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Constraints on families of smooth 4–manifolds from $\mathrm{Pin}^-(2)$–monopole

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Using the Seiberg–Witten monopole equations, Baraglia recently proved that the inclusion $\mathrm{Diff}(X) \hookrightarrow \mathrm{Homeo}(X)$ is not a weak homotopy equivalence for most of simply connected closed smooth 4–manifolds X. We generalize Baraglia's result by using the $\mathrm{Pin}^-(2)$–monopole equations instead. We also give new examples of 4–manifolds X for which $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$ are not surjections.

57R57; 57S05

1 Introduction

T Kato and the authors [8] recently made use of Seiberg–Witten theory for families in order to detect nonsmoothable topological families of 4–manifolds. This argument extracts some homotopical difference between the homeomorphism groups and the diffeomorphism groups of some classes of 4–manifolds. Soon after [8], using Seiberg–Witten theory for families in a different manner, D Baraglia [1] extensively generalized the result in [8] on comparisons between the homeomorphism and diffeomorphism groups of 4–manifolds: he proved in [1, Corollary 1.9] that for every closed, oriented, simply connected, smooth, and indefinite 4–manifold M with $|\sigma(M)| > 8$, the inclusion $\mathrm{Diff}(M) \hookrightarrow \mathrm{Homeo}(M)$ is not a weak homotopy equivalence. Here $\sigma(M)$ denotes the signature of M, and $\mathrm{Diff}(M)$ and $\mathrm{Homeo}(M)$ denote the groups of diffeomorphisms and homeomorphisms, respectively. The proof of this result by Baraglia is based on some constraints on smooth families of 4–manifolds obtained from a finite-dimensional approximation of the families Seiberg–Witten monopole map. The purpose of this paper is to give analogues of arguments in [1] by Baraglia for the $\mathrm{Pin}^-(2)$–monopole equations introduced by the second author in [12], and to make use of the $\mathrm{Pin}^-(2)$–monopole analogues to generalize the above result by Baraglia on comparison between homeomorphism and diffeomorphism groups as follows:

Theorem 1.1 *Let X be a smooth 4–manifold which is homeomorphic to a 4–manifold of the form*

$$M \mathop{\#}_{i=1}^{p} (S^1 \times Y_i) \mathop{\#}_{j=1}^{q} (S^2 \times \Sigma_j), \tag{1}$$

where

- *M is a simply connected, closed, oriented, smooth, and indefinite 4–manifold with $|\sigma(M)| > 8$;*
- *Y_i is an oriented closed 3–manifold, and Σ_j is an oriented closed 2–manifold of positive genus; and*
- *p and q are nonnegative integers, where we interpret $\#_{i=1}^{p}(S^1 \times Y_i)$ as S^4 for $p = 0$, and similarly for $q = 0$.*

Set $n = \min\{b_+(M), b_-(M)\}$. If we fix a homeomorphism between X and a 4–manifold of the form (1), then:

- *If M is nonspin, there exists a nonsmoothable $\mathrm{Homeo}(X)$–bundle*
$$X \to E \to T^n.$$
- *If M is spin, there exists a nonsmoothable $\mathrm{Homeo}(X)$–bundle*
$$X \to E \to T^{n-1}.$$

Here $b_+(M)$ is the maximal dimension of positive-definite subspaces of $H^2(M;\mathbb{R})$ with respect to the intersection form, and $b_-(M) = b_2(M) - b_+(M)$. We say that a $\mathrm{Homeo}(X)$–bundle E is *nonsmoothable* if E does not admit a reduction of structure to $\mathrm{Diff}(X)$.

By standard obstruction theory, we have:

Corollary 1.2 *Let X be a smooth 4–manifold which is homeomorphic to a 4–manifold of the form*

$$M \mathop{\#}_{i=1}^{p} (S^1 \times Y_i) \mathop{\#}_{j=1}^{q} (S^2 \times \Sigma_j),$$

where

- *M is a simply connected, closed, oriented, smooth, and indefinite 4–manifold with $|\sigma(M)| > 8$;*

- *Y_i is an oriented closed 3–manifold, and Σ_j is an oriented closed 2–manifold of positive genus; and*
- *p and q are nonnegative integers.*

Then the inclusion

$$\mathrm{Diff}(X) \hookrightarrow \mathrm{Homeo}(X)$$

is not a weak homotopy equivalence.

More precisely, if we fix a homeomorphism between X and a 4–manifold of the form (1), then:

- *If M is nonspin,*

  $$\pi_k(\mathrm{Diff}(X)) \to \pi_k(\mathrm{Homeo}(X))$$

  is not an isomorphism for some $k \leq \min\{b_+(M), b_-(M)\} - 1$.
- *If M is spin,*

  $$\pi_k(\mathrm{Diff}(X)) \to \pi_k(\mathrm{Homeo}(X))$$

  is not an isomorphism for some $k \leq \min\{b_+(M), b_-(M)\} - 2$.

Remark 1.3 Here we compare Theorem 1.1 and Corollary 1.2 with Baraglia's argument given in [1]:

(1) The case that $p = q = 0$ follows from an argument based on [1, Theorem 1.1].

(2) The case that $p = 0$, $q \leq 2$, and M is spin follows from an argument based on [1, Theorem 1.2].

Instead of a simply connected 4–manifold in M in Theorem 1.1 and Corollary 1.2, we may also consider not simply connected 4–manifolds whose homeomorphism types can be understood well. We give such an example using Enriques surfaces:

Theorem 1.4 *Let X be a smooth 4–manifold which is homeomorphic to a 4–manifold of the form*

$$mS \# M \mathop{\#}_{i=1}^{p} (S^1 \times Y_i) \mathop{\#}_{j=1}^{q} (S^2 \times \Sigma_j),$$

where:

- *S is an Enriques surface and M is a standard simply connected smooth 4–manifold with nonpositive signature. Here M is called standard if M is obtained as the connected sum of finitely many (possibly zero) copies of $\mathbb{CP}^2$, $-\mathbb{CP}^2$, $S^2 \times S^2$, $K3$, and $-K3$. If M is not spin, we assume also that $\sigma(M) < 0$.*
- *Y_i is an oriented closed 3–manifold, and Σ_j is an oriented closed 2–manifold of positive genus.*
- *m is a positive integer, and p and q are nonnegative integers, where we interpret $\#_{i=1}^p (S^1 \times Y_i)$ as S^4 for $p = 0$, and similarly for $q = 0$.*

Set $n = b_+(M) + m$. Then there exists a nonsmoothable $\mathrm{Homeo}(X)$–bundle

$$X \to E \to T^n.$$

Corollary 1.5 *Let X be a smooth 4–manifold which is homeomorphic to a 4–manifold of the form*

$$mS \# M \mathop{\#}_{i=1}^{p} (S^1 \times Y_i) \mathop{\#}_{j=1}^{q} (S^2 \times \Sigma_j),$$

where:

- *S is an Enriques surface and M is a standard simply connected smooth 4–manifold with nonpositive signature. If M is not spin, we assume also that $\sigma(M) < 0$.*
- *Y_i is an oriented closed 3–manifold, and Σ_j is an oriented closed 2–manifold of positive genus.*
- *m is a positive integer, and p and q are nonnegative integers.*

Then the inclusion

$$\mathrm{Diff}(X) \hookrightarrow \mathrm{Homeo}(X)$$

is not a weak homotopy equivalence. More precisely,

$$\pi_k(\mathrm{Diff}(X)) \to \pi_k(\mathrm{Homeo}(X))$$

is not an isomorphism for some $k \le b_+(M) + m - 1$.

As a more specific corollary of Theorem 1.4 than Corollary 1.5, we may give new examples of 4–manifolds X for which $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$ are not surjections:

Corollary 1.6 *Let X be a smooth 4–manifold which is homeomorphic to a 4–manifold of the form*

$$S \# k(-\mathbb{CP}^2) \mathop{\#}_{i=1}^{p} (S^1 \times Y_i) \mathop{\#}_{j=1}^{q} (S^2 \times \Sigma_j),$$

where

- *S is an Enriques surface, Y_i is an oriented closed 3–manifold, and Σ_j is an oriented closed 2–manifold of positive genus; and*
- *k, p and q are nonnegative integers.*

Then

$$\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$$

is not a surjection. Namely, there exists a self-homeomorphism of X which is not topologically isotopic to any self-diffeomorphism of X.

Remark 1.7 The case in Theorem 1.4 and Corollaries 1.5 and 1.6 that $p = q = 0$ can be deduced also from an argument using [1, Theorem 1.1].

The first example of 4–manifolds X for which $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$ are not surjections is a $K3$ surface, proven by Donaldson [5]. One may check the same statement holds also for any homotopy $K3$ surface using the Seiberg–Witten invariants and a result by Morgan and Szabó [10]. We note that examples of 4–manifolds X for which $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$ are not *injections* are known a little more: the first example was given by Ruberman [14], and later additional examples were given by Baraglia and the first author [2], and by Kronheimer and Mrowka [9] recently.

The paper is organized as follows. In Section 2 we recall some basics of $\mathrm{Pin}^{-}(2)$–monopole theory and describe a finite-dimensional approximation of the families $\mathrm{Pin}^{-}(2)$–monopole map. In Section 3 we give constraints on smooth families of 4–manifold using a finite-dimensional approximation of a families $\mathrm{Pin}^{-}(2)$–monopole map. Those constraints are analogues of some constraints by Baraglia [1] obtained from the families Seiberg–Witten monopole map. In Section 4 we give the proofs of Theorems 1.1 and 1.4: we shall construct concrete topological families of 4–manifolds and show the nonsmoothability of them using the constraints obtained in Section 3.

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2 $\mathrm{Pin}^-(2)$–monopole maps for families

First, we briefly review $\mathrm{Pin}^-(2)$–monopole theory. For a thorough treatment, readers are referred to [12; 13].

Let X be an oriented, closed, connected, and smooth 4–manifold. Fix a Riemannian metric g on X. Let $\widetilde{X} \to X$ be an unbranched double cover, and let $\ell = \widetilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, the associated local system with coefficient group $\mathbb{Z}$. We always assume that $\widetilde{X} \to X$ is nontrivial. Let $\ell_{\mathbb{R}} = \ell \otimes \mathbb{R}$ and $i\ell_{\mathbb{R}} = \ell \otimes \sqrt{-1}\mathbb{R}$. Set $b_j^{\ell}(X) = \operatorname{rank} H^j(X;\ell)$ for $j \geq 0$, and set $b_+^{\ell}(X) = \operatorname{rank} H^+(X;\ell)$, where $H^+(X;\ell)$ denotes a maximal-dimensional positive-definite subspace of $H^2(X;\ell)$ with respect to the intersection form of X. Define the Lie groups $\mathrm{Pin}^-(2)$, and $\mathrm{Spin}^{c-}(4)$ by $\mathrm{Pin}^-(2) = \mathrm{U}(1) \cup j\mathrm{U}(1) \subset \mathrm{Sp}(1)$ and $\mathrm{Spin}^{c-}(4) = \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$. Note that $\mathrm{Spin}^{c-}(4)/\mathrm{Spin}^c(4) \cong \{\pm 1\}$ and $\mathrm{Spin}^{c-}(4)/\mathrm{Pin}^-(2) \cong \mathrm{SO}(4)$. A Spin^{c-}*–structure* on $\widetilde{X} \to X$ is defined as a triple $\mathfrak{s} = (P, \sigma, \tau)$, where

- P is a principal $\mathrm{Spin}^{c-}(4)$–bundle over X,
- $\sigma\colon \widetilde{X} \to P/\mathrm{Spin}^c(4)$ is an isomorphism of $\{\pm 1\}$–bundles, and
- $\tau\colon \mathrm{Fr}(X) \to P/\mathrm{Pin}^-(2)$ is an isomorphism of $SO(4)$–bundles, where $\mathrm{Fr}(X)$ denotes the frame bundle of X.

The associated $\mathrm{O}(2)$–bundle $L = P/\mathrm{Spin}(4)$ is called the *characteristic bundle* of a Spin^{c-}–structure $\mathfrak{s} = (P, \sigma, \tau)$. We denote the ℓ–coefficient Euler class of L by $\tilde{c}_1(\mathfrak{s}) \in H^2(X;\ell)$.

Some notions associated to Spin^{c-}–structures are very similar to those of Spin^c–structures: a Spin^{c-}–structure $\mathfrak{s}$ on $\widetilde{X} \to X$ gives rise to the positive and negative spinor bundles $S^{\pm}$ over X and the Clifford multiplication $\rho\colon \Omega^1(X; i\ell_{\mathbb{R}}) \to \mathrm{Hom}(S^+, S^-)$. An $\mathrm{O}(2)$–connection A on L induces the Dirac operator $D_A\colon \Gamma(S^+) \to \Gamma(S^-)$. Note that the self-dual part of the curvature F_A^+ is an element of $\Omega^+(X; i\ell_{\mathbb{R}})$. We denote by $q\colon S^+ \to \Omega^+(X; i\ell_{\mathbb{R}})$ the canonical real quadratic map. The $\mathrm{Pin}^-(2)$*–monopole equations* are defined by

$$D_A\phi = 0, \quad \tfrac{1}{2}F_A^+ = q(\phi) \tag{2}$$

for $\mathrm{O}(2)$–connections A on L and positive spinors $\phi \in \Gamma(S^+)$. The equations in (2) are equivariant under the action of the gauge group $\mathcal{G} = \Gamma(\widetilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))$, where $\{\pm 1\}$ acts on $\mathrm{U}(1)$ by complex conjugation.

Choose a reference O(2)–connection A_0 on L. The $\mathrm{Pin}^-(2)$*–monopole map*

$$m\colon \Omega^1(X;i\ell_{\mathbb{R}})\oplus\Gamma(S^+)\to(\Omega^0\oplus\Omega^+)(X;i\ell_{\mathbb{R}})\oplus\Gamma(S^-)$$

is defined by

$$m(a,\phi)=(d^*a,d^+a-q(\phi),D_{A_0+a}\phi).$$

The map m is decomposed into the sum $m=l+c$, where l is the linear map given by $l=(d^*,d^+,D_{A_0})$, and c is the quadratic part given by $c(a,\phi)=\bigl(0,-q(\phi),\frac{1}{2}\rho(a)\phi\bigr)$. As well as usual Seiberg–Witten theory, we consider the Sobolev completions of the domain and the target of m. Choose $k\geq 4$. Let $\mathcal{V}:=L^2_k(\Omega^1(X;i\ell_{\mathbb{R}})\oplus\Gamma(S^+))$ and $\mathcal{W}:=L^2_{k-1}((\Omega^0\oplus\Omega^+)(X;i\ell_{\mathbb{R}})\oplus\Gamma(S^-))$. Then m is extended to a smooth map $m\colon\mathcal{V}\to\mathcal{W}$. The linear part l is a Fredholm map of index

$$\tfrac{1}{4}(\tilde{c}_1(\mathfrak{s})^2-\sigma(X))+b_1^{\ell}(X)-b_+^{\ell}(X),$$

and c is a nonlinear compact map. Note that $b_0^{\ell}(X)=0$ if ℓ is nontrivial.

We take the L^2_{k+1}–completion of the gauge group $\mathcal{G}$, denoted by the same symbol $\mathcal{G}$ to simplify the notation. Then the $\mathcal{G}$–action is smooth. The space

$$\ker\bigl(d^*\colon L^2_k(\Omega^1(X;i\ell_{\mathbb{R}}))\to L^2_{k-1}(\Omega^0(X;i\ell_{\mathbb{R}}))\bigr)$$

is a global slice for the $\mathcal{G}$–action at $(0,0)$, and we have

$$m^{-1}(0)=\{\text{solutions to (2)}\}\cap\ker d^*.$$

The slice $\ker d^*$ still has a remaining gauge symmetry. Let $\mathcal{H}$ be the group of *harmonic* $\{\pm1\}$–equivariant maps $\widetilde{X}\to\mathrm{U}(1)$, which is the kernel of the composition of the maps

$$L^2_{k+1}(\mathcal{G})\xrightarrow{d}L^2_k(\Omega^1(X;i\ell_{\mathbb{R}}))\xrightarrow{d^*}L^2_{k-1}(\Omega^0(X;i\ell_{\mathbb{R}})).$$

Then m is $\mathcal{H}$–equivariant, and we have

$$m^{-1}(0)/\mathcal{H}=\{\text{solutions to (2)}\}/\mathcal{G}.$$

Note that

$$H^1(X;\ell)=\mathbb{Z}_2\oplus\mathbb{Z}^{b_1^{\ell}}$$

if ℓ is nontrivial. Let $r\colon H^1(X;\ell)\to H^1(X;\ell_{\mathbb{R}})$ be the map induced from the natural map $\ell\to\ell_{\mathbb{R}}$ and set $\overline{H}:=\operatorname{Im}r\cong\mathbb{Z}^{b_1^{\ell}}$. Note the exact sequence

$$1\to\{\pm1\}\to\mathcal{H}\to\overline{H}\to 0. \tag{3}$$

Fixing a splitting of the above sequence, we have

$$\mathcal{H}\cong\{\pm1\}\times\overline{H}.$$

Remark 2.1 A way of fixing a splitting of (3) is as follows; cf [12, Section 4.7]. Choose a loop γ in X such that the restriction of ℓ to γ is nontrivial. Let $\mathcal{K}_\gamma$ be the subgroup of $\mathcal{G}$ consisting of $u \in \mathcal{G}$ satisfying that $u|_\gamma$ is homotopic to the constant map with value 1. Then there is an exact sequence

$$1 \to \mathcal{K}_\gamma \to \mathcal{G} \to \{\pm 1\} \to 1.$$

From this we have

$$\mathcal{H} \cap \mathcal{K}_\gamma \cong \overline{H},$$

and this gives a splitting of (3).

Let $J := H^1(X; \ell_{\mathbb{R}})/\overline{H}$. Then J is a b_1^ℓ–dimensional torus. Dividing the harmonic projection

$$\varpi : \mathcal{V} \to H^1(X; i\ell), \quad (a, \phi) \mapsto h(a),$$

by $\overline{H}$, we obtain a Hilbert bundle $\overline{\mathcal{V}} = \mathcal{V}/\overline{H} \to J$. Then dividing the map m by $\overline{H}$, we obtain a fiber-preserving $\{\pm 1\}$–equivariant map $\bar{m}$:

$$\begin{tikzcd} \overline{\mathcal{V}} \arrow[r, "\bar{m}"] \arrow[d, "\varpi"] & J \times \mathcal{W} \arrow[d] \\ J \arrow[r, equal] & J \end{tikzcd} \tag{4}$$

For our later purpose, there is no need for the whole of $\bar{m}$. What we need is only the restriction $\bar{m}|_{\varpi^{-1}(0)}$ of $\bar{m}$ to the fiber over the origin of J. The restriction $\bar{m}|_{\varpi^{-1}(0)}$ is identified with the map m_0 defined by

$$\begin{gathered} \mathcal{V}_0 := L_k^2\big(\mathrm{Im}(d + d^* : (\Omega^0 \oplus \Omega^2)(X; i\ell_{\mathbb{R}}) \to \Omega^1(X; i\ell_{\mathbb{R}})) \oplus \Gamma(S^+)\big), \\ \mathcal{W}_0 := L_{k-1}^2((\Omega^0 \oplus \Omega^+)(X; i\ell_{\mathbb{R}}) \oplus \Gamma(S^-)), \\ m_0 : \mathcal{V}_0 \to \mathcal{W}_0, \quad (a, \phi) \mapsto (d^*a, F_{A_0} + d^+a - q(a), D_{A_0+a}\phi). \end{gathered} \tag{5}$$

Let $\mathrm{Aut}(X, \mathfrak{s})$ be the automorphism group of the Spin^{c-} 4–manifold $(X, \mathfrak{s})$, which consists of pairs $(f, \tilde{f})$ of diffeomorphisms f preserving the isomorphism class of $\mathfrak{s}$ and lifts $\tilde{f}$ of f to Spin^{c-}–bundle automorphisms of the principal Spin^{c-}–bundle P associated to $\mathfrak{s}$. Let B be a compact space. Suppose a smooth $\mathrm{Aut}(X, \mathfrak{s})$–bundle $(X, \mathfrak{s}) \to E \to B$ is given. That is, E is a smooth fiber bundle $E = \coprod_{b \in B}(X_b, \mathfrak{s}_b)$ with fiber a Spin^{c-} 4–manifold such that there is an isomorphism $(X_b, \mathfrak{s}_b) \cong (X, \mathfrak{s})$ of Spin^{c-} 4–manifolds for each b. Let $\mathbb{L} = \coprod_{b \in B} L_b$ be the associated family of O(2)–bundles where each L_b is the characteristic O(2)–bundle of $(X_b, \mathfrak{s}_b)$. Choose a family of Riemannian metrics $\{g_b\}_{b\in B}$ on E. Then we have an associated vector bundle

$$\mathbb{R}^{b_+^\ell} \to H^+(E,\ell) \to B$$

whose fiber over $b \in B$ is the space $H^+(X_b;\ell_b)$ of harmonic self-dual 2–forms on X_b. The isomorphism class of $H^+(E,\ell)$ is independent of the choice of the family of Riemannian metrics on E since the Grassmannian of maximal-dimensional positive-definite subspaces of $H^2(X;\ell_\mathbb{R})$ is contractible.

Choose a family of reference O(2)–connections $\{A_b\}_{b\in B}$ on $\mathbb{L}$. Then we can obtain a family of m_0 given in (5), denoted by

$$\mu_0\colon \widetilde{\mathcal{V}} \to \widetilde{\mathcal{W}},$$

by parametrizing the previous argument over B. Here $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$ are the Hilbert bundles over B with fibers $\mathcal{V}_0$ and $\mathcal{W}_0$, respectively, and μ_0 is a fiber-preserving map whose restriction on each fiber is identified with the map m_0.

By taking a finite-dimensional approximation of μ_0 [3; 4; 6], we obtain a $\{\pm 1\}$–equivariant proper map

$$f\colon V \to W$$

which satisfies the following properties:

- V and W are finite rank subbundles of $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$.
- V and W are decomposed as $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$. The group $\{\pm 1\}$ acts on V_0 and W_0 trivially, and on V_1 and W_1 by fiberwise multiplication.
- $f^{\{\pm 1\}} = f|_{V_0}\colon V_0 \to W_0$ is a fiberwise linear inclusion.
- W_0 is isomorphic to $V_0 \oplus H^+(E,\ell)$.
- The index of the family of the Dirac operators, $\mathrm{ind}\{D_{A_b}\}$, is represented by $[V_1]-[W_1]$ in $K_{\{\pm 1\}}(B)$.

When $\tilde{c}_1(\mathfrak{s}) = 0$, the $\mathrm{Pin}^-(2)$–monopole equations have a larger gauge symmetry given by $\widetilde{\mathcal{G}} = \Gamma(\widetilde{X} \times_{\{\pm 1\}} \mathrm{Pin}^-(2))$ [12, Section 4.3]. Then the whole theory admits the j–action and the resulting finite-dimensional approximation $f\colon V \to W$ is equivariant under the action of the cyclic group C_4 of order 4 generated by j. In this case, C_4 acts on V_0 and W_0 by fiberwise multiplication of $\{\pm 1\}$ via the surjective homomorphism $C_4 \to \{\pm 1\}$, and on V_1 and W_1 by fiberwise multiplication of j. Note that the j–action gives complex structures on V_1 and W_1.

Remark 2.2 As mentioned above, what we need for the proofs of our results is the family μ_0 and its finite-dimensional approximation. More generally, we can construct a parametrized family of the *total* monopole maps $\bar{m}$ of (4) once a family of splittings of (3) is given. We can obtain such a family of splittings if we can choose a family of loops $\{\gamma_b\}_{b \in B}$ such that $\ell|_{\gamma_b}$ is nontrivial. In this case, the family of the monopole maps is parametrized by the total space of a bundle K over B with fiber J.

3 Constraints from $\mathrm{Pin}^-(2)$–monopole

As in Section 2, suppose that we have a smooth $\mathrm{Aut}(X, \mathfrak{s})$–bundle $(X, \mathfrak{s}) \to E \to B$, where B is a compact space.

The following theorem is a $\mathrm{Pin}^-(2)$–monopole analogue of a part of [1, Theorem 1.1] by Baraglia:

Theorem 3.1 *If $w_{b^\ell_+}(H^+(E, \ell)) \neq 0$ in $H^{b^\ell_+}(B; \mathbb{Z}_2)$, then $\tilde{c}_1(\mathfrak{s})^2 \leq \sigma(X)$ holds.*

Proof The proof is parallel to that of [1, Theorem 1.1]. Throughout this proof, the coefficients of cohomology groups are supposed to be $\mathbb{Z}_2$. Let $G = \{\pm 1\}$. Note that the Borel cohomology $H^*_G(pt)$ is isomorphic to $\mathbb{Z}_2[u]$ with $\deg u = 1$. Since G acts on the base space B trivially, we have $H^*_G(B) \cong H^*(B)[u]$. For a vector bundle U over B, denote its disk bundle by $D(U)$, and the sphere bundle by $S(U)$. Choosing a finite-dimensional approximation f of μ_0, we have the commutative diagram

$$\begin{tikzcd}
V = V_0 \oplus V_1 \arrow[r, "f"] & W = W_0 \oplus W_1 \\
V_0 \arrow[u, "\iota_0"] \arrow[r, "f^G"] & W_0 \arrow[u, "\iota_1"']
\end{tikzcd}$$

Note that the vertical arrows and f^G are fiberwise linear inclusions. We also have a relative version of the above diagram for the pairs $(D(V), S(V))$ etc. Applying the H^*_G–functor, we obtain

$$\begin{tikzcd}
H^*_G(D(V), S(V)) \arrow[d, "\iota_0^*"'] & H^*_G(D(W), S(W)) \arrow[l, "f^*"'] \arrow[d, "\iota_1^*"] \\
H^*_G(D(V_0), S(V_0)) & H^*_G(D(W_0), S(W_0)) \arrow[l, "(f^G)^*"']
\end{tikzcd} \tag{6}$$

Note the following facts:

- The Thom isomorphisms, eg $H^*_G(D(V),S(V)) \cong H^*_G(B)\tau_G(V)$, where $\tau_G(V)$ is the G–equivariant Thom class.
- $\iota_0^*\tau_G(V_0\oplus V_1) = e_G(V_1)\tau_G(V_0)$, where $e_G(V_1)$ is the G–equivariant Euler class. Similarly,
$$\begin{aligned}\iota_1^*\tau_G(W_0\oplus W_1) &= e_G(W_1)\tau_G(W_0),\\ (f^G)^*\tau_G(W_0) &= e_G(H^+(E,\ell))\tau_G(V_0).\end{aligned}$$
The last equation follows from that $W_0 \cong V_0 \oplus H^+(E,\ell)$
- There exists a class α in $H^*_G(B)$ such that $f^*\tau_G(W) = \alpha\tau_G(V)$. The class α is called the *cohomological degree* of f.

By the diagram (6), we obtain the relation

$$\alpha e_G(V_1)\tau_G(V_0) = e_G(H^+(E,\ell))e_G(W_1)\tau_G(V_0). \tag{7}$$

Let $m = \operatorname{rank}_{\mathbb{R}} V_1$ and $n = \operatorname{rank}_{\mathbb{R}} W_1$. Then

$$m-n = \operatorname{ind} D_{A_b} = \tfrac{1}{4}(\tilde{c}_1(\mathfrak{s})^2 - \sigma(X)).$$

The G–Euler classes of V_1 and W_1 are given by

$$\begin{aligned}e_G(V_1) &= w_m(V_1) + w_{m-1}(V_1)u + \cdots + w_1(V_1)u^{m-1} + u^m,\\ e_G(W_1) &= w_n(W_1) + w_{n-1}(W_1)u + \cdots + w_1(W_1)u^{n-1} + u^n.\end{aligned}$$

Since G acts on $H^+(E,\ell)$ trivially, we have $e_G(H^+(E,\ell)) = w_{b_+^\ell}(H^+(E,\ell))$. By (7), $e_G(H^+(E^+,\ell))e_G(W_1)$ is divisible by $e_G(V_1)$. If

$$e_G(H^+(E,\ell)) = w_{b_+^\ell}(H^+(E,\ell)) \neq 0,$$

then $m-n \leq 0$. Finally we obtain $\tilde{c}_1(\mathfrak{s})^2 \leq \sigma(X)$. □

Using the relation (7), we can obtain additional constraints on V_1 and W_1. Let us recall the notation of the Stiefel–Whitney class of virtual vector bundles. For an integer $i \geq 0$ and vector bundles V and W over a common base space, define $w_i([W]-[V])$ as the component of $w(V)^{-1}w(W)$ in degree i, where $w(V)$ denotes the total Stiefel–Whitney class of V.

Corollary 3.2 *For i with $i > n-m$, $w_i([W_1]-[V_1])e(H^+(E,\ell)) = 0$.*

Proof In $H^*(B)[u,u^{-1}]$, the equality (7) implies that

$$\alpha = e_G(H^+(E^+,\ell))e_G(W_1)e_G(V_1)^{-1}.$$

Since α is in $H^*(B)[u]$, the right-hand side has no terms of negative degree in u. □

Remark 3.3 In the proofs of Theorem 3.1 and Corollary 3.2, we used the $\mathbb{Z}_2$–coefficient Borel cohomology. We can obtain similar constraints using the Borel cohomology with local coefficient $\mathbb{Z}_{w_1(H^+(E;\ell))}$. In this case, the constraints are given in terms of Chern classes of V_1 and W_1 with local coefficient.

The following theorem is a $\mathrm{Pin}^-(2)$–monopole analogue of [1, Theorem 1.2]:

Theorem 3.4 *Suppose $\tilde{c}_1(\mathfrak{s}) = 0$ for the family $(X, \mathfrak{s}) \to E \to B$. If*

$$w_{b_+^\ell}(H^+(E,\ell)) \neq 0 \quad \text{or} \quad w_{b_+^\ell - 1}(H^+(E,\ell)) \neq 0,$$

then $\sigma(X) \geq 0$.

Proof Recall that a finite-dimensional approximation f is C_4–equivariant when $\tilde{c}_1(\mathfrak{s}) = 0$. Let $G = C_4$. Also in this proof, the coefficients of cohomology groups are supposed to be $\mathbb{Z}_2$. Then we have $H^*_G(pt) = \mathbb{Z}_2[u, v]/u^2$ with $\deg u = 1$ and $\deg v = 2$. The surjective homomorphism $G \to \{\pm 1\}$ induces the homomorphism

$$H^*_{\{\pm 1\}}(pt) = \mathbb{Z}_2[u] \to H^*_G(pt) = \mathbb{Z}_2[u, v]/u^2, \quad u \mapsto u.$$

Regard G as a subgroup of S^1 in an obvious way. Then the inclusion $G \hookrightarrow S^1$ induces the homomorphism

$$H^*_{S^1}(pt) = \mathbb{Z}_2[v] \to H^*_G(pt) = \mathbb{Z}_2[u, v]/u^2, \quad v \mapsto v.$$

By an argument similar to the proof of Theorem 3.1, we obtain the relation (7) for some $\alpha \in H^*_G(B)$. In this case, V_1 and W_1 are complex vector bundles. Let $r := \operatorname{rank}_{\mathbb{C}} V_1$ and $s := \operatorname{rank}_{\mathbb{C}} W_1$. Then

$$r - s = -\tfrac{1}{8}\sigma(X).$$

The G–Euler classes are written as

$$e_G(V_1) = c_r(V_1) + c_{r-1}(V_1)v + \cdots + c_1(V_1)v^{r-1} + v^r,$$
$$e_G(W_1) = c_s(W_1) + c_{s-1}(W_1)v + \cdots + c_1(W_1)v^{s-1} + v^s,$$

where c_i are the (mod 2)–Chern classes. If we regard $H = H^+(E, \ell)$ as a $\{\pm 1\}$–equivariant bundle, then the $\{\pm 1\}$–Euler class of H is given by

$$e_{\{\pm 1\}}(H) = w_b(H) + w_{b-1}u + \cdots + w_1(H)u^{b-1} + u^b,$$

where $b = b_+^\ell$. Noticing $u^2 = 0$ in $H^*_G(B)$, we obtain

$$e_G(H) = w_b(H) + w_{b-1}(H)u.$$

Then, under the assumption that $e_G(H) \neq 0$, the relation (7) implies that

$$-\tfrac{1}{8}\sigma(X) = r - s \leq 0.$$

This proves the Theorem 3.4. □

Remark 3.5 The proofs of [1, Theorem 1.1] and [1, Theorem 1.2] used S^1–symmetry and Pin(2)–symmetry of the monopole maps respectively. It would be worth noting that the above arguments of the proofs of Theorems 3.1 and 3.4 show that $\{\pm 1\}$–symmetry and C_4–symmetry are enough to prove parts of [1, Theorem 1.1] and [1, Theorem 1.2], respectively.

4 Proof of Theorems 1.1 and 1.4

In this section we give the proofs of Theorems 1.1 and 1.4. For this purpose, we first collect some preliminary results. Let X be an oriented connected closed smooth 4–manifold with a double cover $\widetilde{X} \to X$. The following lemma is given in [12]. (See [12, Proposition 11] and the proof of [12, Theorem 37].)

Lemma 4.1 [12] *For each cohomology class $C \in H^2(X;\ell)$, let $[C]_2 \in H^2(X;\mathbb{Z}_2)$ denote the mod 2 reduction of C. If $[C]_2$ satisfies*

$$[C]_2 = w_2(X) + w_1(\ell_{\mathbb{R}})^2,$$

then there exists a Spin^{c-}*–structure $\mathfrak{s}$ on $\widetilde{X} \to X$ such that $\tilde{c}_1(\mathfrak{s}) = C$.*

Note that, as well as usual Spin^c structure, we may define the notion of a *topological* Spin^{c-}*–structure* on a topological manifold and a *families topological* Spin^{c-}*–structure* on a continuous bundle of manifolds, namely a manifold bundle whose structure group is the homeomorphism group of the fiber. (See [3, Section 4.2] for (families) topological Spin^c structures.) Given a continuous bundle of manifolds and a families topological Spin^{c-}–structure on it, if the manifold bundle is smoothable, then the families topological Spin^{c-}–structure induces a families Spin^{c-}–structure in the usual sense.

Lemma 4.2 *For $i = 1, \dots, n$, let X_i be an oriented closed 4–manifold, $\widetilde{X}_i \to X_i$ be a double cover, $\mathfrak{s}_i$ be a* Spin^{c-}*–structure on $\widetilde{X}_i \to X_i$, f_i be a self-diffeomorphism of X_i preserving orientation of X_i and the isomorphism class of $\mathfrak{s}_i$. Suppose that each f_i*

has a fixed ball B_i embedded in X_i, and extend f_i to a self-diffeomorphism of X by identity outside X_i. Define the connected sums $X = X_1 \# \cdots \# X_n$ and $\mathfrak{s} = \mathfrak{s}_1 \# \cdots \# \mathfrak{s}_n$ by gluing around B_i. Then there exist commuting lifts $\tilde{f}_1, \dots, \tilde{f}_n$ in $\mathrm{Aut}(X, \mathfrak{s})$ of the commuting diffeomorphisms $f_1, \dots, f_n$.

Moreover, a similar statement holds also for topological Spin^{c-}–structures.

Proof The proof of the case for topological Spin^{c-}–structures is similar to the smooth case, so we give the proof only for the smooth case. Note that we have an exact sequence

$$1 \to \mathscr{G}(X) \to \mathrm{Aut}(X, \mathfrak{s}) \to \mathrm{Diff}(X, [\mathfrak{s}]) \to 1,$$

where $\mathscr{G}(X)$ is the gauge group of the Spin^{c-}–structure $\mathfrak{s}$ and $\mathrm{Diff}(X, [\mathfrak{s}])$ is the group of diffeomorphisms preserving the isomorphism class of $\mathfrak{s}$. Take a lift $\hat{f}_i$ in $\mathrm{Aut}(X, \mathfrak{s})$ of f_i. Since f_i is supported inside $X_i \setminus B_i$, we have that

$$\hat{f}_i|_{X \setminus (X_i \setminus B_i)} \in \mathscr{G}(X \setminus (X_i \setminus B_i)).$$

Set $u_i = \hat{f}_i|_{X \setminus (X_i \setminus B_i)}$. To complete the proof of the Lemma 4.2, it suffices to show that there exists an extension of each u_i to an element of $\mathscr{G}(X)$, since then the lifts $\tilde{f}_i := u_i^{-1} \cdot \hat{f}_i$ of f_i satisfy the desired property.

To see that $u_i \in \mathscr{G}(X \setminus (X_i \setminus B_i))$ can be extended to an element of $\mathscr{G}(X)$, note that we may assume that $\widetilde{X}_i \to X_i$ is the trivial double cover around B_i and that $\mathfrak{s}$ is a trivial Spin^{c-}–structure around B_i. Then, as noted in [13, Remark 2.8], we may regard $u_i|_{\partial B_i}$ as a map $u_i|_{\partial B_i} : S^3 \to U(1)$, which can be deformed continuously to the constant map onto the identity element in $U(1)$ since $\pi_3(U(1)) = 0$. This implies that u_i can be extended as we desired. □

We can now start the proof of Theorem 1.1. Some of ideas of the construction of a nonsmoothable family E with fiber X are based on [1, Theorem 10.3; 8, Theorem 4.1; 11, Sections 3 and 4; 12, Section 2; 13, Section 1].

Proof of Theorem 1.1 Let X be as in the statement of Theorem 1.1. Set

$$N = \mathop{\#}_{i=1}^{p} (S^1 \times Y_i) \mathop{\#}_{j=1}^{q} (S^2 \times \Sigma_j). \tag{8}$$

Since the assertion of Theorem 1.1 is invariant under reversing orientation of M, we may assume that $\sigma(M) < 0$ without loss of generality. Then we have $n = b_+(M)$. Note that, since M is assumed to be indefinite, we have $b_+(M) > 0$.

A local system ℓ^N on N is constructed in [13, Section 1.2]. We recall the construction.

For a connected double cover $\widetilde{S}^1 \to S^1$, taking a product with Y_i for each $i = 1, \dots, p$, we have a connected double cover $\widetilde{S}^1 \times Y_i \to S^1 \times Y_i$.

Let $\widetilde{T}^2 \to T^2$ be a nontrivial double cover. For each $j = 1, \dots, q$, consider Σ_j as a connected sum $\Sigma_j = T^2 \# \cdots \# T^2$. Taking a fiber sum of $\widetilde{T}^2 \to T^2$, we obtain a double cover $\widetilde{\Sigma}_j \to \Sigma_j$.

Let $\widetilde{N} \to N$ be a fiber sum of $\widetilde{S}^1 \times Y_i \to S^1 \times Y_i$ ($i = 1, \dots, p$) and $\widetilde{\Sigma}_j \to \Sigma_j$ ($j = 1, \dots, q$). We define the local system ℓ^N by $\ell^N = \widetilde{N} \times_{\pm 1} \mathbb{Z}$. Let $\ell^N_{\mathbb{R}} = \widetilde{N} \times_{\pm 1} \mathbb{Z}$. Then we have

(9) $$b_2^{\ell^N}(N) = 0 \quad \text{and} \quad w(\ell^N_{\mathbb{R}})^2 = 0.$$

Let $\widetilde{X} \to X$ be the fiber sum of the trivial double cover $M \sqcup M \to M$ and $\widetilde{N} \to N$. Set $\ell = \widetilde{X} \times_{\pm 1} \mathbb{Z}$ and $\ell_{\mathbb{R}} = \widetilde{X} \times_{\pm 1} \mathbb{R}$. Then we have

(10) $$H^2(X; \ell) \cong H^2(M; \mathbb{Z}) \oplus H^2(N; \ell^N)$$

and

(11) $$w_1(\ell_{\mathbb{R}})^2 = (0, w_1(\ell^N_{\mathbb{R}})^2)$$

through (10), and also have

$$b_+^{\ell}(X) = b_+(M) = n.$$

It follows from (9) and (11) that

(12) $$w_2(X) + w_1(\ell_{\mathbb{R}})^2 = w_2(M)$$

since $w_2(N) = 0$. Below we consider the case that M is spin and that M is nonspin separately.

First, let us consider the case that M is spin. In this case, M is homeomorphic to

(13) $$2m(-E_8) \# nS^2 \times S^2$$

for some m by Freedman's theorem, where $-E_8$ denotes the negative-definite E_8–manifold. Note that we have $m > 0$ since we have assumed that $\sigma(M) < 0$ (actually we also have $n \geq 2m + 1$ by Furuta's $10/8$–inequality, but this fact is not necessary here). Henceforth we shall identify M with (13) as topological manifold.

As noted in [11, Example 3.3], one may easily find an orientation-preserving self-diffeomorphism $\varrho \colon S^2 \times S^2 \to S^2 \times S^2$ satisfying the following two properties:

- There exists a 4–ball B embedded in $S^2 \times S^2$ such that the restriction of ϱ on a neighborhood of B is the identity map.
- ϱ reverses orientation of $H^+(S^2 \times S^2)$.

Let $f_1, \dots, f_{n-1}$ be copies of ϱ on each connected summand of $(n-1)(S^2 \times S^2)$, and let us extend them as homeomorphisms of M and X by identity over the other connected sum factors. Since $f_1, \dots, f_{n-1}$ commute with each other, we can form the multiple mapping torus

$$X \to E \to T^{n-1}$$

of $f_1, \dots, f_{n-1}$. This family E is a $\mathrm{Homeo}(X)$–bundle, for which we shall show nonsmoothability. We argue by contradiction and suppose the family $X \to E \to T^{n-1}$ has a reduction of structure group to $\mathrm{Diff}(X)$.

Let $M \to E_M \to T^{n-1}$ denote the multiple mapping torus of $f_1, \dots, f_{n-1}$ restricted to M. Then the family E is the fiberwise connected sum of E_M and the trivialized bundle $T^{n-1} \times N \to N$. As in the proof of [1, Theorem 10.3], it is easy to see that $w_{n-1}(H^+(E_M)) \neq 0$. This nonvanishing together with (9) and (10) implies that

$$w_{n-1}(H^+(E, \ell)) \neq 0 \quad \text{in } H^{n-1}(B; \mathbb{Z}_2). \tag{14}$$

Since now we have $w_2(M) = 0$, it follows from Lemma 4.1 and (12) that there exists a Spin^{c-}–structure $\mathfrak{s}$ on $\widetilde{X} \to X$ such that $\tilde{c}_1(\mathfrak{s}) = 0$. More precisely, we may take $\mathfrak{s}$ to be trivial on the connected summand M in X. Here we note the following lemma:

Lemma 4.3 *The family E has a lift of structure group to* $\mathrm{Aut}(X, \mathfrak{s})$, *provided that E has a reduction of structure group to* $\mathrm{Diff}(X)$.

Proof Since the Spin^{c-}–structure $\mathfrak{s}$ on the connected summand M in X is trivial, each f_i obviously preserves the isomorphism class of the restriction of the topological Spin^{c-}–structure $\mathfrak{s}$ on the i^{th} connected summand of $n(S^2 \times S^2)$. Therefore this lemma follows from Lemma 4.2. □

We can now complete the proof of Theorem 1.1 in the case that M is spin. By (14) and Lemma 4.3, the family $X \to E \to T^{n-1}$ satisfies the assumption of Theorem 3.4; thus $\sigma(X) \geq 0$. However $\sigma(X) = \sigma(M)$ holds and we assumed that $\sigma(M) < 0$. This is a contradiction, and hence E is nonsmoothable.

Next, let us consider the case that M is not spin. Some of arguments here are very similar to the spin case above. Denote by $-\mathbb{CP}^2_{\text{fake}}$ the closed simply connected topological 4–manifold whose intersection form is (-1) and whose Kirby–Siebenmann class does not vanish. Then M is homeomorphic to

$$m(-\mathbb{CP}^2)\,\#\,(-E_8)\,\#\,(-\mathbb{CP}^2_{\text{fake}})\,\#\,n(S^2\times S^2)$$

for some $m\geq 0$ and $n>0$. Let $f_1,\ldots,f_n$ be the commuting self-diffeomorphisms of $n(S^2\times S^2)$ obtained as copies of ϱ above as well as the spin case, and extending them as self-homeomorphisms of X by identity, we may obtain a continuous family $X\to E\to T^n$ as the multiple mapping torus. Similar to the spin case, we argue by contradiction and suppose that the family $X\to E\to T^n$ has a reduction of structure group to $\mathrm{Diff}(X)$.

Let $M\to E_M\to T^n$ denote the multiple mapping torus of $f_1,\ldots,f_n$ restricted to M. Then it is easy to see that $e(H^+(E_M,\mathbb{Z}_{w_1(H^+(E_M))}))\neq 0$, where $\mathbb{Z}_{w_1(H^+(E_M))}$ denotes the local system with coefficient group $\mathbb{Z}$ determined by $w_1(H^+(E_M))$. This observation together with (9) and (10) implies that

$$w_n(H^+(E,\ell))\neq 0 \quad\text{in } H^n(B;\mathbb{Z}_2). \tag{15}$$

Let $C\in H^2(X;\ell)$ be a cohomology class expressed as

$$C=(e_1,\ldots,e_m,0,e,0,0)$$

under the direct sum decomposition of $H^2(X;\ell)$ into

$$H^2(-\mathbb{CP}^2;\mathbb{Z})^{\oplus m}\oplus H^2(-E_8;\mathbb{Z})\oplus H^2(-\mathbb{CP}^2_{\text{fake}};\mathbb{Z})\oplus H^2(n(S^2\times S^2);\mathbb{Z})\oplus H^2(N;\ell^N),$$

where e_i and e denote a generator of $H^2(-\mathbb{CP}^2;\mathbb{Z})$ and that of $H^2(-\mathbb{CP}^2_{\text{fake}};\mathbb{Z})$, respectively. Then C satisfies that $[C]_2=w_2(M)$. Therefore it follows from Lemma 4.1 and (12) that there exists a Spin^{c-}–structure $\mathfrak{s}$ on $\tilde{X}\to X$ such that $\tilde{c}_1(\mathfrak{s})=C$.

As well as Lemma 4.3, the structure group of E lifts to $\mathrm{Aut}(X,\mathfrak{s})$ provided that E is smoothable. Therefore by (15) we may apply Theorem 3.1 to this family, and thus we have $\tilde{c}_1(\mathfrak{s})^2\leq\sigma(X)$. However it follows from a direct calculation that

$$\tilde{c}_1(\mathfrak{s})^2=C^2=-m-1 \quad\text{and}\quad \sigma(X)=\sigma(M)=-m-9.$$

This is a contradiction, and hence E is nonsmoothable. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.4 The proof is very similar to that of Theorem 1.1 above. Let X be as in the statement of Theorem 1.4 and $M'=mS\,\#\,M$. Define N by (8). By an argument in the proof of [7, Theorem 3] by Hambleton and Kreck, it turns out that an Enriques surface S is homeomorphic to $-E_8 \# (S^2 \times S^2) \# W$, where W is a nonspin topological rational homology 4–sphere with $\pi_1(W) \cong \mathbb{Z}/2$ and with nontrivial Kirby–Siebenmann invariant. Hence mS is homeomorphic to

$$m(-E_8) \# mS^2 \times S^2 \# mW.$$

Note that M can be topologically decomposed as follows:

- If M is nonspin, M is homeomorphic to $a(S^2 \times S^2) \# b(-\mathbb{CP}^2)$ for some $a \geq 0$ and $b > 0$. Here we have used the assumption that $\sigma(M) < 0$ if M is nonspin.
- If M is spin, M is homeomorphic to $a(S^2 \times S^2) \# 2b(-E_8)$ for some $a, b \geq 0$.

Let us repeat the argument in the proof of Theorem 1.1 until getting (12) under replacing M with M'.

First, let us assume that M is spin. Then M' is homeomorphic to

$$(m+2b)(-E_8) \# nS^2 \times S^2 \# mW,$$

where $n = a+m$. Let $f_1, \dots, f_n$ be the commuting self-diffeomorphisms of $n(S^2 \times S^2)$ obtained as copies of ϱ given in the proof of Theorem 1.1, and extending them as self-homeomorphisms of X by identity, we may obtain a continuous family $X \to E \to T^n$ as the multiple mapping torus. We argue by contradiction and suppose that the family $X \to E \to T^n$ has a reduction of structure group to $\mathrm{Diff}(X)$. First, note that we again obtain (15) similarly. Let $\alpha \in H^2(S;\mathbb{Z})$ be the cohomology class given by $\alpha = (0,1) \in H^2(S;\mathbb{Z})$ under the direct sum decomposition

$$H^2(S;\mathbb{Z}) \cong H^2(-E_8 \# S^2 \times S^2;\mathbb{Z}) \oplus H^2(W;\mathbb{Z}),$$

where $H^2(W;\mathbb{Z})$ is known to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and $1 \in H^2(W;\mathbb{Z})$ denotes the unique nontrivial element. Let $C \in H^2(X;\ell)$ be the cohomology class given by

$$C = (0, \alpha_1, \dots, \alpha_m, 0)$$

under the decomposition of $H^2(X;\ell)$ into

$$H^2((m+2b)(-E_8) \# nS^2 \times S^2;\mathbb{Z}) \oplus H^2(W;\mathbb{Z})^{\oplus m} \oplus H^2(N;\ell^N),$$

where α_i are copies of α. Then C satisfies that $[C]_2 = w_2(M')$. We can deduce from an argument similar to the proof of Theorem 1.1 that $C^2 \leq \sigma(X)$ using Theorem 3.1. However it follows from a direct calculation that $C^2 = 0$ and $\sigma(X) = -8(m+2b)$. This is a contradiction, and hence E is nonsmoothable. This completes the proof of Theorem 1.4 in the spin case.

Next, let us assume that M is nonspin. The proof is similar to the spin case above. First, note that M' is homeomorphic to

$$m(-E_8)\,\#\,n(S^2\times S^2)\,\#\,b(-\mathbb{CP}^2)\,\#\,mW,$$

where $n=a+m$. As well as the spin case, let $f_1,\dots,f_n$ be the commuting self-diffeomorphisms of $n(S^2\times S^2)$ obtained as copies of ϱ, and extending them as self-homeomorphisms of X by identity, we may obtain a continuous family $X\to E\to T^n$ from $f_1,\dots,f_n$. Suppose that the family $X\to E\to T^n$ has a reduction of structure group to $\mathrm{Diff}(X)$. We again obtain (15) similarly. Let $\bar{e}$ be a generator of $H^2(-\mathbb{CP}^2;\mathbb{Z})$. Let $C\in H^2(X;\ell)$ be the cohomology class given by

$$C=(0,\bar{e}_1,\dots,\bar{e}_b,\alpha_1,\dots,\alpha_m,0)$$

under the decomposition of $H^2(X;\ell)$ into

$$H^2(m(-E_8)\,\#\,n(S^2\times S^2);\mathbb{Z})\oplus H^2(b(-\mathbb{CP}^2);\mathbb{Z})\oplus H^2(W;\mathbb{Z})^{\oplus m}\oplus H^2(N;\ell^N),$$

where $\bar{e}_j$ are copies of $\bar{e}$. Then C satisfies that $[C]_2=w_2(M')$, and we can deduce that $C^2\le\sigma(X)$ using Theorem 3.1. However it follows from a direct calculation that $C^2=-b$ and $\sigma(X)=-8m-b$. Since $b>0$ in the nonspin case, this is a contradiction. Hence E is nonsmoothable. This completes the proof of Theorem 1.4 in the nonspin case. □

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