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The extrinsic primitive torsion problem

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Let P_k be the subgroup generated by k^{th} powers of primitive elements in F_r, the free group of rank r. We show that F_2/P_k is finite if and only if k is 1, 2 or 3. We also fully characterize F_2/P_k for $k=2,3,4$. In particular, we give a faithful 9–dimensional representation of F_2/P_4 with infinite image.

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1 Introduction

Let G be a group and r be a cardinality. We say that $g \in G$ is *r–primitive* if it is part of a generating set of G with r elements. The *rank* of a group G is the cardinality of a generating set of minimal size, and an element of G is called *primitive* if it is r–primitive with r equal to the rank of G. Denote the rank r free group by F_r. This paper concerns the following collection of questions.

Question 1 (the extrinsic primitive torsion problems) *Fix positive integers r and k. Let Γ be an image of F_r such that the image of every r–primitive element in F_r has order dividing k.*

(a) *Is Γ necessarily finite?*

(b) *Is Γ necessarily virtually nilpotent?*

(c) *Is Γ necessarily virtually solvable?*

(d) *Is Γ necessarily finitely presented?*

What if Γ is as above and also residually finite?

Observe that a positive answer to Question 1(a) or (b) implies a positive answer to Question 1(d).

The extrinsic primitive torsion problems are topological variants of the classical Burnside problem posed by William Burnside in 1902 [2]. This problem has led to many important discoveries: the classical Jordan–Schur theorem, A Y Olshansky's outrageous

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subgroup	index in F_2	quotient $G_k = F_2/P_k$
P_2	4	the Klein four-group
P_3	27	$H(\mathbb{Z}/3\mathbb{Z})$
P_4	∞	virtually a 5–dimensional image of $H(\mathbb{Z}) \times H(\mathbb{Z})$
P_5	∞	*we conjecture this is virtually solvable*
P_k with $k \geq 6$	∞	*we conjecture the quotient is not finitely presented*

Table 1: Results and conjectures on P_k and F_2/P_k.

Monster groups [17] and the fundamental Golod–Shafarevich theorem [9]. As such, Question 1 is intrinsically motivated through group theory (moreover, it increases our understanding of new characteristic subgroups of free groups). The case of $r = 2$ has direct ties to geometric questions about square-tiled surfaces; please see the appendix.

There has been significant progress made on the primitive torsion problem for some sufficiently large k; see Koberda and Santharoubane [13] and Malestein and Putman [15]. This paper answers Question 1(a) in the case $r = 2$, and also Question 1(a)–(d) in the cases $r = 2$ and $k \in \{2, 3, 4\}$.

We succinctly state our findings in Table 1. Let $P_{r,k} \subset F_r$ be the subgroup generated by k^{th} powers of primitive elements in F_r (observe that the answer to Question 1(a), (b) or (c) is affirmative if and only if the respective answer to (a), (b) or (c) is affirmative for $\Gamma = F_r/P_{r,k}$). Use P_k to denote $P_{2,k}$ and use $H(R)$ to denote the Heisenberg group over a ring R.

In resolving the cases $k = 4$ we show that F_2/P_4 is isomorphic to the matrix group generated by the two matrices

$$\operatorname{diag}(1,-1,-i,-i;-1,1,i,i;1), \qquad \left(\begin{array}{cccc|cccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

For $k \geq 5$, we develop tools for constructing and refining new infinite linear representations of F_2/P_k. These tools allow us to answer Question 1(a), and we hope they will be useful in future work.

Instead of speaking of primitivity in a free group, we can phrase an intrinsic version of Question 1.

Question 2 (the intrinsic (restricted) primitive torsion problems) *Fix positive integers r and k. Let Γ be a* (*residually finite*) *group of rank r such that every primitive element has order dividing k. Which questions from Question 1 have affirmative answers*?

The primitive torsion problems are natural variants of the original *bounded Burnside problem*. There has been great progress in understanding the quotients arising from these problems; see for instance Coulon and Gruber [5]. Moreover, studying laws other than the power law in restricted Burnside problems is a very active area; see Bradford and Thom [1] and Kozma and Thom [14] for the state of the art.

Question 3 (the bounded Burnside problem) *Fix $r, k \in \mathbb{Z}$. Let G be a group generated by r elements. Let B_k be the group in G generated by elements of the form g^k, where $g \in G$. Is G/B_k necessarily finite*?

We note that when G/P_k is virtually solvable, the resulting group G/B_k is necessarily finite. Thus, our work recovers the well-known result that F_2/B_4 is finite. If our conjecture that F_2/P_5 is virtually solvable is correct, then it follows that F_2/B_5 is finite, which is unknown.

Outline of article

In Section 2, we describe normal generators for P_k. We produce finite lists of normal generators for $k \in \{2, 3, 4, 5\}$. The generators in these cases correspond to the vertices of the triangular dihedron, the tetrahedron, the octahedron and the icosahedron. We use our list of generators to show that the quotients F_2/P_2 and F_2/P_3 are as listed in the introduction. Running out of platonic solids with triangular faces, our techniques would give a infinite collection of normal generators for P_k for $k \geq 6$, and so we conjecture that F_2/P_k is not finitely presented for $k \geq 6$.

In Section 3, we produce highly symmetric representations of F_2/P_k into $\mathrm{GL}(n, \mathbb{C})$ with infinite image when $k \geq 4$. Our technique involves deforming a representation into $\mathrm{GL}(n, \mathbb{C})$ inside a bigger group, namely $\mathrm{GL}(N, \mathbb{C})$ for $N > n$. We take highly symmetric representations of $F_2/P_k \to \mathrm{GL}(n, \mathbb{C})$ which factor through a finite group and then deform them in such a way that the representations develop an infinite image in $\mathrm{GL}(N, \mathbb{C})$ while remaining highly symmetric. This allows us to prove that F_2/P_k

is infinite for $k \geq 4$. Also, the process leads to new highly symmetric representations of F_2/P_k. In the case of $k=4$, we repeat this process twice (with a tensor product in the middle) to produce the representation $F_2/P_4 \to \mathrm{GL}(9,\mathbb{C})$ which was mentioned in the introduction.

In Section 4, we prove that our representation $F_2/P_4 \to \mathrm{GL}(9,\mathbb{C})$ is faithful and proves F_2/P_4 has the form mentioned in the introduction.

The appendix discusses the relationship between this work and the geometry of square-tiled surfaces.

2 Normal generators for P_k

2.1 Primitive elements of F_2

Let F_2 denote the free group $\langle a,b\rangle$. The reader will recall or quickly observe the following facts about primitive elements of F_2:

(1) If $c \in F_2$ is primitive then there is a $\phi \in \mathrm{Aut}(F_2)$ such that $\phi(a)=c$.

(2) If $c \in F_2$ is primitive then so is every element of its conjugacy class $[c] = \{gcg^{-1} : g \in F_2\}$.

In particular, we will say a conjugacy class is *primitive* if it consists of primitive elements of F_2.

The observation that there is a short exact sequence

$$1 \to F_2 \to \mathrm{Aut}(F_2) \xrightarrow{D} \mathrm{GL}(2,\mathbb{Z}) \to 1$$

dates back to Jakob Nielsen's 1913 thesis. Here the map $F_2 \to \mathrm{Aut}(F_2)$ sends an element of F_2 to its corresponding inner automorphism and thus $\mathrm{GL}(2,\mathbb{Z})$ is isomorphic to the outer automorphism group $\mathrm{Out}(F_2) = \mathrm{Aut}(F_2)/\mathrm{Inn}(F_2)$. The map $D\colon \mathrm{Aut}(F_2) \to \mathrm{GL}(2,\mathbb{Z})$ may be defined by using the abelianization homomorphism $\mathrm{ab}\colon F_2 \to \mathbb{Z}^2$, which we choose to satisfy $a \mapsto (1,0)$ and $b \mapsto (0,1)$. Then $D(\phi) \in \mathrm{GL}(2,\mathbb{Z})$ is determined by the condition that $D(\phi)\circ \mathrm{ab}(g) = \mathrm{ab}\circ\phi(g)$ for all $g \in F_2$.

An automorphism of F_2 either preserves the conjugacy class of the commutator $[a,b]$ or sends it to the conjugacy class of $[b,a]$. Thus there is a natural homomorphism $\mathrm{Aut}(F_2) \to C_2$, where we identify C_2 with the permutation group of these conjugacy classes. We set $\mathrm{Aut}_+(F_2)$ to be the kernel which consists of automorphisms preserving the conjugacy class of the commutator $[a,b]$. We use $\mathrm{Aut}_-(F_2)$ to denote $\mathrm{Aut}(F_2)\smallsetminus\mathrm{Aut}_+(F_2)$.

The group $\mathrm{Out}_+(F_2)=\mathrm{Aut}_+(F_2)/\mathrm{Inn}(F_2)$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ via the map D above. The following elements of $\mathrm{Aut}_+(F_2)$ have images in $\mathrm{Out}_+(F_2)$ which generate it:

$$\begin{aligned}\psi_0(a)&=b, & \psi_0(b)&=b^{-1}a^{-1},\\ \psi_1(a)&=b, & \psi_1(b)&=a^{-1},\\ \psi_2(a)&=a, & \psi_2(b)&=ab.\end{aligned}\tag{1}$$

We will use $\bar\psi_0$, $\bar\psi_1$ and $\bar\psi_2$ to denote the outer automorphism classes of these elements. It may be observed that

$$\psi_0\circ\psi_2=\psi_1,\quad \psi_0^3=\psi_1^4=1,\quad [\bar\psi_1^2,\bar\psi_0]=[\bar\psi_1^2,\bar\psi_2]=1.\tag{2}$$

Recall that outer automorphisms act on conjugacy classes. We will use $[g]$ to denote the conjugacy class of $g\in F_2$. We have the following:

Lemma 2.1 (primitive conjugacy classes) *An element $g\in F_2$ is primitive if and only if it lies in the conjugacy class $\bar\psi([a])$ for some $\bar\psi\in\mathrm{Out}_+(F_2)$.*

Proof If $g\in F_2$ is primitive then by (1) above there is a $\psi\in\mathrm{Aut}(F_2)$ such that $\psi(a)=g$. Then, by possibly precomposing with the automorphism $\psi_-\in\mathrm{Aut}_-(F_2)$ determined by $\psi_-(a)=a$ and $\psi_-(b)=b^{-1}$, we can assume that $\psi\in\mathrm{Aut}_+(F_2)$. Let $\bar\psi\in\mathrm{Out}_+(F_2)$ be the class containing ψ. Then $[g]=\bar\psi([a])$. The converse is clear since primitivity is a conjugacy invariant and is invariant under automorphisms. □

It follows that the conjugacy classes of primitive elements are naturally identified with $\mathrm{Out}_+(F_2)$ modulo the stabilizer of the conjugacy class $[a]$. This stabilizer is $\langle\bar\psi_2\rangle$.

The primitive conjugacy classes come naturally in pairs: if $g\in F_2$ is primitive, then we call the conjugacy classes $[g]$ and $[g^{-1}]$ *opposites*. We will denote the collection of unions of opposite pairs of conjugacy classes by $\mathcal{P}$. Opposites are related by the action of the central involution $\bar\psi_1^2$ of $\mathrm{Out}_+(F_2)$:

Proposition 2.2 *If $[g]$ is a primitive conjugacy class then its opposite $[g^{-1}]$ is $\bar\psi_1^2([g])$.*

Proof From the lemma above we have $[g]=\bar\psi([a])$ for some $\bar\psi\in\mathrm{Out}_+(F_2)$. Since $\psi_1^2(a)=a^{-1}$ we have $\bar\psi_1^2([a])=[a^{-1}]$ and $[g^{-1}]=\bar\psi\circ\bar\psi_1^2([a])$. Since $\bar\psi_1^2$ is central in $\mathrm{Out}_+(F_2)$ we have $[g^{-1}]=\bar\psi_1^2\circ\bar\psi([a])=\bar\psi_1^2([g])$. □

Since $\langle\overline{\psi}_2\rangle$ is the stabilizer of $[a]$ and $\overline{\psi}_1^2$ acts as above, there is a bijective correspondence from the coset space $\mathcal{C} = \mathrm{Out}_+(F_2)/\langle\overline{\psi}_2, \overline{\psi}_1^2\rangle$ to $\mathcal{P}$ given by

$$\mathcal{C} \to \mathcal{P}, \quad \overline{\psi}\langle\overline{\psi}_2, \overline{\psi}_1^2\rangle \mapsto \overline{\psi}([a]) \cup \overline{\psi}([a^{-1}]). \tag{3}$$

The group $\mathrm{SL}(2,\mathbb{Z})/\pm I$ has a well-known action on the upper half-plane by Möbius transformations with $-I$ acting trivially. Here the matrix

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \text{acts by} \quad z \mapsto \frac{m_{11}z+m_{12}}{m_{21}z+m_{22}}.$$

This is useful for organizing the pairs of primitive conjugacy classes. Observe that $\langle D(\overline{\psi}_2), D(\overline{\psi}_1)\rangle$ is the stabilizer in $\mathrm{SL}(2,\mathbb{Z})$ of the point $\frac{1}{0}$. The $\mathrm{SL}(2,\mathbb{Z})$ orbit of $\frac{1}{0}$ is $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\frac{1}{0}\}$. Thus, we have:

Lemma 2.3 *There are bijections $\mathcal{C}\colon \widehat{\mathbb{Q}} \to \mathcal{C}$ and $\mathcal{P}\colon \widehat{\mathbb{Q}} \to \mathcal{P}$ compatible with (3) such that for any $\frac{p}{q} \in \widehat{\mathbb{Q}}$ we have:*

- *The class $\mathcal{C}(\frac{p}{q})$ is the collection of $\overline{\psi} \in \mathrm{Out}_+(F_2)$ such that $D(\overline{\psi})(\frac{1}{0}) = \frac{p}{q}$.*
- *The union of the pair of conjugacy classes $\mathcal{P}(\frac{p}{q})$ consists of all primitive elements $g \in F_2$ such that $\mathrm{ab}(g) = \pm(p,q)$ (where $p, q \in \mathbb{Z}$ are taken to be relatively prime).*

The *Farey triangulation* $\mathcal{F}$ is an $\mathrm{SL}(2,\mathbb{Z})$–invariant triangulation of the upper half-plane with vertices in $\widehat{\mathbb{Q}}$. We depict $\mathcal{F}$ in Figure 1. The group $\mathrm{PSL}(2,\mathbb{Z})$ is the orientation-preserving symmetry group of $\mathcal{F}$. It is useful to think of the three spaces $\widehat{\mathbb{Q}}$, $\mathcal{C}$ and $\mathcal{P}$ as in bijective correspondence to the vertices in this triangulation.

2.2 Symmetries and images of primitive elements

Having a power of a primitive element in a normal subgroup N guarantees that some corresponding elements of $\mathrm{Out}(F_2)$ stabilize N. It suffices to consider the case when a power of a lies in N.

Lemma 2.4 *Suppose $N \subset F_2$ is a normal subgroup containing a^k for some $k \geq 1$. Then $\psi_2^k(N) = N$. Furthermore, the induced action of ψ_2^k on F_2/N given by*

$$hN \mapsto \psi_2^k(h)N$$

is trivial.

Proof Assume $a^k \in N$ and $N \subset F_2$ is normal. Observe that the action of ψ_2^k satisfies

$$\psi_2^k(a) = a, \quad \psi_2^k(a^{-1}) = a^{-1}, \quad \psi_2^k(b) = a^k b \quad \text{and} \quad \psi_2^k(b^{-1}) = b^{-1}a^{-k}.$$

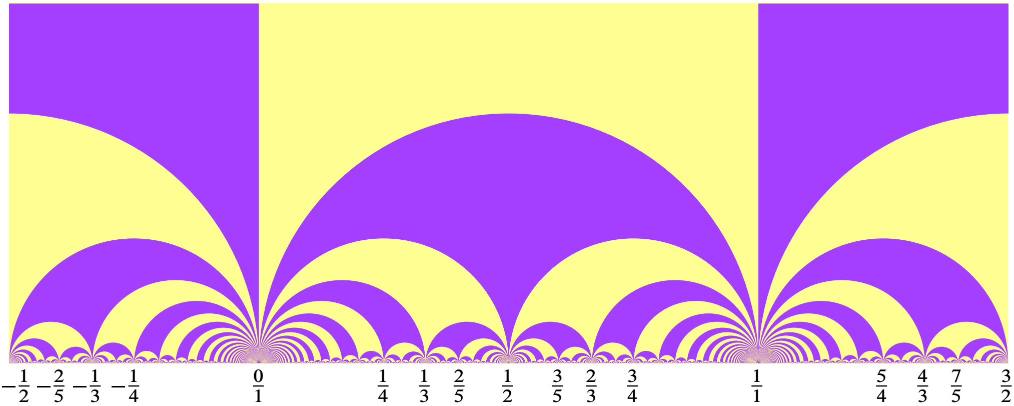


Figure 1: A portion of the Farey triangulation $\mathcal{F}$ of the hyperbolic plane with some rational points at infinity marked. The top endpoint of the vertical edges is $\frac{1}{0}$.

Let $h \in F_2$ and consider h as a word in $\{a, a^{-1}, b, b^{-1}\}$. From the above description of ψ_2^k we see that $\psi_2^k(h)$ is formed from h by inserting copies of a^k and a^{-k} into the word representing h. Let n be the number of such insertions. Then we can write

$$h = \psi_2^k(h) g_1 g_2 \cdots g_n,$$

where each g_i is a conjugate of either a^{-k} or a^k selected to remove an inserted copy of a^k or a^{-k}. Since $a^k \in N$ and N is normal, each $g_i \in N$. It follows that $h \in N$ if and only if $\psi_2^k(h) \in N$. Thus $\psi_2^k(N) = N$. Finally we see that for any $hN \in F_2/N$,

$$\psi_2^k(hN) = \psi_2^k(h)N = h g_n^{-1} \cdots g_2^{-1} g_1^{-1} N = hN. \qquad \square$$

We get the following if a power of a primitive element lies in a normal subgroup of F_2:

Corollary 2.5 *Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$, let $g \in \mathcal{P}\left(\frac{p}{q}\right)$ and let $\psi\colon F_2 \to F_2$ be an automorphism such that the associated outer automorphism $\overline{\psi}$ lies in $\mathcal{C}\left(\frac{p}{q}\right)$. Then, for any $k \geq 2$ and for any normal subgroup $N' \subset F_2$ containing g^k, $\psi \circ \psi_2^k \circ \psi^{-1}(N') = N'$ and the induced action of $\psi \circ \psi_2^k \circ \psi^{-1}$ on F_2/N' is trivial.*

Proof From Lemma 2.3, we note that $\overline{\psi}([a] \cup [a^{-1}]) = [g] \cup [g^{-1}]$. Set $N = \overline{\psi}^{-1}(N')$. By normality, we see that $a^k \in N$. Thus Lemma 2.4 tells us that $\psi_2^k(N) = N$ and ψ_2^k acts trivially on F_2/N. It follows that $\psi \circ \psi_2^k \circ \psi^{-1}$ stabilizes N and acts trivially on F_2/N'. $\square$

Recall that $P_k \subset F_2$ is the subgroup generated by k^{th} powers of primitive elements of F_2. This subgroup is clearly characteristic, and thus there is a well-defined homomorphism

$$\epsilon\colon \mathrm{Aut}(F_2) \to \mathrm{Aut}(F_2/P_k), \quad \epsilon(\phi)(gP_k) = \phi(g)P_k.$$

Inner automorphism of F_2 are sent by ϵ to inner automorphisms of F_2/P_k, thus ϵ induces a well-defined map between outer automorphism groups,

$$\bar{\epsilon}\colon \mathrm{Out}(F_2) \to \mathrm{Out}(F_2/P_k).$$

Let $\mathcal{O}_k \subset \mathrm{Out}_+(F_2)$ denote the subgroup normally generated by $\bar{\psi}_2^k$. The lemma above guarantees:

Corollary 2.6 *The subgroup $\mathcal{O}_k$ is contained in* $\ker \bar{\epsilon}$.

Proof We must show that for each $\bar{\psi} \in \mathrm{Out}_+(F_2)$ we have $\bar{\psi} \circ \bar{\psi}_2^k \circ \bar{\psi}^{-1} \in \ker \bar{\epsilon}$. Since $\ker \bar{\epsilon}$ is a normal subgroup, we may take $\bar{\psi} = 1$, and that $\bar{\psi}_2^k \in \ker \bar{\epsilon}$ follows from Lemma 2.4. □

In order to better understand $\mathcal{O}_k$ we make use of $D\colon \mathrm{Out}_+(F_2) \to \mathrm{SL}(2,\mathbb{Z})$ and the Möbius action on the Farey triangulation $\mathcal{F}$. Note that $\mathrm{SL}(2,\mathbb{Z})$ is the group of orientation-preserving symmetries of $\mathcal{F}$ which permute the triangles. Thus, covering space theory identifies each subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ bijectively with the (possibly orbifold) quotient $\mathcal{F}/\Gamma$ which is tiled by triangles (possibly including some quotients of triangles by their order 3 rotation groups). These quotients are intermediate between $\mathcal{F}$ and the modular surface $\mathcal{F}/\mathrm{SL}(2,\mathbb{Z})$ (which has a vertex added at the cusp since $\mathcal{F}$ includes vertices). The *valence* of a vertex in a triangulation is the number of vertices of triangles that are identified to make that point. The valence of a vertex may be a positive integer or infinity.

The following gives a concrete understanding of the quotient $\mathcal{F}_k = \mathcal{F}/D\mathcal{O}_k$:

Proposition 2.7 *The orbifold $\mathcal{F}_k$ is the unique simply connected triangulated surface such that all vertices have valence k. In particular, the combinatorial type of the triangulated surface $\mathcal{F}_k$ can be described as follows:*

- *If $k \in \{2,3,4,5\}$, then $\mathcal{F}_k$ is a sphere. Specifically, $\mathcal{F}_2$ is a triangle doubled across its boundary, $\mathcal{F}_3$ is a tetrahedron, $\mathcal{F}_4$ is an octahedron, and $\mathcal{F}_5$ is an icosahedron.*

- *The quotient $\mathcal{F}_6$ is the plane tiled by equilateral triangles.*
- *For $k \geq 7$, the quotient $\mathcal{F}_k$ is the hyperbolic plane tiled by equilateral triangles each of whose angles measures $\frac{2\pi}{k}$.*

Proof First observe that $D\psi_2$ acts as the Möbius transformation $z \mapsto z+1$, and thus sends each triangle of $\mathcal{F}$ incident to ∞ to the adjacent triangle in the counterclockwise direction about ∞. Thus, if $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ contains $D\psi_2^k$, the corresponding quotient $\mathcal{F}/\Gamma$ has valence dividing k at the vertex in the image of ∞ under the covering $\mathcal{F} \to \mathcal{F}/\Gamma$.

Now suppose Γ contains all of $D\mathcal{O}_k$. Since $\mathrm{SL}(2,\mathbb{Z})$ acts transitively on $\widehat{\mathbb{Q}}$ and $D\mathcal{O}_k$ is normal in $\mathrm{SL}(2,\mathbb{Z})$, it follows that each vertex of $\mathcal{F}/\Gamma$ has valence dividing k.

Now consider moving from orbifolds to groups. Let S be a connected combinatorial orbifold built by identifying in pairs the edges of some collection of triangles and quotients of a triangle modulo the order 3 rotation. Such an S is covered by the Farey triangulation, and, fixing such a covering map $\pi\colon \mathcal{F} \to S$, covering space theory associates the deck group

$$\Gamma = \{M \in \mathrm{SL}(2,\mathbb{Z}) : \pi \circ M = \pi\}.$$

We observe that if each vertex of S has valence dividing k, then $D\mathcal{O}_k \subset \Gamma$.

We conclude from the previous paragraph that the quotients of $\mathcal{F}$ described in the proposition are of the form $\mathcal{F}/\Gamma$ for some Γ containing $D\mathcal{O}_k$. To see $\Gamma = D\mathcal{O}_k$, recall from covering space theory that the surface $\mathcal{F}_k$ (branched) covers any $\mathcal{F}/\Gamma$ with $D\mathcal{O}_k \subset \Gamma$. But, since the surfaces described in the proposition are simply connected and have all vertices of valence precisely k, they exhibit no (nontrivial) branched covers such that all vertices of the cover have valence dividing k. □

The same mechanism can be used to shorten the list of group elements needed to normally generate P_k.

Theorem 2.8 *Let $k \geq 2$. Let $\{p_i/q_i : i \in \Lambda\}$ be a subset of $\widehat{\mathbb{Q}}$ containing one representative of each preimage of a vertex of $\mathcal{F}_k$ under the covering map $\mathcal{F} \to \mathcal{F}_k$. For each $i \in \Lambda$ choose a primitive element $g_i \in \mathcal{P}(p_i/q_i)$ and an outer automorphism $\overline{\psi}_i \in \mathcal{C}(p_i/q_i)$. If*

$$\{\overline{\psi}_i \circ \overline{\psi}_2^k \circ \overline{\psi}_i^{-1} : i \in \Lambda\}$$

generates $\mathcal{O}_k$ then P_k is normally generated by $\{g_i^k : i \in \Lambda\}$.

Proof Fix the quantities above and assume all hypotheses are satisfied. Let Q be the subgroup of F_2 normally generated by $\{g_i^k : i \in \Lambda\}$. Clearly $Q \subset P_k$ since each g_i is primitive. We will show $P_k \subset Q$.

As a consequence of Corollary 2.5 we know that $\bar{\psi}_i \circ \bar{\psi}_2^k \circ \bar{\psi}_i^{-1}$ stabilizes Q for all $i \in \Lambda$. Then from the hypotheses we know each element of $\mathcal{O}_k$ stabilizes Q.

To show $P_k \subset Q$, it suffices to show that if $g \in F_2$ is primitive then $g^k \in Q$. Fix g. Then there is a $\frac{p}{q} \in \hat{\mathbb{Q}}$ such that $\mathcal{P}(\frac{p}{q}) = [g] \cup [g^{-1}]$. From our hypothesis on $\{p_i/q_i\}$ we know there is an $i \in \Lambda$ and a $\bar{\psi} \in \mathcal{O}_k$ such that $D\bar{\psi}(p_i/q_i) = \frac{p}{q}$. Then $\bar{\psi}([g_i] \cup [g_i^{-1}]) = [g] \cup [g^{-1}]$. By definition of Q we know that the conjugacy classes $[g_i^k]$ and $[g_i^{-k}]$ are contained in Q. Since Q is $\mathcal{O}_k$–invariant and $g^k \in \bar{\psi}([g_i^k] \cup [g_i^{-k}])$ we have $g^k \in Q$, as desired. □

The following describes a combinatorial way to find the generators:

Corollary 2.9 *Fix $k \geq 2$. Let $T \subset \mathcal{F}_k$ be a tree in the 1–skeleton of $\mathcal{F}_k$ whose vertex set coincides with the collection of all vertices of the triangulation. Let $\tilde{T}$ be a lift of T to $\mathcal{F}$ and let $\{p_i/q_i : i \in \Lambda\}$ be the vertices of $\tilde{T}$. Then $P_k = \langle\langle g_i^k \rangle\rangle_{i \in \Lambda}$, where each $g_i \in \mathcal{P}(p_i/q_i)$ is chosen arbitrarily as in Theorem 2.8.*

Proof We must check the hypotheses of Theorem 2.8. Define $\{p_i/q_i\}$ and $\{g_i\}$ as in the statement of the corollary and $\{\bar{\psi}_i\}$ as in Theorem 2.8. Since the vertices of T include all vertices of $\mathcal{F}_k$, we see that $\{p_i/q_i\}$ contains one preimage of each vertex of $\mathcal{F}_k$. Let $Q = \langle \bar{\psi}_i \circ \bar{\psi}_2^k \circ \bar{\psi}_i^{-1} \rangle \subset \mathcal{O}_k$. We need to show $Q = \mathcal{O}_k$.

Associated to the chain of subgroups $\{1\} \subset Q \subset \mathcal{O}_k$ is the sequence of spaces related by covering maps branched at the vertices of the triangulations,

$$\mathcal{F} \to \mathcal{F}/DQ \xrightarrow{\pi} \mathcal{F}_k.$$

Proving that $Q = \mathcal{O}_k$ is equivalent to proving that π is the trivial covering. Note that triviality will follow from Proposition 2.7 if all vertices of $\mathcal{F}/DQ$ have valence dividing k, so this is what we will prove.

Let $T_Q \subset \mathcal{F}/DQ$ denote the image of $\tilde{T}$ under the covering map $\mathcal{F} \to \mathcal{F}/DQ$. Then T_Q is a tree because $\pi(T_Q) = T$. Observe that each vertex of T_Q is incident to k triangles because such a vertex is the image of some $p_i/q_i \in \tilde{T}$ and the action of $D(\bar{\psi}_i \circ \bar{\psi}_2^k \circ \bar{\psi}_i^{-1})$ on $\mathcal{F}$ rotates by k triangles about p_i/q_i. Thus it suffices to prove that every vertex of $\mathcal{F}/DQ$ is a vertex of the tree T_Q. If this were not the case then there would be an edge of a triangle of $\mathcal{F}/DQ$ with one vertex in T_Q and the other not in T_Q. We will show this doesn't happen.

A key observation is the following. Say that the *link* of a vertex of a triangulated surface is the union of the vertex with the interiors of incident edges and triangles. The *link lifting observation* is the observation that π restricted to the link of a vertex $v_Q \in T_Q \subset \mathcal{F}/DQ$ is a bijection to the link of the image vertex $v = \pi(v_Q) \in T \subset \mathcal{F}_k$ since both v_Q and v are incident to k triangles.

Now we return to the proof. Suppose $e_Q = \overrightarrow{v_Q w_Q}$ is an oriented edge of a triangle of $\mathcal{F}/DQ$ initiating at a vertex v_Q of T_Q. We will show that the terminating vertex w_Q also is a vertex of T_Q. Let $e = \overrightarrow{vw}$ be $\pi(e_Q)$. We break into two cases.

First, it could be that e is an edge of T. Since $v_Q \in T_Q$, by the link lifting observation we know that e has a unique lift to $\mathcal{F}_Q$ initiating at v_Q. Since T_Q is a lift of T and e is an edge of T, this means that e_Q must be an edge of T_Q. Thus, w_Q is also a vertex of T_Q, as desired.

Now suppose that e is not an edge of T. Since T is a spanning tree, both v and w are vertices of T. As T is a tree, there is a unique oriented path p in T joining v to w. Let $v = p_0, p_1, \dots, p_n = w$ be the sequence of vertices passed through by p. We will inductively prove p has a unique lift to $\mathcal{F}/DQ$ starting at v_Q. This involves checking that for each $j \in \{1, \dots, n\}$ there is a unique lift of the path $p_0, \dots, p_j$ denoted $\tilde{p}_0, \dots, \tilde{p}_j$ such that $\tilde{p}_0 = v_Q$ and $\pi(\overrightarrow{\tilde{p}_i \tilde{p}_{i+1}}) = \overrightarrow{p_i p_{i+1}}$ for $i \in \{0, \dots, j-1\}$. This is true for $j = 1$ because $v_Q \in T_Q$ using the unique lifting provided by the observation above. Now we will argue the inductive step. Suppose the lift is unique up through index $j < n$. Then, since p is a path in T and $\pi(T_Q) = T$, we must have that all vertices of the lift so far lie in T_Q. From the link lifting observation we know that there is a unique lift of the next edge $\overrightarrow{p_j p_{j+1}}$, completing the inductive step.

Now observe that since $\mathcal{F}_k$ is a triangulation of a simply connected surface, $p \cup e$ bounds a topological disk Δ. Since $v_Q \in T_Q$, by the link lifting observation again there is a unique lift $\tilde{\Delta}$ of Δ to $\mathcal{F}/DQ$ such that v lifts to v_Q. From the previous paragraph, the path p in the boundary of Δ lifts to a path $\tilde{p}$ in the boundary of $\tilde{\Delta}$ and contained in the tree T_Q. Again by the assumption, the edge e in the boundary of Δ lifts to e_Q in the boundary of $\tilde{\Delta}$. Thus, e_Q joins the initial point v_Q of $\tilde{p}$ to the terminal point w_Q of $\tilde{p}$. Since $\tilde{p}$ is contained in T_Q, we see that $w_Q \in T_Q$, as desired. $\square$

Conjecture 1 *The normal generators for P_k provided by Corollary 2.9 are a minimal set of normal generators. In particular, for $k \geq 6$, the group F_2/P_k is not finitely presented.*

2.3 Normal generators for P_k with $k \leq 5$

We describe normal generators for P_k when $k \leq 5$ because these are the cases where Corollary 2.9 yields a finite set of normal generators. These cases are finite because Proposition 2.7 tells us that $\mathcal{F}_k$ is a triangulated sphere.

The case $k = 2$

The triangulated sphere $\mathcal{F}_2$ is the double of a triangle across its boundary. In Figure 2 we depict a tree T in an unfolding of $\mathcal{F}_2$. We have lifted T to a tree $\widetilde{T}$ in the Farey triangulation and labeled the vertices of T by their lifts as elements of $\widehat{\mathbb{Q}}$. Following Theorem 2.8 and Corollary 2.9, we have converted these elements of $\widehat{\mathbb{Q}}$ to normal generators of P_2.

Proposition 2.10 *The quotient F_2/P_2 is isomorphic to the Klein four-group.*

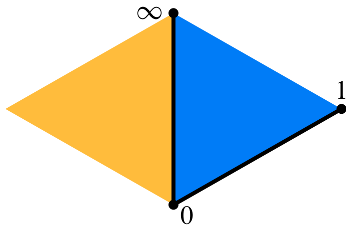
Proof Since all elements of the Klein four-group $K = \langle a, b \mid a^2, b^2, [a,b] \rangle$ have order 2, K is a quotient of F_2/P_2. Therefore, it suffices to prove the defining relations hold in F_2/P_2. Clearly $a^2 = b^2 = 1$ in F_2/P_2 since a and b are primitive in F_2. Thus, $a = a^{-1}$ and $b = b^{-1}$. It follows that $[a,b] = (ab)^2 = 1$ since ab is primitive in F_2. □

The case $k = 3$

The triangulated sphere $\mathcal{F}_3$ is a tetrahedron. We depict a tree T in an unfolded copy of the tetrahedron in Figure 3. We have lifted T to a tree $\widetilde{T}$ in the Farey triangulation and labeled the vertices of T by their lifts as elements of $\widehat{\mathbb{Q}}$. Following Theorem 2.8 and Corollary 2.9, we have converted these elements of $\widehat{\mathbb{Q}}$ to normal generators of P_3.

Proposition 2.11 *The quotient F_2/P_3 is isomorphic to $H(\mathbb{Z}/3\mathbb{Z})$.*

Proof In $H(\mathbb{Z}/3\mathbb{Z})$, all elements have order 3. Thus, $H(\mathbb{Z}/3\mathbb{Z})$ is a quotient of F_2/P_3 and so it suffices to prove that relations defining $H(\mathbb{Z}/3\mathbb{Z})$ are satisfied



vertex	generator of P_2
∞	a^2
0	b^2
1	$(ab)^2$

Figure 2: The triangulated sphere $\mathcal{F}_2$ and the normal generators of P_2 corresponding to the vertices.

in F_2/P_3. We work with the presentation

$$H(\mathbb{Z}) = H(\mathbb{Z}/3\mathbb{Z}) = \langle a, b \mid a^3, b^3, [a,b]^3, [a,[a,b]], [b,[a,b]] \rangle.$$

Since a and b are primitive, we have $a^3 = b^3 = 1$ in F_2/P_3. Also we have

$$\begin{aligned} [a,[a,b]] &= a^{-1}(b^{-1}a^{-1}ba)a(a^{-1}b^{-1}ab) \\ &= a^{-1}b^{-1}a^{-1}bab^{-1}ab \\ &= (a^{-1}b^{-1})^2 b^2 ab^{-1}ab. \end{aligned}$$

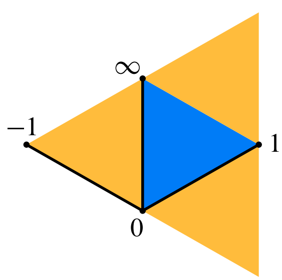
Since $a^{-1}b^{-1}$ is primitive in F^2, we have $(a^{-1}b^{-1})^3 = 1$ and thus, continuing,

$$[a,[a,b]] = bab^2ab^{-1}ab. = bab^{-1}ab^{-1}ab. = b(ab^{-1})^3ba^{-1}ab = b^2a^{-1}ab = 1.$$

Further, since P_3 is characteristic, we get $[b,[a,b]] = 1$. It follows that $[a,b]$ is central, thus $[a,b]^3 = [a^3, b] = 1$ via commutator identities, completing the proof. □

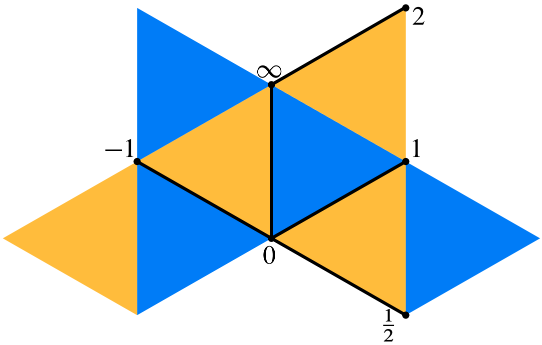
The case $k = 4$

The triangulated sphere $\mathcal{F}_4$ is an octahedron. We depict a tree T in an unfolded copy of the octahedron in Figure 4. We have lifted T to a tree $\widetilde{T}$ in the Farey triangulation



vertex	generator of P_3
∞	a^3
0	b^3
1	$(ab)^3$
-1	$(ab^{-1})^3$

Figure 3: The triangulated sphere $\mathcal{F}_3$ and normal generators of P_3.



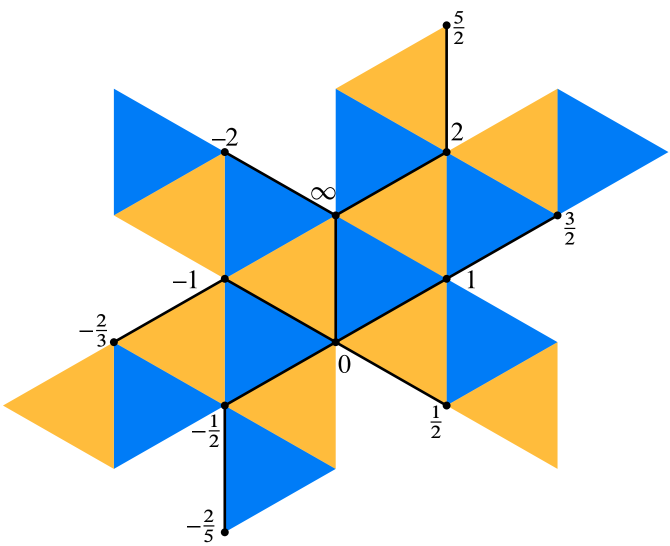
vertex	generator of P_4
∞	a^4
0	b^4
1	$(ab)^4$
-1	$(ab^{-1})^4$
2	$(a^2b)^4$
$\frac{1}{2}$	$(ab^2)^4$

Figure 4: The triangulated sphere $\mathcal{F}_4$ and normal generators of P_4.

and labeled the vertices of T by their lifts as elements of $\widehat{\mathbb{Q}}$. Following Theorem 2.8 and Corollary 2.9, we have converted these elements of $\widehat{\mathbb{Q}}$ to normal generators of P_4.

The case $k = 5$

The triangulated sphere $\mathcal{F}_5$ is an icosahedron. We depict a tree T in an unfolded copy of the icosahedron in Figure 5. We have lifted T to a tree $\widetilde{T}$ in the Farey triangulation and labeled the vertices of T by their lifts as elements of $\widehat{\mathbb{Q}}$. Following Theorem 2.8 and Corollary 2.9, we have converted these elements of $\widehat{\mathbb{Q}}$ to normal generators of P_5. These generators are listed in Figure 5.



vertex	generator of P_5
∞	a^5
0	b^5
1	$(ab)^5$
-1	$(ab^{-1})^5$
2	$(a^2b)^5$
$\frac{1}{2}$	$(ab^2)^5$
-2	$(a^2b^{-1})^5$
$-\frac{1}{2}$	$(ab^{-2})^5$
$\frac{3}{2}$	$(a^2bab)^5$
$-\frac{2}{3}$	$(ab^{-1}ab^{-2})^5$
$\frac{5}{2}$	$(a^3ba^2b)^5$
$-\frac{2}{5}$	$(ab^{-2}ab^{-3})^5$

Figure 5: The triangulated sphere $\mathcal{F}_5$ and normal generators of P_5.

3 Characteristic representations

3.1 Definition and a criterion

We say that a homomorphism $\rho\colon F_2 \to \mathrm{GL}(n,\mathbb{C})$ is a *characteristic representation* if for any $\psi \in \mathrm{Aut}(F_2)$ there is a $\Psi \in \mathrm{Aut}(\mathrm{GL}(n,\mathbb{C}))$ such that

$$(4) \qquad \Psi \circ \rho \circ \psi^{-1}(g) = \rho(g) \quad \text{for all } g \in F_2.$$

The following should be clear:

Proposition 3.1 *The kernel of a characteristic representation is a characteristic subgroup of F_2.*

Recall from Section 2.1 that $\mathrm{Aut}(F_2) = \mathrm{Aut}_+(F_2) \cup \mathrm{Aut}_-(F_2)$. Our automorphisms of $\mathrm{GL}(n,\mathbb{C})$ will have one of two forms corresponding to this partition. If $M \in \mathrm{GL}(n,\mathbb{C})$ then we define

$$(5) \quad \Psi_M, \bar{\Psi}_M \in \mathrm{Aut}(\mathrm{GL}(n,\mathbb{C})) \quad \text{by} \quad \Psi_M(X) = MXM^{-1}, \quad \bar{\Psi}_M(X) = M\bar{X}M^{-1}.$$

We call the map $\bar{\Psi}_M$ a *conjugate inner automorphism.*

We say $\rho\colon F_2 \to \mathrm{GL}(n,\mathbb{C})$ is an *oriented characteristic representation* if the following two statements hold:

- (+) For each $\psi \in \mathrm{Aut}_+(F_2)$ there is an $M \in \mathrm{GL}(n,\mathbb{C})$ such that (4) holds with $\Psi = \Psi_M$.
- (–) For each $\psi \in \mathrm{Aut}_-(F_2)$ there is an $M \in \mathrm{GL}(n,\mathbb{C})$ such that (4) holds with $\Psi = \bar{\Psi}_M$.

We will be working exclusively with oriented characteristic representations.

Based on properties of the tensor product, it can be observed:

Proposition 3.2 *If $\rho_1\colon F_2 \to \mathrm{GL}(n_1,\mathbb{C})$ and $\rho_2\colon F_2 \to \mathrm{GL}(n_2,\mathbb{C})$ are oriented characteristic representations, then so is their tensor product $\rho_1 \otimes \rho_2\colon F_2 \to \mathrm{GL}(n_1 n_2,\mathbb{C})$ and so is the complex-conjugate representation $\bar{\rho}_1$.*

We will now give an elementary method to prove that a homomorphism ρ is an oriented characteristic representation. We single out elements $\psi_1, \psi_2 \in \mathrm{Aut}_+(F_2)$ and

$\psi_- \in \mathrm{Aut}_-(F_2)$ whose images in $\mathrm{Out}(F_2)$ generate $\mathrm{Out}(F_2)$:

$$\begin{aligned} \psi_1(a) &= b, & \psi_1(b) &= a^{-1}, \\ \psi_2(a) &= a, & \psi_2(b) &= ab, \\ \psi_-(a) &= a^{-1}, & \psi_-(b) &= b. \end{aligned} \tag{6}$$

We have the following criterion for checking if a representation is oriented characteristic:

Proposition 3.3 *Let ρ: $F_2 \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism. Then ρ is an oriented characteristic representation if and only if the following statements are satisfied*:

(1) *There is an $M_1 \in \mathrm{GL}(n, \mathbb{C})$ such that $M_1 = \rho(a) M_1 \rho(b)$ and $M_1 \rho(a) = \rho(b) M_1$.*

(2) *There is an $M_2 \in \mathrm{GL}(n, \mathbb{C})$ such that $M_2 \rho(a) = \rho(a) M_2$ and $M_2 \rho(b) = \rho(ab) M_2$.*

(–) *There is an $M_- \in \mathrm{GL}(n, \mathbb{C})$ such that $M_- = \rho(a) M_- \overline{\rho(a)}$ and $M_- \overline{\rho(b)} = \rho(b) M_-$.*

We remark that the equations in the respective statements above are simple algebraic manipulations of (4) in the special cases where (ψ, Ψ) is taken to be one the pairs (ψ_1, Ψ_{M_1}), (ψ_2, Ψ_{M_2}) or $(\psi_-, \bar{\Psi}_{M_-})$ and g is restricted to a pair of generators of F_2. (For (1) and (–) we use generators a and b, while in (2) we use a and ab.) Thus the "only if" direction is clear.

Proof of "if" direction Assume statements (1), (2) and (–) of the proposition hold. We must prove statements (+) and (–) of the definition of oriented characteristic definition. Let

$$\begin{aligned} \Delta_+ &= \{(M, \psi) \in \mathrm{GL}(n, \mathbb{C}) \times \mathrm{Aut}_+(F_2) : (4) \text{ holds with } \Psi = \Psi_M\}, \\ \Delta_- &= \{(M, \psi) \in \mathrm{GL}(n, \mathbb{C}) \times \mathrm{Aut}_-(F_2) : (4) \text{ holds with } \Psi = \bar{\Psi}_M\}. \end{aligned} \tag{7}$$

Observe that $\Delta = \Delta_+ \sqcup \Delta_-$ is a group, though the group operation needs adjustment. If $(M', \psi') \in \Delta_s$ with $s \in \{+, -\}$, we define

$$(M, \psi) \cdot (M', \psi') = \begin{cases} (MM', \psi \circ \psi') \in \Delta_s & \text{if } (M, \psi) \in \Delta_+, \\ (M\overline{M'}, \psi \circ \psi') \in \Delta_{-s} & \text{if } (M, \psi) \in \Delta_-. \end{cases}$$

(This choice is made to be compatible with composition of inner automorphisms and conjugate inner automorphisms.) We must prove that the projection of Δ to $\mathrm{Aut}(F_2)$ is surjective.

First consider the inner automorphisms of F_2, which have the form $\psi_h(g) = hgh^{-1}$ for some $h \in F_2$. By manipulating (4) it can be observed that $(\rho(h), \psi_h) \in \Delta_+$ for all h.

Now consider ψ_1 and ψ_2. Observe that (4) holds for all $g \in F_2$ if and only if it holds for a set of generators of F_2. As indicated above this proof, by manipulating (4) in each case, it follows that $(M_1, \psi_1), (M_2, \psi_2) \in \Delta_+$. The elements ψ_1 and ψ_2 together with the inner automorphisms generate $\mathrm{Aut}_+(F_2)$, so $\mathrm{Aut}_+(F_2)$ is in the image of the projection of Δ_+.

Similarly, consider ψ_-. Again by considering (4), we see that $(M_-, \psi_-) \in \Delta_-$. The collection $\{\psi_-\} \sqcup \mathrm{Aut}_+(F_2)$ generates $\mathrm{Aut}(F_2)$, so it must be that $\mathrm{Aut}(F_2)$ is in the image of the projection of Δ, as desired. □

Remark 3.4 (orientation-reversing elements) For the main goals of the paper, it would suffice to work with $\mathrm{Aut}_+(F_2)$ rather than all of $\mathrm{Aut}(F_2)$, since $\mathrm{Aut}_+(F_2)$ already acts transitively on primitive elements of F_2, and one could define a notion of oriented characteristic representation omitting (–) from the definition. However, all the representations we found have this extra symmetry, and our algorithm for "improvement" of representations described in Section 3.3 respects this additional symmetry. So, we have opted to consider $\mathrm{Aut}_-(F_2)$ throughout this section for aesthetic reasons at the cost of some minor increase in the complexity of some of our arguments.

3.2 Some characteristic representations with finite image

We will now give some finite oriented characteristic representations.

We define $\rho_2\colon F_2 \to \mathrm{GL}(3, \mathbb{C})$ by

$$\rho_2(a) = \mathrm{diag}(-1,-1,1) \quad \text{and} \quad \rho_2(b) = \mathrm{diag}(1,-1,-1). \tag{8}$$

Proposition 3.5 *The image $\rho_2(F_2)$ is isomorphic to the Klein four-group, $C_2 \times C_2$. The representation ρ_2 is oriented characteristic.*

Proof The image $\rho(F_2)$ can easily be seen to consist of four elements: $\rho_2(a)$, $\rho_2(b)$, the identity and $\rho(ab) = \mathrm{diag}(-1,1,-1)$. By inspection, the image is isomorphic to the Klein four-group. By an elementary calculation it can be observed that the statements of Proposition 3.3 are satisfied for the choices of

$$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $M_- = I$. □

For odd numbers $k \geq 3$ define $\rho_k\colon F_2 \to \mathrm{GL}(k,\mathbb{C})$ by

$$\rho_k(a) = \mathrm{diag}(1,\omega,\omega^2,\dots,\omega^{k-1}) \quad \text{and} \quad \rho_k(b) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{9}$$

where $\omega = e^{2\pi i/k}$. Here $\rho_k(b)$ is a permutation matrix of order k.

Proposition 3.6 *The image $\rho_k(F_2)$ is isomorphic to the Heisenberg group $H(\mathbb{Z}/k\mathbb{Z})$. The representation is oriented characteristic: It satisfies the hypotheses of Proposition 3.3 with the matrix M_1 given by*

$$(M_1)_{i,j} = \omega^{(i-1)(j-1)} \quad \text{for } i,j \in \{1,\dots,k\},$$

with M_2 given by the diagonal matrix with entries $(M_2)_{i,i} = \omega^{-(i-1)(i-2)/2}$ and with $M_- = I$.

Proof To see the image is the Heisenberg group, recall that

$$H(\mathbb{Z}/k\mathbb{Z}) = \langle a,b \mid a^k, b^k, [a,b]^k, [a,[a,b]], [b,[a,b]]\rangle.$$

First we will check that ρ_k factors through $H(\mathbb{Z}/k\mathbb{Z})$. It should be clear that a^k and b^k lie in $\ker\rho_k$. By computation we see $\rho_k([a,b]) = \omega^{-1}I$. Thus $[a,b]$ is central in the image and $[a,b]^k \in \ker\rho_k$. This shows that the image $\rho_k(F_2)$ is isomorphic to a quotient of $H(\mathbb{Z}/k\mathbb{Z})$. The image must be isomorphic to $H(\mathbb{Z}/k\mathbb{Z})$ because the homomorphism restricts to an isomorphism of the center of $H(\mathbb{Z}/k\mathbb{Z})$.

The statements of Proposition 3.3 for the matrices M_1, M_2 and M_- listed can be verified by a direct computation (calculation carried out by hand, and checked for various values of k with SageMath [19]). □

Observe that the images of ρ_k are matrices with entries in $\mathbb{Z}[\omega]$. Later we will need the following observation:

Proposition 3.7 *Fix an odd $k \geq 3$. Let $M_{k,k}$ denote the additive group of $k \times k$ matrices with entries in $\mathbb{Z}[\omega]$. The subgroup of $M_{k,k}$ generated by $\{\rho_k(g) : g \in F_2\}$ has finite index.*

Proof Let $E_{i,j}$ denote the matrix with a 1 in the entry in row i and column j but with all other entries equal to zero. It suffices to show that $k\omega^n E_{i,j}$ is in the generated subgroup for all $i,j\in\{1,\dots,k\}$ and all $n\in\{0,\dots,k-1\}$. By direct computation we observe

$$kE_{1,1}=\sum_{\ell=0}^{k-1}\rho_k(a^\ell).$$

Utilizing the action of $\rho_k(b)$ as a permutation matrix we can then see

$$E_{i,j}=\rho_k(b^{1-i})\cdot E_{1,1}\cdot\rho_k(b^{j-1}).$$

Thus, $kE_{i,j}$ is in this generated subgroup as well. Finally, to get the powers of ω observe that $\rho_k([b,a])=\omega I$. □

Corollary 3.8 *For odd $k\geq 3$, the representation ρ_k is irreducible.*

Proof Any subspace of $\mathbb{C}^k$ which is invariant under ρ_k must be mapped into itself by all elements of the subgroup of $M_{k,k}$ generated by $\{\rho_k(g):g\in F_2\}$. The previous proposition implies that there is no such nonzero proper subspace. □

3.3 Improving characteristic representations

We will now explain a process which can take an oriented characteristic representation $\rho\colon F_2\to\mathrm{GL}(n,\mathbb{C})$ and produce a new oriented characteristic representation $\widetilde{\rho}\colon F_2\to\mathrm{GL}(\widetilde{n},\mathbb{C})$ where $\widetilde{n}\geq n$ and hopefully the $\ker\widetilde{\rho}$ is strictly smaller than $\ker\rho$.

Fix ρ for this subsection. We will consider deformations of ρ into the affine group $\mathrm{Aff}(n)=\mathbb{C}^n\rtimes\mathrm{GL}(n,\mathbb{C})$, where the product in $\mathrm{Aff}(n)$ is given by

$$(\boldsymbol{v},M)\cdot(\boldsymbol{w},N)=(\boldsymbol{v}+M\boldsymbol{w},MN). \tag{10}$$

The group $\mathrm{GL}(n,\mathbb{C})$ is isomorphic to a subgroup $\mathrm{Aff}(n)$ via the map $M\mapsto(\mathbf{0},M)$, and this explains how to multiply elements of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{Aff}(n)$. Let $\pi_1\colon\mathrm{Aff}(n)\to\mathbb{C}^n$ and $\pi_2\colon\mathrm{Aff}(n)\to\mathrm{GL}(n,\mathbb{C})$ be the natural projections (noting that π_1 is not a homomorphism). We will say that an *affable* representation $\widehat{\rho}\colon F_2\to\mathrm{Aff}(n)$ is a homomorphism for which $\pi_2\circ\widehat{\rho}=\rho$. We use $\mathcal{A}$ to denote the collection of all affable representations. Observe:

Proposition 3.9 *The collection $\mathcal{A}$ is a vector space over $\mathbb{C}$ when endowed with the operations of addition and scalar multiplication defined by*

$$(\widehat{\rho}_1+\widehat{\rho}_2)(g)=(\pi_1\circ\widehat{\rho}_1(g)+\pi_1\circ\widehat{\rho}_2(g),\rho(g)),\quad(\lambda\widehat{\rho}_1)(g)=(\lambda\pi_1\circ\widehat{\rho}_1(g),\rho(g))$$

for all $\widehat{\rho}_1,\widehat{\rho}_2\in\mathcal{A}$, all $\lambda\in\mathbb{C}$ and all $g\in F_2$. In particular, for any g the map $\mathrm{eval}_g\colon\mathcal{A}\to\mathbb{C}^n$ defined by $\mathrm{eval}_g(\widehat{\rho})=\pi_1\circ\widehat{\rho}(g)$ is linear.

Discussion of proof The operations are clearly linear in nature, but it must be checked that $\hat{\rho}_1 + \hat{\rho}_2$ and $\lambda\hat{\rho}_1$ define group homomorphisms (assuming $\hat{\rho}_1$ and $\hat{\rho}_2$ are group homomorphisms). We leave this elementary check to the reader. □

Proposition 3.10 *Recall a and b denote the generators of F_2. The map*

$$\mathrm{eval}_a \times \mathrm{eval}_b \colon \mathcal{A} \to \mathbb{C}^n \times \mathbb{C}^n$$

is a vector space isomorphism.

Proof It should be clear that this defines a homomorphism between vector spaces by definition of the operations in Proposition 3.9. It is an isomorphism because the images of the generators determine the homomorphism; the inverse map sends $(\boldsymbol{a}, \boldsymbol{b})$ to the homomorphism determined by the following images of the generators of F_2:

$$a \mapsto (\boldsymbol{a}, \rho(a)) \quad \text{and} \quad b \mapsto (\boldsymbol{b}, \rho(b)). \tag{11}$$ □

Let $\mathrm{conj}\colon \mathbb{C}^n \times \mathcal{A} \to \mathcal{A}$ be the action defined by postconjugation by $\mathbb{C}^n \subset \mathrm{Aff}(n)$,

$$\mathrm{conj}_{\boldsymbol{v}}(\hat{\rho})(g) = (\boldsymbol{v}, I) \cdot \hat{\rho}(g) \cdot (-\boldsymbol{v}, I) \quad \text{for all } g \in F_2, \tag{12}$$

where I denotes the identity element of $\mathrm{GL}(n, \mathbb{C})$. When $\mathcal{A}$ is viewed as isomorphic to $\mathbb{C}^{2n}$, we see that each $\mathrm{conj}_{\boldsymbol{v}}$ acts by translation on $\mathcal{A}$ (ie $\mathrm{conj}_{\boldsymbol{v}}(\hat{\rho}) - \hat{\rho}$ does not depend on $\hat{\rho}$):

Proposition 3.11 *For each $\boldsymbol{v} \in \mathbb{C}^n$, each $\hat{\rho} \in \mathcal{A}$ and each $g \in F_2$, we have*

$$(\mathrm{conj}_{\boldsymbol{v}}(\hat{\rho}) - \hat{\rho})(g) = \big((I - \rho(g))\boldsymbol{v}, \rho(g)\big). \tag{13}$$

We call $\mathrm{conj}_{\boldsymbol{v}}(\hat{\rho}) - \hat{\rho}$ the *translation vector* of $\mathrm{conj}_{\boldsymbol{v}}$.

Proof This follows from the computation in $\mathrm{Aff}(n)$,

$$\mathrm{conj}_{\boldsymbol{v}}(\hat{\rho})(g) = (\boldsymbol{v}, I)\cdot(\pi_1 \circ \hat{\rho}(g), \rho(g))\cdot(-\boldsymbol{v}, I) = (\boldsymbol{v} + \pi_1 \circ \hat{\rho}(g) + \rho(g)(-\boldsymbol{v}), \rho(g)).$$ □

Let $\sim$ denote the equivalence relation on $\mathcal{A}$ where

$$\hat{\rho}_1 \sim \hat{\rho}_2 \quad \text{if there is a } \boldsymbol{v} \in \mathbb{C}^n \text{ satisfying } \mathrm{conj}_{\boldsymbol{v}}(\hat{\rho}_1) = \hat{\rho}_2. \tag{14}$$

Corollary 3.12 *The quotient $\mathcal{A}/\sim$ is a complex vector space with operations induced by those of $\mathcal{A}$.*

Proof It needs to be observed that the operations of addition and scalar multiplication induce well-defined actions on $\mathcal{A}/\sim$. This follows from linearity of the translation vector of (13) in $\boldsymbol{v} \in \mathbb{C}^n$. □

Recall that ρ is a fixed homomorphism. Recall the definition of $\Delta = \Delta_+ \sqcup \Delta_-$ in (7) from the proof of Proposition 3.3 and recall that ρ is an oriented characteristic representation if and only if the projection of $\Delta = \Delta_+ \sqcup \Delta_-$ to $\mathrm{Aut}(F_2)$ is surjective.

We view $\mathrm{GL}(n,\mathbb{C})$ as a subgroup of $\mathrm{Aff}(n)$. Conjugation by an element of $\mathrm{GL}(n,\mathbb{C})$ induces an automorphism of $\mathrm{Aff}(n)$.

We use $\mathrm{GL}(\mathcal{A})$ to denote the group of linear automorphisms of $\mathcal{A}$ and $\overline{\mathrm{GL}}(\mathcal{A})$ to denote the collection of conjugate-linear automorphisms. Together, $\mathrm{GL}(\mathcal{A}) \cup \overline{\mathrm{GL}}(\mathcal{A})$ forms a group. We have the following:

Lemma 3.13 *There is a homomorphism* $N\colon \Delta \to \mathrm{GL}(\mathcal{A}) \cup \overline{\mathrm{GL}}(\mathcal{A})$ *such that:*

(+) *If* $(M,\psi) \in \Delta_+$ *and* $\hat{\rho} \in \mathcal{A}$, *then* $N_{M,\psi} \in \mathrm{GL}(\mathcal{A})$ *and*

$$N_{M,\psi}(\hat{\rho})(g) = M \cdot (\hat{\rho} \circ \psi^{-1}(g)) \cdot M^{-1} \quad \text{for all } g \in F_2.$$

(–) *If* $(M,\psi) \in \Delta_-$ *and* $\hat{\rho} \in \mathcal{A}$, *then* $N_{M,\psi} \in \overline{\mathrm{GL}}(\mathcal{A})$ *and*

$$N_{M,\psi}(\hat{\rho})(g) = M \cdot (\overline{\hat{\rho} \circ \psi^{-1}(g)}) \cdot M^{-1} \quad \text{for all } g \in F_2.$$

Each $N_{M,\psi}$ *sends* $\sim$*–equivalence classes to* $\sim$*–equivalence classes and so induces an automorphism* $N^{\sim}_{M,\psi} \in \mathrm{GL}(\mathcal{A}/\sim) \cup \overline{\mathrm{GL}}(\mathcal{A}/\sim)$. *Furthermore, the induced map*

$$N^{\sim}\colon \Delta \to \mathrm{GL}(\mathcal{A}/\sim) \cup \overline{\mathrm{GL}}(\mathcal{A}/\sim) \quad \text{given by} \quad (M,\psi) \mapsto N^{\sim}_{M,\psi}$$

is a homomorphism.

Proof Since $M \in \mathrm{GL}(n,\mathbb{C})$ and $\psi \in \mathrm{Aut}(F_2)$, it should be clear that the definitions provided for $N_{M,\psi}(\hat{\rho})$ give a homomorphism $F_2 \to \mathrm{Aff}(n)$. Writing $\hat{\rho}(g) = (\pi_1 \circ \hat{\rho}(g), \rho(g))$ (using affability of $\hat{\rho}$) we see that when $(M,\psi) \in \Delta_+$ we have

$$\begin{aligned} N_{M,\psi}(\hat{\rho})(g) &= \big(M \cdot \pi_1 \circ \hat{\rho} \circ \psi^{-1}(g), M \cdot (\rho \circ \psi^{-1}(g)) \cdot M^{-1}\big) \\ &= (M \cdot \pi_1 \circ \hat{\rho} \circ \psi^{-1}(g), \rho(g)) \end{aligned} \tag{15}$$

with the last step given by definition of Δ_+ in (7). To see linearity observe that $\pi_1 \circ \hat{\rho} \circ \psi^{-1}(g)$ varies linearly in $\hat{\rho}$ by Proposition 3.9 and we are simply postcomposing with the linear action of $M \in \mathrm{GL}(n, \mathbb{C})$. Similarly, if $(M, \psi) \in \Delta_-$,

$$N_{M,\psi}(\hat{\rho})(g) = (M \cdot \overline{\pi_1 \circ \hat{\rho} \circ \psi^{-1}(g)}, M \cdot \overline{\rho \circ \psi^{-1}(g)} \cdot M^{-1}) \tag{16}$$
$$= (M \cdot \overline{\pi_1 \circ \hat{\rho} \circ \psi^{-1}(g)}, \rho(g)).$$

Observe that $N_{M,\psi}$ is conjugate-linear in this case.

Now we must check that the linear action respects $\sim$–equivalence classes. Suppose $\hat{\rho}_1 \sim \hat{\rho}_2$. By Proposition 3.11, this is true if and only if there is a $\boldsymbol{v} \in \mathbb{C}^n$ such that

$$(\hat{\rho}_1 - \hat{\rho}_2)(g) = \big((I - \rho(g))\boldsymbol{v}, \rho(g)\big) \quad \text{for all } g \in F_2. \tag{17}$$

Fix such a $\boldsymbol{v}$ and let $\hat{\rho}_{\boldsymbol{v}} \in \mathcal{A}$ be defined as in the right side of (17). Then, by linearity or conjugate-linearity of $N_{M,\psi}$, we have

$$N_{M,\psi}(\hat{\rho}_1) - N_{M,\psi}(\hat{\rho}_2) = N_{M,\psi}(\hat{\rho}_{\boldsymbol{v}}).$$

By (15), if $(M, \psi) \in \Delta_+$, we have

$$N_{M,\psi}(\hat{\rho}_{\boldsymbol{v}})(g) = \big(M \cdot (I - \rho \circ \psi^{-1}(g))\boldsymbol{v}, \rho(g)\big) = \big((I - \rho(g))M\boldsymbol{v}, \rho(g)\big),$$

where we are using the identity $M \cdot (\rho \circ \psi^{-1}(g)) \cdot M^{-1} = \rho(g)$ again in the second step. Then Proposition 3.11 tells us that $N_{M,\psi}(\hat{\rho}_1) \sim N_{M,\psi}(\hat{\rho}_2)$. Similarly, if $(M, \psi) \in \Delta_-$, we have

$$N_{M,\psi}(\hat{\rho}_{\boldsymbol{v}})(g) = \big(M \cdot \overline{(I - \rho \circ \psi^{-1}(g))\boldsymbol{v}}, \rho(g)\big) = \big((I - \rho(g))M\overline{\boldsymbol{v}}, \rho(g)\big),$$

and again Proposition 3.11 tells us that $N_{M,\psi}(\hat{\rho}_1) \sim N_{M,\psi}(\hat{\rho}_2)$. □

It will be useful later to note that inner automorphisms act trivially on $\mathcal{A}/\sim$:

Proposition 3.14 *Let $\psi_h \in \mathrm{Aut}(F_2)$ denote the inner automorphism $g \mapsto hgh^{-1}$. Then, for all $z \in \mathbb{C} \smallsetminus \{0\}$, we have*

$$(z\rho(h), \psi_h) \in \Delta_+ \quad \text{and} \quad N^{\sim}_{z\rho(h),\psi_h}([\hat{\rho}]) = z[\hat{\rho}] \quad \text{for all } [\hat{\rho}] \in \mathcal{A}/\sim.$$

In particular, $N^{\sim}_{\rho(h),\psi_h}$ acts trivially on $\mathcal{A}/\sim$.

Proof Fix $z \in \mathbb{C} \smallsetminus \{0\}$ and fix $h \in F_2$. Recall that $(z\rho(h), \psi_h) \in \Delta_+$ if and only if

$$(z\rho(h)) \cdot (\rho \circ \psi_h^{-1}(g)) \cdot (z\rho(h))^{-1} = \rho(g) \quad \text{for all } g \in G.$$

The z and z^{-1} cancel and the left side simplifies as

$$(z\rho(h)) \cdot (\rho \circ \psi_h^{-1}(g)) \cdot (z\rho(h))^{-1} = \rho(h)\rho(h^{-1}gh)\rho(h)^{-1} = \rho(g).$$

Now fix any $\hat{\rho} \in \mathcal{A}$ and observe $\hat{\rho} \circ \psi_h^{-1}(g) = \hat{\rho}(h^{-1})\hat{\rho}(g)\hat{\rho}(h)$. Choose $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^n$ satisfying $\hat{\rho}(g) = (\boldsymbol{v}, \rho(g))$ and $\hat{\rho}(h) = (\boldsymbol{w}, \rho(h))$. Then $\hat{\rho}(h)^{-1} = (-\rho(h)^{-1}\boldsymbol{w}, \rho(h)^{-1})$ and thus

$$\begin{aligned}\hat{\rho} \circ \psi_h^{-1}(g) &= (-\rho(h)^{-1}\boldsymbol{w}, \rho(h)^{-1}) \cdot (\boldsymbol{v}, \rho(g)) \cdot (\boldsymbol{w}, \rho(h)) \\ &= \big(\rho(h)^{-1}(\rho(g) - I)\boldsymbol{w} + \rho(h)^{-1}\boldsymbol{v}, \rho(h^{-1}gh)\big).\end{aligned}$$

By definition,

$$N_{z\rho(h),\psi_h}(\hat{\rho})(g) = (z\rho(h)) \cdot (\hat{\rho} \circ \psi_h^{-1}(g)) \cdot (z\rho(h))^{-1}.$$

By combining with the above we see

$$[N_{z\rho(h),\psi_h}(\hat{\rho}) - z\hat{\rho}](g) = \big((I - \rho(g))(-z\boldsymbol{w}), \rho(g)\big),$$

and so by Proposition 3.11 $N_{z\rho(h),\psi_h}(\hat{\rho}) \sim z\hat{\rho}$. □

Fix an integer $k \geq 2$. Recall $P_k \subset F_2$ denotes the subgroup generated by the k^{th} powers of primitive elements in F_2. Assume $P_k \subset \ker\rho$. The collection of *k–affable representations* is

$$\mathcal{A}_k = \{\hat{\rho} \in \mathcal{A} : P_k \subset \ker\hat{\rho}\}. \tag{18}$$

As a consequence of Proposition 3.9, $\mathcal{A}_k$ is a linear subspace of $\mathcal{A}$: it is the intersection of the kernels of the linear maps eval_{p^k} taken over all primitive $p \in F_2$.

We have:

Proposition 3.15 (1) *Each $\sim$–equivalence class is either contained in or disjoint from $\mathcal{A}_k$.*

(2) *For each $(M, \psi) \in \Delta$, $\mathcal{A}_k$ is invariant under $N_{M,\psi}$.*

Proof Since $P_k \subset F_2$ is characteristic, if $f : \mathcal{A} \to \mathcal{A}$ is such that $\ker f(\hat{\rho})$ differs from $\ker\hat{\rho}$ by an automorphism of F_2 for every affable $\hat{\rho}$, then $\mathcal{A}_k$ is invariant under f. This holds in the cases of f given by $\mathrm{conj}_{\boldsymbol{v}}$ and $N_{M,\psi}$, and these cases cover the respective cases of the proposition. □

Summarizing the results above, we see that $\mathcal{A}_k/{\sim}$ is a linear subspace of $\mathcal{A}/{\sim}$, and $N^{\sim}_{M,\psi}(\mathcal{A}_k/{\sim}) = \mathcal{A}_k/{\sim}$ for all $(M, \psi) \in \Delta$.

Choose any subspace $\mathcal{I} \subset \mathcal{A}_k/{\sim}$ which is invariant under the action of $N^{\sim}_{M,\psi}$ for $(M, \psi) \in \Delta$. Ideally we would take $\mathcal{I} = \mathcal{A}_k/{\sim}$ to get the largest invariant space possible. (Later in the proof of Theorem 3.18 we do not prove that our choice of $\mathcal{I}$ is all of $\mathcal{A}_k/{\sim}$.)

Let $m = \dim \mathcal{I}$. Choose $\widehat{\rho}_1, \dots, \widehat{\rho}_m \in \mathcal{A}_k$ such that the images in $\mathcal{A}_k/{\sim}$ form a basis for $\mathcal{I}$. In block matrix form we define

$$\widetilde{\rho}\colon F_2 \to \mathrm{GL}(n+m,\mathbb{C}), \quad g \mapsto \begin{pmatrix} \rho(g) & Q(g) \\ 0 & I \end{pmatrix} \in \mathrm{GL}(n+m,\mathbb{C}), \tag{19}$$

where

$$Q(g) = \begin{pmatrix} \pi_1 \circ \widehat{\rho}_1(g) & \pi_1 \circ \widehat{\rho}_2(g) & \dots & \pi_1 \circ \widehat{\rho}_m(g) \end{pmatrix}.$$

Here each $\pi_1 \circ \widehat{\rho}_i(g)$ is interpreted as the i^{th} column vector of $Q(g)$. Then:

Theorem 3.16 *Assume $\rho\colon F_2 \to \mathrm{GL}(n,\mathbb{C})$ is an oriented characteristic representation with $P_k \subset \ker \rho$. Define $\mathcal{I}$, m, $\widehat{\rho}_1, \dots, \widehat{\rho}_m$ and $\widetilde{\rho}$ as above. Then $\widetilde{\rho}$ is also an oriented characteristic representation with $P_k \subset \ker \widetilde{\rho}$. Furthermore, there is a short exact sequence of the form*

$$1 \to \widetilde{\rho}(\ker \rho) \to F_2/\ker \widetilde{\rho} \to F_2/\ker \rho \to 1,$$

and $\widetilde{\rho}(\ker \rho)$ is a torsion-free abelian group.

Proof First we will show that $\widetilde{\rho}$ is a group homomorphism. Considering the block form of the image, observe that it suffices to understand the top-right block (since we are given that ρ is a homomorphism). Checking that $\widetilde{\rho}(g_1 g_2) = \widetilde{\rho}(g_1)\widetilde{\rho}(g_2)$ then reduces to checking that

$$Q(g_1 g_2) = Q(g_1) + \rho(g_1) Q(g_2).$$

Checking this for column i amounts to checking that

$$\pi_1 \circ \widehat{\rho}_i(g_1 g_2) = \pi_1 \circ \widehat{\rho}_i(g_1) + \rho(g_1) \cdot \pi_1 \circ \widehat{\rho}_i(g_2),$$

which is true because $\widehat{\rho}_i$ is a homomorphism to $\mathrm{Aff}(n)$, which has product rule as in (10).

From (19) and by definition of $\mathcal{A}_k$, we see that $\widetilde{\rho}(g^k) = I$ for each primitive $g \in F_2$, guaranteeing that $P_k \subset \ker \widetilde{\rho}$.

Exactness of the provided sequence should be clear. The group $\widetilde{\rho}(\ker \rho)$ is torsion-free and abelian because for each $g \in \ker \rho$ we have

$$\widetilde{\rho}(g) = \begin{pmatrix} I & Q(g) \\ 0 & I \end{pmatrix}.$$

In particular, $\widetilde{\rho}(\ker \rho)$ is an additive subgroup of $\mathbb{C}^{mn}$.

It remains to show that $\widetilde{\rho}$ is an oriented characteristic representation. Choose any $\psi \in \mathrm{Aut}(F_2)$. Let $s \in \{+,-\}$ be such that $\psi \in \mathrm{Aut}_s(F_2)$. Define

$$\widetilde{\rho}_0\colon F_2 \to \mathrm{GL}(n+m,\mathbb{C})$$

by

$$\widetilde{\rho}_0(g) = \begin{cases} \widetilde{\rho} \circ \psi^{-1}(g) & \text{if } s = +, \\ \overline{\widetilde{\rho} \circ \psi^{-1}(g)} & \text{if } s = -. \end{cases}$$

We need to show that $\widetilde{\rho}_0$ is conjugate by an element of $\mathrm{GL}(m+n,\mathbb{C})$ to $\widetilde{\rho}$. We will demonstrate this by applying a sequence of conjugations.

First, since ρ is an oriented characteristic representation, there is a matrix $M \in \mathrm{GL}(n,\mathbb{R})$ such that $(M,\psi) \in \Delta_s$. This guarantees that either

$$M \cdot [\rho \circ \psi^{-1}(g)] \cdot M^{-1} = \rho(g) \quad \text{or} \quad M \cdot \overline{\rho \circ \psi^{-1}(g)} \cdot M^{-1} = \rho(g) \tag{20}$$

for all $g \in F_2$ depending on the sign s. Define $\widetilde{\rho}_1$ to be a conjugate of $\widetilde{\rho}_0$ formed as follows:

$$\widetilde{\rho}_1(g) = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \cdot \widetilde{\rho}_0(g) \cdot \begin{pmatrix} M^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Recall from Lemma 3.13 there is an $N_{M,\psi}$ in $\mathrm{GL}(\mathcal{A})$ or $\overline{\mathrm{GL}}(\mathcal{A})$ (depending on s) describing the conjugation action on $\mathcal{A}$. Let $\mathrm{Col}_i(X)$ denote the i^{th} column of a matrix X. By (20) and the description of $N_{M,\psi}$, we have

$$\widetilde{\rho}_1(g) = \begin{pmatrix} \rho(g) & Q_1(g) \\ 0 & I \end{pmatrix}, \quad \text{where } \mathrm{Col}_i(Q_1(g)) = \pi_1(N_{M,\psi}(\widehat{\rho}_i)(g))$$

for $i \in \{1,\dots,m\}$.

Let $\widetilde{\mathcal{I}} \subset \mathcal{A}_k$ denote the preimage of $\mathcal{I}$ under the quotient map $\mathcal{A}_k \to \mathcal{A}_k/{\sim}$; this is the union of the equivalence classes in $\mathcal{I}$. Recall that $\mathcal{I} \subset \mathcal{A}_k/{\sim}$ is $N_{\widetilde{M,\psi}}$–invariant. In addition, $\sim$–equivalence classes are permuted by $N_{M,\psi}$; see Lemma 3.13. It follows that $\widetilde{\mathcal{I}}$ is $N_{M,\psi}$–invariant. We therefore have that $N_{M,\psi}(\widehat{\rho}_i)$ lies in $\widetilde{\mathcal{I}}$ for $i \in \{1,\dots,m\}$. Let $\mathcal{I}_L = \mathrm{span}\{\widehat{\rho}_1,\dots,\widehat{\rho}_m\}$. From our choice of $\widehat{\rho}_1,\dots,\widehat{\rho}_m$, the space $\mathcal{I}_L$ is a lift of $\mathcal{I}$; ie the quotient map $\mathcal{A}_k \to \mathcal{A}_k/{\sim}$ restricts to an isomorphism $\mathcal{I}_L \to \mathcal{I}$. Recall that the $\sim$–equivalence classes are $\mathrm{conj}_{\boldsymbol{v}}$ orbits; see (14). Thus, for each $i \in \{1,\dots,m\}$ there is a unique vector $\boldsymbol{v}_i$ such that

$$\mathrm{conj}_{\boldsymbol{v}_i} \circ N_{M,\psi}(\widehat{\rho}_i) \in \mathcal{I}_L.$$

We now define a homomorphism $\widetilde{\rho}_2\colon F_2 \to \mathrm{GL}(n+m,\mathbb{C})$ conjugate to $\widetilde{\rho}_1$ by

$$\widetilde{\rho}_1(g) = \begin{pmatrix} I & V \\ 0 & I \end{pmatrix} \cdot \widetilde{\rho}_1(g) \cdot \begin{pmatrix} I & -V \\ 0 & I \end{pmatrix}, \quad \text{where } V = \begin{pmatrix} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_m \end{pmatrix}.$$

By definition of conj in (12) we see that

$$\widetilde{\rho}_2(g) = \begin{pmatrix} \rho(g) & Q_2(g) \\ 0 & I \end{pmatrix}, \quad \text{where } \mathrm{Col}_i(Q_2(g)) = \pi_1(\mathrm{conj}_{\boldsymbol{v}_i} \circ N_{M,\psi}(\widehat{\rho}_i))$$

for all $i \in \{1,\dots,m\}$.

Let p denote the isomorphism $\mathcal{I}_L \to \mathcal{I}$ mentioned above. Since this is an isomorphism, there is an $N_L \in \mathrm{GL}(\mathcal{I}_L) \cup \overline{\mathrm{GL}}(\mathcal{I}_L)$ such that $p \circ N_L = N_{\widetilde{M},\psi}|_{\mathcal{I}} \circ p$. Then, in particular, we have

$$\mathrm{conj}_{\boldsymbol{v}_i} \circ N_{M,\psi}(\widehat{\rho}_i) = N_L(\widehat{\rho}_i) \quad \text{for all } i \in \{1,\dots,m\}.$$

As a consequence we see that $\{\mathrm{conj}_{\boldsymbol{v}_i} \circ N_{M,\psi}(\widehat{\rho}_i) : i = 1,\dots,m\}$ is a basis for $\mathcal{I}_L$. Thus there is a matrix $R = (R_{i,j}) \in \mathrm{GL}(m,\mathbb{C})$ such that

$$\widehat{\rho}_j = \sum_{i=1}^{m} R_{i,j} \mathrm{conj}_{\boldsymbol{v}_i} \circ N_{M,\psi}(\widehat{\rho}_i) \quad \text{for all } j \in \{1,\dots,m\}.$$

Then, by linearity of the evaluation maps (see Proposition 3.9), for each $g \in F_2$ we have

$$\pi_1(\widehat{\rho}_j(g)) = \sum_{i=1}^{m} R_{i,j} \pi_1(\mathrm{conj}_{\boldsymbol{v}_i} \circ N_{M,\psi}(\widehat{\rho}_i)(g)) \quad \text{for all } j \in \{1,\dots,m\}. \tag{21}$$

Recall the left side gives the columns of $Q(g)$, which is the top-right submatrix of $\widetilde{\rho}(g)$; see (19). Thus, this equation expresses column j of $Q(g)$ as a linear combination of columns of $Q_2(g)$ with weights given by entries in the j^{th} column of R. Thus we have that $Q_2(g) \cdot R = Q(g)$ for all g. We define the conjugate $\widetilde{\rho}_3$ of $\widetilde{\rho}_2$ by

$$\widetilde{\rho}_3(g) = \begin{pmatrix} I & 0 \\ 0 & R^{-1} \end{pmatrix} \cdot \widetilde{\rho}_2(g) \cdot \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} = \begin{pmatrix} \rho(g) & Q_2(g)R \\ 0 & 1 \end{pmatrix} = \widetilde{\rho}(g).$$

Since $\widetilde{\rho}_3 = \widetilde{\rho}$, we have produced the desired conjugacy. □

3.4 Case $k = 6$

We define ρ_6 to be the tensor product $\rho_2 \otimes \rho_3$, where ρ_2 and ρ_3 are defined as in (8) and (9). Then ρ_6 may be thought of as a homomorphism $F_2 \to \mathrm{GL}(9,\mathbb{C})$. Letting

$\omega = e^{2\pi i/3}$, we have the formulas

$$\rho_6(a) = \operatorname{diag}(-1, -\omega, -\omega^2; -1, -\omega, -\omega^2; 1, \omega, \omega^2),$$

$$\rho_6(b) = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right).$$

Applying the improvement algorithm of Theorem 3.16 to ρ_6 can be shown by calculation to give rise to the representation $\tilde{\rho}_6\colon F_2 \to \mathrm{GL}(12, \mathbb{C})$ defined as block matrices as

$$\tilde{\rho}_6(a) = \begin{pmatrix} \rho_6(a) & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \tilde{\rho}_6(b) = \begin{pmatrix} \rho_6(b) & B \\ 0 & I \end{pmatrix}, \quad \text{where } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will not present the computational proof that $\tilde{\rho}_6$ arises from ρ_6 by applying Theorem 3.16 with $\mathcal{I} = \mathcal{A}_6$. However we will demonstrate that it is an oriented characteristic representation:

Proposition 3.17 *The homomorphism $\tilde{\rho}_6$ is an oriented characteristic representation. The kernel $\ker\tilde{\rho}_6$ contains P_6 and is of infinite index in F_2. Furthermore, there is a short exact sequence of groups of the form*

$$1 \to \mathbb{Z}^{18} \to F_2/\ker\tilde{\rho}_6 \to C_2 \times C_2 \times H(\mathbb{Z}/3\mathbb{Z}) \to 1.$$

Proof To see the representation is oriented characteristic, observe that the criterion of Proposition 3.3 is satisfied with the choices of matrices $M_- = I$ and M_1 and M_2 as below:

$$M_1 = \left(\begin{array}{ccccccccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & -2\omega-1 & 0 \\
0 & 0 & 0 & 1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 2\omega+1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & \omega^2 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\omega-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0
\end{array}\right),$$

$$M_2 = \left(\begin{array}{ccccccccc|ccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2
\end{array}\right).$$

Thus $\ker\widetilde{\rho}_6$ is a characteristic subgroup of F_2 by Proposition 3.1. Observe that $\widetilde{\rho}_6(a^6) = I$, and thus $P_6 \subset \ker\widetilde{\rho}_6$.

The top-left 9×9 block is isomorphic to the representation $\rho_6 = \rho_2 \otimes \rho_3$, where these representations were taken from Section 3.2. Thus, the image $\rho_6(F_2)$ is isomorphic to $C_2 \times C_2 \times H(\mathbb{Z}/3\mathbb{Z})$, and therefore this map induces the surjective map $F_2/\ker\widetilde{\rho}_6 \to C_2 \times C_2 \times H(\mathbb{Z}/3\mathbb{Z})$ in the short exact sequence. The kernel of this map is isomorphic to the image $\widetilde{\rho}_6(\ker\rho_6)$. Thus we get our exact sequence as described but with $\mathbb{Z}^{18}$ replaced by $\widetilde{\rho}_6(\ker\rho_6)$. The group $\widetilde{\rho}_6(\ker\rho_6)$ is an abelian group because matrices in $\widetilde{\rho}_6(\ker\rho_6)$ have a 2×2 block form with copies of the identity matrix along the diagonal and a zero matrix in the lower-left block. In this subgroup, multiplication is the same as addition in the top-right block, and thus $\widetilde{\rho}_6(\ker\rho_6)$ is naturally a subgroup of $\mathbb{Z}[\omega]^{27}$

since $\widetilde{\rho}_6$ takes values in $\mathrm{GL}(12,\mathbb{Z}[\omega])$. This shows that $\widetilde{\rho}_6(\ker\rho_6)$ is a finite-rank free abelian group. To figure out this rank, we observe using [18] that $C_2\times C_2\times H(\mathbb{Z}/3\mathbb{Z})$ can be written as a quotient of the rank 2 free group $\langle a,b\rangle$ by

$$\ker\rho_6=\langle\!\langle a^6,b^6,[a,b]^3,[a,[a,b]],[b,[a,b]]\rangle\!\rangle.$$

We already know that $a^6,b^6\in\ker\widetilde{\rho}_6$ and can check that $[a,b]^3\in\ker\widetilde{\rho}_6$. Thus the abelian image $\widetilde{\rho}_6(\ker\rho_6)$ is generated by elements of the form

$$\widetilde{\rho}_6(g[a,[a,b]]g^{-1})\quad\text{and}\quad\widetilde{\rho}_6(g[b,[a,b]]g^{-1})\tag{22}$$

with $g\in F_2$. Note that if $\rho_6(g_1)=\rho_6(g_2)$, then $g_1g_2^{-1}\in\ker\rho_6$, and since $\widetilde{\rho}_6(\ker\rho_6)$ is abelian, we have

$$\begin{aligned}\widetilde{\rho}_6(g_2[a,[a,b]]g_2^{-1})&=\widetilde{\rho}_6(g_1g_2^{-1})\widetilde{\rho}_6(g_2[a,[a,b]]g_2^{-1})\widetilde{\rho}_6(g_1g_2^{-1})^{-1}\\&=\rho_6(g_1[a,[a,b]]g_1^{-1}).\end{aligned}$$

Similarly, for such g_1 and g_2, we have

$$\widetilde{\rho}_6(g_1[b,[a,b]]g_1^{-1})=\widetilde{\rho}_6(g_2[b,[a,b]]g_2^{-1}).$$

Since $\rho_6(F_2)\cong C_2\times C_2\times H(\mathbb{Z}/3\mathbb{Z})$, to generate $\widetilde{\rho}_6(\ker\rho_6)$ it suffices to take elements from (22) with one g taken from each preimage $\rho_6^{-1}(M)$, where M varies over elements of $\rho_6(F_2)$. Since $\rho_6(F_2)\cong C_2\times C_2\times H(\mathbb{Z}/3\mathbb{Z})$, this amounts to a list of 108 pairs of generators. This reduces the computation of the rank of $\widetilde{\rho}_6(\ker\rho_6)$ to a finite computation which can be done on the computer. Using SageMath [19], we computed $\operatorname{rank}\widetilde{\rho}_6(\ker\rho_6)=18$, so $\widetilde{\rho}_6(\ker\rho_6)\cong\mathbb{Z}^{18}$. □

3.5 Odd $k\geq 5$

Let $k\geq 5$ be odd and define $\rho_k\colon F_2\to\mathrm{GL}(k,\mathbb{C})$ as in (9). We define $\omega=e^{2\pi i/k}$.

We will define an extension $\widetilde{\rho}_k\colon F_2\to\mathrm{GL}\big(k+\frac12(k-3),\mathbb{C}\big)$, which we found by applying the method of Theorem 3.16. (In the next proof, we will show that $\widetilde{\rho}_k$ arises from ρ_k by this method.) Let B denote the $k\times\frac12(k-3)$ matrix whose column vectors are given by

$$\boldsymbol{b}_j=\boldsymbol{e}_{j+1}-\boldsymbol{e}_{k-j}\quad\text{for integers } j \text{ with } 1\leq j\leq\tfrac12(k-3),\tag{23}$$

where $\boldsymbol{e}_i$ denotes the standard basis vector with a 1 in position i. We define $\widetilde{\rho}_k$ in block form by

$$\widetilde{\rho}_k(a)=\begin{pmatrix}\rho_k(a)&0\\0&I\end{pmatrix}\quad\text{and}\quad\widetilde{\rho}_k(b)=\begin{pmatrix}\rho_k(b)&B\\0&I\end{pmatrix}.\tag{24}$$

Theorem 3.18 *For each odd $k \geq 5$ the homomorphism $\widetilde{\rho}_k$ is an oriented characteristic representation. The kernel $\ker \widetilde{\rho}_k$ contains P_k and is of infinite index in F_2. Furthermore, there is a short exact sequence of groups of the form*

$$1 \to \mathbb{Z}^d \to F_2/\ker \widetilde{\rho}_k \to H(\mathbb{Z}/k\mathbb{Z}) \to 1,$$

where $d = k \cdot \frac{1}{2}(k-3) \cdot [\mathbb{Q}(\omega) : \mathbb{Q}]$.

Proof Fix an odd $k \geq 5$. To simplify notation, we will use ρ to denote ρ_k and use $\widetilde{\rho}$ to denote $\widetilde{\rho}_k$ as defined in (24). In this proof, we will show that $\widetilde{\rho}$ is derivable from ρ as described by Theorem 3.16. The theorem then implies that $\widetilde{\rho}$ is an oriented characteristic representation and that $P_k \subset \ker \widetilde{\rho}$.

Verifying that the theorem applies requires working through Section 3.3. We will begin by setting up notation and applying some results from Section 3.3 to our setting. We will then define a subspace $\mathcal{I}_L$ of the space $\mathcal{A}$ of affable representations. We will check that $\mathcal{I}_L$ is contained in $\mathcal{A}_k$ and that its quotient in $\mathcal{A}_k/\sim$ is $N_{\widetilde{M},\psi}$–invariant. Then we will observe that $\widetilde{\rho}$ as defined above coincides with the definition in (19) used in Theorem 3.16 with an appropriate choice of basis. Finally we must check that the group $\widetilde{\rho}(\ker \rho)$ is isomorphic to $\mathbb{Z}^d$ with d as in the statement of the theorem.

Recall that $\mathrm{eval}_a \times \mathrm{eval}_b$ gives an isomorphism $\mathcal{A} \to \mathbb{C}^k \times \mathbb{C}^k$; see Proposition 3.10. We'll find it useful to use coordinates provided by the inverse map

$$R = (\mathrm{eval}_a \times \mathrm{eval}_b)^{-1} \colon \mathbb{C}^k \times \mathbb{C}^k \to \mathcal{A}.$$

The image of $(\boldsymbol{a}, \boldsymbol{b})$ is defined as in (11).

The subgroup $\mathbb{C}^k \subset \mathrm{Aff}(k)$ acts on $\mathcal{A}$ by conjugation. For $\boldsymbol{v} \in \mathbb{C}^k$ we used $\mathrm{conj}_{\boldsymbol{v}}$ to denote this action, and wrote $\widehat{\rho}_1 \sim \widehat{\rho}_2$ if there is a $\boldsymbol{v}$ such that $\mathrm{conj}_{\boldsymbol{v}}(\widehat{\rho}_1) = \widehat{\rho}_2$. The space $\mathcal{A}/\sim$ is a vector space. By Proposition 3.11, we know that $\mathrm{conj}_{\boldsymbol{v}}$ acts by translation on $\mathcal{A}$ and this translation vector depends linearly on $\boldsymbol{v}$. Thus the natural map $C \colon \mathcal{A} \to \mathcal{A}/\sim$ is linear and the kernel $\mathcal{T} = \ker C$ is the collection of translation vectors. By applying the formula in Proposition 3.11 to the standard basis vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k \in \mathbb{C}^k$ and our particular ρ, we see

$$\mathcal{T} = \mathrm{span}_{\mathbb{C}}\big(\{R(\boldsymbol{0}, \boldsymbol{e}_1 - \boldsymbol{e}_k)\} \cup \{R((1-\omega^{j-1})\boldsymbol{e}_j, \boldsymbol{e}_j - \boldsymbol{e}_{j-1}) : 2 \leq j \leq k\}\big). \tag{25}$$

In particular, for each $\widehat{\rho} \in \mathcal{A}$ there is a unique $\boldsymbol{v} \in \mathbb{C}^k$ satisfying

$$\mathrm{conj}_{\boldsymbol{v}}(\widehat{\rho}) = R(c_1 \boldsymbol{e}_1, c_2 \boldsymbol{e}_2 + c_3 \boldsymbol{e}_3 + \cdots + c_k \boldsymbol{e}_k) \tag{26}$$

for some choice of $c_1, \dots, c_k \in \mathbb{C}$. This gives a standard representative for each conjugacy class. Let $\mathcal{S} \subset \mathcal{A}$ denote those representations which can be written in the form on the right side of (26). Then $\mathcal{S}$ is a section for C in the sense that the restriction $C|_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{A}/\sim$ is an isomorphism of vector spaces, and there is a linear map $P\colon \mathcal{A} \to \mathcal{S}$ with kernel $\mathcal{T}$ (defined by $S = C|_{\mathcal{S}}^{-1} \circ C$) which stabilizes points in $\mathcal{S}$. That is, P is the projection to $\mathcal{S}$ with leaves parallel to $\mathcal{T}$.

Now consider the subspace $\mathcal{A}_k \subset \mathcal{A}$ consisting of those $\hat{\rho} \in \mathcal{A}$ such that $P_k \subset \ker \hat{\rho}$, as originally defined in (18). Define

$$\mathcal{I}_L = \operatorname{span}_{\mathbb{C}}\left\{R(\mathbf{0}, \boldsymbol{b}_j) : 1 \le j \le \tfrac{1}{2}(k-3)\right\}, \tag{27}$$

where the vectors $\boldsymbol{b}_j$ are defined as in (23). Note $\mathcal{I}_L \subset \mathcal{S}$. Define $\mathcal{I} = C(\mathcal{I}_L)$. Then $\mathcal{I} = \mathcal{I}_L/\sim$ is a subspace of $\mathcal{A}/\sim$. We make the following claims:

Claim 1 *$\mathcal{I}$ is $N^{\sim}_{M,\psi}$–invariant for all $(M, \psi) \in \Delta$.*

Claim 2 *$\mathcal{I} \subset \mathcal{A}_k$.*

This will verify the hypotheses of Theorem 3.16 providing a new oriented characteristic representation $\tilde{\rho}$ with $P_k \subset \ker \tilde{\rho}$. Let

$$\hat{\rho}_j = R(\mathbf{0}, \boldsymbol{b}_j) \quad \text{for } j \in \left\{1, \dots, \tfrac{1}{2}(k-3)\right\}. \tag{28}$$

We obtain the matrix representation for $\tilde{\rho}$ using (19).

We will see that $\mathcal{I}$ has algebraic significance which explains the invariance in Claim 1.

First consider the kernel of the natural projection $\pi_2\colon \Delta \to \operatorname{Aut}(F_2)$. This subgroup consists of those pairs (M, id) such that M commutes with every $\rho(g)$. Since ρ is irreducible (Corollary 3.8), Schur's lemma implies that only the center of $\mathrm{GL}(k, \mathbb{C})$ commutes with all of $\rho(F_2)$. Thus,

$$\ker \pi_2 = \left\{(zI, \mathrm{id}) : z \in \mathbb{C} \setminus \{0\}\right\}. \tag{29}$$

Let π_2' denote the natural map $\Delta \to \operatorname{Out}(F_2)$. If $(M, \psi) \in \ker \pi_2'$, then there is an h such that $\psi(g) = hgh^{-1}$ for all $g \in F_2$. Fix this ψ for this discussion. Observe that one M which satisfies $(M, \psi) \in \Delta$ is $\rho(h)$ (see Proposition 3.14). The other solutions differ by multiplication by an element of $\ker \pi_2$, so we have $(M, \psi) \in \Delta$ if and only if $M = z\rho(h)$ for some $z \in \mathbb{C} \setminus \{0\}$. Then, by recalling Proposition 3.14, we conclude that for any $(M,\psi)\in\Delta$ with ψ an inner automorphism, there is a $z\in\mathbb{C}\smallsetminus\{0\}$ such that

$$N^{\sim}_{M,\psi}([\hat{\rho}]) = z[\hat{\rho}] \quad \text{for all } [\hat{\rho}]\in\mathcal{A}/{\sim}. \tag{30}$$

Let $\psi_1\in\mathrm{Aut}(F_2)$ be as in (6). Let $M_1\in\mathrm{GL}(k,\mathbb{C})$ be the matrix in Proposition 3.6. Then we have $(M_1,\psi_1)\in\Delta$. We claim that every element of $N^{\sim}_{\Delta}$ preserves the eigenspaces of $N^{\sim}(M_1,\psi_1)^2$. (In this proof, we will use $N^{\sim}(M,\psi)$ to denote $N^{\sim}_{M,\psi}$ and $N(M,\psi)$ to denote $N_{M,\psi}$ to avoid double subscripts.) This has to do with the fact that the outer automorphism class of ψ_1^2 represents $-I$ in the identification of $\mathrm{Out}(F_2)$ with $\mathrm{GL}(2,\mathbb{Z})$, and thus lies in the center of $\mathrm{Out}(F_2)$. To understand these eigenspaces are invariant, first recall that ψ_1^4 is the trivial automorphism of F_2. Thus, by (29), M_1^4 is a nonzero scalar multiple of the identity. Note that $N^{\sim}(M_1,\psi_1)^4$ also scales by the same amount. It follows (say by considering Jordan canonical form) that $N^{\sim}(M_1,\psi_1)^2$ is diagonalizable and has eigenvalues in the set $\{\pm z_1\}$ for some $z_1\in\mathbb{C}\smallsetminus\{0\}$. Let d_+ and d_- denote the dimensions of the eigenspaces with eigenvalue z_1 and $-z_1$, respectively. Since $\dim(\mathcal{A}/{\sim})=k$ (which follows from (26)), we have $d_++d_-=k$. Since k is odd it follows that $d_+\neq d_-$. To verify that these eigenspaces are preserved, pick any $(M,\psi)\in\Delta$. Since the image of ψ_1^2 in $\mathrm{Out}(F_2)$ is central, we know that the commutator $[\psi^{-1},\psi_1^{-2}]$ is an inner automorphism. Thus, by (30), $N^{\sim}([\psi^{-1},\psi_1^{-2}],[M^{-1},M_1^{-2}])$ scales elements of $\mathcal{A}/{\sim}$ by some $z\in\mathbb{C}\smallsetminus\{0\}$, and by simplifying we get

$$N^{\sim}(M,\psi)\circ N^{\sim}(M_1,\psi_1)^2\circ N^{\sim}(M,\psi)^{-1} = zN^{\sim}(M_1^2,\psi_1^2). \tag{31}$$

Observe that the left side above is conjugate to $N^{\sim}(M_1,\psi_1)^2$ and so has eigenspaces of dimension $d_\pm$ with corresponding eigenvalues of $\pm z_1$. The right-hand side, however, has eigenvalues of dimension $d_\pm$ with corresponding eigenvalues of $\pm zz_1$. It follows that $z=1$, and then (31) gives centrality of $N^{\sim}(M_1,\psi_1)^2$ in $N^{\sim}_{\Delta}$ and this centrality implies that the eigenspaces of $N^{\sim}(M_1,\psi_1)^2$ must be preserved by elements of $N^{\sim}_{\Delta}$.

We will now find a basis of eigenvectors for $N^{\sim}(M_1,\psi_1)^2$ to show that $\mathcal{I}$ is an eigenspace. Observe that $\psi_1^2\colon F_2\to F_2$ is the automorphism satisfying

$$\psi_1^2(a)=a^{-1} \quad \text{and} \quad \psi_1^2(b)=b^{-1}$$

and thus ψ_1^2 is an involution. Also, the entries of M_1^2 are given by

$$(M_1^2)_{i,j}=\sum_{\ell=1}^{k}\omega^{(i-1)(\ell-1)}\omega^{(\ell-1)(j-1)}$$

$$= \sum_{\ell=1}^{k} \omega^{(j+i-2)(\ell-1)}$$
$$= \begin{cases} k & \text{if } j+i-2 \equiv 0 \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

(32) $$M_1^2 \boldsymbol{e}_j = k \boldsymbol{e}_i, \quad \text{where } i \text{ is such that } j+i-2 \equiv 0 \pmod{k}.$$

Recalling the notation in the paragraph including (25) and (26), we define the map $N_1^2 \colon \mathcal{S} \to \mathcal{S}$ by

(33) $$N_1^2 = C|_{\mathcal{S}}^{-1} \circ N^{\sim}(M_1, \psi_1)^2 \circ C|_{\mathcal{S}} = P \circ N(M_1, \psi_1)^2|_{\mathcal{S}}.$$

Equality of these two expressions follows from the facts that

$$C \circ N(M_1, \psi_1)^2 = N^{\sim}(M_1, \psi_1)^2 \circ C$$

(ie that $N^{\sim}(M_1, \psi_1)^2$ is the action on $\mathcal{A}/\sim$ induced by $N(M_1, \psi_1)^2$) and that $P = C|_{\mathcal{S}}^{-1} \circ C$, as noted in the paragraph cited above. We will evaluate N_1^2 using the rightmost identity in (33), by applying $N(M_1, \psi_1)^2$ followed by the projection $P \colon \mathcal{A} \to \mathcal{S}$ which has fibers parallel to $\mathcal{T}$. We will show that a list of eigenvalues and eigenvectors of N_1^2 is given by:

(a) The vectors $\hat{\rho}_j = R(\boldsymbol{0}, \boldsymbol{b}_j)$ for $j \in \{1, \dots, \frac{1}{2}(k-3)\}$ are eigenvectors with eigenvalue k.

(b) The vectors $R(\boldsymbol{0}, \boldsymbol{e}_{j+1} + \boldsymbol{e}_{k-j})$ for $j \in \{1, \dots, \frac{1}{2}(k-3)\}$ are eigenvectors with eigenvalue $-k$.

(c) The vectors $R(\boldsymbol{e}_1, \boldsymbol{0})$, $R(\boldsymbol{0}, \boldsymbol{e}_{(k+1)/2})$ and $R(\boldsymbol{0}, \boldsymbol{e}_k)$ are eigenvectors with eigenvalue $-k$.

The reader will observe that the vectors listed above span $\mathcal{S}$ and the eigenspace formed by the span of the eigenvectors in case (a) coincides with $\mathcal{I}_L$. Thus, by proving these statements we will have verified Claim 1.

Before proving (a)–(c) we need to understand the action of $N(M_1, \psi_1)^2$. Let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^k \times \mathbb{C}^k$ and $\hat{\rho} = R(\boldsymbol{a}, \boldsymbol{b})$. We have, by definition of N,

$$\begin{aligned} N(M_1, \psi_1)^2(\hat{\rho})(a) &= M_1^2 \cdot (-\rho(a^{-1})\boldsymbol{a}, \rho(a^{-1})) \cdot M_1^{-2} \\ &= (-M_1^2 \rho(a^{-1})\boldsymbol{a}, M_1^2 \rho(a^{-1}) M_1^{-2}). \end{aligned}$$

Since $(M_1^2, \psi_1^2) \in \Delta$, we know that $M_1^2 \rho(a^{-1}) M_1^{-2} = \rho(a)$ and thus

$$N(M_1, \psi_1)^2(\hat{\rho})(a) = (-\rho(a) M_1^2 \boldsymbol{a}, \rho(a)).$$

Similarly, we have $N(M_1, \psi_1)^2(\hat{\rho})(b) = (-\rho(b) M_1^2 \boldsymbol{b}, \rho(a))$. Putting these two together, we see that

$$N(M_1, \psi_1)^2 \circ R(\boldsymbol{a}, \boldsymbol{b}) = R(-\rho(a) M_1^2 \boldsymbol{a}, -\rho(b) M_1^2 \boldsymbol{b}).$$

We specialize this using our understanding of M_1^2 and $\rho(a)$ and $\rho(b)$ into some useful special cases. We have

$$N(M_1, \psi_1)^2 \circ R(\boldsymbol{e}_1, \boldsymbol{0}) = R(-\rho(a) M_1^2 \boldsymbol{e}_1, \boldsymbol{0}) = R(-k\rho(a)\boldsymbol{e}_1, \boldsymbol{0}) = R(-k\boldsymbol{e}_1, \boldsymbol{0}). \tag{34}$$

For $j > 1$, we have

$$\begin{aligned} N(M_1, \psi_1)^2 \circ R(\boldsymbol{0}, \boldsymbol{e}_j) &= R(\boldsymbol{0}, -\rho(b) M_1^2 \boldsymbol{e}_j) = R(\boldsymbol{0}, -k\rho(b)\boldsymbol{e}_{k+2-j}) \\ &= R(\boldsymbol{0}, -k\boldsymbol{e}_{k+1-j}). \end{aligned} \tag{35}$$

Now we will check (a)–(c). First consider (c). That $N_1^2 \circ R(\boldsymbol{e}_1, \boldsymbol{0}) = -kR(\boldsymbol{e}_1, \boldsymbol{0})$ follows from (34). Similarly, that $R(\boldsymbol{0}, \boldsymbol{e}_{(k+1)/2})$ has eigenvalue $-k$ follows from (35) with $j = \frac{1}{2}(k+1)$. Again by (35), $N_1^2 \circ R(\boldsymbol{0}, \boldsymbol{e}_k)$ is the projection of $R(\boldsymbol{0}, -k\boldsymbol{e}_1)$ to $\mathcal{S}$ along $\mathcal{T}$. Since $R(\boldsymbol{0}, \boldsymbol{e}_1 - \boldsymbol{e}_k) \in \mathcal{T}$, we have $N_1^2 \circ R(\boldsymbol{0}, \boldsymbol{e}_k) = R(\boldsymbol{0}, -k\boldsymbol{e}_k)$, finishing the proof of (c). Now consider (a). Recall that $\boldsymbol{b}_j = \boldsymbol{e}_{j+1} - \boldsymbol{e}_{k-j}$ for $j = 1, \dots, \frac{1}{2}(k-3)$, and using (35), we observe

$$N(M_1, \psi_1)^2 \circ R(\boldsymbol{0}, \boldsymbol{b}_j) = R(\boldsymbol{0}, -k\boldsymbol{e}_{k-j} + k\boldsymbol{e}_{j+1}) = kR(\boldsymbol{0}, \boldsymbol{b}_j),$$

which verifies (a). Finally consider (b). For $j \in \{1, \dots, \frac{1}{2}(k-3)\}$, we have

$$N(M_1, \psi_1)^2 \circ R(\boldsymbol{0}, \boldsymbol{e}_{j+1} + \boldsymbol{e}_{k-j}) = R(\boldsymbol{0}, -k\boldsymbol{e}_{k-j} - k\boldsymbol{e}_{j+1}) = -kR(\boldsymbol{0}, \boldsymbol{e}_{j+1} + \boldsymbol{e}_{k-j}).$$

This completes the proof of Claim 1.

Finally, we need to prove Claim 2 that $\mathcal{I} \subset \mathcal{A}_k/{\sim}$. From the above we know that $\mathcal{I}$ is $N^{\sim}$–invariant, so it suffices to prove that $\hat{\rho}(a^k) = 1$ for each $\hat{\rho} \in \mathcal{I}_L$. We clearly have this since each $\hat{\rho} = R(\boldsymbol{0}, \boldsymbol{v})$ for some $\boldsymbol{v} \in \mathbb{C}^k$; see (27). This proves Claim 2.

Since we have proven the claims, we obtain the representation $\tilde{\rho}$ as discussed surrounding (28). It remains to show that $\tilde{\rho}(\ker \rho) \cong \mathbb{Z}^d$ with $d = k \cdot \frac{1}{2}(k-3) \cdot n$ with $n = [\mathbb{Q}(\omega) : \mathbb{Q}]$ as stated in the theorem. Observe that $\tilde{\rho}$ is a representation from F_2 into $\mathrm{GL}\big(k + \frac{1}{2}(k-3), \mathbb{Z}[\omega]\big)$. For $g \in \ker \rho$, the matrix $\tilde{\rho}(g)$ has a block form as in (19), with the identity appearing in the diagonal blocks, zero in the bottom left, and a $k \times \frac{1}{2}(k-3)$ matrix $Q(g)$ in the top right. We will show that the rank d is as large as possible: as large as the rank $k \cdot \frac{1}{2}(k-3) \cdot n$ of the additive group of $k \times \frac{1}{2}(k-3)$ matrices with entries in $\mathbb{Z}[\omega]$. We claim that it suffices to find a $g \in \ker \rho$ such that the top-right block $Q(g)$ has linearly independent columns. We will prove this suffices, and then give such a g below. Observe that given any $h \in F_2$, we have $hgh^{-1} \in \ker \rho$ and a computation shows that $Q(hgh^{-1}) = \rho(h)Q(g)$. It follows that

$$Q(\ker \rho) \supset \Lambda Q(g),$$

where Λ is the additive group of matrices generated by $\rho(F_2)$. Proposition 3.7 guarantees that the additive group $\boldsymbol{M}$ of $k \times k$ matrices with entries in $\mathbb{Z}[\omega]$ contains Λ as a finite-index subgroup. Thus we can find matrices $M_1, \dots, M_{k^2 n} \in \Lambda$ which generate the space of $k \times k$ matrices with entries in $Q(\omega)$ as a $\mathbb{Q}$–vector space. Define the map Φ to send a $k \times k$ matrix M with entries in $\mathbb{Q}(\omega)$ to the product $MQ(g)$. Then Φ is $\mathbb{Q}(\omega)$–linear, so we have

$$\dim_{\mathbb{Q}(\omega)}(\ker \Phi) + \dim_{\mathbb{Q}(\omega)}(\operatorname{img} \Phi) = \dim_{\mathbb{Q}(\omega)} \boldsymbol{M} = k^2.$$

The kernel of Φ consists of those $M \in \boldsymbol{M}$ such that the rows of M are perpendicular to each column of $Q(g)$. Since the columns of $Q(g)$ are linearly independent, the rows of matrices in $\ker \Phi$ can be taken from a $\mathbb{Q}(\omega)$–linear subspace of codimension $\frac{1}{2}(k-3)$. We conclude $\dim_{\mathbb{Q}(\omega)}(\ker \Phi) = k\bigl(k - \frac{1}{2}(k-3)\bigr)$ and it follows that

$$\dim_{\mathbb{Q}(\omega)}(\operatorname{img} \Phi) = k \cdot \tfrac{1}{2}(k-3) \quad \text{and so} \quad \dim_{\mathbb{Q}}(\operatorname{img} \Phi) = k \cdot \tfrac{1}{2}(k-3) \cdot n.$$

The images $\Phi(M_1), \dots, \Phi(M_{k^2 n})$ of our $\mathbb{Q}$–basis of matrices span the image of Φ as a $\mathbb{Q}$–vector space, so we can find $k \cdot \frac{1}{2}(k-3) \cdot n$ such images which are linearly independent over $\mathbb{Q}$. These images freely generate a free abelian group, which lies in $\Lambda Q(g)$ and therefore also in $Q(\ker \rho)$. We conclude that the rank of $Q(\ker \rho)$ is at least $k \cdot \frac{1}{2}(k-3) \cdot n$, as desired.

We carry out this calculation for $g = [a^{-1}, [a, b^{-1}]] = aba^{-1}b^{-1}a^{-1}bab^{-1} \in \ker(\rho)$. The columns of $Q(g)$ are given by $\pi_1 \circ \hat{\rho}_j(g)$ for $j \in \bigl\{1, \dots, \frac{1}{2}(k-3)\bigr\}$. It may be computed that

$$\begin{aligned}
\hat{\rho}_j(aba^{-1}b^{-1}) &= ((\omega^j - \omega^{-1})\boldsymbol{e}_{j+1} + (\omega^{-1} - \omega^{-j-1})\boldsymbol{e}_{k-j}, \omega^{-1} I),\\
\hat{\rho}_j(a^{-1}bab^{-1}) &= ((\omega^{-j} - \omega^{1})\boldsymbol{e}_{j+1} + (\omega^{1} - \omega^{j+1})\boldsymbol{e}_{k-j}, \omega I),\\
\hat{\rho}_j(g) &= ((\omega^j - 1)(1 - \omega^{-j-1})(\boldsymbol{e}_{j+1} - \boldsymbol{e}_{k-j}), I).
\end{aligned}$$

(This calculation was done by hand and independently verified using [19] for several values of k.) Observe that the coefficient $(\omega^j-1)(1-\omega^{-j-1})$ is never zero for the range of j under consideration. Also the vectors are linearly independent since the positions of nonzero entries never coincide. Thus the above argument gives us the rank we claimed. □

3.6 Case $k=4$

We define $\rho_4\colon F_2\to \mathrm{GL}(2,\mathbb{C})$ by

$$\rho_4(a)=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix} \quad\text{and}\quad \rho_4(b)=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Proposition 3.19 *The image $\rho_4(F_2)$ is isomorphic to the quaternion group Q of order 8. The representation ρ_4 is oriented characteristic and $P_4\subset \ker\rho_4$. The additive subgroup of 2×2 matrices generated by the image of ρ_4 consists of those matrices of the form*

$$M_{x,y}=\begin{pmatrix} x & -\overline{y}\\ y & \overline{x}\end{pmatrix} \quad \textit{with } x,y\in\mathbb{Z}[i].$$

Proof Let $\boldsymbol{M}=\{M_{x,y}:x,y\in\mathbb{C}\}$ and observe that $\boldsymbol{M}$ is closed under multiplication. Thus $\mathrm{SL}(2,\mathbb{C})\cap\boldsymbol{M}$ is a multiplicative group containing $\rho_4(a)$ and $\rho_4(b)$. Furthermore, $\det M_{x,y}=|x|^2+|y|^2$, so there are exactly eight matrices in $\mathrm{SL}(2,\mathbb{C})\cap\boldsymbol{M}$. Observe by inspection that $\mathrm{SL}(2,\mathbb{C})\cap\boldsymbol{M}$ is isomorphic to Q and that $\rho_4(a)$ and $\rho_4(b)$ generate. Also observe that the matrices $M_{1,0},M_{i,0},M_{0,1},M_{0,i}\in\rho_4(F_2)$ generate $\boldsymbol{M}$ as an additive group. To see ρ_4 is oriented characteristic, observe it satisfies Proposition 3.3 with the choice of matrices

$$M_1=\begin{pmatrix} 1 & i\\ i & 1\end{pmatrix},\quad M_2=\begin{pmatrix} i & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad M_-=I.$$

Also we have $P_4\subset\ker\rho_4$ since the kernel is characteristic and $\rho_4(a)^4=I$. □

Let $\widetilde{\rho}_4\colon F_2\to\mathrm{GL}(4,\mathbb{C})$ be defined by

$$\widetilde{\rho}_4(a)=\left(\begin{array}{cc|cc} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right) \quad\text{and}\quad \widetilde{\rho}_4(b)=\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right).$$

This representation was produced by following the argument of Theorem 3.16 with $\mathcal{I}=\mathcal{A}_4/{\sim}$, though we will not prove this. We do show:

Proposition 3.20 *The homomorphism $\widetilde{\rho}_4$ is an oriented characteristic representation. The kernel of $\widetilde{\rho}_4$ contains P_4 and is of infinite index in F_2. We have*

$$\widetilde{\rho}_4(\ker\rho_4) = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & z & -\overline{w} \\ 0 & 1 & w & \overline{z} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) : (w,z) \in \Lambda \right\},$$

where Λ is the kernel of the map

$$\mathbb{Z}[i]^2 \to \mathbb{Z}/2\mathbb{Z} \quad \text{given by} \quad (a+ib, c+id) \mapsto a+b+c+d \pmod 2.$$

Thus there is a short exact sequence of groups of the form

$$1 \to \mathbb{Z}^4 \to F_2/\ker\widetilde{\rho}_4 \to Q \to 1.$$

Proof To see $\widetilde{\rho}$ is oriented characteristic, apply Proposition 3.3 with $M_- = I$,

$$M_1 = \left(\begin{array}{cc|cc} 2 & 2i & i-1 & -i-1 \\ 2i & 2 & -i+1 & -i-1 \\ \hline 0 & 0 & 2i-2 & 0 \\ 0 & 0 & 0 & -2i-2 \end{array}\right) \quad \text{and} \quad M_2 = \left(\begin{array}{cc|cc} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{array}\right).$$

Then, to see that $P_4 \subset \ker\widetilde{\rho}_4$, it suffices to observe that $\widetilde{\rho}_4(a^4) = I$.

Define $\gamma\colon \mathbb{Z}[i]^2 \to \mathrm{GL}(4,\mathbb{C})$ by

$$\gamma(z,w) = \left(\begin{array}{cc|cc} 1 & 0 & z & -\overline{w} \\ 0 & 1 & w & \overline{z} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{36}$$

The proposition claims that $\widetilde{\rho}_4(\ker\rho_4) = \gamma(\Lambda)$. Recall that the quaternion group has a presentation of the form

$$Q = \langle a, b \mid a^4 = b^4 = a^2b^2 = ab^{-1}ab = 1\rangle.$$

Since $\widetilde{\rho}_4(a^4) = \widetilde{\rho}_4(b^4) = I$, it follows that $\widetilde{\rho}_4(\ker\rho_4)$ is generated by images under $\widetilde{\rho}_4$ of conjugates of a^2b^2 and $ab^{-1}ab$. We compute

$$\widetilde{\rho}_4(a^2b^2) = \gamma(-1,1) \quad \text{and} \quad \widetilde{\rho}_4(ab^{-1}ab) = \gamma(0, i+1).$$

Now we will consider $\widetilde{\rho}_4(ga^2b^2g^{-1})$ for $g \in F_2$. Let P be the top-right 2×2 submatrix of $\widetilde{\rho}_4(a^2b^2)$ above. Conjugates $\widetilde{\rho}_4(ga^2b^2g^{-1})$ have top-right submatrix given by $\rho_4(g) \cdot P$. Thus $\tilde\rho_4(\ker \rho_4)$ contains all the matrices $M_{x,y}P$ where $M_{x,y}$ is in the additive group generated by $\rho_4(g)$ which was described by Proposition 3.19 in terms of a vector $(x,y) \in \mathbb{Z}[i]^2$. We have

$$M_{x,y}P = \begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -x-\overline{y} & -x+\overline{y} \\ -y+\overline{x} & -y-\overline{x} \end{pmatrix}. \tag{37}$$

Varying (x,y) over $\{(1,0),(i,0),(0,1),(0,i)\}$ gives generators for the normal subgroup of $\tilde\rho_4(F_2)$

$$N_1 = \langle \tilde\rho_4(ga^2b^2g^{-1}) \mid g \in F_2 \rangle.$$

Namely we see that

$$N_1 = \gamma(\Lambda_1), \quad \text{where } \Lambda_1 = \langle (-1,1),(-i,-i),(-1,-1),(i,-i)\rangle \subset \mathbb{Z}[i]^2.$$

A similar calculation shows that the normal subgroup

$$N_2 = \langle \tilde\rho_4(gab^{-1}abg^{-1}) \mid g \in F_2 \rangle$$

is given by

$$N_2 = \gamma(\Lambda_2), \quad \text{where } \Lambda_1 = \langle (0,i+1),(0,1-i),(-1-i,0),(-1+i,0)\rangle \subset \mathbb{Z}[i]^2.$$

A simple calculation shows that

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle (-1,1),(-i,-i),(-1,-1),(0,i+1)\rangle,$$

which is a subgroup of $\mathbb{Z}[i]^2$ with index 2. Observe that $\Lambda = \langle \Lambda_1, \Lambda_2\rangle$ and from the discussion above we have $\tilde\rho_4(\ker \rho_4) = \gamma(\Lambda)$.

The short exact sequence follows from the fact that $\gamma(\Lambda)$ is a free abelian group of rank 4. □

Given $\tilde\rho_4$ and ρ_4 as above we may consider the tensor product $\tilde\rho_4' = \overline{\rho}_4 \otimes \tilde\rho_4$, which is also an oriented characteristic representation by Proposition 3.2. We have $\ker \tilde\rho_4' = \ker \tilde\rho_4$ and we can view $\tilde\rho_4'$ as a homomorphism to $\mathrm{GL}(8,\mathbb{C})$.

Define the homomorphism $\tilde{\tilde\rho}_4 \colon F_2 \to \mathrm{GL}(9,\mathbb{C})$ so that

$$\tilde{\tilde\rho}_4(a) = \mathrm{diag}(1,-1,-i,-i;-1,1,i,i;1), \tag{38}$$

$$\widetilde{\widetilde{\rho}}_4(b)=\left(\begin{array}{cccc|cccc|c} 0&0&0&0&0&1&1&0&0\\ 0&0&0&0&-1&0&0&1&0\\ 0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&1&0\\ \hline 0&-1&-1&0&0&0&0&0&0\\ 1&0&0&-1&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&-1\\ \hline 0&0&0&0&0&0&0&0&1 \end{array}\right). \tag{39}$$

The top-left 8×8 submatrices of images of $\widetilde{\widetilde{\rho}}_4$ realize $\widetilde{\rho}_4'$. The representation $\widetilde{\widetilde{\rho}}_4$ was found by applying the approach of Theorem 3.16 to $\widetilde{\rho}_4'$ but we will not prove this. We have:

Proposition 3.21 *The homomorphism $\widetilde{\widetilde{\rho}}_4$ is an oriented characteristic representation. The kernel of $\widetilde{\widetilde{\rho}}_4$ contains P_4. Furthermore, there is a short exact sequence of groups of the form*

$$1\to\mathbb{Z}^d\to F_2/\ker\widetilde{\widetilde{\rho}}_4\to F_2/\ker\widetilde{\rho}_4\to 1,$$

where $d\geq 1$.

It will follow from later work that $\ker\widetilde{\widetilde{\rho}}_4=P_4$ and that $d=1$ in the statement above. See Theorem 4.2.

Proof That $\widetilde{\widetilde{\rho}}_4$ is oriented characteristic follows from Proposition 3.3 with

$$M_1=\left(\begin{array}{cccc|cccc|c} 2&2i&i-1&-i-1&-2i&2&i+1&i-1&i-1\\ 2i&2&-i+1&-i-1&2&-2i&-i-1&i-1&i-1\\ 0&0&2i-2&0&0&0&2i+2&0&2\\ 0&0&0&-2i-2&0&0&0&2i-2&-2\\ \hline -2i&2&i+1&i-1&2&2i&i-1&-i-1&-i-1\\ 2&-2i&-i-1&i-1&2i&2&-i+1&-i-1&i+1\\ 0&0&2i+2&0&0&0&2i-2&0&-2\\ 0&0&0&2i-2&0&0&0&-2i-2&-2\\ \hline 0&0&0&0&0&0&0&0&4 \end{array}\right),$$

$$M_2=\mathrm{diag}(1,-i,-i,1,i,1,1,i,1)$$

and $M_-=I$. Again we have $P_4\subset\ker\widetilde{\widetilde{\rho}}_4$ because $\ker\widetilde{\widetilde{\rho}}_4(a^4)=I$.

It may be observed that the upper-left 8×8 submatrix of $\widetilde{\widetilde{\rho}}(g)$ is a matrix representation of $\overline{\rho}_4(g)\otimes\widetilde{\rho}_4(g)$. Since $\ker\widetilde{\rho}_4\subset\ker\rho_4$, we have that $\ker(\overline{\rho}_4\otimes\widetilde{\rho}_4)=\ker\widetilde{\rho}_4$. Matrices in $\tilde{\tilde{\rho}}(\ker \tilde{\rho}_4)$ therefore have the block form

$$\begin{pmatrix} I & \boldsymbol{v} \\ 0 & 1 \end{pmatrix},$$

where I is the 8×8 identity matrix and $\boldsymbol{v}$ is an 8×1 matrix with entries in $\mathbb{Z}[i]$. Thus $\tilde{\tilde{\rho}}(\ker \tilde{\rho}_4)$ is isomorphic to an additive subgroup of $\mathbb{Z}[i]^8$. Let $d = \operatorname{rank} \tilde{\tilde{\rho}}(\ker \tilde{\rho}_4)$. We compute

$$\tilde{\tilde{\rho}}_4([a,b]^2) = \left(\begin{array}{cccc|cccc|c} 1&0&0&0&0&0&0&0&2i \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ \hline 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&2i \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ \hline 0&0&0&0&0&0&0&0&1 \end{array}\right). \tag{40}$$

Thus $[a,b]^2$ lies in $\ker \tilde{\rho}_4$ and its image generates a copy of $\mathbb{Z}$ in $\tilde{\tilde{\rho}}_4(F_2)$. This shows $d \geq 1$. Finally observe that we have the natural short exact sequence

$$1 \to \tilde{\tilde{\rho}}(\ker \tilde{\rho}_4) \to \tilde{\tilde{\rho}}_4(F_2) \to (\bar{\rho}_4 \otimes \tilde{\rho}_4)(F_2) \to 1.$$

Here, the map $\tilde{\tilde{\rho}}_4(F_2) \to (\bar{\rho}_4 \otimes \tilde{\rho}_4)(F_2)$ is the map that takes a matrix in $\tilde{\tilde{\rho}}_4(F_2)$ to its top-left 8×8 block. We have $\tilde{\tilde{\rho}}_4(F_2) \cong F_2/\ker \tilde{\tilde{\rho}}_4$, and $(\bar{\rho}_4 \otimes \tilde{\rho}_4)(F_2) \cong F_2/\ker \tilde{\rho}_4$ from the discussion above. This yields the exact sequence in the proposition. □

4 Characterizing F_2/P_4

A *polycyclic group* is a group that admits a subnormal series with cyclic factors. Any group that is virtually nilpotent is polycyclic. The *Hirsch length* of a polycyclic group is the number of infinite factors in any subnormal series with cyclic factors. For any polycyclic group G, we will refer to the Hirsch length as the *dimension* of the group, and denote it by $\dim(G)$. A fact that we will use repeatedly in this section is that for any normal subgroup $N \subset G$, we have (see for instance [3, Theorem 4.7] for a proof)

$$\dim(G) - \dim(N) = \dim(G/N).$$

In particular, if G is torsion-free and N is nontrivial, then $\dim(G) > \dim(G/N)$.

For this section let $G = F_2/P_4$. The following proposition tells us that G is virtually torsion-free nilpotent of dimension equal to 5. We will use this result to prove that the representation $\widetilde{\widetilde{\rho}}_4$ is faithful. For this, we will need to record the generators of the torsion free subgroup we find.

Proposition 4.1 *Let N be the subgroup of G generated by*

$$a_1 = E^{-2}, \quad a_2 = A^4 D^2, \quad a_3 = AE^{-2}A^{-4}BA^{-1}B^{-1},$$
$$a_4 = A^9CA^{-1}C^{-1}BAB^{-1}A^{-1},$$

where

$$A = ba^{-1}b^{-1}a, \quad B = b^{-1}aba^{-1}, \quad C = b^{-1}a^{-1}ba,$$
$$D = a^2b(a^{-1}b^{-1})^2a^{-1}ba, \quad E = b^{-1}(ab)^2a^{-3}b^{-1}a.$$

Then N is a 5–dimensional torsion-free nilpotent subgroup of index 2^{12} in F_2/P_4 that is isomorphic to $H(\mathbb{Z}) \times H(\mathbb{Z})$ with one nontrivial added relator.

Proof Let G_2 be the second term of the derived series for G, where G is described in terms of the relations provided by the table in Figure 4. Using GAP [18], we can confirm that G_2 is a subgroup of finite index in G. Moreover, GAP gives us the following presentation for G_2 (the F_i notation follows GAP's output):

$$\begin{aligned}\langle F_1, F_2, F_3, F_4, F_5 \mid\ & F_3^{-1}F_1^{-1}F_3F_1 = F_2^{-1}F_3^{-1}F_2F_3 = F_2^{-1}F_4F_2F_4^{-1} = 1,\\ & F_1^{-1}F_2F_1F_2^{-1} = F_4F_5F_4^{-1}F_5^{-1} = F_5^{-1}F_2F_5F_2^{-1} = 1,\\ & F_4F_1^{-1}F_5F_4^{-1}F_1F_5^{-1} = F_4^{-1}F_5^{-1}F_3F_5F_4F_3^{-1}\\ & \qquad\qquad\qquad\qquad\quad = F_5^{-1}F_3F_1F_5F_3F_1^{-1} = 1,\\ & F_4F_1F_3F_4^{-1}F_1^{-1}F_3^{-1} = F_3^{-1}F_2F_1^{-1}F_4F_3F_2^{-1}F_1F_4^{-1} = 1,\\ & F_5^{-1}F_3F_5F_3^{-1}F_5F_3F_5^{-1}F_3^{-1} = 1,\\ & F_1^{-1}F_2^{-1}F_5F_3^{-1}F_5^{-1}F_1F_2^{-2}F_3F_2^{-1} = F_2F_4F_1F_4^{-1}F_2^3F_1^{-1} = 1\rangle.\end{aligned}$$

From this presentation and computations in [18], we see that G_2 satisfies the following:

(1) The first homology of G_2 is $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^4$.

(2) G_2 has index 1024.

Let N be the group generated by F_1, F_3, F_4 and F_5. Using GAP [18], we can check that N has index 2^{12} and has the desired generators. Moreover, GAP gives that N has a presentation of the form

$$N = \langle a_1, a_2, a_3, a_4 \mid R \rangle,$$

where

$$R=\{[a_1,a_2],[a_3,a_4],[a_1,a_3],[a_4^2a_3,a_1^{-1}a_2],[a_2,a_4],[a_4a_3,a_1^{-2}a_2],\\ [a_4a_3,(a_1^{-1}a_2)^{-1}a_4(a_1^{-1}a_2)]\}.$$

This is a quotient of the right-angled Artin group $F_2\times F_2$ with the three added relators

$$[a_4^2a_3,a_1^{-1}a_2],\quad [a_4a_3,a_1^{-2}a_2],\quad [a_4a_3,(a_1^{-1}a_2)^{-1}a_4(a_1^{-1}a_2)].$$

Viewing the group as $F_2\times F_2=\langle a_1,a_4\rangle\times\langle a_2,a_3\rangle$, we can simplify the relations to

$$([a_4^2,a_1^{-1}],[a_3,a_2]),\quad ([a_4,a_1^{-2}],[a_3,a_2]),\quad \gamma:=([a_4,a_1a_4a_1^{-1}],1).$$

Using suitable conjugations, we further simplify the relations to

$$([a_1,a_4^2],[a_3,a_2]),\quad ([a_1^2,a_4],[a_3,a_2]),\quad \gamma:=([a_4,a_1a_4a_1^{-1}],1).$$

Then N is the group $(F_2\times F_2)/K$, where K is the normal subgroup generated by the elements above.

The last relator gives that $[a_1,a_4]$ and a_4 commute. By the two other relators, we have $([a_1,a_4^2],1)=([a_1^2,a_4],1)$. This equality is equivalent to

$$[a_1,a_4][a_1,a_4]^{a_4}=[a_1,a_4]^{a_1}[a_1,a_4].$$

Hence, $([a_1,a_4],1)$ is central in N. Moreover, since $([a_1,a_4^2],[a_3,a_2])$ is a relator, $(1,[a_3,a_2])$ is also central in N.

Let H_1 be the image of $F_2\times 1$ in N and H_2 the image of $1\times F_2$. Since $([a_1,a_4],1)$ and $(1,[a_3,a_2])$ are central, the groups H_1 and H_2 are both quotients of $H(\mathbb{Z})$ (in fact, they are both isomorphic to $H(\mathbb{Z})$). It follows that N must be $H(\mathbb{Z})\times H(\mathbb{Z})$ with a relation identifying the square of a central generator of $H(\mathbb{Z})\times 1$ with one of $1\times H(\mathbb{Z})$. It is now clear that N has infinite center. Thus, N has Hirsch length equal to 5 and is torsion-free, as desired. □

Recall the definition of $\tilde{\tilde{\rho}}_4\colon F_2\to \mathrm{GL}(9,\mathbb{C})$ described by (38) and (39). From Proposition 3.21, $P_4\subset\ker\tilde{\tilde{\rho}}_4$, thus we can consider $\tilde{\tilde{\rho}}_4$ to be a homomorphism from G to $\mathrm{GL}(9,\mathbb{C})$.

Theorem 4.2 *The representation $\tilde{\tilde{\rho}}_4\colon F_2/P_4\to\mathrm{GL}(9,\mathbb{C})$ is faithful. We have $d=1$ in Proposition 3.21.*

Proof First we will show that $d=1$ using a dimension argument. For this proof, consider ρ_4, $\widetilde{\rho}_4$ and $\widetilde{\widetilde{\rho}}_4$ to be homomorphisms from F_2/P_4 and their kernels to be subgroups of F_2/P_4. We can compute that the generators of N lie in $\ker\rho_4$, and we conclude $N\subset\ker\rho_4$. Propositions 3.20 and 3.21 tell us that $\widetilde{\rho}_4(\ker\rho_4)$ is isomorphic to $\mathbb{Z}^4$ and $\widetilde{\widetilde{\rho}}_4(\ker\rho_4)$ is a further $\mathbb{Z}^d$–extension for $d\geq 1$. It follows that $\widetilde{\widetilde{\rho}}_4(\ker\rho_4)$ is polycyclic. Moreover, $\dim\widetilde{\widetilde{\rho}}_4(\ker\rho_4)=4+d\geq 5$. Since dimension nonstrictly drops under surjective homomorphisms, we have $\dim N\geq\dim\widetilde{\widetilde{\rho}}_4(N)$, and since N is of finite index inside of F_2/P_4, we have $\dim\widetilde{\widetilde{\rho}}_4(N)=\dim\widetilde{\widetilde{\rho}}_4(F_2/P_4)$. Putting this all together we have

$$5=\dim N\geq\dim\widetilde{\widetilde{\rho}}_4(N)=\dim\widetilde{\widetilde{\rho}}_4(F_2/P_4)=4+d\geq 5.$$

We conclude that all expressions in the above line are 5, and therefore $d=1$. Since nontrivial quotients of N have strictly smaller dimension, we also get that the restriction of $\widetilde{\widetilde{\rho}}_4$ to N is injective. Thus the faithfulness claimed in the theorem will follow if we can prove that subgroup indices satisfy

$$[\widetilde{\widetilde{\rho}}_4(F_2/P_4):\widetilde{\widetilde{\rho}}_4(N)]=[F_2/P_4:N].$$

We already know that $[F_2/P_4:N]=2^{12}$. It suffices to prove that $[\widetilde{\widetilde{\rho}}_4(G):\widetilde{\widetilde{\rho}}_4(N)]\geq 2^{12}$ since index cannot grow under group homomorphisms.

First observe that $[\rho_4(G):\rho_4(N)]=2^3$ since $N\subset\ker\rho_4$ and $\rho_4(G)$ is isomorphic to the quaternion group.

Now consider the index $[\widetilde{\rho}_4(G):\widetilde{\rho}_4(N)]$. Let a_1, a_2, a_3 and a_4 denote the generators for N listed in Proposition 4.1. Define $\gamma\colon\mathbb{Z}[i]^2\to\mathrm{GL}(4,\mathbb{C})$ as in (36). By Proposition 3.20, $\widetilde{\rho}_4(\ker\rho_4)=\gamma(\Lambda)$, where $\Lambda\subset\mathbb{Z}[i]^2$ is a subgroup of index 2. We compute

$$\begin{aligned}\widetilde{\rho}_4(a_1)&=\gamma(-2i-2,-2i+2), &\quad \widetilde{\rho}_4(a_2)&=\gamma(2i-2,2i+2),\\ \widetilde{\rho}_4(a_3)&=\gamma(4,0), & \widetilde{\rho}_4(a_4)&=\gamma(0,4i).\end{aligned}\tag{41}$$

Thus $\widetilde{\rho}_4(N)=\gamma(\Lambda')$, where

$$\Lambda'=\langle(-2-2i,-2i+2),(2i-2,2i+2),(4,0),(0,4i)\rangle.$$

Based on this, we observe $\Lambda'\subset\Lambda$ and we can compute that $[\mathbb{Z}[i]^2:\Lambda']=2^7$ and thus $[\Lambda:\Lambda']=2^6$. It follows that

$$[\widetilde{\rho}_4(\ker\rho_4):\widetilde{\rho}_4(N)]=2^6\quad\text{and}\quad[\widetilde{\rho}_4(F_2/P_4):\widetilde{\rho}_4(N)]=2^{6+3}.$$

Finally, we consider the index $[\tilde{\tilde{\rho}}_4(G) : \tilde{\tilde{\rho}}_4(N)]$. From the above, we know that $F_2/\ker\tilde{\tilde{\rho}}_4$ is a $\mathbb{Z}$–extension of $F_2/\ker\tilde{\rho}_4$. We have $[a,b]^2 \in \ker\tilde{\rho}_4$ but $\tilde{\tilde{\rho}}_4([a,b]^2) \neq I$ (see (40)). Since the images under $\tilde{\rho}_4$ of the four generators a_i freely generate the image $\tilde{\rho}_4(N)$, which is isomorphic to $\mathbb{Z}^4 \cong N/[N,N]$, it follows that $N \cap \ker\tilde{\rho}_4 = [N,N]$. Since N is two-step nilpotent, this commutator subgroup is generated by commutators of the generators of N. We compute

$$\tilde{\tilde{\rho}}_4([a_1,a_2]) = \tilde{\tilde{\rho}}_4([a_3,a_4]) = I.$$

For other pairs of generators of N we have

$$\tilde{\tilde{\rho}}_4([a_3,a_1]) = \tilde{\tilde{\rho}}_4([a_4,a_1]) = \tilde{\tilde{\rho}}_4([a_2,a_3]) = \tilde{\tilde{\rho}}_4([a_4,a_2]) = \tilde{\tilde{\rho}}_4([a,b]^2)^8.$$

Thus the central copy of $\mathbb{Z}$ in $\tilde{\tilde{\rho}}_4(F_2/P_4)$ contains $\tilde{\tilde{\rho}}_4(N \cap \ker\tilde{\rho}_4)$ with index at least 2^3. Consequently, $[\tilde{\tilde{\rho}}_4(F_2/P_4) : \tilde{\tilde{\rho}}_4(N)] \geq 2^{3+6+3}$, as desired. □

Appendix Relation to square-tiled surfaces

A *translation surface* is a surface equipped with an atlas of coordinate charts to the plane such that all transition functions are restrictions of translations.

Let $\mathbb{T}$ denote the 2–torus $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{T}^\circ = \mathbb{T} \smallsetminus \{\mathbf{0}\}$ be the once-punctured torus. A *square-tiled surface* (or *origami*) is a cover of $\mathbb{T}^\circ$ endowed with the pullback translation structure. Here we allow the cover to be finite or infinite. See [20] for a survey discussing translation surfaces including square-tiled surfaces.

Fix a translation surface S. Given a vector $(u,v) \in \mathbb{R}^2$ the *straight-line flow determined by* (u,v) is the flow $F^t \colon S \to S$ given in local coordinates by

$$F^t(x,y) = (x,y) + t(u,v).$$

The straight-line flow of a point will not be defined for all time if under the projection to $\mathbb{T}$ the flow hits the puncture at $\mathbf{0}$. We call such a straight-line trajectory *singular*.

Let $(u,v) \in \mathbb{Z}^2$ and assume u and v are relatively prime. Then the straight-line flow determined by (u,v) on the torus $\mathbb{T}$ is periodic with all points having period 1. Let S be a square-tiled surface. For a positive integer k we say S is k*–periodic* if for all relatively prime $(u,v) \in \mathbb{Z}^2$, every nonsingular straight-line trajectory determined by (u,v) is periodic with period dividing k.

We take $\left(\frac{1}{2}, \frac{1}{2}\right)$ to be the basepoint of $\mathbb{T}^\circ$ and say that a *square-tiled surface with basepoint* is a square-tiled surface S with the choice of a basepoint s such that the covering map to $\mathbb{T}^\circ$ maps s to $\left(\frac{1}{2}, \frac{1}{2}\right)$. If S_1 and S_2 are two square-tiled surfaces with basepoints s_1 and s_2, respectively, and $\pi_i\colon S_i \to \mathbb{T}^\circ$ are the associated covering maps, we say that S_1 *covers* S_2 if there is a covering map $\pi\colon S_1 \to S_2$ satisfying $\pi(s_1) = \pi(s_2)$ and $\pi_2 \circ \pi = \pi_1$.

This paper originated with the following observation:

Proposition A.1 *For any $k \geq 1$ there is a k–periodic square-tiled surface with basepoint U_k such that U_k covers any other k–periodic square-tiled surface with basepoint.*

We call U_k the *universal k–periodic square-tiled surface.*

Covering space theory associates a square-tiled surface S with basepoint to a subgroup Γ_S of the fundamental group $\pi_1\left(\mathbb{T}^\circ, \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. Note that this fundamental group is isomorphic to the free group F_2. For purposes of this appendix consider $\pi_1\left(\mathbb{T}^\circ, \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ to be the same as F_2. Following Herrlich we call S *characteristic* if Γ_S is a characteristic subgroup of F_2. Characteristic square-tiled surfaces S are maximally symmetric: they have a deck group acting transitively on the lifts of any point of $\mathbb{T}^\circ$ and each element of $\mathrm{GL}(2, \mathbb{Z})$ stabilizes S (through the action of $\mathrm{GL}(2, \mathbb{R})$ on the space of translation surfaces).

Some finite characteristic square-tiled surfaces which are k–periodic have attained an almost mythical status in the subject of translation surfaces, serving up numerous counterexamples in the field. Especially famous are the fantastically named *eierlegende Wollmilchsau* discovered independently in [6] and [11] and the *ornithorynque* first described in [7]. These surfaces were studied further in [8; 16]. If this article were written more geometrically, the Heisenberg origamis studied by Herrlich in [10] would play a leading role.

Two facts combine to give a proof of Proposition A.1:

(1) From basic covering space theory, the square-tiled surface with basepoint S_1 covers the square-tiled surface S_2 with basepoint if and only if $\Gamma_{S_2} \subset \Gamma_{S_1}$.

(2) A conjugacy class in F_2 represents a homotopy class of curves containing closed geodesics on $\mathbb{T}^\circ$ if and only if the conjugacy class consists of primitive elements in F_2. This observation dates back to Nielsen's 1913 thesis.

It follows that a square-tiled surface with basepoint S is k–periodic if and only if it is covered by the square-tiled surface U_k defined so that $\Gamma_{U_k} = P_k$, where $P_k \subset F_2$ denotes the subgroup generated by k^{th} powers of primitive elements as in this paper.

From work in this paper we obtain an understanding of $U_1, \dots, U_4$:

(1) We have $U_1 = \mathbb{T}^\circ$.

(2) The surface U_2 is $(\mathbb{R}/2\mathbb{Z})^2$ punctured at the integer points.

(3) The surface U_3 is the Heisenberg origami denoted by $O_{3,3}$ in [10] jointly discovered by Herrlich, Möller and Weitze-Schmithüsen.

The eierlegende Wollmilchsau mentioned above is the square-tiled surface W defined so that Γ_W is the kernel of the surjective homomorphism $F_2 \to Q$, where Q is the quaternion group. The surface W is 4–periodic. From our understanding in this paper of P_4 and in particular knowledge of the representation $\tilde{\tilde{\rho}}_4$ of Section 3.6, which is faithful by Theorem 4.2, we see:

Theorem A.2 *The surface U_4 is an infinite area square-tiled surface and is a torsion-free 5–dimensional 2–step nilpotent cover of the eierlegende Wollmilchsau.*

It is particularly interesting that U_4 is a geometrically natural example of an infinite nilpotent cover of a compact translation surface, because some methods are available to study the dynamics of the straight-line flow on such a surface; see for instance [4]. It is a consequence of [12, Theorem G.3, Remark 4.1] and $\mathrm{GL}(2,\mathbb{Z})$–invariance of U_4 that:

Corollary A.3 *There is a dense subset E of the unit circle in $\mathbb{R}^2$ with Hausdorff dimension larger than $\frac{1}{2}$ such that for any $(u,v) \in E$ the straight-line flow determined by (u,v) on U_4 is ergodic.*

As a consequence of the universality of U_4 it follows that the straight-line flow determined by each $(u,v) \in E$ is ergodic on each 4–periodic square-tiled surface. This motivates:

Question 4 *Is the straight-line flow determined by (u,v) ergodic on U_4 whenever $\frac{u}{v} \notin \mathbb{Q}$?*

The kernels of the representations $\tilde{\rho}_k$ for odd $k \geq 5$ determine characteristic k–periodic origamis O_k which are infinite free abelian covers of the Heisenberg origamis of Herrlich. The conclusions of Corollary A.3 then hold for the surfaces O_k and we similarly wonder what the answer to Question 4 would be in these cases.

This paper shows that P_k is of infinite index in F_2 when $k \geq 4$ and it follows that for $k \geq 4$ the surface U_k is infinite. Virtual nilpotence of F_2/P_k is necessary to apply [12, Theorem G.3], so an affirmative answer to Question 1(b) in the case of $r = 2$ and $k \geq 5$ would extend Corollary A.3 to cover the corresponding U_k. Even in the absence of this, the method of Section 3 can be iterated to produce other characteristic multistep nilpotent covers of compact square-tiled surfaces when applied multiple times in the cases of $k \geq 5$ as with our construction of the representation $\tilde{\tilde{\rho}}_4\colon F_2/P_4 \to \mathrm{GL}(9, \mathbb{C})$.

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