msp

Algebraic & Geometric Topology 20 (2020) 3205–3218

Correction to the article Floer homology and splicing knot complements

EAMAN EFTEKHARY

This note corrects the mistakes in the splicing formulas of the paper "Floer homology and splicing knot complements" (Algebr. Geom. Topol. 15 (2015) 3155–3213). The mistakes are the result of the incorrect assumption that for a knot K inside a homology sphere Y, the involution on $\widehat{\mathrm{HFK}}(K)$ which corresponds to moving the basepoints by one full twist around K is trivial. The correction implicitly involves considering the contribution from this (possibly nontrivial) involution in a number of places.

57M27; 57R58

We first describe the incorrect assumption about the triviality of the involution on $\widehat{\mathrm{HFK}}(K)$, which corresponds to moving the basepoints by one full twist around a knot K inside a homology sphere Y. The incorrect assumption that this map is trivial first appears in [1, Section 3.3]. It is then used in [1, Section 5.3] to give a simplified matrix presentation of the splicing formula. For the trefoils, the involution map happens to be trivial, and thus the computations of Section 6.1 remain unchanged. Some modifications are necessary to the computations of Section 6.2. Finally, there are some changes that should be made to the matrix presentations in the appendix. The corrected Sections $3.3'$, $5.3'$ and $6.2'$, as well as the corrected form of the appendix, appear in this note.

The original form of the splicing formulas, which are based on the aforementioned wrong assumption, are in fact not incorrect. However, the arguments of this paper are not sufficient for showing this and the corrections of this note are thus very crucial. The issue is further discussed in the sequel [2].

The author would like to thank Adam Levine for pointing out the mistake, and for some very helpful discussions.

The involution of $\widehat{\mathrm{HFK}}$

In Section 3.3 of [1], the author considers the isomorphisms $\tau_\bullet\colon \mathbb{H}_\bullet(K) \to \mathbb{H}_\bullet(K)$ for $\bullet \in \{0, 1, \infty\}$ coming from changing the role of the two punctures z_1 and z_2 in the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; z_1, z_2)$. It was claimed that $\tau_\bullet^2$ is the identity map. However, the isomorphism $\tau_\bullet^2$ is in fact the involution map $\varsigma_\bullet$ studied by Sarkar [7] and

Published: 8 December 2020 DOI: 10.2140/agt.2020.20.3205

by Hendricks and Manolescu [4], which is not necessarily trivial. This map corresponds to moving the punctures z_1 and z_2 one full twist along the knot K and its square is the identity map. The map ς_∞ is in general nontrivial, eg for the figure-eight knot. Correspondingly, it is not a priori true that

$$\overline{\mathfrak{f}}_0 = \tau_\infty \circ \mathfrak{f}_0 \circ \tau_1, \quad \overline{\mathfrak{f}}_1 = \tau_0 \circ \mathfrak{f}_1 \circ \tau_\infty \quad \text{and} \quad \overline{\mathfrak{f}}_\infty = \tau_1 \circ \mathfrak{f}_\infty \circ \tau_0.$$

Without the above inequalities, many of the matrix computations in Section 5.3 need significant changes and the computation of Section 6.3 should also be modified. Some changes are also necessary in the later parts of the appendix. The revised forms of these parts follow.

Nevertheless, the three equalities

$$\overline{\mathfrak{f}}_0 = \tau_\infty \circ \mathfrak{f}_0 \circ \tau_1, \quad \overline{\mathfrak{f}}_1 = \tau_0 \circ \mathfrak{f}_1 \circ \tau_\infty \quad \text{and} \quad \overline{\mathfrak{f}}_\infty = \tau_1 \circ \mathfrak{f}_\infty \circ \tau_0$$

are true for nontrivial reasons that will not be discussed here and are not relevant for the purposes of this paper. A detailed discussion of this claim appears in [2].

3.3′ Some properties of the maps $\mathfrak{f}_\bullet(K)$ and $\overline{\mathfrak{f}}_\bullet(K)$

Our first observation is that changing the orientation of the knot K, and correspondingly that of K_1 and K_0, corresponds to changing the markings u, v and w with $\overline{u}$, $\overline{v}$ and $\overline{w}$ in Figure 1. Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; z_1, z_2)$ represents $K_\bullet$, meaning that an oriented longitude for $K_\bullet$ is constructed from gluing an oriented arc on Σ from z_1 to z_2 in the complement of $\boldsymbol{\alpha}$ and an oriented arc on Σ from z_2 to z_1 in the complement of $\boldsymbol{\beta}$. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; z_2, z_1)$ is a Heegaard diagram for $-K_\bullet$ (the knot $K_\bullet$ with the reverse orientation) while $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}; z_2, z_1)$ is a Heegaard diagram for $K_\bullet$. The chain complexes associated with the above three Heegaard diagrams are identical. Heegaard moves give chain homotopy equivalences

$$\tau_\bullet(K)\colon \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; z_1, z_2) = \widehat{\mathrm{CF}}(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}; z_2, z_1) \to \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; z_1, z_2).$$

These chain homotopy equivalences induce the isomorphisms

$$\tau_\bullet(K)\colon \mathbb{H}_\bullet(K) \to \mathbb{H}_\bullet(K), \quad \bullet \in \{0, 1, \infty\}.$$

In terms of these isomorphisms,

$$\begin{aligned} \overline{\mathfrak{f}}_0(K) &= \tau_\infty(K) \circ \mathfrak{f}_0(K) \circ \tau_1(K)^{-1}, \\ \overline{\mathfrak{f}}_1(K) &= \tau_0(K) \circ \mathfrak{f}_1(K) \circ \tau_\infty(K)^{-1}, \\ \overline{\mathfrak{f}}_\infty(K) &= \tau_1(K) \circ \mathfrak{f}_\infty(K) \circ \tau_0(K)^{-1}. \end{aligned} \tag{4}$$

Note however, that the equality $\overline{\theta}(K) = \tau_\infty(K)\theta(K)\tau_0(K)^{-1}$ is only satisfied for the induced maps from $\mathrm{Ker}(\overline{\mathfrak{f}}_\infty(K))$ to $\mathrm{Coker}(\overline{\mathfrak{f}}_0(K))$.

The exactness of the sequences in [1, (3)] implies that, in appropriate decompositions

$$\begin{aligned}
\mathbb{H}_0(K) &= \frac{\mathbb{H}_0(K)}{\mathrm{Ker}(\mathfrak{f}_\infty(K))} \oplus \mathrm{Ker}(\mathfrak{f}_\infty(K)) =: \mathbb{A}_\infty(K) \oplus \mathbb{A}_1(K),\\
\mathbb{H}_1(K) &= \frac{\mathbb{H}_1(K)}{\mathrm{Ker}(\mathfrak{f}_0(K))} \oplus \mathrm{Ker}(\mathfrak{f}_0(K)) =: \mathbb{A}_0(K) \oplus \mathbb{A}_\infty(K),\\
\mathbb{H}_\infty(K) &= \frac{\mathbb{H}_\infty(K)}{\mathrm{Ker}(\mathfrak{f}_1(K))} \oplus \mathrm{Ker}(\mathfrak{f}_1(K)) =: \mathbb{A}_1(K) \oplus \mathbb{A}_0(K),
\end{aligned} \tag{5}$$

we have

$$\mathfrak{f}_\bullet(K) = \begin{pmatrix} 0 & 0 \\ I_{a_\bullet(K)} & 0 \end{pmatrix},$$

where $a_\bullet(K)$ denotes the rank of $\mathbb{A}_\bullet(K)$ for every $\bullet \in \{0, 1, \infty\}$. In this basis we may present the matrices $\tau_\bullet(K)$ as

$$\tau_\bullet(K) = \begin{pmatrix} A_\bullet(K) & B_\bullet(K) \\ C_\bullet(K) & D_\bullet(K) \end{pmatrix}, \quad \bullet \in \{0, 1, \infty\}.$$

The map $B_0(K)$ corresponds to the induced map

$$\tau_0(K)\colon \mathrm{Ker}(\mathfrak{f}_\infty(K)) \to \frac{\mathbb{H}_0(K)}{\mathrm{Ker}(\mathfrak{f}_\infty(K))}.$$

The decomposition $\mathbb{H}_0(K) = \mathbb{A}_\infty(K) \oplus \mathbb{A}_1(K)$ may be modified using a change of basis of the form $P_X = \left(\begin{smallmatrix} I & 0 \\ -X & I \end{smallmatrix}\right)$, which does not change the block presentations of the maps $\mathfrak{f}_\infty(K)$ and $\mathfrak{f}_1(K)$. In the new basis, $\tau_0(K)$ has the presentation

$$\begin{aligned}
\tau_0(K) &= \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} A_0(K) & B_0(K) \\ C_0(K) & D_0(K) \end{pmatrix} \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}\\
&= \begin{pmatrix} A_0(K) - B_0(K)X & B_0(K) \\ \star & -XB_0(K) + D_0(K) \end{pmatrix}.
\end{aligned}$$

If $B_0(K)$ is injective, we may thus assume that $D_0(K) = 0$, while if $B_0(K)$ is surjective, we may assume that $A_0(K) = 0$. With a similar reasoning, if $B_\bullet(K)$ is injective, we may assume that $D_\bullet(K) = 0$, while if $B_\bullet(K)$ is surjective, we may assume that $A_\bullet(K) = 0$.

In the above decompositions for $\mathbb{H}_\bullet(K)$, the map $\theta(K)\colon \mathbb{H}_0(K) \to \mathbb{H}_\infty(K)$ takes the form

$$\theta(K) = \begin{pmatrix} X & I \\ Z & Y \end{pmatrix},$$

since the induced map from $\mathbb{A}_1(K)\subset\mathbb{H}_0(K)$ to $\mathbb{A}_1(K)\subset\mathbb{H}_\infty(K)$ is the inverse of the map induced by $\mathfrak{f}_1(K)$, ie the identity. Moreover, since the rank of $\theta(K)$ is the same as the rank of $\mathfrak{f}_1(K)$, we conclude that $Z=YX$. Applying the change of basis P_Y on $\mathbb{H}_0(K)$ and the corresponding change of basis P_X on $\mathbb{H}_\infty(K)$, $\theta(K)$ takes the form

$$\begin{pmatrix} I & 0\\ -Y & I\end{pmatrix}\begin{pmatrix} X & I\\ YX & Y\end{pmatrix}\begin{pmatrix} I & 0\\ -X & I\end{pmatrix}=\begin{pmatrix} 0 & I\\ 0 & 0\end{pmatrix}.$$

It is thus possible to choose the above decompositions so that $\theta(K)=\left(\begin{smallmatrix} 0 & I\\ 0 & 0\end{smallmatrix}\right)$. If this is the case, the 2×2 presentation of $\tau_\infty(K)^{-1}\overline{\theta}(K)\tau_0(K)$ will be of the form

$$\tau_\infty(K)^{-1}\overline{\theta}(K)\tau_0(K)=\begin{pmatrix} M & I\\ Q & P\end{pmatrix},$$

and since the ranks of $\theta(K)$ and $\overline{\theta}(K)$ are the same, we find $Q=PM$.

5.3′ Simplifications of the splicing formula

We now apply [1, Lemma 2.3] to the splicing formula of [1, Proposition 5.3] and make some cancellations. The first cancellation comes from setting $C=\mathbb{H}$, $A=\mathbb{H}_{1,1}$ and

$$B=(\mathbb{H}_{\infty,\infty}\oplus\mathbb{H}_{1,\infty}\oplus\mathbb{H}_{\infty,1})\oplus(\mathbb{H}_{0,1}\oplus\mathbb{H}_{1,0}\oplus\mathbb{H}_{0,0})=\mathbb{E}_1\oplus\mathbb{E}_2.$$

We thus have $\widehat{\mathrm{HF}}(Y)=H_*(B,d_B)$, where

$$d_B=\begin{pmatrix} 0 & \mathfrak{f}_0^1\otimes I & I\otimes\mathfrak{f}_0^2 & \theta^1\otimes\overline{\mathfrak{f}}_0^2 & \overline{\mathfrak{f}}_0^1\otimes\theta^2 & \Gamma\\ 0 & 0 & 0 & \Phi & I\otimes(\mathfrak{f}_0^2\circ\overline{\mathfrak{f}}_\infty^2) & \mathfrak{f}_\infty^1\otimes\overline{\theta}^2\\ 0 & 0 & 0 & (\mathfrak{f}_0^1\circ\overline{\mathfrak{f}}_\infty^1)\otimes I & \Psi & \overline{\theta}^1\otimes\mathfrak{f}_\infty^2\\ 0 & 0 & 0 & 0 & 0 & I\otimes\overline{\mathfrak{f}}_\infty^2\\ 0 & 0 & 0 & 0 & 0 & \overline{\mathfrak{f}}_\infty^1\otimes I\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix},$$

with $\Gamma=\overline{\mathfrak{f}}_0^1\circ\mathfrak{f}_\infty^1\otimes\overline{\mathfrak{f}}_0^2\circ\mathfrak{f}_\infty^2+\theta^1\otimes\overline{\theta}^2+\overline{\theta}^1\otimes\theta^2$, $\Phi=\overline{\mathfrak{f}}_\infty^1\otimes\mathfrak{f}_0^2+\mathfrak{f}_\infty^1\otimes\overline{\mathfrak{f}}_0^2$ and $\Psi=\mathfrak{f}_0^1\otimes\overline{\mathfrak{f}}_\infty^2+\overline{\mathfrak{f}}_0^1\otimes\mathfrak{f}_\infty^2$.

The dimension of the $\mathbb{F}$–vector space $H_*(B,d_B)$ only depends on the rank of the kernel and the cokernel of the matrix d_B. Define a pair of matrices M_1 and M_2 to be equivalent if $\mathrm{Ker}(M_1)\simeq\mathrm{Ker}(M_2)$ and $\mathrm{Coker}(M_1)\simeq\mathrm{Coker}(M_2)$. For a matrix M let

$$\imath(M):=\mathrm{Ker}(M)\oplus\mathrm{Coker}(M)\quad\text{and}\quad i(M):=\mathrm{rnk}(\imath(M)).$$

If M_1 and M_2 are equivalent matrices then $\imath(M_1) \simeq \imath(M_2)$ and $i(M_1) = i(M_2)$.

We make a change of basis for $\mathbb{E}_2$ which is given by the matrix

$$\begin{pmatrix} \tau_0(K_1)\otimes\tau_1(K_2) & 0 & 0 \\ 0 & \tau_1(K_1)\otimes\tau_0(K_2) & 0 \\ 0 & 0 & \tau_0(K_1)\otimes\tau_0(K_2) \end{pmatrix}.$$

The matrix $d_{\mathcal{B}}$ is thus equivalent to the matrix

$$d'_{\mathcal{B}} = \begin{pmatrix} 0 & \mathfrak{f}_0^1\otimes I & I\otimes\mathfrak{f}_0^2 & \theta^1\tau_0^1\otimes\tau_\infty^2\mathfrak{f}_0^2 & \tau_\infty^1\mathfrak{f}_0^1\otimes\theta^2\tau_0^2 & \Gamma \\ 0 & 0 & 0 & \Phi & \tau_1^1\otimes\mathfrak{f}_0^2\tau_1^2\mathfrak{f}_\infty^2 & \mathfrak{f}_\infty^1\tau_0^1\otimes\overline{\theta}^2\tau_0^2 \\ 0 & 0 & 0 & \mathfrak{f}_0^1\tau_1^1\mathfrak{f}_\infty^1\otimes\tau_1^1 & \Psi & \overline{\theta}^1\tau_0^1\otimes\mathfrak{f}_\infty^2\tau_0^2 \\ 0 & 0 & 0 & 0 & 0 & I\otimes\mathfrak{f}_\infty^2 \\ 0 & 0 & 0 & 0 & 0 & \mathfrak{f}_\infty^1\otimes I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Phi = \tau_1^1\mathfrak{f}_\infty^1\otimes\mathfrak{f}_0^2\tau_1^2 + \mathfrak{f}_\infty^1\tau_0^1\otimes\tau_\infty^2\mathfrak{f}_0^2,$$
$$\Psi = \mathfrak{f}_0^1\tau_1^1\otimes\tau_1^2\mathfrak{f}_\infty^2 + \tau_\infty^1\mathfrak{f}_0^1\otimes\mathfrak{f}_\infty^2\tau_0^2,$$
$$\Gamma = \overline{\mathfrak{f}}_0^1\mathfrak{f}_\infty^1\tau_0^1\otimes\overline{\mathfrak{f}}_0^2\mathfrak{f}_\infty^2\tau_0^2 + \theta^1\tau_0^1\otimes\overline{\theta}^2\tau_0^2 + \overline{\theta}^1\tau_0^1\otimes\theta^2\tau_0^2.$$

Let us use the decompositions of (5) for K_1 and K_2 to obtain a 24×24 block decomposition of $d'_{\mathcal{B}}$. Moreover, following the discussion at the end of [1, Section 3.3] we may assume that, in the corresponding decompositions,

$$\theta^i = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \quad\text{and}\quad (\tau_\infty^i)^{-1}\overline{\theta}^i\tau_0^i = \begin{pmatrix} M^i & I \\ P^iM^i & P^i \end{pmatrix}.$$

Each entry in the above 6×6 decomposition for $d'_{\mathbb{B}}$ corresponds to a 4×4 submatrix of the aforementioned 24×24 decomposition. For instance, the $(1,4)$ entry $\theta^1\tau_0^1\otimes\tau_\infty^2\mathfrak{f}_0^2$ corresponds to

$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_0^1 & B_0^1 \\ C_0^1 & D_0^1 \end{pmatrix}\otimes\begin{pmatrix} A_\infty^2 & B_\infty^2 \\ C_\infty^2 & D_\infty^2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} = \begin{pmatrix} C_0^1 & D_0^1 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} B_\infty^2 & 0 \\ D_\infty^2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} C_0^1\otimes B_\infty^2 & 0 & D_0^1\otimes B_\infty^2 & 0 \\ C_0^1\otimes D_\infty^2 & 0 & D_0^1\otimes D_\infty^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For another instance, note that the $(3,5)$ entry corresponds to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_1^1 \otimes B_1^2 & 0 & B_1^1 \otimes B_1^2 & 0 \\ A_1^1 \otimes D_1^2 & 0 & B_1^1 \otimes D_1^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_\infty^1 \otimes A_0^2 & B_\infty^1 \otimes B_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D_\infty^1 \otimes A_0^2 & D_\infty^1 \otimes B_0^2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_\infty^1 \otimes A_0^2 & B_\infty^1 \otimes B_0^2 & 0 & 0 \\ A_1^1 \otimes B_1^2 & 0 & B_1^1 \otimes B_1^2 & 0 \\ A_1^1 \otimes D_1^2 + D_\infty^1 \otimes A_0^2 & D_\infty^1 \otimes B_0^2 & B_1^1 \otimes D_1^2 & 0 \end{pmatrix}.$$

The aforementioned 24×24 decomposition includes identity matrices as the entries determined by the block coordinates

$$(2,9), \quad (3,5), \quad (4,6), \quad (14,21), \quad (16,23) \quad \text{and} \quad (20,22).$$

We use the above six identity matrices for cancellation to obtain an equivalent matrix $d = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$ over $\mathbb{B}_1 \oplus \mathbb{B}_2$, where $\mathbb{A}_{\bullet\star} = \mathbb{A}_\bullet(K_1) \otimes \mathbb{A}_\star(K_2)$ and

$$\mathbb{B}_1 = \mathbb{A}_{11} \oplus \mathbb{A}_{\infty 1} \oplus \mathbb{A}_{\infty 0} \oplus \mathbb{A}_{1\infty} \oplus \mathbb{A}_{0\infty} \oplus \mathbb{A}_{00},$$
$$\mathbb{B}_2 = \mathbb{A}_{\infty 0} \oplus \mathbb{A}_{10} \oplus \mathbb{A}_{\infty\infty} \oplus \mathbb{A}_{0\infty} \oplus \mathbb{A}_{01} \oplus \mathbb{A}_{11}.$$

Rearrange the rows and the columns of the matrix D, so that D corresponds to the rows 11, 7, 8, 10, 12 and 1 and the columns 19, 13, 15, 17, 18 and 24 in the above 24×24 decomposition to obtain the matrix

$$\begin{pmatrix} B_1^1 \otimes B_1^2 & B_1^1 \otimes A_1^2 & 0 & A_1^1 \otimes B_1^2 & 0 & 0 \\ 0 & A_0^1 \otimes B_\infty^2 & B_0^1 \otimes B_\infty^2 & 0 & 0 & B_0^1 \otimes (A_\infty^2 + B_\infty^2 P^2) \\ D_1^1 \otimes B_1^2 & D_1^1 \otimes A_1^2 + A_0^1 \otimes D_\infty^2 & B_0^1 \otimes D_\infty^2 & C_1^1 \otimes B_1^2 & 0 & B_0^1 \otimes (C_\infty^2 + D_\infty^2 P^2) \\ 0 & 0 & 0 & B_\infty^1 \otimes A_0^2 & B_\infty^1 \otimes B_0^2 & (A_\infty^1 + B_\infty^1 P^1) \otimes B_0^2 \\ B_1^1 \otimes D_1^2 & B_1^1 \otimes C_1^2 & 0 & D_\infty^1 \otimes A_0^2 + A_1^1 \otimes D_1^2 & D_\infty^1 \otimes B_0^2 & (C_\infty^1 + D_\infty^1 P^1) \otimes B_0^2 \\ 0 & C_0^1 \otimes B_\infty^2 & D_0^1 \otimes B_\infty^2 & B_\infty^1 \otimes C_0^2 & B_\infty^1 \otimes D_0^2 & \Gamma \end{pmatrix}$$

with $\Gamma = B_\infty^1 \bar{B}_1^1 B_0^1 \otimes B_\infty^2 \bar{B}_1^2 B_0^2 + (A_\infty^1 + B_\infty^1 P^1) \otimes D_0^2 + D_0^1 \otimes (A_\infty^2 + B_\infty^2 P^2)$.

Here, we assume that

$$(\tau_\bullet^i)^{-1} = \begin{pmatrix} \bar{A}_\bullet^i & \bar{B}_\bullet^i \\ \bar{C}_\bullet^i & \bar{D}_\bullet^i \end{pmatrix}, \quad \bullet \in \{0, 1, \infty\},\ i = 1, 2,$$

are the corresponding block decompositions for the inverses of $\tau_\bullet^i$.

This matrix is in turn equivalent to the matrix $\mathfrak{D}=\mathfrak{D}(K_1,K_2)$ below, which is obtained by adding $I\otimes P^2$ times the third column and $P^1\otimes I$ times the fifth column to the last column of the above matrix:

$$\mathfrak{D}=\begin{pmatrix} B_1^1\otimes B_1^2 & C_1^1\otimes A_1^2 & 0 & A_1^1\otimes B_1^2 & 0 & 0 \\ 0 & A_0^1\otimes B_\infty^2 & B_0^1\otimes B_\infty^2 & 0 & 0 & B_0^1\otimes A_\infty^2 \\ D_1^1\otimes B_1^2 & D_1^1\otimes A_1^2+A_0^1\otimes D_\infty^2 & B_0^1\otimes D_\infty^2 & C_1^1\otimes B_1^2 & 0 & B_0^1\otimes C_\infty^2 \\ 0 & 0 & 0 & B_\infty^1\otimes A_0^2 & B_\infty^1\otimes B_0^2 & A_\infty^1\otimes B_0^2 \\ B_1^1\otimes D_1^2 & B_1^1\otimes C_1^2 & 0 & D_\infty^1\otimes A_0^2+A_1^1\otimes D_1^2 & D_\infty^1\otimes B_0^2 & C_\infty^1\otimes B_0^2 \\ 0 & C_0^1\otimes B_\infty^2 & D_0^1\otimes B_\infty^2 & B_\infty^1\otimes C_0^2 & B_\infty^1\otimes D_0^2 & \Psi \end{pmatrix},$$

where $\Psi=A_\infty^1\otimes D_0^2+D_0^1\otimes A_\infty^2+X^1\otimes X^2$ and $X^i=X(K_i)=B_\infty^i\overline{B}_1^iB_0^i$ for $i=1,2$. Combining [1, Proposition 5.3] with the above observations we find:

Proposition 5.5 *Let $K_i\subset Y_i$ denote null-homologous knots for $i=1,2$ and let $Y(K_1,K_2)$ denote the three-manifold obtained by splicing the complement of K_1 with the complement of K_2. With the above definition of $\mathfrak{D}(K_1,K_2)$,*

$$\widehat{\mathrm{HF}}(Y(K_1,K_2),\mathbb{F})\simeq\imath(\mathfrak{D}(K_1,K_2)).$$

Corollary 5.6 *The splicing formula of* [1, Proposition 5.3] *is independent of the choice of extensions θ^i and $\overline{\theta}^i$.*

Proof The fact that the matrices P^i and M^i do not appear in the matrix $\mathfrak{D}(K_1,K_2)$ implies that the choice of the extensions $\theta^i,\overline{\theta}^i\colon\mathbb{H}_0^i\to\mathbb{H}_\infty^i$ does not change the rank of the homology group in the splicing formula of [1, Proposition 5.3 or Theorem 1.1]. □

With the above corollary in place, the proof of [1, Theorem 1.1] is now complete.

Definition 5.7 For a pair of knots $K_i\subset Y_i$ for $i=1,2$, define

$$\chi(K_1,K_2):=(h_1(K_1)-h_\infty(K_1))(h_1(K_2)-h_\infty(K_2))\\ -(h_1(K_1)-h_0(K_1))(h_1(K_2)-h_0(K_2)).$$

Note that $\chi(K_1,K_2)$ is in fact the difference between the ranks of $\mathbb{B}_1=\mathbb{B}_1(K_1,K_2)$ and $\mathbb{B}_2=\mathbb{B}_2(K_1,K_2)$. In the corresponding $\mathbb{Z}/2\mathbb{Z}$–grading on $\mathbb{B}_1\oplus\mathbb{B}_2$, $\chi(K_1,K_2)$ is thus the Euler characteristic of the chain complex $(\mathbb{B}_1\oplus\mathbb{B}_2,d)$.

Corollary 5.8 *With the above notation fixed,*

$$\mathrm{rnk}\big(\widehat{\mathrm{HF}}(Y(K_1,K_2))\big)\geq|\chi(K_1,K_2)|.$$

Proof It is enough to note that

$$\chi(K_1, K_2) = \operatorname{rnk}\big(\operatorname{Ker}(\mathfrak{D}(K_1, K_2))\big) - \operatorname{rnk}\big(\operatorname{Coker}(\mathfrak{D}(K_1, K_2))\big). \qquad \square$$

Consider the matrices

$$P_L = \begin{pmatrix} I\otimes \bar{A}^2_1 & 0 & 0 & 0 & I\otimes \bar{B}^2_1 & 0 \\ 0 & I\otimes \bar{A}^2_\infty & I\otimes \bar{B}^2_\infty & 0 & 0 & 0 \\ 0 & I\otimes \bar{C}^2_\infty & I\otimes \bar{D}^2_\infty & 0 & 0 & 0 \\ 0 & 0 & 0 & I\otimes \bar{A}^2_0 & 0 & I\otimes \bar{B}^2_0 \\ I\otimes \bar{C}^2_1 & 0 & 0 & 0 & I\otimes \bar{D}^2_1 & 0 \\ 0 & 0 & 0 & I\otimes \bar{C}^2_0 & 0 & I\otimes \bar{D}^2_0 \end{pmatrix}$$

and

$$P_R = \begin{pmatrix} \bar{D}^1_1\otimes I & 0 & 0 & \bar{C}^1_1\otimes I & 0 & 0 \\ 0 & \bar{A}^1_0\otimes I & \bar{B}^1_0\otimes I & 0 & 0 & 0 \\ 0 & \bar{C}^1_0\otimes I & \bar{D}^1_0\otimes I & 0 & 0 & 0 \\ \bar{B}^1_1\otimes I & 0 & 0 & \bar{A}^1_1\otimes I & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{D}^1_\infty\otimes I & \bar{C}^1_\infty\otimes I \\ 0 & 0 & 0 & 0 & \bar{B}^1_\infty\otimes I & \bar{A}^1_\infty\otimes I \end{pmatrix}.$$

Both P_R and P_L are invertible and $\mathfrak{D}(K_1, K_2)$ is equivalent to $\mathfrak{D}'(K_1, K_2) = P_L\mathfrak{D}(K_1, K_2)P_R$. The matrix $\mathfrak{D}'(K_1, K_2)$ has the block presentation

$$\begin{pmatrix} D^1_\infty \bar{B}^1_1\otimes \bar{B}^2_1 A^2_0 & B^1_1 \bar{A}^1_0\otimes I & B^1_1 \bar{B}^1_0\otimes I & D^1_\infty \bar{A}^1_1\otimes \bar{B}^2_1 A^2_0 & I\otimes \bar{B}^2_1 B^2_0 & 0 \\ I\otimes \bar{B}^2_\infty B^2_1 & D^1_1 \bar{A}^1_0\otimes \bar{B}^2_\infty A^2_1 & D^1_1 \bar{B}^1_0\otimes \bar{B}^2_\infty A^2_1 & 0 & B^1_0 \bar{B}^1_\infty\otimes I & B^1_0 \bar{A}^1_\infty\otimes I \\ I\otimes \bar{D}^2_\infty B^2_1 & \Psi_1 & D^1_1 \bar{B}^1_0\otimes \bar{D}^2_\infty A^2_1 & 0 & 0 & 0 \\ B^1_\infty \bar{B}^1_1\otimes I & 0 & I\otimes \bar{B}^2_0 B^2_\infty & B^1_\infty \bar{A}^1_1\otimes I & \Gamma_1 & \Gamma_2 \\ D^1_\infty \bar{B}^1_1\otimes \bar{D}^2_1 A^2_0 & 0 & 0 & \Psi_2 & I\otimes \bar{D}^2_1 B^2_0 & 0 \\ 0 & 0 & I\otimes \bar{D}^2_0 B^2_\infty & 0 & \Gamma_3 & \Gamma_4 \end{pmatrix}$$

with

$$\begin{aligned} \Psi_1 &= I\otimes I + D^1_1 \bar{A}^1_0 \otimes \bar{D}^2_\infty A^2_1, \\ \Psi_2 &= I\otimes I + D^1_\infty \bar{A}^1_1 \otimes \bar{D}^2_1 A^2_0, \\ \Gamma_1 &= D^1_0 \bar{B}^1_\infty \otimes \bar{B}^2_0 A^2_\infty + X^1 \bar{B}^1_\infty \otimes \bar{B}^2_0 X^2, \\ \Gamma_2 &= D^1_0 \bar{A}^1_\infty \otimes \bar{B}^2_0 A^2_\infty + X^1 \bar{A}^1_\infty \otimes \bar{B}^2_0 X^2, \end{aligned}$$

$$\Gamma_3 = D_0^1 \bar{B}_\infty^1 \otimes \bar{D}_0^2 A_\infty^2 + X^1 \bar{B}_\infty^1 \otimes \bar{D}_0^2 X^2,$$
$$\Gamma_4 = I \otimes I + D_0^1 \bar{A}_\infty^1 \otimes \bar{D}_0^2 A_\infty^2 + X^1 \bar{A}_\infty^1 \otimes \bar{D}_0^2 X^2,$$

and is easier to use in actual computations. Note that

$$\iota(\mathfrak{D}'(K_1, K_2)) \simeq \iota(\mathfrak{D}(K_1, K_2)) \simeq \widehat{\mathrm{HF}}(Y(K_1, K_2), \mathbb{F}).$$

6.2′ Splicing a knot complement with the complement of a trefoil

For a knot $K \subset Y$, let $Y(R, K)$ denote the three-manifold obtained by splicing the complement of $K \subset Y$ with the complements of the right-handed trefoil. We study the rank $r_r(K)$ of $\widehat{\mathrm{HF}}(Y(R, K))$ in this subsection. With the notation of Section 5.3′, $r_r(K) = i(\mathfrak{D}'(R, K))$. Replacing the block forms of [1, (21)] in $\mathfrak{D}'(R, K)$, we find

$$\mathfrak{D}'(R, K) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & I & 0 & \bar{B}_1 B_0 & 0 & 0 \\ \bar{B}_\infty B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & \bar{B}_\infty B_1 & 0 & 0 & \bar{B}_\infty A_1 & 0 & 0 & I & 0 & 0 \\ \bar{D}_\infty B_1 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{D}_\infty B_1 & 0 & I & \bar{D}_\infty A_1 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & \bar{B}_0 B_\infty & 0 & 0 & 0 & 0 & \bar{B}_0 X \\ 0 & 0 & 0 & 0 & 0 & \bar{B}_0 B_\infty & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & \bar{D}_1 B_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{D}_0 B_\infty & 0 & 0 & 0 & I & \bar{D}_0 X \\ 0 & 0 & 0 & 0 & 0 & \bar{D}_0 B_\infty & 0 & 0 & 0 & I \end{pmatrix},$$

where $A_\bullet = A_\bullet(K)$, $B_\bullet = B_\bullet(K)$, $C_\bullet = C_\bullet(K)$, $D_\bullet = D_\bullet(K)$, $\bar{A}_\bullet = \bar{A}_\bullet(K)$, $\bar{B}_\bullet = \bar{B}_\bullet(K)$, $\bar{C}_\bullet = \bar{C}_\bullet(K)$, $\bar{D}_\bullet = \bar{D}_\bullet(K)$ and $X = X(K)$ for $\bullet \in \{0, 1, \infty\}$. Doing a series of cancellations that correspond to the identity matrices which appear as the

$$(1,6), \quad (3,8), \quad (4,3), \quad (5,4), \quad (6,1), \quad (8,7), \quad (9,9) \quad \text{and} \quad (10,10)$$

entries in the above block presentation, we obtain the equivalent matrix

$$R_r(K) := \begin{pmatrix} 0 & \bar{B}_0 X \bar{B}_\infty \\ \bar{X} B_\infty \bar{B}_1 & \bar{X} B_\infty \bar{A}_1 + \bar{D}_0 X \bar{B}_\infty \end{pmatrix}, \tag{22}$$

where $\bar{X} = \bar{X}(R) = \bar{B}_\infty(R) B_1(R) \bar{B}_0(R)$.

Corollary 6.1 *For a knot $K \subset Y$, let $Y(R, K)$ denote the three-manifold obtained by splicing the complement of K and the complement of the trefoil. Then*

$$\widehat{\mathrm{HF}}(Y(R, K)) = \iota(R_r(K)). \tag{23}$$

Proof The claim follows immediately from the above discussion. □

For the trefoils, our computations imply that

$$\begin{aligned}\overline{X}(R)B_\infty(R) = X(R)\overline{B}_\infty(R) = \overline{X}(L)B_\infty(L) = X(L)\overline{B}_\infty(L) = 0 \\ \Longrightarrow \quad R_r(R) = R_r(L) = 0 \\ \Longrightarrow \quad |\widehat{\mathrm{HF}}(Y(R,R))| = 7 \quad \text{and} \quad |\widehat{\mathrm{HF}}(Y(R,L))| = 9.\end{aligned}$$

The above computations agree with the computations of Hedden and Levine [3].

Corollary 6.2 *For every knot K in a homology sphere Y, we have*

$$\begin{aligned}|\widehat{\mathrm{HF}}(Y(R,K))| &\geq (a_0(K)+a_1(K)+2a_\infty(K)) - 4\min\{a_0(K),a_1(K),a_\infty(K)\} \\ &= 4\max\{h_0(K),h_1(K),h_\infty(K)\} - (h_0(K)+h_1(K)+2h_\infty(K)).\end{aligned}$$

Moreover, if $Y(R,K)$ is a homology sphere L–space, then K is trivial and Y is a homology sphere L–space.

Proof Let $M = M(K) = X(K)\overline{B}_\infty(K)$ and $\overline{M} = \overline{M}(K) = \overline{X}(K)B_\infty(K)$ and note that

$$\begin{aligned}\mathrm{rnk}(R_r(K)) &= \mathrm{rnk}\begin{pmatrix} 0 & \overline{B}_0(K)M \\ \overline{M}B_1(K) & \overline{M}A_1(K)+\overline{D}_0(K)M \end{pmatrix} \\ &\leq \mathrm{rnk}\begin{pmatrix} \overline{M}B_1(K) & \overline{M}A_1(K) \end{pmatrix} + \mathrm{rnk}\begin{pmatrix} \overline{B}_0(K)M \\ \overline{D}_0(K)M \end{pmatrix} \\ &= \mathrm{rnk}(M) + \mathrm{rnk}(\overline{M}).\end{aligned}$$

For every knot $K \subset Y$ as above, note that the ranks of $M = M(K)$ and $\overline{M} = \overline{M}(K)$ are at most equal to the minimum of the sizes of the matrices $B_0(K)$, $B_1(K)$ and $B_\infty(K)$, which is

$$\min\{a_0(K),a_1(K),a_\infty(K)\}.$$

Since $R_r(K)$ is of size $h_0(K)\times h_1(K) = (a_1(K)+a_\infty(K))\times(a_0(K)+a_\infty(K))$ this proves the first part of the corollary.

Let us assume that $\mathrm{rnk}\big(\widehat{\mathrm{HF}}(Y(R,K))\big) = 1$. From here we find

$$\begin{aligned}&(a_0(K)+a_1(K)+2a_\infty(K)) - 4\min\{a_0(K),a_1(K),a_\infty(K)\} \\ &\qquad = (a_0(K)+a_1(K)+2a_\infty(K)) - 2\,\mathrm{rnk}(M) - 2\,\mathrm{rnk}(\overline{M}) = 1.\end{aligned}$$

Since $a_1(K)$ and $a_\infty(K)$ have the same parity while the parity of $a_0(K)$ is different from the parity of both $a_1(K)$ and $a_\infty(K)$, one can easily conclude that $a_0(K)-1 = a_1(K) = a_\infty(K)$. Let a denote the common value $a_1(K) = a_\infty(K)$. Then the rank of M and $\overline{M}$ is a, and $B_0(K)$, $X(K)$ and $\overline{X}(K)$ are all invertible. We may thus

assume that $A_0(K) = D_0(K) = 0$. Since

$$\operatorname{rnk}(\mathfrak{f}_\infty(K) + \overline{\mathfrak{f}}_\infty(K)) = \operatorname{rnk}\begin{pmatrix} B_1(K)A_0(K) & B_1(K)B_0(K) \\ I + D_1(K)A_0(K) & D_1(K)B_0(K) \end{pmatrix} = 2a,$$

the three-manifold Y is an L–space. Since splicing K with the trefoil is also a homology sphere L–space, we conclude that K is trivial, by [3, Theorem 1]. □

Appendix′ Bordered Floer homology for knot complements

The first draft of this paper appeared while the theory of bordered Floer homology was being developed. With bordered Floer homology conventions widely known to the Heegaard Floer community, the referee recommended the inclusion of an appendix which addresses the contribution of this paper within the realm of bordered Floer homology.

Let $K \subset Y$ denote a null-homologous knot inside the three-manifold Y, and let $H = (\Sigma, \boldsymbol{\alpha}, \widehat{\boldsymbol{\beta}} \cup \{\lambda, \mu\}; z)$ denote a special Heegaard diagram for K, as constructed in [1, Lemma 4.1]. In particular, H is a nice Heegaard diagram for the bordered three-manifold Y_K determined by $K \subset Y$ in the sense of [6]. The bordered Floer complex $\widehat{\mathrm{CFD}}(Y_K)$ may then be constructed from the chain complexes $M = M(K)$ and $L = L(K)$ (which are described in [1, Proposition 5.1] as the mapping cones of $\overline{\mathfrak{f}}_\infty(K)\colon C_0(K) \to C_1(K)$ and $\mathfrak{f}_0(K)\colon C_1(K) \to C_\infty(K)$, respectively) and the chain maps $\Phi = \Phi(K)\colon L \to M$ and $\Psi_i = \Psi_i(K)\colon M \to L$ for $i = 1, 2, 3$.

More precisely, and following the notation of [5, Section 4.2], the idempotents ι_0 and ι_1, and the chords ρ_1, ρ_2, ρ_3, $\rho_{12} = \rho_1\rho_2$, $\rho_{23} = \rho_2\rho_3$ and $\rho_{123} = \rho_1\rho_2\rho_3$ form an $\mathbb{F}$–basis for the differential graded algebra associated with the torus boundary,

$$\mathcal{A}(T^2, 0) = \left\langle \iota_0 \bullet \underset{\rho_3}{\overset{\rho_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\rho_2}}}} \bullet \iota_1 \right\rangle / (\rho_2\rho_1 = \rho_3\rho_2 = 0).$$

The module $\widehat{\mathrm{CFD}}(Y_K)$ is generated (over $\mathcal{A}(T^2, 0)$) by the generators of M and L. For a generator $\boldsymbol{x}$ of L we have

$$I(\boldsymbol{x}) = \iota_0 \quad \text{and} \quad \partial(\boldsymbol{x}) = d_L(\boldsymbol{x}) + \rho_1\Psi_1(\boldsymbol{x}) + \rho_3\Psi_2(\boldsymbol{x}) + \rho_{123}\Psi_3(\boldsymbol{x}), \tag{24}$$

while for a generator $\boldsymbol{y}$ of M we have

$$I(\boldsymbol{y}) = \iota_1 \quad \text{and} \quad \partial(\boldsymbol{y}) = d_M(\boldsymbol{y}) + \rho_2\Phi(\boldsymbol{y}). \tag{25}$$

The splicing formula of [1, (17)] is then just the gluing formula for bordered Floer homology, ie [6, Theorem 1.3]. A related discussion is carried over in [6, Section 8].

Definition A.1 The chain complexes $(C_\bullet(K), d_\bullet)$, $\bullet \in \{0, 1, \infty\}$ and the chain maps $f_\bullet(K)$ and $\bar{f}_\bullet(K)$ for $\bullet \in \{0, \infty\}$ are called *admissible data* associated with the knot K if they satisfy the following conditions:

- The homology of the complex $(C_\bullet(K), d_\bullet)$ is $\mathbb{H}_\bullet(K)$.
- The maps induced by $f_\bullet(K)$ and $\bar{f}_\bullet(K)$ in homology (under the identification of the homology of $(C_\bullet(K), d_\bullet)$ with $\mathbb{H}_\bullet(K)$) are $\mathfrak{f}_\bullet(K)$ and $\bar{\mathfrak{f}}_\bullet(K)$, respectively.
- We have $f_0(K) \circ f_\infty(K) = \bar{f}_0(K) \circ \bar{f}_\infty(K) = 0$.
- The corresponding maps

  $$\theta(K)\colon \operatorname{Ker}(\mathfrak{f}_\infty(K)) \to \operatorname{Coker}(\mathfrak{f}_0(K)), \quad \bar{\theta}(K)\colon \operatorname{Ker}(\bar{\mathfrak{f}}_\infty(K)) \to \operatorname{Coker}(\bar{\mathfrak{f}}_0(K))$$

  are isomorphisms, and are the inverses of the maps induced by $\mathfrak{f}_1(K)$ and $\bar{\mathfrak{f}}_1(K)$, respectively.

The proof of [1, Theorem 1.1] implies that $(C^i_\bullet, d^i_\bullet)$ and the chain maps $f^i_\bullet$ and $\bar{f}^i_\bullet$ for $\bullet \in \{0, \infty\}$ and $i = 1, 2$ in [1, (17)] may be replaced by other admissible data corresponding to the knots K_1 and K_2. Correspondingly, the bordered Floer complex associated with any knot $K \subset Y$ may be constructed from admissible data associated with K. More precisely, we have the following proposition:

Proposition A.2 *Suppose that the chain complexes $(C_\bullet(K), d_\bullet)$ for $\bullet \in \{0, 1, \infty\}$ and the chain maps $f_\bullet = f_\bullet(K)$ and $\bar{f}_\bullet = \bar{f}_\bullet(K)$ for $\bullet \in \{0, \infty\}$ are admissible data associated with the knot $K \subset Y$ and set*

$$M(K) = C_0(K) \oplus C_1(K), \quad L(K) = C_1(K) \oplus C_\infty(K).$$

The bordered Floer complex $\widehat{\mathrm{CFD}}(Y_K)$ may then be constructed as the left module over the differential graded algebra $\mathcal{A}(T^2, 0)$ which is generated by $\iota_0.L(K)$ and $\iota_1.M(K)$, and equipped with the differential $\partial\colon \widehat{\mathrm{CFD}}(Y_K) \to \widehat{\mathrm{CFD}}(Y_K)$ defined by

$$\partial\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} d_0(\boldsymbol{x}) \\ \bar{f}_\infty(\boldsymbol{x}) + d_1(\boldsymbol{y}) \end{pmatrix} + \begin{pmatrix} \rho_1 f_\infty(\boldsymbol{x}) \\ \rho_3 \bar{f}_0(\boldsymbol{y}) + \rho_1\rho_2\rho_3 \bar{f}_0(f_\infty(\boldsymbol{x})) \end{pmatrix} & \text{if } \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in M(K), \\ \begin{pmatrix} d_1(\boldsymbol{x}) \\ f_0(\boldsymbol{x}) + d_\infty(\boldsymbol{y}) \end{pmatrix} + \rho_2 . \begin{pmatrix} 0 \\ \boldsymbol{x} \end{pmatrix} & \text{if } \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in L(K). \end{cases} \tag{26}$$

In particular, let the $\mathbb{F}$–modules $\mathbb{A}_\bullet = \mathbb{A}_\bullet(K)$ for $\bullet \in \{0, 1, \infty\}$ and the matrices $A_\bullet = A_\bullet(K)$, $B_\bullet = B_\bullet(K)$, $C_\bullet = C_\bullet(K)$ and $D_\bullet = D_\bullet(K)$ be defined as in [1, Section 3.3]. Set

$$(C_0(K), d_0) = (\mathbb{A}_\infty \oplus \mathbb{A}_1, 0), \quad (C_\infty(K), d_\infty) = (\mathbb{A}_1 \oplus \mathbb{A}_0, 0),$$

$$C_1(K) = \mathbb{A}_1 \oplus \mathbb{A}_0 \oplus \mathbb{A}_\infty \oplus \mathbb{A}_1 \quad \text{and} \quad d_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_{\mathbb{A}_1} & 0 & 0 & 0 \end{pmatrix}.$$

Correspondingly, define

$$f_\infty(K) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix}, \qquad f_0(K) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

$$\tau_1(K) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1 & B_1 & 0 \\ 0 & C_1 & D_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{\tau}_1(K) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{A}_1 & \overline{B}_1 & 0 \\ 0 & \overline{C}_1 & \overline{D}_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and set $\overline{f}_\infty(K) = \tau_1(K) f_\infty(K) \tau_0^{-1}(K)$ and $\overline{f}_0(K) = \tau_\infty(K) f_0(K) \overline{\tau}_1(K)$. The data associated with K consisting of $(C_\bullet(K), d_\bullet)$ and $f_\bullet(K)$ and $\overline{f}_\bullet(K)$ for $\bullet \in \{0, \infty\}$ is then admissible.

Corresponding to the above admissible data and associated with $K \subset Y$ we may construct the bordered Floer complex for K via

$$M(K) = C_0(K) \oplus C_1(K) = \mathbb{A}_\infty \oplus \mathbb{A}_1 \oplus \mathbb{A}_1 \oplus \mathbb{A}_0 \oplus \mathbb{A}_\infty \oplus \mathbb{A}_1,$$
$$L(K) = C_1(K) \oplus C_\infty(K) = \mathbb{A}_1 \oplus \mathbb{A}_0 \oplus \mathbb{A}_\infty \oplus \mathbb{A}_1 \oplus \mathbb{A}_1 \oplus \mathbb{A}_0,$$

$$d_M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_1\overline{A}_0 & B_1\overline{B}_0 & 0 & 0 & 0 & 0 \\ D_1\overline{A}_0 & D_1\overline{B}_0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad d_L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{pmatrix},$$

$$\Phi(K) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{pmatrix}, \quad \Psi_1(K) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Psi_2(K) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_\infty\overline{A}_1 & B_\infty\overline{B}_1 & 0 \\ 0 & 0 & 0 & D_\infty\overline{A}_1 & D_\infty\overline{B}_1 & 0 \end{pmatrix} \quad \text{and} \quad \Psi_3(K) = \Psi_2(K)\Phi(K)\Psi_1(K).$$

as the left module over the differential graded algebra $\mathcal{A}(T^2, 0)$ generated by $\iota_0.L$ and $\iota_1.M$ and equipped with the differential $\partial\colon \widehat{\mathrm{CFD}}(Y_K) \to \widehat{\mathrm{CFD}}(Y_K)$ defined by the equations (24) and (25).

Remark A.3 Simultaneous computation of the matrices $\tau_\bullet(K) = \left(\begin{smallmatrix} A_\bullet & B_\bullet \\ C_\bullet & D_\bullet \end{smallmatrix}\right)$ is a priori quite difficult, as we observed in the case of trefoils in [1, Section 6]. This makes the above description of the bordered Floer homology hard to use even for knots $K \subset Y$ where we have complete understanding of the Heegaard Floer complex associated with K. However, it is possible to construct admissible data associated with $K \subset Y$ completely in terms of the filtered chain complex $\mathrm{CF}^\infty(Y, K; \mathbb{F})$, as will be discussed in the revision of [2].

References

[1] **E Eftekhary**, *Floer homology and splicing knot complements*, Algebr. Geom. Topol. 15 (2015) 3155–3213 MR

[2] **E Eftekhary**, *Bordered Floer homology and existence of incompressible tori in homology spheres*, Compos. Math. 154 (2018) 1222–1268 MR

[3] **M Hedden**, **A S Levine**, *Splicing knot complements and bordered Floer homology*, J. Reine Angew. Math. 720 (2016) 129–154 MR

[4] **K Hendricks**, **C Manolescu**, *Involutive Heegaard Floer homology*, Duke Math. J. 166 (2017) 1211–1299 MR

[5] **R Lipshitz**, **P S Ozsváth**, **D P Thurston**, *Notes on bordered Floer homology*, from "Contact and symplectic topology" (F Bourgeois, V Colin, A Stipsicz, editors), Bolyai Soc. Math. Stud. 26, János Bolyai Math. Soc., Budapest (2014) 275–355 MR

[6] **R Lipshitz**, **P S Ozsváth**, **D P Thurston**, *Bordered Heegaard Floer homology*, Mem. Amer. Math. Soc. 1216, Amer. Math. Soc., Providence, RI (2018) MR

[7] **S Sarkar**, *Moving basepoints and the induced automorphisms of link Floer homology*, Algebr. Geom. Topol. 15 (2015) 2479–2515 MR

School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
Tehran, Iran

`eaman@ipm.ir`

`http://math.ipm.ir/~eftekhary`

Received: 28 October 2017 Revised: 21 January 2020