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Fundamental groups of formal Legendrian and horizontal embedding spaces

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We compute the fundamental group of each connected component of the space of formal Legendrian embeddings in $\mathbb{R}^3$. We use it to show that previous examples in the literature of nontrivial loops of Legendrian embeddings are already nontrivial at the formal level. Likewise, we compute the fundamental group of the different connected components of the space of formal horizontal embeddings into the standard Engel $\mathbb{R}^4$. We check that the natural inclusion of the space of horizontal embeddings into the space of formal horizontal embeddings induces an isomorphism at π_1–level.

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1 Introduction

The computation of the homotopy type of the space of Legendrian embeddings into a contact 3–fold has a long history. For a while, it was thought that the computation could be made at the formal level. By that we mean that the inclusion of the space of Legendrian embeddings into the space of formal Legendrian embeddings, ie the space of pairs of a smooth embedding and a formal Legendrian derivative, was a weak homotopy equivalence. This was proven to be wrong in the key article of D. Bennequin [4], in which it was shown that the formal space associated to the standard contact $\mathbb{R}^3$ possesses some connected components that are not representable by Legendrian knots. In other words, the restriction of the induced map of the inclusion at π_0–level was not surjective. This was the first hint of rigidity phenomena in contact topology. Later on, there developed an industry in checking how far the inclusion map is at π_0–level from being injective or surjective; see for example the work of Chekanov [7], Ding and Geiges [8], Eliashberg and Fraser [10], Etnyre and Honda [13] or Osváth, Szabó and Thurston [25].

The next step was the study of higher homotopy groups. This was developed by Kálmán [22] in dimension 3 using pseudoholomorphic curve invariants and by Sabloff and Sullivan [27] in dimensions $2n+1$, $n>1$, using generating function invariants. However, they just checked that several nontrivial loops in the space of Legendrian embeddings were trivial as elements in the fundamental group of the space of smooth embeddings. We show that all Kálmán's examples are nontrivial in the space of formal

Legendrian embeddings; see Section 4. This makes the use of sophisticated invariants to compute these examples unnecessary. In order to do that, we compute the fundamental group of the space of formal Legendrian embeddings. This is the content of Section 3.

Moreover, the analogous problem for horizontal embeddings into Engel manifolds satisfies an h–principle at π_0–level; see eg Adachi [1], Geiges [15] or del Pino and Presas [26]. In Section 5, we compute the fundamental group of the space of formal horizontal embeddings into the standard Engel $\mathbb{R}^4$. We check that it is $\mathbb{Z} \oplus \mathbb{Z}_2$, where the first component is a rotation invariant that captures the formal immersion class of the loop and the second component captures the formal embedding class of the loop. In Section 6, we provide a way of combinatorially computing this $\mathbb{Z}_2$–invariant.

Finally, in Section 7, we check that the formal fundamental group is isomorphic to the fundamental group, ie the $\mathbb{Z}_2$–invariant and the rotation invariant completely classify the elements of the fundamental group of horizontal embeddings. Independently, Casals and del Pino have shown that there is a full h–principle for horizontal embeddings [6]. Although our proof is more specific, it has the advantage of not using the topology of the space of smooth embeddings. Therefore, we reprove as a corollary that the space of smooth embeddings of circles into $\mathbb{R}^4$ is simply connected; see Budney [5].

The methods developed in this article may allow computation of higher rank homotopy groups of the space of embeddings of the circle in $\mathbb{R}^4$. The strategy that we are pursuing in a forthcoming project is based on the following facts. First, we compute the higher homotopy groups of the space of horizontal embeddings by the geometrical method developed in this article, heavily simplified thanks to the work of Igusa [21]. Secondly, we use [6] to state that the previous computations also compute homotopy groups of the formal horizontal embedding space. Finally, by using obstruction theory (see Hatcher [19, Chapter 4.3]) we isolate the homotopy groups of the space of smooth embeddings.

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2 Spaces of embeddings of the circle into euclidean space

Denote by $\mathfrak{Emb}(N, M)$ the space of embeddings of a manifold N into a manifold M equipped with the C^r–topology, $r \geq 5$.

2.1 The space $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$

Theorem 2.1.1 (Hatcher [17, Appendix, equivalence (15)]) *The space of parametrized unknotted circles in $\mathbb{R}^3$ has the homotopy type of $\mathrm{SO}(3)$.*

The group $\mathrm{SO}(4)$ acts freely on the connected component

$$\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3) \subseteq \mathfrak{Emb}(\mathbb{S}^1, \mathbb{S}^3)$$

of the parametrized (p, q) *torus knots* as

$$\mathrm{SO}(4) \times \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3) \to \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3), \quad (A, \gamma) \mapsto A \cdot \gamma.$$

Thus, we have an inclusion $\mathrm{SO}(4) \hookrightarrow \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)$. The following result holds:

Theorem 2.1.2 (Hatcher [18, Theorem 1]) *The inclusion $\mathrm{SO}(4) \hookrightarrow \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)$ is a homotopy equivalence.*

As a consequence of these results we obtain:

Corollary 2.1.3 *Let $\mathfrak{Emb}_0(\mathbb{S}^1, \mathbb{R}^3), \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3) \subseteq \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ respectively be the connected component of the parametrized **unknots** and of the parametrized (p, q) **torus knots**. The fundamental groups of these spaces are given by*

- $\pi_1(\mathfrak{Emb}_0(\mathbb{S}^1, \mathbb{R}^3)) \cong \mathbb{Z}_2$,
- $\pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3)) \cong G_{p,q} \rtimes \mathbb{Z}_2$,

where $G_{p,q}$ is the knot group of the (p, q) torus knot.

Proof Theorem 2.1.1 proves the case of the connected component $\mathfrak{Emb}_0(\mathbb{S}^1, \mathbb{R}^3)$.

We need to study the connected component $\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3)$ to conclude the proof. Consider the space

$$\mathcal{S}tereo_{p,q} = \{(\gamma, x) : x \notin \mathrm{Image}(\gamma)\} \subseteq \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3) \times \mathbb{S}^3.$$

We have two natural fibrations associated to the projection maps

$$\begin{tikzcd} \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3) \arrow[r, hook] & \mathcal{S}tereo_{p,q} \arrow[d] \\ & \mathbb{S}^3 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^3 \setminus K_{p,q} \arrow[r, hook] & \mathcal{S}tereo_{p,q} \arrow[d] \\ & \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3) \end{tikzcd}$$

where $K_{p,q}$ is the image of the standard (p,q) torus knot in $\mathbb{S}^3$. From the first fibration we obtain

$$\pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3)) \cong \pi_1(\mathcal{S}tereo_{p,q}).$$

Moreover, from the second one and the fact that $\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)$ has the homotopy type of $\mathrm{SO}(4)$ (Theorem 2.1.2), we obtain that the sequence

$$0 \to G_{p,q} \to \pi_1(\mathcal{S}tereo_{p,q}) \to \mathbb{Z}_2 \to 0$$

is exact. Now it is a simple exercise to check that this sequence is right split. □

2.2 The space $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$

A *long embedding* of $\mathbb{R}$ into $\mathbb{R}^4$ is an embedding $\gamma\colon \mathbb{R} \to \mathbb{R}^4$ that coincides with the standard inclusion $\mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$ outside a compact neighborhood of the origin. Let $\mathfrak{LEmb}(\mathbb{R}, \mathbb{R}^4)$ denote the space of long embeddings of $\mathbb{R}$ into $\mathbb{R}^4$.

Lemma 2.2.1 (Budney) *$\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ is homotopy equivalent to*

$$\mathbb{S}^3 \times \mathbb{S}^2 \times \mathfrak{LEmb}(\mathbb{R}, \mathbb{R}^4).$$

Proof It follows from [5, Proposition 2.2] that $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ is homotopy equivalent to $(\mathrm{SO}(4) \times \widetilde{\mathcal{S}tereo})/\,\mathrm{SO}(3)$, where

$$\widetilde{\mathcal{S}tereo} = \{(p, f) : p \notin \mathrm{Image}(f)\} \subseteq \mathbb{R}^4 \times \mathfrak{LEmb}(\mathbb{R}, \mathbb{R}^4).$$

Observe that the quaternion structure in $\mathbb{R}^4(i,j,k)$ induces a homotopy equivalence

$$\mathbb{S}^3 \times \widetilde{\mathcal{S}tereo} \to (\mathrm{SO}(4) \times \widetilde{\mathcal{S}tereo})/\,\mathrm{SO}(3), \quad (v, (p, f)) \mapsto [((v|iv|jv|kv), (p, f))].$$

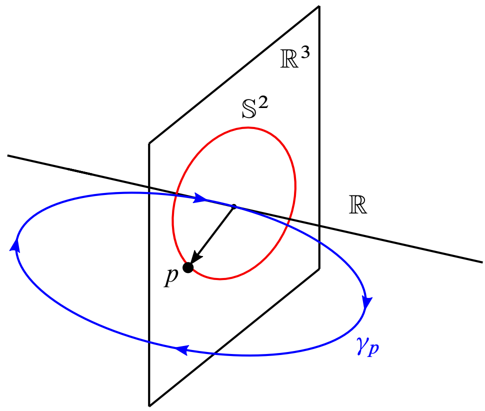

Figure 1: Construction of S_U (schematically).

Finally, the natural fibration $\widetilde{Stereo} \mapsto \mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4)$ has fiber homotopy equivalent to $\mathbb{S}^2$, since the space $\mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4)$ is connected and $\mathbb{R}^4 \setminus \mathbb{R} \overset{\text{h.e.}}{\sim} \mathbb{S}^2$. Moreover, the fibration homotopically splits since any family $\varphi\colon \mathbb{S}^k \to \mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4)$ admits a lift given by

$$\varphi_\varepsilon\colon \mathbb{S}^k \to \widetilde{Stereo}, \quad z \mapsto (\varphi(z)(0) + \varepsilon \cdot i\varphi(z)'(0), \varphi(z))$$

for some $\varepsilon > 0$ small enough. □

Let us geometrically explain the homotopy equivalence stated in the last lemma. Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}^4$ and write $\mathbb{R}^4 = \mathbb{R}\langle e_1\rangle \times \mathbb{R}^3\langle e_2, e_3, e_4\rangle$. From a long embedding we obtain an embedding of $\mathbb{S}^1$ into $\mathbb{R}^4$, closing it in the plane $\langle e_1, p\rangle$, where $p \in \mathbb{S}^2 \subseteq \mathbb{R}^3\langle e_2, e_3, e_4\rangle$. The $\mathbb{S}^3$ factor acts by quaternionic multiplication on a fixed embedding.

It follows that

$$\pi_2(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^4)) \cong \pi_2(\mathbb{S}^2) \oplus \pi_2(\mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4)) \cong \mathbb{Z} \oplus \pi_2(\mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4)).$$

We provide an explicit construction of the generator of the $\mathbb{Z}$ factor in this decomposition. For every point $p \in \mathbb{S}^2 \subseteq \mathbb{R}^3\langle e_2, e_3, e_4\rangle$, take the standard parametrized *unknot* in $\operatorname{span}\{e_1, p\}$ centered at p and tangent to the line generated by e_1 at 0. This gives us a 2–parameter family of knots, which we denote by S_U, whose homotopy class is the generator $(1,0)$ of $\pi_2(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^4))$. Explicitly, S_U is defined as

$$S_U\colon \mathbb{S}^2 \to \mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^4), \quad p \mapsto \gamma_p(t),$$

where, if $p = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{S}^2$,

$$\gamma_p(t) = \sin(2\pi t)e_1 + (1 - \cos(2\pi t))(\lambda_1 e_2 + \lambda_2 e_3 + \lambda_3 e_4).$$

Remark 2.2.2 In [5, Proposition 3.9(4)], Budney shows that the space $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ is simply connected and $\pi_2(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Moreover, he provides an explicit construction of the second generator of the second homotopy group [5, Theorem 3.13].

In Section 7 we provide an alternative proof of the fact that $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ is simply connected based on the techniques developed in this paper.

3 Formal Legendrian embeddings in $\mathbb{R}^3$

Let ξ denote the standard contact structure in $\mathbb{R}^3(x, y, z)$ given by $\xi = \ker(dz - y\,dx)$. Throughout the section we fix the Legendrian framing ∂_y.[1]

3.1 Formal Legendrian embeddings in $\mathbb{R}^3$

Definition 3.1.1 An immersion $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^3$ is said to be *Legendrian* if $\gamma'(t) \in \xi_{\gamma(t)}$ for all $t \in \mathbb{S}^1$. If γ is an embedding, we say it is a *Legendrian embedding*.

Definition 3.1.2 (a) A *formal Legendrian immersion* in $\mathbb{R}^3$ is a pair (γ, F) such that:

- (i) $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^3$ is a smooth map.
- (ii) $F\colon \mathbb{S}^1 \to \gamma^*(T\mathbb{R}^3 \setminus \{0\})$ satisfies $F(t) \in \xi_{\gamma(t)}$.

(b) A *formal Legendrian embedding* in $\mathbb{R}^3$ is a pair (γ, F_s) satisfying:

- (i) $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^3$ is an embedding.
- (ii) $F_s\colon \mathbb{S}^1 \to \gamma^*(T\mathbb{R}^3 \setminus \{0\})$, $s \in [0, 1]$, is a 1–parameter family such that $F_0 = \gamma'$ and $F_1(t) \in \xi_{\gamma(t)}$.

Use the framing $\langle \partial_y \rangle$ to trivialize the contact distribution understood as a bundle. This provides a bundle isomorphism $\xi \simeq \mathbb{R}^2$. From now on, we will understand the map $F\colon \mathbb{S}^1 \to \mathbb{S}^1 \equiv \mathbb{S}^2 \cap \mathbb{R}^2$ and the family $F_s\colon \mathbb{S}^1 \to \mathbb{S}^2$ with $F_1\colon \mathbb{S}^1 \to \mathbb{S}^1 \equiv \mathbb{S}^2 \cap \mathbb{R}^2$. We say that an immersion is *strict* if it is a noninjective map.

[1] Observe that ξ is naturally (co)oriented by the contact form $dz - y\,dx$; thus, ∂_y determines a unique oriented framing up to homotopy.

Denote by $\mathfrak{LegImm}(\mathbb{R}^3)$ the space of Legendrian immersions in $\mathbb{R}^3$, by $\mathfrak{Leg}(\mathbb{R}^3)$ the space of Legendrian embeddings in $\mathbb{R}^3$ and by

$$\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3) = \mathfrak{LegImm}(\mathbb{R}^3) \setminus \mathfrak{Leg}(\mathbb{R}^3)$$

the space of strict Legendrian immersions. Denote also by $\mathfrak{FLegImm}(\mathbb{R}^3)$ the space of formal Legendrian immersions and by $\mathfrak{FLeg}(\mathbb{R}^3)$ the space of formal Legendrian embeddings. These definitions make sense for immersions and embeddings of the interval. We define $\mathfrak{LegImm}([0,1],\mathbb{R}^3)$, $\mathfrak{Leg}([0,1],\mathbb{R}^3)$, $\mathfrak{LegImm}^{\textbf{strict}}([0,1],\mathbb{R}^3)$, $\mathfrak{FLegImm}([0,1],\mathbb{R}^3)$ and $\mathfrak{FLeg}([0,1],\mathbb{R}^3)$ analogously.

Remark 3.1.3 The space of C^r–maps $\mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1,\mathbb{R}^3)$ endowed with the C^r–topology is a Banach vector space. It has an open submanifold $\mathfrak{Imm}^r(\mathbb{S}^1,\mathbb{R}^3)$ that is the space of immersions. Denote by α the standard contact form in $\mathbb{R}^3$; ie $\ker(\alpha) = \xi$. There is a smooth submersion $\widetilde{\alpha}\colon \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1,\mathbb{R}^3) \to \Omega^1(\mathbb{S}^1)$ given by $\gamma \mapsto \gamma^*(\alpha)$. We have that $\mathfrak{LegImm}^r(\mathbb{R}^3)$ is a Banach submanifold of $\mathfrak{Imm}^r(\mathbb{S}^1,\mathbb{R}^3)$; see Lang [23, Corollary 5.7, page 19]. Whenever we speak about families of curves in the space of Legendrians we are implicitly assuming this Banach structure. This means that our maps are C^r instead of smooth but all the definitions in the paper make sense for C^r–maps whenever $r \geq 5$. To simplify the notation we will be writing smooth maps instead of C^5–maps, unless stated otherwise. The bound $r \geq 5$ comes from the fact that we are considering 3–dimensional families of Legendrians. In order to locally classify them we need to consider up to 4 derivatives of their front projection (5 of the original Legendrians).

It is important to note that any $\gamma \in \mathfrak{LegImm}^r(\mathbb{R}^3)$ admits a particular type of immersed chart φ_γ, called a *Weinstein chart*, which identifies via an immersed contactomorphism a tubular neighborhood of the zero section of the jet space $J^1(\mathbb{S}^1)$ (see Geiges [16, Example 2.5.11]) with a tubular neighborhood of $\gamma(\mathbb{S}^1)$. This construction provides the local structure produced by the implicit function theorem applied in the last paragraph. We can just check that the exponential map in a C^r–neighborhood of γ is defined as the following construction: take $f \in \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1,\mathbb{R})$ and a map $\widetilde{f} \in \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1,\mathbb{S}^1)$ such that $\widetilde{f}(\theta) = \theta + f(\theta)$. Then the map is defined as

$$\mathcal{C}\colon \mathcal{U} \subseteq \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1,\mathbb{R}) \times \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1,\mathbb{R}) \to \mathcal{V} \subseteq \mathfrak{LegImm}^r(\mathbb{R}^3), \qquad (f,g) \mapsto \varphi_\gamma^{-1} \circ j^1(g \circ \widetilde{f}). \tag{1}$$

Note that this is a local chart (ie a diffeomorphism). However, if we consider $r = \infty$, the previous map is not surjective in the reparametrizations direction since it is well known that the action of the Lie algebra of $\mathrm{Diff}(\mathbb{S}^1)$ is not surjective on $\mathrm{Diff}(\mathbb{S}^1)$. On the other hand, if we quotient out $\mathfrak{LegImm}^{\infty}(\mathbb{R}^3)$ by the action of $\mathrm{Diff}(\mathbb{S}^1)$, ie we consider nonparametrized Legendrians, the map is a local diffeomorphism. This is an obvious consequence of the characterization of the Legendrian curves in $J^1(\mathbb{S}^1)$ as graphs of smooth functions. We use $r < \infty$ in order to make sure that the parametrizations are surjective. We do not forget about reparametrizations because they are important in our algebraic topology arguments.

From now on we will skip the index $r = 5$ in all the statements, unless it is not clear from the context.

All the spaces of Legendrians are equipped with the C^r–topology. On the other hand, the spaces of formal Legendrians are equipped with the product topology that is the C^r–topology for the first factor (the smooth immersion/embedding) and the C^{r-1}–topology for the second factor (the formal derivative).

Remark 3.1.4 It is well known that the h–principle holds for Legendrian immersions; see for instance Eliashberg and Mishachev [11]. Hence, $\pi_0(\mathfrak{LegImm}(\mathbb{R}^3)) \cong \mathbb{Z}$, $\pi_1(\mathfrak{LegImm}(\mathbb{R}^3)) \cong \mathbb{Z}$ and $\pi_k(\mathfrak{LegImm}(\mathbb{R}^3)) = 0$ for all $k \geq 2$. The connected components of $\mathfrak{LegImm}(\mathbb{R}^3)$ are given by the *rotation number*. The rotation number of an Legendrian immersion γ is $\mathrm{Rot}(\gamma) = \deg(\gamma' \colon \mathbb{S}^1 \to \mathbb{S}^1)$. Let us explain the group $\pi_1(\mathfrak{LegImm}(\mathbb{R}^3)) \cong \mathbb{Z}$. Take a loop γ^θ in $\mathfrak{LegImm}(\mathbb{R}^3)$; the integer is just $\mathrm{Rot}_L(\gamma^\theta) = \deg(\theta \mapsto (\gamma^\theta)'(0))$ and we call this number *rotation number of the loop*. These invariants make sense in the formal case and the definitions are the obvious ones.

3.2 The space $\mathfrak{FLeg}(\mathbb{R}^3)$

Consider the space $\widehat{\mathfrak{FLeg}}(\mathbb{R}^3) = \{(\gamma, F) \mid \gamma \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3),\ F \in \mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^1)\}$. We have a natural fibration $\mathfrak{FLeg}(\mathbb{R}^3) \to \widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$. In order to compute the homotopy groups of $\mathfrak{FLeg}(\mathbb{R}^3)$, take $\gamma \in \mathfrak{Leg}(\mathbb{R}^3)$ and fix (γ, γ') as basepoint. The fiber over this point is $\mathcal{F} = \mathcal{F}_{(\gamma,\gamma')} = \Omega_{\gamma'}(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^2))$. We have the following exact sequence of homotopy groups associated to the fibration:

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_2(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \arrow[dll] \\
\pi_1(\mathcal{F}) \arrow[r] & \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \arrow[r] & \pi_1(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \arrow[dll] \\
\pi_0(\mathcal{F}) \arrow[r] & \pi_0(\mathfrak{FLeg}(\mathbb{R}^3)) \arrow[r] & \pi_0(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \arrow[r] & 0
\end{tikzcd}$$

Notice that $\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$ has the homotopy type of $\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)\times\mathbb{S}^1\times\mathbb{Z}$. Hence, $\pi_0(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3))\cong\pi_0(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3))\oplus\mathbb{Z}$, where the integer is the rotation number. Moreover, $\pi_1(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3))\cong\pi_1(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3))\oplus\mathbb{Z}$ and the $\mathbb{Z}$ factor is given by the rotation number of the loop. Finally, $\pi_k(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3))\cong\pi_k(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3))$ for all $k\geq 2$.

The homotopy groups of $\mathcal{F}$ are easily computable. Just observe that there is a fibration $\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2)\to\mathbb{S}^2$ defined via the *evaluation map*, with fiber over $p\in\mathbb{S}^2$ given by $\Omega_p(\mathbb{S}^2)$. As every element $[f]\in\pi_n(\mathbb{S}^2)$ can be lifted to an element $[f_n]\in\pi_n(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))$, defined as

$$f_n(p)(t)=f(p),\quad t\in\mathbb{S}^1,\ p\in\mathbb{S}^n,$$

all the diagonal maps in the associated exact sequence are zero. This implies that there are short exact sequences $\pi_n(\Omega_p(\mathbb{S}^2))\to\pi_n(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))\to\pi_n(\mathbb{S}^2)$ for $n\geq 1$. In particular, since $\mathbb{S}^2$ is simply connected, we obtain that

$$\pi_0(\mathcal{F})\cong\pi_1(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))\cong\pi_1(\Omega_p(\mathbb{S}^2))\cong\pi_2(\mathbb{S}^2)\cong\mathbb{Z}.$$

Moreover, these sequences are right split and, thus, split for $n>2$, since the groups involved are abelian. So we have

$$\pi_1(\mathcal{F})\cong\pi_2(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))\cong\pi_2(\mathbb{S}^2)\oplus\pi_2(\Omega_p(\mathbb{S}^2))\cong\pi_2(\mathbb{S}^2)\oplus\pi_3(\mathbb{S}^2)\cong\mathbb{Z}\oplus\mathbb{Z},$$
$$\pi_2(\mathcal{F})\cong\pi_3(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))\cong\pi_3(\mathbb{S}^2)\oplus\pi_3(\Omega_p(\mathbb{S}^2))\cong\pi_3(\mathbb{S}^2)\oplus\pi_4(\mathbb{S}^2)\cong\mathbb{Z}\oplus\mathbb{Z}_2.$$

Lemma 3.2.1 $\pi_0(\mathfrak{FLeg}(\mathbb{R}^3))\cong\pi_0(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3))\oplus\mathbb{Z}\oplus\mathbb{Z}$.

Proof It is sufficient to show that every element in

$$\pi_1(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3))\cong\pi_1(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3))\oplus\pi_1(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^1))\cong\pi_1(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3))\oplus\mathbb{Z}$$

can be lifted to an element in $\pi_1(\mathfrak{FLeg}(\mathbb{R}^3))$.

Take a loop $(\gamma^\theta,F_1^\theta)$ in $\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$. Let $F_0=(\gamma^\theta)'\colon\mathbb{S}^1\times\mathbb{S}^1(\theta,t)\to\mathbb{S}^2$ be the derivative $F_0(\theta,t)=(\gamma^\theta)'(t)$. We need to show that $F_0=(\gamma^\theta)'$ is homotopic to the map $F_1\colon\mathbb{S}^1\times\mathbb{S}^1\to\mathbb{S}^2$. Observe that the homotopy classes of maps from $\mathbb{S}^1\times\mathbb{S}^1$ to $\mathbb{S}^2$ are classified by the degree, and $\deg(F_1)=0$, so we just need to show that $\deg(F_0)=0$ to complete the proof.

Indeed, the map

$$G_\varepsilon(\theta,t)=\begin{cases}(\gamma^\theta)'(t) & \text{if } \varepsilon=0,\\ \dfrac{\gamma^\theta(t+\varepsilon)-\gamma^\theta(t)}{\|\gamma^\theta(t+\varepsilon)-\gamma^\theta(t)\|} & \text{if } 0<\varepsilon<1,\\ -(\gamma^\theta)'(t) & \text{if } \varepsilon=1,\end{cases}\tag{2}$$

is well-defined, because γ^θ for $\theta \in \mathbb{S}^1$ is an embedding. Thus, $F_0 = (\gamma^\theta)'$ is homotopic to $-F_0 = -(\gamma^\theta)'$, and $\deg(F_0) = \deg(\gamma^\theta)' = 0$. $\square$

3.3 Classification of formal Legendrian embeddings in $\mathbb{R}^3$

We have checked that $\pi_0(\mathfrak{FLeg}(\mathbb{R}^3)) \cong \pi_0(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \oplus \mathbb{Z}$. The first $\mathbb{Z}$ corresponds to the *rotation number* and we will show that the second one corresponds to the *Thurston–Bennequin invariant*.

Let us refine the definition of formal Legendrian embedding to extend the definition of the *Thurston–Bennequin invariant* to the formal case.

Definition 3.3.1 A *formal extended Legendrian embedding* in $\mathbb{R}^3$ is a pair (γ, G_s) satisfying:

(i) $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^3$ is a embedding.

(ii) $G_s\colon \mathbb{S}^1 \to \mathrm{SO}(3)$ is a smooth family in the parameter $s \in [0, 1]$ such that $G_0 = \mathrm{Id}$ and $G_1(\gamma') \in \xi_{\gamma(t)}$.

We denote by $\mathfrak{FELeg}(\mathbb{R}^3)$ the space of formal extended Legendrian embeddings in $\mathbb{R}^3$ equipped with the C^r–topology in the first factor and the C^{r-1}–topology in the second.

Remark 3.3.2 The natural fibration

$$t\colon \mathfrak{FELeg}(\mathbb{R}^3) \to \mathfrak{FLeg}(\mathbb{R}^3), \quad (\gamma, G_s) \mapsto (\gamma, G_s(\gamma')),$$

has contractible fibers. Thus, this map is a weak homotopy equivalence.

Given $(\gamma, G_s) \in \mathfrak{FELeg}(\mathbb{R}^3)$ we have a well-defined *formal contact framing* $\mathcal{F}_{\mathrm{FCont}}$ of $\nu(G_1(\gamma'))$ given by the Legendrian condition $G_1(\gamma') \subseteq \xi_{\gamma(t)}$. Then $G_1^{-1}(\mathcal{F}_{\mathrm{FCont}})$ defines a framing of the normal bundle ν of γ. On the other hand, we have a *topological framing* $\mathcal{F}_{\mathrm{Top}}$ of ν given by a Seifert surface of γ.

Definition 3.3.3 Let $(\gamma, G_s) \in \mathfrak{FELeg}(\mathbb{R}^3)$. The *Thurston–Bennequin invariant* is

$$\mathrm{tb}(\gamma, G_s) = \mathrm{tw}_\nu(G_1^{-1}(\mathcal{F}_{\mathrm{FCont}}), \mathcal{F}_{\mathrm{Top}}),$$

ie the twisting of $G_1^{-1}(\mathcal{F}_{\mathrm{FCont}})$ with respect to $\mathcal{F}_{\mathrm{Top}}$.

The Thurston–Bennequin invariant is defined over

$$\widehat{\mathfrak{FELeg}}(\mathbb{R}^3) = \{(\gamma, G) \mid \gamma \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3), G \in \mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathrm{SO}(3)), G(\gamma') \in \xi_{\gamma(t)}\}.$$

Furthermore, since the unique oriented $\mathbb{S}^1$–bundle over $\mathbb{S}^1$ is the trivial one,

$$\pi_0(\widehat{\mathfrak{FELeg}}(\mathbb{R}^3)) \cong \pi_0(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

The first $\mathbb{Z}$ is just the *rotation number* and the second one corresponds to the *Thurston–Bennequin invariant*.

Now we can state the main result of this section, which is well-known in the literature.

Theorem 3.3.4 *Formal Legendrian embeddings are classified by their parametrized knot type, rotation number and Thurston–Bennequin invariant.*

The proof follows directly using the fibration $\widehat{F}\colon \mathfrak{FELeg}(\mathbb{R}^3) \to \widehat{\mathfrak{FELeg}}(\mathbb{R}^3)$ and the fact that the map $t\colon \mathfrak{FELeg}(\mathbb{R}^3) \to \mathfrak{FLeg}(\mathbb{R}^3)$ is a weak homotopy equivalence. Note also that the fibration $\widehat{F}$ has connected fiber, because its π_0 is given by $\pi_2(\mathrm{SO}(3)) = 0$. This completes the proof. However, to get a more geometric picture, we will express the isomorphism in more concrete terms. Clearly the isomorphism preserves the rotation invariant, ie the rotation number (γ, G_s) is sent to the rotation invariant of $(\gamma, G_s(\gamma'))$. To understand the rest of the isomorphism we fix a basepoint $(\gamma, F = \gamma')$ in $\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$ with $\mathrm{Rot}(\gamma, \gamma') = 0$, ie we declare the basepoint to be a Legendrian embedding with zero rotation. Now, given an element of the fiber, ie (γ, F^s) with $F^0 = F^1 = \gamma'$, we claim that for $[(\gamma, 0, k)] \in \pi_0(\mathfrak{FLeg}(\mathbb{R}^3))$, the isomorphism $\pi_0(t)$ is given by $\mathrm{tb}(\pi_0(t)^{-1}(\gamma, 0, k)) = \mathrm{tb}(\gamma, \gamma') - 2k$. In other words, it depends on the choice of basepoint. This is obvious if we check that given a double stabilization — see Definition 7.2.3 for a precise definition — of the Legendrian knot, the value of the degree invariant in the fiber increases by 1, and it is a simple computation to check that the tb decreases by 2.

3.4 Fundamental group of formal Legendrian embeddings in $\mathbb{R}^3$

As a consequence of Lemma 3.2.1 we have that the following sequence is exact:

$$\begin{tikzcd}
& \cdots \arrow[r] & \pi_2(\mathfrak{FLeg}(\mathbb{R}^3)) \arrow[r] & \pi_2(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \arrow[dll] & \\
& \pi_1(\mathcal{F}) \arrow[r] & \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \arrow[r] & \pi_1(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \arrow[r] & 0
\end{tikzcd}$$

Take a 2–sphere (γ^z, F_1^z) in $\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$; the diagonal map $d\colon \pi_2(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \to \pi_1(\mathcal{F})$ measures the obstruction to lifting (γ^z, F_1^z) to a 2–sphere (γ^z, F_s^z) in $\mathfrak{FLeg}(\mathbb{R}^3)$, ie the obstruction to finding a homotopy between the derivative map

$$F_0 = (\gamma^z)'\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2, \quad (z,t) \mapsto F_0(z,t) = (\gamma^z)'(t),$$

and the map $F_1\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$. Note that by the Legendrian condition F_1 is null-homotopic, since it is not surjective. The first obstruction to finding this homotopy is just the degree of $F_0(z,0) = (\gamma^z)'(0)$, and corresponds to the first $\mathbb{Z}$ factor of $\pi_1(\mathcal{F}) \cong \pi_2(\mathbb{S}^2) \oplus \pi_3(\mathbb{S}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. In particular, since $(\gamma^z)'(0)$ is homotopic to $-(\gamma^z)'(0)$, this obstruction vanishes; see equation (2).

Theorem 3.4.1 *For $m \geq 0$, the sequence*

$$0 \to \mathbb{Z} \oplus \mathbb{Z}_m \to \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \to \pi_1(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \to 0$$

*is exact. In particular, if we fix the connected component $\mathfrak{Emb}'(\mathbb{S}^1, \mathbb{R}^3) \subseteq \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ of the parametrized **unknot** or of the parametrized (p,q) **torus knot**, we have that $m = 0$ and so*

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \to \pi_1(\mathfrak{Emb}'(\mathbb{S}^1, \mathbb{R}^3)) \oplus \mathbb{Z} \to 0$$

is exact.

Proof We only need to check the particular cases mentioned above. For the connected component $\mathfrak{Emb}_0(\mathbb{S}^1, \mathbb{R}^3)$ of the parametrized unknot the result follows from Theorem 2.1.1, since $\pi_2(\widehat{\mathfrak{FLeg}}_0(\mathbb{R}^3)) = \pi_2(\mathfrak{Emb}_0(\mathbb{S}^1, \mathbb{R}^3)) = \pi_2(\mathrm{SO}(3)) = 0$, where $\widehat{\mathfrak{FLeg}}_0(\mathbb{R}^3)$ stands for a formal Legendrian connected component of the smooth unknot.

On the other hand, fix the connected component $\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3)$ of the parametrized (p,q) torus knot and consider the commutative diagram

$$\begin{tikzcd} \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3) \arrow[r, hook] \arrow[d, hook] & \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3) \arrow[d, hook] \\ \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3) \arrow[r, hook] & \mathfrak{Imm}(\mathbb{S}^1, \mathbb{S}^3) \end{tikzcd}$$

defined by the natural inclusions, where $\mathfrak{Imm}(N, M)$ denotes the space of immersions of a manifold N into a manifold M equipped with the C^r–topology for $r \geq 5$.

By the Smale–Hirsch theorem for immersions (see [20] or [11]), $\mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ has the homotopy type of $\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^2)$, and $\mathfrak{Imm}(\mathbb{S}^1, \mathbb{S}^3)$ has the homotopy type of

$\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^3)\times\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2)$. Moreover, the map induced by the inclusion

$$\mathfrak{Emb}_{p,q}(\mathbb{S}^1,\mathbb{R}^3)\hookrightarrow\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3)$$

at π_2–level sends the homotopy class $[\gamma^z]\in\pi_2(\mathfrak{Emb}_{p,q}(\mathbb{S}^1,\mathbb{R}^3))$ to the homotopy class $[(\gamma^z)']\in\pi_2(\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3))\cong\pi_2(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))$, ie coincides with the diagonal map $d\colon\pi_2(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3))\to\pi_1(\mathcal{F})\cong\pi_2(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1,\mathbb{S}^2))$.

Consider the induced commutative diagram at π_2–level:

$$\begin{tikzcd}
\pi_2(\mathfrak{Emb}_{p,q}(\mathbb{S}^1,\mathbb{R}^3)) \arrow[r] \arrow[d] & \pi_2(\mathfrak{Emb}_{p,q}(\mathbb{S}^1,\mathbb{S}^3)) \arrow[d] \\
\pi_2(\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3)) \arrow[r] & \pi_2(\mathfrak{Imm}(\mathbb{S}^1,\mathbb{S}^3))
\end{tikzcd}$$

Since $\pi_2(\mathfrak{Emb}_{p,q}(\mathbb{S}^1,\mathbb{S}^3))$ is trivial (see Theorem 2.1.2), it is sufficient to show that the homomorphism $\pi_2(\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3))\to\pi_2(\mathfrak{Imm}(\mathbb{S}^1,\mathbb{S}^3))$ is injective to conclude the proof; but this is clear by using the h–principle for immersions, since the degree of the induced map $\mathbb{S}^2\times\mathbb{S}^1$ to $\mathbb{S}^3$ is zero and the induced map for the derivative from $\mathbb{S}^2\times\mathbb{S}^1$ to $\mathbb{S}^2$ is sent to itself by the inclusion. □

4 An application: Kálmán's loop

4.1 Kálmán's loop

Kálmán [22] has constructed a series of examples of loops of Legendrian positive torus knots which are noncontractible in the space $\mathfrak{Leg}(\mathbb{R}^3)$, though contractible in $\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$. Let us prove that Kálmán's examples are nontrivial even as loops of formal Legendrian embeddings; that is, in the space $\mathfrak{FLeg}(\mathbb{R}^3)$. We will prove that they are not contractible for any choice of parametrization, thus they are not contractible as loops of unparametrized oriented knots. Consider a Legendrian positive (p,q) torus knot; a loop is described in Figure 2. The loop takes the p strands of the knot to their cyclic rotation. This geometrically corresponds to a $2\pi/p$ rotation along the core of the defining torus. Let us consider $2p$ concatenations of this loop. Thus, it is generated by two full rotations along the core of the torus.

4.1.1 Simplified position First, we will deform the initial loop through formal loops into a formal loop in a "simplified" position.

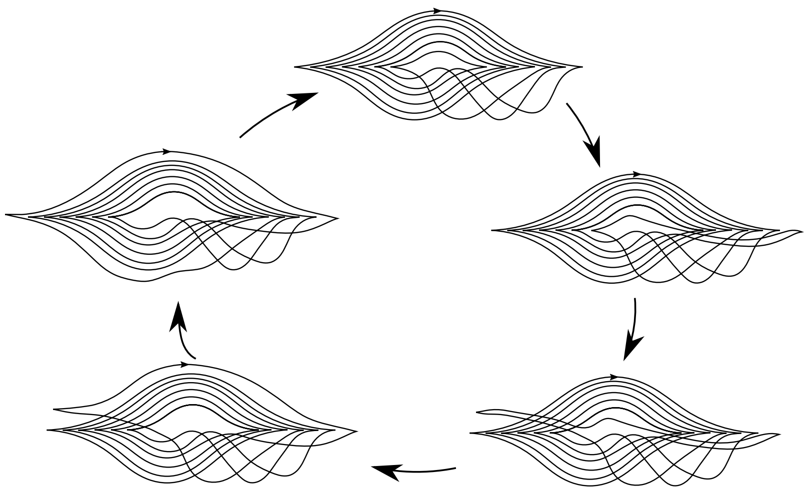

Figure 2: The loop in front projection $(p=3, q=7)$. Note that we have to cycle $2p=6$ times.

Step 1 Consider the contactomorphism $f(x,y,z)=(x/r, y/r, z/r^2)$ in the standard $(\mathbb{R}^3, \ker(dz-y\,dx))$. By using it, we assume that the defining torus for the loop has arbitrarily small meridional radius. Therefore, the knot is C^1–close to the core β of the torus. We are not using the standard notion of C^1–closeness, but a weaker one, ie we mean that a sequence of immersions $\widehat{\gamma}_k$ is C^1– close to an immersion $\widetilde{\gamma}$ if for any $\varepsilon>0$ and for any point $t\in\mathbb{S}^1$, we have the following: for every $k\in\mathbb{Z}$ large enough, there exists a point $\tau(t)\in\mathbb{S}^1$ such that

$$|\widehat{\gamma}_k(t)-\widetilde{\gamma}(\tau(t))|\le\varepsilon \quad \text{and} \quad \left|\frac{\widehat{\gamma}_k'(t)}{\|\widehat{\gamma}_k'(t)\|}-\frac{\widetilde{\gamma}'(\tau(t))}{\|\widetilde{\gamma}'(\tau(t))\|}\right|\le\varepsilon.$$

Moreover, by further shrinking, the knot and the core β may be assumed to be arbitrarily close to $\ker dz$, ie C^1–close to their Lagrangian projections.

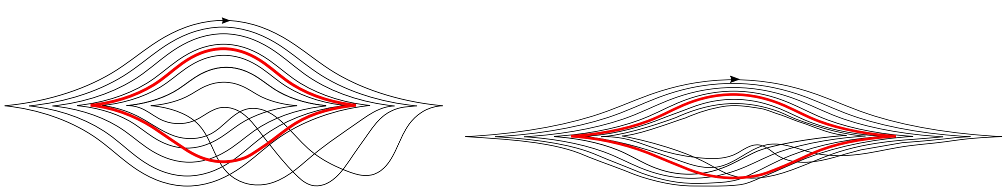

Figure 3: C^1–approximation of the knot to the core, shown in the front projection. Left: front projection of the knot and the core (in red). Right: knot C^1–close to the core.

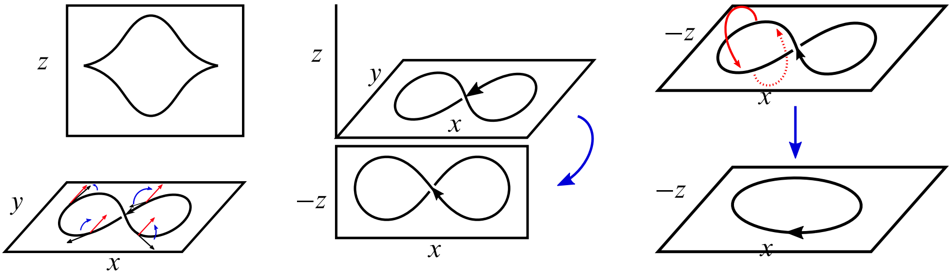

Figure 4: Construction of the path of loops. We represent the moves of the core β.

Step 2 Denote by γ^θ the initial loop of Legendrian embeddings. Understood as a formal loop, it is written as $(\gamma^\theta, F_s^\theta)$, where $F_s^\theta = (\gamma^\theta)'$. Let us construct a 1–parameter family of formal loops $(\gamma^{\theta,u}, F_s^{\theta,u})$ for $u \in [0,1]$, defined as follows:

(i) $\gamma^{\theta,u} = \gamma^\theta$,

(ii) $F_s^{\theta,u} = (1-s)(\gamma^\theta)' + s((1-u)(\gamma^\theta)' + u\,\partial_y)$.

This is a family of formal loops, since $(\gamma^\theta)'$ is never a negative multiple of ∂_y.[2] To check that, note that β is C^1–close to γ^θ.

Step 3 We consider the family of rotations $\{r_v \in \mathrm{SO}(3)\}_{v\in[0,1]}$ taking the quadrant XY to the quadrant $-ZX$; see Figure 4. Construct a family of formal loops $(\gamma^{\theta,u}, F_s^{\theta,u})$ for $u \in [1,2]$ as follows:

(i) $\gamma^{\theta,u} = r_{u-1}\cdot\gamma^\theta$,

(ii) $F_s^{\theta,u} = (1-s)(\gamma^{\theta,u})' + s\partial_y$.

Again, this is a family of formal loops because $(\gamma^{\theta,u})'$ is never a negative multiple of ∂_y. Note that we are using that γ^θ is C^1–close to β.

Step 4 Finally, we turn over the left lobe (see Definition 5.1.3) of the unknot core $r_1\cdot\beta$ by an isotopy defined as follows. Take polar coordinates (r,φ) in the plane YZ and $\varepsilon > 0$ small enough. We define the isotopy as

$$f_u(x,r,\varphi) = \begin{cases} (x,r,\varphi + u\cdot\chi(x)\cdot\pi) & \text{if } x \le \varepsilon, \\ (x,r,\varphi) & \text{if } \varepsilon \le x, \end{cases}$$

[2] We are assuming an orientation of the Legendrians. The argument with the opposite orientation runs in the same way by changing ∂_y to $-\partial_y$.

where $0 \le u \le 1$ and $\chi\colon \mathbb{R} \to \mathbb{R}$ is a nondecreasing smooth function satisfying

- $\chi(x) = 1$ for all $x \le 0$,
- $\chi(x) = 0$ for all $x \ge \varepsilon$.

We apply the isotopy to $r_1 \cdot \gamma^\theta$; see Figure 4. Again, the derivative is never tangent to $-\partial_y$ and thus we can interpolate to ∂_y.

We have proven that our initial loop of Legendrian embeddings is homotopic to the loop of formal Legendrian embeddings $(\tilde{\gamma}^\theta, \tilde{F}_s^\theta)$ defined as follows:

(i) $\tilde{\gamma}^\theta$ is the loop of parametrized (p,q) torus knots supported in the torus associated to the unknot contained in the plane XZ. The loop is obtained by a rotation of 4π radians of the standard (p,q)–embedding in the direction of the parallel of the supporting torus.

(ii) $\tilde{F}_s^\theta = (1-s)(\tilde{\gamma}^\theta)' + s\partial_y$.

4.2 Set of parametrizations of the family of loops

As an outcome of the previous discussion, we may assume that our formal Legendrian parametrized (p,q) torus knot can be written as (γ, F_s), where

$$\gamma(t) = \begin{pmatrix} (\cos(2\pi pt)+2)\cos(2\pi qt) \\ \sin(2\pi pt) \\ (\cos(2\pi pt)+2)\sin(2\pi qt) \end{pmatrix} \quad \text{and} \quad F_s(t) = s\partial_y + (1-s)(\gamma)'(t).$$

One particular parametrization of the loop can be written as $(\gamma^\theta, F_s^\theta)$, where

$$\gamma^\theta(t) = \begin{pmatrix} \cos(4\pi\theta) & 0 & -\sin(4\pi\theta) \\ 0 & 1 & 0 \\ \sin(4\pi\theta) & 0 & \cos(4\pi\theta) \end{pmatrix} \begin{pmatrix} (\cos(2\pi pt)+2)\cos(2\pi qt) \\ \sin(2\pi pt) \\ (\cos(2\pi pt)+2)\sin(2\pi qt) \end{pmatrix},$$

$$F_s^\theta(t) = s\partial_y + (1-s)(\gamma^\theta)'(t).$$

We will show that any possible parametrization of the loop gives rise to a nontrivial loop of parametrized formal Legendrian embeddings. Up to homotopy, the possible parametrizations of the formal Legendrian loop are given by

$$\gamma^{\theta,k}(t) = \gamma^\theta(t+k\theta) \quad \text{and} \quad F_s^{\theta,k}(t) = F_s^\theta(t+k\theta) = s\partial_y + (1-s)(\gamma^{\theta,k})'(t),$$

where $k \in \mathbb{Z}$. This is because $\pi_1(\mathrm{Diff}^+(\mathbb{S}^1)) = \pi_1(\mathrm{SO}(2)) = \mathbb{Z}$.[3]

[3] Remember that the Legendrian knots are oriented.

We will prove the following statement:

Proposition 4.2.1 *The loop of formal Legendrian embeddings* $(\gamma^{\theta,k}, F_s^{\theta,k})$ *is nontrivial for any* $k \in \mathbb{Z}$.

This proves that the loop is nontrivial as a loop of nonparametrized formal Legendrian knots.

4.3 Proof of Proposition 4.2.1

It follows from the previous discussion that the loop $\gamma^{\theta,k}(t)$ of smooth embeddings lies in $\mathrm{SO}(4) \subset \mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)$, ie $\gamma^{\theta,k}(t) = A_{\theta,k}\gamma(t)$, where $A_{\theta,k} \in \mathrm{SO}(4)$. More specifically, on $\mathbb{S}^3(\sqrt{2})$, the $\sqrt{2}$ radius sphere in $\mathbb{C}^2$, we have

$$\gamma^{\theta,0}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{4\pi i\theta} \end{pmatrix} \begin{pmatrix} e^{2\pi ipt} \\ e^{2\pi iqt} \end{pmatrix}.$$

Thus, in these coordinates the other parametrizations are given by

$$\gamma^{\theta,k}(t) = \gamma^{\theta,0}(t + k\theta) = \begin{pmatrix} e^{2\pi ipk\theta} & 0 \\ 0 & e^{2\pi i(2+qk)\theta} \end{pmatrix} \begin{pmatrix} e^{2\pi ipt} \\ e^{2\pi iqt} \end{pmatrix}.$$

By Theorem 2.1.2 the parametrized loop $\gamma^{\theta,k}$ is trivial in $\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)$ if and only if $2 + k(p+q)$ is even. From now on we will assume that this is the case. Thus, there is a family $\{\widetilde{A}_{(r,\theta),k}\}_{(r,\theta)\in\mathbb{D}}$ such that $\widetilde{A}_{(1,\theta),k} = A_{\theta,k}$. Since $\pi_2(\mathrm{SO}(4)) = 0$, the disk $\widetilde{A}_{(r,\theta),k}$ is unique up to homotopy fixing the boundary, and the same holds for the disk $\widetilde{A}_{(r,\theta),k} \cdot \gamma$ in $\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)$.

By Theorem 3.4.1, we have the exact sequence

$$0 \to \pi_1(\mathcal{F}) \xrightarrow{m_1} \pi_1(\mathfrak{FLeg}(\mathbb{R}^3)) \xrightarrow{m_2} \pi_1(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)) \to 0. \tag{3}$$

Thus, in order to prove that our loop $\ell_k = [(\gamma^{\theta,k}, F_s^{\theta,k})] \in \pi_1(\mathfrak{FLeg}(\mathbb{R}^3))$ is nontrivial we distinguish two cases:

Case 1 $(k \neq 0)$ We claim that $m_2(\ell_k) \neq 0$, ie $[\gamma^{\theta,k}] \neq 0 \in \pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3))$.[4]

Recall from Corollary 2.1.3 that $\pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{R}^3)) \cong \pi_1(\mathcal{S}tereo_{p,q})$ and that we have an exact sequence

$$0 \to G_{p,q} \to \pi_1(\mathcal{S}tereo_{p,q}) \to \pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3)) \to 0.$$

[4] Observe that $\mathrm{Rot}_L(\gamma^{\theta,k}, F_s^{\theta,k}) = 0$.

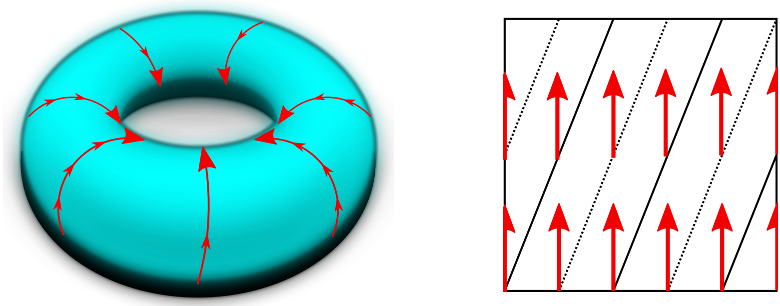

Figure 5: Visualization of the loop $\gamma^{\theta,k}$ for a $(5,2)$ torus knot $(k \neq 0)$.

Thus, we must show that $[(A_{\theta,k}\gamma, \infty)] \in \pi_1(\mathcal{S}tereo_{p,q})$ is nontrivial; ie the family of loops that is trivial by hypothesis in $\pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3))$ does not admit a capping disk whose evaluation map avoids $\infty \in \mathbb{S}^3$. We check it by composing with the 1–parameter family of loops $\widetilde{A}^{-1}_{(r,\theta),k} \in \mathrm{SO}(4)$, and obtain a 1–parameter family of loops $(\widetilde{A}^{-1}_{(r,\theta),k} A_{\theta,k}\gamma, \widetilde{A}^{-1}_{(r,\theta),k}\infty)$ for $r \in [0,1]$. For $r=0$ we obtain the initial loop and for $r=1$ we obtain the loop $(\gamma, A^{-1}_{\theta,k}\infty)$. Thus, these two loops represent the same element of $\pi_1(\mathcal{S}tereo_{p,q})$. Moreover, $[(\gamma, A^{-1}_{\theta,k}\infty)]$ can be lifted to $G_{p,q}$, since it lies on the fiber defined by the element $[\gamma] \in \pi_1(\mathfrak{Emb}_{p,q}(\mathbb{S}^1, \mathbb{S}^3))$. So we are reduced to checking whether $[A^{-1}_{\theta,k}\infty] \in G_{p,q}$ is trivial. The knot group of the (p,q) torus knot is $G_{p,q} = \langle a, b : a^p = b^q \rangle$. Thus, $[A^{-1}_{\theta,k}\infty] = b^{pk} \neq 0$, since b is a nontorsion element of $G_{p,q}$.

Case 2 $(k=0)$ Since $m_2(\ell_0) \in \pi_1(\widehat{\mathfrak{FLeg}}(\mathbb{R}^3))$ is zero, there exists $A \in \pi_1(\mathcal{F})$ such that $m_1(A) = \ell_0$. We are going to geometrically check that $A \neq 0$, from which, by the injectivity of m_1 provided by the sequence (3), the nontriviality of ℓ_0 follows.

Write $(\gamma^\theta, F^\theta_s) = (\gamma^{\theta,0}, F^{\theta,0}_s)$. In $(\mathbb{R}^3(x,y,z), \ker(dz - y\,dx))$ the parametrized loop is written as

$$\gamma^\theta(t) = B_\theta\gamma(t) = \begin{pmatrix} \cos(4\pi\theta) & 0 & -\sin(4\pi\theta) \\ 0 & 1 & 0 \\ \sin(4\pi\theta) & 0 & \cos(4\pi\theta) \end{pmatrix} \begin{pmatrix} (\cos(2\pi pt)+2)\cos(2\pi qt) \\ \sin(2\pi pt) \\ (\cos(2\pi pt)+2)\sin(2\pi qt) \end{pmatrix},$$

$$F^\theta_s(t) = s\partial_y + (1-s)(\gamma^\theta)'(t).$$

Moreover, γ^θ is bounded by $\tilde{\gamma}^{(r,\theta)}(t) = \widetilde{B}_{(r,\theta)}\gamma(t)$, where $\widetilde{B}_{(r,\theta)} \in \mathrm{SO}(3)$ is such that $\widetilde{B}_{(1,\theta)} = B_\theta$ and is homotopic to $\widetilde{A}_{(r,\theta)}$ inside $\mathrm{SO}(4)$. Thus, we have a disk $\mathcal{D}(r,\theta) = \{(\widetilde{B}_{r,\theta}\gamma, \partial_y)\}$ in $\widehat{\mathfrak{FLeg}}(\mathbb{R}^3)$ that bounds $m_2(\ell_0)$. We try to lift it to a disk in $\mathfrak{FLeg}(\mathbb{R}^3)$ that bounds ℓ_0. There is no homotopical obstruction to lifting it for the

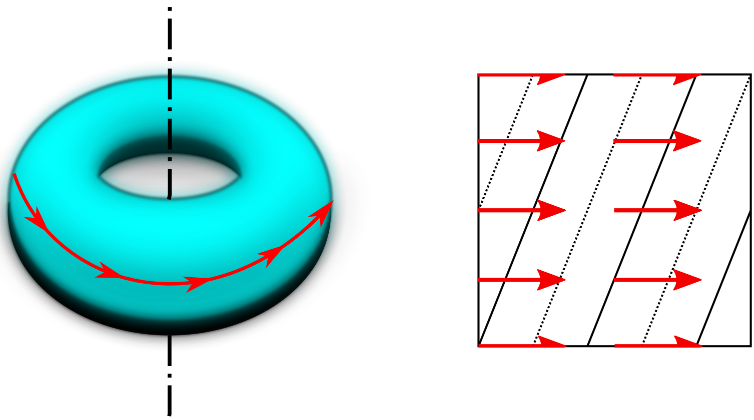

Figure 6: Visualization of the loop $\gamma^{\theta,0}$ for a $(5,2)$ torus knot $(k=0)$.

punctured disk $\mathcal{D}(r,\theta)|_{r>0}$. Therefore, the homotopy obstruction is represented by an element of $\pi_1(\mathcal{F})$. So we obtain a loop $(\gamma_0, \widetilde{F}_s^\theta)$ over the fiber of (γ_0, ∂_y) where $\gamma_0 = B_0\gamma$. It follows by construction that $A = [(\gamma_0, \widetilde{F}_s^\theta)]$. We compute:

$$\widetilde{F}_s^\theta = \begin{cases} \widetilde{B}_{2s,\theta}(\gamma') & \text{if } 0 \le s \le \frac{1}{2}, \\ F_{2s-1}^\theta = (2s-1)\partial_y + (2-2s)B_\theta(\gamma') & \text{if } \frac{1}{2} \le s \le 1. \end{cases} \tag{4}$$

Observe that $\widetilde{F}_0^\theta = \gamma_0'$ and $\widetilde{F}_1^\theta = \partial_y$. Thus, we can understand $\widetilde{F}_s^\theta$ as a map $\widetilde{F}\colon \mathbb{S}^2 \to \mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^2)$. It follows that

$$A = [\widetilde{F}] \in \pi_1(\mathcal{F}) \cong \pi_2(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^2)) \cong \pi_2(\mathbb{S}^2) \oplus \pi_2(\Omega_p(\mathbb{S}^2)),$$

that is, the fundamental group of the fiber. We already computed this group in Theorem 3.4.1. Moreover, we also showed that the morphism to $\pi_1(\mathfrak{FLeg}(\mathbb{R}^3))$ induced by the inclusion is injective.

To conclude the proof we must check that $[\widetilde{F}] \neq 0$. In order to see this, we will verify that $\deg(\widetilde{F}_s^\theta(0)\colon \mathbb{S}^2 \to \mathbb{S}^2)$ is nonzero. This degree is the first coordinate of $[\widetilde{F}] \in \mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(\mathcal{F}) \cong \pi_2(\mathbb{S}^2) \oplus \pi_2(\Omega_p(\mathbb{S}^2))$.

We give an explicit description of $\{\widetilde{B}_{r,\theta}\}$ in $\mathrm{SO}(3)$. Identify $\mathrm{SO}(3)$ with $\mathbb{RP}^3$ in the usual way; ie understand $\mathbb{RP}^3$ as the 3–ball of radius π with its boundary points identified via the antipodal map. Then a point $p = (p_1, p_2, p_3)$ in the described 3–ball corresponds in $\mathrm{SO}(3)$ to the rotation of angle $\sqrt{p_1^2 + p_2^2 + p_3^2}$ in $\mathbb{R}^3(x,y,z)$ around the axis described by its position vector p. In these coordinates — see the left drawing

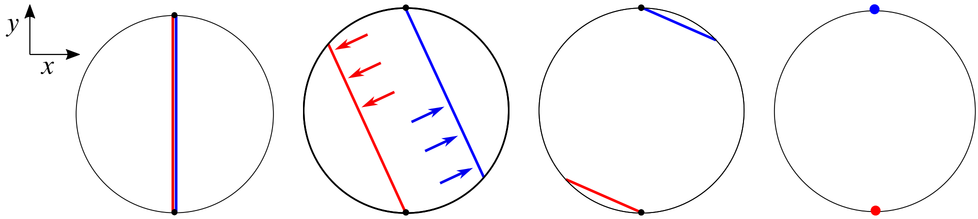

Figure 7: Explicit construction of the capping disk.

in Figure 7 — the loop B_θ is given by

$$B_\theta = \begin{cases} (0, 4\pi\theta, 0) & \text{if } 0 \le \theta \le \frac{1}{4}, \\ (0, -2\pi + 4\pi\theta, 0) & \text{if } \frac{1}{4} \le \theta \le \frac{3}{4}, \\ (0, -4\pi + 4\pi\theta, 0) & \text{if } \frac{3}{4} \le \theta \le 1. \end{cases}$$

Define the disk $\{\widetilde{B}_{r,\theta}\}$ as the intersection of the plane $\{z = 0\}$ with the π–radius ball. It produces an $\mathbb{RP}^2 = \mathbb{RP}^3 \cap \{z = 0\} \subseteq \mathbb{RP}^3$. We have $\widehat{B} = \{B_\theta : \theta \in \mathbb{S}^1\} \subseteq \mathbb{RP}^2$. We obtain $\mathbb{RP}^2 \setminus \widehat{B}$ is an embedded 2–disk. See Figure 7.

We have that $\gamma'(0) = 2\pi p\partial_y + 6\pi q\partial_z$. Substituting in equation (4) for $t = 0$, we get

$$\widetilde{F}_s^\theta(0) = \begin{cases} \widetilde{B}_{2s,\theta}(2\pi p\partial_y + 6\pi q\partial_z) & \text{if } 0 \le s \le \frac{1}{2}, \\ F_{2s-1}^\theta = (2s-1)\partial_y + (2-2s)B_\theta(2\pi p\partial_y + 6\pi q\partial_z) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

Moreover, for $u \in [0, 1]$, the maps

$$G(s, \theta, u) = \begin{cases} \widetilde{B}_{2s,\theta}((1-u)(2\pi p\partial_y + 6\pi q\partial_z) + u\partial_z) & \text{if } 0 \le s \le \frac{1}{2}, \\ F_{2s-1}^\theta = (2s-1)\partial_y + (2-2s)B_\theta((1-u)(2\pi p\partial_y + 6\pi q\partial_z) + u\partial_z) & \text{if } \frac{1}{2} \le s \le 1, \end{cases}$$

are always nonzero. Thus, the map $\widetilde{F}_s^\theta(0) = G(s, \theta, 0)$ is homotopic to

$$G(s, \theta) = G(s, \theta, 1) = \begin{cases} \widetilde{B}_{2s,\theta}(\partial_z) & \text{if } 0 \le s \le \frac{1}{2}, \\ (2s-1)\partial_y + (2-2s)B_\theta(\partial_z) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

In order to compute the degree of $G(s, \theta)$, we write

$$G(s, \theta) = (g_x(s, \theta), g_y(s, \theta), g_z(s, \theta)) \in \mathbb{S}^2(\partial_x, \partial_y, \partial_z)$$

and we check that $\#G^{-1}(-\partial_y) = 1$:

- If $\frac{1}{2} \le s \le 1$, then $G(s, \theta)$ is a linear combination with positive coefficients between ∂_y and $B_\theta(\partial_z) \in \mathbb{S}^1(\partial_x, \partial_z) \subseteq \mathbb{S}^2(\partial_x, \partial_y, \partial_z)$, thus $g_y(s, \theta) \ge 0$.

- If $0 \leq s \leq \frac{1}{2}$, then $G(s,\theta)$ is just a rotation around an axis in the XY–plane acting over ∂_z. The rotation of angle $\frac{\pi}{2}$ around the X–axis is the unique rotation that sends ∂_z to $-\partial_y$. Thus, $G^{-1}(-\partial_y) = \{(s_0,\theta_0)\}$, where (s_0,θ_0) is the only point such that $\widetilde{B}_{2s_0,\theta_0}$ is the mentioned rotation.

The map G is a local diffeomorphism in a neighborhood of the point (s_0,θ_0). Thus, $-\partial_y$ is a regular value for G and $|\deg(\widetilde{F}_s^\theta(0))| = |\deg(G(s,\theta))| = 1 \neq 0$.

5 Formal horizontal embeddings in $\mathbb{R}^4$

In this section, we denote by $\mathcal{D}$ the standard Engel structure in $\mathbb{R}^4(x,y,z,w)$ given by $\mathcal{D} = \ker(dy - z\,dx) \cap \ker(dz - w\,dx)$. Throughout the section we fix the framing of $\mathcal{D}$ given by the *kernel* of the Engel structure $\mathcal{W} = \langle \partial_w \rangle$.

5.1 Formal horizontal embeddings in $\mathbb{R}^4$

Definition 5.1.1 An immersion $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^4$ is said to be *horizontal* if $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in \mathbb{S}^1$. When γ is an embedding, we say it is a *horizontal embedding*.

Remark 5.1.2 The projection

$$\pi_G\colon \mathbb{R}^4(x,y,z,w) \to \mathbb{R}^3(x,z,w)$$

is called the *Geiges projection*. Notice that the Geiges projection maps horizontal embeddings to Legendrian immersions in $(\mathbb{R}^3(x,z,w), \ker(dz - w\,dx))$. Moreover, since the front projection for $(\mathbb{R}^3(x,z,w), \ker(dz-w\,dx))$ is the Lagrangian projection for $(\mathbb{R}^3(x,y,z), \ker(dy - z\,dx))$, the Geiges projection γ_G of a horizontal embedding $\gamma \in \mathfrak{Hor}(\mathbb{R}^4)$ is a Legendrian immersion which satisfies the area condition

$$\int_\gamma z\,dx = 0.$$

The composition of the Geiges projection and the front projection is called the *front Geiges projection*. The front Geiges projection of γ is denoted by γ_{FG}. The front projection of a Legendrian immersion β is denoted by β_F and the Lagrangian projection is denoted by β_L.

Definition 5.1.3 Let $\gamma \in \mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^2)$. By a *lobe of* γ we mean any segment of the curve that encloses a topological disk in $\mathbb{R}^2 \setminus \gamma(\mathbb{S}^1)$.

When we do not specify it explicitly, by a lobe of a horizontal embedding γ we mean a lobe of γ_{FG}.

Definition 5.1.4 (a) A *formal horizontal immersion* in $\mathbb{R}^4$ is a pair (γ, F) such that:

(i) $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^4$ is a smooth map.

(ii) $F\colon \mathbb{S}^1 \to \gamma^*(T\mathbb{R}^4 \setminus \{0\})$ satisfies $F(t) \in \mathcal{D}_{\gamma(t)}$.

(b) A *formal horizontal embedding* in $\mathbb{R}^4$ is a pair (γ, F_s) satisfying:

(i) $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^4$ is an embedding.

(ii) $F_s\colon \mathbb{S}^1 \to \gamma^*(T\mathbb{R}^4 \setminus \{0\})$ for $s \in [0,1]$ is a 1–parameter family such that $F_0 = \gamma'$ and $F_1(t) \in \mathcal{D}_{\gamma(t)}$.

Use $\langle \partial_w \rangle$ to identify $\mathcal{D} \equiv \mathbb{R}^2$. From now on, we will understand the family $F_s\colon \mathbb{S}^1 \to \mathbb{S}^3$ with $F_1\colon \mathbb{S}^1 \to \mathbb{S}^1 \equiv \mathbb{S}^3 \cap \mathbb{R}^2$.

Denote by $\mathfrak{HorImm}(\mathbb{R}^4)$ the space of horizontal immersions and by $\mathfrak{Hor}(\mathbb{R}^4)$ the space of horizontal embeddings in $\mathbb{R}^4$. All these spaces are endowed with the C^r–topology. Denote by $\mathfrak{FHorImm}(\mathbb{R}^4)$ the space of formal horizontal immersions in $\mathbb{R}^4$ and by $\mathfrak{FHor}(\mathbb{R}^4)$ the space of formal horizontal embeddings in $\mathbb{R}^4$. All these spaces are endowed with the (C^r, C^{r-1})–topology. These definitions make sense for immersions and embeddings of the interval. We define $\mathfrak{HorImm}([0,1], \mathbb{R}^4)$, $\mathfrak{Hor}([0,1], \mathbb{R}^4)$, $\mathfrak{FHorImm}([0,1], \mathbb{R}^4)$ and $\mathfrak{FHor}([0,1], \mathbb{R}^4)$ analogously.

Remark 5.1.5 In the same vein as in the Legendrian case (see Remark 3.1.3) we can equip the spaces $\mathfrak{HorImm}(\mathbb{R}^4)$, $\mathfrak{Hor}(\mathbb{R}^4)$, $\mathfrak{FHorImm}(\mathbb{R}^4)$ and $\mathfrak{FHor}(\mathbb{R}^4)$ with a structure of Banach manifolds.

Like in the Legendrian case it is important to note that any $\gamma \in \mathfrak{HorImm}^r(\mathbb{R}^4)$ that is transverse to the kernel $\mathcal{W}$ of the Engel structure admits a particular type of immersed chart φ_γ which identifies, via an immersed Engel morphism, a tubular neighborhood of the zero section of the jet space $J^2(\mathbb{S}^1)$ (see [26, Lemma 1]) with a tubular neighborhood of $\gamma(\mathbb{S}^1)$. This construction provides local charts in the C^r–topology just by defining the map

$$\begin{aligned} \mathcal{C}\colon \mathcal{U} \subseteq \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1, \mathbb{R}) \times \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1, \mathbb{R}) &\to \mathcal{V} \subseteq \mathfrak{HorImm}^r(\mathbb{R}^4), \\ (f, g) &\mapsto \varphi_\gamma^{-1} \circ j^2(g \circ \tilde{f}). \end{aligned} \tag{5}$$

This proves that the open subset of horizontal immersed curves that are transverse to the kernel of the Engel structure is a Banach manifold.

Remark 5.1.6 Del Pino and Presas [26] proved that the inclusion $\mathfrak{HorImm}(\mathbb{R}^4) \hookrightarrow \mathfrak{FHorImm}(\mathbb{R}^4)$ is a weak homotopy equivalence. Thus, $\pi_0(\mathfrak{HorImm}(\mathbb{R}^4)) \cong \mathbb{Z}$, $\pi_1(\mathfrak{HorImm}(\mathbb{R}^4)) \cong \mathbb{Z}$ and $\pi_k(\mathfrak{HorImm}(\mathbb{R}^4)) = 0$ for all $k \geq 2$. As in the Legendrian case the connected components of $\mathfrak{HorImm}(\mathbb{R}^4)$ are classified by a *rotation number*. The rotation number of $\gamma \in \mathfrak{HorImm}(\mathbb{R}^4)$ is $\mathrm{Rot}(\gamma) = \deg(\gamma' \colon \mathbb{S}^1 \to \mathbb{S}^1)$. In the same way, the homotopy type of a loop γ^θ in $\mathfrak{HorImm}(\mathbb{R}^4)$ is determined by the integer $\mathrm{Rot}_L(\gamma^\theta) = \deg(\theta \mapsto (\gamma^\theta)'(0))$, which is called *rotation number of the loop*. Moreover, it is known that horizontal embeddings in $\mathbb{R}^4$ are also classified by their rotation number. This result was proven by Adachi [1] and Geiges [15]. Note that these invariants are defined in the formal setting in the obvious way.

5.2 The space $\mathfrak{FHor}(\mathbb{R}^4)$

Consider the natural fibration

$$\mathfrak{FHor}(\mathbb{R}^4) \to \widehat{\mathfrak{FHor}}(\mathbb{R}^4) = \{(\gamma, F) : \gamma \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4),\ F \in \mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^1)\}.$$

Take $\gamma \in \mathfrak{Hor}(\mathbb{R}^4)$ and fix $(\gamma, \gamma') \in \widehat{\mathfrak{FHor}}(\mathbb{R}^4)$ as basepoint to compute the homotopy of $\mathfrak{FHor}(\mathbb{R}^4)$. The fiber over this point is $\mathcal{F} = \mathcal{F}_{(\gamma,\gamma')} = \Omega_{\gamma'}(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3))$. To compute some homotopy groups of $\mathcal{F}$ use the fibration $\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3) \to \mathbb{S}^3$ defined by the *evaluation map*. Observe that, since every element $[f] \in \pi_n(\mathbb{S}^3)$ can be lifted to an element $[f_n] \in \pi_n(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3))$ defined as

$$f_n(p)(t) = f(p), \quad t \in \mathbb{S}^1,\ p \in \mathbb{S}^n,$$

all the diagonal maps in the associated exact sequence are zero and the associated short exact sequences are right split and, thus, split for $n > 1$. Since $\mathbb{S}^3$ is 2–connected we conclude that

$$\pi_0(\mathcal{F}) \cong \pi_1(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3)) = 0,$$
$$\pi_1(\mathcal{F}) \cong \pi_2(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3)) \cong \pi_2(\Omega_p(\mathbb{S}^3)) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z},$$
$$\pi_2(\mathcal{F}) \cong \pi_3(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3)) \cong \pi_3(\mathbb{S}^3) \oplus \pi_3(\Omega_p(\mathbb{S}^3)) \cong \pi_3(\mathbb{S}^3) \oplus \pi_4(\mathbb{S}^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Furthermore, the space $\widehat{\mathfrak{FHor}}(\mathbb{R}^4)$ has the homotopy type of $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4) \times \mathbb{S}^1 \times \mathbb{Z}$. Thus (see Lemma 2.2.1),

$$\begin{aligned}
&\pi_0(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)) \cong \mathbb{Z},\\
&\pi_1(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)) \cong \pi_1(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)) \oplus \mathbb{Z},\\
&\pi_2(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)) \cong \pi_2(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)) \cong \mathbb{Z}[S_U] \oplus \pi_2(\mathfrak{LEmb}(\mathbb{R}, \mathbb{R}^4)).
\end{aligned}$$

The exact sequence associated to the fibration $\mathfrak{FHor}(\mathbb{R}^4) \to \widehat{\mathfrak{FHor}}(\mathbb{R}^4)$ takes the form

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathbb{Z} \oplus \pi_2(\mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4)) \arrow[dll] & \\
\mathbb{Z} \arrow[r, hook] & \pi_1(\mathfrak{FHor}(\mathbb{R}^4)) \arrow[r] & \pi_1(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^4)) \oplus \mathbb{Z} \arrow[dll] & \\
0 \arrow[r, hook] & \pi_0(\mathfrak{FHor}(\mathbb{R}^4)) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

In particular, this proves that *formal horizontal embeddings are classified by their rotation number.*

We can state the next result concerning the fundamental group of each connected component of $\mathfrak{FHor}(\mathbb{R}^4)$:

Lemma 5.2.1 *The sequence*

$$0 \to \mathbb{Z}_2 \to \pi_1(\mathfrak{FHor}(\mathbb{R}^4)) \to \pi_1(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^4)) \oplus \mathbb{Z} \to 0$$

is exact.

Proof It is sufficient to show that the image of the diagonal map $d\colon \pi_2(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)) \to \pi_1(\mathcal{F})$ is $2\mathbb{Z}$. Observe that d measures the obstruction to lifting an element $[(\gamma^z, F_1^z)] \in \pi_2(\widehat{\mathfrak{FHor}}(\mathbb{R}^4))$ to $\pi_2(\mathfrak{FHor}(\mathbb{R}^4))$. In other words, let $F_0 = (\gamma^z)'\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$, $(z,t) \mapsto F_0(z,t) = (\gamma^z)'(t)$, be the derivative map. The homomorphism d measures the obstruction to finding a homotopy between the derivative map $F_0 = (\gamma^z)'$ and the map $F_1\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$, $(z,t) \mapsto F_1^z(t)$. Since the map F_1 is null-homotopic, we conclude that $d[(\gamma^z, F_1^z)] = \deg(F_0)$.[5]

Recall that $\pi_2(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)) \cong \pi_2(\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^4)) \cong \mathbb{Z}[S_U] \oplus \pi_2(\mathfrak{LEmb}(\mathbb{R},\mathbb{R}^4))$; see Lemma 2.2.1. We have that $d[S_U] = 2$. Indeed, $d[S_U]$ is the degree of the map

$$\begin{aligned}
F_\gamma = \frac{\gamma'}{\|\gamma'\|}&\colon \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3,\\
&((\lambda_1,\lambda_2,\lambda_3),t) \mapsto \cos(2\pi t)e_1 + \sin(2\pi t)(\lambda_1 e_2 + \lambda_2 e_3 + \lambda_3 e_4),
\end{aligned}$$

where $\{e_1,e_2,e_3,e_4\}$ is the canonical basis of $\mathbb{R}^4 \supseteq \mathbb{S}^3$. It follows that

$$F_\gamma^{-1}(e_4) = \left\{q_1 = \left((0,0,1),\tfrac{1}{4}\right),\ q_2 = \left((0,0,-1),\tfrac{3}{4}\right)\right\}.$$

[5] We are arguing as in Lemma 3.2.1, but now the antipodal map in $\mathbb{S}(\mathbb{R}^4) = \mathbb{S}^3$ has degree 1.

We want to compute the orientation class of the image of the tangent space in the tangent space. We understand $\mathbb{S}^2 \times \mathbb{S}^1$ as a submanifold of $\mathbb{R}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3$ as a submanifold of $\mathbb{R}^4$. We extend the tangent spaces of those two submanifolds by adding the outward normal vectors to both spheres $T\mathbb{S}^2 \times T\mathbb{S}^1 \subset T(\mathbb{S}^2 \oplus r\,\partial r) \times T\mathbb{S}^1$ and $T\mathbb{S}^3 \subset T(\mathbb{S}^3 \oplus r\,\partial r)$. We declare that the orientations in the submanifolds are induced from the orientations in $\mathbb{R}^3 \times \mathbb{S}^1$ and $\mathbb{R}^4$. In order to compute the degree of the differential at the two points, we realize that F_γ linearly extends to $\overline{F}_\gamma\colon \mathbb{R}^3 \times \mathbb{S}^1 \to \mathbb{R}^4$. The degree of F_γ is computed by that of $\overline{F}_\gamma$. We obtain that

$$\det(dF_\gamma) = \det \begin{pmatrix} 0 & 0 & 0 & -2\pi \sin(2\pi t) \\ \sin(2\pi t) & 0 & 0 & 2\pi \cos(2\pi t)\lambda_1 \\ 0 & \sin(2\pi t) & 0 & 2\pi \cos(2\pi t)\lambda_2 \\ 0 & 0 & \sin(2\pi t) & 2\pi \cos(2\pi t)\lambda_3 \end{pmatrix} = 2\pi(\sin(2\pi t))^4$$

is nonzero and positive in $(F_\gamma)^{-1}(e_4)$. Thus, e_4 is a regular value for F_γ and $d[S_U] = \deg F_\gamma = 2$.

To conclude the proof we must check that any 2–sphere $\gamma^z \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ which comes from the space of long embeddings satisfies $d[\gamma^z] \in 2\mathbb{Z}$. In order to see this observe that if $\gamma(t) = (x(t), y(t), z(t), w(t))^\mathsf{T} \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ comes from a long embedding, then we may assume the following:

(i) $\gamma\big(\big(\tfrac{1}{4}, \tfrac{3}{4}\big)\big) \subseteq \operatorname{Int}((-1,1)^4)$,

(ii) $\gamma\big(\tfrac{1}{4}\big) = -e_1$, $\gamma\big(\tfrac{3}{4}\big) = e_1$ and $\gamma'\big(\tfrac{1}{4}\big) = \gamma'\big(\tfrac{3}{4}\big) = e_1$,

(iii) $\gamma\big(\mathbb{S}^1 \setminus \big(\tfrac{1}{4}, \tfrac{3}{4}\big)\big)$ is contained in the plane spanned by e_1 and e_2.

(iv) $\gamma|_{\mathbb{S}^1 \setminus (\frac{1}{8}, \frac{7}{8})}$ parametrizes the semicircle of radius R and center $R \cdot e_2$ joining $\gamma\big(\tfrac{1}{8}\big) = -R \cdot e_1 + R \cdot e_2$ and $\gamma\big(\tfrac{7}{8}\big) = R \cdot e_1 + R \cdot e_2$ which passes through $\gamma(0) = 2R \cdot e_2$,

where $R \gg 1$ is a constant independent of γ.

Let γ^z be any 2–sphere which comes from the space of long embeddings. The associated map $\gamma' = (\gamma^z)'$ is homotopic to

$$\gamma^h(z,t) = \frac{\gamma^z(t+h) - \gamma^z(t)}{\|\gamma^z(t+h) - \gamma^z(t)\|}$$

for any $h \in (0,1)$. Thus, γ' is homotopic to

$$\gamma^{\widetilde{h}}(z,t) = \frac{\gamma^z(t+\widetilde{h}(t)) - \gamma^z(t)}{\|\gamma^z(t+\widetilde{h}(t)) - \gamma^z(t)\|},$$

where $\widetilde{h}\colon [0,1] \to (0,1)$ is any continuous map which satisfies $\widetilde{h}(0) = \widetilde{h}(1)$. Consider the family of functions $\widetilde{h}_\varepsilon$, $0 < \varepsilon < \frac{1}{4}$, defined as

$$\widetilde{h}_\varepsilon(t) = \begin{cases} \varepsilon & \text{if } t \in \left[0, \frac{1}{4} - \varepsilon\right] \cup \left[\frac{7}{8} - \varepsilon, 1\right], \\ \frac{7}{8} - t & \text{if } t \in \left[\frac{1}{4}, \frac{7}{8} - \varepsilon\right], \\ \left(1 - \frac{t - \frac{1}{4} + \varepsilon}{\varepsilon}\right)\varepsilon + \frac{t - \frac{1}{4} + \varepsilon}{\varepsilon}\frac{5}{8} & \text{if } t \in \left[\frac{1}{4} - \varepsilon, \frac{1}{4}\right]. \end{cases}$$

It follows that $e_3 \notin \operatorname{Image}(\gamma^{\widetilde{h}_\varepsilon})$. Hence, $d[\gamma^z] = \deg(\gamma') = \deg(\gamma^{\widetilde{h}_\varepsilon}) = 0 \in 2\mathbb{Z}$. □

5.3 The $\mathbb{Z}_2$ factor in $\pi_1(\mathfrak{FHor}(\mathbb{R}^4))$

Let us explain the $\mathbb{Z}_2$ factor in the exact sequence described in Lemma 5.2.1. Observe that $\mathbb{Z}_2$ is just the *kernel* of the homomorphism

$$f\colon \pi_1(\mathfrak{FHor}(\mathbb{R}^4)) \to \pi_1(\widehat{\mathfrak{FHor}}(\mathbb{R}^4))$$

induced by the fibration $\mathfrak{FHor}(\mathbb{R}^4) \to \widehat{\mathfrak{FHor}}(\mathbb{R}^4)$, and $\ker(f) \cong \pi_1(\mathcal{F})/d(\pi_2(\widehat{\mathfrak{FHor}}))$; see Lemma 5.2.1. Consider the unique isomorphism

$$\mu\colon \ker(f) \to \mathbb{Z}_2.$$

Definition 5.3.1 The μ–invariant of a loop $(\gamma^\theta, F_s^\theta) \in \ker(f)$ is $\mu(\gamma^\theta, F_s^\theta) \in \mathbb{Z}_2$.

Take a loop $(\gamma^\theta, F_s^\theta)$ in $\mathfrak{FHor}(\mathbb{R}^4)$ and assume that $(\gamma^\theta, F_1^\theta)$ is trivial in $\widehat{\mathfrak{FHor}}(\mathbb{R}^4)$, ie γ^θ is trivial as a loop of smooth embeddings and $\operatorname{Rot}_L(\gamma^\theta, F_s^\theta) = 0$. Then there is a disk (γ^z, F_1^z) in $\widehat{\mathfrak{FHor}}(\mathbb{R}^4)$, with $z \in \mathbb{D}$, such that $(\gamma^{e^{2\pi i\theta}}, F_1^{e^{2\pi i\theta}}) = (\gamma^\theta, F_1^\theta)$. Now we try to lift it to a disk (γ^z, F_s^z) in $\mathfrak{FHor}(\mathbb{R}^4)$. This can be done for all the values of the parameter except for $z = 0$, where we have $\mathbb{S}^1$ different limits. So we have a loop $(\gamma^0, \widetilde{F}_s^\theta)$ in the fiber of (γ^0, F_1^0). We define $\widetilde{F}$ as

$$\widetilde{F}\colon [0,1] \times \mathbb{S}^1 \to \mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3), \quad (s,\theta) \mapsto \widetilde{F}(s,\theta) = \widetilde{F}_s^\theta.$$

$\widetilde{F}_0^\theta = (\gamma^0)'$ and $\widetilde{F}_1^\theta = F_1^0$, so we can understand $\widetilde{F}$ as a map from $\mathbb{S}^2$ to $\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3)$, ie $[\widetilde{F}] \in \pi_1(\mathcal{F}) \cong \pi_2(\mathcal{M}\mathfrak{aps}(\mathbb{S}^1, \mathbb{S}^3)) \cong \pi_3(\mathbb{S}^3)$. The μ–invariant measures the *degree mod* 2 of the associated map from $\mathbb{S}^2 \times \mathbb{S}^1$ to $\mathbb{S}^3$, ie $\mu(\gamma^\theta, F_s^\theta) = \deg_2(\widetilde{F})$. This is because the elements in $\pi_1(\mathcal{F}) \cong \pi_3(\mathbb{S}^3)$ are classified by the degree but the construction of $\widetilde{F}$ depends on the choice of disk inside $\widehat{\mathfrak{FHor}}(\mathbb{R}^4)$ bounding $(\gamma^\theta, F_1^\theta)$. The difference between two disks is measured by the diagonal map $d\colon \pi_2(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)) \to \pi_2(\mathcal{F})$, ie $\widetilde{F}$ is unique, up to homotopy, inside $\pi_1(\mathcal{F})/d(\pi_2(\widehat{\mathfrak{FHor}}(\mathbb{R}^4)))$. It follows from

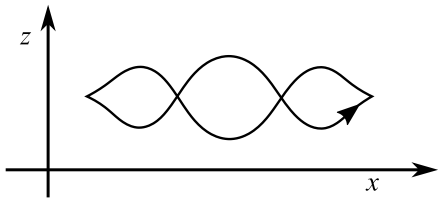

Figure 8: The basepoint (a horizontal knot) of the area twist loop in the front Geiges projection.

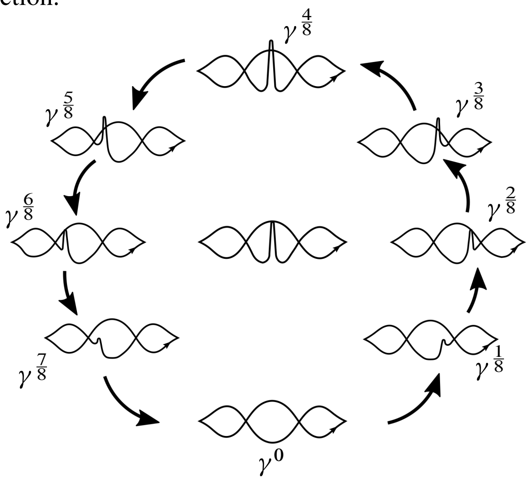

Figure 9: The area twist loop in the front Geiges projection with the standard capping disk through horizontal immersions. Note that the unique strict horizontal immersion in the disk is the center of it.

Lemma 5.2.1 that the elements $\pi_1(\mathcal{F})/d(\pi_2(\widehat{\mathfrak{F}\mathfrak{Hor}}(\mathbb{R}^4))) \cong \mathbb{Z}_2$ are classified by the degree mod 2 of the associated map.

Let us explicitly describe $\ker(f)$. The first element of $\ker(f)$ is just the trivial loop and the second one is given by the homotopy class of a loop γ^θ of horizontal embeddings, which we name *area twist loop* and which is described in Figure 9; we give a precise definition below. Observe that this loop is trivial as a loop of horizontal immersions.

Let γ be a horizontal immersion with exactly one generic self-intersection. We are going to construct a loop of horizontal embeddings that is supported on a very small neighborhood of the self-intersection. Informally speaking,[6] on a neighborhood of γ inside the space of horizontal immersions, the subset of strict horizontal immersions has codimension 2. Thus, there exists a disk $\mathbb{D}^2 \to \mathfrak{Hor}\mathfrak{Imm}(\mathbb{R}^4)$ that intersects the

[6]It could be completely formalized by adapting Section 6.1 to the horizontal case; in particular we would need the equivalent of Lemma 6.1.5

strict immersion just at the center γ. There is a natural orientation on it and we take the loop $\partial\mathbb{D}^2$ with the induced orientation. Let us give a more hands-on approach.

Assume that the self-intersection times are t_0, t_1 and fix $t_1 = 0$. Now select $\varepsilon, \delta > 0$ such that $\delta \ll \varepsilon < 1$. By [26, Lemma 1] we may assume that

$$\gamma_{FG}(t) = \begin{cases} (t - t_0, 0) & \text{for } t \in \mathcal{O}p(\{t_0\}), \\ (t, -t^2) & \text{for } t \in (-2\varepsilon, 2\varepsilon). \end{cases}$$

For each $(u, v) \in [-\delta, \delta]^2$, define $\gamma^{u,v} \in \mathfrak{HorImm}(\mathbb{R}^4)$ in terms of the front Geiges projection $\gamma^{u,v}_{FG}(t) = (x^{u,v}(t), z^{u,v}(t))$ as follows:

- $\gamma^{0,0} = \gamma$.
- $\gamma^{u,v}_{FG}(t) = \gamma_{FG}(t)$ for $t \in \mathbb{S}^1 \setminus (-2\varepsilon, 2\varepsilon)$.
- $x^{u,v} = x^{0,0}$.
- $z^{u,v}(t)$ for $t \in (-2\varepsilon, 2\varepsilon)$ is defined by the following conditions:

(i) $z^{u,v}(t) = v - (t-u)^2$ for $t \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$.

(ii) $z^{u,v}(t) = -t^2$ for $t > |\varepsilon|$.

(iii) $\int_{-\varepsilon}^{\varepsilon} z^{u,v}(t)\,dt = \int_{-\varepsilon}^{\varepsilon} z^{0,0}(t)\,dt$.

(iv) $\int_{-\varepsilon}^{u} z^{u,v}(t)\,dt < \int_{-\varepsilon}^{0} z^{0,0}(t)\,dt$ for $0 < u \leq \delta$.

(v) $\int_{-\varepsilon}^{u} z^{u,v}(t)\,dt > \int_{-\varepsilon}^{0} z^{0,0}(t)\,dt$ for $-\delta \leq u < 0$.

Observe that conditions (iii), (iv) and (v) imply $\gamma^{u,v} \in \mathfrak{Hor}(\mathbb{R}^4)$ for $(u, v) \neq (0, 0)$.

Definition 5.3.2 An *area twist loop* is any loop of horizontal embeddings which is homotopic to $\gamma^\theta := \gamma^{\delta\cos(2\pi\theta), \delta\sin(2\pi\theta)}$.

Observe that the notion of *area twist loop* is well-defined since the space of choices in the construction of $\gamma^{u,v}$ is contractible.

The next lemma states that the area twist loop is actually the second element of $\ker(f)$.

Lemma 5.3.3 *Let γ^θ be an area twist loop. Then*

$$(\gamma^\theta, (\gamma^\theta)') \in \ker(f) \quad \textit{and} \quad \mu(\gamma^\theta, (\gamma^\theta)') = 1.$$

Proof We prove the result for the area twist γ^θ described in Figure 9. The general case is analogous.

Consider the 1–parameter family of formal horizontal embeddings $(\gamma^{\theta,u}, F_s^{\theta,u})$ for $u \in [0,1]$ defined by

(i) $\gamma^{\theta,u} = \gamma^{\theta}$,

(ii) $F_s^{\theta,u} = (1-s)(\gamma^{\theta})' + s((1-u)(\gamma^{\theta})' + u\partial_w)$.

Note that $F_0^{\theta,u} = (\gamma^{\theta})'$ for $u \in [0,1]$, and $(\gamma^{\theta})'$ is never a positive multiple of $-\partial_w$. Hence we can assume that the area twist loop in $\mathfrak{FHor}(\mathbb{R}^4)$ is $(\gamma^{\theta}, F_s^{\theta,1})$. To compute the $\mathbb{Z}_2$–invariant of the area twist loop we are going to construct a disk

$$\{(\tilde{\gamma}^{r,\theta}, \partial_w) : r \in [0,1],\ \theta \in \mathbb{S}^1\}$$

in $\widehat{\mathfrak{FHor}}(\mathbb{R}^4)$ so that $\tilde{\gamma}^{1,\theta} = \gamma^{\theta}$.

For each $\lambda \in \mathbb{R}$ consider the 3–plane $\Pi_\lambda \subseteq \mathbb{R}^4$ with equation $x = \lambda$. The plane Π_λ intersects each embedding γ^{θ} in zero, one or two points. Consider the segment $G = \{\lambda : \#\Pi_\lambda \cap \gamma^{\theta} = 1 \text{ or } 2\}$ and write $\Pi_\lambda \cap \gamma^{\theta} = \{A_\lambda^{\theta}, B_\lambda^{\theta}\}$ for all $\lambda \in G$. Denote by $\overline{A_\lambda^{\theta} B_\lambda^{\theta}}$ the unique segment (in $\mathbb{R}^4$) defined by the pair of points $A_\lambda^{\theta}, B_\lambda^{\theta}$. The union of all these segments allows us to construct an embedded 2–disk $\mathbb{D}^{\theta} = \bigcup_{\lambda \in G} \overline{A_\lambda^{\theta} B_\lambda^{\theta}}$ whose boundary is γ^{θ}. The foliation in the so-constructed 2–disk provided by the segments we call the "ruling".

Assume that the left cusp of γ^{θ} happens at time $t_0 \in \mathbb{S}^1$. Observe that $\gamma^{\theta}(t) = \gamma^0(t)$ for $t \in \mathcal{O}p(\{t_0\})$. Thus, by construction, the intersection $\mathbb{D}^{\theta} \cap \mathcal{O}p(\{\gamma^0(t_0)\}) = \mathbb{D}$ is independent of θ. Fix $q_0 \in \mathrm{Int}(\mathbb{D})$ and carefully take polar coordinates

$$\mathbb{D}^{\theta} = \{p_\theta^{\rho,t} : \rho \in [0,1],\ t \in \mathbb{S}^1\}$$

such that

- $p_\theta^{1,t} = \gamma^{\theta}(t)$,
- $p_\theta^{0,t} = q_0$,
- for any ρ and θ fixed, the loop $\{p_\theta^{\rho,t}\}_{t \in \mathbb{S}^1}$ intersects the "ruling" provided by $\overline{A_\lambda^{\theta} B_\lambda^{\theta}}$ in zero, one or two points. In other words, the curve $\{p_\theta^{\rho,t}\}_{t \in \mathbb{S}^1}$ becomes tangent to the ruling just two times.

Moreover, fix $\varepsilon > 0$ small enough and let $Q_0 \colon \mathbb{D}(\varepsilon) \to \mathrm{Int}(\mathbb{D})$ be an embedding such that $Q_0(0) = q_0$. Assume that $Q_0(\rho e^{2\pi i t}) = p_\theta^{\rho,t}$ for $0 \le \rho \le \varepsilon$ and $\theta \in \mathbb{S}^1$. Finally, consider a smooth increasing function $\tilde{\rho} \colon [0,1] \to [\varepsilon, 1]$ such that $\tilde{\rho}(0) = \varepsilon$ and $\tilde{\rho}(1) = 1$. Then the disk $\{\tilde{\gamma}^{r,\theta}\}$ is defined as $\tilde{\gamma}^{r,\theta}(t) = p_\theta^{\tilde{\rho}(r),t}$.

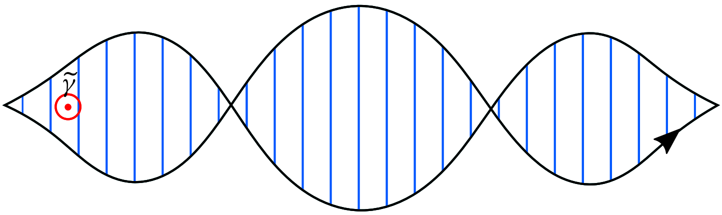

Figure 10: Disk $\mathbb{D}^\theta$ with $\widetilde{\gamma} = \widetilde{\gamma}^{0,\theta}$ in its first lobe in the front Geiges projection.

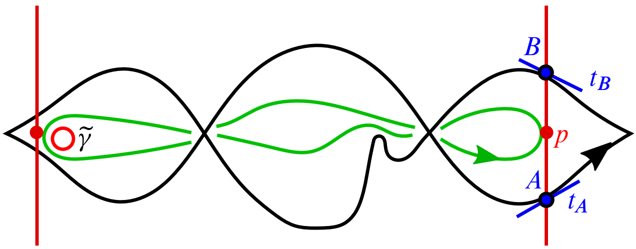

Figure 11: Example where the first condition is fulfilled for a point p but not the other two.

Observe that the tangent space at each point of $\mathbb{D}^\theta$ is given by

$$\langle \partial_x + \epsilon, \delta\partial_z + \beta\partial_y + \alpha\partial_w \rangle, \tag{6}$$

where $\epsilon \in \operatorname{span}\{\partial_y, \partial_z, \partial_w\}$ and $\delta\partial_z + \beta\partial_y + \alpha\partial_w = B_\lambda^\theta - A_\lambda^\theta$ as vectors in $\mathbb{R}^3$. Obviously the embedding $\widetilde{\gamma}^{r,\theta}$ is tangent to $\mathbb{D}^\theta$.

Consider the map $\widetilde{F}\colon [0,2] \times \mathbb{S}^1 \times \mathbb{S}^1 (s,\theta,t) \to \mathbb{S}^3$ defined by

$$\widetilde{F}(s,\theta,t) = \begin{cases} (\widetilde{\gamma}^{s,\theta})'(t) & \text{if } s \in [0,1], \\ F_{s-1}^{\theta,1}(t) & \text{if } s \in [1,2]. \end{cases}$$

Since $\widetilde{F}(0,\theta,-) = (\widetilde{\gamma})'(-)$ and $\widetilde{F}(1,\theta,-) = \partial_w$ for $\theta \in \mathbb{S}^1$, the map $\widetilde{F}$ can be regarded as a map from $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$. The μ–invariant of the area twist is given by the degree mod 2 of this map. In order to compute the degree of this map we compute the preimages of ∂_z. Since $\partial_z \notin \mathcal{D}$ and $F(s,\theta,t) \in \mathcal{D}$ for $s \geq 1$, the equality $F(s,\theta,t) = \partial_z$ is only possible if $s \in [0,1)$, ie we need to study the equation $(\widetilde{\gamma}^{s,\theta})'(t) = \partial_z$ for $s \in [0,1)$. This equality implies that:

(a) $(\widetilde{\gamma}^{s,\theta})'(t) \in \ker(dx)$, ie the front Geiges projection of $(\widetilde{\gamma}^{r,\theta})$ has either a cusp or a vertical tangency at time t. In other words, $\widetilde{\gamma}^{r,\theta}$ becomes tangent to the "ruling"; see equation (6).

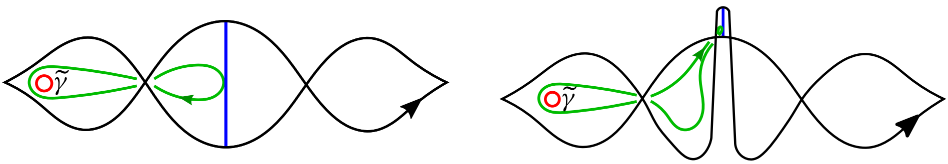

Figure 12: Cases where the three conditions are fulfilled: on the left, the tangent is $-\partial_z$, on the right, the tangent is ∂_z.

(b) $(\widetilde{\gamma}^{s,\theta})'(t) \in \ker(dy)$, ie $y(B^{\theta}_{\lambda}) - y(A^{\theta}_{\lambda}) = 0$. In other words, the area enclosed by the front Geiges projection of γ^{θ} between A^{θ}_{λ} and B^{θ}_{λ} is zero.

(c) $(\widetilde{\gamma}^{s,\theta})'(t) \in \ker(dw)$, ie $w(B^{\theta}_{\lambda}) - w(A^{\theta}_{\lambda}) = 0$. In other words, the slopes of the tangent lines of the front Geiges projection of γ^{θ} at A^{θ}_{λ} and B^{θ}_{λ} are the same.

After these observations, it is clear that the μ–invariant of the area twist loop is 1; see Figure 12. □

6 The area invariant

In this section we define a $\mathbb{Z}_2$–invariant, called *area invariant*, which in the horizontal case coincides with the μ–invariant defined in the previous section; ie for any loop of horizontal embeddings γ^{θ}, the *area invariant* satisfies $\mathrm{Area}(\gamma^{\theta}) = \mu(\gamma^{\theta}, (\gamma^{\theta})') \in \mathbb{Z}_2$. This invariant is defined studying the space $\mathfrak{Hor}(\mathbb{R}^4)$ as a subspace of the space of horizontal immersions $\mathfrak{HorImm}(\mathbb{R}^4)$, which is a flexible space [26]. The key point is that the strict horizontal immersions are encoded by means of the zeros of a "suitable" function, called the *area function*. We keep the same notation as in Section 5.

6.1 Local models of Legendrians

Our goal is to understand generic local models of families of Legendrian immersions. Before studying the different models of Legendrian immersions depending on the codimension, we first define a proper subset of Legendrian immersions which has infinite codimension in the space $\mathfrak{LegImm}(\mathbb{R}^3)$ and, thus, does not intersect a generic finite-dimensional family. A point in a Legendrian is *noninjective* if the preimage of the image has cardinality greater than one. The set of *noninjective images* is the image of the set of noninjective points. A Legendrian is *special* if the cardinality of the noninjective images is infinite. We denote the space of special Legendrian immersions by $\mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$.

If a Legendrian is special, there exists a Weinstein chart (x,y,z) such that the Legendrian has at least two branches on the chart, one of them being the curve $(t,0,0)$ and the other one is $(t, f(t), f'(t))$, where $f(t)$ is a function with infinite zeros such that $t=0$ is a zero and is limit of a sequence of zeros. This implies that f has an infinite-order zero at $t=0$. This is an infinite codimension condition in the space of $\mathfrak{LegImm}(\mathbb{R}^3)$. Therefore, the set $\mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$ is a subset of an infinite codimension stratified submanifold of the space of Legendrian immersions. We can dismiss this set when considering finite-dimensional families, since any finite-dimensional family can be generically perturbed to be disjoint from an infinite codimension subset.

First, consider the space

$$\mathfrak{SLegImm}(\mathbb{R}^3)=$$
$$\{(\gamma,v,t_0,t_1)\in(\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)\setminus\mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3))\times\mathbb{R}^3\times\mathbb{S}^1\times\mathbb{S}^1:\gamma(t_0)=\gamma(t_1)=v\}$$

equipped with the natural inclusion $\mathfrak{SLegImm}(\mathbb{R}^3)\subseteq\mathfrak{LegImm}(\mathbb{R}^3)\times\mathbb{R}^3\times\mathbb{S}^1\times\mathbb{S}^1$. Denote by $\pi\colon\mathfrak{LegImm}(\mathbb{R}^3)\times\mathbb{R}^3\times\mathbb{S}^1\times\mathbb{S}^1\to\mathfrak{LegImm}(\mathbb{R}^3)$ the projection onto the first factor. Observe that $\pi(\mathfrak{SLegImm}(\mathbb{R}^3))=\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)\setminus\mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$.

The projection $\pi_{|\mathfrak{SLegImm}(\mathbb{R}^3)}\colon\mathfrak{SLegImm}(\mathbb{R}^3)\to\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)$ is a local diffeomorphism. We just need to check that the fiber of π is transverse to $\mathfrak{SLegImm}(\mathbb{R}^3)$; ie given $\alpha=(\gamma,v,t_0,t_1)\in\mathfrak{SLegImm}(\mathbb{R}^3)$, then $T_\alpha\mathcal{F}_\gamma\cap T_\alpha\mathfrak{SLegImm}(\mathbb{R}^3)=\{0\}$, where $\mathcal{F}_\gamma=\pi^{-1}(\gamma)$. This is obvious since we are considering Legendrians with isolated noninjective points. Thus, we have the following diagram:

$$\begin{tikzcd}
\mathfrak{SLegImm}(\mathbb{R}^3) \arrow[r, hook] \arrow[d, "\text{loc. diff.}"] & \mathfrak{LegImm}(\mathbb{R}^3)\times\mathbb{R}^3\times\mathbb{S}^1\times\mathbb{S}^1 \arrow[d] \\
\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3) \arrow[r, hook] & \mathfrak{LegImm}(\mathbb{R}^3)
\end{tikzcd}$$

The space $\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)\setminus\mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$ is not a stratified submanifold, while $\mathfrak{SLegImm}(\mathbb{R}^3)$ is. Thus, the first set can be regarded as the image by a local diffeomorphism of a stratified submanifold. We can now compute local models of R–versal deformations (see Arnold, Gusein-Zade and Varchenko [3, page 147]) upstairs and translate them downstairs just by using the local diffeomorphism.

Definition 6.1.1 [3, page 49] Consider a germ of a smooth real function $f\colon\mathbb{R}\to\mathbb{R}$ satisfying $f(0)=0$. Write $f\in\Sigma^{1,\overset{k}{\ldots},1}$ if $f^{(j)}(0)=0$ for $j\le k$. If $f^{(k+1)}(0)\neq 0$, then we write $f\in\Sigma^{1,\overset{k}{\ldots},1,0}\subseteq\Sigma^{1,\overset{k}{\ldots},1}$.

Fix $\gamma(t_0) = \gamma(t_1)$ a noninjective image for a curve $\gamma \in \mathfrak{LegImm}(\mathbb{R}^3)$. By *Weinstein's tubular neighborhood theorem,* we can assume that there is a chart in which the branch of γ through t_0 becomes the zero section in the Weinstein model. So we can locally regard the other branch as a smooth real map with an order 2 zero at $t = 0$.

Let us now introduce some concepts that will be useful in the upcoming discussion.

Remark 6.1.2 [3, pages 147–148] Given $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ a smooth map-germ, we say that a deformation with parameter space $K = \mathbb{R}^k$ is a germ at the origin of a map-germ $F\colon (\mathbb{R}^n \times \mathbb{R}^k, 0) \to \mathbb{R}$ such that $F(x, 0) \equiv f(x)$. We say that a deformation G is R–equivalent to F if

$$G(x,k) \equiv F(h(x,k),k)$$

with $h\colon (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^m, 0)$ a smooth germ with $h(x, 0) \equiv x$. We say that a deformation F of f is R–versal if any other deformation G can be expressed as

$$G(x,\kappa) \equiv F(h(x,\kappa),\varphi(\kappa)) \quad \text{with } h(x,0) \equiv x \text{ and } \varphi(0) = 0.$$

In order to understand the local models that we will produce, we consider the space of Legendrians up to reparametrizations. Note that this is equivalent to saying that the local model representing the noninjective point of the Legendrian that is given by a smooth function has to be considered up to R–versal deformations.

We can then say that a point $(\gamma, v, t_0, t_1) \in \mathfrak{SLegImm}(\mathbb{R}^3)$ belongs to Σ^I if and only if when we express γ as the germ of a smooth real function around the origin near the tangency (t_0, t_1), this germ of function belongs to Σ^I. Note that this is well defined.

Let us introduce some notation. Define the configuration space of two points in $\mathbb{S}^1$ as $\mathrm{Conf}^2(\mathbb{S}^1) = \{(\theta_0, \theta_1) \in \mathbb{S}^1 \times \mathbb{S}^1 : \theta_0 \neq \theta_1\}$. If $e\colon \mathbb{D}^k \to \mathfrak{LegImm}(\mathbb{R}^3)$ is a k–disk of Legendrian immersions, we want to check that *Thom's transversality theorem* can be applied to e in order to obtain a disk which is transverse to Σ^I. By a map transversal to a stratified set we mean a map that is transversal to each stratum. We say that a property P for maps is generic if the set of maps that satisfy the property is a Baire set in the total space of maps. We are going to check that a property of the map $e\colon \mathbb{D}^k \to \mathfrak{LegImm}(\mathbb{R}^3)$ is generic. Thus, by a standard compactness argument, we can further assume that the disk is very small; ie taking any finite covering for which we check that the property is satisfied in a Baire subset, then we are able to check the property in the initial disk since the finite intersection of Baire subsets is Baire.

By the previous discussion, there is an associated map $\overline{e}\colon \mathbb{D}^k \times \mathbb{S}^1 \to \mathbb{R}$ that is constructed by identifying a tubular neighborhood of $e(0)$ with $J^1(\mathbb{S}^1, \mathbb{R})$, where the image of $e(0)$ corresponds to the zero section. Thus, $\overline{e}(z,-)$ is a function whose associated 1–jet represents the Legendrian immersion $e(z)$ in a Weinstein chart. Define $\widehat{e}\colon \mathbb{D}^k \times \mathrm{Conf}^2(\mathbb{S}^2) \to \mathbb{R}$ by $(z,t_0,t_1) \mapsto \overline{e}(z,t_0) - \overline{e}(z,t_1)$, considered as a k–parameter family of real functions. Now fix $r \in \mathbb{Z}^+$, the amount of regularity of the Banach manifold of maps in which we are working; ie we are working in the space of C^r Legendrian immersions. Consider the holonomic lift of $\widehat{e}$ into $J^r(\mathbb{D}^k \times \mathrm{Conf}^2(\mathbb{S}^1), \mathbb{R})$ and denote it by $j^r\widehat{e}\colon \mathbb{D}^k \times \mathrm{Conf}^2(\mathbb{S}^1) \to J^r(\mathbb{D}^k \times \mathrm{Conf}^2(\mathbb{S}^1), \mathbb{R})$. We have a stratified submanifold inside $J^r(\mathbb{D}^k \times \mathrm{Conf}^2(\mathbb{S}^1), \mathbb{R})$ defined by $F \in \Sigma^{\overset{k}{1,\dots,1}}$ with $k<r$, defined by the intersection of hyperplanes

$$F_{\theta_0}^{(j)} = F_{\theta_1}^{(j)} \quad \text{for } j \in \{0,1,\dots,k\}.$$

In particular, the holonomic lift of $\widehat{e}$ intersects $\Sigma^{\overset{k}{1,\dots,1}}$ if there exist $z \in \mathbb{D}^k$ and $(t_0,t_1) \in \mathrm{Conf}^2(\mathbb{S}^1)$ such that $e(z)^{(j)}(t_0) = e(z)^{(j)}(t_1)$ for $j \in \{0,1\dots,k\}$.

Now we further reduce our domain. For this denote by $\mathcal{U}_\Delta$ an open neighborhood of the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$. Thus, $j^r\widehat{e}_{|\mathbb{D}^k \times (\mathcal{U}_\Delta \setminus \Delta)}$ does not intersect Σ^I. So, by compactness, we may assume that there is a finite covering by squares of $\mathrm{Conf}^2(\mathbb{S}^1) \setminus (\mathcal{U}_\Delta \setminus \Delta)$,

$$\bigl\{S_j = [t_0^j-\varepsilon, t_0^j+\varepsilon] \times [t_1^j-\varepsilon, t_1^j+\varepsilon] : [t_0^j-\varepsilon, t_0^j+\varepsilon] \cap [t_0^j-\varepsilon, t_1^j+\varepsilon] = \varnothing,\ j \in \{1,\dots,N\}\bigr\}.$$

Now, we try to perturb $\widehat{e}$ along the domains $\mathbb{D}^k \times S_j$ in such a way that $\widehat{e}|_{\mathbb{D}^k \times S_j}$ becomes transverse to Σ^I. But in that domain perturbations of $\widehat{e}$ are in one-to-one correspondence with perturbations of $\overline{e}$. Therefore, we can apply the standard *Thom's transversality theorem* (see [11, Theorem 2.3.2]) to conclude that the space of perturbations transverse to the stratified submanifold is a Baire set.

Thus, we have proven the following statement:

Lemma 6.1.3 *Assume $0 \le k \le r-2$. Let $e\colon \mathbb{D}^k \to \mathfrak{LegImm}(\mathbb{R}^3) \setminus \mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$ be a disk. Then there is a C^r–perturbation $\widetilde{e}\colon \mathbb{D}^k \to \mathfrak{LegImm}(\mathbb{R}^3) \setminus \mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$ of e such that*

(i) *$\widetilde{e}$ is transverse to $\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$, and*

(ii) *the lift of $\widetilde{e}$ to $\mathfrak{SLegImm}(\mathbb{R}^3)$ is transverse to each Σ^I with $|I| \le r$.*

From the previous discussion we conclude that the classification of generic local models of self-intersections of Legendrians is tantamount to classifying local germs of smooth

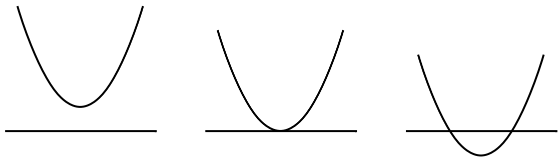

Figure 13: Local model in codimension 1 representing a simple crossing.

real maps. In fact, we obtain a systematic method for classifying local models of generic k–dimensional disks of Legendrians by taking into account Arnold's classification of local germs of smooth functions; see [3]. In order to be precise we need to work on $\mathfrak{SLegImm}(\mathbb{R}^3)$.

- **Codimension 0** A generic germ of a smooth function vanishing at the origin has generically nonzero first derivative at the origin. This just reflects the fact that Legendrian immersions are generically embeddings.

- **Codimension 1** A generic 1–parameter disk of smooth maps at the origin intersects $\Sigma^{1,0}$ at isolated points and $\Sigma^{1,1}$ in the empty set. By an application of *Hadamard's lemma* (see Milnor [24, Lemma 2.1]) we can assume that the intersection with this stratum takes place at the germ at the origin of the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. We have that an R–versal deformation of x^2 at 0 is given by $x^2 + a$, where $a \in \mathcal{O}p(\{0\})$; see [3, page 151] for further details. This corresponds to the local model represented in Figure 13.

- **Codimension 2** A generic 2–parameter family $\mathbb{D}$ of smooth maps at the origin intersects Σ^1 at curves and $\Sigma^{1,1,0}$ at isolated points. Fixing a point in $\Sigma^{1,1,0} \cap \mathbb{D}$, we have that the crossing can be identified with the germ at the origin of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $t \mapsto t^3$. We know that an R–versal deformation of t^3 at 0 is given by $\beta(t,a,b) = t^3 + at + b$, where $(a,b) \in \mathbb{D}^2(\varepsilon)$ for $\varepsilon > 0$ small enough; see [3, page 151] for details. Realize that the intersection of the disk with the stratum Σ^1 is given by the curve (seen as a curve of germs of functions at the origin) determined by the equations

$$\beta(t,a,b) = 0 \iff t^3 + at + b = 0,$$

$$\frac{d}{dt}\beta(t,a,b) = 0 \iff 3t^2 + a = 0,$$

with solutions $t = \pm\sqrt{-a/3}$ and the parameters satisfying the relationship

$$a^3(3^{-1/2} - 3^{-3/2})^2 + b^2 = 0,$$

which describes a cuspidal curve. This local model is represented in Figure 14.

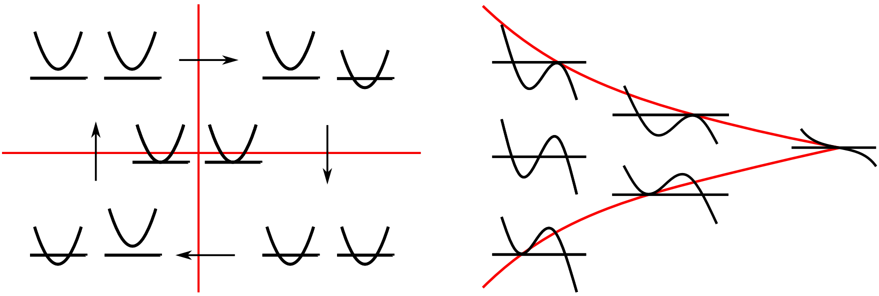

Figure 14: Local models in codimension 2: two simultaneous simple tangencies, left, and cuspidal curve, right.

- **Codimension 3** A generic 3–parameter family $\mathbb{D}^3$ intersects Σ^1 at surfaces, $\Sigma^{1,1}$ at curves and $\Sigma^{1,1,1,0}$ at isolated points. A point in $\Sigma^{1,1,1,0} \cap \mathbb{D}^3$ can be identified with the germ at the origin of the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^4$. An R–versal deformation of x^4 at 0 is given by $x^4 + ax^2 + bx + c$, where $(a,b,c) \in \mathbb{D}^3(\varepsilon)$; see [3, page 151] for further details. This gives rise to the local model represented in Figure 15; see [3, pages 47–48].

So as to understand deformations around points of $\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3) \setminus \mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$, we project the models from $\mathfrak{LegImm}(\mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1$ to $\mathfrak{LegImm}(\mathbb{R}^3)$. Since the projection is a local diffeomorphism, we get the same local models, but different branches upstairs may come together downstairs. So we have the same models but with the addition of cartesian product models that show up at points $\gamma \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ that lie at the intersection of 2 or more branches. For instance, in codimension 2 we get a new model represented in Figure 14.

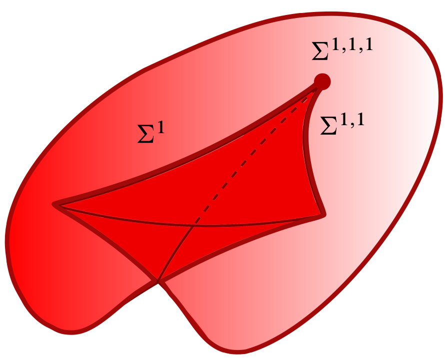

Figure 15: Local model in codimension 3.

We have shown that any continuous disk in the space of Legendrian immersions can be perturbed to be in generic position. However, in this article we usually have a disk of horizontal immersions and we want to check that a perturbation of this disk renders a generic perturbation of its Geiges projection inside the space of Legendrian immersions.

Lemma 6.1.4 *Assume that $0 \leq k \leq r-2$. Let $e\colon \mathbb{D}^k \to \mathfrak{HorImm}(\mathbb{R}^4)$ be a disk such that $e_G = \pi_G \circ e\colon \mathbb{D}^k \to \mathfrak{LegImm}(\mathbb{R}^3) \setminus \mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^4)$. Then there is a C^r–perturbation $\tilde{e}\colon \mathbb{D}^k \to \mathfrak{HorImm}(\mathbb{R}^4)$ of e such that*

(i) $\tilde{e}_G\colon \mathbb{D}^k \to \mathfrak{LegImm}(\mathbb{R}^3) \setminus \mathcal{S}p\mathfrak{LegImm}(\mathbb{R}^3)$,

(ii) *$\tilde{e}_G$ is transverse to $\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)$, and*

(iii) *the lift of $\tilde{e}_G$ to $\mathfrak{SLegImm}(\mathbb{R}^3)$ is transverse to each Σ^I with $|I| \leq r$.*

Proof It follows directly from Lemmas 6.1.3 and 7.2.1. This last lemma will be proven later on and it allows us to lift multiparameter families of Legendrian immersions. □

For later use, we want to check that:

Lemma 6.1.5 *$\Sigma^{1,0} \subseteq \mathfrak{LegImm}(\mathbb{R}^3)$ is a locally closed codimension-1 immersed Banach submanifold.*

Note that the lift of $\Sigma^{1,0}$ to $\mathfrak{SLegImm}(\mathbb{R}^3)$ is, in fact, an embedded submanifold. However, it does not help our purposes.

Proof We already noted that $\mathfrak{LegImm}(\mathbb{R}^3)$ is an immersed Banach submanifold of $\mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$; see Remark 3.1.3. In order to check that $\Sigma^{1,0}$ is a codimension-1 locally closed immersed submanifold we just need to define an atlas of $\mathfrak{LegImm}(\mathbb{R}^3)$ with charts satisfying that the pullback by the chart map of the intersection of $\Sigma^{1,0}$ with the chart is a closed codimension-1 Banach subspace.

For a point $\gamma \in \Sigma^{1,0}$, the exponential map is a local diffeomorphism because its differential at γ is the identity; see [23, Theorem 5.2, page 15]. By using a Weinstein chart around γ we just define the exponential by the formula (1). We can simply check that given a vector $(f, g) \in \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1, \mathbb{R}) \times \mathcal{M}\mathfrak{aps}^r(\mathbb{S}^1, \mathbb{R})$, the condition for it to be tangent to $\Sigma^{1,0}$ reads as $f(\theta_0) = 0$, where θ_0 is the self-intersection time. This is obviously a closed codimension-1 subspace; this concludes the proof. □

6.2 The area invariant

Fix a loop γ^θ in $\mathfrak{Hor}(\mathbb{R}^4)$ with $\mathrm{Rot}_L(\gamma^\theta)=0$. This means that γ^θ is trivial *as a loop of horizontal immersions*. Thus, there exists a disk $\mathbb{D}=\{\widetilde{\gamma}^z\}$ in $\mathfrak{HorImm}(\mathbb{R}^4)$ such that $\widetilde{\gamma}^{e^{2\pi i\theta}}=\gamma^\theta$. We want to study the obstruction for that disk to belong to $\mathfrak{Hor}(\mathbb{R}^4)$.

By Lemma 6.1.4 we can assume that $\mathbb{D}_G$ is in generic position, ie $\mathbb{D}_G$ contains curves (1–dimensional connected manifolds), smooth away from the cusps, of strict Legendrian immersions which may intersect in $\mathrm{Int}(\mathbb{D}_G)$. Moreover, we may assume that in $\partial\mathbb{D}_G\cap\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ we have only elements in $\Sigma^{1,0}$, hence the number of strict Legendrian immersions in $\partial\mathbb{D}_G$ is even. See Figure 16.

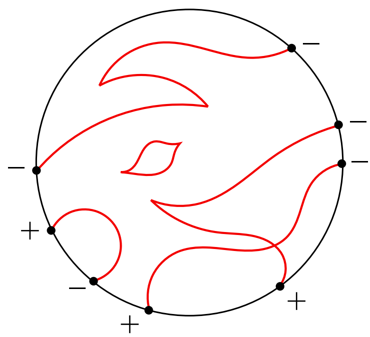

Figure 16: Generic disk $\mathbb{D}_G$. The red curves represent strict Legendrian immersions.

Let $\widetilde{\gamma}^z_G\in\mathbb{D}_G$ be a strict Legendrian immersion. Write $\widetilde{\gamma}^z_G(t)=(x(t),z(t),w(t))^\mathsf{T}$ and assume that $p=\widetilde{\gamma}^z_G(t_0)=\widetilde{\gamma}^z_G(t_1)$ is a transverse self-intersection point, ie of type $\Sigma^{1,0}$, with $t_0,t_1\in\mathbb{S}^1$. We define

$$\varepsilon_A(\widetilde{\gamma}^z_G,p)=\delta\int_{t_0}^{t_1}z(t)x'(t)\,dt,$$

where the factor $\delta\in\{\pm1\}$ corresponds to the following sign convention: *we always integrate along a segment that starts at the lower branch of the curve in the front projection and follow the orientation of the curve.*[7] Notice that this convention is well-defined since we are working with self-intersections of type $\Sigma^{1,0}$. If $\widetilde{\gamma}^z_G$ has only one tangency point, then we write $\varepsilon_A(\widetilde{\gamma}^z_G)=\varepsilon_A(\widetilde{\gamma}^z_G,p)$. We call ε_A the *area function*. The sign of ε_A over a curve of strict Legendrian immersions is reversed when it passes through a cusp (a point of type $\Sigma^{1,1,0}$). Indeed, note that when crossing these points,

[7] The tangent space of the two branches in $\mathbb{R}^3$ at the intersection point $p\in\mathbb{R}^3$ defines a framing of ξ_p. The choice just ensures that the orientation of the framing is always positive.

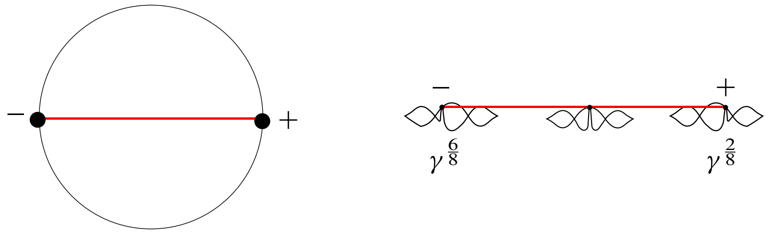

Figure 17: Geiges projection of the standard capping disk through horizontal immersions of the area twist loop. Left: capping disk in $\mathfrak{LegImm}(\mathbb{R}^3)$. Right: curve of strict Legendrian immersions in the front Geiges projection.

the lower branch becomes the upper branch and vice versa, thus producing a change of sign in the area function. In particular, the absolute value of the area function is well-defined over a cusp. Note that, by genericity, we may assume that ε_A is nonzero over the cusp points. Hence, the area function is continuous over the curves of strict Legendrian immersions except at the cusp points. Moreover, if $\widetilde{\gamma}^z_G$ is a strict Legendrian immersion, then $\widetilde{\gamma}^z$ is a horizontal embedding if and only if $\varepsilon_A(\widetilde{\gamma}^z_G, \cdot)$ is nonzero over all tangency points of $\widetilde{\gamma}^z_G$. Thus, to study the obstruction for $\mathbb{D}$ to belong to $\mathfrak{Hor}(\mathbb{R}^4)$ we need to study the zero set of the area function. Observe that, since $\partial\mathbb{D}_G$ is the Geiges projection of $\gamma^\theta \in \mathfrak{Hor}(\mathbb{R}^4)$, we obtain that $\varepsilon_A(\widetilde{\gamma}^{e^{2\pi i\theta}}_G)$ is nonzero. We define the sign of $\widetilde{\gamma}^z_G$ as the sign of $\varepsilon_A(\widetilde{\gamma}^z_G)$. Denote by $\mathcal{P}^+(\mathbb{D}_G)$ the set of positive points in $\partial\mathbb{D}_G$. The following lemma is just a simple exercise:

Lemma 6.2.1 *Let $C \subseteq \mathbb{D}_G$ be a curve of strict Legendrian immersions. Assume that one of the following three conditions holds:*

(i) *The boundary points of C have the same sign and the number of cusps in C is odd.*

(ii) *The boundary points of C have different signs and the number of cusps in C is even.*

(iii) *The boundary of C is empty and the number of cusps in C is odd.*

Then $\mathbb{D}$ contains at least one strict horizontal immersion.

Definition 6.2.2 Let $\mathcal{C}\mathfrak{usp}(\mathbb{D}_G)$ be the set of cusps of $\mathbb{D}_G$. The *area* of $\mathbb{D}$ is

$$\mathrm{Area}(\mathbb{D}) = (\#\mathcal{P}^+(\mathbb{D}_G) + \#\mathcal{C}\mathfrak{usp}(\mathbb{D}_G)) \mod 2.$$

Lemma 6.2.3 *Assume that* $\mathrm{Area}(\mathbb{D}) = 1$. *Then $\mathbb{D}$ contains at least one strict horizontal immersion.*

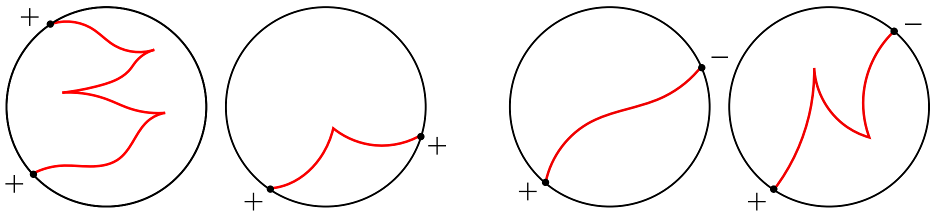

Figure 18: Some examples of curves of strict Legendrian immersions that contain the Geiges projection of a strict horizontal immersion. Left: curves with the same sign in the boundary and an odd number of cusps. Right: curves with different signs in the boundary and an even number of cusps.

Proof We proceed by induction over the number of strict Legendrian immersions in the boundary of $\mathbb{D}_G$. Observe that, since $\mathrm{Area}(\mathbb{D}) = 1$, there has to be a curve of strict Legendrian immersions in $\mathbb{D}_G$.

0 points in the boundary $\mathrm{Area}(\mathbb{D}) = 1$ implies that $\#\mathcal{C}\mathfrak{usp}(\mathbb{D}_G)$ must be odd. Hence, there exists a closed curve with an odd number of cusps. By Lemma 6.2.1(iii), we conclude this case.

2 points in the boundary We have two cases:

(a) Assume that the two points in the boundary have the same sign. Further assume there exists a curve of strict Legendrian immersions with the same sign in the boundary and an odd number of cusps. Use Lemma 6.2.1(i) to conclude. Otherwise, there is a closed curve of strict Legendrian immersions with an odd number of cusps, and by Lemma 6.2.1(iii) we conclude this case.

(b) Assume that there exist two points in the boundary which have different signs. If the curve of strict Legendrian immersions that connects them has an even number of cusps, we conclude by Lemma 6.2.1(ii). Otherwise, there must be a closed curve with an odd number of cusps. By Lemma 6.2.1(iii) we conclude this case.

$2n + 2$ points in the boundary We have two cases:

(a) Assume that the points in the boundary connected by a curve of strict Legendrian immersions have the same sign. Further assume there exists a curve with the same sign in the boundary and an odd number of cusps. We conclude by Lemma 6.2.1(i). Otherwise, there exists a closed curve with an odd number of cusps by Lemma 6.2.1(iii) and we conclude this case.

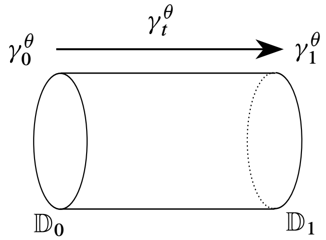

Figure 19: Solid cylinder in $\mathfrak{HorImm}(\mathbb{R}^4)$ connecting $\mathbb{D}_0$ and $\mathbb{D}_1$.

(b) Assume that we have a curve C with opposite signs in the boundary. Then, if the number of cusps is even, we apply Lemma 6.2.1(ii) to conclude. On the other hand, if the number of cusps is odd, consider a new finite set of curves given by the old one minus C, and apply the induction hypothesis. $\square$

We want to show that the assignment $\gamma^\theta \mapsto \mathrm{Area}(\mathbb{D})$ defines a homotopy invariant. Thus, let γ_0^θ and γ_1^θ be two homotopic loops of horizontal embeddings with rotation number zero. For that, choose two disks $\mathbb{D}_0 = \{\tilde{\gamma}_0^z\}$ and $\mathbb{D}_1 = \{\tilde{\gamma}_1^z\}$ in $\mathfrak{HorImm}(\mathbb{R}^4)$ such that $\tilde{\gamma}_0^{e^{2\pi i\theta}} = \gamma_0^\theta$ and $\tilde{\gamma}_1^{e^{2\pi i\theta}} = \gamma_1^\theta$. We need to study the relationship between the configuration of curves of strict Legendrian immersions in $(\mathbb{D}_0)_G$ and $(\mathbb{D}_1)_G$. Denote by γ_t^θ the homotopy between γ_0^θ and γ_1^θ. Since $\pi_2(\mathfrak{HorImm}(\mathbb{R}^4)) = 0$ we can assume that we have a solid cylinder $\mathcal{C}yl = [0,1] \times \mathbb{D}$ in $\mathfrak{HorImm}(\mathbb{R}^4)$ such that $\{t\} \times \partial\mathbb{D}$ coincides with γ_t^θ, $\mathbb{D}_0 = \{0\} \times \mathbb{D}$ and $\mathbb{D}_1 = \{1\} \times \mathbb{D}$. Let $\mathcal{C}yl_G = [0,1] \times \mathbb{D}_G$ be the Geiges projection of the cylinder. Denote by $\tilde{L} = [0,1] \times \partial\mathbb{D}$ the lateral boundary of the cylinder and $L = \pi_G(\tilde{L})$ its Geiges projection. In particular, every point in $L = [0,1] \times \partial\mathbb{D}_G$ is a Legendrian embedding. By genericity, we can assume that every Legendrian in $\mathcal{C}yl_G \subseteq \mathfrak{LegImm}(\mathbb{R}^3)$ is the projection of an element in $\mathfrak{SLegImm}(\mathbb{R}^3)$ via the canonical projection

$$\pi\colon \mathfrak{LegImm}(\mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathfrak{LegImm}(\mathbb{R}^3).$$

In addition, we can assume that $S_0 := \pi(\Sigma^{1,1,1,0}) \cap \mathcal{C}yl_G$ is a set of isolated points, $S_1 := \pi(\Sigma^{1,1,0}) \cap \mathcal{C}yl_G$ a set of immersed curves and $S_2 := \pi(\Sigma^{1,0}) \cap \mathcal{C}yl_G$ a set of immersed surfaces. In addition, by genericity, we have that different branches of S_1 do not intersect, different branches of S_1 and S_2 intersect in isolated points $S_{12} = S_1 \cap S_2$, and different branches of S_2 intersect in a set of embedded curves S_{22}. We can also assume that

(i) $S_0 \cap L = \varnothing$,

(ii) $S_{1L} := S_1 \cap L$ is a finite set of points,

(iii) $S_{2L} := S_2 \cap L$ is a set of embedded curves.

Assume that the "height function" for the cylinder $h\colon \mathcal{C}yl_G = [0,1] \times \mathbb{D}_G \to [0,1]$ is a Morse function for S_1, S_{2L} and S_{22}. With these assumptions, the 1–parameter family of slices $\{t\} \times \mathbb{D}_G$ for $t \in [0,1]$ induces a movie of curves defined by the intersections of $\pi(\Sigma^{1,0})$ with each slice. Let us discuss the topology of the configuration of $\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ at each slice; see Figure 20. The changes of the topology of the curves happen at specific times due to the following phenomena:

- **Points in S_{12}** If such a point lies on a slice $\{t\} \times \mathbb{D}_G$ the movie around that time corresponds to the crossing of a curve with a cusp with another immersed curve; see Figure 20, first elementary change.

- **Points in S_{22}** This corresponds to the crossing of two immersed curves; see Figure 20, second elementary change.

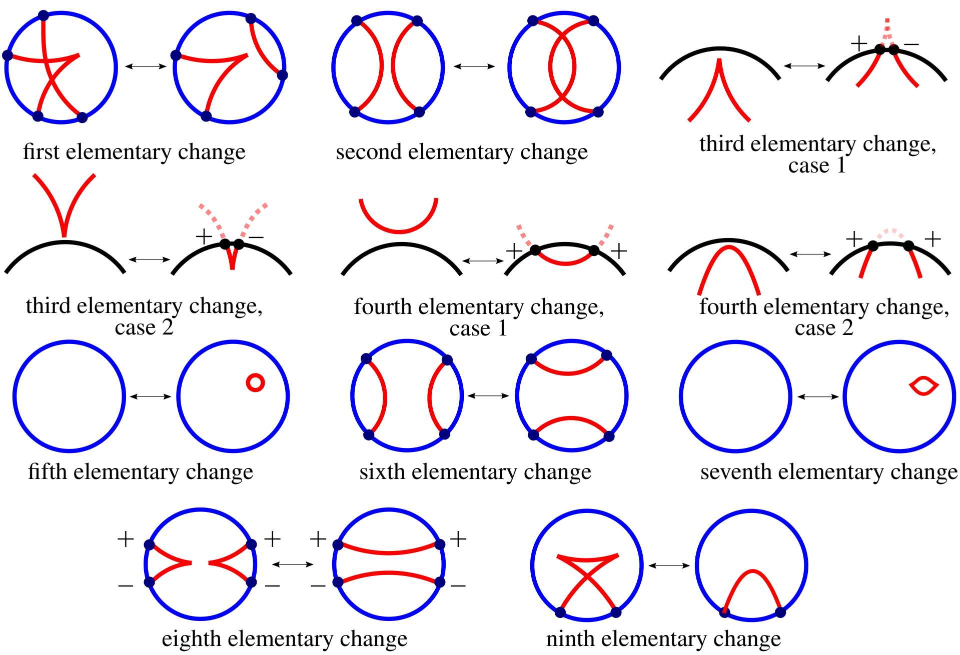

Figure 20: List of elementary changes. The boundary of the cylinder is represented in black. Blue circles represent local neighborhoods of the slices. The signs in the boundary points are defined by the area function. Of course the same pictures with signs reversed could happen.

- **Points in S_{1L}** This corresponds to the crossing of a cusp with the boundary; see Figure 20, third elementary change, cases 1 and 2.
- **Appearance (disappearance) of points in $S_{2L} \cap (\{t\} \times \mathbb{D}_G)$ for some $t \in [0,1]$** This generically corresponds to the appearance (or disappearance) of an immersed curve during the 1–parameter family. The time $t \in [0,1]$ represents the time where this curve appears in the movie of slices (fourth elementary change, cases 1 and 2).
- **Critical points of h for S_2** A critical point in S_2 for h represents either the appearance/disappearance of a closed embedded curve, case of a minimum/maximum; or an index-1 surgery in the set of curves, case of an index-1 critical point. See Figure 20, fifth and sixth elementary changes.
- **Critical points of h for S_1** This corresponds to the appearance/disappearance of an embedded curve (respectively minimum/maximum). What is special is that the curve has a double cusp. See Figure 20, seventh elementary change and eighth elementary change.
- **Points in S_0** This corresponds to the codimension-3 swallowtail singularity; see Figure 20, ninth elementary change.

Corollary 6.2.4 *Let γ_0^θ and γ_1^θ be two homotopic loops of horizontal embeddings with rotation number zero. Let $\mathbb{D}_0 = \{\tilde{\gamma}_0^z\}$ and $\mathbb{D}_1 = \{\tilde{\gamma}_1^z\}$ be disks in $\mathfrak{HorImm}(\mathbb{R}^4)$ such that $\tilde{\gamma}_0^{e^{2\pi i\theta}} = \gamma_0^\theta$ and $\tilde{\gamma}_1^{e^{2\pi i\theta}} = \gamma_1^\theta$. Then*

$$\operatorname{Area}(\mathbb{D}_0) = \operatorname{Area}(\mathbb{D}_1).$$

Proof By the previous discussion, any generic homotopy between two disks is a composition of the nine elementary changes: they come with suitable signs in order to be geometrically realizable; see Figure 20. Since these nine elementary changes preserve the area, the result follows. □

We claim:

Theorem 6.2.5 *Let γ^θ be a loop in $\mathfrak{Hor}(\mathbb{R}^4)$ such that $\operatorname{Rot}_L(\gamma^\theta) = 0$. Let $\mathbb{D} = \{\tilde{\gamma}^z\}$ be some disk in $\mathfrak{HorImm}(\mathbb{R}^4)$ such that $\tilde{\gamma}^{e^{2\pi i\theta}} = \gamma^\theta$. The assignment*

$$\gamma^\theta \mapsto \operatorname{Area}(\gamma^\theta) := \operatorname{Area}(\mathbb{D})$$

*defines a homotopy $\mathbb{Z}_2$–invariant, called the **area invariant**. Alternatively, we define the **area invariant** as the number of strict horizontal immersions in $\mathbb{D}$ mod 2. The two definitions coincide.*

Proof We are left to check that the area invariant is computed by the number of strict horizontal immersions mod 2. We proceed by induction on the number of zeros of the area function over the strict immersed curves. If there are no zeros, then Lemma 6.2.1 implies that the area invariant is zero. Assume that it is true for $k = 0, \dots, n$. Then we want to study a configuration with $n+1$ zeros. If all of them lie in the same curve, we have that Lemma 6.2.1 implies the result. If not, the curves can be divided into two subsets such that they have $n \geq k_1 > 0$ and $n \geq k_2 > 0$ zeros each with $n+1 = k_1 + k_2$. By induction, we conclude the proof. $\square$

Remark 6.2.6 Let us check that the area invariant coincides with the μ–invariant defined in the previous section in the horizontal case; ie for any loop of horizontal embeddings γ^θ, the *area invariant* satisfies

$$\mathrm{Area}(\gamma^\theta) = \mu(\gamma^\theta, (\gamma^\theta)') \in \mathbb{Z}_2.$$

The μ–invariant describes the $\mathbb{Z}_2$ factor given by the *kernel* of the homomorphism

$$f\colon \pi_1(\mathfrak{F}\mathfrak{Hor}(\mathbb{R}^4)) \to \pi_1(\widehat{\mathfrak{F}\mathfrak{Hor}}(\mathbb{R}^4)).$$

Consider the natural inclusion $i\colon \mathfrak{Hor}(\mathbb{R}^4) \hookrightarrow \mathfrak{F}\mathfrak{Hor}(\mathbb{R}^4)$ and let

$$\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_f := \pi_1(i)^{-1}(\ker(f)).$$

In order to prove that the area invariant computes the μ–invariant in the formal case, we must check that the next diagram commutes:

$$\begin{tikzcd}
\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_f \arrow[r, "\pi_1(i)"] \arrow[dr, "\mathrm{Area}"'] & \ker(f) \arrow[d, "\mu"] \\
& \mathbb{Z}_2
\end{tikzcd}$$

Let γ^θ be a loop of horizontal embeddings which lies in $\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_f$. Let $\mathbb{D} = \{\tilde{\gamma}^z\}$ be a disk in $\mathfrak{Hor}\mathfrak{Imm}(\mathbb{R}^4)$ such that $\tilde{\gamma}^{e^{2\pi i\theta}} = \gamma^\theta$. Assume that $\mathbb{D}$ has exactly n strict horizontal immersions $\beta^1, \dots, \beta^n$. Let $\mathbb{D}_G$ be the Geiges projection of $\mathbb{D}$; note that by genericity, we can assume that $\beta^1_G, \dots, \beta^n_G$ belong to $\Sigma^{1,0}$. Because of Definition 5.3.2, any small disk whose diagrams of strict Legendrians is as in Figure 17 has as boundary circle an area twist. Then collapsing the disk into small neighborhoods of the curves β^j_G, as we show in Figure 21, γ^θ is homotopic to a concatenation of n area twist loops. Observe that the area invariant and the μ–invariant of an area twist loop is one. Thus, $\mathrm{Area}(\gamma^\theta) = \mu(\gamma^\theta, (\gamma^\theta)') = n \mod 2$; see Lemma 5.3.3. This concludes the argument.

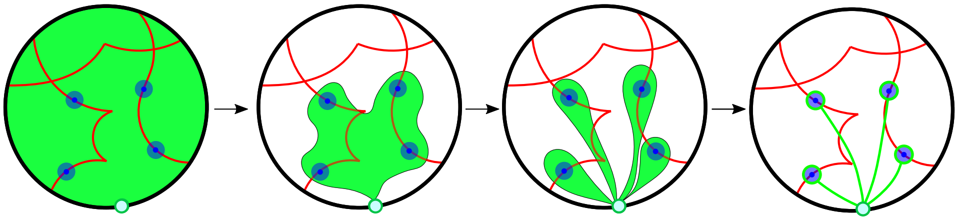

Figure 21: Homotopy between γ^θ and the concatenation of n area twist loops for the case $n=4$ represented in $\mathbb{D}_G$. We have drawn the zeros of the area function in blue. The concatenation of 4 area twist loops is represented by the green curve in the last step.

7 h–principle at π_1–level for horizontal embeddings

We keep the same notation as in Sections 5 and 6.

7.1 The main theorem

Theorem 7.1.1 *The inclusion $\mathfrak{Hor}(\mathbb{R}^4) \hookrightarrow \mathfrak{FHor}(\mathbb{R}^4)$ induces an isomorphism of fundamental groups.*

We are assuming a basepoint in the previous inclusion. We claim that the isomorphism works for any choice of basepoint, ie for all the connected components: they are given by the rotation number of the horizontal embedding; see Remark 5.1.6. To prove this result we proceed as follows. Consider the homomorphism

$$\mathrm{Rot}_L\colon \pi_1(\mathfrak{Hor}(\mathbb{R}^4)) \to \mathbb{Z}, \quad [\gamma^\theta] \mapsto \mathrm{Rot}_L(\gamma^\theta),$$

and define $\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0$ as the *kernel* of Rot_L. The homomorphism Rot_L is surjective. Indeed, it is enough to check that for any horizontal embedding $\beta \in \mathfrak{Hor}(\mathbb{R}^4)$ and any integer $k \in \mathbb{Z}$, there exists a loop of horizontal embeddings β^θ, based at β, with $\mathrm{Rot}_L(\beta^\theta) = k$. Define β^θ_G as the Legendrian immersion obtained by a rotation of angle $2\pi k\theta$ of β_G in the Lagrangian projection. The lift β^θ of β^θ_G to $\mathfrak{Hor}(\mathbb{R}^4)$ satisfies the desired property.

It follows from Theorem 6.2.5 that the area invariant defines a homomorphism

$$\mathrm{Area}\colon \pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0 \to \mathbb{Z}_2.$$

Moreover, we will prove the following:

Theorem 7.1.2 *The area invariant defines an isomorphism*

$$\text{Area}\colon \pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0 \to \mathbb{Z}_2. \tag{7}$$

In particular, $\pi_1(\mathfrak{Hor}(\mathbb{R}^4)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Moreover, the techniques developed in this section will allow us to prove, at the end of Section 7.2:

Lemma 7.1.3 *The homomorphism* $\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0 \to \pi_1(\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4))$, *induced by the inclusion* $\mathfrak{Hor}(\mathbb{R}^4) \hookrightarrow \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$, *is surjective.*

Thus, since the two elements in $\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0$ (the trivial loop and the area twist loop) are trivial as loops of smooth embeddings and the space $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ is path-connected, we conclude the following:

Corollary 7.1.4 *The space* $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$ *is simply connected.*

This is a "Legendrian" proof of a well-known result; see [5, Proposition 3.9(4)]. We expect to be able to compute higher-rank homotopy groups of this space generalizing the discussion below, since we know the space of Legendrian immersions behaves nicely, ie the crossing types correspond to tangencies of smooth functions with the horizontal line. It is known that these tangencies can be assumed to be either Morse or birth–death; see [21]. For higher-dimensional families this is a rather nontrivial result. In a forthcoming project, we will use this fact to collect nontrivial information about the high-rank homotopy groups of the space $\mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$.

It is clear that Theorem 7.1.2 and Corollary 7.1.4, together with the exact sequence stated in Lemma 5.2.1, imply Theorem 7.1.1.

The purpose of this section is to reduce the proof of Theorem 7.1.2 to a technical cancellation theorem (see Proposition 8.1.22), which is of independent interest and is proven in Section 8. This section presents several results needed in order to reduce and to develop the proof of the cancellation theorem.

An overview of what follows is:

- In Section 7.2, we introduce several results: Lemma 7.2.1 states that the area function over a connected component (see below for a precise definition of *connected component*) can be easily deformed without affecting the other connected components;

then we focus on deforming the immersed Legendrians produced by Geiges projection. The main result states that the only obstruction to having an h–principle for embeddings of Legendrians lies in the fact that the double stabilization of a Legendrian (the main tool to approximate a smooth curve) changes the embedding class of the Legendrian. The key remark (Lemma 7.2.4) shows that this is not the case in the horizontal case. We give a multiparametric proof of the fact that (double) stabilizations allow to approximate any smooth curve: Theorem 7.2.5. To complete a full h–principle, what is left is to deal with the positions of the cusp points. This is done just in 1–parameter families, which is enough for the purposes of this article. The proof contained in [6] controls those cups in higher-dimensional families.

- In Section 7.3, we start studying the injectivity of the morphism (7). This will be the task of the remaining part of the article. For that we need to show that the parity of the zeros (area invariant) of a capping disk on $\mathfrak{HorImm}(\mathbb{R}^4)$ for a loop on $\mathfrak{Hor}(\mathbb{R}^4)$ fully recovers the homotopy class of the loop. Thus, we need to cancel the zeros by pairs. In this subsection, we assume that the two zeros to be canceled do lie in the same connected component. In that case, a careful use of Lemma 7.2.1 is all we need to conclude.

- In Section 8.1, we explain how to change the diagram of the capping disk associated to an element in $\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0$ in order to place two zeros in the same connected component. We prove in these pages a previous key result (Proposition 8.1.22) explaining how to deform a path connecting the two zeros through curves that, except for a finite number of points, are Legendrian embeddings in Geiges projection. The idea is to build a map of the square into $\mathfrak{Hor}(\mathbb{R}^4)$. The bottom side of the square is the initial path. The other three edges correspond to a "much better" path that connects the two zeros by a curve of strict immersions; ie we have found a curve making them live in the same connected component. Finally, we explain how to exploit the success in order to use the previous square to change the diagram of curves in a suitable way (creating an index-1 surgery in the diagram).

- In Section 8.2, we take a suitable path joining the two zeros, then the index-1 surgery previously built allows us to *connect* the connected components of the two different zeros.

7.2 Creating flexibility

Lemma 7.2.1 *Let K be a compact parameter space and $\gamma^s \in \mathfrak{HorImm}([0,1],\mathbb{R}^4)$ for $s \in K$. Let $A\colon K \to \mathbb{R}$ be a continuous map. Let $p^s \in [0,1]$ for $s \in K$ be any* *continuous family of regular points for the front Geiges projection of* γ^s*, and* $\varepsilon > 0$ *sufficiently small. Then there exists a* 1*–parameter family* $\gamma^{s,u} \in \mathfrak{HorImm}([0,1],\mathbb{R}^4)$*, with* $u \in [0,1]$*, satisfying:*

(i) $\gamma^{s,0} = \gamma^s$.

(ii) *The Geiges projection of* γ^s *and* $\gamma^{s,u}$ *are* C^0*–close and*[8]

$$\gamma^{s,u} - \gamma^s = \begin{cases} (0,0,0,0) & \text{in } [0, p^s - \varepsilon], \\ (0, uA(s), 0, 0) & \text{in } [p^s + \varepsilon, 1]. \end{cases}$$

Moreover, if γ^s_G *is an embedding, then* $\gamma^{s,u}_G$ *is also an embedding.*

Proof Define $\gamma^{s,u} = \gamma^s$ in $[0, p^s - \varepsilon]$ and $\gamma^{s,u}_G = \gamma^s_G$ in $[p^s + \varepsilon, 1]$. It is enough to define the front Geiges projection of $\gamma^{s,u}$ in $[p^s - \varepsilon, p^s + \varepsilon]$. Let $N \in \mathbb{Z}^+$, consider the equispaced partition $\{a^s_0 = p^s - \varepsilon < a^s_1 < \dots < a^s_{N+1} = p^s + \varepsilon\}$ of $[p^s - \varepsilon, p^s + \varepsilon]$. Define $\gamma^{s,u}_{FG}$ in $[p^s - \varepsilon, p^s + \varepsilon]$ just by adding a Reidemeister I move of area $uA(s)/N$ to γ^s_{FG} at each point $\gamma^s_{FG}(a^s_i)$ for $i = 1, \dots, N$; see Figure 22. The C^0–closeness follows from choosing N large enough. □

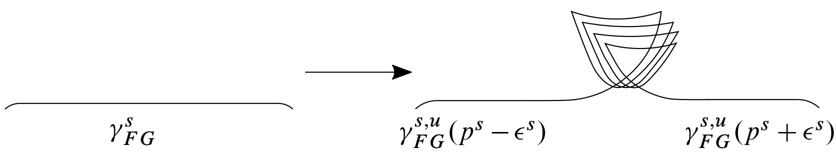

Figure 22: Construction of $\gamma^{s,u}$ via its front Geiges projection.

Remark 7.2.2 Lemma 7.2.1 also applies to horizontal loops, provided that the zero total area condition is preserved. We do it as follows: let $\gamma \in \mathfrak{HorImm}(\mathbb{R}^4)$ be a horizontal immersion, $p, n \in \mathbb{S}^1$ different regular points of the front Geiges projection of γ, and $A \in \mathbb{R}$. We write

$$\gamma \# \delta_p(A) \# \delta_n(-A) \in \mathfrak{HorImm}(\mathbb{R}^4)$$

to denote the horizontal immersions obtained from γ when, using Lemma 7.2.1, we *add area* A near the point $\gamma(p)$ and $-A$ near the point $\gamma(n)$. Moreover, if $\beta \in \mathfrak{LegImm}(\mathbb{R}^3)$ satisfies the zero total area condition, ie $\beta = \gamma_G$ for $\gamma \in \mathfrak{HorImm}(\mathbb{R}^4)$, and $p, n \in \mathbb{S}^1$ are regular points of the front projection of β, we write

$$\beta \# \delta_p(A) \# \delta_n(-A) \in \mathfrak{LegImm}(\mathbb{R}^3)$$

to denote the Geiges projection of $\gamma \# \delta_p(A) \# \delta_n(-A) \in \mathfrak{HorImm}(\mathbb{R}^4)$.

[8] We work in $\mathcal{D} = \ker(dy - z\,dx) \cap \ker(dz - w\,dx)$, the standard Engel structure in $\mathbb{R}^4(x,y,z,w)$.

Definition 7.2.3 Let $\gamma \in \mathfrak{LegImm}(\mathbb{R}^3)$. A double stabilization (DS) of γ is a 1–parameter family $\gamma^u \in \mathfrak{LegImm}(\mathbb{R}^3)$ with $u \in [0,1]$ such that

(i) $\gamma^0 = \gamma$,

(ii) γ^1 is γ stabilized once negatively and once positively.

The homotopy γ^u with $u \in [0,1]$ is explicitly described in Figure 23.

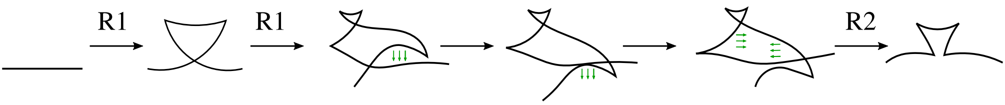

Figure 23: Sequence of moves in the front projection of γ creating a double stabilization.

We can define the DS of a horizontal immersion γ via the Geiges projection, just being careful enough not to change the total area condition. To do this pick two regular points $t_0, t_1 \in \mathbb{S}^1$ of the front Geiges projection of γ. Use t_0 to add a DS and use Lemma 7.2.1 to add the necessary area over $\gamma(t_1)$ to ensure that the area condition is fulfilled. We can do it even better, as the next key result shows.

Lemma 7.2.4 *The double stabilization is a well-defined operation for horizontal embeddings.*

Proof The total area enclosed by the only tangency point created in a DS is nonzero. □

We do not claim that this construction is unique. In fact, it is possible to produce two paths not formally homotopic, connecting the end of the DS with the initial loop: just draw a different picture movie with a different area balance and then it is clear that the concatenation of the first path with the inverse of the second one is a loop with nontrivial area invariant. However, this is enough for our purposes.

The next result shows that DSs can be used in order to approximate families of smooth curves by Legendrian curves.

Theorem 7.2.5 *Given* $K = [0,1] \times [0,1]$, $C = [0,1] \times \{0\} \subseteq K$ *and a family* $\gamma^k \colon [0,1] \to \mathbb{R}^3$ *for* $k \in K$ *such that*

(i) γ^k *are embeddings,*

(ii) $\gamma^k(t)$ *are Legendrian for* $t \in \mathcal{O}p\{0,1\}$,

(iii) γ^k *are Legendrian for every* $k \in C$,

then there exists a family of curves $\gamma^{k,u}\colon [0,1] \to \mathbb{R}^3$ *with* $k \in K$, $u \in [0,1]$, *such that*

(i) $\gamma^{k,0} = \gamma^k$,

(ii) $\gamma^{k,u}$ *are immersions and* C^0*–close to* γ^k,

(iii) $\gamma^{k,1}$ *are Legendrian embeddings,*

(iv) $\gamma^{k,u}(t) = \gamma^k(t)$ *for* $t \in \mathcal{O}p\{0,1\}$,

(v) *there exist a finite sequence* $0 < u_1 < u_2 < \cdots < u_N < 1$ *and small enough* $\delta > 0$ *such that*

  (a) $\gamma^{k,u}$ *are Legendrian embeddings for every* $k \in C$ *and* $u \neq u_1, \dots, u_N$,

  (b) $\gamma^{k,u}$ *for* $u \in [u_i - \delta, u_i + \delta]$ *is a DS of* $\gamma^{k,u_i - \delta}$ *for every* $k \in C$ *and* $i = 1, \dots, N$.

This theorem together with Lemma 7.2.1 easily implies the following:

Corollary 7.2.6 *Given* $K = [0,1] \times [0,1]$, $C = [0,1] \times \{0\} \subseteq K$ *and a family* $\gamma^k\colon [0,1] \to \mathbb{R}^4$ *with* $k \in K$ *such that*

(i) γ^k_G *are embeddings,*

(ii) $\gamma^k(t)$ *are horizontal for* $t \in \mathcal{O}p\{0,1\}$,

(iii) γ^k *are horizontal for every* $k \in C$,

then there exists a family of curves $\gamma^{k,u}\colon [0,1] \to \mathbb{R}^4$ *with* $k \in K$, $u \in [0,1]$, *such that*

(i) $\gamma^{k,0} = \gamma^k$,

(ii) $\gamma^{k,u}$ *are embeddings and* C^0*–close to* γ^k,

(iii) $\gamma^{k,1}$ *are horizontal embeddings,*

(iv) $\gamma^{k,u}(t) = \gamma^k(t)$ *for* $t \in \mathcal{O}p\{0,1\}$.

Proof Apply the previous theorem to the family γ^k_G to obtain a family $\gamma^{k,u}_G$ with $u \in [0,1]$ of immersed curves in $\mathbb{R}^3$. To conclude the proof we just need to approximate the area coordinate (the w–coordinate). This is easily done by using Lemma 7.2.1. □

We can relax the hypothesis (i) for later use. We will prove a particular result just with the interval as the parameter space.

Lemma 7.2.7 *Let $\gamma^s \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ with $s \in [-1,1]$ be a 1–parameter family of immersions such that*

(i) $\gamma^s \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ *for* $s \neq 0$,

(ii) γ^0 *has a generic self-intersection.*

Then there exists a deformation $\gamma^{s,u} \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ with $u \in [0,1]$ satisfying

(a) $\gamma^{s,0} = \gamma^s$,

(b) $\gamma^{s,u} \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ *for* $s \neq 0$,

(c) $\gamma^{0,u}$ *for* $u \in [0,1]$ *has a generic self-intersection,*

(d) $\gamma^{s,1} \in \mathfrak{LegImm}(\mathbb{R}^3)$ *for* $s \in \mathcal{O}p(\{0\})$,

(e) $\gamma^{s,u} = \gamma^s$ *for* $s \notin \mathcal{O}p(\{0\})$.

Proof Let $t_0, t_1 \in \mathbb{S}^1$ be the self-intersection times of γ^0. After applying an isotopy we may assume that there exists $0 < \varepsilon$ such that:

- There exists a time-dependent vector field v_0, supported near $\gamma^0(t_0)$, such that $\gamma^{0,\pm u}$ for $u \in (0,\varepsilon)$ is obtained from γ^0 by pushing the branch through $\gamma(t_0)$ in the direction of $\pm v^0$.
- $\gamma^0|_{\mathcal{O}p(\{t_0,t_1\})}$ is Legendrian.

Take the two smooth embedded arcs $\gamma^0_{\mathbb{S}^1 \setminus \mathcal{O}p(\{t_0,t_1\})}$ and apply Theorem 7.2.5 to them, to construct a path $\widetilde{\gamma}^{0,u} \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ with $u \in [0,1]$, where $\widetilde{\gamma}^{0,1} \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ has a generic self-intersection. Define a family of vector fields $\widetilde{v}^u$ with $u \in [0,1]$, supported near $\gamma^{0,u}(t_0)$, such that $\widetilde{v}^0 = v^0$ and $\widetilde{v}^1$ is a contact vector field. Define $\widetilde{\gamma}^{\pm s,u} \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ for $(s,u) \in (0,\varepsilon) \times [0,1]$ by pushing the branch through $\widetilde{\gamma}^{0,u}(t_0)$ in the direction of $\pm v^u$. It follows that $\widetilde{\gamma}^{\pm s,1} \in \mathfrak{Leg}(\mathbb{R}^3)$ for $s \neq 0$. Finally, consider a smooth bump function $\chi\colon [-\varepsilon,\varepsilon] \to [0,1]$ such that $\chi|_{\mathcal{O}p(\{0\})} \equiv 1$ and $\chi|_{\mathcal{O}p(\{-\varepsilon,\varepsilon\})} \equiv 0$. The desired family is

$$\gamma^{s,u} = \begin{cases} \gamma^s & \text{if } s \in [-1,1] \setminus [-\varepsilon,\varepsilon] \\ \widetilde{\gamma}^{s,\chi(s)} & \text{if } s \in [-\varepsilon,\varepsilon]. \end{cases}$$

□

Corollary 7.2.8 *Let $\gamma^s\colon [0,1] \to \mathbb{R}^4$ be a 1–parameter family of embeddings such that the curves γ^i for $i \in \{0,1\}$ are horizontal, and $\gamma^s(t)$ for $s \in [0,1]$ are horizontal for $t \in \mathcal{O}p\{0,1\}$. Then there exists a homotopy $\gamma^{s,u}\colon [0,1] \to \mathbb{R}^4$ with $(s,u) \in [0,1]^2$ satisfying:*

(i) $\gamma^{s,0} = \gamma^s$,

(ii) $\gamma^{s,u}$ *are embeddings and* C^0*–close to* γ^s,

(iii) $\gamma^{s,1}$ *are horizontal embeddings,*

(iv) $\gamma^{s,u}(t) = \gamma^s(t)$ *for* $t \in \mathcal{O}p\{0,1\}$,

(v) $\gamma^{i,u}$ *for* $i \in \{0,1\}$ *are horizontal embeddings.*

Proof We may assume that there exist $0 < s_1 < \cdots < s_N < 1$ such that

- γ_G^s are embeddings for $s \in [0,1] \setminus \{s_1, \dots, s_N\}$,
- for $i \in \{1, \dots, N\}$, $\gamma_G^{s_i}$ has a generic self-intersection.

Moreover, after applying Lemma 7.2.7 we may also assume that γ_G^s is Legendrian for $s \in \mathcal{O}p(\{s_1, \dots, s_N\})$. Now the result follows by an application of Corollary 7.2.6 to the paths γ^s for $s \in [0,1] \setminus \mathcal{O}p\{s_1, \dots, s_N\}$. □

Remark 7.2.9 The result can be extended to any compact parameter space K and any closed subset $C \subseteq K$. A direct consequence is a parametric version of the Fuchs–Tabachnikov result [14] that asserts that any two Legendrian embeddings which are smoothly isotopic are, after a finite number of stabilizations, Legendrian isotopic. If we assume that we have a k–disk $\mathbb{D}^k \subseteq \mathfrak{FHor}(\mathbb{R}^4)$ such that $\partial\mathbb{D}^k \subseteq \mathfrak{Hor}(\mathbb{R}^4)$, it follows that we can homotope this ball inside $\mathfrak{FHor}(\mathbb{R}^4)$ into a ball in $\mathfrak{Hor}(\mathbb{R}^4)$. For more details, see [6]. Nevertheless, our argument does not assume that we have a disk of formal horizontal embeddings, but just a disk of horizontal immersions; ie $\mathbb{D}^k \subseteq \mathfrak{HorImm}(\mathbb{R}^4)$. Thus, we need to homotope this disk, through the space of horizontal immersions, into a disk of horizontal embeddings. In other words, we are not using as initial datum the homotopy type of the space of smooth embeddings of the circle into $\mathbb{R}^4$: part of the data provided by having a disk of formal horizontal embeddings. In a sense, we are studying the relative homotopy type of the space of horizontal embedding inside the space of horizontal immersions.

The following result works only for 1–parameter families over the segment. This is the main obstacle, in our proof, to generalizing to a complete h–principle. In [6] a generalization to deal with the cusp points in higher-dimensional families is provided and this proves a complete h–principle. Our result just claims that for a 1–parameter family of Legendrian immersions the cusps can be assumed to be constant (in time and position) for the whole family.

Lemma 7.2.10 *Let γ^s for $s \in [0,1]$ be a path in $\mathfrak{Leg}([0,1],\mathbb{R}^3)$ such that $\mathcal{O}p\{0,1\}$ are regular points for the front projection. Then there exists a 2–parameter family $\gamma^{s,u} \in \mathfrak{Leg}([0,1],\mathbb{R}^3)$ with $s,u \in [0,1]$, satisfying*:

(i) $\gamma^{s,0} = \gamma^s$ *and* $\gamma^{s,u}(t) = \gamma^s(t)$ *for* $t \in \mathcal{O}p\{0,1\}$.

(ii) $\gamma^{s,u}$ *and* γ^s *are* C^0*–close.*

(iii) $\gamma^{s,1}$ *is generated by a sequence of Legendrian Reidemeister moves of (only) types II and III.*

Proof The family γ^s is understood as $\gamma\colon [0,1]\times[0,1] \to \mathbb{R}^3$, $(t,s) \mapsto \gamma^s(t)$. By the *Thom transversality theorem* (see [11, Theorem 2.3.2]) we assume that the sets of cusp points in $[0,1]\times[0,1]$ are embedded curves. Moreover, we assume that the height function $h(t,s) = s$, restricted to these curves, has a finite number of maximum and minimum points, all of them nondegenerate. These points correspond to Reidemeister I moves. Take the point with the lowest height among all the minima. Since there is not any other minimum with a lower height, we can find a curve C joining this point with $\{s=0\}$ that does not intersect any other curve of cusp points and which does not have any critical point. Now, we can remove this minimum adding a (C^0–small) family of Reidemeister I moves over C. Keep going to remove all the minima. To remove the maxima we do likewise. □

Thus we obtain:

Corollary 7.2.11 *Up to reparametrization, there exist $c_1,\dots,c_m \in [0,1]$ such that the cusp points of $\gamma^{s,1}$ are at fixed times $c_1,\dots,c_m$ for all $s \in [0,1]$.*

Proof of Theorem 7.2.5 Assume first that γ^k, $k \in C$, does not have any cusp point. Since the h–principle for smooth immersions works relative to the domain and the parameters and is C^0–dense (see [11]), we can assume that the front projection of each γ^k is immersed. We declare the slope of the formal derivative of the front projection γ_F of γ to be the value of the coordinate z (here ∂_z is the projection direction). If γ_F is the projection of a Legendrian this definition of formal derivative coincides with the derivative of the front projection. Therefore, this equips the whole family γ^k with a formal Legendrian structure.

Take an equidistant partition

$$u_0 = 0 < u_1 = \frac{1}{N+1} < \cdots < u_N = \frac{N}{N+1} < u_{N+1} = 1$$

of $[0,1]$. Consider the collection of intervals $I_i = [u_i, u_{i+1}]$ for $i = 0, \dots, N$, and let $K(N) \in \mathbb{Z}^+$. Define $\gamma^{k,u}$ inductively over $\{I_i\}$ as follows:

(i) In I_0 and I_N define $\gamma^{k,u} = \gamma^k$.

(ii) In I_i for $i = 1, \dots, N-1$, define $\gamma^{k,u}$ to be $K(N)$ DSs of γ^{k,u_i} approximating the formal derivative of γ^{k,u_i} at time $t = (2i+1)/2(N+1)$ as described in Figures 24 and 25 (for $K(N) = 2$). Let us provide the details of the construction. The depicted blue segment represents $\gamma_F^k(I_i)$, which can be assumed to be (C^∞–close to) a straight segment for N large enough. The green segments represent the formal derivative, which again for N large enough is C^0–close to a constant. We further assume that the formal derivative has angle zero with respect to the horizontal axis, but we are reduced to that case composing with a linear transformation in $\mathbb{R}^2(x, y)$ to make the formal derivative zero, ie the formal derivative is defined as $(1, z(t))$, we just change coordinates by using the unique linear transformation

$$A_{z(t)}\colon \mathbb{R}^2 \to \mathbb{R}^2$$

satisfying $A_{z(t)}\binom{0}{1} = \binom{0}{1}$ and $A_{z(t)}\binom{1}{z(t)} = \binom{1}{0}$. This canonically lifts to a contact transformation. This allows us to assume that the formal derivative is zero. This is the reason why we are depicting only the case of horizontal formal derivative.

Coming back to the figure, the number below each image is the angle between the curve (straight segment) and the formal derivative. The black curves are the $K(N)$ DSs that C^0–approximate γ^k with prescribed derivative given by the green curve (slope zero in our pictures). The geometric idea is to perform first the negative stabilizations[9] and later on the positive ones. The rule is for the case in which the curve to be approximated points downwards, we make the negative stabilizations small and the positive ones big. In particular, for $K(N)$ sufficiently large, we can make the derivative of the approximating curve arbitrarily close to the formal one (horizontal).

(iii) The rule for the case in which the angle between the curve to be approximated and the formal derivative (horizontal in our pictures) tends to π radians is the same. First, negative stabilizations, and afterwards, positive ones. The key point is that the front projected curve cannot remain an embedding when it reaches π, increasing the angle. The reason is that the y coordinate of the ending point of the curve for an angle slightly over π is below the y coordinate of the starting point. Since the end of the

[9] We follow the standard convention that a negative stabilization is an upward zigzag, and vice versa for the positive ones.

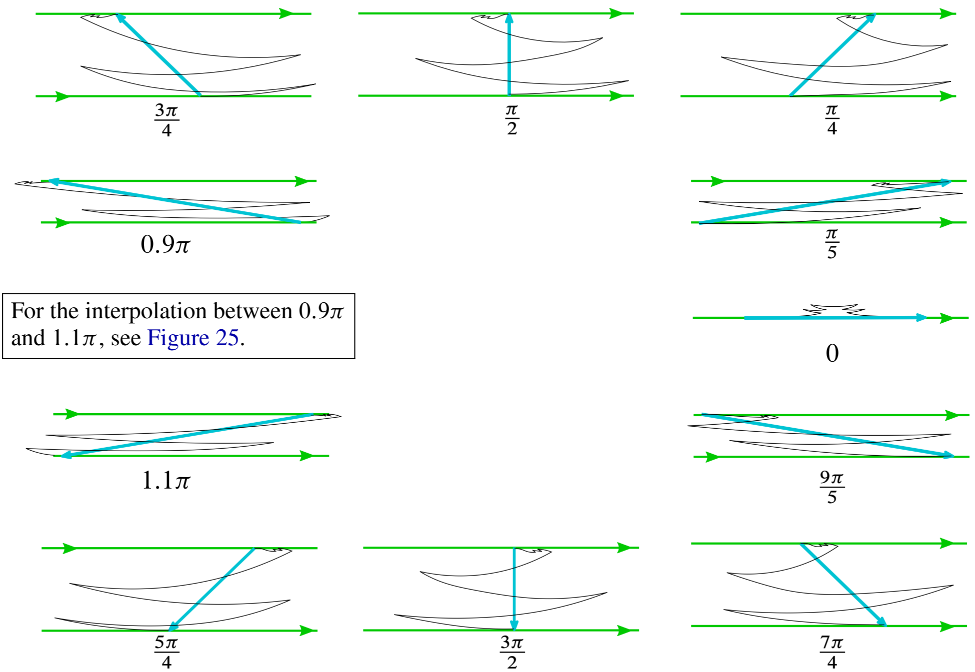

Figure 24: Interpolation. Note that the orientation of the formal derivative (green) is determined by the Legendrian segments in C.

curve is decreasing (positive stabilizations), there is a moment in which the positive stabilization has to cross (through $K(N)-1$ Reidemeister type II moves) over the curve that connects the negative with the positive stabilizations. Note that no front tangencies are created. This is done in order to be able to balance the DSs and place them in the middle at angle π, and then push them to the beginning when going over π radians. This makes it possible to proceed symmetrically when coming from angles bigger than π. In other words, the approximation of a curve with angle α is symmetric with respect to the y axis to the one with angle $2\pi-\alpha$.

The construction is clearly continuous in the angle, therefore it can be done in a canonical way. This readily implies that it works parametrically. Moreover, the approximation coincides with the formal derivative at the beginning and the end of the interval I_i, thus the constructed approximations are smooth.

We claim that the $\gamma^{k,1}$ are embeddings. The segments $\gamma^{k,1}|_{I_i}$ are clearly embedded: just by visual inspection and since they just possess Reidemeister II front moves. However, intersections may show up between the end of $\gamma^{k,1}|_{I_i}$ and the beginning of $\gamma^{k,1}|_{I_{i+1}}$. By sufficiently increasing N, we assume that the angle of the two consecutive

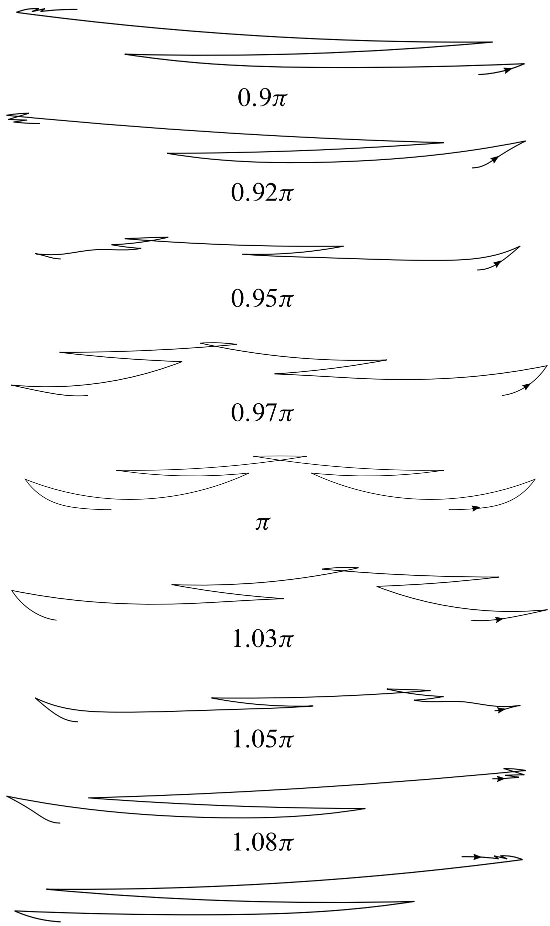

Figure 25: Interpolation between 0.9π and 1.1π in Figure 24.

segments is almost the same. In other words, we are in the same position in the clock figure (Figure 24). There are no tangencies between the two consecutive segments if the angle is away from π radians, since the front curve of the union is clearly embedded. In the neighborhood of π radians, we need to be much more careful and depict the precise movie of the two consecutive segments. The problem is that there are intersections in the front projection between the end of the first approximating curve and the beginning of the next one. In Figure 26, $\gamma^{k,1}|_{I_i}$ is depicted in red and $\gamma^{k,1}|_{I_{i+1}}$ in black. The key point is to make sure that the big-sized stabilization boundary, ie the beginning of the approximating curve for angle smaller than π (respectively the end for angle bigger than π) has the shape of a scimitar with a fat blade. The fatness of the blade allows it to

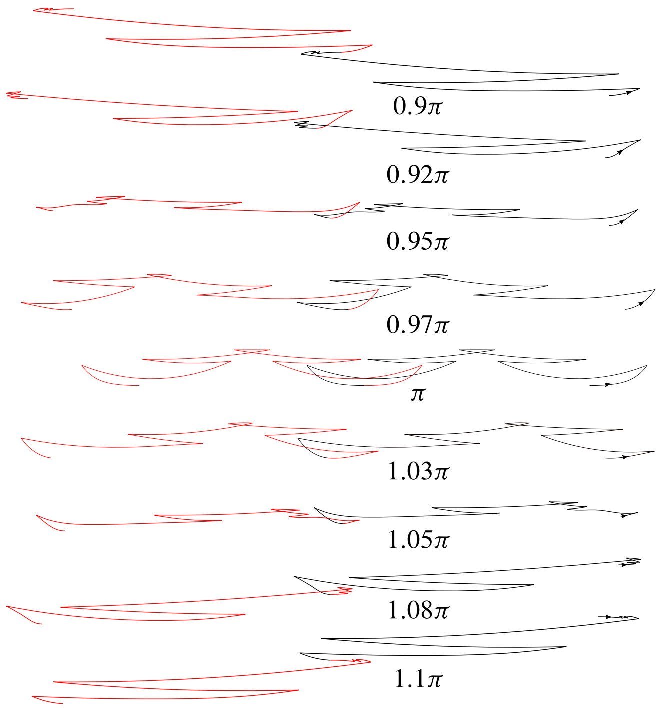

Figure 26: Checking that the gluing process does not create tangency points.

fit inside its bell the whole end of the previous approximating curve (consisting of very small stabilizations) and the convexity of the bottom part of the scimitar blade allows one to cross with Reidemeister II moves to the exterior part of the scimitar. Once there, the small stabilizations run away from the danger area. It is left to check the crossing of the first stabilization of the second curve with the last stabilization of the first one. They do not create tangencies thanks to the convexity of the scimitar shape.

In the general case, apply Corollary 7.2.11 to assume that the cusp points of each γ^k for $k \in C$ are at times $c_1, \dots, c_m$. It is very simple to deform each γ^k, $k \in K$, at small neighborhoods of times $c_1, \dots, c_m$, making the family Legendrian with a cusp point over each $c_1, \dots, c_m$. Finally, add $c_0 = 0$ and $c_{m+1} = 1$ and apply the first case over the family $\gamma^k|_{[c_i, c_{i+1}]}$ for $i = 0, \dots, m$ to conclude the proof. $\square$

We are able to complete the following:

Proof of Lemma 7.1.3 Let γ be a horizontal embedding such that γ_G is a Legendrian embedding and let $\gamma^s \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^4)$, $s \in [0, 1]$, be any loop based at γ, ie $\gamma^0 = \gamma^1 = \gamma$.

Without loss of generality we may assume that the Geiges projection of γ^s is embedded except for a finite sequence of times $0 < s_1 < \cdots < s_N < 1$ and the strict immersions $\gamma_G^{s_i}$ have a generic self-intersection. By Corollary 7.2.8 there exists a loop $\widehat{\gamma}^s$ of horizontal embeddings with $\widehat{\gamma}^0 = \widehat{\gamma}^1 = \gamma$ homotopic to γ^s. Note that the total zero-area condition for a closed horizontal curve can be easily achieved with Lemma 7.2.1, since the map $A\colon [0,1] \to \mathbb{R}$, $s \mapsto \int (\widehat{\gamma}^s)^* z\, dx$ is continuous. Moreover the approximation procedure fixes the cusps of γ and, thus, the rotation number of this loop is zero, so the result follows. □

7.3 Cancellation by pairs of zeros in the same connected component

For a generic 2–disk of Legendrians $\mathbb{D}$, the intersection of the disk with the space of strict immersions is a set of curves with crossings. They come from a set of curves in $\mathfrak{SLegImm}(\mathbb{R}^3)$ by projection. We declare a connected component of the set $\mathbb{D} \cap \mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)$ to be a connected component of the set seen in $\mathfrak{SLegImm}(\mathbb{R}^3)$. In other words, the two branches of a crossing are (possibly) in different connected components.

Let $\varphi\colon \mathbb{D} \to \mathfrak{HorImm}(\mathbb{R}^4)$ defined by $z \mapsto \gamma^z$ be a disk of horizontal immersions satisfying:

- $\varphi(\partial\mathbb{D}) \subseteq \mathfrak{Hor}(\mathbb{R}^4)$,
- $\pi_G \circ \varphi\colon \mathbb{D} \to \mathfrak{LegImm}(\mathbb{R}^3)$ is in generic position,
- the preimage $(\pi_G \circ \varphi)^{-1}(\mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3))$ is a finite union of connected components $C_1, \dots, C_n$,
- C_1 contains at least two zeros of the area function.

Note that Lemma 6.1.4 implies that there is no loss of generality in these assumptions. Let $\bar{C} \subseteq C_1$ be a connected arc containing exactly two zeros of the area function. Parametrize the curve $(\pi_G \circ \varphi)(\bar{C})$ as $\{\gamma_G^s : s \in [0,1]\}$ such that $\gamma_G^{1/4}$ and $\gamma_G^{3/4}$ are the two zeros of the area function. Moreover, we can assume that $\{\gamma_G^s : s \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]\}$ is smooth (does not contain any cusp) in $\mathfrak{LegImm}(\mathbb{R}^3)$.

For $\varepsilon > 0$ small enough, consider the closed ε–neighborhood K_ε of $\bar{C}$ in $\mathbb{D}$ such that K_ε does not contain any other zero of the area function for a different connected component and such that $K_\varepsilon \cap \partial\mathbb{D} = \varnothing$; see Figure 27. We have a well-defined continuous map $\bar{\Gamma}\colon \bar{C} \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $s \mapsto (t_0^s, t_1^s)$, where the components of the ordered pair (t_0^s, t_1^s) are the tangency times of the front projection of $\gamma_G^s(t) = (x^s(t), z^s(t), w^s(t))^{\mathsf{T}}$

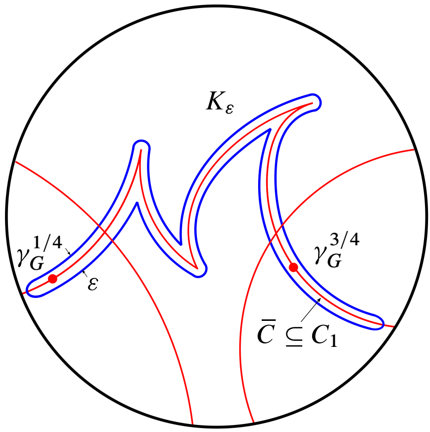

Figure 27: Compact neighborhood K_ε of $\overline{C}$ in $\mathbb{D}$.

for $s \in [0,1]$, and the order of the pair is determined by the following rule: (t_0^0, t_1^0) are chosen with the usual rule to define the area function ε_A, and the pair of points (t_0^s, t_1^s) are extended in the unique continuous way. In order to apply Lemma 7.2.1 to cancel the zeros $\gamma_G^{1/4}$ and $\gamma_G^{3/4}$ it is necessary to find an extension $\Gamma\colon K_\varepsilon \to \mathbb{S}^1 \times \mathbb{S}^1$ sending $k \mapsto (t_0^k, t_1^k)$, of $\overline{\Gamma}$. Just take any retraction $r\colon K_\varepsilon \to \overline{C}$ and define $\Gamma := r^*\overline{\Gamma} = \overline{\Gamma} \circ r$. Write $K = K_\varepsilon$.

We state the main result of this section.

Proposition 7.3.1 *There exists a* 1*–parameter family* $\varphi_u\colon \mathbb{D} \to \mathfrak{HorImm}(\mathbb{R}^4)$ *with* $u \in [0,1]$*, satisfying:*

(i) $\varphi_0 = \varphi$,

(ii) $\varphi_u(z) = \varphi(z)$ *for* $z \in \mathbb{D} \setminus K$,

(iii) $(\pi_G \circ \varphi_u)^{-1}(\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)) = (\pi_G \circ \varphi)^{-1}(\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3))$,

(iv) $\varepsilon_A(\pi_G(\varphi_1(z))) \neq 0$ *for* $z \in \overline{C} \subseteq K$.

Proof Assume that the area function over γ_G^s is positive in $s \in \left[0, \frac{1}{4}\right)$. Define

$$\widetilde{\varepsilon}_A(s) = \widetilde{\varepsilon}_A(\widetilde{\gamma}_G^s) = \int_{t_0^s}^{t_1^s} z^s(t)(x^s)'(t)\,dt.$$

Observe that $\widetilde{\varepsilon}_A$ is just the continuous version of the area function over $\overline{C}$ and satisfies $|\varepsilon_A| = |\widetilde{\varepsilon}_A|$. The function $\widetilde{\varepsilon}_A\colon \overline{C} \to \mathbb{R}$ is depicted in Figure 28.

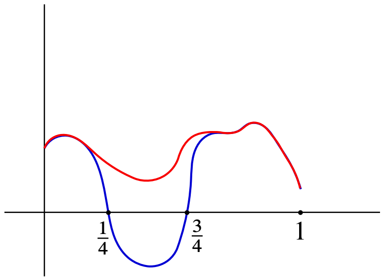

Figure 28: Possible graph of $\widetilde{\varepsilon}_A$ before (blue) and after (red) adding area using Lemma 7.2.1.

Define $\varphi_u = \varphi$ over $\mathbb{D} \setminus K$. The problem is clearly to define φ_u over K. Proceed as follows: In order to *add area* choose two points $p^k, n^k \in \mathbb{S}^1$ in the domain of γ_G^k for $k \in K$. We require these points to have the following properties:

- p^k and n^k are regular for the front projection of γ_G^k.
- If γ_G^k has more tangency points different from (t_0^k, t_1^k), the area function over these points remains unchanged after adding area over p^k and n^k.

For a given γ_G^k, choose two regular points p^k and n^k very close to t_0^k, one to the left and another one to the right of t_0^k. These two points satisfy the properties above. By compactness, there exists a finite open covering $\{U^i\}$ of K and an election $(p^i, n^i) \in \mathbb{S}^1 \times \mathbb{S}^1$ compatible with each γ_G^k for $k \in U^i$. Take a partition of unity $\{\phi^i\}$ subordinate to $\{U^i\}$ and a continuous function $\eta\colon K \to [0,1]$ such that

- $\eta \equiv 0$ in ∂K, and
- $\eta \equiv 1$ in $(\pi_G \circ \varphi)^{-1}\big(\{\gamma_G^s : s \in \big[\frac{1}{4}, \frac{3}{4}\big]\}\big) \cap K$.

Finally, fix some positive number $A > \max\{|\widetilde{\varepsilon}_A(\widetilde{\gamma}_G^s)| : s \in [0,1]\}$. Define $\varphi_u(k)$ for $k \in K$ as the lift of

$$\gamma_G^k \#_i \delta_{p^i}(u\phi^i(k)\eta(k)A) \#_i \delta_{n^i}(-u\phi^i(k)\eta(k)A)$$

to $\mathfrak{HorImm}(\mathbb{R}^4)$. The family φ_u satisfies the required properties. $\square$

8 The core of the proof

We explain now how for a given capping disk of horizontal immersions, we are able to cancel zeros of the area function by deforming the disk relative to the boundary. In particular, we prove that any pair of zeros can be canceled out without creating new zeros: there will be no assumptions about the zeros. This completes the proof of the main theorem.

8.1 Changing the diagram of curves of strict Legendrian immersions

The goal is to create a 2–square (2–disk) in the space of $\mathfrak{LegImm}(\mathbb{R}^3)$ such that the bottom of the square is a path γ^s in the disk $\mathbb{D}_G$ whose endpoints are two strict Legendrian immersions γ^0 and γ^1. Moreover, there exists a finite set of values of the parameter $0 < s_1 < \cdots < s_\nu < 1$ such that γ^{s_j} is also a strict immersed Legendrian. The rest of the points γ^s for $s \in (0,1) \setminus \{s_1, \ldots, s_\nu\}$ are Legendrian embeddings. The other three sides of the square will produce a path of strict Legendrian immersions that has the property that the self-intersection points $\hat{\gamma}^s(t_0) = \hat{\gamma}^s(t_1)$ form a continuous family. In other words, we are constructing a curve of Legendrians that makes the two initial Legendrians live in the same connected component of strict immersions. This is the key point and is the content of Proposition 8.1.15. Afterwards, we play with the result: given an embedded curve in the disk whose ends live in two different connected components in the disk, we find a square transverse to the disk $\mathbb{D}_G$ that creates on the other three sides a curve of strict immersions joining the two strict Legendrian immersions. Then we take a small open neighborhood of it, removing the interior in order to produce a new disk (see Figure 39), in which the curves of strict Legendrian immersions have changed their topology (see Figure 38).[10] Proposition 8.1.13 explains how to build the disk in the smooth category. Finally, we just use Theorem 7.2.5 to conclude in the Legendrian setting.

Denote by $\mathfrak{LegImm}_0^{\textbf{strict}}(\mathbb{R}^3) \subseteq \mathfrak{LegImm}^{\textbf{strict}}(\mathbb{R}^3)$ the open stratum of simple transverse self-intersections.

8.1.1 Legendrian fingers

Definition 8.1.1 Fix $\gamma \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ and $b \in \mathbb{S}^1$. A *finger bone* is an embedded path $\beta\colon [0,1] \to \mathbb{R}^3$ such that $\beta(0) = \gamma(b)$ and the intersection is transverse. Moreover, $\beta(0,1] \cap \gamma(\mathbb{S}^1)$ is either one point or the empty set. In the former case, the intersection must be transverse.

[10] The picture looks like a bypass picture in contact topology, though geometrically it has nothing to do with a bypass. Formally, we play the same game: our initial disk plays the formal role of the convex surface, the strict immersions play the formal role of the dividing sets. The bypass changes the strict immersions/dividing set combinatorial structure.

Definition 8.1.2 Fix $\gamma \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ and $b \in \mathbb{S}^1$. A *finger germ* is a pair (β, v) satisfying:

(a) $\beta\colon [0,1] \to \mathbb{R}^3$ is a finger bone such that $\beta(0) = \gamma(b)$,

(b) $v \in \beta^* T\mathbb{R}^3$ is such that

  (i) $v(t)$ and $\beta'(t)$ are linearly independent,

  (ii) $v(0) = \gamma'(b)$.

Without loss of generality, assume that γ is geodesic on a very small neighborhood of b. For $\varepsilon > 0$ small enough, fix an embedding $Q\colon [-\varepsilon, \varepsilon] \times [0,1] \to \mathbb{R}^3$ defined as $Q(x,z) = \exp_{\beta(z)}(x \cdot v(z))$. Choose a bump function $\chi\colon [-\varepsilon, \varepsilon] \to [0,1]$ such that

- χ is an even function,
- χ is increasing in $[-\varepsilon, 0]$,
- $\chi(-\varepsilon) = 0$, $\chi^{(j)}(-\varepsilon) = 0$ for all $j \in \mathbb{Z}^+$, and
- $\chi(0) = 1$ and $\chi^{(j)}(0) = 0$ for all $j \in \mathbb{Z}^+$.

Definition 8.1.3 The *finger deformation* $\gamma \# (\beta, v) \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ associated to the finger germ (β, v) is given by the curve

$$\gamma \# (\beta, v)(t) = \begin{cases} \gamma(t) & \text{for } t \notin (b-\varepsilon, b+\varepsilon), \\ Q(t-b, \chi(t-b)) & \text{for } t \in [b-\varepsilon, b+\varepsilon]. \end{cases}$$

This is a well-defined operation that works for compact families of curves and finger germs, provided that $\varepsilon > 0$ is chosen small enough and uniform for the whole family.

Note that γ and $\gamma \# (\beta, v)$ are homotopic in $\mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$. Denote by $(\beta_u, v_u) = (\beta_{|[0,u]}, v_{|[0,u]})$ the obvious homotopy of germs of fingers. We say that $\gamma \# (\beta_u, v_u)$ is obtained from γ by *adding the finger germ* (β_u, v_u).

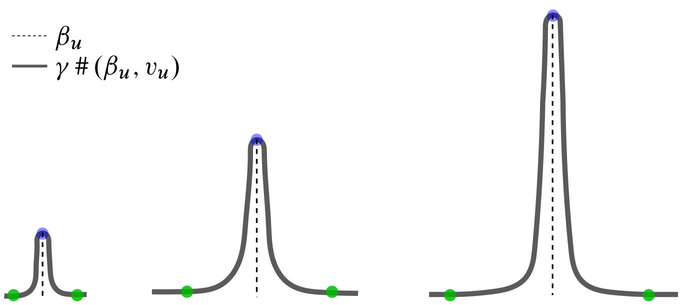

Figure 29: Finger addition process $\gamma \# (\beta_u, v_u)$ for $u \in [0,1]$.

Definition 8.1.4 Fix $\gamma \in \mathfrak{Leg}(\mathbb{R}^3)$ and (β, v) a finger germ such that

(a) β is a transverse embedding, ie $\beta'(t) \notin \xi_{\beta(t)}$ for each $t \in [0,1]$,

(b) v is Legendrian, ie $v(t) \in \xi_{\beta(t)}$ for each $t \in [0,1]$.

We say that (β, v) is a *pre-Legendrian finger germ.*

Let us explain how to add the *Legendrian finger* associated to a pre-Legendrian finger germ. We are working on the standard $(\mathbb{R}^3(x,y,z), \xi = \ker(dz - y\,dx))$ with the trivialization of ξ given by the Legendrian framing $\langle \partial_y \rangle$. Take a contactomorphism $\Phi\colon (C_\varepsilon, \xi) \to (\mathbb{D}^2(0;\varepsilon) \times [0,1](x,y,z), \xi = \ker(dz - y\,dx))$. The domain C_ε, which we will call a cylindrical shape, is a sufficiently small tubular neighborhood of β, and fixing $\varepsilon > 0$ small enough, we have that

- $\Phi \circ \beta(t) = (0,0,t)$,
- $\Phi_* v = \partial_x$,
- $\Phi \circ \gamma(t) = (t-b, 0, 0)$ for $|t-b| \le \varepsilon$.

The existence of this contactomorphism is a direct consequence of the *tubular neighborhood theorem for contact submanifolds* [16, Theorem 2.5.15]. From now on we work in these coordinates. The *front projection* in $(\mathbb{D}^2(0;\varepsilon) \times [0,1](x,y,z), \xi = \ker(dz - y\,dx))$ is given by the projection in the XZ–plane. The front projection of the original Legendrian in a neighborhood of the base of the finger $\gamma|_{\mathcal{O}p(\{b\})}$ is the horizontal axis in front coordinates. The front of the transverse finger bone β is the vertical axis; see Figure 30. To approximate the finger bone by a Legendrian finger just add $N = N(\varepsilon) \in \mathbb{Z}_{\ge 0}$ DSs to the Legendrian γ over $\gamma(b)$ as in Figure 30, in such a way that the derivative of the new Legendrian $\gamma \,\#\, (\beta, v)$ at time b is $v(1)$.

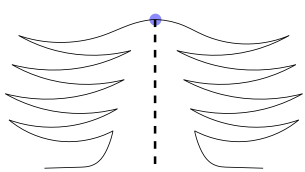

Figure 30: Legendrian finger construction process. The key tool is to create a huge *tower* of double stabilizations.

This operation is well defined up to Legendrian immersion homotopy and works for compact families. Note that this follows just taking the radius ε of the tubular neighborhood and the number N of DSs in the finger addition process constant in the whole family. This means that given a family $(\gamma^k, (\beta^k, v^k), C_\varepsilon^k)$ with $k \in K$, we obtain a continuous family of Legendrian finger deformations $\gamma^k \,\#\, (\beta^k, v^k)$. In particular, if there is some $k_0 \in K$ such that the pre-Legendrian finger germ (β^{k_0}, v^{k_0}) is trivial, ie

$$(\beta^{k_0}, v^{k_0}) \equiv (\gamma^{k_0}(b), (\gamma^{k_0})'(b)),$$

and $N \neq 0$, then the Legendrians γ^{k_0} and $\gamma^{k_0} \,\#\, (\beta^{k_0}, v^{k_0})$ are different. In other words, to interpolate between a trivial pre-Legendrian finger germ and a nontrivial one we need to add DSs to the trivial finger. Another important observation is that the DSs operation is well-defined for horizontal embeddings (see Lemma 7.2.4); this is relevant since we are going to work with the Geiges projection of families of horizontal curves.

Remark 8.1.5 Another possible definition of Legendrian finger uses Reidemeister I moves instead of DSs. In both cases, the transverse finger bone is approximated by *half of the Legendrian finger*, which is a very stabilized Legendrian curve. Moreover, the definition using Reidemeister I moves does not work well for our purposes because the two *halves* of the finger are linked, and in the later argument it is very important to not have linking between the two halves.

A stabilization of a transverse curve is an operation that changes the curve in such a way that in front projection a small arc is replaced by a 2π rotation, as in Figure 31. The stabilization does not depend on the small segment chosen; see Etnyre [12]. So we can speak of a k–stabilization of a transverse curve β_0, which we will denote by β_k.

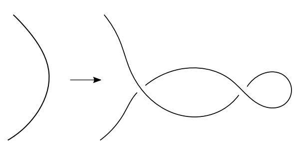

Figure 31: Stabilization of a transverse curve.

The next lemma will be useful in the next subsection, and ensures that we can interpolate between a small finger with finger bone a small Reeb chord, and another one with finger bone a very stabilized Reeb chord.

Lemma 8.1.6 *Let $\gamma \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ be a Legendrian immersion with a self-intersection at times $t_0, t_1 \in \mathbb{S}^1$, and $\tilde{\gamma} \in \mathfrak{Leg}(\mathbb{R}^3)$ a Legendrian embedding such that*

(i) *there is a small Reeb chord $\beta\colon [0,1] \to \mathbb{R}^3$ between $\tilde{\gamma}(b)$ and $\tilde{\gamma}(e)$ for some times $b, e \in \mathbb{S}^1$,*

(ii) $\gamma = \widetilde{\gamma}\,\#\,(\beta, v)$, *where* $v = \partial_x + y\,\partial_z$.[11]

Then for any $k \in \mathbb{Z}^+$, *there exists a* 1*–parameter* $\gamma^u \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ *with* $u \in [0,1]$, *with a self-intersection at times* $t_0, t_1 \in \mathbb{S}^1$, *such that* $\gamma^0 = \gamma$ *and* $\gamma^1 = \widetilde{\gamma}\,\#\,(\beta_k, v)$. *Moreover, the family is constructed from* γ *just by adding a sequence of DSs to the initial finger, ie the number* N *of DSs in the definition of Legendrian finger can be chosen in such a way that* $\widetilde{\gamma}\,\#\,(\beta, v) = \widetilde{\gamma}\,\#\,(\beta_k, v)$.

Proof Since β_k is C^0–close to β we may assume that the finger associated to (β_k, v) is contained in the tubular neighborhood C_ε of β. Moreover, (C_ε, ξ) is contactomorphic to $(\mathbb{R}^3, \xi = \ker(dz - y\,dx))$ by Eliashberg's classification of contact structures in $\mathbb{R}^3$ [9]. In this sense the right side of the fingers associated to (β, v) and (β_k, v), denoted by R and R_k respectively, can be understood as positively stabilized *long* Legendrian unknots. In the same way the left side, denoted as L and L_k, can be understood as negatively stabilized long Legendrian unknots. Thus, there exist two nonnegative numbers $M, M_k \in \mathbb{Z}_{\geq 0}$ such that

- $S_+^M(R)$ is Legendrian isotopic to $S_+^{M_k}(R_k)$,
- $S_-^M(R)$ is Legendrian isotopic to $S_-^{M_k}(R_k)$,

where S_+ (S_-) denotes a positive (negative) stabilization of a Legendrian curve. This follows from the classification of Legendrian unknots given in [10]. The lemma follows from the fact that the number of positive and negative stabilizations are the same. □

Remark 8.1.7 Another possible proof of the previous lemma is to apply Theorem 7.2.5 to each half of the finger, and then slide the DSs over the finger nails.

A Legendrianization of a transverse knot/curve (with fixed framings at the ends) is an induced Legendrian knot/curve that is created by drawing the characteristic foliation of the boundary of a small solid torus fixed around the transverse knot for a radius such that the slope is rational, and picking one integral curve. The transverse push-off of a Legendrianization is always transverse-isotopic to the original one; see [12].

The next lemma ensures the existence of 2–parameter families of pre-Legendrian finger germs, generalizing the 1–parameter result explained in Lemma 3.3.3 in [16]. It is the key lemma that we will use to produce controlled deformations on our families of curves.

[11] We may assume that $\widetilde{\gamma}'(b) = v_{\gamma(b)}$.

Lemma 8.1.8 *Fix $\beta^{s,u} \in \mathfrak{Emb}([0,1],\mathbb{R}^3)$ for $(s,u) \in [0,1]\times[0,1]$, assume that $\beta^{s,0}$ is transverse and $\beta^{s,u}|_{\mathcal{O}p(\{0,1\})}$ is also transverse. Then there exist a constant $k>0$ and a family of transverse embeddings $\widetilde{\beta}^{s,u}\colon [0,1]\to\mathbb{R}^3$, which is C^0–close to $\beta^{s,u}$, such that $\widetilde{\beta}^{s,0}=\beta^{s,0}_k$ and $\widetilde{\beta}^{s,u}|_{\mathcal{O}p(\{0,1\})}=\beta^{s,u}|_{\mathcal{O}p(\{0,1\})}$.*

Proof Perform a Legendrianization of $\beta^{s,0}$ that we call $\gamma^{s,0}$, which extends to a family $\gamma^{s,u}$ that is smoothly isotopic to $\beta^{s,u}$. Further assume that we Legendrianize the small segments $\beta^{s,u}|_{t\in\mathcal{O}p(0)\cup\mathcal{O}p(1)}$. Abusing notation, we still call $\gamma^{s,u}$ a perturbation of the original family in order to make it Legendrian on a very small neighborhood of a time t_0.

Now apply Theorem 7.2.5 to the two segments $\gamma^{s,u}|_{t\in[0,t_0]}$ and $\gamma^{s,u}|_{t\in[t_0,1]}$. This produces a family $\widehat{\gamma}^{s,u}$ that is Legendrian. The process introduces a high number of DSs on the family $\gamma^{s,0}$. We denote by k the number of introduced DSs.

Construct a new family $\widetilde{\gamma}^{s,0}$ that it is just built by performing k DSs at time t_0 on $\gamma^{s,0}$. It is well known that DSs can be moved around a Legendrian (they are Legendrian homotopic [12]). This, in particular, implies that $\widetilde{\gamma}^{s,0}$ and $\widehat{\gamma}^{s,0}$ are homotopic through Legendrian embeddings $\gamma^{s,0}_r$ with $r\in[0,1]$, ie $\gamma^{s,0}_0=\widetilde{\gamma}^{s,0}$ and $\gamma^{s,0}_1=\widehat{\gamma}^{s,0}$. So we have proven that

$$\overline{\gamma}^{s,u}=\begin{cases}\gamma^{s,0}_{2u} & \text{for } u\in\left[0,\tfrac12\right],\\ \widehat{\gamma}^{s,2u-1} & \text{for } u\in\left[\tfrac12,1\right]\end{cases}$$

is a family of Legendrians approximating $\gamma^{s,u}$ such that $\widetilde{\gamma}^{s,0}=\overline{\gamma}^{s,0}$.

Perform a transverse push-off to obtain a family $\widetilde{\beta}^{s,u}$ of transverse knots. Since the transverse push-off of k DSs of a Legendrian knot corresponds to k stabilizations, and since the transverse push-off of a Legendrianization coincides with the original transverse knot, we have shown that $\widetilde{\beta}^{s,0}=\beta^{s,0}_k$. □

The *unknotting number* of an embedding is finite; see Adams [2]. We can think of the unknotting number as the number of fingers that we need to add to the knot in order to make it the unknot. Moreover, the necessary number of fingers in order to unlink two embeddings is also finite. We summarize this information as:

Proposition 8.1.9 *Fix $\gamma,\eta\in\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$ disjoint embeddings. There exists a parametrized unknot $\widetilde{\gamma}\in\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$, obtained from γ just by adding a finite number of finger germs, which is unlinked from η.*

We are interested in applying this result to the two *resolutions* of a strict immersion with exactly one (generic) self-intersection. More precisely, let $\gamma \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ be an immersion with exactly one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$. Assume without loss of generality that there exists $\varepsilon > 0$ small enough that

$$\gamma((t_0 - \varepsilon, t_0 + \varepsilon) \cup (t_1 - \varepsilon, t_1 + \varepsilon))$$

is contained in the affine plane $\gamma(t_0) + \operatorname{span}\{\gamma'(t_0), \gamma'(t_1)\}$. Take coordinates $(x, y) \in \mathbb{R}^2$ in the plane $\gamma(t_0) + \operatorname{span}\{\gamma'(t_0), \gamma'(t_1)\}$ such that $\gamma(t_0 + r) = (r, 0)$ and $\gamma(t_1 + r) = (0, r)$ for $r \in (-\varepsilon, \varepsilon)$. Finally, fix any smooth nonincreasing function $\chi\colon [-\varepsilon, \varepsilon] \to [0, 1]$ such that $\chi(-\varepsilon) = 1$ and $\chi(\varepsilon) = 0$.

Definition 8.1.10 The *resolution* of γ over $[t_0, t_1]$ is the embedding $\gamma_{\mathrm{Res}(t_0,t_1)} \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ defined as

$$\gamma_{\mathrm{Res}(t_0,t_1)}(t) = \begin{cases} \gamma(t) & \text{if } t \in (t_0 + \varepsilon, t_1 - \varepsilon), \\ \chi(t - t_0)(0, t - t_0) + (1 - \chi(t - t_0))(t - t_0, 0) & \text{if } t \in [t_0, t_0 + \varepsilon), \\ \chi(t - t_1)(0, t - t_1) + (1 - \chi(t - t_1))(t - t_1, 0) & \text{if } t \in (t_1 - \varepsilon, t_1]. \end{cases}$$

where $\mathbb{S}^1 = [t_0, t_1]/\sim$.

Remark 8.1.11 The resolution $\gamma_{\mathrm{Res}(t_0,t_1)}$ is well-defined up to isotopy. Moreover, observe that $\gamma_{\mathrm{Res}(t_0,t_1)}(\mathbb{S}^1) \cap \gamma_{\mathrm{Res}(t_1,t_0)}(\mathbb{S}^1) = \varnothing$.

In these terms, Proposition 8.1.9 applied over γ just says that, up to adding a finite number of fingers to γ, we may assume that the two resolutions $\gamma_{\mathrm{Res}(t_0,t_1)}$ and $\gamma_{\mathrm{Res}(t_1,t_0)}$ are unlinked and one of them is an unknot. In fact, we can add fingers in order to obtain two unlinked unknots.

In the same fashion we can define the resolution of a strict Legendrian immersion $\gamma \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$. Assume that $\gamma(t_0) = \gamma(t_1)$. Take adapted coordinates such that the front $\gamma_F(t_0 + r) = (r, 0)$ and $\gamma_F(t_1 + r) = (\pm r, r^2)$ for $r \in (-\varepsilon, \varepsilon)$. Fix a nonincreasing smooth function $\chi\colon [-\varepsilon, 0] \to [0, 1]$ such that $\chi(-\varepsilon) = 1$ and $\chi(0) = 0$. Finally, take a decreasing smooth function $\eta\colon [-\varepsilon, 0] \to [0, \delta]$, where $\delta > 0$ is very small, satisfying $\eta(-\varepsilon) = \delta$ and $\eta^{j)}(0) = 0$ for $j \in \mathbb{Z}^+ \cup \{0\}$.

Definition 8.1.12 The *resolution* of γ over $[t_0, t_1]$ is the Legendrian embedding $\gamma_{\mathrm{Res}(t_0,t_1)} \in \mathfrak{Leg}(\mathbb{R}^3)$ defined as follows:

(i) If $\gamma_F(t_1 + r) = (r, r^2)$ for $r \in (-\varepsilon, \varepsilon)$ then

$$\gamma_{\mathrm{Res}(t_0,t_1),F}(t) = \begin{cases} \gamma_F(t) & \text{if } t \in [t_0, t_1 - \varepsilon), \\ (r, \chi(r) r^2 + (1 - \chi(r))\eta(r)) & \text{if } t = t_1 + r \in (t_1 - \varepsilon, t_1]. \end{cases}$$

(ii) If $\gamma_F(t_1+r) = (-r, r^2)$ for $r \in (-\varepsilon, \varepsilon)$ then

$$\gamma_{\mathrm{Res}(t_0,t_1),F}(t) = \begin{cases} \gamma_F(t) & \text{if } t \in [t_0, t_1-\varepsilon), \\ (-r, \chi(r)r^2 + (1-\chi(r))r^{3/2}) & \text{if } t = t_1 + r \in (t_1-\varepsilon, t_1]. \end{cases}$$

Here $\mathbb{S}^1 = [t_0, t_1]/\sim$.

Figure 32: Resolutions in Definition 8.1.12. $\gamma_{\mathrm{Res}(t_0,t_1)}$ is represented in red.

We have the following result for 1–parameter families of smooth curves.

Proposition 8.1.13 *Fix $\gamma^s \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ with $s \in [0,1]$ such that*:

(i) *γ^s is an embedding for $s \in (0,1) \setminus \{s_1, \dots, s_\nu, 1\}$, where $0 < s_1 < \dots < s_\nu < 1$ for $\nu \geq 0$.*

(ii) *γ^i for $i \in \{0,1\}$ has exactly one (generic) self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and γ^{s_j} for $j \in \{s_1, \dots, s_\nu\}$ has exactly one (generic) self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$. The times $t_0, t_1, \tau^0_{s_j}, \tau^1_{s_j}$ for $j \in \{1, \dots, \nu\}$ are pairwise different.*

Then there exist k 1–parameter families $\{\beta^s_1, \dots, \beta^s_k\}$ of finger germs and a family $\gamma^{s,u} \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3)$ with $(s,u) \in [0,1] \times [0,1]$ such that $\gamma^{s,0} = \gamma^s$ and $\gamma^{s,1} = \gamma^s \# \beta^s_1 \# \dots \# \beta^s_k$ — ie the 1–parameter family $\gamma^{s,1}$ is obtained from γ^s by adding in consecutive times the finite sequence of fingers — satisfying:

(i) *$\gamma^{s,1}$ for $s \in [0,1]$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$. This self-intersection is unique for $s \in [0,1] \setminus \{s_1, \dots, s_\nu\}$.*

(ii) *$\gamma^{i,u}$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^{s_j,u}$ for $j \in \{1, \dots, \nu\}$ has one self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

(iii) *There exists a finite sequence $0 < u_1 < \dots < u_N < 1$ such that*

$$\gamma^{s,u_j} \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3) \setminus \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$$

for each $j \in \{1, \dots, N\}$ and $s \in (0,1)$.

(iv) *$\gamma^{s,u} \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ for $u \neq 1, u_1, \dots, u_N$ and $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.*

Moreover, if there is a $c \in (t_0, t_1)$ such that $\gamma^s|_{\mathcal{O}p(\{c\})}$ is constant, then the construction can be done relative to $\mathcal{O}p(\{\gamma^s(c)\})$; ie $\gamma^{s,u}|_{\mathcal{O}p(\{c\})}$ is constant.

Note that we start with a segment γ^s and we create a square $\gamma^{s,u}$ in such a way that one of the edges is the original segment and the other three edges constitute a curve of self-intersections. Therefore, this is the key result in the smooth category and we will adapt it to the Legendrian case in Proposition 8.1.15.

Proof of Proposition 8.1.13 We consider two cases:

Case 1 $(\{s_1,\dots,s_\nu\}=\varnothing)$ The proof is divided into three steps.

Step I (unknot $\gamma^0_{\mathrm{Res}(t_0,t_1)}$) For γ^0 and γ^1, we declare t_0 to be the point in the lower branch in front projection.[12] Apply Proposition 8.1.9 and create a sequence of finger germs

$$(\beta^0_1,v^0_1),\dots,(\beta^0_K,v^0_k)$$

to unknot and unlink $\gamma^0_{\mathrm{Res}(t_0,t_1)}$ from $\gamma^0_{\mathrm{Res}(t_1,t_0)}$. Let $(b^0_1,e^0_1)\in\mathbb{S}^1\times\mathbb{S}^1$ be the basepoint and the endpoint of the finger (β^0_1,v^0_1); ie the finger germ (β^0_1,v^0_1) joins $\gamma^0(b^0_1)$ with $\gamma^0(e^0_1)$. We extend this finger germ to a whole family of finger germs for the family γ^s. In order to do this, choose any continuous map $F\colon [0,1]\to\mathbb{S}^1\times\mathbb{S}^1$ sending $s\mapsto(b^s_1,e^s_1)$ such that $b^s_1\neq e^s_1$ and $b^1_1,e^1_1\neq t_i$ for $i\in\{0,1\}$. This provides a continuous family (β^s_1,v^s_1) of finger germs joining $\gamma^s(b^s_1)$ with $\gamma^s(e^s_1)$. This produces a family $\gamma^{s,u}$, with $s\in[0,1]$ and $u\in[0,1/(3K)]$, where $\gamma^{s,1/(3K)}=\gamma^s\,\#\,\beta^s_1$.

Proceed in the same way with the rest of the finger germs to construct a family $\gamma^{s,u}$, with $s\in[0,1]$ and $u\in\left[0,\frac{1}{3}\right]$, satisfying:

(i) $\gamma^{s,0}=\gamma^s$.

(ii) $\gamma^{s,1/3}=\gamma^s\,\#\,\beta^s_1\,\#\cdots\#\,\beta^s_K$.

(iii) $\gamma^{i,u}$ for $i\in\{0,1\}$ has one self-intersection at times $(t_0,t_1)\in\mathbb{S}^1\times\mathbb{S}^1$.

(iv) There exists a finite sequence $0<u_1<\cdots<u_K<\frac{1}{3}$ such that

$$\gamma^{s,u_j}\in\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3)\setminus\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$$

for each $j\in\{1,\dots,K\}$ and $s\in(0,1)$.

(v) $\gamma^{s,u}\in\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$ for $u\neq u_1,\dots,u_K$ and $s\in(0,1)$.

(vi) $\gamma^{0,1/3}_{\mathrm{Res}(t_0,t_1)}$ is a parametrized unknot and is unlinked from $\gamma^{0,1/3}_{\mathrm{Res}(t_1,t_0)}$.

[12] To be precise we should write $t_0(0)$ and $t_0(1)$, since the intersection times of γ^0 and γ^1 could be different. We avoid this to simplify the notation.

Step II (unknot $\gamma^{1,1/3}_{\mathrm{Res}(t_0,t_1)}$) Do the same as in Step I, but be careful in order to add families of fingers which restricted to $s=0$ end up in $\gamma^{0,1/3}_{\mathrm{Res}(t_1,t_0)}$ and to not relink this resolution from the other resolution $\gamma^{0,1/3}_{\mathrm{Res}(t_0,t_1)}$. More precisely, let

$$(\beta^1_{K+1}, v^1_{K+1}), \dots, (\beta^1_N, v^1_N)$$

be the sequence of finger germs obtained from Proposition 8.1.9 to unknot and unlink $\gamma^{1,1/3}_{\mathrm{Res}(t_0,t_1)}$ from $\gamma^{1,1/3}_{\mathrm{Res}(t_1,t_0)}$. As in the previous step, begin with $(b^1_{K+1}, e^1_{K+1}) \in \mathbb{S}^1 \times \mathbb{S}^1$, the base and the endpoints of the finger $(\beta^1_{K+1}, v^1_{K+1})$, and extend this finger to the whole family via a map $F\colon [0,1] \to \mathbb{S}^1 \times \mathbb{S}^1$ sending $s \mapsto (b^s_{K+1}, e^s_{K+1})$. We need to make sure that the finger germ $(\beta^1_{K+1}, v^1_{K+1})$ extends to a 1–parameter family $(\beta^s_{K+1}, v^s_{K+1})$ such that

- $\beta^0_{K+1}(0) = \gamma^{0,1/3}(b^0_{K+1})$, $\beta^0_{K+1}(1) = \gamma^{0,1/3}(e^0_{K+1}) \in \gamma^{0,1/3}(\mathbb{S}^1 \setminus (t_0,t_1))$, ie $b^0_{K+1}, e^0_{K+1} \in \mathbb{S}^1 \setminus (t_0,t_1)$, and
- β^0_{K+1} is unlinked from $\gamma^{0,1/3}_{\mathrm{Res}(t_0,t_1)}$.

In other words, denote by $\mathbb{D}$ an embedded 2–disk whose boundary is given by $\gamma^{0,1/3}_{\mathrm{Res}(t_0,t_1)}$. Then the condition $\beta^0_{K+1}[0,1] \cap \mathbb{D} = \varnothing$ is sufficient to keep the unlinking property. We say that a 1–parameter family of finger germs that satisfies this condition is *valid*. Hence, we must show that there always exists a *valid* 1–parameter family of finger germs which extends $(\beta^1_{K+1}, v^1_{K+1})$.

Assume the family $(\beta^s_{K+1}, v^s_{K+1})$ does not satisfy this condition, ie $\beta^0_{K+1}[0,1] \cap \mathbb{D} = \{p_1, \dots, p_k\}$. Let $\delta > 0$ be small enough and let $\widetilde{\mathbb{D}} \supsetneq \mathbb{D}$ be a small deformation of $\mathbb{D}$ such that $\widetilde{\mathbb{D}} \cap \gamma^{\delta,1/3}(\mathbb{S}^1) = \gamma^{\delta,1/3}[t_0,t_1] \subsetneq \partial\widetilde{\mathbb{D}}$. Take any $\phi \in \mathrm{Diff}(\widetilde{\mathbb{D}}, \partial\widetilde{\mathbb{D}})$, isotopic to the identity, such that $\phi(p_i) \in \widetilde{\mathbb{D}} \setminus \mathbb{D}$ for $i \in \{1,\dots,k\}$, and extend it to a diffeomorphism Φ of $\mathbb{R}^3$ with compact support close enough to $\widetilde{\mathbb{D}}$ and isotopic to the identity. Let $\varepsilon > 0$ be small enough and let Φ_s with $s \in [0,1]$ be any isotopy between $\Phi_0 = \Phi$ and $\Phi_1 = \mathrm{Id}$ such that $\Phi_s = \mathrm{Id}$ for $s \geq \varepsilon$.

The family of finger germs

$$(\widetilde{\beta}^s_{K+1}, \widetilde{v}^s_{K+1}) = (\Phi_s)_*(\beta^s_{K+1}, v^s_{K+1}) = (\Phi_s \circ \beta^s_{K+1}, d\Phi_s \circ v^s_{K+1})$$

satisfies the condition $\widetilde{\beta}^0_{K+1} \cap \mathbb{D} = \varnothing$ and extends $(\beta^1_{K+1}, v^1_{K+1})$, so we find a *valid* family of finger germs which extends $(\beta^1_{K+1}, v^1_{K+1})$; see Figure 33.

In conclusion, we may assume that the family of finger germs $(\beta^s_{K+1}, v^s_{K+1})$ which extends $(\beta^1_{K+1}, v^1_{K+1})$ is valid. Use this family to construct $\gamma^{s,u}$ for $s \in [0,1]$ and $u \in \left[0, \frac{1}{3} + 1/3(N-K)\right]$ so that $\gamma^{s,1/3+1/3(N-K)} = \gamma^{s,1/3} \# \beta^s_{K+1}$.

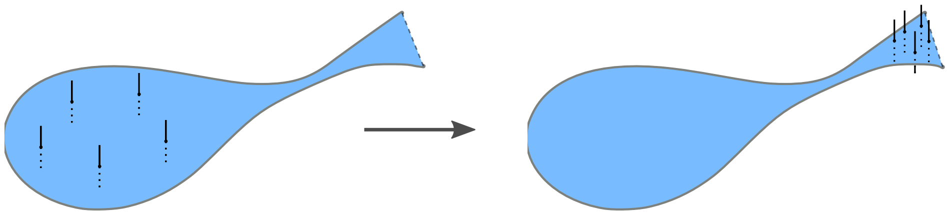

Figure 33: Family of finger bones before (β^s_{K+1}) and after ($\widetilde{\beta}^s_{K+1} = \Phi \circ \beta^s_{K+1}$) applying the described isotopy.

Proceed in the same way with the rest of finger germs ($N-K$ in total), ie extending them to *valid* families of finger germs, to obtain a 2–parameter family $\gamma^{s,u}$ with $s \in [0,1]$ and $u \in \left[0, \frac{2}{3}\right]$, such that:

(i) $\gamma^{s,0} = \gamma^s$.

(ii) $\gamma^{s,2/3} = \gamma^s \# \beta^s_1 \# \cdots \# \beta^s_N$.

(iii) $\gamma^{i,u}$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$.

(iv) There exists a finite sequence $0 < u_1 < \cdots < u_N < \frac{2}{3}$ such that

$$\gamma^{s,u_j} \in \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3) \setminus \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$$

for each $j \in \{1, \ldots, N\}$ and $s \in (0,1)$.

(v) $\gamma^{s,u} \in \mathfrak{Emb}(\mathbb{S}^1, \mathbb{R}^3)$ for $u \neq u_1, \ldots, u_N$ and $s \in (0,1)$.

(vi) The resolution $\gamma^{i,2/3}_{\mathrm{Res}(t_0,t_1)}$ for $i \in \{0,1\}$ is a parametrized unknot and is unlinked from the other resolution $\gamma^{i,2/3}_{\mathrm{Res}(t_1,t_0)}$.

Step III (create a 1–parameter family of self-intersections using $\gamma^{s,2/3}$) We will further extend the family $\gamma^{s,u}$ with $u \in \left[0, \frac{2}{3}\right]$ to a new family $\gamma^{s,u}$ with $u \in [0,1]$.

There exists a disk which bounds $\gamma^{i,2/3}_{\mathrm{Res}(t_0,t_1)}$. Interpret it as a band $B_i \colon [0,1] \times [0,1] \to \mathbb{R}^3$ for $i \in \{0,1\}$ satisfying

(i) $B_i(1,h) = \gamma^{i,2/3}(t_0) = \gamma^{i,2/3}(t_1)$,

(ii) $B_i(0,h) = \gamma^{i,2/3}\left(\frac{1}{2}(t_0+t_1)\right)$,

(iii) $B_i(t,1) = \gamma^{i,2/3}\left(\frac{1}{2}(t_0+t_1) + t \cdot \frac{1}{2}(t_1-t_0)\right)$,

(iv) $B_i(t,0) = \gamma^{i,2/3}\left(\frac{1}{2}(t_0+t_1) - t \cdot \frac{1}{2}(t_1-t_0)\right)$,

(v) $B_i((0,1)\times(0,1)) \cap \gamma^{i,2/3}(\mathbb{S}^1) = \varnothing$.

Then there exists a 1–parameter family of bands $B_s\colon [0,1]\times[0,1]\to\mathbb{R}^3$ interpolating between B_0 and B_1, satisfying the properties (ii), (iii), (iv), (v) and

(i)′ $B_s(1,0)=\gamma^{s,2/3}(t_0)$, $B_s(1,1)=\gamma^{s,2/3}(t_1)$, $B_s(\{1\}\times(0,1))\cap\gamma^{s,2/3}(\mathbb{S}^1)=\varnothing$.

This is just built by continuously collapsing the two initial bands into small bands $\widetilde{B}^i_\delta\colon [0,1]\times[0,1]\to\mathbb{R}^3$ defined as $\widetilde{B}^i_\delta(t,h)=B^i(\delta\cdot t,h)$. From there, it is simple to interpolate between $\widetilde{B}^0_\delta$ and $\widetilde{B}^1_\delta$ for $\delta>0$ small enough.

In particular,

$$\left(\beta^s_{N+1}(-),v^s_{N+1}(-)\right)=\left(B_s(1,-),\frac{\partial B_s}{\partial t}(1,-)\right)$$

defines a continuous family of finger germs for $\gamma^{s,2/3}$. Use this family to create the required self-intersections; ie define $\gamma^{s,u}$ for $s\in[0,1]$ and $u\in[0,1]$ so that

(i) $\gamma^{s,0}=\gamma^s$,

(ii) $\gamma^{s,1}=\gamma^{s,2/3}\#\beta^s_{N+1}=\gamma^s\#\beta^s_1\#\cdots\#\beta^s_{N+1}$,

(iii) $\gamma^{i,u}$ for $i\in\{0,1\}$ has one self-intersection at times $(t_0,t_1)\in\mathbb{S}^1\times\mathbb{S}^1$,

(iv) $\gamma^{s,1}$ has exactly self-intersection at times $(t_0,t_1)\in\mathbb{S}^1\times\mathbb{S}^1$,

(v) there exists a finite sequence $0<u_1<\cdots<u_N<1$ such that

$$\gamma^{s,u_j}\in\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3)\setminus\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$$

for each $j\in\{1,\dots,N\}$ and $s\in(0,1)$,

(vi) $\gamma^{s,u}\in\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$ for $u\neq 1,u_1,\dots,u_N$ and $s\in(0,1)$.

This completes the argument to prove the first part of the proposition. For the relative construction observe that since $\mathbb{S}^1\setminus\{c\}$ is connected, all the finger germs β^s_j for $j\in\{1,\dots,N+1\}$ can be chosen so that $\beta^s_j(0),\beta^s_j(1)\neq\gamma^s(c)$, ie $b^s_j,e^s_j\in\mathbb{S}^1\setminus\{c\}$.

Case 2 $(\{s_1,\dots,s_\nu\}\neq\varnothing)$ Observe that in the proof of the first case all the deformations over the family of immersions $\gamma^{s,u}\colon\mathbb{S}^1\to\mathbb{R}^3$ take place in a neighborhood of at most two points in the domain $z_0(s,u),z_1(s,u)\in\mathbb{S}^1$. There is a finite number of steps at times $0<u_1<\cdots u_N\le 1$ with $N\ge 0$ such that $\gamma^{s,u}\in\mathfrak{Imm}(\mathbb{S}^1,\mathbb{R}^3)\setminus\mathfrak{Emb}(\mathbb{S}^1,\mathbb{R}^3)$ for $s\in[0,1]$. We may assume in all the steps that the points $z_0(s_j,u_k),z_1(s_j,u_k),\tau^0_{s_j},\tau^1_{s_j}$ for $(j,k)\in\{1,\dots,\nu\}\times\{1,\dots,N\}$ are pairwise different. Observe that a generic choice of the times $z_0(s,u)$, $z_1(s,u)$ satisfies this assumption. Thus, this case follows immediately from the previous one. □

8.1.2 Construction of the square Theorem 7.2.5 and Lemma 8.1.8 allow us to state the last proposition in the Legendrian setting, which provides the answer that we are looking for. To simplify the statement we introduce the following definition.

Definition 8.1.14 Let γ^s, $s \in [0,1]$, be a path of Legendrian immersions such that:

(i) γ^s is a Legendrian embedding for $s \in [0,1] \setminus \{0, s_1, \dots, s_\nu, 1\}$, where $0 < s_1 < \cdots < s_\nu < 1$ for $\nu \geq 0$.

(ii) $\gamma^i \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $i \in \{0,1\}$ has a self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^{s_j} \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $j \in \{1, \dots, \nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$.

A homotopy $\gamma^{s,u}$, $(s,u) \in [0,1] \times [0,1]$, of paths of Legendrian immersions between $\gamma^{s,0} = \gamma^s$ and $\gamma^{s,1}$ is *admissible* if there exists a finite sequence $0 < u_1 < \cdots < u_N \leq 1$ such that:

(i) $\gamma^{s,u_j} \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $j \in \{1, \dots, N\}$ and $s \in (0,1)$.

(ii) $\gamma^{s,u} \in \mathfrak{Leg}(\mathbb{R}^3)$ for $u \neq u_1, \dots, u_N$ and $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.

(iii) $\gamma^{i,u} \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ has a self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$ for $i \in \{0,1\}$ and $u \in [0,1]$.

(iv) $\gamma^{s_j,u} \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $j \in \{1, \dots, \nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$ and $u \in [0,1]$. Moreover, the intersection times of γ^{s_j,u_k} for $k \in \{1, \dots, N\}$ do not coincide, ie the real numbers $t_0, t_1, \tau^0_{s_j}, \tau^1_{s_j}$ are pairwise different.

(v) Moreover, the segments $\gamma^{i,u} \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ have self-intersections of type $\Sigma^{1,0}$ except at an even finite number of values $u \in (0,1)$ in which the self-intersections will be of type $\Sigma^{1,1,0}$.

Let us detail some coorientation issues in $\Sigma^{1,0}$. There is a standard orientation of the normal bundle of $\Sigma^{1,0} \setminus \Sigma^{1,1,0}$. It is defined at $\gamma \in \Sigma^{1,0} \setminus \Sigma^{1,1,0}$ as follows. Denote by $\gamma(t_0) = \gamma(t_1)$ the self-intersection point. Let us order the two branches. We declare that the branch through $\gamma(t_0)$ goes first and the branch through $\gamma(t_1)$ second if and only if $\langle \gamma'(t_0), \gamma'(t_1) \rangle$ is a positive basis of $\xi_{\gamma(t_0)}$. We want to define the normal vector to $\Sigma^{1,0} \setminus \Sigma^{1,1,0}$ at γ that is a field in $T_\gamma \mathfrak{LegImm}(\mathbb{R}^3)$, ie a vector field on $\gamma^* T\mathbb{R}^3$. We define it as a vector field supported around $\gamma(t_1)$ which extends to a contact vector field in a small ball $\widetilde{B}$ centered at $\gamma(t_1) \in \mathbb{R}^3$. This contact vector field has as associated Hamiltonian function a cut-off function $\chi\colon \widetilde{B} \to [0,1]$ satisfying $\chi(0) = 1$, $d\chi(0) = 0$

and $\chi|_{\mathcal{O}p(\partial \widetilde{B})} = 0$. In other words, we are pushing the second branch through the Reeb vector field.

This definition does not extend continuously to $\Sigma^{1,1,0}$. However, it provides a coorientation for $\Sigma^{1,0} \setminus \Sigma^{1,1,0}$. We call this coorientation the *standard* one. Let γ^s be the path of Legendrian immersions such that $\gamma^i \in \Sigma^{1,0}$ for $i \in \{0,1\}$. The vectors

$$v_i = (-1)^{i-1} \frac{d\gamma^s}{ds}\bigg|_{s=i} \in T_{\gamma^i} \mathfrak{LegImm}(\mathbb{R}^3)$$

for $i \in \{0,1\}$ determine two orientations of the normal bundle of $\Sigma^{1,0} \setminus \Sigma^{1,1,0}$ at γ^0 and γ^1. We need the coorientation defined by these two vectors to coincide with the standard one.

In these terms, the main result of the section is the following:

Proposition 8.1.15 *Let γ^s for $s \in [0,1]$ be a path of Legendrian immersions such that:*

(i) *γ^s is a Legendrian embedding for $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.*

(ii) *$\gamma^i \in \mathfrak{LegImm}_0^{\textbf{strict}}(\mathbb{R}^3)$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^s \in \mathfrak{LegImm}_0^{\textbf{strict}}(\mathbb{R}^3)$ for $i \in \{s_1, \dots, s_\nu\}$ has a self-intersection at times $(\tau_{s_j}^0, \tau_{s_j}^1) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

(iii) *Moreover, $v_i = (-1)^{i-1}(d\gamma_t/dt)|_{t=i}$ defines the standard coorientation at γ^i for $i \in \{0,1\}$.*

Then, there exists an admissible homotopy $\gamma^{s,u} \in \mathfrak{LegImm}(\mathbb{R}^3)$, $(s,u) \in [0,1] \times [0,1]$, with $\gamma^{s,0} = \gamma^s$ such that $\gamma^{s,1} \in \mathfrak{LegImm}_0^{\textbf{strict}}(\mathbb{R}^3)$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$.

We need a sequence of simple lemmas. We assume them to be true and we will prove them later.

The following lemma is fairly general in homotopy theory and says that if you have a retraction $K \to L$, then by precomposing with the retraction, any map on L extends to K. For our purpose $K = [0,1] \times [0,1]$ and $L \subseteq K$ such that the K deformation retracts to L, and we have a map $L \to \mathfrak{LegImm}(\mathbb{R}^3)$. This allows us to extend local deformations in a path of Legendrian immersions to produce a complete square.

Lemma 8.1.16 *Let $0 < \delta < s_1$ be a positive number. Fix $\gamma^{s,u} \in \mathfrak{LegImm}(\mathbb{R}^3)$ for $(s,u) \in L = [0,\delta] \times [0,1] \cup [0,1] \times [0,\delta]$. Assume that*

(i) $\gamma^{i,u} \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ *for* $i \in \{0,1\}$,

(ii) $\gamma^{s,u} \in \mathfrak{Leg}(\mathbb{R}^3)$ *for* $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.

Then there exists a family $\widetilde{\gamma}^{s,u} \in \mathfrak{LegImm}(\mathbb{R}^3)$ with $(s,u) \in [0,1] \times [0,1]$ satisfying

(i) $\widetilde{\gamma}^{s,u} = \gamma^{s,u}$ *if* $(s,u) \in L$,

(ii) $\widetilde{\gamma}^{s,u} \in \mathfrak{Leg}(\mathbb{R}^3)$ *if* $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$,

(iii) $\widetilde{\gamma}^{1,u} = \gamma^{1,\delta}$ *and* $\widetilde{\gamma}^{s_j,u} = \gamma^{s_j,\delta}$ *for* $j \in \{1, \dots, \nu\}$ *if* $u \in [\delta, 1]$.

The following lemma allows us to assume, up to reparametrization, that there is a fixed cusp in any family of Legendrian immersions parametrized by the segment (we need to add parametric Reidemeister I Legendrian moves, see the proof for the details).

Lemma 8.1.17 *Let γ^s be a path of Legendrian immersions such that*

(i) γ^s *for* $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$ *is a Legendrian embedding,*

(ii) $\gamma^i \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ *for $i \in \{0,1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^s \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $s \in \{s_1, \dots, s_\nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

There exists an admissible homotopy $\gamma^{s,u}$ for $(s,u) \in [0,1] \times [0,1]$, with $\gamma^{s,0} = \gamma$ satisfying:

(i) $\gamma^{s,u} \in \mathfrak{Leg}(\mathbb{R}^3)$ *for* $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.

(ii) *There exists a sequence of times $c_1, \dots, c_{2k} \in \mathbb{S}^1$ such that all the cusps of $\gamma^{s,1}$ are at times c_i.*

The following lemma plays the role of Steps I and II in the proof of Proposition 8.1.13 and solves our problem modulo the creation of the interpolating bands.

Lemma 8.1.18 *Let γ^s be a path of Legendrian immersions such that*

(i) γ^s *for* $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$ *is a Legendrian embedding,*

(ii) $\gamma^i \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ *for $i \in \{0,1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^s \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $s \in \{s_1, \dots, s_\nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

There exists an admissible homotopy $\gamma^{s,u}$ *for* $(s,u) \in [0,1] \times [0,1]$, *with* $\gamma^{s,0} = \gamma^s$ *such that*:

(i) *The self-intersection times* $(t_0(i,u), t_1(i,u))$ *of* $\gamma^{i,u}$ *for* $i \in \{0,1\}$ *satisfy that* $t_1(i,1) < \tau^0_{s_j}, \tau^1_{s_j} < t_0(i,0)$ *for* $j \in \{1,\dots,\nu\}$.

(ii) $\gamma^{s,1} \in \mathfrak{Leg}(\mathbb{R}^3)$ *for* $s \in (0,1) \setminus \{s_1,\dots,s_\nu\}$,

(iii) $\gamma^{i,1}_{\mathrm{Res}(t_0,t_1)}$ *for* $i \in \{0,1\}$ *is a Legendrian (stabilized) unknot and is unlinked from* $\gamma^{i,1}_{\mathrm{Res}(t_1,t_0)}$.

We will need to extend a strict Legendrian immersion into a path in which the self-intersection points cross a positive (negative) stabilization. This will be key to changing the formal invariants associated to the resolutions $\gamma_{\mathrm{Res}(t_0,t_1)}$ and $\gamma_{\mathrm{Res}(t_1,t_0)}$, since we will add/subtract a stabilization to/from each of the resolutions. The price that you pay is that the new path has two cusp points ($\Sigma^{1,1,0}$). We are being precise here: *moving the self-intersection point through two cusps in the front projection creates two cusps in the induced curve in the moduli of strict Legendrian immersions.*

Lemma 8.1.19 *Assume that there is a 2–disk* $\varphi\colon \mathbb{D} \to \mathfrak{LegImm}(\mathbb{R}^3)$ *which possesses just one connected curve* C_0, *parametrized as* γ^s, $s \in [0,1]$, *of strict immersions which is a properly embedded segment on the disk. Denote by* t_0 *and* t_1 *the self-intersection times. Moreover, assume that* $\gamma^s \in C_0 \subset \mathfrak{LegImm}^{\mathbf{strict}}_0(\mathbb{R}^3)$ *possesses a positive DS at time* $t_0 + \delta$ *for* $\delta > 0$ *small enough. Then*:

(i) *There is a local* C^0 *deformation of the disk* $\varphi_t\colon \mathbb{D} \to \mathfrak{LegImm}(\mathbb{R}^3)$, $t \in [0,1]$, *relative to the boundary, such that for* $t = 1$, *the curve* C_0 *is deformed into a curve* C_1 *with* 4 *cusps: points of type* $\Sigma^{1,1,0}$ *(a DS in the moduli). Moreover,* $d_{C^0}(\varphi, \varphi^t) = O(\delta)$.

(ii) *Parametrize the curve* C_1 *by* $\tilde{\gamma}^s$, $s \in [0,1]$. *Assume that* $\tilde{\gamma}^{1/2}$ *lies between the second and the third cusp in the moduli, and denote its self-intersection times by* t_0 *and* t_1. *Then* $(\tilde{\gamma}^{1/2})^1_{\mathrm{Res}(t_0,t_1)}$ *inherits a negative stabilization and* $(\tilde{\gamma}^{1/2})^1_{\mathrm{Res}(t_1,t_0)}$ *inherits a positive stabilization with respect to the resolutions associated to* $\gamma^{1/2}$.

The *standard* normal vector for a strict Legendrian immersion of type $\Sigma^{1,0}$ is preserved when we cross over a cusp (a point of type in $\Sigma^{1,1,0}$) in the moduli space (see Figure 37), so we can easily assume that the orientations are preserved in the previous construction, ie the *standard* normal vector is the same as the vector field normal to the curve in the

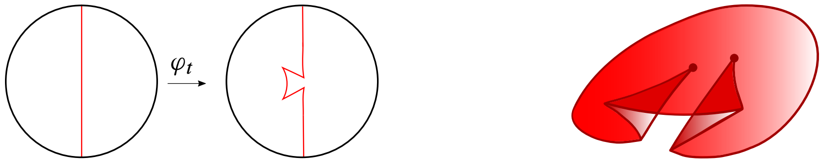

Figure 34: The 1–parameter family of disks $\varphi_t\colon \mathbb{D} \to \mathfrak{LegImm}(\mathbb{R}^3)$ for $t \in [0,1]$, described in Lemma 8.1.19. Left: Change in the topology of the disk in the moduli space via φ_t. Right: Birth of four cusps in the moduli space.

1–parameter family constructed by the rule of avoiding tangencies with the curve of strict immersions.

The next lemma provides a way to match the formal invariants of the resolutions of two generic strict Legendrian immersions. This is a necessary condition to join two generic strict Legendrian immersions by a path of generic strict Legendrian immersions. In other words we want the ends of the path to be stabilized Legendrian unknots, with the same formal invariants. Therefore, by the classification theorem of Legendrian unknots [10], they are the same Legendrian knot. Thanks to that fact, it makes sense to mimic the smooth proof and look for a 1–parameter family of fingers interpolating between both of them.

Lemma 8.1.20 *Let γ^s be a path of Legendrian immersions such that*:

(i) *γ^s is a Legendrian embedding for $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.*

(ii) *$\gamma^i \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0,t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^s \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $s \in \{s_1, \dots, s_\nu\}$ has a self-intersection at times $(\tau_{s_j}^0, \tau_{s_j}^1) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

There exists an admissible homotopy $\gamma^{s,u}$, $(s,u) \in [0,1] \times [0,1]$, with $\gamma^{s,0} = \gamma^s$ satisfying

(i) *$\gamma^{s,1} \in \mathfrak{Leg}(\mathbb{R}^3)$ for $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$,*

(ii) *$\mathrm{tb}(\gamma^{0,1}_{\mathrm{Res}(t_0,t_1)}) = \mathrm{tb}(\gamma^{1,1}_{\mathrm{Res}(t_0,t_1)})$ and $\mathrm{Rot}(\gamma^{0,1}_{\mathrm{Res}(t_0,t_1)}) = \mathrm{Rot}(\gamma^{1,1}_{\mathrm{Res}(t_0,t_1)})$.*

The next lemma provides the last step in the proof: we show how to mimic the band construction in the Legendrian setting. It allows us to connect two strict Legendrian immersions by a path of strict Legendrian immersions. Thus, to prove Proposition 8.1.15 we just need to deform our initial path, via an admissible homotopy, to a path which satisfies the hypotheses of this lemma.

Lemma 8.1.21 *Let γ^s, $s \in [0,1]$, be a path of Legendrian immersions such that*:

(i) *γ^s is a Legendrian embedding for $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$.*

(ii) *$\gamma^i \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$, and $\gamma^s \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $s \in \{s_1, \dots, s_\nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$. Moreover, $t_1 < \tau^0_{s_j}, \tau^1_{s_j} < t_0$ for $j \in \{1, \dots, \nu\}$.*

(iii) *There exists a time $c_1 \in (t_0, t_1) \subseteq \mathbb{S}^1$ such that γ^s has a cusp at time c_1 and $\gamma^s|_{\mathcal{O}p(\{c_1\})}$ is constant.*

(iv) *For $i \in \{0,1\}$, $\gamma^i_{\mathrm{Res}(t_0,t_1)}$ is a Legendrian (stabilized) unknot and is unlinked from $\gamma^i_{\mathrm{Res}(t_1,t_0)}$.*

(v) *$\gamma^0_{\mathrm{Res}(t_0,t_1)}$ and $\gamma^1_{\mathrm{Res}(t_0,t_1)}$ are formally isotopic.*

Then there exists an admissible homotopy $\gamma^{s,u}$, $(s,u) \in [0,1] \times [0,1]$, with $\gamma^{s,0} = \gamma^s$, such that

(i) *$\gamma^{s,u}|_{\mathcal{O}p(\{c_1\})}$ is constant,*

(ii) *$\gamma^{s,u} \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ for $(s,u) \in \{0,1\} \times [0,1] \cup [0,1] \times \{1\}$ has one self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$.*

Proof of Proposition 8.1.15 The proof is divided into four steps.

Step I (fix the cusp times) Apply Lemma 8.1.17 to produce a family $\gamma^{s,u}$, with $(s,u) \in [0,1] \times \left[0, \frac{1}{4}\right]$, such that

(i) $\gamma^{s,0} = \gamma^s$,

(ii) $\gamma^{s,u} \in \mathfrak{Leg}(\mathbb{R}^3)$ for $s \in (0,1) \setminus \{s_1, \dots, s_\nu\}$,

(iii) $\gamma^{i,u} \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ has a self-intersection at times $(t_0, t_1) \in \mathbb{S}^1 \times \mathbb{S}^1$ for $i \in \{0,1\}$, and $\gamma^{s,u} \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$, $s \in \{s_1, \dots, s_\nu\}$, has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j}) \in \mathbb{S}^1 \times \mathbb{S}^1$,

(iv) there exists a sequence of times $c_1, \dots, c_{2k} \in \mathbb{S}^1$ such that the cusps of $\gamma^{s,1/4}$ are at times c_i.

Assume that $c_1 \in (t_0, t_1)$ and that $\gamma^{s,1/4}|_{\mathcal{O}p(\{c_1\})}$ is constant. This property will be preserved all over throughout the construction.

Step II (unknotting process) Apply Lemma 8.1.18 to find an extended family $\gamma^{s,u}$, with $(s,u) \in [0,1] \times \left[0, \frac{2}{4}\right]$, such that:

(i) $\gamma^{s,u}$ is an admissible homotopy.

(ii) $\gamma^{i,u}$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0(i,u), t_1(i,u))$, and $\gamma^{s,u}$ for $s \in \{s_1,\dots,s_\nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j})$. Moreover, $t_1(i,1) < \tau^0_{s_j}, \tau^1_{s_j} < t_0(i,0)$ for $(i,j) \in \{0,1\} \times \{1,\dots,\nu\}$.

(iii) $\gamma^{s,u}|_{\mathcal{O}p(\{c_1\})}$ is constant for $u \geq \frac{1}{4}$.

(iv) The resolution $\gamma^{i,2/4}_{\mathrm{Res}(t_0,t_1)}$ for $i \in \{0,1\}$ is a (stabilized Legendrian) unknot unlinked from the other one.

Step III (match the formal invariants of the resolutions $\gamma^{0,2/4}_{\mathrm{Res}(t_0,t_1)}$ and $\gamma^{1,2/4}_{\mathrm{Res}(t_0,t_1)}$) Use Lemma 8.1.20 to extend our family to $\gamma^{s,u}$ for $(s,u) \in [0,1] \times \left[0, \frac{3}{4}\right]$ so that:

(i) $\gamma^{s,u}$ is an admissible homotopy.

(ii) $\gamma^{i,u}$ for $i \in \{0,1\}$ has one self-intersection at times $(t_0(i,u), t_1(i,u))$, and $\gamma^{s,u}$ for $s \in \{s_1,\dots,s_\nu\}$ has a self-intersection at times $(\tau^0_{s_j}, \tau^1_{s_j})$. Moreover, $t_1(i,1) < \tau^0_{s_j}, \tau^1_{s_j} < t_0(i,0)$ for $(i,j) \in \{0,1\} \times \{1,\dots,\nu\}$.

(iii) $\gamma^{s,u}|_{\mathcal{O}p(\{c_1\})}$ is constant for $u \geq \frac{1}{4}$.

(iv) The resolution $\gamma^{i,3/4}_{\mathrm{Res}(t_0,t_1)}$ for $i \in \{0,1\}$ is an unknot unlinked from the other one.

(v) $\gamma^{0,3/4}_{\mathrm{Res}(t_0,t_1)}$ and $\gamma^{1,3/4}_{\mathrm{Res}(t_0,t_1)}$ are formally isotopic Legendrian embeddings (stabilized unknots).

Step IV (create a 1–parameter family of self-intersections) Finally, observe that $\gamma^{s,3/4}$ satisfies the hypotheses of Lemma 8.1.21. So apply this lemma to extend our family to $\gamma^{s,u}$ with $(s,u) \in [0,1] \times [0,1]$. The family $\gamma^{s,u}$ satisfies the required properties. □

8.1.3 Proofs of Lemmas 8.1.16–8.1.21

There are two cases.

Case 1 ($\{s_1,\dots,s_\nu\} = \varnothing$) We assume that the set of points $\{s_1,\dots,s_\nu\}$ is empty. We explain afterwards the reason why the same proofs work for the general case.

Proof of Lemma 8.1.16 Take a retraction $r\colon [0,1] \times [0,1] \to L$ such that

(i) $r(1,u) = (1,\delta)$ for all $u \in [\delta, 1]$,

(ii) $r((0,1) \times [0,1]) \subseteq (0,1) \times [0,1]$.

Then define $\widetilde{\gamma}^{s,u} = \gamma^{r(s,u)}$. □

Proof of Lemma 8.1.17 Lemma 7.2.10 and Corollary 7.2.11 also work for γ^s. □

Proof of Lemma 8.1.18 The first two steps in the proof of Proposition 8.1.13 readily imply, via Lemma 8.1.8, the result. □

Proof of Lemma 8.1.19 Draw in the Lagrangian projection a picture with two intersecting branches: one of the branches with slope -1 and orientation from right to left, and the other one horizontal with a stabilization on it. The stabilization lies to the right of the intersection point. We continuously move the intersection point from left to right, making sure that the area of the intersection point is zero. So we get an actual intersection in the Legendrian above. Visual inspection shows that if the track intersection point has area zero, all the other intersections in the Lagrangian projection have nonzero area, and therefore they are not actual intersections in the Legendrian. There are 2 times $t_0 < \tau_0 < \tau_1 < t_1$ in which the branch becomes tangent. Those times correspond to cusps in the moduli space: curves of type $\Sigma^{1,1,0}$. See Figure 35 for a description of half of the deformation (crossing of the first stabilization), and Figure 34 for a description of the 2–dimensional family as a subset of a 3–dimensional ball of Legendrian immersions.

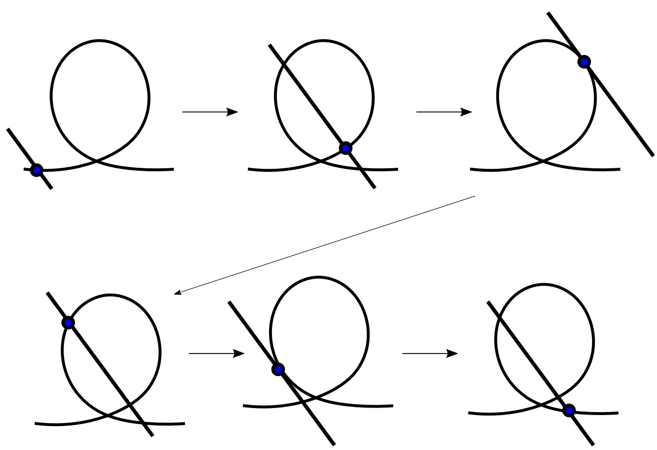

Figure 35: Description of the movie in Lagrangian projection.

The last statement follows if we are able to extend the standard orientation over the cusps. We want to create a vector field $v_s \in T_{\gamma_s}\mathfrak{LegImm}(\mathbb{R}^3)$ transverse to the curve that defines the orientation in the moduli space given by Figure 37. Flowing along this vector, we obtain the required band. First, observe that away from the two cusp points this just corresponds to the standard orientation. Denote by $\gamma_s|_{s\in[0,1/2]}$ the part of the curve that crosses the first two cusps in the moduli and fix $\gamma_{1/6}$ and $\gamma_{2/6}$ to be those cusps. For $s \in [0, \varepsilon]$, with $\varepsilon > 0$ small enough, we interpolate between the standard normal vector for $s = 0$ and, for $s = \varepsilon$, the vector defined by a Hamiltonian function which is constant and equal to 1 in the image of a segment joining two points $t_0', t_1' \in \mathbb{S}^1$ and supported on an arbitrarily small neighborhood of it. We require $t_0 < t_0' < t_1 < t_1' < t_0$. This last condition allows us to homotope those two vectors through transverse vectors. The vector v_ε is clearly equivalent to the one obtained by adding a positive bump (area) to the Lagrangian projection at t_0', and a negative bump at t_1'. Thus, the deformation has support in $[t_0', t_1']$. The algorithm to extend the deformation to the cusp points of the moduli space is depicted in Figure 36 and it corresponds to changing the bump function from t_i' to $t_i' + 2\pi$ for $i \in \{0, 1\}$, creating a family $t_{i,s}'$ with $s \in \left[0, \frac{1}{2}\right]$ such that $t_{i,s}'$ is strictly increasing with s and such that $t_{0,1/6}' = t_0$ and $t_{0,2/6}' = t_1$. Likewise, $t_{1,1/6}' = t_0$ and $t_{1,2/6}' = t_1$. See Figure 36. This clearly produces a continuous family of the deformations that removes the self-intersections. See Figure 37 to visualize the deformation as a tangent vector in the moduli space. $\square$

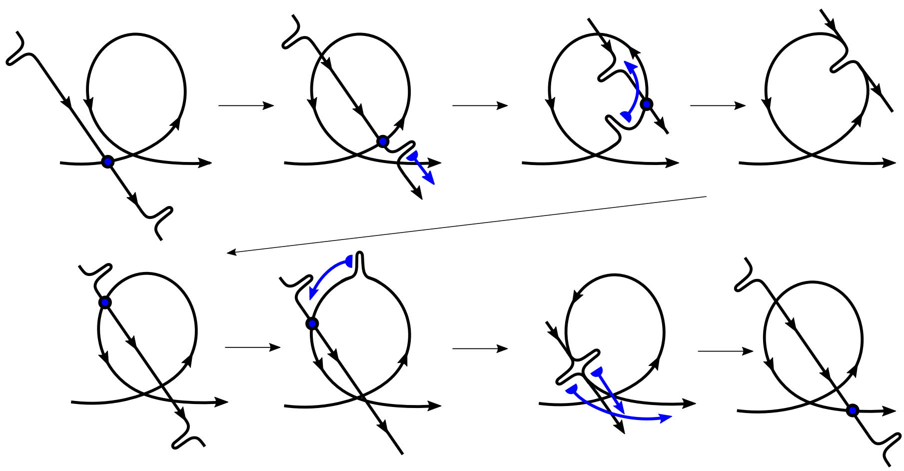

Figure 36: Algorithm in Lagrangian projection to extend the deformation to the cusp points of the moduli space.

Proof of Lemma 8.1.20 Define $\gamma^{s,u}$ for $(s, u) \in [0, 1] \times [0, 1]$ as follows:

(i) If the parities of $\mathrm{tb}(\gamma^0_{\mathrm{Res}(t_0,t_1)})$ and $\mathrm{tb}(\gamma^1_{\mathrm{Res}(t_0,t_1)})$ coincide, define $\gamma^{s,u} = \gamma^s$ for $u \in \left[0, \frac{1}{3}\right]$. Otherwise add one DS in γ^s so that for $s = 0, 1$ it is inserted on the segment (t_0, t_1) for $u \in \left[0, \frac{1}{6}\right]$. Then apply Lemma 8.1.19 to create a band $\widehat{\gamma}^{s',v}$ for $s' \in [0, 1]$ and $v \in [-\varepsilon, \varepsilon]$ such that $\widehat{\gamma}^{0,v} = \gamma^{v,1/6}$ for $v \in [0, \varepsilon]$. Thus we define $\gamma^{s,u} = \widehat{\gamma}^{6(u-1/6),s}$ for

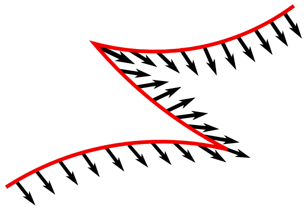

Figure 37: Deformation in the moduli space.

$s \in [0, \varepsilon]$ and $u \in \left[\frac{1}{6}, \frac{1}{3}\right]$. Finally, apply Lemma 8.1.16 to extend $\gamma^{s,u}$ for all $s \in [0,1]$ and $u \in \left[\frac{1}{6}, \frac{1}{3}\right]$. Thus, we have made sure that $\mathrm{tb}(\gamma^{0,1/3}_{\mathrm{Res}(t_0,t_1)}) - \mathrm{tb}(\gamma^{1,1/3}_{\mathrm{Res}(t_0,t_1)}) = 2k$ for $k \in \mathbb{Z}$. Recall from [12, Section 2.7] that the Thuston–Bennequin invariant decreases by 1 after a stabilization and that the rotation number increases (decreases) by 1 whenever the stabilization is positive (negative).

(ii) If $k > 0$, for each $i \in \{1, \dots, k\}$, take a path $B_i \colon [0,1] \to \mathbb{S}^1$, $s \mapsto B_i(s)$, such that $B_i(0) \in (t_0, t_1)$ and $B_i(1) \in \mathbb{S}^1 \setminus [t_0, t_1]$. Recursively define $\gamma^{s,u}$ in $[0,1] \times \left[\frac{1}{3}, \frac{1}{3} + \frac{i}{3k}\right]$ creating a DS on $\gamma^{s,1/3+(i-1)/3k}$ at time $B_i(s)$.

Observe that a DS decreases the Thurston–Bennequin invariant by 2. Thus, we obtain

$$\mathrm{tb}(\gamma^{0,2/3}_{\mathrm{Res}(t_0,t_1)}) = \mathrm{tb}(\gamma^{1,2/3}_{\mathrm{Res}(t_0,t_1)}). \tag{8}$$

Do the symmetric sequence of moves in case $k < 0$ to again obtain (8).

For any $\gamma \in \mathfrak{Leg}(\mathbb{R}^3)$ we have $\mathrm{tb}(\gamma) + \mathrm{Rot}(\gamma) \in 2\mathbb{Z} + 1$; see [16, Proposition 3.5.23]. Thus, since $\mathrm{tb}(\gamma^{0,2/3}_{\mathrm{Res}(t_0,t_1)}) = \mathrm{tb}(\gamma^{1,2/3}_{\mathrm{Res}(t_0,t_1)})$, we conclude that

$$\mathrm{Rot}(\gamma^{0,2/3}_{\mathrm{Res}(t_0,t_1)}) - \mathrm{Rot}(\gamma^{1,2/3}_{\mathrm{Res}(t_0,t_1)}) = 2m. \tag{9}$$

Assume that $m \geq 0$; the other case is totally analogous. Add a DS in (t_0, t_1) and consecutively apply Lemma 8.1.19 and Lemma 8.1.16. This adds a square $\gamma^{s,u}$ with $s \in [0,1]$ and $u \in \left[\frac{2}{3}, \frac{2}{3} + \frac{1}{6m}\right]$. Now add a DS in (t_1, t_0) and consecutively apply Lemma 8.1.19 and Lemma 8.1.16. This adds a second square. In the new top $\gamma^{s,2/3+1/3m}$, the difference (9) has decreased by 2. Repeat $m-1$ more times to make $m = 0$ in that equation.

It follows that $\gamma^{0,1}_{\mathrm{Res}(t_0,t_1)}$ and $\gamma^{1,1}_{\mathrm{Res}(t_0,t_1)}$ have the same formal invariants. □

Proof of Lemma 8.1.21 The proof is divided into several steps.

Step 0 (preparation) By hypothesis, the standard coorientations are matched, so we may assume that the self-intersection times t_0 and t_1 are such that $\{(\gamma^i)'(t_0), (\gamma^i)'(t_1)\}$

for $i \in \{0,1\}$ is a positive basis of ξ at the point of self-intersection $\gamma^i(t_0) = \gamma^i(t_1)$, and that there exists $\varepsilon > 0$ such that

- γ^ε is obtained from γ^0 by pushing $\gamma^0(\mathcal{O}p(\{t_1\}))$ in the Reeb direction for time ε,
- $\gamma^{1-\varepsilon}$ is obtained from γ^1 by pushing $\gamma^1(\mathcal{O}p(\{t_1\}))$ in the Reeb direction for time ε.

We may also think that $\gamma^i = \hat{\gamma}^i \# (\beta_0^i, v_0)$, where $\hat{\gamma}^i$ is a Legendrian embedding and β_0^i is a very small Reeb chord between $\hat{\gamma}^i(t_0)$ and $\hat{\gamma}^i(t_1)$. Moreover, the hypothesis about the coorientations allows us to further assume

(A.1) $\gamma^s = \hat{\gamma}^0 \# \big(\beta_0^0\big(\big(1-\frac{s}{\varepsilon}\big)t\big), v_0\big(\big(1-\frac{s}{\varepsilon}\big)t\big)\big)$ for $s \in [0,\varepsilon]$,

(A.1)′ $\gamma^{1-s} = \hat{\gamma}^1 \# \big(\beta_0^1\big(\big(1-\frac{s}{\varepsilon}\big)t\big), v_0\big(\big(1-\frac{s}{\varepsilon}\big)t\big)\big)$ for $s \in [0,\varepsilon]$,

(A.2) $\gamma^s \equiv \hat{\gamma}^0$ is constant for $s \in [\varepsilon, 2\varepsilon]$,

(A.2)′ $\gamma^{1-s} \equiv \hat{\gamma}^1$ is constant for $s \in [\varepsilon, 2\varepsilon]$.

Assume that the fixed cusp point of the curves γ^s is placed at time $c_1 = \frac{1}{2}(t_0 + t_1)$ in the whole family. Choose a capping disk for $\mathbb{D}_i$ for $\gamma^i_{\mathrm{Res}(t_0,t_1)}$, $i \in \{0,1\}$. Choose small neighborhoods U_i of each disk. By [9], U_1 is contactomorphic to the standard contact $\mathbb{R}^3$ and there is a contactomorphism $\Phi\colon U_0 \to U_1$ such that $\Phi(\gamma^0(t_0)) = \gamma^1(t_0)$ and $\Phi(\gamma^0(c_1)) = \gamma^1(c_1)$. By [10] we can further assume that $\gamma^1_{\mathrm{Res}(t_0,t_1)} = \Phi \circ \gamma^0_{\mathrm{Res}(t_0,t_1)}$. In other words, the construction that we are going to do is symmetric (there is a symmetry $s \to 1-s$ for $s \in [0,2\varepsilon]$); ie all the constructions that we are going to do near $s=0$ immediately translate to $s=1$ by means of the contactomorphism Φ. We do not detail the construction in a neighborhood of $s=1$, just declaring it to be the symmetric construction to the one that we provide for $s=0$.

Use the disk $\mathbb{D}_0$ to construct a band $B_0\colon [0,1]^2 \to \mathbb{R}^3$ with the following properties:

- $B_0(0,v) = \beta_0^0(v)$,
- $B_0(1,v) = \hat{\gamma}^0(c_1)$,
- $B_0(\sigma,0) = \hat{\gamma}^0((1-\sigma)t_0 + \sigma c_1)$,
- $B_0(\sigma,1) = \hat{\gamma}^0((1-\sigma)t_1 + \sigma c_1)$,
- $B_0((0,1)^2) \cap \hat{\gamma}^0(\mathbb{S}^1) = \varnothing$,
- for each $\sigma \in (0,1]$, the map $B_0^\sigma\colon [0,1] \to \mathbb{R}^3$, $y \mapsto B_0(\sigma,v)$ is an embedding.

Let $\delta > 0$ be small enough in such a way that $\gamma^s_{[c_1-\delta, c_1+\delta]}$ is constant (independent of s). Consider the linear interpolation

$$\chi\colon [0,1] \to [t_0, c_1-\delta], \quad s \mapsto (1-s)t_0 + s(c_1-\delta).$$

The family $\widetilde{\beta}^s_0 = B_0^{\chi(s)}$ defines a family of smooth finger bones joining $\widehat{\gamma}^0(\chi(s))$ with $\widehat{\gamma}^0(t_1+t_0-\chi(s))$. Note that $\widetilde{\beta}^0_0 = \beta^0_0$. Use Lemma 8.1.8 to homotope this family through finger bones to a family of transverse finger bones β^s_k. Observe that $\beta^0_k = \widetilde{\beta}^0_k$ by the construction of the interpolation. There is no obstruction to finding a family of vector fields v_s in $(\beta^s)^*\xi$ such that $v_0 = v$ and (β_s, v_s) is a pre-Legendrian finger germ for $\widehat{\gamma}$.

Finally, by Lemma 8.1.6 there exists a nonnegative integer $K \in \mathbb{Z}_{\geq 0}$ such that we can interpolate through strict Legendrian immersions between $\widehat{\gamma}^0 \# (\beta^0_0, v_0)$ and $\widehat{\gamma}^0 \# (\beta^0_k, v_0)$ just by adding a tower of K DSs to the initial finger (by using Lemma 8.1.19). This number K is going to be fixed for the rest of the proof.

Step I (adding DSs to the family) Define $\gamma^{s,u}$ for $(s,u) \in [0,1] \times \left[0, \frac{1}{3}\right]$ by adding a tower of K DSs (see Figure 29) to $\gamma^{s,0} = \gamma^s$ based at the point $\gamma^s(T(s))$, where $T(s)$ satisfies

- $T(s) \equiv t_0 + \delta$ for $s \in [0, \varepsilon]$ and, symmetrically, for $s \in [1-\varepsilon, \varepsilon]$,
- $T(s)$ linearly interpolates between $t_0 + \delta$ and $c_1 - \delta$ for $s \in [\varepsilon, 2\varepsilon]$ and, symmetrically, for $s \in [1-2\varepsilon, 1-\varepsilon]$,[13]
- $T(s) \equiv c_1 - \delta$ for $s \in [2\varepsilon, 1-2\varepsilon]$.

Step II (crossing the first half of the tower of DSs) Use Lemma 8.1.19 to define $\gamma^{s,u}$ for $(s,u) \in [0,1] \times \left[\frac{1}{3}, \frac{2}{3}\right]$ just by crossing half of the tower of DSs through the self-intersection of $\gamma^{i,1/3}$ for $i \in \{0,1\}$ at time t_0. Moreover, do this in such a way that $\gamma^{s,u} = \gamma^{s,1/3}$ for $(s,u) \in [\varepsilon, 1-\varepsilon] \times \left[\frac{1}{3}, \frac{2}{3}\right]$.

Step III (creating the self-intersections) To define $\gamma^{s,u}$ for $(s,u) \in [0,1] \times \left[\frac{2}{3}, 1\right]$, proceed as follows:

- Define $\gamma^{s,u}$ for $s \in [0,\varepsilon]$ in such a way that $\gamma^{s,1} \in \mathfrak{LegImm}_0^{\mathbf{strict}}(\mathbb{R}^3)$ for $s \in [0,\varepsilon]$ interpolates between $\gamma^{0,2/3}$ and $\widehat{\gamma}^0 \# (\beta^0_k, v_0)$. This is a direct application of Lemma 8.1.6 and the fact that we have a tower of K DSs to produce this interpolation. Define $\gamma^{s,u}$ for $s \in [1-\varepsilon, 1]$ symmetrically.

[13] Note that the path of Legendrians is constant over $[\varepsilon, 2\varepsilon]$ and over $[1-2\varepsilon, 1-\varepsilon]$.

- Use the family of pre-Legendrian finger germs (β_k^s, v^s) to define $\gamma^{s,u}$ for $s \in [\varepsilon, 2\varepsilon]$. Do the symmetric construction to define $\gamma^{s,u}$ for $s \in [1-2\varepsilon, 1-\varepsilon]$. Note that the self-intersection of $\gamma^{2\varepsilon,1}$ and, symmetrically, of $\gamma^{1-2\varepsilon,1}$, happens at times $c_1 - \delta$ and $c_1 + \delta$.
- Define $\gamma^{s,u}$ for $s \in [2\varepsilon, 1-2\varepsilon]$ just by using the pre-Legendrian finger germ (β_k^1, v^1) to create a self-intersection at times $c_1 - \delta$ and $c_1 + \delta$. Note that this is trivial because the family is constant near the cusp point.

This concludes the construction of the required family. □

Case 2 $(\{s_1, \dots, s_\nu\} \neq \varnothing)$ We reason as in the smooth case. Let us provide the details. In all the previous proofs we have made deformations in the moduli space $\mathfrak{LegImm}(\mathbb{R}^3)$. These deformations over a point $\gamma^{s,u}\colon \mathbb{S}^1 \to \mathbb{R}^3$ of the moduli space change the type of Legendrian by creating tangencies at a finite number of values of the parameter $0 < u_1 < \cdots < u_N \leq 1$ with $N \geq 0$, ie $\gamma^{s,u_j} \in \mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)$ for $s \in [0,1]$. The tangencies are created identifying at most two points in the domain $z_0(s,u_j), z_1(s,u_j) \in \mathbb{S}^1$. We assume in all the steps that the points $z_0(s_j,u_k)$, $z_1(s_j,u_k)$, $\tau_{s_j}^0$, $\tau_{s_j}^1$ for $(j,k) \in \{1,\dots,\nu\} \times \{1,\dots,N\}$ are pairwise different. This implies that there is no interference between the deformation and the pair of points $\tau_{s_j}^0$ and $\tau_{s_j}^1$. Observe that a generic choice of the times $z_0(s,u)$ and $z_1(s,u)$ satisfies this assumption. So we need to change nothing if we add the set $\{s_1,\dots,s_\nu\}$ in the proofs, except for the fact that we need to make the choices of times (mainly for the finger creation process) generic in the previous sense.

8.1.4 Index-1 surgeries

Proposition 8.1.22 *Let $\mathbb{D}(x,y) \subseteq \mathbb{R}^2$ be the disk of radius 2, and let $\varphi\colon \mathbb{D}(x,y) \to \mathfrak{Hor}(\mathbb{R}^4)$ be a map into the horizontal embeddings space such that*

(i) *the configuration of $\pi_G \circ \varphi$ is given by $\nu+2$ vertical segments, ie*

$$(\pi_G \circ \varphi)^{-1}(\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)) = (\{x=0\} \cup \{x=s_1\} \cup \cdots \cup \{x=1\}) \cap \mathbb{D}(x,y)$$

for $\nu \geq 0$ and $0 < s_1 < \cdots < s_\nu < 1$,

(ii) *the area function has the same sign over the first and the last segments.*

Then there exists a 1–parameter family of disks $\varphi_t\colon \mathbb{D} \to \mathfrak{Hor}(\mathbb{R}^4)$ with $t \in [0,1]$ satisfying

(i) $\varphi_0 = \varphi$,

(ii) $\varphi_t|_{\partial\mathbb{D}} = \varphi_0|_{\partial\mathbb{D}}$,

(iii) *the configuration of $\pi_G \circ \varphi_1$ is given by the right diagram of Figure 38.*

The changes in the configuration of curves of strict immersions in $\pi_G \circ \varphi_t$ moving from $\pi_G \circ \varphi_0$ to $\pi_G \circ \varphi_1$ correspond to an ambient index-one surgery whose attaching points are the middle points $(0,0)$ and $(1,0)$ of the two vertical segments $\{x=0\}$ and $\{x=1\}$. The attached disk is given by $[0,1]\times[-\varepsilon,\varepsilon]$ with $\varepsilon>0$ small enough, and the surgery replaces the pair of 1–disks $\{0,1\}\times[-\varepsilon,\varepsilon]$ by the pair of 1–disks $[0,1]\times\{\pm\varepsilon\}$. The new 1–disks, which are curves of strict immersions, have an even number of cusp points and they do not add new zeros to their the area functions. The other segments $\{x=s_1\},\dots,\{x=s_\nu\}$ remain untouched in the diagram. Moreover, there are a finite number of closed circular components of type $S_n = \big\{(x,y)\in\mathbb{D} : \big(x-\frac{1}{2}\big)^2+y^2=r_n^2\big\}$ with $0<r_1<\cdots<r_n<c$. These new closed components do not have cusp points and have positive area function everywhere.

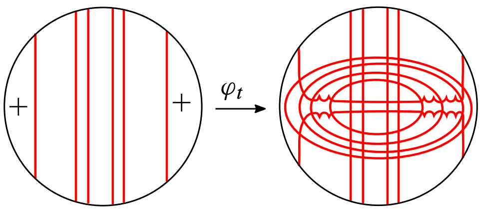

Figure 38: Description of the homotopy φ_t. Observe that the cusp points appear in pairs, therefore no zeros of the area function appear.

Proof Denote the two curves of strict Legendrian immersions in $\mathbb{D}_G = \pi_G\circ\varphi(\mathbb{D})$ by C_0 and C_1. Let $i\in\{0,1\}$ and fix $\gamma^i\in C_i$. Choose a path γ^s, $s\in[-1,2]$, transversal to the strict immersion curves at C_i such that $C_i\cap\{\gamma^s : s\in[-1,2]\}=\{\gamma^i\}$. A neighborhood $K=\{\gamma_r^s : s\in[-1,2], r\in[-1,1]\}\subseteq\mathbb{D}$ of the path γ^s may be assumed to satisfy $\gamma_r^s\equiv\gamma^s$ for $s\in[-1,2]$.

Apply Proposition 8.1.15 to find a disk $E=\{\gamma^{s,u} : s,u\in[0,1]\}$ such that:

- $\gamma^{s,0}=\gamma^s$ for $s\in[0,1]$.
- $C=\big\{\gamma^{s,u}\in\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3) : (s,u)\in\{0,1\}\times[0,1]\cup[0,1]\times\{1\}\big\}$ is a continuous curve of strict Legendrian immersions, with self-intersection times $t_0^{s,u}, t_1^{s,u}\in\mathbb{S}^1$ with $t_0^{s,u}\neq t_1^{s,u}$.
- $\mathcal{S}_j=\{\gamma^{s_j,u} : u\in[0,1]\}$ for $j\in\{1,\dots,\nu\}$ is a curve of strict Legendrian immersions.

- There is a finite number of curves $\mathcal{C}_n$ of strict Legendrian immersions defined by
$$\mathcal{C}_n = \{\gamma^{s,u} : (s,u) \in [0,1] \times \{u_n\}\},$$
where $0 < u_1 < \cdots < u_n < 1$. The self-intersection times of γ^{s,u_n} are $\tau_0^{s,u_n}, \tau_1^{s,u_n} \in \mathbb{S}^1$, with $\tau_0^{s,u_n} \neq \tau_1^{s,u_n}$. These curves do not have cusp points.
- $\tau_i^{s,u_n} \neq t_j^{s,u_n}$ for each $i, j \in \{0,1\}$ and $(s,u_n) \in C \cap \mathcal{C}_n$.

Apply Lemma 7.2.1 to lift E into a disk of horizontal immersions. Now, apply again Lemma 7.2.1 over C in $t_0^{s,u} - \varepsilon$ and $t_0^{s,u} + \varepsilon$, for $\varepsilon > 0$ small enough in order to increase the area function at will. Do likewise over the self-intersection points in $\mathcal{C}_n$. Note that there is no interaction between the two deformations because the self-intersection times are different.

By construction the outward normal vector field X to $C = \partial E$ coincides with the standard orientation. Push C in the direction of X to obtain an extended disk $\widetilde{E} = \{\gamma^{s,u} : s \in [-1,2], u \in [0,2]\} \supseteq E$ in $\mathfrak{LegImm}(\mathbb{R}^3)$, which lifts to $\mathfrak{Hor}(\mathbb{R}^4)$, such that $\gamma^{s,0} = \gamma^s$ for each $s \in [-1,2]$. We define the map
$$\psi \colon [-1,2] \times [0,2] \times [-1,1] \to \mathfrak{Hor}(\mathbb{R}^4), \quad (s,u,r) \mapsto \gamma_r^{s,u} = \gamma^{s,u}.$$

See Figure 39 to visualize the piecewise linear disk that we are about to define in $[-1,2] \times [0,2] \times [-1,1] \subseteq \mathbb{R}^3$. First, we define the base (a closed piecewise linear curve in $\mathbb{R}^2$):
$$\Delta_B = \{(s,r) \in [-1,2] \times [-1,1] : |r| \leq g(s)\} \quad \text{for } g(s) = \begin{cases} 1+s & \text{if } s \in [-1,0], \\ 1 & \text{if } s \in [0,1], \\ 2-s & \text{if } s \in [1,2]. \end{cases}$$

And now the piecewise linear graph in $\mathbb{R}^3$ as
$$\Delta = \{(s, u(s), r) \in [-1,2] \times [1,2] \times [-1,1] : (s,r) \in \Delta_B\},$$
where
$$u(s) = \begin{cases} 2+s-|r| & \text{if } s \in [-1,0], \\ 2-|r| & \text{if } s \in [0,1], \\ 3-s-|r| & \text{if } s \in [1,2]. \end{cases}$$

We define a new (piecewise linear) disk
$$\widehat{\Delta} = (\partial(\Delta_B) \times [0,1]) \cup \Delta \subseteq [-1,2] \times [0,2] \times [-1,1].$$

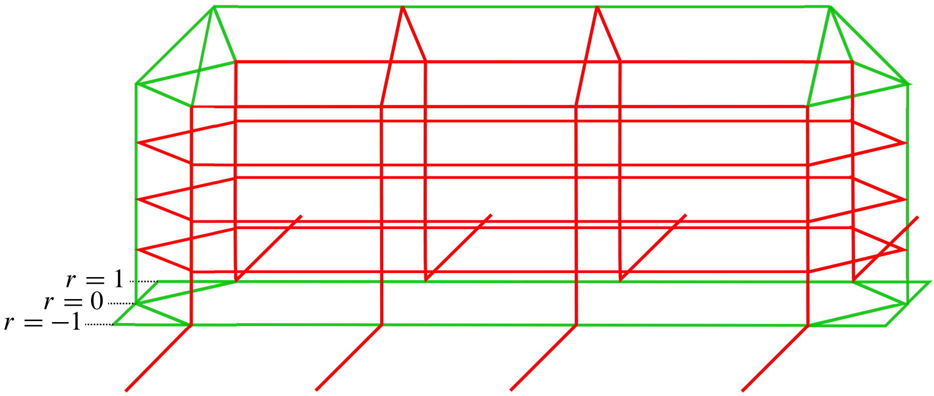

Figure 39: Construction of $\widetilde{\mathbb{D}}$.

Observe that the boundary of $\widehat{\Delta}$ coincides with $\partial\Delta_B$, therefore we can define a new disk $\widetilde{\mathbb{D}} = (\mathbb{D} \setminus \Delta_B) \cup \widehat{\Delta}$ gluing along the boundaries. We can define a new morphism

$$\varphi_1\colon \widetilde{\mathbb{D}} \to \mathfrak{Hor}(\mathbb{R}^4), \qquad p \mapsto \begin{cases} \varphi_0(p) & \text{if } p \in (\mathbb{D} \setminus \Delta_B), \\ \psi(p) & \text{if } p \in \widehat{\Delta}. \end{cases}$$

It remains to check that φ_0 and φ_1 are homotopic. This is performed by building any homotopy between the two disks Δ_B and $\widehat{\Delta}$ relative to the boundary $\partial\Delta_B$ inside $[-1,2] \times [0,2] \times [-1,1]$. □

8.2 Globalization

We use the theory developed in the last two subsections to prove the main theorem of Section 7. We first adapt a previous construction.

Lemma 8.2.1 (Corollary of Lemma 8.1.19) *Let $\varphi\colon \mathbb{D}(x,y) \to \mathfrak{HorImm}(\mathbb{R}^4)$ be a disk of horizontal immersions such that*

(i) $\varphi(\partial\mathbb{D}) \subseteq \mathfrak{Hor}(\mathbb{R}^4)$,

(ii) $(\pi_G \circ \varphi)^{-1}(\mathfrak{LegImm}^{\text{strict}}(\mathbb{R}^3)) = C$ *is a connected curve with a unique zero of the area function $\beta \in C$.*

Then there exists a 1–parameter family of disks $\varphi^t\colon \mathbb{D} \to \mathfrak{HorImm}(\mathbb{R}^4)$, $t \in [0,1]$, satisfying:

(i) $\varphi^0 = \varphi$.

(ii) $\varphi^t|_{\partial\mathbb{D}} = \varphi^0|_{\partial\mathbb{D}}$.

(iii) *There is a C^0 deformation C_t, $t \in [0,1]$, of $C = C_0$ and a family of connected curves S_t such that*:

(a) $(\pi_G \circ \varphi^t)^{-1}(\mathfrak{LegImm}^{\mathbf{strict}}(\mathbb{R}^3)) = C_t \cup S_t$.

(b) *$\beta \in C_t$ is the unique zero of the area function in φ^t and the curves S_t encircle β.*

(c) *$C_t = C_0$ for $t \in \left[0, \frac{1}{2}\right)$.*

(d) *C_1 is a DS of $C_{1/2}$ in the moduli* (see Lemma 8.1.19) *and β lies between the first and the second cusp.*

(e) *$S_t = \varnothing$ for $t \in \left[0, \frac{1}{2}\right)$.*

(f) *$S_t = \left\{x^2 + y^2 = \left(\frac{1}{2} - t\right)^2\right\}$ for $t \in \left[\frac{1}{2}, \frac{3}{4}\right]$.*

(g) *$S_t = S_{3/4}$ for $t \in \left[\frac{3}{4}, 1\right]$.*

(See Figure 40.)

Proof Since the problem is local we may assume that the disk is small and the self-intersection times for the Legendrian projections in the curve C are constant and equal to t_0 and t_1. Choose $t_2 \in \mathbb{S}^1$ a time value different from t_0 and t_1. Use polar coordinates $\mathbb{D} = \mathbb{D}(r,\theta)$ and define parametrically for each horizontal immersion $\varphi(r,\theta)$ a DS that is denoted $B = \mathbb{D} \times \left[0, \frac{1}{2}\right] \to \mathfrak{HorImm}(\mathbb{R}^4)$, where $B(r,\theta,h)$ for $h \in \left[0, \frac{1}{2}\right]$ is the DS of $\varphi(r,\theta)$.

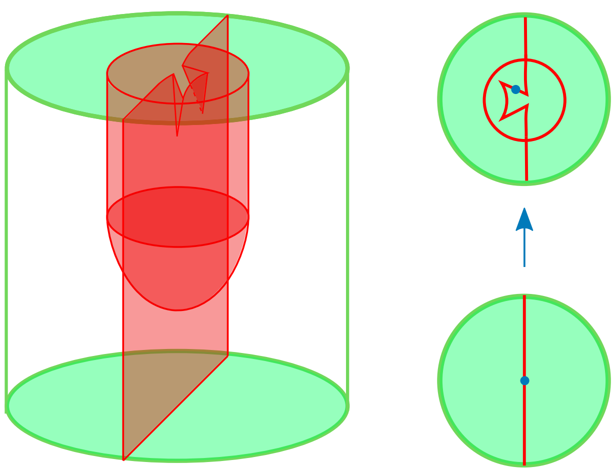

Figure 40: Schematic description of Lemma 8.2.1. The blue point in the middle of the circle represents β and the red middle circle in the cylinder the self-intersection (with nonzero area) created in the DS.

Now we apply Lemma 8.1.19 in order to further deform and create 4 cusp points in the moduli: just sliding one branch (t_0) over the other (t_1). Apply Lemma 7.2.1 to place the zero in the required position. This second deformation produces a family of disks $B(r,\theta,h)$ with $h \in \left[\frac{1}{2}, 1\right]$.

Define

$$L = \left\{B(r,\theta,h) : r = \tfrac{1}{2}, h \in [0,1]\right\} \subseteq \mathrm{Image}(B),$$
$$T = \mathbb{D}\left(r \le \tfrac{1}{2}\right) \times \{h = 1\} \subseteq \mathrm{Image}(B),$$

which builds a piecewise linear disk $\mathbb{D}' = L \cup T$ whose boundary coincides with the boundary of $\mathbb{D}\left(r \le \frac{1}{2}\right) \subseteq \mathbb{D}$. Replace $\mathbb{D}\left(r \le \frac{1}{2}\right)$ by $\mathbb{D}'$, ie define

$$\widetilde{\mathbb{D}} = \left(\mathbb{D} \setminus \mathbb{D}\left(r \le \tfrac{1}{2}\right)\right) \cup \mathbb{D}'.$$

The associated diagram of strict immersions confirms the statement of the lemma. See Figure 40. □

Proof of Theorem 7.1.2 Let γ^θ be a loop of horizontal embeddings which lies in $\pi_1(\mathfrak{Hor}(\mathbb{R}^4))_0$, ie $\mathrm{Rot}_L(\gamma^\theta) = 0$. Assume that $\mathrm{Area}(\gamma^\theta) = 0$; we must check that γ^θ is trivial.

Let $\mathbb{D}$ be a disk in $\mathfrak{HorImm}(\mathbb{R}^4)$ whose boundary is given by $\{\gamma^\theta\}$. Apply successively Proposition 7.3.1 to assume that each curve of strict Legendrian immersions in $\mathbb{D}_G$ has at most one zero of the area function. Since $\mathrm{Area}(\gamma^\theta) = 0$, the number of zeros is even. If $\mathbb{D}_G$ has no zeros then we are done. Hence, assume that $\mathbb{D}_G$ has a positive number of zeros and denote them by $\beta_0, \dots, \beta_{2n-1}$.

Let $C_k \subseteq \mathbb{D}_G$ be the curve of strict Legendrian immersions that contains β_k for $k \in \{0, \dots, 2n-1\}$. It is sufficient to show that we can do a surgery along any two curves C_0 and C_1 to produce a new curve of strict Legendrian immersions that contains the zeros β_0 and β_1 without creating any new zero. Application of Proposition 7.3.1 will provide the required result.

To formalize the previous discussion, note that C_i is naturally oriented by the gradient vector g_i of the area function at β_i. Together with the standard coorientation of $\Sigma^{1,0}$ at β_i, denoted by w_i, it provides an oriented basis at $T_{\beta_i}\mathbb{D}$, namely $\{g_i, w_i\}$. We can assume that the orientations induced at the two points β_0 and β_1 are the same. If it is not the case, apply Lemma 8.2.1 in order to change the orientation of the curve C_0. This works because the area function changes signs when crossing a cusp in the curve C_0 and thus the gradient of the area function over the zero gets multiplied by -1.

Select a pair of points α_i arbitrarily close to β_i whose area function is positive. Fix an embedded curve $\alpha\colon [0,1] \to \mathbb{D}$ such that $\alpha(i) = \alpha_i$ and it is transverse to C_i. Moreover, we assume that $(d\alpha_t/dt)|_{t=0}$ defines the standard coorientation at $\alpha_0 \in \Sigma^{1,0}$. Respectively, $(d\alpha_t/dt)|_{t=1}$ defines the opposite orientation to the standard one (see Figure 41 to learn how to deal with the wrong coorientation case).

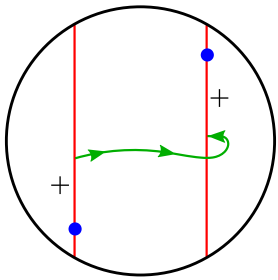

Figure 41: Embedded path connecting α_0 and α_1 conforming to the coorientations rule.

The hypotheses of Proposition 8.1.22 are satisfied by a small tubular neighborhood $\mathbb{D}_\alpha$ of the path. This implies that we can get a deformed disk with the same number of zeros such that the deformation is performed with compact support on the disk $\mathbb{D}_\alpha$, following the diagram explained in Figure 38. This new diagram makes the two zeros β_i be part of the same connected component of curves of strict immersions; see Figure 42. This is ensured by the compatibility of the orientations that we have enforced in the basis $\{g_i, w_i\}$ of $T_{\beta_i}\mathbb{D}$. Apply Proposition 7.3.1 to cancel the two zeros. □

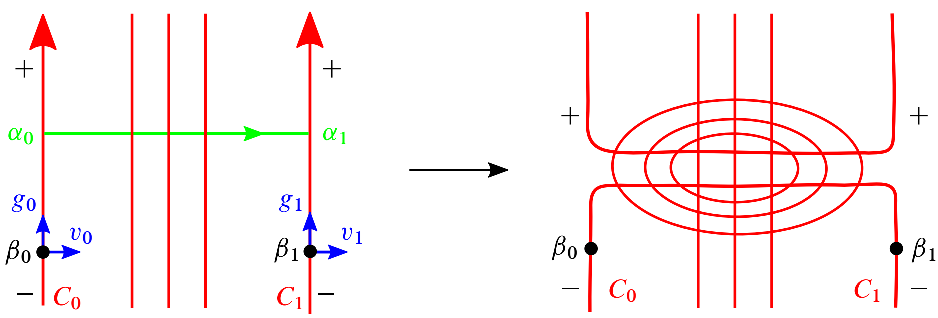

Figure 42: Schematic description of the neighborhood of $\alpha(t)$. The cusps between the concentric circles are not drawn to make the picture easier to understand.

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