldsymbol{a}) \subset T$ for the identity component of the intersection of $\ker \chi_{a_1}, \ldots, \ker \chi_{a_m}$, which is of codimension m. Given an R–module M, we also write $M/\boldsymbol{a} = M/(a_1, \ldots, a_m)M$.

Let X be a T–space. We say that X is *nice* if it is Hausdorff, second-countable, finite-dimensional, locally compact and locally contractible; see [1, Sections 3.1 and 4.1]. For instance, X can be a T–manifold or T–orbifold or a complex algebraic variety with an algebraic action of $(\mathbb{C}^\times)^r$. We additionally assume that $H^*(X)$ is finite-dimensional and that only finitely many subtori of T occur as identity components of isotropy groups in X. In the examples just mentioned, this last condition is redundant; see [9, Theorem 7.7].[1] (We have implicitly used this in the introduction already.)

Let $k \geq 0$. A finitely generated R–module M is called a kth *syzygy* if any regular sequence in R of length at most k is also M–regular. (See [1, Section 2.3] for equivalent definitions of syzygies.) If M is a syzygy of order k, but not of order $k+1$, then we say that the *syzygy order* of M equals k.

In our topological context, it is enough to consider sequences of linear elements.

Lemma 2.1 *Let X be a nice T–space and let $k \geq 0$. Then $H_T^*(X)$ is a kth syzygy if and only if every linearly independent sequence in $H^2(BT; \mathbb{Z})$ of length at most k is $H_T^*(X)$–regular.*

Proof This is implicit in [1, Theorem 5.7]. There it is shown that $H_T^*(X)$ is a kth syzygy if and only if it is free over all subrings $H^*(BT'') \subset R$ where T'' is a quotient of T of rank $\leq k$ (equivalently, equal to k if $k \leq r$). These are exactly the subrings of R that are generated by linearly independent sequences in $H^2(BT; \mathbb{Z})$ of length $\leq k$.

Let $\boldsymbol{a}$ be such a sequence and let T'' be the corresponding quotient of T. Because the graded module $M = H_T^*(X)$ is bounded below, it is free over $R'' = H^*(BT'')$ if

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[1] The algebraic case reduces to the one for manifolds: algebraic varieties have finite Betti sum and can be decomposed into finitely many smooth varieties, stable with respect to an algebraic action.

and only if a is M–regular; see [14, Lemma on page 5]. (The argument given there remains valid for modules that are not finitely generated.)   □

Lemma 2.2 *Let X be a T–space, and let a be a sequence in $H^2(BT; \mathbb{Z})$. If a is $H_T^*(X)$–regular, then the restriction map $H_T^*(X) \to H_{T(a)}^*(X)$ induces an isomorphism*

$$H_{T(a)}^*(X) \cong H_T^*(X)/a.$$

Proof This is again contained in the proof of [1, Theorem 5.7]. By induction, we may assume that a consists of a single element $0 \neq a \in H^2(BT; \mathbb{Z})$. We may also assume that it is not divisible by any integer > 1.

Let $C_T^*(X)$ be the singular Cartan model for X; see [1, Section 3.2]. We may assume that a is contained in the basis of $H^2(BT; \mathbb{Z})$ chosen in the definition of the singular Cartan model. We then have a short exact sequence

(2-1) $$0 \to C_T^*(X) \xrightarrow{\cdot a} C_T^*(X) \to C_{T''}^*(X) \to 0.$$

Because a is $H_T^*(X)$–regular, this induces the exact sequence

(2-2) $$0 \to H_T^*(X) \xrightarrow{\cdot a} H_T^*(X) \to H_{T''}^*(X) \to 0,$$

proving the claim.   □

For any $x \in X$, the kernel of the restriction map $p_x\colon H^2(BT) \to H^2(BT_x)$ has dimension equal to the codimension of T_x in T. We say that a subset $S \subset H^2(BT)$ is *k–localizing for X* if for any $x \in X$ at least $\min(k, \operatorname{codim} T_x)$ linearly independent elements from S lie in $\ker p_x$. This notion behaves well with respect to subtori:

Lemma 2.3 *Let X be a T–space, and let $S \subset H^2(BT)$ be k–localizing for X for some $k \geq 0$. For any subtorus $T' \subset T$ of codimension $l \leq k$, the image of S in $H^2(BT')$ is $(k-l)$–localizing for X, considered as a T'–space.*

Proof Let $x \in X$ and consider the commutative diagram of surjections

(2-3)
$$\begin{array}{ccc}
H^2(BT) & \xrightarrow{p_x} & H^2(BT_x) \\
\pi \downarrow & & \downarrow \\
H^2(BT') & \xrightarrow{p_x'} & H^2(BT_x')
\end{array}$$

If S contains a basis for $\ker p_x$, then $\pi(S)$ contains one for $\ker p_x'$. If S contains k linearly independent elements from $\ker p_x$, then $\pi(S)$ contains $k - l$ linearly

independent elements from $\ker p'_x$. In either case, we have found enough linearly independent elements in $\pi(S) \cap \ker p'_x$, which proves the claim. □

Theorem 2.4 Let X be a nice T–space, and let $S \subset H^2(BT;\mathbb{Z})$ be k–localizing for X for some $k \geq 0$. Then $H^*_T(X)$ is a kth syzygy over R if and only if any linearly independent sequence in S of length at most k is $H^*_T(X)$–regular.

Proof The "only if" direction follows from the definition of syzygies given above. We prove the converse by induction on k. Note that we may assume $0 \notin S$.

We consider first the case $k = 1$. Because S is 1–localizing for X, we can, for any $x \notin X^T$, find an element in S lying in the kernel of the restriction map $H^*(BT) \to H^*(BT_x)$. By the localization theorem in equivariant cohomology [4, Theorem 3.2.6], this implies that the bottom arrow in the commutative diagram

(2-4)
$$
\begin{array}{ccc}
H^*_T(X) & \longrightarrow & H^*_T(X^T) \\
\downarrow & & \downarrow \\
\widehat{S}^{-1} H^*_T(X) & \longrightarrow & \widehat{S}^{-1} H^*_T(X^T)
\end{array}
$$

is an isomorphism, where $\widehat{S} \subset R$ is the multiplicative subset generated by S.

By assumption, no element in $\widehat{S}$ is a zero-divisor for $H^*_T(X)$, so the left localization map in the diagram is injective. It follows that the top arrow is also injective, meaning that the equivariant cohomology of X embeds into that of the fixed-point set. Since $H^*_T(X^T) \cong H^*(X^T) \otimes R$ is free over R, $H^*_T(X)$ must be torsion-free.

We now consider the case $k > 1$ and assume that $M := H^*_T(X)$ is not a kth syzygy. By Lemma 2.1 this means that there is an R–regular sequence $\boldsymbol{a}$ of length at most k in $H^2(BT;\mathbb{Z})$ that is not M–regular. We are going to show that there is another such sequence contained in S. If $\boldsymbol{a} = (a_1, \dots, a_m)$ is of length $m < k$ or if $m = k$ and $\boldsymbol{a}' = (a_1, \dots, a_{k-1})$ is not M–regular, then M is not a syzygy of order $k - 1$, and we are done by induction.

So we can assume that $m = k$ and that $\boldsymbol{a}'$ is M–regular. We write $T' = T(\boldsymbol{a}')$, $R' = H^*(BT') = R/\boldsymbol{a}'$ and π' for the canonical projection $R \to R'$. By Lemma 2.2 we have an isomorphism

(2-5)
$$
M' := H^*_{T'}(X) \cong M/\boldsymbol{a}',
$$

and $\pi'(a_k) \in R'$ is a zero-divisor for this module. Moreover, Lemma 2.3 implies that $\pi'(S) \subset R'$ is 1–localizing for the T'–space X. By the already established base case,

there is a zero-divisor $\pi'(b) \neq 0$ in $\pi'(S)$, and hence $0 \neq b \in S$ is also a zero-divisor for M'. Therefore, the sequence (a', b) is not M–regular.

We may assume that b is not a zero-divisor for M for otherwise we would be done as M would not be a first syzygy. Because M is graded and bounded below and the sequence (a', b) is made of homogeneous elements, we can rearrange it [14, Proposition on page 1] to obtain (b, a'), which is again R–regular, but not M–regular. Since b is not a zero-divisor for M, this means that a' is not regular for $M'' = M/b$.

We write $T'' = T(b)$ and define R'' and π'' accordingly. Again by Lemma 2.3, $\pi''(S)$ is $(k{-}1)$–localizing for X, considered as a T''–space. Appealing once more to Lemma 2.2, we get an isomorphism

$$(2\text{-}6) \qquad\qquad M'' \cong H_{T''}(X).$$

Given that a' is not M''–regular, M'' cannot be a $(k{-}1)^{\text{st}}$ syzygy over R''. By induction, we can therefore find a sequence $\pi''(c)$ of length at most $k - 1$ in $\pi''(S)$ that is regular for R'', but not for M''. Thus, (b, c) is an R–regular sequence in S of length at most k that is not M–regular, as desired. □

3 Big polygon spaces

Let $r \geq 1$. We write $[r] = \{1, \ldots, r\}$ and Δ for the simplex with vertex set $[r]$, considered as a simplicial complex. We call a length vector $\ell \in \mathbb{R}^r$ *strongly generic* if $\ell(\sigma) \neq \ell(\tau)$ for any two distinct simplices σ and τ in Δ, where

$$(3\text{-}1) \qquad\qquad \ell(\sigma) = \sum_{j \in \sigma} \ell_j.$$

Two generic length vectors are called *equivalent* if they induce the same notion of "long" and "short" on subsets of $[r]$. The equivalence classes of generic length vectors ℓ are open polyhedral cones in $\mathbb{R}^r$ which are the connected components of the complement of a hyperplane arrangement. Because strong genericity means that certain additional hyperplanes are avoided, any generic length vector is equivalent to a strongly generic one. Two equivalent generic length vectors give rise to equivariantly diffeomorphic big polygon spaces, hence to isomorphic equivariant cohomologies. Moreover, there is no loss of generality if one assumes ℓ to be positive and weakly increasing; see [8, Section 2]. In this case, nonequivalent generic length vectors give rise to big polygon spaces which even nonequivariantly are not diffeomorphic [8, Proposition 3.7]. For the rest of this section, $\ell \in \mathbb{R}^r$ denotes a strongly generic length vector with positive coordinates.

For any R–algebra $\bar{R}$, we write $\mathcal{C}(\Delta; \bar{R})$ for the Koszul complex with coefficients in $\bar{R}$. That is, $\mathcal{C}(\Delta; \bar{R})$ is a free $\bar{R}$–module with basis Δ and differential

$$(3\text{-}2) \qquad d\gamma = \sum_{j \in \gamma} \pm t_j^q (\gamma \setminus j)$$

for $\gamma \in \Delta$; see [8, Section 5]. (Note that we sometimes omit braces, as in $\gamma \setminus j$.) We introduce a grading by giving each generator $t_i \in R$ the degree 2 and each $\gamma \in \Delta$ the degree $(2p + 2q - 1) \cdot \#\gamma$. The differential (3-2) then has degree $1 - 2p$.

Let $S \subset \Delta$ be a subset. We define S_+ and S_- to be the set of ℓ–long and ℓ–short simplices in S, respectively. We write $\mathcal{C}(S; \bar{R})$ for the $\bar{R}$–submodule of $\mathcal{C}(\Delta; \bar{R})$ with basis S so that

$$(3\text{-}3) \qquad \mathcal{C}(\Delta; \bar{R}) = \mathcal{C}(S; \bar{R}) \oplus \mathcal{C}(\Delta \setminus S; \bar{R})$$

as $\bar{R}$–modules. If S is a simplicial subcomplex of Δ, then $\mathcal{C}(S; \bar{R})$ and $\mathcal{C}(S_-; \bar{R})$ are subcomplexes of $\mathcal{C}(\Delta; \bar{R})$, but $\mathcal{C}(S_+; \bar{R})$ is not in general. For any S we define the subcomplex

$$(3\text{-}4) \qquad \mathcal{D}(S; \bar{R}) = \mathcal{C}(S; \bar{R}) + d\,\mathcal{C}(S; \bar{R}) \subset \mathcal{C}(\Delta; \bar{R}).$$

For any $c = \sum_{\sigma \in \Delta} c_\sigma \sigma \in \mathcal{C}(\Delta; \bar{R})$, we write

$$(3\text{-}5) \qquad \operatorname{supp} c = \{\sigma \in \Delta \mid c_\sigma \neq 0\}$$

for its support and, assuming $c \notin \mathcal{C}(\Delta_+)$,

$$(3\text{-}6) \qquad \ell(c) = \min\{\ell(\sigma) \mid \sigma \in \operatorname{supp} c \text{ is short}\}.$$

Lemma 3.1 *Consider the differential as a map*

$$f_\ell \colon \mathcal{C}(\Delta_+; R) \to \mathcal{C}(\Delta; R)/\mathcal{C}(\Delta_+; R) \cong \mathcal{C}(\Delta_-; R), \quad \gamma \mapsto d\gamma.$$

Then there is a short exact sequence of graded R–modules

$$0 \to \operatorname{coker} f_\ell \to H_T^*(X(\ell)) \to (\ker f_\ell)[-2p] \to 0.$$

In particular, the syzygy order of $H_T^*(X(\ell))$ over R equals that of $\operatorname{coker} f_\ell$.

Here $[-2p]$ denotes a degree shift by $2p$ downwards. The sequence actually splits by a result of Puppe [16, Lemma 3.12].

Proof See [8, Section 4, Lemma 6.2]. Note that we have indexed the basis elements in a form more convenient for our purposes. $\qquad \square$

For any $\gamma \in \Delta_+$ we define

(3-7) $$\sigma_\ell(\gamma) = \#\{j \in \gamma \mid \gamma \setminus j \text{ is } \ell\text{–short}\}$$

and

(3-8) $$\mu(\ell) = \min\{\sigma_\ell(\gamma) \mid \gamma \in \Delta_+ \text{ and } \sigma_\ell(\gamma) > 0\} \geq 1$$

as in [8, Equations (6.6)–(6.7)].

Theorem 3.2 *The syzygy order of $H_T^*(X(\ell))$ over R is $\mu(\ell) - 1$.*

In [8, Proposition 6.3] it is shown that $\mu(\ell) - 1$ is an upper bound for the syzygy order, and it was conjectured that one has equality [8, Conjecture 6.6].

Proof   According to Lemma 3.1, the syzygy order of $H_T^*(X(\ell))$ equals that of

(3-9) $$M(\ell) = \mathcal{C}(\Delta; R)/\mathcal{D}(\Delta_+; R).$$

By what we have just said, we only have to show that $M(\ell)$ is a syzygy of order at least $\mu(\ell) - 1$.

The isotropy subgroups appearing in $X(\ell) = X_{p,q}(\ell)$ are the coordinate subtori of $T = (S^1)^r$. Hence for any k the set $S = \{t_1, \ldots, t_r\} \subset H^2(BT; \mathbb{Z})$ is k–localizing for X. By Theorem 2.4, it suffices to show that for any $k < \mu(\ell)$ and any pairwise distinct elements $i_1, \ldots, i_k$ the sequence $(t_{i_1}, \ldots, t_{i_k})$ is $M(\ell)$–regular.

We proceed by induction on k, the case $k = 0$ being void. For $k > 0$, we know by induction that the sequence $t_{i_1}, \ldots, t_{i_{k-1}}$ is $M(\ell)$–regular. It remains to show that t_{i_k} is not a zero-divisor in $N = M(\ell)/(t_{i_1}, \ldots, t_{i_{k-1}})$ or, equivalently, that $t_{i_k}^q$ is not a zero-divisor in N. (Recall that $M(\ell)$ is graded and bounded below, so that $t_{i_k} N \neq N$.) We write $I = \{i_1, \ldots, i_{k-1}\}$ and $i = i_k$.

We start by observing that

(3-10) $$\begin{aligned} N &= \mathcal{C}(\Delta; R)/(\mathcal{D}(\Delta_+; R) + (t_{i_1}, \ldots, t_{i_{k-1}}) \mathcal{C}(\Delta; R)) \\ &= \mathcal{C}(\Delta; \bar{R})/\mathcal{D}(\Delta_+; \bar{R}), \end{aligned}$$

where

(3-11) $$\bar{R} = R/(t_{i_1}, \ldots, t_{i_{k-1}}) = \Bbbk[t_j \mid j \notin I].$$

In this case the differential (3-2) takes the form

(3-12) $$d\gamma = \sum_{j \in \gamma \setminus I} \pm t_j^q (\gamma \setminus j).$$

Assume that our claim is false. Then there is a $c \in \mathcal{C}(\Delta; \mathbb{R})$ such that $t_i^q c$ is contained in $\mathcal{D}(\Delta_+; \bar{R})$ while c itself is not. We can write $t_i^q c = a + db$ for some $a, b \in \mathcal{C}(\Delta_+; \bar{R})$. Since

$$(3\text{-}13) \qquad (a + t_i^q a') + d(b + t_i^q b') = t_i^q(c + a' + db'),$$

we may assume a and b to be t_i^q–*free*. By this we mean that in the canonical monomial basis for $\bar{R}$, no monomial divisible by t_i^q appears in the nonzero coefficients a_γ and b_γ of a and b with respect to the basis Δ of $\mathcal{C}(\Delta; \bar{R})$. It implies that c is t_i^q–free.

Let $\Gamma \subset \Delta$ be the simplicial complex of the facet not containing i, so that Δ is the cone over Γ with vertex i. Recall that we have a bijection between the simplices $\gamma \in \Gamma$ and those in $\Delta \setminus \Gamma$, given by $\gamma \mapsto \hat{\gamma} := \gamma \cup i$.

Using (3-3), we can decompose a, b and c as

$$(3\text{-}14) \qquad c = c_1 + c_2 \in \mathcal{C}(\Gamma; \bar{R}) \oplus \mathcal{C}(\Delta \setminus \Gamma; \bar{R}),$$

$$(3\text{-}15) \qquad a = a_1 + a_2, \; b = b_1 + b_2 \in \mathcal{C}(\Gamma_+; \bar{R}) \oplus \mathcal{C}(\Delta_+ \setminus \Gamma_+; \bar{R}).$$

By inspection of (3-12) we see that $db_1 \in \mathcal{C}(\Gamma; \bar{R})$ is t_i^q–free and

$$(3\text{-}16) \qquad db_2 = e_1 + e_2 \in \mathcal{C}(\Gamma; \bar{R}) \oplus \mathcal{C}(\Delta \setminus \Gamma; \bar{R}),$$

where e_1 is divisible by t_i^q and e_2 is t_i^q–free. Hence

$$(3\text{-}17) \qquad t_i^q c_1 = \underbrace{a_1 + db_1}_{t_i^q\text{–free}} + \underbrace{e_1}_{\text{div. by } t_i^q} \quad \text{and} \quad t_i^q c_2 = \underbrace{a_2 + e_2}_{t_i^q\text{–free}}.$$

This implies

$$(3\text{-}18) \qquad t_i^q c_1 = e_1, \quad a_1 + db_1 = 0, \quad c_2 = a_2 + e_2 = 0.$$

Hence $c \in \mathcal{C}(\Gamma; \bar{R})$, and we can write it in the form

$$(3\text{-}19) \qquad t_i^q c = a + db, \quad a, b \in \mathcal{C}(\Delta_+ \setminus \Gamma_+; \bar{R}).$$

We assume also that c is a counterexample maximizing $\ell(c)$. The simplex $\sigma \in \operatorname{supp} c$ realizing $\ell(c)$ is necessarily short since $c \notin \mathcal{C}(\Delta_+; \bar{R}) \subset \mathcal{D}(\Delta_+; \bar{R})$. We finally require that among all these counterexamples we pick one with the fewest monomials appearing in $c_\sigma \in \bar{R}$.

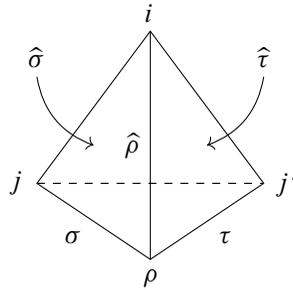Figure 1 may help the reader to visualize the simplices constructed in the following arguments.

Figure 1: The simplex $\widehat{\beta}$. The face ρ is of codimension 3.

Being short, σ cannot be contained in $\operatorname{supp} a$. Hence $\widehat{\sigma} \in \operatorname{supp} b$ and $b_{\widehat{\sigma}} = \pm c_\sigma$ by (3-19) and (3-12). In particular, $\widehat{\sigma}$ is long. Since σ is short and $\mu(\ell) \geq k+1$, we conclude that $\widehat{\sigma}$ has $k+1$ short facets. Hence there is a short facet of the form

$$(3\text{-}20) \qquad \widehat{\rho} = \widehat{\sigma} \setminus j, \quad j \notin I \cup i,$$

given that $\#I = k-1$. Let us write

$$(3\text{-}21) \qquad \sigma = \{j_0, \dots, j_m\} \quad \text{with } \ell_{j_0} < \cdots < \ell_{j_m}$$

(where we have used our assumption that ℓ is strongly generic) and

$$(3\text{-}22) \qquad j_u = \max(\sigma \setminus I).$$

By (3-21), we may assume $j = j_u$ in (3-20) because replacing j by j_u can only decrease the length of $\widehat{\rho}$.

Looking at (3-12), we have $\widehat{\rho} \in \operatorname{supp} d\widehat{\sigma}$ since $j_u \notin I$. Given that $\widehat{\rho}$ is short and not contained in Γ, it cannot appear in $db = t_i^q c - a$. Hence there must be a (necessarily long) simplex $\widehat{\tau} \neq \widehat{\sigma}$ appearing in b and having $\widehat{\rho}$ as a facet.

We have

$$(3\text{-}23) \qquad \widehat{\tau} = \widehat{\rho} \cup j' \quad \text{for some } j' \notin \widehat{\sigma}.$$

The contribution of $d(b_{\widehat{\sigma}}\widehat{\sigma}) = \pm d(c_\sigma \widehat{\sigma})$ to the coefficient $(db)_{\widehat{\rho}}$ of $\widehat{\rho}$ in db is $\pm t_{j_u}^q c_\sigma$, and that of $d(b_{\widehat{\tau}}\widehat{\tau})$ likewise is $\pm t_{j'}^q b_{\widehat{\tau}}$. Since all monomials appearing in $(dc_\sigma \widehat{\sigma})_{\widehat{\rho}}$ must somehow be compensated for by other simplices appearing in b, we may choose $\widehat{\tau}$ such that $t_{j'}^q$ divides a monomial appearing in c_σ. Because τ appears in db and σ is the shortest simplex appearing there, we additionally have $\ell(\tau) > \ell(\sigma)$ or, in other words, $\ell_{j'} > \ell_{j_u}$, again by strong genericity.

Now $\hat{\tau}$ is a long facet of

$$(3\text{-}24) \qquad \hat{\beta} = \hat{\sigma} \cup j' \quad \text{where } \beta = \sigma \cup j'.$$

The other facets of $\hat{\beta}$ different from β are obtained from $\hat{\tau}$ by substituting j_u for some $j_v \in \tau$. If $v < u$, we get another long facet by (3-21). Therefore, $\hat{\beta}$ has at most $\#I + 1 = k$ short facets by (3-22), including possibly β. But $\mu(\ell) \geq k + 1$, so all facets of $\hat{\beta}$ are long.

Since we have

$$(3\text{-}25) \qquad 0 = d\,d\hat{\beta} = d\left(\pm t_i^q \beta \pm \sum_{j \in \hat{\beta} \backslash i} t_j^q (\hat{\beta} \backslash j)\right),$$

we can write $t_i^q \tilde{c} = d\tilde{b}$ with

$$(3\text{-}26) \qquad \tilde{c} = d\beta = \sum_{j \in \beta \backslash I} \pm t_j^q (\beta \backslash j) \in \mathcal{C}(\Gamma; \bar{R}),$$

$$(3\text{-}27) \qquad \tilde{b} = \pm \sum_{j \in \beta \backslash I} t_j^q (\hat{\beta} \backslash j) \in \mathcal{C}(\Delta_+ \backslash \Gamma_+; \bar{R}).$$

Consider now all monomials appearing in c_σ that are divisible by $t_{j'}^q$ and write their sum as $t_{j'}^q x$ with $x \in \bar{R}$. Then $x \neq 0$ by our choice of $\hat{\tau}$, and no monomial appearing in it is divisible by t_i^q since c is t_i^q–free. The preceding discussion implies

$$(3\text{-}28) \qquad t_i^q(c + x\tilde{c}) = a + d(b + x\tilde{b}),$$

where both a and $b + x\tilde{b} \in \mathcal{C}(\Delta_+ \backslash \Gamma_+; \bar{R})$ are t_i^q–free and $c + x\tilde{c} \in \mathcal{C}(\Gamma; \bar{R})$. In particular, $c + x\tilde{c}$ is another counterexample of the form (3-19) to our claim that t_i^q is not a zero-divisor in $M(\ell)$.

Since $\ell_{j'} > \ell_{j_u}$ and $j_v \in I$ for $u < v \leq m$, the simplex $\sigma = \beta \backslash j'$ is the shortest one appearing in the sum (3-26). Hence

$$(3\text{-}29) \qquad \ell(c + x\tilde{c}) \geq \ell(c).$$

The coefficient of σ in $c + x\tilde{c}$ is of the form

$$(3\text{-}30) \qquad (c + x\tilde{c})_\sigma = c_\sigma - t_{j'}^q x.$$

If it vanishes, then we have a strict inequality in (3-29) since ℓ is strongly generic. This would contradict our choice of c with maximal $\ell(c)$. If it does not vanish, then it is still obtained from c_σ by removing certain monomials. As such, it contains fewer monomials than c_σ, again contradicting our choice of c.

We conclude that no counterexample exists. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Classification of high syzygies

Using the result of the previous section, we can extend the classification of big polygon spaces with high syzygies in their equivariant cohomology. Throughout this section, $\ell \in \mathbb{R}^r$ denotes a generic length vector with positive and weakly increasing coefficients.

Maximal syzygies, that is, those of order m for $r = 2m + 1$ odd or $r = 2m + 2$ even were determined in [8]. We are going to rephrase the proof in our setting and extend the result to syzygies of order $m - 1$.

Lemma 4.1 *If there is a long subset $J \subset [r]$ of size $\#J = \mu(\ell)$, then*

$$\ell \sim (0, \ldots, 0, \underbrace{1, \ldots, 1}_{2\mu(\ell)-1}).$$

Proof We may assume ℓ to be strongly generic and set $\mu(\ell) = k$. Note that all subsets $I \subset [r]$ with fewer than k elements are short for otherwise we would get the contradiction

$$(4\text{-}1) \qquad\qquad k = \mu(\ell) \leq \sigma_\ell(I) = \#I < k$$

for a (necessarily nonempty) long set I of minimal size.

Among all long subsets $J \subset [r]$ of size k, we pick the one with minimal $\ell(J)$. We set $j_{\min} = \min(J)$ and $j_{\max} = \max(J)$. By what we have just said, $J \setminus j_{\max}$ is short.

Let $I \subset [r]$ be the set of those values $j \notin J \setminus j_{\max}$ such that $(J \setminus j_{\max}) \cup j$ is long. This set contains $j_{\max}$ and therefore is nonempty. Hence $\#I \geq k$ and $\#(I \setminus j_{\max}) \geq k - 1$.

For any $i \in I \setminus j_{\max}$, the set $J_i = (J \setminus j_{\max}) \cup i$ is long and of size k, and $J_i \setminus i$ is short. Hence

$$(4\text{-}2) \qquad\qquad k \leq \sigma_\ell(J_i) \leq \#J_i = k.$$

This implies $\ell(J_i) > \ell(J)$ for otherwise $\ell(J)$ would not be minimal. In summary, I consists of $j_{\max}$ and $k - 1$ values larger than $j_{\max}$.

Consider the remaining $r - \#(J \cup I) = r - 2k + 1$ elements of $[r] \setminus (J \cup I)$. If one of them were greater than $j_{\min}$, then we would have $\ell([r] \setminus J) > \ell(J)$, contradicting the assumption that J is long. These elements therefore are smaller than $j_{\min}$, and we conclude that

$$(4\text{-}3) \qquad J = \{j_{\min}, \ldots, j_{\max}\} = \{r - 2k + 2, \ldots, r - k + 1\}.$$

This implies that any subset of $[r]$ containing k values $\geq j_{\min} = r - 2k + 2$ is long. These sets are exactly the long sets for the length vector

(4-4)
$$\ell' = (\underbrace{0, \ldots, 0}_{r-2k+1}, \underbrace{1, \ldots, 1}_{2k-1}).$$

We conclude that they comprise half of all subsets and therefore that ℓ and ℓ' induce the same notion of "long" and "short". □

Proposition 4.2 [8, Corollary 6.4] *Let $r \geq 1$.*

(i) *Assume that $r = 2m + 1$ is odd. Then $H_T^*(X(\ell))$ is a syzygy of order m if and only if $\ell \sim (1, \ldots, 1)$.*

(ii) *Assume that $r = 2m + 2$ is even. Then $H_T^*(X(\ell))$ is a syzygy of order m if and only if $\ell \sim (0, 1, \ldots, 1)$.*

Proof By Theorem 3.2, the condition on the syzygy order translates into $\mu(\ell) = m + 1$. In both cases it is immediate to check that this is satisfied by the given length vectors. It remains to show the "only if" direction.

As in the previous proof, any subset with fewer than $\mu(\ell) = m + 1$ elements is short. Hence there must be a long subset of size $m + 1$ for otherwise more than half of all subsets would be short. The claim now follows from Lemma 4.1. □

Proposition 4.3 *Let $r \geq 3$.*

(i) *Assume that $r = 2m + 1$ is odd. Then $H_T^*(X(\ell))$ is a syzygy of order $m - 1$ if and only if $\ell \sim (0, 0, 1, \ldots, 1)$.*

(ii) *Assume that $r = 2m + 2$ is even. Then $H_T^*(X(\ell))$ is a syzygy of order $m - 1$ if and only if $\ell \sim (0, 0, 0, 1, \ldots, 1)$ or $\ell \sim (1, 1, 1, 2, \ldots, 2)$.*

We can restrict ourselves to $r \geq 3$ here since $H_T^*(X(\ell))$ is always torsion for $r \leq 2$.

Proof This time the condition on the syzygy order translates into $\mu(\ell) = m$. In all cases it is elementary to verify that it is satisfied by the given length vectors. It remains to show the "only if" direction.

Let $J \subset [r]$ be a long subset of minimal size. Since half of all subsets are long, we have $1 \leq \#J \leq m + 1$ and also $m = \mu(\ell) \leq \sigma_\ell(J) = \#J$, and hence $m \leq \#J \leq m + 1$.

Assume that $r = 2m + 1$ is odd. If $\#J = m + 1$, then all subsets of size at most m are short. Since these are already half of all subsets, those having at least $m+1$ elements are long. This implies $\mu(\ell) = m + 1$, contrary to our assumption. Hence $\#J = m = \mu(\ell)$, and we can appeal to Lemma 4.1.

Now let $r = 2m + 2$ be even. The case $\#J = m$ is dealt with as before. So we can assume that long sets have at least $m + 1$ elements, and we have to show that $\ell \sim \ell' = (1, 1, 1, 2, \ldots, 2)$.

For the purpose of this proof, call a subset $J \subset [r]$ *distinguished* if it is of size $m + 1$ and contains 2 and 3. Assume that there is a distinguished long set J. We claim that in this case all distinguished sets are long.

In order to prove this, choose a $j \notin J$ such that $j > 3$. (This is possible because there are $m + 1 \geq 2$ elements not in J.) Then both $J' = J \cup j$ and $J = J' \setminus j$ are long, and hence so are $J' \setminus 2$ and $J' \setminus 3$. Thus,

$$(4\text{-}5) \qquad\qquad \sigma_\ell(J') \leq (m + 2) - 3 = m - 1.$$

Since $\mu(\ell) = m$, this implies $\sigma_\ell(J') = 0$. In other words, replacing any element of J by an element $J \not\ni j > 3$ leads to another long set. Applying this procedure repeatedly, we can transform J into any other distinguished set while keeping it long, which proves the claim.

Given a distinguished set, we can also replace 2 and 3 by larger elements without making the set short. So we see that any subset $J \subset [r]$ of size $m + 1$ not containing 1 is long, as are all subsets of larger size (because their complements, being of size at most m, are short). These sets are exactly the long subsets for $\ell'' = (0, 1, \ldots, 1)$ and therefore all ℓ–long subsets. But this is impossible as $\mu(\ell'') = m + 1 \neq m$.

We conclude that any distinguished set is short. Hence so is any subset J of size $m + 1$ such that $\#(J \cap \{1, 2, 3\}) \geq 2$. Together with the subsets of smaller size, these are exactly the ℓ'–short subsets. So they are also exactly the ℓ–short subsets, which shows $\ell \sim \ell'$. □

There seems to be no easy description of syzygies of lower order. For instance, computer experiments show that for $r = 9 = 2 \cdot 4 + 1$ there are, up to equivalence, 5 length vectors ℓ (out of 175,428) such that $H_T^*(X(\ell))$ has syzygy order $2 = 4 - 2$, and for $r = 10 = 2 \cdot 4 + 2$ there are 18 (out of 52,980,624). (The numbers of nonequivalent length vectors can be found in [13, Section 10.3.1].)

5 The real case

Let $\ell \in \mathbb{R}^r$ be a generic length vector, and let $p, q \geq 1$. The *real big polygon space* $Y(\ell) = Y_{p,q}(\ell)$ is defined by restricting all variables to the reals, that is, by

$$(5\text{-}1) \quad \|u_j\|^2 + \|z_j\|^2 = 1 \quad \text{for any } 1 \leq j \leq r \qquad \text{and} \qquad \ell_1 u_1 + \cdots + \ell_r u_r = 0,$$

where $u_1, \ldots, u_r \in \mathbb{R}^p$ and $z_1, \ldots, z_r \in \mathbb{R}^q$. It is the fixed-point set of $X(\ell)$ under complex conjugation of all variables, hence again smooth. The 2–torus $G = (\mathbb{Z}_2)^r$ of rank r acts on $Y(\ell)$ by reversing the signs of the z–variables. We assume now that $\Bbbk$ is a field of characteristic 2. The G–equivariant cohomology of real big polygon spaces (and more general spaces) with coefficients in $\Bbbk$ has been studied by Puppe [16], following the work of Hausmann [13, Section 10.3].

The fixed-point set $Y(\ell)^G$ is empty for $p = 1$, by the very definition of a generic length vector. By the localization theorem, this implies that $H_G^*(Y(\ell))$ is a torsion module over $R = H^*(BG)$ in this case. For $p > 1$, however, the theory parallels the one for the complex case. In particular, the equivariant cohomology $H_G^*(Y(\ell))$ is given by a formula analogous to Lemma 3.1, which allows us to proceed in the same way as before. We content ourselves with sketching the arguments.

Lemma 5.1 *Grade* $\mathcal{C}(\Delta; R)$ *by setting* $|t_i| = 1$ *for each generator* $t_i \in R$ *and setting* $|\gamma| = (p + q - 1) \cdot \#\gamma$ *for each* $\gamma \in \Delta$*. With* f_ℓ *defined as in Lemma 3.1, there is a short exact sequence of graded* R*–modules,*

$$0 \to \operatorname{coker} f_\ell \to H_T^*(X(\ell)) \to (\ker f_\ell)[-p] \to 0.$$

In particular, the syzygy order of $H_T^*(X(\ell))$ *over* R *equals that of* $\operatorname{coker} f_\ell$*.*

Proof See [16, Section 3], in particular Proposition 3.11 and Equations (3.19)–(3.22). Alternatively, one could adapt the proof given in [8]. $\qquad\square$

Let X be a G–space, and for $x \in X$, let $p_x \colon H^1(BG) \to H^1(BG_x)$ be the restriction map. For $0 \leq k \leq r$, we call a subset $S \subset H^1(BG)$ k–*localizing* if for any $x \in X$ at least $\min(k, \operatorname{corank} G_x)$ linearly independent elements from S lie in $\ker p_x$.

We need the following analogue of Lemma 2.1; see [3, Theorem 10.2]. A *nice* G–space is defined in analogy with the torus case.

Lemma 5.2 *Let X be a nice G–space and let $k \geq 0$. Then $H_G^*(X)$ is a kth syzygy if and only if every linearly independent sequence in $H^1(BG; \mathbb{F}_2)$ of length at most k is $H_G^*(X)$–regular.*

Arguing as in Sections 2 and 3, we obtain the following.

Theorem 5.3 *Let X be a nice G–space, and let $S \subset H^1(BG; \mathbb{F}_2)$ be k–localizing for X for some $k \geq 0$. Then $H_G^*(X)$ is a kth syzygy over R if and only if any linearly independent sequence in S of length at most k is $H_G^*(X)$–regular.*

Theorem 5.4 *Assume $p > 1$. Then the syzygy order of $H_G^*(Y(\ell))$ over R is $\mu(\ell) - 1$. In particular, it is the same as the syzygy order of $H_T^*(X(\ell); \mathbb{Q})$ over $H^*(BT; \mathbb{Q})$.*

As a consequence, the characterizations of high syzygies derived in Section 4 carry over to real big polygon spaces. The analogue of Proposition 4.2 has already been established by Puppe [16, Proposition 3.14].

References

[1] **C Allday**, **M Franz**, **V Puppe**, *Equivariant cohomology, syzygies and orbit structure*, Trans. Amer. Math. Soc. 366 (2014) 6567–6589 MR

[2] **C Allday**, **M Franz**, **V Puppe**, *Equivariant Poincaré–Alexander–Lefschetz duality and the Cohen–Macaulay property*, Algebr. Geom. Topol. 14 (2014) 1339–1375 MR

[3] **C Allday**, **M Franz**, **V Puppe**, *Syzygies in equivariant cohomology in positive characteristic*, preprint (2020) arXiv

[4] **C Allday**, **V Puppe**, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics 32, Cambridge Univ. Press, Cambridge (1993) MR

[5] **G E Bredon**, *The free part of a torus action and related numerical equalities*, Duke Math. J. 41 (1974) 843–854 MR

[6] **T Chang**, **T Skjelbred**, *The topological Schur lemma and related results*, Ann. of Math. 100 (1974) 307–321 MR

[7] **M Farber**, **V Fromm**, *The topology of spaces of polygons*, Trans. Amer. Math. Soc. 365 (2013) 3097–3114 MR

[8] **M Franz**, *Big polygon spaces*, Int. Math. Res. Not. 2015 (2015) 13379–13405 MR

[9] **M Franz**, *Syzygies in equivariant cohomology for non-abelian Lie groups*, from "Configuration spaces" (F Callegaro, F Cohen, C De Concini, E M Feichtner, G Gaiffi, M Salvetti, editors), Springer INdAM Ser. 14, Springer (2016) 325–360 MR

[10] **M Franz**, *A quotient criterion for syzygies in equivariant cohomology*, Transform. Groups 22 (2017) 933–965 MR

[11] **M Franz**, **V Puppe**, *Freeness of equivariant cohomology and mutants of compactified representations*, from "Toric topology" (M Harada, Y Karshon, M Masuda, T Panov, editors), Contemp. Math. 460, Amer. Math. Soc., Providence, RI (2008) 87–98 MR

[12] **M Goresky**, **R Kottwitz**, **R MacPherson**, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998) 25–83 MR

[13] **J-C Hausmann**, *Mod two homology and cohomology*, Springer (2014) MR

[14] **M Hochster**, *Cohen–Macaulay rings*, unpublished course notes (2014) Available at `http://www.math.lsa.umich.edu/~hochster/615W14/CM.pdf`

[15] **J Huang**, *Syzygy order of big polygon spaces*, PhD thesis, University of Western Ontario (2018) Available at `http://ir.lib.uwo.ca/etd/5643`

[16] **V Puppe**, *Equivariant cohomology of $(\mathbb{Z}_2)^r$ –manifolds and syzygies*, Fund. Math. 243 (2018) 55–74 MR

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