Algebraic & Geometric Topology 20 (2020) 2677–2685

Null-homologous exotic surfaces in 4–manifolds

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We exhibit infinite families of embedded tori in 4–manifolds that are topologically isotopic but smoothly distinct. The interesting thing about these tori is that they are topologically trivial in the sense that each bounds a topologically embedded solid handlebody. This implies that there are stably ribbon surfaces in 4–manifolds that are not ribbon.

57Q45, 57R40, 57R52, 57R57; 57M25

1 Introduction

Just as a 4–manifold can have many inequivalent smooth structures, there can be many different smooth embeddings of surfaces into a 4–manifold which are topologically isotopic, but smoothly distinct. Any surface which admits more than one smooth embedding in a topological isotopy class will be said to admit *exotic embeddings*.

In this paper we will show that null-homologous tori first discovered by Fintushel and Stern in their knot surgery construction in fact provide examples of exotically embedded tori. Specifically:

Theorem 1.1 *Let X be a smooth 4–manifold with $b_2 \geq |\sigma| + 6$, nontrivial Seiberg–Witten invariant and embedded torus T of self-intersection 0 such that $\pi_1(X \setminus T) = 1$. Then X contains an infinite family of distinct tori $\{T_i\}$ that are topologically isotopic to the unknotted torus* (a torus that bounds a solid handlebody in X), *but no diffeomorphism of X exists taking T_i to T_j if $i \neq j$.*

The first examples of orientable exotic embeddings come from Fintushel and Stern's "rim surgery" technique [4]. Their surfaces all have simply connected complement. A variation on rim surgery was given by Kim [8] and Kim and Ruberman [10; 9], which works in the case that the complement has nontrivial fundamental group. Tom Mark has used Heegaard Floer homology to show that these constructions are also effective for constructing exotic embeddings of surfaces with negative self-intersection [11]. On the other hand, all of these constructions involve surfaces whose complement has

Published: 4 November 2020 DOI: 10.2140/agt.2020.20.2677

finite first homology, and moreover all of these constructions essentially begin with symplectically embedded surfaces in a symplectic 4–manifold. Such surfaces can never be null-homologous. The significance of our examples is that they are null-homologous and moreover topologically trivial.

One of the features of this theorem is that there are numerous tractable examples where the theorem can be applied. For example, any elliptic surface contains such a torus and has nontrivial Seiberg–Witten invariant by virtue of being a symplectic manifold.

The strategy of proof is as follows: The knot surgery construction of Fintushel and Stern produces an infinite family of exotic smooth structures on a 4–manifold through a series of log transforms on null-homologous tori. These are the tori we will focus on. Using Seiberg–Witten theory, we will define a gauge-theoretic invariant of null-homologous tori to distinguish the tori smoothly. Finally, we will show that all such tori are topologically isotopic by a theorem of the second author:

Theorem 1.2 [14, Theorem 7.2] *Let Σ_0 and Σ_1 be locally flat embedded surfaces of the same genus in a simply connected 4–manifold X. The surfaces are topologically isotopic when $\pi_1(X \setminus \Sigma_i) = \mathbb{Z}$ and $b_2 \geq |\sigma| + 6$.*

Note that a trivially embedded surface in any 4–manifolds will satisfy $\pi_1(X \setminus \Sigma) = \mathbb{Z}$.

One might wonder how robust these exotic embeddings are. That is, what does it take to make any of the exotically embedded topologically trivial surfaces constructed here smoothly equivalent again? In [1], Inanç Baykur and the second author show that these tori become smoothly equivalent once one increases the genus of each of these surfaces in the most trivial possible way. Namely, tubing any one of the topologically trivial surfaces of Theorem 1.1 to a smoothly trivial torus results in a smoothly trivial surface.

It would be interesting to know what the simplest examples of exotic embeddings are. For example, this paper provides context for the following two natural questions, which have motivated work on exotically embedded surfaces:

Question Do there exist exotically embedded surfaces in S^4? In particular, is there an embedded S^2 that is topologically isotopic to the unknot but not smoothly isotopic to the unknot?

The examples in this paper can be seen as prototypes for answering this sort of question, as answering this question in other manifolds can be seen as a first step to better understanding the S^4 question. At the same time, the examples of this paper can better inform an attack on the question above.

Recently, it has been claimed in [7] that a surface in S^4 that is stably ribbon is in fact ribbon. Our examples show that such an argument cannot be generalized to arbitrary 4–manifolds. We conclude the introduction with the following corollary to Theorem 1.1:

Corollary 1.3 *There exists a 4–manifold X that contains an exotically embedded torus T such that T is a stably ribbon surface, but not ribbon.*

Proof The examples in this paper identified in Theorem 1.1 (referred to as $\mathcal{T}_{K_j}$ in the proof) are stably ribbon, because they are stably smoothly trivial (by [1]). But the tori are not ribbon because log transforms on these tori have a different effect than log transforms on trivial tori (see Section 2.1). However, Theorem 8.3 of [14] implies that log transforms on ribbon tori are equivalent to those on trivial tori. □

Acknowledgements Both authors would like to thank the Max Planck Institute for Mathematics for hosting them while they worked on this project, and Danny Ruberman and Tom Mark for their comments on an early draft of this paper. We are also especially grateful to Inanç Baykur, who provided extensive comments on a later version. Hoffman was partially supported by a grant from the Simons Foundation (#524123 to Neil R Hoffman) during later revisions of this paper.

2 Constructing the tori

Let T be an embedded torus with self-intersection zero in a 4–manifold X such that $\pi_1(X \setminus T) = 1$. We will not construct exotic embeddings of T (any such torus is necessarily homologically essential since it will have a dual), but rather we will find exotic embeddings of nearby null-homologous tori which arise in the "knot surgery" construction of Fintushel and Stern [5; 3]. Knot surgery along torus T using a knot $K \subset S^3$ is most straightforwardly defined as $X_K = (X \setminus \nu(T)) \cup (S^1 \times S^3 \setminus \nu(K))$, where the union is formed by taking the longitude of K to the meridian of T (apart from this requirement, the gluing is not, strictly speaking, well defined, and X_K may depend on the gluing map, but this ambiguity does not factor into our argument below). Fintushel and Stern proved that X is homeomorphic to X_K under the assumption that the complement of T is simply connected, and they further proved that their Seiberg–Witten invariants are related by $\mathrm{SW}_{X_K} = \mathrm{SW}_X \cdot \Delta_K(2[T])$, where Δ_K is the

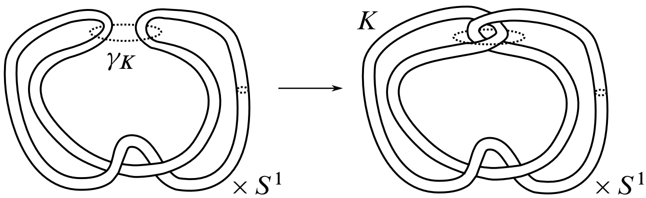

Figure 1: The left figure represents νT, which can be thought of as the complement of the unknot in S^3 crossed with S^1. Performing a $(+1)$–log transform on $\mathcal{T}_K = S^1 \times \gamma_K$ gives $S^1 \times (S^3 \setminus \nu K)$, depicted on the right.

Alexander polynomial for K. Therefore, by varying K, one can construct infinitely many smooth structures on X.

The Seiberg–Witten formula is proved by viewing knot surgery as a series of log transforms on null-homologous tori. That is, rather than cutting out $\nu(T) = S^1 \times (S^1 \times D^2)$ and replacing it with $S^1 \times S^3 \setminus \nu(K)$, we can view knot surgery as a series of log transforms in $S^1 \times (S^1 \times D^2)$ which eventually lead to $S^1 \times S^3 \setminus \nu(K)$. Forgetting the extra S^1 direction for the moment, one can go from $S^3 \setminus \nu(K)$ to $(S^1 \times D^2)$, the complement of the unknot, by doing ± 1 surgery along crossings of K to unknot it. Crossing this whole picture with S^1 gives the log transforms needed for knot surgery. It is these null-homologous tori that are needed to do the knot surgery that will turn out to be the exotically unknotted tori that we are looking for in our theorem.

We'll focus on the following particular situation: Suppose that K is a knot of unknotting number 1, and T is a torus in X with trivial normal bundle and simply connected complement. Then knot surgery is the result of doing a single log transform on the null-homologous torus $\mathcal{T}_K = S^1 \times \gamma_K$ in $\nu T \subset X$ (see Figure 1). The proof of our theorem will show that $\mathcal{T}_K$ is topologically unknotted, but smoothly nontrivial. To determine the topological isotopy class of $\mathcal{T}_K$ we will now compute the fundamental group of $X \setminus \mathcal{T}_K$: Note that since $\pi_1(X \setminus T) = 1$, we have that the inclusion $\pi_1(\partial \nu T) \to \pi_1(X \setminus \nu T)$ is trivial. Therefore, by repeated applications of van Kampen's theorem,

$$\begin{aligned}
\pi_1(X \setminus \mathcal{T}_K) &= \pi_1((X \setminus \nu T) \cup (\nu T \setminus \mathcal{T}_K)) \\
&= \frac{\pi_1(\nu T \setminus \mathcal{T}_K)}{\pi_1(\partial \nu T)} \\
&= \frac{\pi_1\bigl(S^1 \times (S^3 \setminus (\nu U \cup \gamma_K))\bigr)}{\langle S^1 \rangle \times \pi_1(\partial \nu U)} \\
&= \pi_1(S^3 \setminus \gamma_k) = \mathbb{Z}.
\end{aligned}$$

The second-to-last line is just a change of notation, since $\nu T \setminus \mathcal{T}_K$ is just the same thing as $S^1 \times (S^3 \setminus (\nu U \cup \gamma_K))$, where U is the unknot (see Figure 1). And the final equality is true because γ_k is necessarily unknotted in S^3.

2.1 Examples of exotically embedded tori

Already we see that this gives at least one exotically embedded torus. Specifically, $\mathcal{T}_K$ is topologically standard by Theorem 1.2, and, moreover, performing a log transform on $\mathcal{T}_K$ will give an exotic smooth structure on X, whereas performing a log transform on the standardly embedded torus (ie the one that bounds a solid handlebody in $D^4 \subset X$) will not (see for example [12, Corollary 5.6]). Therefore these tori are smoothly distinct, but by Theorem 1.2 they must be topologically isotopic.

To construct infinite families of exotic surfaces, we need to be more careful. For example suppose K_1 and K_2 are knots with associated null-homologous tori $\mathcal{T}_{K_1}$ and $\mathcal{T}_{K_2}$. Then it is conceivable that one might be able to construct both X_{K_1} and X_{K_2} by some surgery on $\mathcal{T}_{K_1}$. In this circumstance we would not be able to distinguish $\mathcal{T}_{K_1}$ from $\mathcal{T}_{K_2}$ as we did above. To resolve this issue, we have to look more deeply at how the Seiberg–Witten invariant changes under log transforms on $\mathcal{T}_K$, and restrict ourselves to certain classes of knots.

3 Smooth invariants of null-homologous tori

The Seiberg–Witten invariant of a 4–manifold X is a map $\mathrm{SW}_X\colon \mathcal{S} \to \mathbb{Z}$, where $\mathcal{S}$ is the set of isomorphism classes of spin^c structures on X. The *basic classes* of X are defined to be the spin^c structures that map to nonzero integers. It is a well-known property of the Seiberg–Witten invariant that a closed 4–manifold has only a finite number of basic classes. Below, we will often not distinguish between a spin^c structure and its first Chern class or even the Poincaré dual of its first Chern class.

We will distinguish our null-homologous tori by computing an invariant that is, in a technical sense clarified below, related to the Seiberg–Witten basic classes of the complement of the tori. To do this we will need to understand how the Seiberg–Witten invariant of a 4–manifold is affected by log transforms. Suppose we are given a 4–manifold with T^3 boundary, eg $X \setminus \nu T$, and suppose $H_1(T^3) = \mathbb{Z}[a,b,c]$. Denote the log-transformed 4–manifold constructed by gluing on a $D^2 \times T^2$, where $[\partial D^2]$ is glued to $[pa + qb + rc]$, as $X_T(p,q,r)$ or sometimes just $X_{(p,q,r)}$, and denote the core torus in the $D^2 \times T^2$ part of this manifold as $T_{(p,q,r)}$.

A formula of Morgan, Mrowka and Szabó from [13] gives a formula relating the Seiberg–Witten invariants of various log transforms:

$$(1)\quad \sum_i \mathrm{SW}_{X_T(p,q,r)}(k_{(p,q,r)} + i[T_{(p,q,r)}]) = p\sum_i \mathrm{SW}_{X_T(1,0,0)}(k_{(1,0,0)} + i[T_{(1,0,0)}]) + q\sum_i \mathrm{SW}_{X_T(0,1,0)}(k_{(0,1,0)} + i[T_{(0,1,0)}]) + r\sum_i \mathrm{SW}_{X_T(0,0,1)}(k_{(0,0,1)} + i[T_{(0,0,1)}]),$$

where the spin^c structures agree away from the log-transformed tori, ie

$$(2)\quad k_{(p,q,r)}|_{X_{(p,q,r)}\setminus T_{(p,q,r)}} = k_{(1,0,0)}|_{X_{(1,0,0)}\setminus T_{(1,0,0)}} = k_{(0,1,0)}|_{X_{(0,1,0)}\setminus T_{(0,1,0)}} = k_{(0,0,1)}|_{X_{(0,0,1)}\setminus T_{(0,0,1)}}.$$

Suppose that $T_{(p,q,r)}$ is null-homologous. Then the left side of (1) has only one term. Moreover, since $X_{(1,0,0)}$ has only a finite number of basic classes $k_{(1,0,0)} + i[T_{(1,0,0)}]$, we see that (2) implies that there are only a fixed finite number of possible basic classes $k_{(p,q,r)}$ for $X_{T(p,q,r)}$, and these possibilities depend only on the choice of T, not on (p,q,r). To put this another way, there are only a finite number of spin^c structures on $X \setminus \nu T$ that can be extended to basic classes on $X_T(p,q,r)$ when $[T_{(p,q,r)}] = 0$. Therefore, the following invariant is well defined:

Definition 3.1 Let T be a null-homologous torus in X (defined up to smooth isotopy). Define $B(X,T)$ to be the maximum divisibility of the difference between any two basic classes of $X_T(p,q,r)$ for any (p,q,r) such that $[T_{(p,q,r)}] = 0$.

Remark It is important to use the divisibility of the *difference* of basic classes rather than just the divisibility of the basic classes because, very often, after performing knot surgery all of the basic classes have divisibility 1.

4 Families of unknotting number 1 knots, and the proof of Theorem 1.1

Now that we have a better understanding of the smooth invariants needed to distinguish potential infinite families of smooth tori, we can describe an explicit family of knots that will give rise to smoothly distinct $\mathcal{T}_K$. For the invariant $B(X,T)$ to be useful,

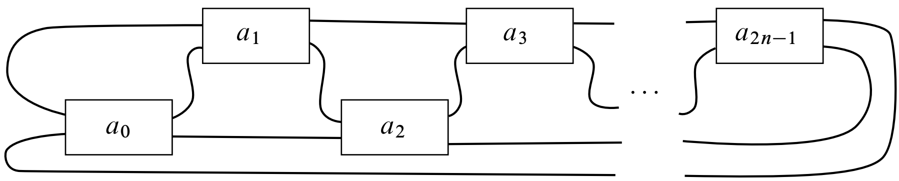

Figure 2: The two-bridge knot $C(a_0, \dots, a_{2n-1})$.

we will need to find a family of knots with unknotting number 1 whose Alexander polynomials have arbitrarily high degree. All two-bridge knots can be given in the form of Figure 2, where a_i is the number of right half-twists when i is odd, and left half-twists when i is even. We refer to two-bridge knots using Conway's notation, $C(a_0, \dots, a_m)$, and we note that it is well known (see [2, Section 12B] for instance) that two 2–bridge knots are equivalent if and only if $[a_0, \dots, a_m]$ and $[a'_0, \dots, a'_{m'}]$ are continued fraction expansions of the same rational number.

Proposition 4.1 (Kanenobu and Murakami [6]) *A two-bridge knot has unknotting number* 1 *if and only if it can be expressed as*

$$C(b, b_1, b_2, \dots, b_k, \pm 2, -b_k, \dots, -b_2, -b_1).$$

The following proposition of Burde and Zieschang tells us how to compute the relevant polynomial invariants:

Proposition 4.2 (Burde and Zieschang [2, Proposition 12.23]) *The Conway polynomial of a two-bridge knot expressed as* $C(a_0, \dots, a_{2n-1})$ *has degree* $\sum_{j=0}^{n-1} |a_{2i}|$.

Remark The notation in [2] is different than that used in Proposition 4.1. To reconcile the conventions, the two-bridge knot diagram in Figure 2 can be converted to a 4–plat diagram as in [2] by pulling the inner strand on the right-hand side of the figure over the outer strand. This has the effect of adding a new crossing (ie $a_{2n} = +1$) and adjusting a_{2n-1} by $+1$.

Lemma 4.3 *There exists an infinite family of unknotting number* 1 *knots whose Alexander polynomials have arbitrarily high degree.*

Proof Combining Propositions 4.1 and 4.2 shows that there exists an infinite family of two-bridge knots of unknotting number 1 such that the Conway polynomial has arbitrarily high degree. The lemma is thus immediate from the fact that the Conway polynomial is related to the Alexander polynomial by the formula $\nabla(t - t^{-1}) = \Delta(t^2)$. □

Proof of Theorem 1.1 Let $\{K_j\}$ be a sequence of knots of unknotting number 1 such that the degree of their Alexander polynomials goes to infinity, and let $\mathcal{T}_{K_j}$ be the associated (topologically trivial) tori from Section 2.

Since there is a log transform on $\mathcal{T}_{K_j}$ that gives X_{K_j}, we have that

$$\lim_{j\to\infty} B(X,\mathcal{T}_{K_j}) \geq \lim_{j\to\infty}\begin{pmatrix}\text{max divisibility of the difference}\\ \text{between any two basic classes of } X_{K_j}\end{pmatrix}$$
$$\geq \lim_{j\to\infty} 4\deg(\Delta_{K_j}) = \infty.$$

The second inequality follows because the knot surgery formula,

$$\mathrm{SW}_{X_K} = \mathrm{SW}_X \cdot \Delta_K(2[T]),$$

allows us to determine the basic classes of X_K from those of X. Specifically, if κ is a basic class of X and the degree of Δ_{K_j} is n, then X_{K_j} has $\kappa+2n[T]$ and $\kappa-2n[T]$ as basic classes (among others), and the divisibility of the difference of this pair of basic classes is $4n$, which serves as a lower bound for $B(X,\mathcal{T}_{K_j})$. Therefore, after possibly passing to a subsequence of the $\{X_{K_j}\}$, there are an infinite number of the $\mathcal{T}_{K_j}$ that are smoothly distinguished by their B–invariant. □

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Received: 1 August 2019 Revised: 23 September 2019