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Recognition of connective commutative algebra spectra through an idempotent quasiadjunction

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A recognition principle for ∞–loop pairs of spaces of connective commutative algebra spectra over connective commutative ring spectra is proved. This is done by generalizing the classical recognition principle for connective commutative ring spectra using relative operads. The machinery of weak Quillen quasiadjunctions, a generalization of Quillen adjunctions, is used to handle the model theoretical aspects of the proof.

55P43, 55P48; 55P42, 55P47, 55P60, 55P65

1 Introduction

A recognition principle is a specification of conditions for a space to be of the weak homotopy type of an N–loop space. Stasheff showed in [31; 32] that a pointed space is of the weak homotopy type of a 1–loop space if and only if X is a grouplike $\mathcal{A}_\infty$–space. By the work of May in [21; 22] and homological computations by Cohen in [6], for $2 \leq N \leq \infty$ a pointed space is of the weak homotopy type of an N–loop space if and only if X is a grouplike $\mathcal{E}_N$–space. The proof of the recognition principle for ∞–loop spaces gives an equivalence between the homotopy category of grouplike E_∞–spaces and the homotopy category of connective spectra. Due to reasons we explain shortly, an interesting feature of this equivalence is that it is not induced by a Quillen adjunction as is usual in a model theoretical setting.

In [23] May defines actions between operads, which encode distributive properties and provide a natural definition of E_∞–rings. The canonical multiplicative operad is the linear isometries operad $\mathscr{L}$, which induces a nonunital monoidal structure on the category of spectra, and thus a definition of E_∞–spectra. In [9] Elmendorf, Kriz, Mandell and May show that the category $\mathtt{Mod}_{\mathbb{S}}$ of sphere modules is a monoidal model category of spectra. Commutative monoids in $\mathtt{Mod}_{\mathbb{S}}$ form the category $\mathtt{CRingSp}$ of commutative ring spectra, which is Quillen equivalent to the category of E_∞–ring spectra. The monoidal structure also provides a convenient setting to define module spectra and algebra spectra over commutative ring spectra. As explained by May in [25] the recognition principle can then be extended to an equivalence between ringlike E_∞–rings and connective commutative ring spectra.

In [15] Hoefel, Livernet and Stasheff show that relative 1–loop spaces are recognized by A_∞–actions, which are pairs of spaces acted on by a resolution of the relative operad of actions of a monoid on a space. In [37] I extended the recognition of relative N–loop spaces for the cases $3 \leq N \leq \infty$. In particular, for $N = \infty$ this states that the homotopy category of grouplike E_∞^{rel}–pairs, which are algebras over a relative operad homotopy equivalent to the infinite-dimensional Swiss-cheese relative operad, is equivalent to the homotopy category of degree 1 maps between connective spectra.

All these recognition theorems share one feature: they can be formulated as equivalences between homotopy categories of model categories induced by loop space functors and delooping functors that are not adjoint. In [37] I introduced the notion of a weak Quillen quasiadjunction, a generalization of Quillen adjunctions that allows for units and counits to exist only up to functorial resolutions. In the same vein I defined a generalization of Quillen idempotent (co)monads, which induce left (right) Bousfield localizations of model structures. Through these we have a natural definition of idempotent quasiadjunctions, which induce equivalences between the associated homotopy subcategories. This machinery provides a natural model theoretical axiomatization of the essential elements of May's original proof of the recognition principle of ∞–loop spaces.

A less direct model categorical treatment of the recognition principle combines the Quillen equivalence between grouplike E_∞–spaces and very special Γ–spaces proved by Santhanam in [27], with the latter category shown to be Quillen equivalent to the category of connective spectra by Mandell, May, Schwede and Shipley in [20]. See Ando, Blumberg, Gepner, Hopkins and Rezk [1] for further model categorical considerations about the recognition principle.

In this article I introduce a relative version of actions between operads which provides a natural definition of E_∞–algebra spaces over E_∞–ring spaces, referred to simply as $E_\infty^{\rightarrow}$–algebras. The machinery of idempotent quasiadjunctions is used to prove the main result, Theorem 4.6.1, a recognition principle for ∞–loop pairs of spaces of commutative algebra spectra over commutative ring spectra. Explicitly it states that the homotopy category of algebralike[1] $E_\infty^{\to}$–algebras is equivalent to the homotopy category of connective commutative algebra spectra over connective commutative ring spectra. This result is a consequence of the intermediary Theorems 4.3.1 and 4.3.2, which constitute a recognition principle for ∞–loop pairs of spaces of spectra maps.

We finish this introduction by sketching how May's original proof of the recognition principle of ∞–loop spaces can be framed as an equivalence induced by an idempotent quasiadjunction, which provides a blueprint for the main proofs in this article.

The category $\mathtt{Sp}_{\mathbb{N}}$ of sequential prespectra — see Lima [18] — consists of sequences of spaces $\langle Y_N \rangle \in \prod_{\mathbb{N}} \mathtt{Top}_*$ equipped with structural maps $\sigma_N^M : Y_M \wedge \mathbb{S}^{N-M} \to Y_N$ for $M \le N$ satisfying compatibility conditions. The Ω–spectra are the prespectra whose adjoint structural maps $\tilde{\sigma}_N^M$ are all weak equivalences, which by Brown representability represent (co)homology theories [5]. Spectra are prespectra whose dual structural maps $\tilde{\sigma}_N^M$ are all homeomorphisms; see for instance [9]. In this article we will work exclusively in the category of prespectra, so from now on we will simply refer to prespectra as spectra. From $\mathtt{Sp}_{\mathbb{N}}$ we can define via filtered colimits over the dual structural maps the ∞–loop spaces functor

$$\Omega^\infty : \mathtt{Sp}_{\mathbb{N}} \to \mathtt{Top}_*, \quad \Omega^\infty Y := \operatorname*{colim}_{N \in \mathbb{N}} Y_N^{\mathbb{S}^N}.$$

The ∞–loop spaces $\Omega^\infty Y$ are homotopy commutative H–spaces, but such description ignores a lot of information. In order to describe the algebraic structure completely we require an E_∞–operad $\mathcal{E}$, a gadget used to describe topological spaces with operations that are associative and commutative up to coherent homotopy [21, Definitions 1.1 and 3.5].

For $\mathbb{S}$, the category of finite sets and bijections, a topological operad is a contravariant functor equipped with composition maps and an abstract identity element

$$\mathcal{P} : \mathbb{S}^{\mathrm{op}} \to \mathtt{Top}; \quad \circ : \mathcal{P}A \times \textstyle\prod_A \mathcal{P}B^a \to \mathcal{P}\Sigma_A B^a, \quad \mathrm{id} \in \mathcal{P}\underline{1},$$

with $\mathcal{P}\varnothing = *$ satisfying invariance, associativity and unitary laws. We can interpret points in the underlying spaces as abstract multivariable functions with inputs indexed

[1] Semialgebras and semirings are like algebras and rings without the assumption that additive inverses exist, ie we have an additive commutative monoid instead of an additive abelian group. An $E_\infty^{\to}$–algebra is algebralike if the connected components of the underlying pair of spaces form an algebra over a ring, not only a semialgebra over a semiring.

by the set A. Operads induce monads via the coend construction (see Loregian [19])

$$P\colon \mathtt{Top}_* \to \mathtt{Top}_*, \quad PX := \int^{\mathbb{S}^{\mathrm{inj}}} \mathcal{P}A \times X^{\times A};$$
$$\eta x := [\mathrm{id}, x], \quad \mu[\alpha, \langle[\beta^a, \langle x^{ab}\rangle]\rangle] := [\alpha\langle\beta^a\rangle, \langle x^{ab}\rangle].$$

The category $\mathcal{P}[\mathtt{Top}]$ of $\mathcal{P}$–spaces consists of pointed spaces $X \in \mathtt{Top}_*$ equipped with structural P–algebra maps $\xi\colon PX \to X$, which we interpret as an instantiation of the abstract operations of $\mathcal{P}$.

An important family of operads are the embeddings operads Emb_N for $N \in \mathbb{N}$ with

$$\mathrm{Emb}_N\, A := \{\alpha = \langle\alpha_a\rangle \in (\mathbb{R}^N)^{\sqcup_A \mathbb{R}^N} \mid \alpha \text{ is an embedding}\}.$$

There are natural inclusions $\mathrm{Emb}_M \hookrightarrow \mathrm{Emb}_N$ and we define

$$\mathrm{Emb}_\infty := \mathrm{colim}_{N\in\mathbb{N}}\, \mathrm{Emb}_N\, .$$

All N–loop spaces are naturally Emb_N–spaces with

$$\alpha\langle\gamma^a\rangle := \left(\vec{u} \mapsto \begin{cases} \gamma^a\alpha_a^{-1}\vec{u} & \text{if } \vec{u} \in \alpha_a\mathbb{R}^N \\ * & \text{if } \vec{u} \notin \alpha \sqcup_A \mathbb{R}^N \end{cases}\right),$$

and these induce Emb_∞–space structures on ∞–loop spaces.

An E_∞–operad is an operad $\mathcal{E}$ with each underlying space $\mathcal{E}A$ a contractible free $\mathbb{S}A$–space. For the purpose of studying ∞–loop spaces, we further require E_∞–operads to be equipped with an operad map $\psi\colon \mathcal{E} \to \mathrm{Emb}_\infty$, which induces by pullback a functor $\Omega^\infty\colon \mathtt{Sp} \to \mathcal{E}[\mathtt{Top}]$. This functor is not a right adjoint since any abelian group G is an $\mathcal{E}$–space, and the strictness of the operations in G implies any $\mathcal{E}$–map $\varphi \in \mathcal{E}[\mathtt{Top}](G, \Omega^\infty Y)$ must be trivial; therefore no unit of adjunction can be constructed.

In May's recognition theorem the solution was to consider the resolution of $\mathcal{E}$–spaces by the bar construction

$$\bar{B}\colon \mathcal{E}[\mathtt{Top}] \to \mathcal{E}[\mathtt{Top}], \quad \bar{B}X := |B_-(E, E, X)|,$$

which comes equipped with a natural weak equivalence $\eta'\colon \bar{B} \Rightarrow \mathrm{Id}$.

The maps $\psi\colon \mathcal{E} \to \mathrm{Emb}_\infty$ induce by pullback a suboperad filtration $\mathcal{E}_N$ on $\mathcal{E}$. If each underlying space $\mathcal{E}_N A$ is equivariantly homotopy equivalent to the configuration space of A elements in $\mathbb{R}^N$ then we can define the ∞–delooping functor

$$B^\infty\colon \mathcal{E}[\mathtt{Top}] \to \mathtt{Sp}, \quad B^\infty X := \langle|B_-(\Sigma^N, E_N, X)|\rangle$$

such that there is a natural transformation $\eta\colon \overline{B} \Rightarrow \Omega^\infty B^\infty$, with η_X a weak equivalence if and only if X is grouplike, meaning that $\pi_0 X$ is not only a monoid but also a group.

Dually there is no counit map. There is a spectrification functor[2]

$$\widetilde{\Omega}\colon \mathtt{Sp} \to \mathtt{Sp}, \quad \widetilde{\Omega} Y := \langle \operatorname*{colim}_{M \le N} \widetilde{Y}_N^{\$^{N-M}} \rangle,$$

where $\widetilde{Y}$ is a certain inclusion prespectrum constructed from Y, such that we have a natural stable weak equivalence $\epsilon'\colon \mathrm{Id} \Rightarrow \widetilde{\Omega}$. This functor plays an important role in the construction of the stable model structures of spectra. There is a natural transformation $\epsilon\colon B^\infty \Omega^\infty \Rightarrow \widetilde{\Omega}$ such that the equation $\Omega^\infty \epsilon \eta_{\Omega^\infty} = \Omega^\infty \epsilon' \eta'_{\Omega^\infty}$ holds in $\mathcal{E}[\mathtt{Top}]$ and we have a homotopy equivalence $\epsilon_{B^\infty} B^\infty \eta_X \simeq \epsilon'_{B^\infty X} B^\infty \eta'_X$ in $\mathtt{Sp}$:

$$\begin{array}{ccc} B^\infty \overline{B} X & \xrightarrow{B^\infty \eta_X} & B^\infty \Omega^\infty B^\infty X \\ {\scriptstyle B^\infty \eta'_X}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \epsilon_{B^\infty X}} \\ B^\infty X & \xrightarrow[\epsilon'_{B^\infty X}]{\sim} & \widetilde{\Omega} B^\infty X \end{array} \qquad \begin{array}{ccc} \overline{B} \Omega^\infty Y & \xrightarrow{\eta_{\Omega^\infty Y}} & \Omega^\infty B^\infty \Omega^\infty Y \\ {\scriptstyle \eta'_{\Omega^\infty Y}}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \Omega^\infty \epsilon_Y} \\ \Omega^\infty Y & \xrightarrow[\Omega^\infty \epsilon'_Y]{\sim} & \Omega^\infty \widetilde{\Omega} Y \end{array}$$

Note the similarity of these equations to the ones for an adjunction. Indeed if $\overline{B}$, $\widetilde{\Omega}$, η' and ϵ' were substituted by identities and both equations held strictly we would have an adjunction in the regular sense.

Adapting May's original proof of the recognition principle, we can show that we have a weak Quillen quasiadjunction

$$(B^\infty \dashv_{\overline{B}, \widetilde{\Omega}} \Omega^\infty)\colon \mathcal{E}[\mathtt{Top}] \leftrightarrows \mathtt{Sp}_{\mathbb{N}}$$

which is idempotent and induces an equivalence between the homotopy category of grouplike $\mathcal{E}$–spaces and the homotopy category of connective spectra.[3]

1.1 Structure of the article

In Section 2 we review the definition of weak Quillen quasiadjunctions, idempotent quasimonads and idempotent quasiadjunctions. Our main theorem will be a particular case of the fact that idempotent quasiadjunctions induce equivalences between the associated homotopy subcategories.

[2]The spectrification functor $\widetilde{\Omega}$ is the left adjoint to the inclusion of the category of spectra in the sense used in [9] into the category of prespectra, hence the name.

[3]For details see the proof of the relative recognition theorem in [37] ignoring the open coloring. The proofs of Theorems 4.3.1 and 4.3.2 ignoring the codomain coloring is a coordinate-free version of this result.

In Section 3 we present the definition of $E^{\to}_{\infty}$–algebras through relative operads. A detailed description of relative sets, filtered rooted relative trees and operations on them will be required to construct bar resolutions and delooping spectra, as well as describe their algebraic structures. We then briefly review relative operads, $E^{\to}_{\infty}$–operads and the bar resolution of $E^{\to}_{\infty}$–pairs. Relative operad actions are then introduced. They provide an account of distributivity laws between multiplicative and additive relative operad actions, and are central in the definition of the category $(\mathscr{E}^{\to}, \mathscr{L}^{\to})[\mathtt{Top}]$ of $E^{\to}_{\infty}$–algebras. We also show how the mixed model structure on $(\mathscr{E}^{\to}, \mathscr{L}^{\to})[\mathtt{Top}]$ is transferred from the one on $\mathtt{Top}^2_{\mathbb{S}0}$.

The main theorems are in Section 4. The basics of coordinate-free spectra and the construction of the stable mixed model structure are presented. The recognition principle for ∞–loop pairs of spaces of spectra maps is proved via an idempotent quasiadjunction in Theorems 4.3.1 and 4.3.2, which imply the homotopy category of grouplike $E^{\to}_{\infty}$–pairs is equivalent to the homotopy category of spectra maps between connective spectra. After a review of the basics of $\mathbb{S}$–modules and commutative algebra spectra, including the construction of the stable mixed model structure, the main result, Theorem 4.6.1, is proved.

1.2 Notation and terminology

We denote by $\mathtt{Set}$ the category of sets and functions, by $\mathbb{S}^{\mathrm{inj}}$ the subcategory of finite sets and injections, and by $\mathbb{S}$ the subcategory of finite sets and bijections. We will use the notation $\underline{m}$ for the sets $\{1, \dots, m\}$, with $\underline{0} = \varnothing$.

Given a class A and a family of classes $\langle B^a \rangle$ indexed by A, the dependent sum $\Sigma_A B^a$ is the class of pairs (a, b) with $a \in A$ and $b \in B^a$; the dependent product $\Pi_A B^a$ is the class of sequences $\langle b^a \rangle$ indexed on A with $b^a \in B^a$ for each $a \in A$, or equivalently it is the class of sections of the natural surjection $\Sigma_A B^a \to A$.

We denote by $\mathtt{POSet}$ the category of ordered sets and monotone functions, and Δ the full subcategory on $\underline{m}_* = \langle 0 < 1 < \cdots < m \rangle$ for $m \in \mathbb{N}$. This category is generated by the coface injections $\partial_i : \underline{m-1}_* \to \underline{m}_*$, with $i \notin \partial_i \underline{m-1}_*$, and codegeneracy surjections $\delta_i : \underline{m+1}_* \to \underline{m}_*$, with $\delta_i i = \delta_i(i+1)$, for all $i \in \underline{m}_*$. For $\langle \underline{m}^a_* \rangle \in \Pi_A \Delta$ we define the set $\vee_A \underline{m}^a_* := \{0\} \sqcup \Sigma_A \underline{m}^a$.

Let $\mathtt{Top}$ be the cartesian closed category of compactly generated weakly Hausdorff spaces as presented by Strickland [35]. We will make extensive use of mapping spaces

Y^X and will express their elements as $x \mapsto \Phi$ for some expression Φ which may use the variable x. For X a set (or space) equipped with an equivalence relation $\sim$ we will denote the equivalence classes of $x \in X$ using square brackets $[x] \in X/_{\sim}$. We use the notation $K \subset_{\mathrm{cpct}} X$ to indicate K is a compact subspace of X. We denote by I the interval $[0,1] \subset \mathbb{R}$, and for $N \in \mathbb{N}$ we denote by $\mathbb{S}^N$ the one-point compactification of $\mathbb{R}^N$.

We denote by $\mathtt{Top}_*$ the closed monoidal category of pointed spaces (X, x_0) equipped with the smash product $X \wedge Y := X \times Y/_{X \times \{y_0\} \cup \{x_0\} \times Y}$ and unit $(\{0,1\}, 0)$. For $X \in \mathtt{Top}$ we denote by $X_+ \in \mathtt{Top}_*$ the pointed space obtained by adjoining a disjoint basepoint. We also denote by $\mathtt{Top}_{\mathbb{S}^0}$ the category of spaces with two distinguished points (X, x_0, x_1).

The theory of model categories in Goerss and Jardine [13], Hirschhorn [14], and Hovey [16] is assumed, and also the theory of monads, their algebras and the bar construction in [21, Section 9]. In diagrams in a model category $\mathcal{T}$ the morphisms in the class of weak equivalences W are denoted by arrows marked with a tilde $\xrightarrow{\sim}$, the ones in the class of cofibrations C by hooked arrows $\hookrightarrow$, and the ones in the class of fibrations F by double headed arrows $\twoheadrightarrow$. The functorial weak factorization systems are denoted by $(\mathrm{Fat}_{C,F_t}; C-, F_t-)$ and $(\mathrm{Fat}_{C_t,F}; C_t-, F-)$ such that a morphism $f \in \mathcal{T}(X,Y)$ is factored for instance as $X \xhookrightarrow{Cf} \mathrm{Fat}_{C,F_t} f \xrightarrow[\sim]{F_t f} Y$.

The notations $\mathfrak{C}\colon \mathcal{T} \to \mathcal{T}$ and $\mathrm{cof}\colon \mathfrak{C} \Rightarrow \mathrm{Id}$ are used for the cofibrant resolution functor and the associated natural trivial fibration, and the notations $\mathfrak{F}\colon \mathcal{T} \to \mathcal{T}$ and $\mathrm{fib}\colon \mathrm{Id} \Rightarrow \mathfrak{F}$ are used for the fibrant resolution functor and the associated natural trivial cofibration. The homotopy category $\mathcal{H}o\mathcal{T}$ of $\mathcal{T}$ is the category with objects the bifibrant objects of $\mathcal{T}$ and morphisms the sets $\mathcal{T}(X,Y)/_{\simeq}$ of homotopy classes of maps [16, Section 1.2]. If a functor $S\colon \mathcal{T} \to \mathcal{A}$ is left derivable, meaning it preserves cofibrant objects and weak equivalences between them, its left derived functor $\mathbb{L}S$ is defined on objects as $\mathbb{L}SX := \mathfrak{F}SX$, and dually if $\Lambda\colon \mathcal{A} \to \mathcal{T}$ is right derivable its right derived functor $\mathbb{R}\Lambda$ is defined on objects as $\mathbb{R}\Lambda Y := \mathfrak{C}\Lambda Y$.

The closed cartesian category $\mathtt{Top}$ of compactly generated weakly Hausdorff spaces admits three monoidal model structures:

For all $X \in \mathtt{Top}$ its cylinder is $X \times I$ and its cone is $CX := X \times I/_{(x,0) \sim (x',0)}$. We then have the cofibrantly generated *Quillen model structure* [26] with weak equivalences the weak homotopy equivalences (q–equivalences), ie the maps that induce isomorphisms of all homotopy groups; fibrations the Serre fibrations (q–fibrations), ie the maps satisfying the homotopy lifting property with respect to the cylinder inclusions of cones of spheres $\mathrm{in}_0 \in \mathtt{Top}(C\mathbb{S}^N, C\mathbb{S}^N \times I)$ for all $N \in \mathbb{N}$; and cofibrations retracts of inclusions of relative CW–complexes (q–cofibrations). This is a cofibrantly generated model structure with factorization systems induced by the small object argument. In this model structure all spaces are fibrant and the cofibrant spaces are the retracts of CW–complexes.

We also have the Hurewicz/Strøm model structure [36] with weak equivalences the homotopy equivalences (h–equivalences), ie the maps that admit an inverse up to homotopy; fibrations the Hurewicz fibrations (h–fibrations), ie the maps satisfying the homotopy lifting property with respect to all cylinder inclusions $\mathrm{in}_0 \in \mathtt{Top}(X, X \times I)$; and cofibrations the Hurewicz cofibrations[4] (h–cofibrations), ie the maps satisfying the homotopy extension property with respect to all evaluation fibrations $\mathrm{ev}_1 \in \mathtt{Top}(Y^I, Y)$. The weak factorization system can be constructed through (co)monads as described by Barthel and Riehl in [2]. For any $X \in \mathtt{Top}$ let the space of Moore paths in X be

$$MX := \Sigma_{t\in[0,\infty)}\{\gamma \in X^{[0,\infty]} \mid r \geq t \implies \gamma r = \gamma t\},$$

which comes equipped with the natural fibration $\mathrm{ev}_\infty \in \mathtt{Top}(MX, X)$. The factorization systems are then defined on every $\phi \in \mathtt{Top}(X, Y)$ as

$$(\Gamma\phi = X \times_Y MY; \quad C_t\phi x := (x, 0, r \mapsto \phi x), \quad F\phi(x, t, \gamma) := \gamma t),$$
$$(E\phi := \Gamma\phi \times [0, \infty] \sqcup_{\Gamma\phi} Y; \quad C\phi x := (x, 0, r \mapsto \phi x, 0), \quad F_t\phi(x, t, \gamma, s) := \gamma s).$$

In this model structure all objects are bifibrant.

These model structures can be combined into the *mixed model structure* — see Cole [7] — with weak equivalences the q–equivalences; fibrations the h–fibrations; and cofibrations the m–cofibrations, ie the maps that can be factored as a q–cofibration followed by an h–equivalence. A space is m–cofibrant if it is of the homotopy type of a CW–complex.

We denote by $\mathcal{T}^{\to}$ the category of morphisms $f\colon X_d \to X_c$ in $\mathcal{T}$ as objects and commutative squares as morphisms. For notational convenience we denote elements of categories of pairs $\mathcal{T}^2$ as $X = (X_d, X_c)$, and we will consider relative operads colored on the set $\{d, c\}$, with d being the "domain" color and c the "codomain" color.

Let $\mathtt{Inn}$ denote the topological category of finite or countably infinite-dimensional real inner product spaces and linear maps, with the topology defined as the colimit

[4] We note that in the category of compactly generated weakly Hausdorff spaces all Hurewicz cofibrations are closed.

of the finite-dimensional subspaces. Let $\mathscr{I}$ be the subcategory with the same objects and with morphisms the linear isometries. Both $\mathtt{Inn}$ and $\mathscr{I}$ are monoidal under direct sums. For $\mathbb{U} \in \mathtt{Inn}$ we denote by $\mathscr{A}_{\mathbb{U}}$ the set of finite-dimensional subspaces of $\mathbb{U}$, partially ordered by inclusion, and for $U \in \mathscr{A}_{\mathbb{U}}$ we define $\mathscr{A}_U := \{V \in \mathscr{A}_{\mathbb{U}} \mid U \leq V\}$. For $\mathbb{U} = \mathbb{R}^\infty$ we simply write $\mathscr{A} := \mathscr{A}_{\mathbb{R}^\infty}$. For $\langle f_a\rangle \in \mathscr{I}(\oplus_A \mathbb{U}^a, \mathbb{V})$ and $\langle \vec{u}^a\rangle \in \oplus_A \mathbb{U}^a$ we use the Einstein summation convention $f_a \vec{u}^a := \sum_A f_a \vec{u}^a$. For $\mathbb{U} \in \mathscr{I}$ and $U < \mathbb{U}$ we use the notation $U^\perp := \{\vec{v} \in \mathbb{U} \mid \forall \vec{u} \in U : \vec{v} \cdot \vec{u} = 0\}$ for the orthogonal complement.

For any $U < \mathbb{U}$ we define $\pi_U \in \mathtt{Inn}(\mathbb{U}, \mathbb{U})$ as the orthogonal projection onto U, and for $f \in \mathtt{Inn}(\mathbb{U}, \mathbb{V})$ define $f|_U \in \mathtt{Inn}(U, \mathbb{V})$ as the restriction of f on U. For $f \in \mathscr{I}(\mathbb{U}, \mathbb{V})$ its adjoint is $f^* := f^{-1}\pi_{f\mathbb{U}} \in \mathscr{I}(\mathbb{V}, \mathbb{U})$. For all $U \in \mathscr{A}_{\mathbb{U}}$ let $\mathbb{S}^U$ be the one point compactification of U obtained by adding a point ∞ at infinity and for $(U, V) \in \Sigma_{\mathscr{A}} \mathscr{A}_U$ let $V - U := V \cap U^\perp$.

Consider the cosimplicial space of partitions of the interval $\mathrm{Part}^- \in \mathtt{Top}^\Delta$ with

$$
\begin{aligned}
\mathrm{Part}^{\underline{m}_*} &:= \mathtt{POSet}(\underline{m}, I);\\
\partial_i \cdot \langle t^j\rangle &:= \begin{cases} \left\langle \begin{cases} 0 & \text{if } j' = 1 \\ t^{j'-1} & \text{if } j' > 1 \end{cases} \right\rangle & \text{if } i = 0,\\ \left\langle \begin{cases} t^{j'} & \text{if } j' \leq i \\ t^{j'-1} & \text{if } j' > i \end{cases} \right\rangle & \text{if } 0 < i < m,\\ \left\langle \begin{cases} t^{j'} & \text{if } j' < m \\ 1 & \text{if } j' = m \end{cases} \right\rangle & \text{if } i = m, \end{cases}\\
\delta_i \cdot \langle t^j\rangle &:= \left\langle \begin{cases} t^{j'} & \text{if } j' \leq i \\ t^{j'+1} & \text{if } j' > i \end{cases} \right\rangle,
\end{aligned}
$$

with $\mathrm{Part}^{\underline{m}_*}$ topologized as a subspace of $I^{\underline{m}}$. For each $\langle\langle t^{ai}\rangle\rangle \in \Pi_A \mathrm{Part}^{\underline{m}^a_*}$ the order of the points t^{ai} in I induces an order on $\vee_A \underline{m}^a_*$, and so an element

$$\lhd_A \langle t^{ai}\rangle \in \mathrm{Part}^{\vee_A \underline{m}^a_*}.$$

For each $\langle\langle t^{ai}\rangle\rangle \in \Pi_A \mathrm{Part}^{\underline{m}^a_*}$ and $a' \in A$ we can define

$$\delta^{a'} \in \Delta(\vee_A \underline{m}^a_*, \underline{m}^{a'}_*), \quad \delta^{a'}(a,i) := \begin{cases} 0 & \text{if } t^{ai} < t^{a'1}, \\ \max_{t^{a'i'} \leq t^{ai}} i' & \text{if } t^{ai} \geq t^{a'1}, \end{cases}$$

such that $\delta^{a'} \cdot \lhd_A t^a = \langle t^{a'i}\rangle \in \mathrm{Part}^{\underline{m}^{a'}_*}$.

For any simplicial space $X^- \in \mathtt{Top}^{\Delta^{\mathrm{op}}}$ its geometric realization $|X^-|$ is defined via the coend construction [19] as

$$|X^-| := \int^\Delta X^{\underline{m}_*} \times \mathrm{Part}^{\underline{m}_*}.$$

The reason we consider the geometric realization via the partitions cosimplicial space instead of the usual homeomorphic cosimplicial space of topological simplexes is that this choice simplifies the algorithm in [21, Theorem 11.5].

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2 Idempotent quasiadjunctions

2.1 Weak Quillen quasiadjunction

The following definition introduced in [37] is a generalization of Quillen adjunctions between model categories. The basic idea is that to construct the unit and counit natural transformations of an adjunction between the homotopy categories it suffices to construct a unit natural span and counit natural cospan at the model categories level, plus some additional compatibility conditions.

Definition 2.1.1 Let $\mathcal{T}$ and $\mathcal{A}$ be model categories. A *weak Quillen quasiadjunction*, or just *quasiadjunction*, between $\mathcal{T}$ and $\mathcal{A}$, denoted by

$$(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A},$$

is a quadruple of functors

$$\mathscr{C} \circlearrowright \mathcal{T} \underset{\Lambda}{\overset{S}{\rightleftarrows}} \mathcal{A} \circlearrowleft \mathscr{F}$$

with S the *left quasiadjoint* and Λ the *right quasiadjoint*, equipped with a natural span in $\mathcal{T}$ and a natural cospan in $\mathcal{A}$

$$\mathrm{Id}_{\mathcal{T}} \overset{\eta'}{\underset{\sim}{\Longleftarrow}} \mathscr{C} \overset{\eta}{\Longrightarrow} \Lambda S, \qquad S\Lambda \overset{\epsilon}{\Longrightarrow} \mathscr{F} \overset{\epsilon'}{\underset{\sim}{\Longleftarrow}} \mathrm{Id}_{\mathcal{A}}$$

such that

(i) S is left derivable;

(ii) Λ is right derivable;

(iii) $\mathscr{C}$ and $\mathscr{F}$ preserve cofibrant and fibrant objects;

(iv) η' and ϵ' are natural weak equivalences;

(v) if $X \in \mathcal{T}$ is cofibrant then $\epsilon_{SX} S\eta_X \simeq \epsilon'_{SX} S\eta'_X$;

(vi) if $Y \in \mathcal{A}$ is fibrant then $\Lambda\epsilon_Y \eta_{\Lambda Y} \simeq \Lambda\epsilon'_Y \eta'_{\Lambda Y}$:

$$\begin{array}{ccc} S\mathscr{C}X & \xrightarrow{S\eta_X} & S\Lambda SX \\ {\scriptstyle S\eta'_X}\downarrow\sim & & \downarrow{\scriptstyle \epsilon_{SX}} \\ SX & \xrightarrow[\epsilon'_{SX}]{\sim} & \mathscr{F}SX \end{array} \qquad \begin{array}{ccc} \mathscr{C}\Lambda Y & \xrightarrow{\eta_{\Lambda Y}} & \Lambda S\Lambda Y \\ {\scriptstyle \eta'_{\Lambda Y}}\downarrow\sim & & \downarrow{\scriptstyle \Lambda\epsilon_Y} \\ \Lambda Y & \xrightarrow[\Lambda\epsilon'_Y]{\sim} & \Lambda\mathscr{F}Y \end{array}$$

Theorem 2.1.2 [37, Theorem 2.1.2] *A quasiadjunction induces an adjunction*

$$(\mathbb{L}S \dashv \mathbb{R}\Lambda): \mathcal{H}o\mathcal{T} \rightleftharpoons \mathcal{H}o\mathcal{A},$$

$$\mathrm{Id}_{\mathcal{H}o\mathcal{T}} \xRightarrow{[\mathrm{cof}\mathfrak{C}\eta']^{-1}} \mathbb{R}\mathscr{C} \xRightarrow{[\mathfrak{C}(\Lambda \mathrm{fib}_S \eta)]} \mathbb{R}\Lambda\mathbb{L}S$$

$$\mathbb{L}S\mathbb{R}\Lambda \xRightarrow{[\mathfrak{F}(\epsilon S \mathrm{cof}_\Lambda)]} \mathbb{L}\mathscr{F} \xRightarrow{[\mathfrak{F}\epsilon' \mathrm{fib}]^{-1}} \mathrm{Id}_{\mathcal{H}o\mathcal{A}}$$

between the homotopy categories.

2.2 Idempotent quasi(co)monads

The following generalization of idempotent Quillen monads [4] was also introduced following the same principle of only requiring the existence of a unit natural span, and they also induce Bousfield localizations.

Definition 2.2.1 Let $\mathcal{T}$ be a right proper model category. A *Quillen idempotent quasimonad on* $\mathcal{T}$, or simply an *idempotent quasimonad*, is a pair of endofunctors $Q, \overline{\mathscr{C}}: \mathcal{T} \to \mathcal{T}$ equipped with a natural span

$$\mathrm{Id}_{\mathcal{T}} \overset{\eta'}{\underset{\sim}{\Longleftarrow}} \overline{\mathscr{C}} \overset{\eta}{\Longrightarrow} Q$$

such that

(i) η' is a natural weak equivalence;

(ii) Q preserves weak equivalences;

(iii) $Q\eta$ and η_Q are natural weak equivalences;

(iv) if $f \in \mathcal{T}(X, B)$, $p \in F(E, B)$ and $\eta_E, \eta_B, Qf \in W$ then $Q(f^*p) \in W$:

$$\begin{array}{ccccccccc} X \times_B E & \xrightarrow{f^*p} & E & \underset{\sim}{\xleftarrow{\eta'_E}} & \overline{\mathscr{C}}E & \underset{\sim}{\xrightarrow{\eta_E}} & QE & \underset{\sim}{\xleftarrow{Q(f^*p)}} & Q(X \times_B E) \\ {\scriptstyle p^*f}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle \overline{\mathscr{C}}p}\downarrow & & {\scriptstyle Qp}\downarrow & & \downarrow{\scriptstyle Q(p^*f)} \\ X & \xrightarrow[f]{} & B & \overset{\sim}{\underset{\eta'_B}{\longleftarrow}} & \overline{\mathscr{C}}B & \overset{\sim}{\underset{\eta_B}{\longrightarrow}} & QB & \overset{\sim}{\underset{Qf}{\longleftarrow}} & QX \end{array}$$

(v) if $\iota \in C(\overline{\mathscr{C}}X, K)$ then $\iota_*\eta' \in W$:

$$\begin{array}{ccc} \overline{\mathscr{C}}X & \xrightarrow[\sim]{\eta'} & X \\ {\scriptstyle\iota}\big\downarrow & & \big\downarrow{\scriptstyle\eta'_*\iota} \\ K & \xrightarrow[\iota_*\eta']{\sim} & K \sqcup_{\overline{\mathscr{C}}X} X \end{array}$$

Theorem 2.2.2 [37, Theorems 2.3.5 and 2.3.6] *An idempotent quasimonad induces a left Bousfield localization*

$$\mathcal{T}_Q = (\mathcal{T}; W_Q := Q^{-1}W, C_Q := C, F_Q := \{p \in F \mid (1) \text{ is a homotopy pullback}\})$$

(1)
$$\begin{array}{ccc} E & \xrightarrow{i_E} & E \sqcup_{\overline{\mathscr{C}}E} QE \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle (p,Qp)} \\ B & \xrightarrow[i_B]{} & B \sqcup_{\overline{\mathscr{C}}B} QB \end{array}$$

The resulting homotopy category is the reflective subcategory

$$\mathcal{H}o\mathcal{T}_Q := \{X \in \mathcal{H}o\mathcal{T} \mid (i_X : X \to X \sqcup_{\overline{\mathscr{C}}X} QX) \in W\}$$

of Q–fibrant objects.

The above definition can be dualized. The resulting idempotent quasicomonads induce right Bousfield localizations and associated coreflective homotopy subcategories.

2.3 Idempotent quasiadjunctions

A quasiadjunction $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda) : \mathcal{T} \rightleftharpoons \mathcal{A}$ induces the following natural span on $\mathcal{T}$ and natural cospan on $\mathcal{A}$:

(2)
$$\mathrm{Id}_{\mathcal{T}} \xLeftarrow[\sim]{\mathrm{cof}\,\eta'_{\mathfrak{C}}} \mathscr{C}\mathfrak{C} \xRightarrow{(\Lambda \mathrm{fib}_S \eta)_{\mathfrak{C}}} \Lambda\mathfrak{F}S\mathfrak{C} \qquad S\mathfrak{C}\Lambda\mathfrak{F} \xRightarrow{(\epsilon S \mathrm{cof}_\Lambda)_{\mathfrak{F}}} \mathscr{F}\mathfrak{F} \xLeftarrow[\sim]{\epsilon'_{\mathfrak{F}}\mathrm{fib}} \mathrm{Id}_{\mathcal{A}}$$

Definition 2.3.1 An *idempotent quasiadjunction* is a quasiadjunction such that the induced span and cospan (2) are respectively an idempotent quasimonad and an idempotent quasicomonad.

Theorem 2.3.2 [37, Theorem 2.3.8] *An idempotent quasiadjunction induces an equivalence between the associated (co)reflective homotopy subcategories*:

$$\mathcal{H}o\mathcal{T} \underset{\mathbb{R}\mathrm{Id}}{\overset{\mathbb{L}\mathrm{Id}}{\rightleftarrows}} \mathcal{H}o\mathcal{T}_{\Lambda\mathfrak{F}S\mathfrak{C}} \underset{\mathbb{R}\Lambda}{\overset{\mathbb{L}S}{\rightleftarrows}} \mathcal{H}o\mathcal{A}_{S\mathfrak{C}\Lambda\mathfrak{F}} \underset{\mathbb{R}\mathrm{Id}}{\overset{\mathbb{L}\mathrm{Id}}{\rightleftarrows}} \mathcal{H}o\mathcal{A}$$

3 $E^{\rightarrow}_{\infty}$–algebras

3.1 Relative sets and filtered rooted relative trees

Relative operads are abstract operations with entries indexed by relative sets. We now give the basic definitions and constructions on these colored sets. We will also require filtered rooted relative trees in the construction of the bar resolutions and delooping spectra, and we provide here the relevant definitions and constructions.

Let $\mathtt{Set}_{\{d,c\}}$ be the *category of relative sets* composed of sets equipped with a coloring on the colors $\{d,c\}$, ie the class of objects

$$\{(A,\mathfrak{c}) \in \Sigma_{\mathtt{Set}}\mathtt{Set}(\{A\} \sqcup A, \{d,c\}) \mid \mathfrak{c}A = d \implies \forall a \in A : \mathfrak{c}a = d\},$$

with $(A,\mathfrak{c})$ usually being denoted simply as A or explicitly as a set of elements in brackets with coloring given by subscripts, eg $\{1_d, 2_d, 3_c, 4_d, 5_c\}_c$. The morphisms sets are

$$\mathtt{Set}_{\{d,c\}}(A,A') := \begin{cases} \{\sigma \in \mathtt{Set}(A,A') \mid \mathfrak{c}a = c \implies \mathfrak{c}'\sigma a = c\}, & \mathfrak{c}A = d \text{ or } \mathfrak{c}'A' = c, \\ \varnothing, & \mathfrak{c}A = c \text{ and } \mathfrak{c}'A' = d. \end{cases}$$

For $\star \in \{d,c\}$ we denote by $\mathtt{Set}_\star \subset \mathtt{Set}_{\{d,c\}}$ the full subcategory of relative sets A such that $\mathfrak{c}A = \star$.

Given $((A,\mathfrak{c}), \langle (B^a, \mathfrak{c}^a)\rangle) \in \Sigma_{\mathtt{Set}_{\{d,c\}}} \Pi_A \mathtt{Set}_{\mathfrak{c}a}$ we have the dependent sum

$$(\Sigma_A B^a, \Sigma_A \mathfrak{c}^a) \in \mathtt{Set}_{\{d,c\}}; \quad \Sigma_A \mathfrak{c}^a(\Sigma_A B^a) := \mathfrak{c}A, \quad \Sigma_A \mathfrak{c}^a(a,b) := \mathfrak{c}^a b.$$

For $\sigma \in \mathtt{Set}_{\{d,c\}}(A,A')$ let

$$\sigma(B^a) \in \mathtt{Set}_{\{d,c\}}(\Sigma_A B^a, \Sigma_{A'} B^{\sigma^{-1}a'}), \quad \sigma(B^a)(a',b) := (\sigma a', b),$$

and for $\langle \tau^a \rangle \in \Pi_A \mathtt{Set}_{\mathfrak{c}(a)}(B^a, B'^a)$ let

$$\Sigma_A \tau^a \in \mathtt{Set}_{\{d,c\}}(\Sigma_A B^a, \Sigma_A B'^a), \quad \Sigma_A \tau^a(a',b) := (a', \tau^{a'} b).$$

We also have the dependent product

$$\Pi_A B^a \in \mathtt{Set}_{\{d,c\}}; \quad \Pi_A \mathfrak{c}^a \Pi_A B^a := \mathfrak{c}A, \quad \Pi_A \mathfrak{c}^a \langle b^a \rangle := \begin{cases} d & \forall a \in A : \mathfrak{c}^a b^a = d, \\ c & \exists a \in A : \mathfrak{c}^a b^a = c. \end{cases}$$

For $\sigma \in \mathtt{Set}_{\{d,c\}}(A,A')$ let

$$\sigma\langle B^a \rangle \in \mathtt{Set}_{\{d,c\}}(\Pi_A B^a, \Pi_{A'} B^{\sigma^{-1}a'}), \quad \sigma\langle B^a \rangle \langle b^a \rangle := \langle b^{\sigma^{-1}a'} \rangle,$$

and for $\langle \tau^a \rangle \in \Pi_A \mathtt{Set}_{\mathfrak{c}a}(B^a, B'^a)$ let

$$\Pi_A \tau^a \in \mathtt{Set}_{\{d,c\}}(\Pi_A B^a, \Pi_A B'^a), \quad \Pi_A \tau^a \langle b^a \rangle := \langle \tau^a b^a \rangle.$$

For every $\langle b^a\rangle \in \Pi_A B^a$ we can form a new relative set $A_{\langle b^a\rangle}$ composed of the pairs (a, b^a) with coloring $\mathfrak{c}_{\langle b^a\rangle} A_{\langle b^a\rangle} = \Pi_A \mathfrak{c}^a \langle b^a\rangle$ and $\mathfrak{c}_{\langle b^a\rangle}(a, b^a) = \mathfrak{c}^a b^a$. This relative set is naturally equipped with $\pi_{\langle b^a\rangle} \in \mathtt{Set}_{\{d,c\}}(A_{\langle b^a\rangle}, A)$ with $\pi_{\langle b^a\rangle}(a, b^a) = a$. Let

$$\nu \in \mathtt{Set}_{\{d,c\}}(\Pi_A \Sigma_{B^a} C^{ab}, \Sigma_{\Pi_A B^a} \Pi_{A_{\langle b^a\rangle}} C^{ab^a}), \quad \nu\langle (b^a, c^a)\rangle := (\langle b^a\rangle, \langle c^a\rangle).$$

This is a key element in distributivity properties.

Let $\mathbb{S}^{\mathrm{inj}}_{\{d,c\}} \subset \mathtt{Set}_{\{d,c\}}$ be the subcategory of $\mathtt{Set}_{\{d,c\}}$ composed of the finite relative sets and the injective functions that preserve coloring, ie

$$\mathbb{S}^{\mathrm{inj}}_{\{d,c\}}(A, A') = \begin{cases} \{\sigma \in \mathtt{Set}_{\{d,c\}}(A, A') \mid \sigma \text{ is injective}, \mathfrak{c}'\sigma a = \mathfrak{c} a\} & \text{if } \mathfrak{c}A = \mathfrak{c}'A', \\ \varnothing & \text{if } \mathfrak{c}A \neq \mathfrak{c}'A'. \end{cases}$$

Let $\mathbb{S}_{\{d,c\}} \subset \mathbb{S}^{\mathrm{inj}}_{\{d,c\}}$ be the subcategory with the same objects and bijections that preserve coloring as morphisms. For $\star \in \{d, c\}$ we denote by $\mathbb{S}^{\mathrm{inj}}_{\star}$ and $\mathbb{S}_{\star}$ the full subcategories of $\mathbb{S}^{\mathrm{inj}}_{\{d,c\}}$ and $\mathbb{S}_{\{d,c\}}$, respectively, composed of relative sets A such that $\mathfrak{c}A = \star$. Define also the subcategory $\mathbb{S}_{\langle d<c\rangle} \subset \mathtt{Set}_{\{d,c\}}$ with objects the finite relative sets and with morphisms the bijections (that don't necessarily preserve coloring). Note that $\mathbb{S}_{\{d,c\}}$ is a subcategory of $\mathbb{S}_{\langle d<c\rangle}$.

Many spaces of interest are built via the two sided bar construction for monads induced by operads, which can be described using filtered rooted relative trees.

Definition 3.1.1 The simplicial category $\mathbb{T}_{\{d,c\}} \in \mathtt{Cat}^{\Delta^{\mathrm{op}}}$ of filtered rooted relative trees has as objects triples

$$T = (\langle V^i\rangle, \langle \phi^i\rangle, \mathfrak{c}) \in \mathbb{T}_{\{d,c\}}\underline{m}_*$$

composed of:

- A sequence of finite sets of *vertices* $\langle V^i\rangle \in \mathbb{S}^{\underline{m+1}}$. We also set $V^0 := \{r\}$, and call r the *root vertex* of T. The vertices in V^{m+1} are called the *leaves* of T.
- A sequence of functions $\langle \phi^i\rangle \in \Pi_{\underline{m}} \mathtt{Set}(V^{i+1}, V^i)$. We also set ϕ^0 the unique function in $\mathtt{Set}(V^1, V^0)$.
- A function in $\{\mathfrak{c} \in \mathtt{Set}(\Sigma_{\underline{m+1}_*} V^i, \{d, c\}) \mid (\phi^i v' = v \text{ and } \mathfrak{c}v = d) \Longrightarrow \mathfrak{c}v' = d\}$, the *coloring* of the vertices.

We sometimes just write $T = (\langle V^i\rangle, \langle \phi^i\rangle)$ and leave the coloring implicit.

Morphisms $\sigma \in \mathbb{T}_{\{d,c\}}\underline{m}_*(T, T')$ are sequences of bijections in

$$\{\langle \sigma^i\rangle \in \Pi_{\underline{m+1}_*} \mathbb{S}(V^i, V'^i) \mid \sigma^i \phi^i = \phi'^i \sigma^{i+1}\ \forall i \in \underline{m}_* \text{ and } \mathfrak{c}'\sigma^i = \mathfrak{c}\ \forall i \in \underline{m+1}_*\}.$$

The simplicial structural functors are defined on objects as

$$T\cdot\partial_i := \left(\left\langle\begin{cases}V^j & \text{if } j\le i\\ V^{j+1} & \text{if } j>i\end{cases}\right\rangle, \left\langle\begin{cases}\phi^j & \text{if } j<i\\ \phi^i\phi^{i+1} & \text{if } j=i\\ \phi^{j+1} & \text{if } j>i\end{cases}\right\rangle\right),$$

$$T\cdot\delta_i := \left(\left\langle\begin{cases}V^j & \text{if } j\le i+1\\ V^{j-1} & \text{if } j>i+1\end{cases}\right\rangle, \left\langle\begin{cases}\phi^j & \text{if } j\le i\\ \mathrm{id}_{V^{i+1}} & \text{if } j=i+1\\ \phi^{j-1} & \text{if } j>i+1\end{cases}\right\rangle\right),$$

with the coloring maps induced naturally from the ones in T.

Define also $\mathbb{T}^{\mathbf{0}}_{\{d,c\}} \in \mathtt{Cat}^{\Delta^{\mathrm{op}}}$ as the full simplicial subcategory of relative trees such that $|V^1| = 1$. Define also the simplicial full subcategories $\mathbb{T}_\star \subset \mathbb{T}_{\{d,c\}}$ for $\star \in \{d,c\}$ of the trees such that $\mathfrak{c}r = \star$. We similarly define the simplicial full subcategories $\mathbb{T}^{\mathbf{0}}_\star \subset \mathbb{T}^{\mathbf{0}}_{\{d,c\}}$.

Note that any $T \in \mathbb{T}_{\{d,c\}}\underline{m}_*$ has a natural partial order structure on the union of the set of vertices induced by the structural functions, with the root r its unique minimal element. For each $v \in V^1$ let $T_{\ge v} \in \mathbb{T}^{\mathbf{0}}_{\{d,c\}}\underline{m}_*$ be the subtree composed of the root vertex and the vertices greater than or equal to v.

For all $T \in \mathbb{T}_{\{d,c\}}\underline{m}_*$ and $(i,v) \in \Sigma_{\underline{m}_*} V^i$ define the relative set

$$\mathrm{in}\,(i,v) := \{v' \in V^{i+1} \mid \phi^i v' = v\}_{\mathfrak{c}v} \in \mathbb{S}_{\{d,c\}}.$$

Note that $\mathbb{S}_{\{d,c\}}$ is isomorphic to $\mathbb{T}_{\{d,c\}}\underline{0}_*$.

We have natural dependent sums and dependent products of filtered rooted trees of a fixed height $\langle T^a\rangle \in \Pi_A \mathbb{T}_{\{d,c\}}\underline{m}_*$ defined as

$$\Sigma_A T^a := (\langle \Sigma_A V^{ai}\rangle, \langle \Sigma_A \phi^{ai}\rangle, \Sigma_A \mathfrak{c}^a), \qquad \Pi_A T^a := (\langle \Pi_A V^{ai}\rangle, \langle \Pi_A \phi^{ai}\rangle, \Pi_A \mathfrak{c}^a).$$

We also have, for all

$$(T, \langle S^e\rangle) = ((\langle V^i\rangle, \langle\phi^i\rangle), \langle(\langle W^{ej}\rangle, \langle\psi^{ej}\rangle)\rangle) \in \Sigma_{\mathbb{T}_{\{d,c\}}\underline{m}_*} \Pi_{V^{m+1}} \mathbb{T}_{\mathfrak{c}e}\underline{n}_*,$$

the grafting

$$T\circ\langle S^e\rangle := \left(\left\langle\begin{cases}V^i & \text{if } i\le m\\ \Sigma_{V^{m+1}} W^{e(i-m-1)} & \text{if } i>m\end{cases}\right\rangle, \left\langle\begin{cases}\phi^i & \text{if } i\le m\\ \Sigma_{V^{m+1}} \psi^{e(i-m-1)} & \text{if } i>m\end{cases}\right\rangle\right)$$

in $\mathbb{T}_{\{d,c\}}\underline{m+n+1}_*$, with the obvious coloring map.

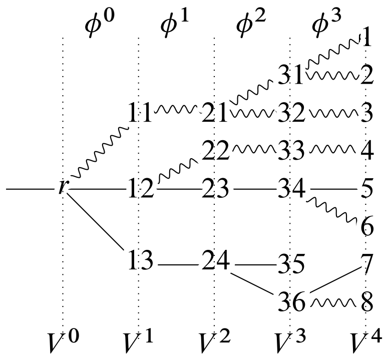

Figure 1: A filtered rooted relative tree in $\mathbb{T}_{\mathfrak{c}}\underline{3}_*$. Edges, oriented from right to left, represent to which element of V^i the function ϕ^i maps the elements of V^{i+1}. Decorations indicate the color of the start vertex, with wiggled edges representing "domain" color and straight edges representing "codomain" color. An extra edge is added to the root vertex to indicate its color.

3.2 Relative operads

We now give a brief review of relative operads, a kind of colored operad introduced by Voronov in [38].

Definition 3.2.1 The category of $\mathbb{S}_{\{d,c\}}$–spaces is the contravariant functor category $\mathtt{Top}^{\mathbb{S}^{\mathrm{op}}_{\{d,c\}}}$. A topological relative operad is an $\mathbb{S}_{\{d,c\}}$–space $\mathcal{P} \in \mathtt{Top}^{\mathbb{S}^{\mathrm{op}}_{\{d,c\}}}$ equipped with elements $\mathrm{id}_\star \in \mathcal{P}\{1_\star\}_\star$ for $\star \in \{d, c\}$ and structural maps

$$\langle \circ_{A,\langle B^a \rangle} \rangle \in \Pi_{\Sigma_{\mathbb{S}_{\{d,c\}}}} \Pi_A \mathbb{S}_{\mathfrak{c}a} \mathtt{Top}(\mathcal{P}A \times \Pi_A \mathcal{P}B^a, \mathcal{P}\Sigma_A B^a)$$

such that $\mathcal{P}\varnothing_\star = \{*_\star\}$ for $\star \in \{d, c\}$ and, using the notation $\alpha\langle\beta^a\rangle := \circ_{A,\langle B^a\rangle}(\alpha, \langle\beta^a\rangle)$, satisfying the equations

- $\alpha\langle\beta^a\langle\gamma^{ab}\rangle\rangle = \alpha\langle\beta^a\rangle\langle\gamma^{ab}\rangle$,
- $\mathrm{id}_{\mathfrak{c}A}\alpha = \alpha = \alpha\langle\mathrm{id}_{\mathfrak{c}a}\rangle$,
- $\alpha \cdot \sigma\langle\beta^a\rangle = \alpha\langle\beta^{\sigma^{-1}a'}\rangle \cdot \sigma(B^a)$,
- $\alpha\langle\beta^a \cdot \tau^a\rangle = \alpha\langle\beta^a\rangle \cdot \Sigma_A \tau^a$.

Operad morphisms are natural transformations that preserve the unit and compositions, and we denote the category of topological relative operad by $\mathtt{Op}_{\{d,c\}}[\mathtt{Top}]$.[5]

[5]Another way to define a relative operad $\mathcal{Q}$ over an operad $\mathcal{P}$ is as an operad in the category of right $\mathcal{P}$–modules; see for instance [11]. For a relative operad $\mathcal{P}$ in the sense of Definition 3.2.1 the spaces

For $X = ((X_d, e_d), (X_c, e_c)) \in \mathtt{Top}_*^2$ define

$$\Pi_- X : \mathbb{S}^{\mathrm{inj}}_{\{d,c\}} \to \mathtt{Top}; \quad \Pi_A X := \Pi_A X_{\mathfrak{c}a}, \quad \sigma \cdot \langle x^a \rangle := \left\langle \begin{cases} e_{\mathfrak{c}a'} & \text{if } a' \notin \operatorname{Im}\sigma \\ x^{\sigma^{-1}a'} & \text{if } a' \in \operatorname{Im}\sigma \end{cases} \right\rangle.$$

The underlying functor of a unital relative operad $\mathcal{P}$ can be extended to a functor on $\mathbb{S}^{\mathrm{inj,op}}_{\{d,c\}}$. For $\sigma \in \mathbb{S}^{\mathrm{inj}}_{\{d,c\}}(A, A')$ the right action $\cdot\sigma \in \mathtt{Top}(\mathcal{P}A', \mathcal{P}A)$ is defined as

$$\alpha \cdot \sigma := \alpha \left\langle \begin{cases} *_{\mathfrak{c}a'} & \text{if } a' \notin \operatorname{Im}\sigma \\ \mathrm{id}_{\mathfrak{c}a'} & \text{if } a' \in \operatorname{Im}\sigma \end{cases} \right\rangle.$$

These morphisms are the *degenerations* of the relative operad.

A relative operad $\mathcal{P}$ induces a monad $(P; \eta, \mu)$ on $\mathtt{Top}_*^2$ with

$$\begin{gathered} PX_\star := \int^{\mathbb{S}^{\mathrm{inj}}_\star} \mathcal{P}A \times \Pi_A X; \\ \eta_\star x := [\mathrm{id}_\star, x], \quad \mu_\star[\alpha, \langle [\beta^a, \langle x^{ab} \rangle] \rangle] := [\alpha\langle \beta^a \rangle, \langle x^{ab} \rangle]. \end{gathered} \tag{3}$$

Definition 3.2.2 Let $\mathcal{P}$ be a relative operad. A *$\mathcal{P}$–space* is a P–algebra, ie a pair of pointed spaces $X \in \mathtt{Top}_*^2$ equipped with structural maps

$$\langle \theta_A \rangle \in \Pi_{\mathbb{S}_{\{d,c\}}} \mathtt{Top}(\mathcal{P}A \times \Pi_A X, X_{\mathfrak{c}A})$$

satisfying, using the notation $\alpha\langle x^a \rangle = \theta_A\langle \alpha, \langle x^a \rangle \rangle$,

$$\alpha\langle \beta^a \langle x^{ab} \rangle \rangle = \alpha\langle \beta^a \rangle \langle x^{ab} \rangle, \quad \mathrm{id}_\star x = x, \quad \alpha \cdot \sigma \langle x^a \rangle = \alpha(\sigma \cdot \langle x^a \rangle).$$

The category of $\mathcal{P}$–spaces is denoted $\mathcal{P}[\mathtt{Top}]$.

The following are the relative operads relevant to the main result.

The terminal relative operad is $\mathrm{Com}^\to$ with underlying $\mathbb{S}_{\{d,c\}}$–space given by

$$\mathrm{Com}^\to(A) := *.$$

The $\mathbb{S}_{\{d,c\}}$ right actions, units and compositions are the unique terminal maps. The $\mathrm{Com}^\to$–spaces are pairs (M_d, M_c) of topological commutative monoids equipped with a continuous homomorphism $\iota : M_d \to M_c$ induced by the unique element in $\mathrm{Com}^\to\{1_d\}_c$.

For $U \in \mathscr{A}$ the relative operad of U–embeddings $\mathrm{Emb}^\to_U$ is

$$\begin{gathered} \mathrm{Emb}^\to_U A := \{\alpha = \langle \alpha_a \rangle \in U^{\sqcup_A U} \mid \langle \alpha_a \rangle \text{ is an embedding}\}; \\ \langle \alpha_{a'} \rangle \cdot \sigma := \langle \alpha_{\sigma a} \rangle, \quad \mathrm{id}_\star := id_U, \quad \alpha\langle \beta^a \rangle := \langle \alpha_a \beta^a_b \rangle. \end{gathered}$$

$\mathcal{P}\{1_d, \ldots, m_d\}_d$ for $m \in \mathbb{N}$ form an operad, and the modules $\coprod_{n \in \mathbb{N}} \mathcal{P}\{1_d, \ldots, n_d, 1_c, \ldots, m_c\}_c$ for $m \in \mathbb{N}$ form a relative operad over the previous operad.

Degenerations delete embeddings.

For $U \in \mathscr{A}$ the loop space map functors image has natural $\mathrm{Emb}_U^{\rightarrow}$–pairs structure, giving us the functor

$$\Omega_2^U : \mathtt{Top}_*^{\rightarrow} \to \mathrm{Emb}_U^{\rightarrow}[\mathtt{Top}], \quad \Omega_2^U(\iota\colon Y_d \to Y_c) := (Y_d^{\mathbb{S}^U}, Y_c^{\mathbb{S}^U}); \tag{4}$$
$$\alpha\langle\gamma^a\rangle := \left(\vec{u} \mapsto \begin{cases} \gamma^a \alpha_a^{-1}\vec{u} & \text{if } \mathfrak{c}a = \mathfrak{c}A \\ \iota\gamma^a\alpha_a^{-1}\vec{u} & \text{if } \mathfrak{c}a \neq \mathfrak{c}A \end{cases}\right).$$

For $(U,V) \in \Sigma_{\mathscr{A}}\mathscr{A}_U$ we have natural inclusion of relative operads

$$i_V^U : \mathrm{Emb}_U^{\rightarrow} \Rightarrow \mathrm{Emb}_V^{\rightarrow}, \quad i_V^U \alpha := \langle \pi_{V-U} + \alpha_a \pi_U \rangle$$

and we define $\mathrm{Emb}_\infty^{\rightarrow} := \mathrm{colim}_{\mathscr{A}} \mathrm{Emb}_U^{\rightarrow}$.

The embeddings operad contains embeddings of configuration spaces, and these embeddings are relevant to the definition of $E^{\rightarrow}$–operads we give here. For each $U \in \mathscr{A}$ define the configurations $\mathbb{S}_{\{d,c\}}$–space

$$\mathrm{Conf}_U^{\rightarrow} : \mathbb{S}_{\{d,c\}}^{\mathrm{op}} \to \mathtt{Top}, \quad \mathrm{Conf}_U^{\rightarrow} A := \{\vec{x} = \langle \vec{x}_a \rangle \in U^A \mid a \neq a' \Longrightarrow \vec{x}_a \neq \vec{x}_{a'}\}.$$

For all $\vec{x} \in \mathrm{Conf}_U^{\rightarrow} A$ let $\min \vec{x} := \min_{a \neq a'} \|\vec{x}_a - \vec{x}_{a'}\|$, and

$$\chi_U \in \mathbb{S}_{\{d,c\}}(\mathrm{Conf}_U^{\rightarrow}, \mathrm{Emb}_U^{\rightarrow}), \quad \chi_U \vec{x} := \left\langle \vec{u} \mapsto \vec{x}_a + \frac{(\min \vec{x})\vec{u}}{(\min \vec{x}) + 2\|\vec{u}\|} \right\rangle.$$

Definition 3.2.3 An $E_\infty^{\rightarrow}$–operad is a relative operad

$$\mathcal{E}^{\rightarrow} \in \mathtt{Op}_{\{d,c\}}[\mathtt{Top}]$$

equipped with a relative operad map

$$\Psi \in \mathtt{Op}_{\{d,c\}}[\mathtt{Top}](\mathcal{E}^{\rightarrow}, \mathrm{Emb}_\infty^{\rightarrow})$$

and, for the induced $\mathscr{A}$–filtration $\mathcal{E}_U^{\rightarrow} := \Psi^{-1}\,\mathrm{Emb}_U$, a $\mathbb{S}_{\{d,c\}}$–space homotopy equivalence

$$\Phi_U \in \mathtt{Top}^{\mathbb{S}_{\{d,c\}}^{\mathrm{op}}}(\mathrm{Conf}_U, \mathcal{E}_U^{\rightarrow})$$

for each $U \in \mathscr{A}$ such that $\Psi|_U \Phi_U = \chi_U$.

By this definition the $\mathcal{E}_U^{\rightarrow}$ are m–cofibrant as $\mathbb{S}_{\{d,c\}}$–spaces and $\mathcal{E}^{\rightarrow}$ is contractible and free. One of the main examples of $E_\infty^{\rightarrow}$–operads we will consider is the Steiner relative operad, composed of paths of embeddings [33].

For all $U \in \mathscr{A}$ define the relative operad $\mathscr{H}^{\rightarrow}_U$ as

$$\begin{aligned}\mathscr{H}^{\rightarrow}_U A := \big\{\alpha = \langle \alpha_a \rangle \in U^{\sqcup_A I \times U} \mid \forall a \in A, t \in I, \vec{u}, \vec{v} \in U : \\ (\vec{u} \mapsto \alpha_a(t,\vec{u})) \in \mathrm{Emb}^{\rightarrow}_U\{a\}, \|\alpha_a(t,\vec{u}) - \alpha_a(t,\vec{v})\| \leq \|\vec{u}-\vec{v}\|, \\ \alpha_a(1,\vec{u}) = \vec{u}, \langle \vec{u} \mapsto \alpha_a(0,\vec{u})\rangle \in \mathrm{Emb}^{\rightarrow}_U A\big\}, \\ \langle \alpha_{a'} \rangle \cdot \sigma := \langle \alpha_{\sigma a} \rangle, \quad \mathrm{id}_\star := (t \mapsto \mathrm{id}), \quad \alpha\langle \beta^a \rangle := \langle t \mapsto \alpha_a(t)\beta^a_b(t)\rangle.\end{aligned}$$

Degenerations delete paths of embeddings.

We have natural inclusions $\iota^U_V : \mathscr{H}^{\rightarrow}_U \Rightarrow \mathscr{H}^{\rightarrow}_V$ for all $(U,V) \in \Sigma_{\mathscr{A}} \mathscr{A}_U$ with

$$\iota^U_V \alpha := \langle t \mapsto \pi_{V-U} + \alpha_a(t)\pi_U \rangle$$

and we define $\mathscr{H}^{\rightarrow}_\infty := \mathrm{colim}_{\mathscr{A}} \mathscr{H}^{\rightarrow}_U$.

The $E^{\rightarrow}_\infty$–structural transformations are

$$\begin{aligned}\Psi &: \mathscr{H}^{\rightarrow}_\infty \Rightarrow \mathrm{Emb}^{\rightarrow}_\infty, & \Psi\alpha &:= \langle \alpha_a(0) \rangle, \\ \Phi_U &: \mathrm{Conf}^{\rightarrow}_U \Rightarrow \mathscr{H}^{\rightarrow}_U, & \Phi_U \vec{x} &:= \big\langle t \mapsto (1-t)(\chi_U \vec{x})_a + t\,\mathrm{id}\big\rangle, \\ \overline{\Phi}_U &: \mathscr{H}^{\rightarrow}_U \Rightarrow \mathrm{Conf}^{\rightarrow}_U, & \overline{\Phi}_U \alpha &:= \langle \alpha_a(0,\vec{0})\rangle.\end{aligned}$$

See [33] for the homotopies $\overline{\Phi}_U \Phi_U \cong \mathrm{id}$ and $\Phi_U \overline{\Phi}_U \cong \mathrm{id}$.

3.3 Bar resolution

For the construction of the quasiadjunctions in our main theorems we will require the *bar resolution* of $\mathcal{E}^{\rightarrow}$–pairs. Recall from [21, Construction 9.6] that for a monad (C,η,μ) in the category $\mathcal{T}$, a C–functor (F,λ) in the category $\mathcal{A}$ and a C–algebra (X,ξ) we have the two sided bar construction $B_{-}(F,C,X) \in \mathcal{A}^{\Delta^{\mathrm{op}}}$ with

$$B_{\underline{m}*}(F,C,X) := FC^m X;$$

$$\delta_i := FC^i \eta_{C^{m-i}}, \quad \partial_i := \begin{cases} \lambda_{C^m} & \text{if } i = 0, \\ FC^{i-1}\mu_{C^{m-i+1}} & \text{if } 0 < i < m, \\ FC^{m-1}\xi & \text{if } i = m. \end{cases}$$

In particular for a relative operad $\mathcal{P}$ we have the monad (P,η,μ) constructed in (3) and the P–functor (P,μ). We then have a natural isomorphism

$$B_{\underline{m}*}(P,P,X)_\star \cong \int^{\mathbb{T}_\star \underline{m}_*} \Pi_{\Sigma_{\underline{m}*} V^i} \mathcal{P}\mathrm{in}\,(i,v) \times \Pi_{V^{m+1}} X_{\mathfrak{ce}};$$

$$[\alpha^r, \langle\alpha^{iv}\rangle, \langle x^e\rangle]_T \cdot \partial_{i'} := \begin{cases} [\alpha^r\langle\alpha^{1v'}\rangle, \langle\alpha^{jw}\rangle, \langle x^e\rangle]_{T\cdot\partial_0}, & i'=0, \\ \left[\alpha^r, \left\langle \begin{cases} \alpha^{jw}, & j<i' \\ \alpha^{i'w}\langle\alpha^{(i'+1)v'}\rangle, & j=i' \\ \alpha^{(j+1)w}, & j>i' \end{cases} \right\rangle, \langle x^e\rangle\right]_{T\cdot\partial_{i'}}, & 0<i'<m, \\ [\alpha^r, \langle\alpha^{jw}\rangle, \langle\alpha^{mw}\langle x^{e'}\rangle\rangle]_{T\cdot\partial_m}, & i'=m, \end{cases}$$

$$[\alpha^r, \langle\alpha^{iv}\rangle, \langle x^e\rangle]_T \cdot \delta_{i'} := \left[\alpha^r, \left\langle \begin{cases} \alpha^{jw}, & j\le i' \\ \mathrm{id}_{\mathfrak{c}w}, & j=i'+1 \\ \alpha^{(j-1)w}, & j>i'+1 \end{cases} \right\rangle, \langle x^e\rangle\right]_{T\cdot\delta_{i'}}.$$

The $\mathcal{E}^{\rightarrow}$–pair structural maps in each dimension are

(5) $$\alpha\langle[\beta^{ar}, \langle\beta^{aiv}\rangle, \langle x^{ae}\rangle]_{T^a}\rangle := [\alpha\langle\beta^{ar}\rangle, \langle\beta^{aiv}\rangle, \langle x^{ae}\rangle]_{\Sigma_A T^a}.$$

The bar resolution of $\mathcal{E}^{\rightarrow}$–pairs is then the geometric realization of this simplicial $\mathcal{E}^{\rightarrow}$–pair functor

$$\bar{B}_2\colon \mathcal{E}^{\rightarrow}[\mathtt{Top}] \to \mathcal{E}^{\rightarrow}[\mathtt{Top}], \quad \bar{B}_2 X_\star := |B_-(E^{\rightarrow}, E^{\rightarrow}, X)_\star|.$$

By the above isomorphism we can intuitively think of points in $\bar{B}_2X$ as equivalence classes of filtered rooted relative trees with vertices decorated with elements of $\mathcal{E}^{\rightarrow}$, leaves decorated with elements of X and we associate an ordered partition of I with the filtration of the inner vertices.

It is not the case in general that the geometric realization of a simplicial C–algebra for a topological monad C is a C–algebra. This is however the case when the monad is the one induced by an operad. The structural maps are induced by the algorithm described in [21, Theorem 11.5]. For a sequence of elements with representatives of distinct dimensions we can systematically determine equivalent representatives of the same dimension, and then apply (5). The $\mathcal{E}^{\rightarrow}$–pair structural maps of $\bar{B}_2X$ are induced by the formula

(6) $$\alpha\langle[[\beta^{ar}, \langle\beta^{aiv}\rangle, \langle x^{ae}\rangle]_{T^a}, \langle t^{ai}\rangle]\rangle$$
$$:= \left[\left[\alpha\langle\beta^{ar}\rangle, \left\langle \begin{cases} \beta^{ajw} & \text{if } j=\delta^a(j,w) \\ \mathrm{id}_{\mathfrak{c}w} & \text{if } j\ne\delta^a(j,w) \end{cases} \right\rangle, \langle x^{ae}\rangle\right]_{\Sigma_A T^a\cdot\delta^a}, \lhd_A \langle t^{ai}\rangle\right],$$

which is illustrated in Figure 2.

This functor can be equipped with the natural transformation

(7) $$\eta'\colon \bar{B}_2 \Rightarrow \mathrm{Id}, \quad \eta'_\star[[\alpha^r, \langle\alpha^{iv}\rangle, \langle x^e\rangle]_T, \langle t^i\rangle] := \alpha\langle x^e\rangle,$$

where α is the composition of all the α^v, including α^r, induced by the operadic composition and the structure of T.

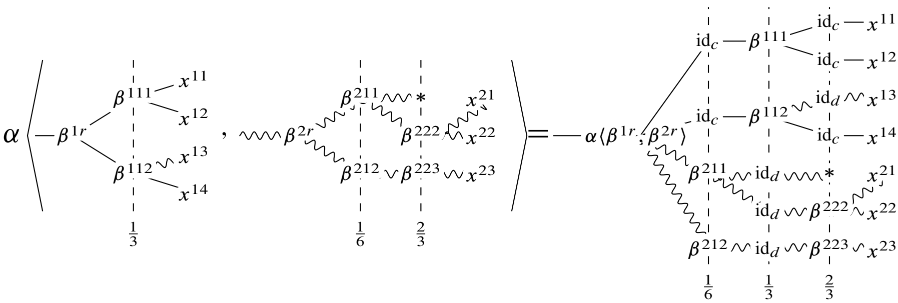

Figure 2: $E^{\rightarrow}_{\infty}$–structure of $\overline{B}_2 X$.

3.4 Relative operad action

Operad actions, introduced in [23, Definition VI.1.6], encode distributive laws between operations defined by operads.[6] The following definition is a relative version of this notion.

Definition 3.4.1 A *relative operad pair* is a pair of relative operads $(\mathcal{P}, \mathcal{G})$ equipped with an extension of $\mathcal{G}$ to $\mathbb{S}^{\mathrm{op}}_{\langle d<c\rangle}$ and an *action* of $\mathcal{G}$ on $\mathcal{P}$ defined by structural maps

$$\langle \ltimes_{A,\langle B^a\rangle} \rangle \in \Pi_{\Sigma_{\mathbb{S}_{\{d,c\}}}} \Pi_A \mathbb{S}_{\mathfrak{c}a} \mathtt{Top}(\mathcal{G}A \times \Pi_A \mathcal{P} B^a, \mathcal{P}\Pi_A B^a)$$

such that, using the notations $\mathcal{P}\varnothing_\star = \{0_\star\}$, $\mathcal{G}\varnothing_\star = \{1_\star\}$, $f \ltimes \langle \alpha^a \rangle := \ltimes_{A,\langle B^a\rangle}(f, \langle \alpha^a \rangle)$ and $f \ltimes \langle \beta^{ab^a} \rangle := f \cdot \pi_{\langle b^a \rangle} \ltimes \langle \beta^{ab^a} \rangle$ the following equations are satisfied:

- $f \ltimes \langle g^a \ltimes \langle \alpha^{ab} \rangle \rangle = f \langle g^a \rangle \ltimes \langle \alpha^{ab} \rangle$.
- $f \ltimes \langle \alpha^a \langle \beta^{ab} \rangle \rangle = f \ltimes \langle \alpha^a \rangle \langle f \ltimes \langle \beta^{ab^a} \rangle \rangle \cdot \nu$.
- $\mathrm{id}_{\mathfrak{c}A} \ltimes \alpha = \alpha$.
- $f \ltimes \langle \mathrm{id}_{\mathfrak{c}a} \rangle = \mathrm{id}_{\mathfrak{c}A}$.
- $f \cdot \sigma \ltimes \langle \alpha^a \rangle = f \ltimes \langle \alpha^{\sigma^{-1}a'} \rangle \cdot \sigma \langle B^a \rangle$.
- $f \ltimes \langle \alpha^a \cdot \tau^a \rangle = f \ltimes \langle \alpha^a \rangle \cdot \Pi_A \tau^a$.
- $1_\star \ltimes * = \mathrm{id}_\star$.
- There exists an $a \in A$ such that $\alpha^a = 0_{\mathfrak{c}a}$ implies $f \ltimes \langle \alpha^a \rangle = 0_{\mathfrak{c}A}$.

[6]The original reference has typos corrected in [24, Definition 1.8]. The reader should keep in mind that the literature on pairings of operads has been plagued by errors, in particular in how it is applied to the study of K–theory through bipermutative categories. Erratas can be found in the appendix of [24] and the introduction of [10].

We refer to the operad $\mathcal{P}$ as the *additive relative operad* and $\mathcal{G}$ as the *multiplicative relative operad* of the pair.

For $X = ((X_d, 0_d, 1_d), (X_c, 0_c, 1_c)) \in \mathtt{Top}^2_{\mathbb{S}^0}$ define

$$X^{\wedge -}\colon \mathbb{S}^{\mathrm{inj}}_{\{d,c\}} \to \mathtt{Top}; \quad X^{\wedge A} := \wedge_A X_{\mathfrak{c}a}, \quad \sigma\cdot[x^a] := \left[\begin{cases} 1_{\mathfrak{c}a'} & \text{if } a' \notin \operatorname{Im}\sigma \\ x^{\sigma^{-1}a'} & \text{if } a' \in \operatorname{Im}\sigma \end{cases}\right],$$

with the zeros as basepoints for the wedge products. We can then define the monad $(G_0;\eta,\mu)$ on $\mathtt{Top}^2_{\mathbb{S}^0}$ as

$$G_0X_\star := \int^{\mathbb{S}^{\mathrm{inj}}_\star} \mathcal{G}A_+ \wedge X^{\wedge A}; \quad \eta_\star x := [\mathrm{id}_\star, x], \quad \mu_\star\big[f,[g^a,[x^{ab}]]\big] := [f\langle g^a\rangle,[x^{ab}]].$$

Definition 3.4.2 A $\mathcal{G}_0$–*space* is a G_0–algebra, ie a pair of $\mathbb{S}^0$–spaces $X \in \mathtt{Top}^2_{\mathbb{S}^0}$ equipped with a structural map $\chi\colon G_0X \to X$ satisfying, using $f[x^a] = \chi_A[f,[x^a]]$, similar equations as in Definition 3.2.2 and also that 0 is an absorbing element, ie

$$\text{there exists } a \in A \text{ such that } x^a = 0_{\mathfrak{c}a} \implies f[x^a] = 0_{\mathfrak{c}A}.$$

The category of $\mathcal{G}_0$–spaces is denoted by $\mathcal{G}_0[\mathtt{Top}]$.

If $\mathcal{G}$ acts on $\mathcal{P}$ then the functor P induces a monad on $\mathcal{G}_0[\mathtt{Top}]$.

Definition 3.4.3 Let $(\mathcal{P},\mathcal{G})$ be a relative operad pair. A $(\mathcal{P},\mathcal{G})$–space is a P–algebra in $\mathcal{G}_0[\mathtt{Top}]$. Equivalently a $(\mathcal{P},\mathcal{G})$–space is a pair of $\mathbb{S}^0$–spaces $X \in \mathtt{Top}^2_{\mathbb{S}^0}$ equipped with a $\mathcal{G}_0$–space structure and a $\mathcal{P}$–space structure with neutral elements the zeros such that

$$f[\alpha^a\langle x^{ab}\rangle] = f \ltimes \langle\alpha^a\rangle\langle f[x^{ab^a}]\rangle.$$

The category of $(\mathcal{P},\mathcal{G})$–spaces is denoted $(\mathcal{P},\mathcal{G})[\mathtt{Top}]$.

There is a natural operad pair structure on $(\mathrm{Com}^{\to},\mathrm{Com}^{\to})$. Denote by $\sum_A \in \mathrm{Com}^{\to} A$ the additive copy of $\mathrm{Com}^{\to}$ and $\prod_A \in \mathrm{Com}^{\to} A$ the multiplicative copy of $\mathrm{Com}^{\to}$. Then in a $(\mathrm{Com}^{\to},\mathrm{Com}^{\to})$–space the distributivity equations and the equality of the additive and multiplicative homomorphisms

$$\begin{aligned}
\textstyle\prod_A \sum_{B^a} x^{ab} &= \textstyle\sum_{\prod_A B^a} \prod_{A_{\langle b^a\rangle}} x^{ab^a} \\
\phi_+ x &= \textstyle\prod_{\{1_c,2_c\}_c} \langle \phi_+ x, 1_c\rangle \\
&= \textstyle\prod_{\{1_c,2_c\}_c} \ltimes \langle \phi_+, \mathrm{id}_c\rangle \prod_{\{1_d,2_c\}_c} \langle x, 1_c\rangle \\
&= \phi.x
\end{aligned}$$

hold. This means that $(\mathrm{Com}^{\to}, \mathrm{Com}^{\to})[\mathtt{Top}]$ is isomorphic to the category of topological commutative semialgebras over commutative semirings.

The main example of multiplicative relative operad we will consider is the *relative linear isometries operad* $\mathscr{L}^{\to}$ with

$$\mathscr{L}^{\to}A := \mathscr{I}(\oplus_A \mathbb{R}^\infty, \mathbb{R}^\infty); \quad f \cdot \sigma := \langle f_{\sigma a}\rangle, \quad \mathrm{id} := id_{\mathbb{R}^\infty}, \quad f\langle g^a\rangle := \langle f_a g^a_b\rangle.$$

The identity maps provide a natural extension of $\mathscr{L}^{\to}$ to $\mathbb{S}^{\mathrm{op}}_{\langle d<c\rangle}$. We have a natural action of $\mathscr{L}^{\to}$ on $\mathrm{Emb}^{\to}_\infty$ given by the formula

$$f \ltimes \langle \alpha^a\rangle := \langle \pi_{f^\perp} + \textstyle\sum_A f_a \alpha^a_{b^a} f^*_a\rangle.$$

This naturally extends to an action on $\mathscr{H}^{\to}_\infty$ given by the formula

$$f \ltimes \langle \alpha^a\rangle := \langle t \mapsto \pi_{f^\perp} + \textstyle\sum_A f_a \alpha^a_{b^a}(t) f^*_a\rangle. \tag{8}$$

Definition 3.4.4 The category of $E^{\to}_\infty$–algebras is $(\mathscr{E}^{\to}, \mathscr{L}^{\to})[\mathtt{Top}]$ for an $E^{\to}_\infty$–operad $\mathscr{E}^{\to}$ equipped with an action by $\mathscr{L}^{\to}$ that is preserved by the structural relative operad morphism $\Psi\colon \mathscr{E}^{\to} \Rightarrow \mathrm{Emb}^{\to}_\infty$.

Steiner's original argument in [33] implies the relative operad $\mathscr{H}^{\to}_\infty$ equipped with the action (8) satisfies the conditions of Definition 3.4.4. Proof of the compatibility with the coloring is straightforward and will be omitted.

Although we give this general definition we note that there is no known nontrivial example of an E_∞–operad equipped with an $\mathscr{L}$–action other then the Steiner operad $\mathscr{H}_\infty$. Having a q–cofibrant, not just mixed Σ–cofibrant example would be interesting and useful, but since we can work in the mixed model structure it is not necessary.

The images of $\overline{B}_2X$ are also $\mathscr{L}^{\to}_0$–pairs with structural maps defined as

$$\begin{aligned} &f\big[[[\alpha^{ar}, \langle \alpha^{aiv}\rangle, \langle x^{ae}\rangle]_{T^a}, \langle t^{ai}\rangle]\big] \\ &:= \left[\left[f \ltimes \langle \alpha^{ar}\rangle, \left\langle f \ltimes \left\langle \begin{cases} \alpha^{ajw^a}, & j = \delta^a(j, w^a) \\ \mathrm{id}_{\mathfrak{c}w^a}, & j \neq \delta^a(j, w^a) \end{cases} \right\rangle\right\rangle, \langle f[x^{ae^a}]\rangle\right]_{\prod_A T^a \cdot \delta^a}, \lhd_A \langle t^{ai}\rangle\right] \end{aligned} \tag{9}$$

which is illustrated in Figure 3.

3.5 Mixed model structure of $E^{\to}_\infty$–algebras

In [3, Theorem 2.1] Berger and Moerdijk construct a q–model structure on categories of algebras over colored operads by transferring cofibrantly generated model structures

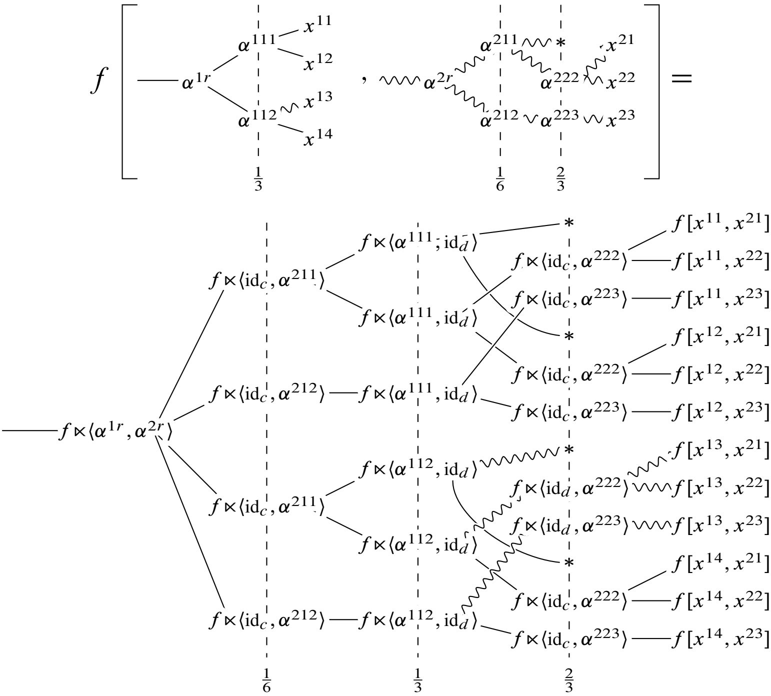

Figure 3: $\mathscr{L}^{\to}$–structure of $\bar{B}_2 X$.

from the underlying monoidal model category. Their method generalizes to the $E_\infty^{\to}$–algebra context, at least in the topological context we are interested in this article. In [2] Barthel and Riehl show how to transfer h–model structures, which gives us a mixed model structure of $E_\infty^{\to}$–algebras.

Let $(S \dashv \Lambda)\colon \mathcal{T} \leftrightarrows \mathcal{A}$ be an adjunction and suppose $\mathcal{T}$ is equipped with a model structure. Define a morphism ϕ of $\mathcal{A}$ to be a weak equivalence or fibration if $\Lambda\phi$ is respectively a weak equivalence or fibration. It is often possible to extend these distinguished classes of morphisms to a model structure on $\mathcal{A}$, and in this case we say the model structure of $\mathcal{A}$ is transferred from the one on $\mathcal{T}$. For instance, an extension exists if the model structure of $\mathcal{T}$ is cofibrantly generated and Crans' transfer criteria are met [8, Theorem 3.3]:

(i) The left adjoint S preserves small objects, ie if $\mathcal{T}(X, -)$ preserves filtered colimits then $\mathcal{A}(SX, -)$ also does.

(ii) Any sequential colimit of pushouts of images under S of the generating trivial cofibrations of $\mathcal{T}$ yields a weak equivalence in $\mathcal{A}$.

The first criterion is often easily verified; for instance it holds if the right adjoint Λ preserves filtered colimits. The second can be harder to verify, but the existence of path-objects yields condition (ii). Recall that a path object of X is a factorization of its diagonal into a weak equivalence followed by a fibration $X \xrightarrow{\sim} \mathrm{Path}(X) \twoheadrightarrow X \times X$. By Quillen's path-object argument, the conditions

(a) $\mathcal{A}$ has a fibrant replacement functor,

(b) $\mathcal{A}$ has functorial path-objects for fibrant objects,

imply Crans' condition (ii) [26, Chapter II, page 4.9; 34, Lemma A.4].

The Quillen model structure of $E_\infty^{\to}$–algebras is transferred from the q–model structure of $\mathtt{Top}^2_{\mathbb{S}0}$ by the adjunction

$$(E^{\to} L_0^{\to} \dashv U)\colon \mathtt{Top}^2_{\mathbb{S}0} \leftrightarrows (\mathcal{E}^{\to}, \mathcal{L}^{\to})[\mathtt{Top}],$$

where $E^{\to} L_0^{\to}$ is the composition of the free $\mathcal{L}_0^{\to}$–algebra functor followed by the free $\mathcal{E}^{\to}$–algebra functor, both left adjoint to forgetful functors.

The forgetful functors preserve filtered colimits, so Crans' condition (i) is satisfied. All objects of $\mathtt{Top}^2_{\mathbb{S}0}$ are fibrant, so the same is true in $(\mathcal{E}^{\to}, \mathcal{L}^{\to})[\mathtt{Top}]$. Now note that for all $X \in (\mathcal{E}^{\to}, \mathcal{L}^{\to})[\mathtt{Top}]$, the pair of Moore spaces $MX := (MX_d, MX_c)$ equipped with the pointwise operations

$$f[(t^a, \gamma^a)] := (\max_A t^a, r \mapsto f[\gamma^a r]), \quad \alpha\langle(t^a, \gamma^a)\rangle := (\max_A t^a, r \mapsto \alpha\langle\gamma^a r\rangle)$$

is an $E_\infty^{\to}$–algebra. The inclusion $\iota\colon X \to MX$ of constant paths is a homotopy equivalence, and the evaluations at the start and end $(\mathrm{ev}_0, \mathrm{ev}_\infty)\colon MX \to X \times X$ is a fibration. We therefore have a functorial construction of path-objects, and so a transferred q–model structure on the category of $E_\infty^{\to}$–algebras.

Even though the h–model structure on $\mathtt{Top}^2_{\mathbb{S}0}$ is not cofibrantly generated it can still be transferred by the adjunction $(E^{\to} L_0^{\to} \dashv U)$, with the h–cofibrations defined as the maps with the left lifting properties against the trivial h–fibrations [2]. For all $\phi \in (\mathcal{E}^{\to}, \mathcal{L}^{\to})[\mathtt{Top}](X, Y)$ the pair $\Gamma\phi := (\Gamma\phi_d, \Gamma\phi_c)$, equipped with the pointwise operations

$$f[(x^a, t^a, \gamma^a)] := (f[x^a], \max_A t^a, r \mapsto f[\gamma^a r]),$$
$$\alpha\langle(x^a, t^a, \gamma^a)\rangle := (\alpha\langle x^a\rangle, \max_A t^a, r \mapsto \alpha\langle\gamma^a r\rangle)$$

is an $E_\infty^{\to}$–algebra. Then $(\Gamma; C_t, F)$ forms an algebraic weak factorization system in $(\mathcal{E}^{\to}, \mathcal{L}^{\to})[\mathtt{Top}]$. On the other hand there doesn't seem to be any natural $E_\infty^{\to}$–algebra structure on $E\phi$ such that the h–cofibration/trivial h–fibration factorization $(E; C, F_t)$ in $\mathtt{Top}^2_{\mathbb{S}^0}$ induces a factorization in $(\mathscr{E}^{\to}, \mathscr{L}^{\to})[\mathtt{Top}]$. We do have an h–cofibration/h–equivalence factorization

$$X \xhookrightarrow{\mathrm{in}_X} X \sqcup_{E^{\to}L_0^{\to}X} E^{\to}L_0^{\to}(\Gamma\phi \times [0,\infty]) \sqcup_{E^{\to}L_0^{\to}\Gamma\phi} Y \xrightarrow[\cong]{(\phi, F_t\phi^{\dagger}, \mathrm{id})} Y,$$

and the fact that $C_t\phi$ has the left lifting property against h–fibrations in $\mathtt{Top}^2_{\mathbb{S}^0}$ induces the left lifting property against h–fibrations in $(\mathscr{E}^{\to}, \mathscr{L}^{\to})[\mathtt{Top}]$ on in_X. The map $(\phi, F_t\phi^{\dagger}, \mathrm{id})$ is an h–equivalence, but it is not necessarily an h–fibration. Applying $(\Gamma; C_t, F)$ then gives us the h–cofibration/trivial h–fibration factorization

$$X \xhookrightarrow{C_t(\phi, F_t\phi^{\dagger}, \mathrm{id})\mathrm{in}_X} \Gamma(\phi, F_t\phi^{\dagger}, \mathrm{id}) \xrightarrow[\cong]{F(\phi, F_t\phi^{\dagger}, \mathrm{id})} Y,$$

which determines the strict Hurewicz/Strøm model structure on $(\mathscr{E}^{\to}, \mathscr{L}^{\to})[\mathtt{Top}]$. We then have a mixed model structure by Cole's construction [7].

4 Recognition of algebra spectra

4.1 Coordinate-free spectra

We give a brief review of coordinate-free spectra [17] and some examples. Let $\mathbb{U} \in \mathscr{I}$ be countably infinite-dimensional (in the context of coordinate-free spectra we refer to $\mathbb{U}$ as a *universe*). The topological category $\mathtt{Sp}_{\mathbb{U}}$ of coordinate-free $\mathbb{U}$–spectra is composed of the class of objects

$$\{Y = (\langle Y_U\rangle, \langle\sigma^U_V\rangle) \in \Sigma_{\Pi_{\mathscr{A}}\mathtt{Top}_*} \Pi_{\Sigma_{\mathscr{A}}\mathscr{A}_U} \mathtt{Top}_*(Y_U \wedge \mathbb{S}^{V-U}, Y_V) \mid \sigma^U_U[y, \vec{0}] = y,\ \sigma^V_W[\sigma^U_V[y, \vec{v}], \vec{w}] = \sigma^U_W[y, \vec{v} + \vec{w}]\}$$

and morphism spaces

$$\mathtt{Sp}_{\mathbb{U}}(Y, Z) := \{\mathfrak{f} = \langle \mathfrak{f}_U\rangle \in \Pi_{\mathscr{A}}\mathtt{Top}_*(Y_U, Z_U) \mid \sigma^U_V[\mathfrak{f}_U y, \vec{v}] = \mathfrak{f}_V\sigma^U_V[y, \vec{v}]\}.$$

We are particularly interested here in the case $\mathbb{U} = \mathbb{R}^{\infty}$, and in this case we use the notation $\mathtt{Sp} := \mathtt{Sp}_{\mathbb{R}^{\infty}}$.

Example 4.1.1 Interesting coordinate-free spectra to keep in mind are the following, with details similar to the equivalent symmetric examples in [29, Section I.2]:

- For each $p \in \mathbb{Z}$ the p–sphere spectrum is defined as

$$\mathbb{S}^p := \begin{cases} \langle \mathbb{S}^{U-\mathbb{R}^{|p|}}\rangle,\ \sigma^U_V[\vec{u}, \vec{v}] := \vec{u} + \pi_{V-\mathbb{R}^{|p|}}\vec{v} & \text{if } p < 0, \\ \langle \mathbb{S}^U\rangle,\ \sigma^U_V[\vec{u}, \vec{v}] := \vec{u} + \vec{v} & \text{if } p = 0, \\ \langle \mathbb{S}^{U\oplus\mathbb{R}^p}\rangle,\ \sigma^U_V[(\vec{u}, \vec{w}), \vec{v}] := (\vec{u} + \vec{v}, \vec{w}) & \text{if } p > 0. \end{cases}$$

We use the notation $\mathbb{S} := \mathbb{S}^0$.

- For each $G \in \mathtt{AbGrp}$, define the *Eilenberg–Mac Lane* spectrum

$$HG := \langle G \otimes F[\mathbb{S}^U]_* \rangle, \quad \sigma_V^U[g_a \otimes \vec{u}^a, \vec{v}] := g_a \otimes \vec{u}^a + \vec{v},$$

where $F[\mathbb{S}^U]_*$ denotes the quotient of the free abelian group generated by the points of the U–sphere by the subgroup generated by ∞, and as in the Einstein convention $g_a \otimes \vec{u}^a$ indicates a finite sum of elements. Note that $g \otimes \infty = 0$.

- For each $U \in \mathscr{A}$ let O_U be the orthogonal group of isometric automorphisms of U. The total space EO_U of the universal principal O_U–bundle is the geometric realization of the simplicial space $O_U^- \in \mathtt{Top}^{\Delta^{\mathrm{op}}}$ with

$$\langle f^j \rangle \cdot \partial_i := \left\langle \begin{cases} f^{j'} & \text{if } j' < i \\ f^i f^{i+1} & \text{if } j' = i \\ f^{j'+1} & \text{if } j' > i \end{cases} \right\rangle, \quad \langle f^j \rangle \cdot \delta_i := \left\langle \begin{cases} f^j & \text{if } j < i+1 \\ \mathrm{id}_U & \text{if } j = i+1 \\ f^{j-1} & \text{if } j > i+1 \end{cases} \right\rangle.$$

The U–spheres admit a left O_U–action by evaluation $f \cdot \vec{u} := f\vec{u}$, and EO_{U+} admits the right O_U–action

$$[\langle g^i \rangle, \langle t^i \rangle] \cdot f := \left[\left\langle \begin{cases} g^i & \text{if } i < m \\ g^m f & \text{if } i = m \end{cases} \right\rangle, \langle t^i \rangle \right].$$

For $(U, V) \in \Sigma_{\mathscr{A}} \mathscr{A}_U$ we have a natural inclusion

$$\iota_V^U : O_U \to O_V, \quad \iota_V^U f := \pi_{V-U} + f \pi_U.$$

The *Thom spectrum* is

$$MO := \langle EO_{U+} \wedge_{O_U} \mathbb{S}^U \rangle, \quad \sigma_V^U[[\langle f^i \rangle, \langle t^i \rangle, \vec{u}], \vec{v}] := [\langle \iota_V^U f^i \rangle, \langle t^i \rangle, \vec{u} + \vec{v}].$$

An Ω–spectrum is a spectrum $Y \in \mathtt{Sp}$ such that the adjoint structural maps $\tilde{\sigma}_V^U \in \mathtt{Top}_*(Y_U, Y_V^{\mathbb{S}^{V-U}})$ are q–equivalences.

The stable homotopy groups of spectra are $\pi_p^S Y := \pi_0 \mathtt{Sp}(\mathbb{S}^p, Y)$. If Y is an Ω–spectrum then

$$\pi_p^S Y \cong \begin{cases} \pi_0 Y_{\mathbb{R}^{|p|}} & \text{if } p < 0, \\ \pi_p Y_0 & \text{if } p \geq 0. \end{cases}$$

Spectra maps that induce isomorphisms of the stable homotopy groups are called *stable weak equivalences*, and spectra $Y \in \mathtt{Sp}$ with $\pi_p^S Y$ trivial for $p < 0$ are called *connective*.

The base space functor is

$$\Lambda^\infty : \mathtt{Sp} \to \mathtt{Top}_*, \quad \Lambda^\infty Y := Y_0,$$

which is right adjoint to the suspension spectrum functor

$$\Sigma^\infty\colon \mathtt{Top}_* \to \mathtt{Sp}; \quad \Sigma^\infty X := \langle X \wedge \mathbb{S}^U \rangle, \quad \sigma_V^U[[x,\vec{u}],\vec{v}] := [x,\vec{u}+\vec{v}],$$

with adjunction unit and counit

$$\eta x := [x,\vec{0}], \quad \epsilon_U[y,\vec{u}] := \sigma_U^0[y,\vec{u}].$$

4.2 Stable mixed model structure of spectra

For any spectrum $Y \in \mathtt{Sp}$ the cylinder spectrum is $Y \wedge I_+ := \langle Y_U \wedge I_+ \rangle$, and the cone spectrum is $CY := Y \wedge I_+/_{[y,1]\sim[y',1]}$.

In the strict Quillen model structure on $\mathtt{Sp}$ a morphism $\mathfrak{f} \in \mathtt{Sp}(X,Y)$ is a weak equivalence if each $\mathfrak{f}_U$ is a q–equivalence, a fibration if it is a Serre fibration, ie if it has the homotopy lifting property with respect to the cylinder inclusions of cones of sphere spectra $\mathrm{in}_0 \in \mathtt{Sp}(C\mathbb{S}^q, C\mathbb{S}^q \wedge I_+)$ for all $q \in \mathbb{Z}$, and a cofibration if it is a retract of a relative cell-spectrum, with cells given by cones of sphere spectra and domain of the attaching maps the boundary sphere spectra [9, Section VII.4]. This is a cofibrantly generated model structure with factorization systems induced by the small object argument. The weak equivalences, fibrations and cofibrations of this model structure are referred to as q–equivalences, q–fibrations and q–cofibrations, respectively.

Homotopy equivalences in $\mathtt{Sp}$ are spectra maps that admit an inverse up to homotopy, with homotopies defined via the cylinder spectra in the usual way. In the strict Hurewicz/Strøm model structure $\mathfrak{f}$ is a weak equivalence if it is a homotopy equivalence, a fibration if it is a Hurewicz fibration, ie if it has the homotopy lifting property with respect to all cylinder inclusions $\mathrm{in}_0 \in \mathtt{Sp}(X, X \wedge I_+)$, and a cofibration if it has the left lifting property against trivial Hurewicz fibrations.

The weak factorization system can be constructed through (co)monads as described in [2]. For any $Y \in \mathtt{Sp}$ let the spectrum of Moore paths in Y be

$$MY := \langle (MY_U, (0, r \mapsto y_{U0})) \rangle, \quad \sigma_V^U[(t,\gamma),\vec{v}] := (t, r \mapsto \sigma_V^U[\gamma r, \vec{v}]).$$

The factorization systems are then defined as $(\Gamma\mathfrak{f}; C_t\mathfrak{f}, F\mathfrak{f}) := (\langle \Gamma\mathfrak{f}_U \rangle; \langle C_t\mathfrak{f}_U \rangle \langle F\mathfrak{f}_U \rangle)$ and $(E\mathfrak{f}; C\mathfrak{f}, F_t\mathfrak{f}) := (\langle E\mathfrak{f}_U \rangle; \langle C\mathfrak{f}_U \rangle \langle F_t\mathfrak{f}_U \rangle)$ for all spectra maps $\mathfrak{f}$.

The weak equivalences, fibrations and cofibrations of this model structure are referred to as h–equivalences, h–fibrations and h–cofibrations, respectively. We then equip $\mathtt{Sp}$ with the mixed model structure as described in [7, Proposition 3.6].

Since the point of spectra is to study stabilization phenomena we are actually interested in inverting the stable weak homotopy equivalences. From the strict model structure the process of Bousfield localization constructs the stable model structure, with stable weak homotopy equivalences as weak equivalence [4; 28]. For every spectrum $Y \in \mathtt{Sp}$ we can functorially define an inclusion spectrum[7] $\widetilde{Y}$ equipped with a quotient map $Y \to \widetilde{Y}$, so we may think of points in $\widetilde{Y}$ as equivalence classes of points in Y [17, Appendix 1]. If Y is already an inclusion spectrum then $\widetilde{Y} = Y$. The *spectrification functor* is

$$\widetilde{\Omega}\colon \mathtt{Sp} \to \mathtt{Sp}; \quad \widetilde{\Omega} Y := \langle \operatorname*{colim}_{\mathscr{A}_U} \widetilde{Y}_V^{\$^{V-U}} \rangle, \quad \sigma_W^U[\gamma, \vec{w}] := [\vec{v} \mapsto \gamma(\vec{v} + \vec{w})],$$

induced by the adjoint structural maps $\tilde{\sigma}$ and with the formula for the structural maps determined by a choice of representative γ with domain $V \in \mathscr{A}_W$. This is a Quillen idempotent monad with structural natural map

$$\epsilon'\colon \mathrm{Id} \Rightarrow \widetilde{\Omega}, \quad \epsilon'_U y := [\vec{v} \mapsto \sigma_V^U[y, \vec{v}]]. \tag{10}$$

The stable model structure on spectra $\mathtt{Sp}_{\widetilde{\Omega}}$ has as weak equivalences the stable weak equivalences, and stable fibrations are $\mathfrak{p} \in \mathtt{Sp}(E, B)$ composed of indexwise Hurewicz fibrations such that the maps

$$(\tilde{\sigma}_V^U, \mathfrak{p}_U)\colon E_U \to E_V^{\$^{V-U}} \times_{B_V^{\$^{V-U}}} B_U$$

are q–equivalences. The fibrant spectra are the Ω–spectra, and the cofibrant spectra are those homotopy equivalent to retracts of q–cofibrant spectra. With the induced stable model structure the adjunction $(\Sigma^\infty \dashv \Lambda^\infty)$ is a Quillen adjunction.

The morphisms category $\mathtt{Sp}^{\to}$ admits a projective stable model structure with $(\mathfrak{f}_d, \mathfrak{f}_c) \in \mathtt{Sp}^{\to}(\mathfrak{i}\colon Y_d \to Y_c, \mathfrak{j}\colon Z_d \to Z_c)$ a weak equivalence or fibration if $\mathfrak{f}_d$ and $\mathfrak{f}_c$ are both stable weak equivalences or stable fibrations, respectively; and it is a cofibration if both $\mathfrak{f}_d$ and $(\mathfrak{f}_c, \mathfrak{j})\colon Y_c \vee_{Y_d} Z_d \to Z_c$ are stable cofibrations.

4.3 Recognition of ∞–loop maps

We can now prove the recognition principle for ∞–loop pairs of spaces of spectra maps. The base pair of spaces functor is

$$\Lambda_2^\infty\colon \mathtt{Sp}^{\to} \to \mathtt{Top}_*^2, \quad \Lambda_2^\infty(\mathfrak{i}\colon Y_d \to Y_c) := (Y_{d0}, Y_{c0}),$$

[7] Inclusion spectra are those with adjoint structural maps $\tilde{\sigma}$ all inclusions.

and the relative suspension functor is

$$\Sigma^\infty_\rightarrow\colon \mathtt{Top}^2_* \to \mathtt{Sp}^\rightarrow, \quad \Sigma^\infty_\rightarrow X := \Sigma^\infty(\mathrm{in}_d\colon X_d \to X_d \vee X_c).$$

We have a Quillen adjunction

$$(\Sigma^\infty_\rightarrow \dashv \Lambda^\infty_2)\colon \mathtt{Top}^2_* \leftrightarrows \mathtt{Sp}^\rightarrow; \quad \eta_\star x := [x, \vec{0}], \quad \epsilon_{\star U}[y,\vec{u}] := \left[\begin{cases} \sigma^0_U[y,\vec{u}] & \text{if } \mathfrak{c}y = \star \\ \sigma^0_U[\mathfrak{i}y,\vec{u}] & \text{if } \mathfrak{c}y \neq \star \end{cases}\right].$$

The spectrification functor $\widetilde{\Omega}$ induces

$$\widetilde{\Omega}_\rightarrow\colon \mathtt{Sp}^\rightarrow \to \mathtt{Sp}^\rightarrow, \quad \widetilde{\Omega}_\rightarrow \mathfrak{i} := (\widetilde{\Omega}\mathfrak{i}\colon \widetilde{\Omega}Y_d \to \widetilde{\Omega}Y_c).$$

The ∞–loop pair of spaces functor is defined as

$$\Omega^\infty_2\colon \mathtt{Sp}^\rightarrow \to \mathcal{E}^\rightarrow[\mathtt{Top}], \quad \Omega^\infty_2 \mathfrak{i} := \Lambda^\infty_2 \widetilde{\Omega}_\rightarrow \mathfrak{i}$$

with structural maps induced by the formula (4) by taking representatives of the γ^a with a common domain.

This functor is not a right adjoint, but it is a weak Quillen right quasiadjoint. The left quasiadjoint functor is defined as follows: We have simplicial pointed maps $B_-(\Sigma^U_\rightarrow, E^\rightarrow_U, X) \in (\mathtt{Top}^\rightarrow_*)^{\Delta^{\mathrm{op}}}$ with

$$B_{\underline{m}*}(\Sigma^U_\rightarrow, E^\rightarrow_U, X)_\star \cong \mathbb{S}^U \wedge \int^{\mathbb{T}^0_\star \underline{m}_*} \Pi_{\Sigma_{\underline{m}} V^i} \mathcal{E}^\rightarrow \mathrm{in}(i,v) \times \Pi_{V^{m+1}} X_{\mathfrak{c}e};$$

$$[\vec{u}, \langle \alpha^{iv} \rangle, \langle x^e \rangle]_T \cdot \partial_{i'} := \begin{cases} \begin{cases} [\alpha^{1,-1}_{v'} \vec{u}, \langle \alpha^{jw} \rangle, \langle x^{e'} \rangle]_{T_{\geq v'}}, & \vec{u} \in \alpha^1_{v'} U \\ \infty, & \vec{u} \notin \alpha^1 \sqcup_{V^2} U \end{cases} & \text{if } i' = 0, \\ \left[\vec{u}, \left\langle \begin{cases} \alpha^{jw}, & j < i' \\ \alpha^{i'w} \langle \alpha^{(i'+1)v'} \rangle, & j = i' \\ \alpha^{(j+1)w}, & j > i' \end{cases} \right\rangle, \langle x^e \rangle \right]_{T \cdot \partial_i} & \text{if } 0 < i' < m, \\ [\vec{u}, \langle \alpha^{jw} \rangle, \langle \alpha^{mw} \langle x^{e'} \rangle \rangle]_{T \cdot \partial_m} & \text{if } i' = m, \end{cases}$$

$$[\vec{u}, \langle \alpha^{iv} \rangle, \langle x^e \rangle]_T \cdot \delta_{i'} := \left[\vec{u}, \left\langle \begin{cases} \alpha^{jw}, & j \leq i' \\ \mathrm{id}_{\mathfrak{c}w}, & j = i'+1 \\ \alpha^{(j-1)w}, & j > i'+1 \end{cases} \right\rangle, \langle x^e \rangle \right]_{T \cdot \delta_{i'}}.$$

Define the relative ∞–delooping functor as

$$B^\infty_\rightarrow\colon \mathcal{E}^\rightarrow[\mathtt{Top}] \to \mathtt{Sp}^\rightarrow, \quad B^\infty_\rightarrow X_\star := \langle |B_-(\Sigma^U_\rightarrow, E^\rightarrow_U, X)_\star| \rangle;$$

$$\sigma^U_V\big[[[\vec{u}, \langle \alpha^{iv} \rangle, \langle x^e \rangle]_T, \langle t^i \rangle], \vec{v}\big] := [[\vec{u} + \vec{v}, \langle \alpha^{iv} \rangle, \langle x^e \rangle]_T, \langle t^i \rangle].$$

Points in $B^\infty_\rightarrow X_{\star U}$ are equivalence classes of decorated filtered rooted relative trees as in the description of the bar resolution $\overline{B}_2 X$, except the root vertex is decorated with a vector in U and the relative operad points decorating the inner vertices must be contained in the suboperad $\mathcal{E}^\rightarrow_U$ of the $\mathscr{A}$–filtration of $\mathcal{E}^\rightarrow$.

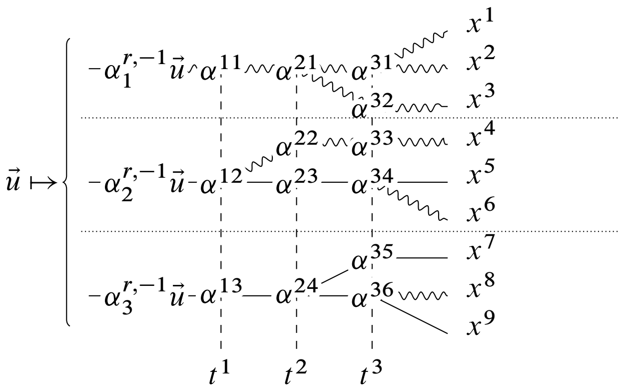

Figure 4: Representative U–loop of $\eta_c[[\alpha^r, \langle\alpha^{iv}\rangle, \langle x^e\rangle]_T, \langle t^i\rangle]$.

Theorem 4.3.1 *For $\mathcal{E}^\rightarrow$ an $E_\infty^\rightarrow$–operad we have a quasiadjunction*

$$(B_\rightarrow^\infty \dashv_{\bar{B}_2, \tilde{\Omega}_\rightarrow} \Omega_2^\infty): \mathcal{E}^\rightarrow[Top] \leftrightharpoons Sp^\rightarrow.$$

Proof The unit span and cospan has η' the natural weak equivalence (7), ϵ' induced by the idempotent monad transformation (10) and η and ϵ are defined by the formulas

$$\eta: \bar{B} \Rightarrow \Omega_2^\infty B_\rightarrow^\infty, \quad \epsilon: B_\rightarrow^\infty \Omega_2^\infty \Rightarrow \tilde{\Omega}_\rightarrow;$$

$$\eta_\star[[\alpha^r, \langle\alpha^{iv}\rangle, \langle x^e\rangle]_T, \langle t^i\rangle] := \left[\vec{u} \mapsto \begin{cases} [[\alpha_{v'}^{r,-1}\vec{u}, \langle\alpha^{iv}\rangle, \langle x^e\rangle]_{T_{\geq v'}}, \langle t^i\rangle], & \vec{u} \in \alpha_{v'}^r U \\ \infty, & \vec{u} \notin \alpha^r \sqcup_{V^1} U \end{cases}\right],$$

$$\epsilon_{\star U}[[\vec{u}, \langle\alpha^{iv}\rangle, \langle\gamma^e\rangle]_T, \langle t^i\rangle] := [\vec{v} \mapsto \alpha\langle\gamma^e\rangle(\vec{u}+\vec{v})].$$

We verify that the conditions for Definition 2.1.1 are satisfied.

(i) By the assumptions on $\mathcal{E}^\rightarrow$ and [37, Proposition 3.2.3] the functor $B_\rightarrow^\infty$ is left derivable.

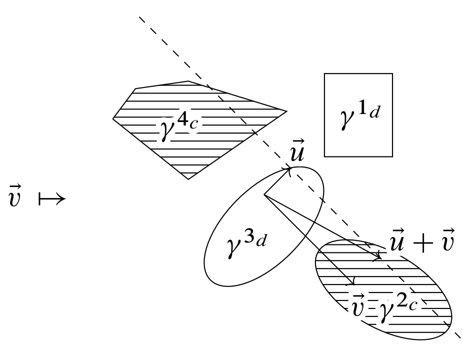

Figure 5: Representative V–loop of $\epsilon_{c,U}[[\vec{u}, \langle\alpha^{iv}\rangle, \langle\gamma^e\rangle]_T, \langle t^i\rangle]$.

(ii) Trivially Ω_2^∞ preserves fibrant objects. Since $\Omega^\infty = \Lambda^\infty\widetilde{\Omega}$ and stable weak equivalences are by definition maps whose images under $\widetilde{\Omega}$ are strict weak equivalences we have that Ω_2^∞ preserves weak equivalences.

(iii) The functor $\overline{B}_2$ trivially preserves fibrant objects, and by [37, Proposition 3.2.3] it preserves cofibrant objects. By definition the functor $\widetilde{\Omega}_\to$ preserves fibrant objects in the mixed stable model structure, and by [12, Section 5.3] it preserves cofibrant objects.

(iv) As a map of topological spaces, η' is a realization of a simplicial strong deformation retract, so it is itself a strong deformation retract of topological spaces and therefore in particular a q–equivalence [21, Theorems 9.10, 9.11 and 11.10]. The map ϵ' is a natural stable weak equivalence by definition.

(v) The natural homotopy H which gives the homotopy commutativity in $\mathtt{Sp}^\to$,

$$\begin{aligned}
&\epsilon_{B_\to^\infty X} B_\to^\infty \eta_X[[\vec{u}, \langle\alpha^{iv}\rangle, \langle[\beta^{er}, \langle\beta^{ejw}\rangle, \langle x^{ef}\rangle]_{S^e}, \langle s^{ej}\rangle]\rangle]_T, \langle r^i\rangle] \\
&= \left[\vec{v} \mapsto \begin{cases} [[(\circ_{<e}\alpha_{v'}^{iv}\beta_{w'}^{er})^{-1}(\vec{u}+\vec{v}), \langle\beta^{ejw}\rangle, \langle x^{ef}\rangle]_{S^e_{\geq w'}}, \langle s^{ej}\rangle], & \vec{u}+\vec{v} \in \circ_{<e}\alpha_{v'}^{iv}\beta_{w'}^{er} V \\ \infty, & \vec{u}+\vec{v} \notin \alpha\langle\beta^{er}\rangle \sqcup_{\Sigma_{V^{m+1}} W^{e1}} V \end{cases}\right] \\
&\qquad\simeq_{H_X} [\vec{v} \mapsto [[\vec{u}+\vec{v}, \langle\alpha^{iv}\rangle, \langle\beta^e\langle x^{ef}\rangle\rangle]_{S^e}, \langle r^i\rangle]] \\
&\qquad= \epsilon'_{B_\to^\infty X} B_\to^\infty \eta'_X[[\vec{u}, \langle\alpha^{iv}\rangle, \langle[\beta^{er}, \langle\beta^{ejw}\rangle, \langle x^{ef}\rangle]_{S^e}, \langle s^{ej}\rangle]\rangle]_T, \langle r^i\rangle]
\end{aligned}$$

is

$$H\colon B_\to^\infty \overline{B}_2 \wedge I_+ \Rightarrow \widetilde{\Omega} B_\to^\infty,$$

$$\begin{aligned}
&H_{XU}([[\vec{u}, \langle\alpha^{iv}\rangle, \langle[\beta^{er}, \langle\beta^{ejw}\rangle, \langle x^{ef}\rangle]_{S^e}, \langle s^{ej}\rangle]\rangle]_T, \langle r^i\rangle], t) \\
&\qquad := \left[\vec{v} \mapsto \left[\left[\vec{u}+\vec{v}, \left\langle \begin{cases} \alpha^{iv} \\ \beta^{ejw} \\ \mathrm{id}_{\mathfrak{c}w} \end{cases} \right\rangle, \langle x^{ef}\rangle\right]_{T\circ\langle S^e\cdot\delta^e\rangle}, \Phi(t, \langle r^i\rangle, \langle s^{ej}\rangle)\right]\right],
\end{aligned}$$

where

$$\Phi(t, \langle r^i\rangle, \langle s^{ej}\rangle) := (1-t)(\circ_{k=m+1}^{m+n+1}\partial_k \cdot \langle r^i\rangle) + t(\circ_{k=0}^{m}\partial_0 \cdot \lhd_{E^m} \langle s^{ej}\rangle),$$

with the conditions in the formula similar to the ones in (6).

(vi) In $\mathcal{E}^\to[\mathtt{Top}]$ we have strict commutativity

$$\begin{aligned}
\Omega_2^\infty \epsilon \eta_{\Omega_2^\infty}[[\alpha^r, \langle\alpha^{iv}\rangle, \langle\gamma^e\rangle]_T, \langle t^i\rangle] &= [\alpha\langle\gamma^e\rangle] \\
&= \Omega_2^\infty \epsilon' \eta'_{\Omega_2^\infty}[[\alpha^r, \langle\alpha^{iv}\rangle, \langle\gamma^e\rangle]_T, \langle t^i\rangle]. \qquad \square
\end{aligned}$$

Theorem 4.3.2 *The quasiadjunction in Theorem 4.3.1 is idempotent and induces an equivalence*

$$(\mathbb{L}B^{\infty}_{\rightarrow} \dashv \mathbb{R}\Omega^{\infty}_{2})\colon \mathcal{H}o\mathcal{E}^{\rightarrow}[\mathit{Top}]_{\mathrm{grp}} \leftrightarrows \mathcal{H}o\mathcal{S}p^{\rightarrow}_{con}$$

between the homotopy categories of grouplike $\mathcal{E}^{\rightarrow}$–pairs and maps between connective spectra.

Proof In $\mathcal{E}^{\rightarrow}[\mathtt{Top}]$ the conditions for Definition 2.2.1 are satisfied and the resulting reflective homotopy subcategory is composed of the grouplike $\mathcal{E}^{\rightarrow}$–pairs:

(i) As we have seen η' is a natural weak equivalence and by definition cof is a natural trivial fibration, so $\mathrm{cof}\,\eta'_{\mathfrak{C}}$ is a weak equivalence.

(ii) Since Ω^{∞}_{2} preserves weak equivalence between fibrant objects and $B^{\infty}_{\rightarrow}$ preserves weak equivalences between cofibrant objects we have that $\Omega^{\infty}_{2}\mathfrak{F}B^{\infty}_{\rightarrow}\mathfrak{C}$ preserves weak equivalences.

(iii) The natural transformation η is a natural group completion, since it is a realization of a simplicial group completion map — see [21, Theorems 2.7, 9.10 and 9.11] and [22, Theorem 2.2] — and the images of $\Omega^{\infty}_{2}\mathfrak{F}B^{\infty}_{\rightarrow}\mathfrak{C}$ are grouplike; therefore $\eta_{\Omega^{\infty}_{2}\mathfrak{F}B^{\infty}_{\rightarrow}\mathfrak{C}}$ is a natural weak equivalence. By naturality $\Omega^{\infty}_{2}\mathfrak{F}B^{\infty}_{\rightarrow}\mathfrak{C}\eta$ is also a group completion, and since the domain and codomain are grouplike this is a natural weak equivalence.

(iv) This condition holds since fibrations are preserved by pullbacks, fibrations induce long exact sequences of homotopy groups and for a fibration $p\colon E \twoheadrightarrow B$ and a map $f\colon X \to B$ the fibers of the pullback $f^*p\colon X \times_B E \to X$ are homeomorphic to the fibers of p.

(v) Pushouts in $\mathcal{E}^{\rightarrow}[\mathtt{Top}]$ by a cofibration whose domain is m–cofibrant in $\mathtt{Top}_*$ is a retract of a transfinite composition of pushouts by m–cofibrations in $\mathtt{Top}_*$ [30, I.4], hence this condition holds since $\mathtt{Top}_*$ with the mixed model structure is left proper and the underlying functor of $\mathcal{E}^{\rightarrow}$ is an m–cofibrant $\mathbb{S}_{\{d,c\}}$–space.

By the characterization of fibrations in the resulting Bousfield localization in [37, Proposition 2.3.6] the fibrations are the group completions and fibrant objects are the grouplike $\mathcal{E}^{\rightarrow}$–pairs.

The dual conditions for Definition 2.2.1 are also satisfied in $\mathtt{Sp}^{\rightarrow}$ and the resulting coreflective homotopy subcategory is composed of the maps between connective spectra. Note that conditions (i), (ii) and (iii) are self dual.

(i) By definition of the stable model structure ϵ' is a natural stable weak equivalence and by definition fib is a natural trivial cofibration, so $\eta'_{\mathfrak{F}}\mathrm{fib}$ is a weak equivalence.

(ii) That $B^{\infty}_{\rightarrow}\mathfrak{C}\Omega^{\infty}_{2}\mathfrak{F}$ preserves weak equivalences follows by the same argument for $\Omega^{\infty}_{2}\mathfrak{F}B^{\infty}_{\rightarrow}\mathfrak{C}$.

(iii) We have that $\eta_{\Omega^{\infty}_{2}}$ is a natural weak equivalence, and since

$$\Omega^{\infty}_{2}\epsilon\eta_{\Omega^{\infty}_{2}} = \Omega^{\infty}_{2}\epsilon'\eta'_{\Omega^{\infty}_{2}}$$

and $\Omega^{\infty}_{2}\epsilon'\eta'_{\Omega^{\infty}_{2}}$ is a natural weak equivalence by the two-out-of-three property $\Omega^{\infty}_{2}\epsilon$ is a natural weak equivalence. Since the images of $\widetilde{\Omega}$ are Ω–spectra by the formula for stable homotopy groups of Ω–spectra we have that ϵ induces isomorphisms on the nonnegative stable homotopy groups, and is therefore a stable weak equivalence on the maps between connective spectra. The images of $B^{\infty}_{\rightarrow}$ are connective by [21, 11.12] and [22, A5]. Therefore $\epsilon_{B^{\infty}_{\rightarrow}\mathfrak{C}\Omega^{\infty}_{2}\mathfrak{F}}$ is a natural weak equivalence. By naturality $B^{\infty}_{\rightarrow}\mathfrak{C}\Omega^{\infty}_{2}\mathfrak{F}\epsilon$ also induces isomorphisms on the nonnegative stable homotopy groups and so is also a natural weak equivalence.

(iv) This condition holds since cofibrations are preserved by pullbacks, spectra cofibrations induce long exact sequences of stable homotopy groups and for any cofibration $\mathfrak{i}\colon A \hookrightarrow X$ and map $\mathfrak{f}\colon A \to Y$ the cofibers of the pushout $\mathfrak{f}_*\mathfrak{i}\colon Y \to X \sqcup_A Y$ are homeomorphic to the cofibers of $\mathfrak{i}$.

(v) The stable model structure of spectra is right proper so the dual of (v) holds.

By the dual of the characterization in [37, Proposition 2.3.6] the cofibrant objects are the spectra maps such that

$$\Gamma((\epsilon B^{\infty}_{\rightarrow}\mathrm{cof}_{\Omega^{\infty}_{2}})_{\mathfrak{F}}\mathfrak{i}) \times_{\widetilde{\Omega}\mathfrak{F}\mathfrak{i}} \mathfrak{i} \to \mathfrak{i}$$

are weak equivalences, which is equivalent to ι being a map of connective spectra. □

4.4 $\mathbb{S}$–modules and commutative algebra spectra

We need to work on the more structured category of sphere modules $\mathtt{Mod}_{\mathbb{S}}$; it admits a monoidal structure that provides a natural definition of spectral algebraic structures [9]. As a first step consider for $A \in \mathbb{S}$ the *external smash product* functor

$$\overline{\wedge}_A : \Pi_A \mathtt{Sp} \to \mathtt{Sp}_{\oplus_A \mathbb{R}^\infty}, \quad \overline{\wedge}_A \langle Y^a \rangle := \langle \wedge_A Y^a_{U^a} \rangle, \quad \sigma^{\langle U^a \rangle}_{\langle V^a \rangle}[[y^a], \langle \vec{v}^a \rangle] := [\sigma^{U^a}_{V^a}[y^a, \vec{v}^a]].$$

The change of universe in this product is formally problematic, and the following construction is used to internalize the smash product in $\mathtt{Sp}$. For $K \subset_{\mathrm{cpct}} \mathscr{L}A$ define the monotone functions

$$\mu \in \mathtt{POSet}(\mathscr{A}_{\oplus_A \mathbb{R}^\infty}, \mathscr{A}), \quad \mu\langle U^a\rangle := \textstyle\sum_K f\langle U^a\rangle,$$
$$\nu \in \mathtt{POSet}(\mathscr{A}, \mathscr{A}_{\oplus_A \mathbb{R}^\infty}), \quad \nu U := \cap_K \langle f_a^* U\rangle,$$

which satisfy

$$\mu\nu U \subset U, \quad \nu\mu\nu U = \nu U, \quad \langle U^a\rangle \subset \nu\mu\langle U^a\rangle, \quad \mu\langle U^a\rangle = \mu\nu\mu\langle U^a\rangle.$$

For all $(\langle U^a\rangle, V) \in \Sigma_{\mathscr{A}_{\oplus_A \mathbb{R}^\infty}} \mathscr{A}_{\mu\langle U^a\rangle}$ we have the associated Thom complex

$$TK_V^{\langle U^a\rangle} := \Sigma_K \mathbb{S}^{V - f\langle U^a\rangle} /_{(f,\infty)\sim(g,\infty)} \in \mathtt{Top}_*,$$

where $\Sigma_K \mathbb{S}^{V-f\langle U^a\rangle}$ is topologized as a subspace of $K \times \mathbb{S}^V$. We will use the notation ${}^{\vec{v}}_{f} := [f, \vec{v}] \in TK_V^{\langle U^a\rangle}$, so that the basepoint is denoted ${}^{\infty}_{f}$ for any $f \in K$.

The *twisted half-smash product* is

$$\mathscr{L}A \ltimes -\colon \mathtt{Sp}_{\oplus_A \mathbb{R}^\infty} \to \mathtt{Sp}, \quad \mathscr{L}A \ltimes Z := \langle \operatorname*{colim}_{K \subset_{\mathrm{cpct}} \mathscr{L}A} TK_U^{\nu U} \wedge Z_{\nu U}\rangle;$$
$$\sigma_V^U[[{}^{\vec{u}}_{f}, z], \vec{v}] := [{}^{\pi_{V - f\nu V}(\vec{u}+\vec{v})}_{f}, \sigma_{\nu V}^{\nu U}[z, f|_{\nu V}^*(\vec{u}+\vec{v})]].$$

The monad $(\mathbb{L}; \eta, \mu)$ on $\mathtt{Sp}$ is

$$\mathbb{L}Y := \mathscr{L}\underline{1} \ltimes Y; \quad \eta y := [{}^{\vec{0}}_{\mathrm{id}}, y], \quad \mu[{}^{\vec{u}}_{f}, [{}^{\vec{v}}_{g}, y]] := [{}^{\vec{u}+f\vec{v}}_{fg}, y].$$

We refer to the $\mathbb{L}$–algebras as $\mathbb{L}$–spectra and for $(Y, \mathfrak{y}) \in \mathbb{L}[\mathtt{Sp}]$ we use the notation ${}^{\vec{u}}_{f} y := \mathfrak{y}[{}^{\vec{u}}_{f}, y]$.

The sphere spectrum $\mathbb{S}$, the Eilenberg–Mac Lane spectra HG and the Thom spectrum MO in Example 4.1.1, as well as the suspensions $\Sigma^\infty X$ of $\mathscr{L}$–spaces, the spectrifications $\widetilde{\Omega} Y$ of $\mathbb{L}$–spectra and deloopings $B^\infty X$ of E_∞–ring spaces are all $\mathbb{L}$–spectra with structural morphisms given as in Table 1.

The A–indexed smash product is

$$\wedge_{\mathscr{L}A}\colon \Pi_A \mathbb{L}[\mathtt{Sp}] \to \mathbb{L}[\mathtt{Sp}], \quad \wedge_{\mathscr{L}A}\langle Y^a\rangle := \left\langle \mathscr{L}A \ltimes \overline{\wedge}_A {Y^a}_U /_{[{}^{\vec{u}}_{f}, [{}^{\vec{v}^a}_{g^a} y^a]] \sim [{}^{\vec{u}+f_a\vec{v}^a}_{f\langle g^a\rangle}, [y^a]]} \right\rangle,$$

with structural maps induced by the ones for the twisted smash product. In order to make explicit the parallel between the smash product of spectra with the tensor product of abelian groups we will use the notation

$$\otimes^{\vec{u}}_{f}[y^a] := [[{}^{\vec{u}}_{f}, [y^a]]] \in \wedge_{\mathscr{L}A}\langle Y^a\rangle,$$

	${}^{\vec{u}}_{f} y$
$\mathbb{S}$	$\vec{u} + f\vec{v}$
HG	$g_a \otimes \vec{u} + f\vec{v}^a$
MO	$[\langle \iota_U^{f\nu U} f g^i f^{-1}\rangle, \langle t^i\rangle, \vec{u} + f\vec{v}]$
$\Sigma^\infty X$	$[fy, \vec{u} + f\vec{v}]$
$\tilde{\Omega} Y$	$[\vec{v} \mapsto {}^{\pi_{f\nu V^\perp}(\vec{u}+\vec{v})}_{f} \gamma f\|^*_{\nu V}(\vec{u}+\vec{v})]$
$B^\infty X$	$[[\vec{u} + f\vec{v}, \langle f \ltimes \alpha^{iv}\rangle, \langle f x^e\rangle]_T, \langle t^i\rangle]$

Table 1

so that the $\mathbb{L}$ structural maps are

$$ {}^{\vec{u}}_{f}\otimes^{\vec{v}}_{g}[y^a] := \otimes^{\vec{u}+f\vec{v}}_{fg}[y^a]. $$

For $A = \underline{2}$ this defines an associative and symmetric smash product

$$ Y^1 \wedge_{\mathscr{L}} Y^2 := \wedge_{\mathscr{L}\underline{2}}\langle Y^1, Y^2\rangle. $$

Associativity follows from the fact that the maps

$$ \Psi_{A,\langle B^a\rangle} \colon \wedge_{\mathscr{L}A}(\wedge_{\mathscr{L}B^a}\langle Y^{ab}\rangle) \to \wedge_{\mathscr{L}\Sigma_A B^a}\langle Y^{ab}\rangle, \quad \Psi \otimes^{\vec{u}}_{f}[\otimes^{\vec{v}^a}_{g^a}[y^{ab}]] := \otimes^{\vec{u}+f_a\vec{v}^a}_{f\langle g^a\rangle}[y^{ab}] $$

are isomorphisms [9, Theorems I.5.4, I.5.5 and I.5.6]. In particular when the B^a are a constant set B we have $\Sigma_A B \cong A \times B$ and therefore also a natural isomorphism

$$ \Phi_{A,B} := \Psi^{-1}_{B,\langle A\rangle}\Psi_{A,\langle B\rangle} \colon \wedge_{\mathscr{L}A}(\wedge_{\mathscr{L}B}\langle Y^{ab}\rangle) \to \wedge_{\mathscr{L}B}(\wedge_{\mathscr{L}A}\langle Y^{ab}\rangle). $$

We set the notation

$$ \otimes^{\vec{v}}_{g}[\otimes^{\vec{u}^b}_{f^b}[y^{ab}]] := \Phi_{A,B} \otimes^{\vec{u}}_{f}[\otimes^{\vec{v}^a}_{g^a}[y^{ab}]]. $$

Symmetry is given by the natural isomorphism

$$ \tau_{Y^1,Y^2} \colon Y^1 \wedge_{\mathscr{L}} Y^2 \xrightarrow{\cong} Y^2 \wedge_{\mathscr{L}} Y^1, \quad \tau \otimes^{\vec{u}}_{f}[y^1, y^2] := \otimes^{\vec{u}}_{f\cdot(12)}[y^2, y^1]. $$

For all $Z \in \mathbb{L}[\mathrm{Sp}]$ set the notation $\Sigma^Z := - \wedge_{\mathscr{L}} Z \colon \mathbb{L}[\mathrm{Sp}] \to \mathbb{L}[\mathrm{Sp}]$.

This smash product almost has as unit the sphere spectrum $\mathbb{S}$, in that there are natural weak equivalences

$$ \rho_Y \colon \Sigma^{\mathbb{S}} Y \xrightarrow{\sim} Y, \quad \rho \otimes^{\vec{u}}_{f}[y, \vec{v}] := \sigma^{f_1\nu U^1}_{U}[{}^{\vec{u}}_{f_1} y, f_2\vec{v}]. $$

	$\rho^{-1} y$
$\mathbb{S}$	$\otimes_f^{\vec{0}}[f_1^{-1}\vec{u}, \vec{0}]$
HG	$\otimes_f^{\vec{0}}[g_a \otimes f_1^{-1}\vec{u}^a, \vec{0}]$
MO	$\otimes_f^{\vec{0}}[\langle f_1^{-1} g^i f_1\rangle, \langle t^i\rangle, f_1^{-1}\vec{u}], \vec{0}]$
$\Sigma^{\mathbb{S}} Y$	$\otimes_{\langle f_1, g_2, g_3\rangle}^{\vec{u}}[y, g_2^* f_2 \vec{v}, g_3^* f_2 \vec{v}]$
$\widetilde{\Omega} Y$	$[\vec{v} \mapsto \rho^{-1} \gamma \vec{v}]$

Table 2

Unfortunately ρ is not in general a natural isomorphism. The category of $\mathbb{S}$–modules is the full subcategory

$$\mathtt{Mod}_{\mathbb{S}} := \{Y \in \mathbb{L}[\mathtt{Sp}] \mid \rho_Y \text{ is an isomorphism}\}.$$

With the same smash product and unit $\mathbb{S}$ the category $\mathtt{Mod}_{\mathbb{S}}$ is a symmetric monoidal category.

From the nontrivial fact that $\mathscr{L}A/_{\mathscr{L}\underline{1}^A}$ has a single equivalence class [9, Theorem I.8.1 and Section XI.2] the sphere spectrum $\mathbb{S}$, the Eilenberg–Mac Lane spectra HG and the Thom spectrum MO in Example 4.1.1, as well as $\Sigma^{\mathbb{S}} Y$ for $Y \in \mathbb{L}[\mathtt{Sp}]$ and spectrifications $\widetilde{\Omega} Y$ for $Y \in \mathtt{Mod}_{\mathbb{S}}$, are all $\mathbb{S}$–modules with inverse maps given as in Table 2, where in the first three lines $f \in \mathscr{L}\underline{2}$ is any linear isometry such that $U \subset f_1 \mathbb{R}^\infty$, in the fourth $\langle g_2, g_3\rangle \in \mathscr{L}\underline{2}$ are chosen such that $\langle f_1, g_2, g_3\rangle \in \mathscr{L}\underline{3}$ and $\operatorname{Im} f_2 = \operatorname{Im}\langle g_2, g_3\rangle$.

The functor $\Sigma^{\mathbb{S}} := - \wedge_{\mathscr{L}} \mathbb{S}$ is the right adjoint of the inclusion of $\mathtt{Mod}_{\mathbb{S}}$ in $\mathbb{L}[\mathtt{Sp}]$. The functor $\Sigma^{\mathbb{S}}$ is also a left adjoint, with right adjoint induced by a closed structure on $\mathbb{L}[\mathtt{Sp}]$ given by an $\mathbb{L}$–mappings functor $F_{\mathscr{L}}$. Details of this construction can be found in [9, Section I.7], but we give an overview to establish notation. The twisted half-smash product $\mathscr{L}A \ltimes -$ admits a right adjoint, the *twisted function spectrum* functor

$$F[\mathscr{L}A, -]\colon \mathtt{Sp} \to \mathtt{Sp}_{\oplus_A \mathbb{R}^\infty}, \quad F[\mathscr{L}(A), Y) := \Big\langle \lim_{K \subset_{\text{cpct}} \mathscr{L}(A)} Y_{\mu\langle U^a\rangle}^{TK_{\mu\langle U^a\rangle}^{\langle U^a\rangle}} \Big\rangle;$$

$$\sigma_{\langle U^a\rangle}^{\langle V^a\rangle}[\varphi, \langle \vec{v}^a\rangle] := \langle_f^{\vec{u}} \mapsto \sigma_{\mu\langle U^a\rangle}^{\mu\langle V^a\rangle}[\varphi_f^{\pi_{\mu\langle U^a\rangle}(\vec{u}+f_a\vec{v}^a)}, \pi_{\mu\langle U^a\rangle^\perp}(\vec{u} + f_a \vec{v}^a)]\rangle.$$

For $U^1 \in \mathscr{A}$ we also have a shift functor

$$-[U^1]\colon \mathtt{Sp}_{\mathbb{R}^\infty \oplus \mathbb{R}^\infty} \to \mathtt{Sp}; \quad Y[U^1] = \langle Y_{U^1, U^2}\rangle, \quad \sigma_{V^2}^{U^2}[y, \vec{v}] := \sigma_{U^1, V^2}^{U^1, U^2}[y, (\vec{0}, \vec{v})].$$

If $Y \in \mathbb{L}[\mathtt{Sp}]$ then $F[\mathscr{L}\underline{2}, Y)[U^1] \in \mathbb{L}[\mathtt{Sp}]$, with structural map

$$\,^{\vec{u}}_{f}\varphi := \langle^{\vec{v}}_{g} \mapsto \varphi^{\vec{v}+g_2\vec{u}}_{\langle g_1, g_2 f\rangle}\rangle.$$

Finally, we can now define

$$F_{\mathscr{L}}(-,-)\colon \mathbb{L}[\mathtt{Sp}]^{\mathrm{op}} \times \mathbb{L}[\mathtt{Sp}] \to \mathbb{L}[\mathtt{Sp}],$$

$$\begin{aligned} &F_{\mathscr{L}}(Z,Y)\\ &:= \langle\{\phi \in \mathbb{L}[\mathtt{Sp}](Z, F[\mathscr{L}\underline{2}, Y)[U^1]) \mid \,^{\vec{u}}_{f}(\phi z^{\vec{v}}_{g}) = \phi z^{\vec{u}+f\vec{v}}_{\langle fg_1, fg_2\rangle}, \phi(^{\vec{u}}_{f} z)^{\vec{v}}_{g} = \phi z^{\vec{v}+g_2\vec{u}}_{\langle g_1, g_2 f\rangle}\}\rangle;\\ &\qquad \sigma^{U^1}_{V^1}[\phi, \vec{v}] := \langle z \mapsto \sigma^{U^1,U^2}_{V^1,U^2}[\phi z, (\vec{v}, \vec{0})]\rangle, \quad \,^{\vec{u}}_{f}\phi := \langle [z, ^{\vec{v}}_{g}] \mapsto \phi z^{\vec{v}+g_1\vec{u}}_{\langle g_1 f, g_2\rangle}\rangle. \end{aligned}$$

The functor $F^{\mathbb{S}} := F_{\mathscr{L}}(\mathbb{S}, -)\colon \mathtt{Mod}_{\mathbb{S}} \to \mathbb{L}[\mathtt{Sp}]$ is right adjoint to $\Sigma^{\mathbb{S}}$.

The monoidal structure of $\mathbb{S}$–modules provides a natural definition of ring spectra, module spectra and algebra spectra.

Definition 4.4.1 A *commutative ring spectrum* R is a commutative monoid in $\mathtt{Mod}_{\mathbb{S}}$, ie an $\mathbb{S}$–module equipped with a *multiplication* map $\mu\colon R \wedge_{\mathscr{L}} R \to R$ and a *unit* map $\eta\colon \mathbb{S} \to R$ satisfying natural associative, unit and commutative laws. The category of commutative ring spectra is denoted $\mathtt{CRingSp}$.

For $R \in \mathtt{CRingSp}$ an *R–module* M is a module over R, ie an $\mathbb{S}$–module equipped with an *action* $\lambda\colon R \wedge_{\mathscr{L}} M \to M$, satisfying natural associativity and unit laws. The category of R–modules is denoted $\mathtt{Mod}_R$.

The category of R–modules admits a symmetric monoidal structure with tensor product the coequalizer

$$M \wedge_R N := \mathrm{Coeq}(M \wedge_{\mathscr{L}} R \wedge_{\mathscr{L}} N \rightrightarrows M \wedge_{\mathscr{L}} N)$$

and unit R. A commutative R–algebra is a commutative monoid in $(\mathtt{Mod}_R, \wedge_R, R)$, and the category of commutative R–algebra is denoted $\mathtt{CAlg}_R$. The category of commutative algebra spectra is

$$\mathtt{CAlgSp} := \Sigma_{\mathtt{CRingSp}} \mathtt{CAlg}_R.$$

As in the classical set theoretical setting there is a natural isomorphism [9, VII.1] $\mathtt{CAlgSp} \cong \mathtt{CRingSp}^{\to}$. Alternatively we have a monad $(\mathbb{P}^{\to}; \eta, \mu)$ on $\mathbb{L}[\mathtt{Sp}]^2$ with

$$\mathbb{P}^{\to} Y_\star := \sqcup \mathbb{S}_\star \wedge_{\mathscr{L}A} \langle Y_{\mathfrak{c}a}\rangle / \mathbb{S}_A;$$

$$\eta_\star y := [\otimes^{\vec{0}}_{\mathrm{id}} y], \quad \mu_\star\big[\otimes^{\vec{u}}_{f}[\otimes^{\vec{v}^a}_{g^a}[y^{ab}]]\big] := [\otimes^{\vec{u}+f_a\vec{v}^a}_{f\langle g^a\rangle}[y^{ab}]],$$

	$\eta\vec{u}$	$\prod_f^{\vec{u}} y^a$
$\mathbb{S}$	$\vec{u}$	$\vec{u} + f_a\vec{v}^a$
HR	$1_R \otimes \vec{u}$	$\prod_{A_{\langle b^a \rangle}} r^a_{b^a} \otimes \vec{u} + f_a\vec{v}^{ab^a}$
MO	$[\mathrm{id}, \varnothing, \vec{u}]$	$\left[\prod_A \langle \iota_U^{f_a \nu U^a} f_a g^{ai} f_a^{-1} \rangle \cdot \delta^a, \lhd_A \langle t^{ai} \rangle, \vec{u} + f_a\vec{v}^a\right]$
$\Sigma^\infty X$	$[1_X, \vec{u}]$	$[f[x^a], \vec{u} + f_a\vec{v}^a]$
$\Sigma^{\mathbb{S}} R$	$\otimes_f^{\pi_f \perp \vec{u}}[\eta f_1^*\vec{u}, f_2^*\vec{u}]$	$\otimes_g^{\vec{v}}\left[\prod_{f^1}^{\vec{u}^1} r^a, \vec{u}^2 + f_a^2\vec{w}^a\right]$
$\widetilde{\Omega} R$	$[\vec{v} \mapsto \sigma_V^U[\eta\vec{u}, \vec{v}]]$	$\left[\vec{v} \mapsto \prod_f^{\pi_{f \nu V} \perp (\vec{u}+\vec{v})} \gamma^a(f_a\|^*_{\nu V^a}(\vec{u}+\vec{v}))\right]$
$B^\infty X$	$[[\vec{u}, \varnothing, 1_X]_{\underline{1}}, \varnothing]$	$\left[\left[\vec{u} + f_a\vec{v}^a, \left\langle f_a \ltimes \left\langle \begin{cases} \alpha^{aiw^a} \\ \mathrm{id}_{\mathfrak{c}w^a} \end{cases} \right\rangle\right\rangle, \langle f_a[x^{ae^a}] \rangle\right]_{\Pi_A T^a \cdot \delta^a}, \lhd_A \langle t^{ai} \rangle\right]$

Table 3

which restricts to a monad on $\mathtt{Mod}_{\mathbb{S}}^2$. The objects of $\mathbb{P}^{\to}[\mathbb{L}[\mathtt{Sp}]^2]$ behave like algebra spectra over ring spectra, except they have units only up to weak equivalence. They are referred to as $E_\infty^{\to}$–algebra spectra, similarly to how algebras in $\mathbb{L}[\mathtt{Sp}]$ over the nonrelative version $\mathbb{P}$ of this monad are called E_∞–ring spectra. By the same argument as in [9, Proposition II.4.5] we have an isomorphism

$$\mathbb{P}^{\to}[\mathtt{Mod}_{\mathbb{S}}^2] \cong \mathtt{CAlgSp}. \tag{11}$$

For $R = ((R_d, R_c); \eta, \mu) \in \mathtt{CAlgSp}$ and $\otimes_f^{\vec{u}}[r^a] \in \wedge_{\mathscr{L}A}\langle R_{\mathfrak{c}a}\rangle$ we will use the notation

$$\prod_f^{\vec{u}} r^a := \mu[\otimes_f^{\vec{u}}[r^a]].$$

The sphere spectrum $\mathbb{S}$, the Eilenberg–Mac Lane spectrum of a commutative ring HR, the Thom spectrum MO, suspensions $\Sigma^\infty X$ of $\mathscr{L}$–spaces, the $\mathbb{S}$–module $\Sigma^{\mathbb{S}} R$ associated to an E_∞–ring spectrum R and spectrifications $\widetilde{\Omega} R$ of commutative ring spectra R are all commutative ring spectra, while deloopings $B^\infty X$ of E_∞–ring spaces are E_∞–ring spectra; see Table 3, where the implicit conditions in the last line are as in (9).

There is a natural isomorphism $\mathtt{CAlg}_{\mathbb{S}} \cong \mathtt{CRingSp}$, which is analogous to the isomorphism between commutative rings and commutative $\mathbb{Z}$–algebras. Moreover $(MO, HR) \in \mathtt{CAlgSp}$ with

$$\prod_f^{\vec{u}}[[\langle g^{1i}\rangle, \langle t^{1i}\rangle, \vec{v}^1], r_b^2 \otimes \vec{v}^{2b}] := r_b^2 \otimes \vec{u} + f_1 \circ_{\underline{m}_*} g^{1i}\vec{v}^1 + f_2\vec{v}^{2b}.$$

4.5 Stable mixed model structure of commutative algebra spectra

The stable mixed model structure of $\mathtt{Mod}_{\mathbb{S}}$ is right transferred from the one in $\mathtt{Sp}$ by the adjunction

$$(\Sigma^{\$}\mathbb{L} \dashv UF^{\$})\colon \mathtt{Sp} \leftrightharpoons \mathtt{Mod}_{\mathbb{S}}$$

as described in [2; 7; 9]; the weak equivalences and fibrations in $\mathtt{Mod}_{\mathbb{S}}$ are those maps whose underlying spectrum maps are q–equivalences and h–fibrations, respectively. The Hurewicz/Strøm strict factorization systems are constructed as in $\mathtt{Sp}$ with the $\mathbb{S}$–module structures of $\Gamma\mathfrak{f}$ and $E\mathfrak{f}$ defined pointwise. The strict mixed model structure of $\mathtt{CAlgSp}$ is right transferred from the one in $\mathtt{Mod}_{\mathbb{S}}^{2}$ by the adjunction

$$(\mathbb{P}^{\rightarrow} \dashv U)\colon \mathtt{Mod}_{\mathbb{S}}^{2} \leftrightharpoons \mathtt{CAlgSp}.$$

The strict Quillen model structure is transferred due to the fact that $\mathtt{CAlgSp}$ has continuous coequalizers and satisfies the "cofibration hypothesis" [9, Theorem VII.4.7]. By a similar argument to the one we used to construct the h–model structure of $E_\infty^{\rightarrow}$–algebras, the strict Hurewicz/Strøm model structure is also transferred. We can define an algebra structure on $\Gamma\mathfrak{f}$ for $\mathfrak{f} \in \mathtt{CAlgSp}$ as

$$\eta\vec{u} := (\eta_X \vec{u}, 0, r \mapsto \eta_Y \vec{u}), \quad \textstyle\prod_f^{\vec{u}} [(x^a, t^a, \gamma^a)] := \big(\prod_f^{\vec{u}} x^a, \max_A t^a, r \mapsto \prod_f^{\vec{u}} \gamma^a r\big).$$

As in $\mathtt{Sp}$, $(\Gamma; C_t, F)$ forms an algebraic weak factorization system in $\mathtt{CAlgSp}$. We have an h–cofibration/h–equivalence factorization

$$X \xhookrightarrow{\mathrm{in}_X} X \wedge_{\mathbb{P}X} \mathbb{P}(\Gamma\mathfrak{f} \wedge [0,\infty]_+) \wedge_{\mathbb{P}\Gamma\mathfrak{f}} Y \xrightarrow[\cong]{(\mathfrak{f}, F_t\mathfrak{f}^\dagger, id)} Y,$$

and applying $(\Gamma; C_t, F)$ then gives us the h–cofibration/trivial h–fibration factorization

$$X \xhookrightarrow{C_t(\mathfrak{f}, F_t\mathfrak{f}^\dagger, \mathrm{id})\mathrm{in}_X} \Gamma(\mathfrak{f}, F_t\mathfrak{f}^\dagger, \mathrm{id}) \xrightarrow[\cong]{F(\mathfrak{f}, F_t\mathfrak{f}^\dagger, \mathrm{id})} Y.$$

This determines the strict Hurewicz/Strøm model structure on $\mathtt{CAlgSp}$, and therefore also the strict mixed model structure. The stable model structure is then induced by the idempotent Quillen monad $\widetilde{\Omega}$ as in $\mathtt{Sp}$.

4.6 Recognition of algebra spectra

Let $\mathcal{E}^{\rightarrow}$ be an $E_\infty^{\rightarrow}$–operad equipped with an $\mathscr{L}^{\rightarrow}$–action. The functors $F^{\$}$ and $\Sigma^{\$}$ induce objectwise adjoint functors $F^{\$}_{\rightarrow}$ and $\Sigma^{\$}_{\rightarrow}$ on the morphism categories. We can then define the functors

$$\Omega_2^{\infty,\$}\colon \mathtt{CAlgSp} \to (\mathscr{L}^{\to}, \mathscr{E}^{\to})[\mathtt{Top}], \quad \Omega_2^{\infty,\$} R := \Omega_2^{\infty} F_{\to}^{\$} \eta;$$
$$f[\phi^a] := [\vec{u}^1 \mapsto \langle [\vec{u}^2, {}^{\vec{v}}_{g}] \mapsto \textstyle\prod_f^{\vec{0}} \phi^a [\vec{u}^1, \vec{u}^2]_g^{\vec{v}} \rangle];$$
$$B_{\to}^{\$,\infty}\colon (\mathscr{L}^{\to}, \mathscr{E}^{\to})[\mathtt{Top}] \to \mathtt{AlgSp}, \quad B_{\to}^{\$,\infty} X := \Sigma_{\to}^{\$} B_{\to}^{\infty} X.$$

Theorem 4.6.1 *There is an idempotent quasiadjunction*

$$(B_{\to}^{\$,\infty} \dashv_{\bar{B}_2, \tilde{\Omega}_{\to}} \Omega_2^{\infty,\$})\colon (\mathscr{H}_{\infty}^{\to}, \mathscr{L}^{\to})[\mathit{Top}] \leftrightharpoons \mathit{CAlgSp}$$

that induces an equivalence

$$(\mathbb{L} B_{\to}^{\$,\infty} \dashv \mathbb{R}\Omega_2^{\infty,\$})\colon \mathcal{H}o(\mathscr{H}_{\infty}^{\to}, \mathscr{L}^{\to})[\mathit{Top}]_{\mathrm{alg}} \leftrightharpoons \mathcal{H}o\mathit{CAlgSp}_{\mathrm{con}}$$

between the homotopy category of algebralike $E_{\infty}^{\to}$–algebras and the homotopy category of connective commutative algebra spectra over connective commutative ring spectra.

Proof The natural weak equivalences η' and ϵ' are the same as those defined in the proof of Theorem 4.3.1. The other natural transformations of the unit span and counit cospan are

$$\eta_\star[[\alpha^r, \langle \alpha^{iv} \rangle, \langle x^e \rangle]_T, \langle t^i \rangle]$$
$$:= \left[\vec{u}^1 \mapsto \begin{cases} \langle [\vec{u}^2, {}^{\vec{v}}_{f}] \mapsto \otimes_f^{\vec{v}} \big[[[\alpha_{v'}^{r,-1} \vec{u}^1, \langle \alpha^{iv} \rangle, \langle x^e \rangle]_{T_{\geq v'}}, \langle t^i \rangle], \vec{u}^2 \big] \rangle & \text{if } \vec{u}^1 \in \alpha_{v'}^r U \\ \infty & \text{if } \vec{u}^1 \notin \alpha^r \sqcup_{V^1} U \end{cases} \right],$$
$$\epsilon_{\star U} \otimes_f^{\vec{u}} \big[[[\vec{v}^1, \langle \alpha^{iv} \rangle, \langle \phi^e \rangle]_T, \langle t^i \rangle], \vec{v}^2 \big] := [\vec{w} \mapsto \alpha \langle \phi^e \rangle [\vec{v}^1 + f_1^* \vec{w}, \vec{v}^2 + f_2^* \vec{w}]_f^{\vec{u} + \pi_{f^\perp} \vec{w}}].$$

The conditions in the first formula are the same as in the proof of Theorem 4.3.1; the domain in the last formula is any $W \in \mathscr{A}_{U + f_1 V}$ with V a common domain of representatives of the loops ϕ^e.

That these maps satisfy the conditions for an idempotent quasiadjunction follows from the fact that $(\Sigma_{\to}^{\$} \dashv F_{\to}^{\$})\colon \mathbb{P}^{\to}[\mathbb{L}[\mathtt{Sp}]^2] \leftrightharpoons \mathtt{CAlgSp}$ is a Quillen equivalence and the same argument as for Theorems 4.3.1 and 4.3.2. □

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