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Time-periodic solutions of Hamiltonian PDEs using pseudoholomorphic curves

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We extend the pseudoholomorphic curve methods from Floer theory to infinite-dimensional phase spaces and use our results to prove the existence of a forced time-periodic solution to a general Hamiltonian PDE with regularizing nonlinearity. In particular, when the nonlinearity is sufficiently regularizing, bounded and time-periodic, we prove an infinite-dimensional version of Gromov–Floer compactness by using ideas from the theory of Diophantine approximations to overcome the small divisor problem. Furthermore, in the case when the infinite-dimensional phase space is a product of a finite-dimensional closed symplectic manifold with linear symplectic Hilbert space, we prove a cup-length estimate for the number of periodic solutions.

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Introduction

It is well known that problems in classical mechanics can be formalized and solved in the language of Hamiltonian systems on finite-dimensional symplectic manifolds, or phase spaces. This led to a rather novel mathematical branch called symplectic topology. Most of the groundbreaking results in symplectic topology rely on the existence of so-called pseudoholomorphic curves, that is, maps from a Riemann surface into the finite-dimensional symplectic manifold equipped with a compatible almost-complex structure, satisfying a Cauchy–Riemann-type equation; see McDuff and Salamon [23]. They were introduced by M Gromov in his seminal paper [17]. In order to prove the existence of time-periodic solutions of Hamiltonian systems on finite-dimensional phase spaces, A Floer has developed the tool of Floer homology, which is based on the study of moduli spaces of so-called Floer curves, which satisfy a Cauchy–Riemann-type equation involving a zeroth-order Hamiltonian term; see [13]. The key technical result on which his theory relies is a compactness result for the moduli space of Floer curves called Gromov–Floer compactness which, among other things, crucially uses compactness of the target manifold; see also Floer and Hofer [14].

Generalizing from point particles to continuous fields, and hence from classical mechanics to classical field theory, one arrives at Hamiltonian systems which are defined on infinite-dimensional phase spaces, such as symplectic Hilbert spaces. More generally, it is known that a number of important partial differential equations, such as the (nonlinear) Schrödinger equation, the (nonlinear) wave equation, the Korteweg-de Vries equation and many more, can be viewed as infinite-dimensional Hamiltonian systems. Such partial differential equations are also called *Hamiltonian partial differential equations*; see Kuksin [22]. We stress that it is a general feature of Hamiltonian PDEs that the Hamiltonian is typically only densely defined on the symplectic Hilbert space. In these cases, the best thing we can expect is an sc–Hamiltonian system in the sense of Hofer, Wysocki and Zehnder [19].

We show how the pseudoholomorphic curve methods of Hamiltonian Floer theory can be generalized to the infinite-dimensional setup of Hamiltonian PDEs with so-called regularizing nonlinearities. As a first step towards generalizing Floer theory from finite dimensions to infinite dimensions, we prove a version of Gromov–Floer compactness and use our result to establish the existence of time-periodic solutions. There has been a lot of great work on finding solutions; see Rabinowitz [25], or Berti [1] for an overview of the current state of the art and Section 9 for more references. Our aim is to show

how pseudoholomorphic methods can be applied to this problem. An obvious problem comes from the fact that the Hamiltonian is only densely defined on the symplectic Hilbert space. As it turns out, one of the main new arising challenges is a small divisor problem: while for generic time and space periods the underlying linear Hamiltonian PDE only has the trivial periodic solution and the return map only has eigenvalues different from one, there is always a subsequence of eigenvalues converging to one.

Apart from showing that the bubbling-off argument still works in order to uniformly bound derivatives, as our main result we show how regularizing the nonlinearity of a Hamiltonian PDE needs to be in order to guarantee that Gromov–Floer compactness still holds. It turns out that this is intimately related with the aforementioned small divisor problem, and ultimately with the theory of diophantine approximations. We define the concept of (weakly) admissible nonlinearities in order to classify the types of nonlinearities for which Gromov–Floer compactness can still be established, and we further study the regularity of the Floer curves and the time-periodic solution in both the flow and time coordinates as well as the extra spatial coordinate.

We want to emphasize that this paper is mainly addressed to researchers with a background in Hamiltonian Floer theory who are interested in the generalization of these techniques to the infinite-dimensional case of Hamiltonian PDEs. While we cite some well-established results from finite-dimensional Floer theory without proof, this paper is written in such a way that we do not assume any prior knowledge about Hamiltonian partial differential equations, small divisor problems and Diophantine approximations. In particular, we make no claim that the results about periodic solutions could not be obtained using different methods. Our ultimate goal is to construct a full Floer homology theory for Hamiltonian PDEs with regularizing nonlinearities. As a first result which needs pseudoholomorphic curve methods, we prove a cup-length estimate for a Hamiltonian system on a phase space which is a product of a closed finite-dimensional symplectic manifold with linear symplectic Hilbert space.

This paper is organized as follows: In Section 1 we give a brief introduction to nonlinear Hamiltonian PDEs and establish notation. There and in Section 2 we give a number of examples. In Section 2 we furthermore discuss the part of the Hamiltonian which contains the differential operator, as well as illustrate the so-called small divisor problem which occurs when passing to infinite dimensions. In the same section and in Section 3, we give a number of admissibility conditions and define the class of Hamiltonians for which our results hold, and we give a counterexample to show that some of these conditions cannot be relaxed.

Section 4 contains a summary of the main results. In Section 5 we recall well-established results from finite-dimensional Floer theory in order to establish the existence of Floer curves in finite-dimensional linear symplectic spaces. In Section 6 we generalize the bubbling-off analysis for finite-dimensional pseudoholomorphic curves and show that the derivatives of the sequence of Floer curves are bounded; this includes a standard elliptic regularity argument to include higher derivatives. Using a series of estimates, Section 7 shows how the higher-dimensional components of Floer curves can be controlled in the presence of the small divisor problem. In Section 8 we complete the proof of the main theorem and subsequently generalize this to a wider class of Hamiltonians in Section 9. Section 10 provides a cup-length estimate for the number of periodic solutions when we consider Hamiltonian systems on the product of a finite-dimensional closed symplectic manifold with linear symplectic Hilbert space. Finally, the appendix introduces sc–Hamiltonian flows.

1 Nonlinear Hamiltonian PDEs

Let us start by describing the framework in which we will study our PDEs. Let $(\mathbb{H},\omega)$ be a separable symplectic Hilbert space, meaning a separable Hilbert space with an antisymmetric bilinear map ω for which $i_\omega\colon \mathbb{H}\to\mathbb{H}^*$ is an isomorphism. Following [21], we fix a complete Darboux basis $\{e_n^\pm\}_{n\in\mathbb{Z}}$ in the sense that $\omega(e_i^+,e_j^-)=\delta_{ij}$. We define an antisymmetric linear operator J by $Je_n^\pm:=\mp e_n^\mp$ so that we can write the symplectic form as $\omega=\langle\,\cdot\,,J\,\cdot\,\rangle$ for some equivalent inner product on $\mathbb{H}$, which we fix from now on. Introducing the complex basis $z_n:=2^{-\frac{1}{2}}(e_n^+ + ie_n^-)$ for $n\in\mathbb{Z}$, the Hilbert space $\mathbb{H}$ can be identified with a subspace of the complexified Hilbert space $\mathbb{H}\otimes\mathbb{C}$ spanned by z_n, $n\in\mathbb{Z}$, where $J=i$ and $\omega=i\,dz\wedge d\bar{z}$. For a smooth Hamiltonian function $H\colon\mathbb{H}\to\mathbb{R}$, the symplectic gradient X_H is defined by $dH(\cdot)=\omega(X_H,\cdot)$ so that $X_H=J\nabla H$. We then write the Hamiltonian equation as

$$\dot{u}=X_H(u)=J\nabla H(u). \tag{1}$$

In this paper, we consider general time-dependent Hamiltonian PDEs of the form

$$\dot{u}=JAu+J\nabla F_t(u)$$

with underlying Hamiltonian

$$H_t(u)=\tfrac{1}{2}\langle Au,u\rangle+F_t(u)=:H_A(u)+F_t(u) \tag{2}$$

for some time-dependent and T–periodic nonlinearity F_t defined in Definition 3.3, and quadratic term H_A (also called the free term) defined by a linear, possibly unbounded, self-adjoint (differential) operator $A\colon \mathrm{Dom}(A) \subset \mathbb{H} \to \mathbb{H}$ where the domain of A is dense in $\mathbb{H}$. We will always assume the nonlinearity to be smooth in the time variable. We restrict to the case where the Hilbert space is a space of functions in one variable, which depends on the specific Hamiltonian PDE. See [1; 6; 21] for examples of Hamiltonian PDEs.

Before we give some examples of Hamiltonian PDEs with Hamiltonian of the form (2), let us first address the problem that the free term H_A is actually only densely defined when the differential operator A is of positive order. However, the free flow $\phi_t^A = e^{tJA}$ is a linear unitary map and hence extends to a linear unitary map on the whole space $\mathbb{H}$. Indeed, we obtain an sc–Hamiltonian flow in the sense of the appendix. Even though the existence of the free flow can be established, we have to be careful about the kind of nonlinearities we allow for the flow of the full Hamiltonian to still be guaranteed: even when F_t is smooth, there need not exist a flow of F_t due to compactness problems. The existence of the flow is a very delicate problem and a large amount of great work has been done in this direction. We avoid this problem by working with nonlinearities for which finite-dimensional approximations exist and are immediate. In particular, below we consider two important examples of Hamiltonian PDEs with regularizing nonlinearities (as in [8; 9]). Such nonlinearities appear in nonlocal (or quasilocal) classical field theories, where fields have nonlocal interactions, and actually lead to a Hamiltonian partial *integrodifferential* equation. In contrast to local models, such nonlocal models, which in many cases model reality even better, are almost never integrable. The models we consider below can have arbitrary nonlocality, or quasilocality [29]. For the relevance of nonlocal Hamiltonian PDEs see [10; 18; 32].

Example 1.1 (nonlinear wave equation (NLW)) We write the nonlinear wave equation with regularizing nonlinearity as

(3) $$\ddot{\varphi} - \varphi_{xx} - \partial_1 g_t(\varphi * \psi, x) * \psi - c_t = 0, \quad \varphi = \varphi(t,x) = \varphi(t, x+X),$$

for $x \in S^1 = \mathbb{R}/X\mathbb{Z}$, with $\psi \in C^h(S^1)$ for $h > 0$ and $g_{t+T} = g_t$ being bounded and smooth in both components and having bounded derivatives in the first component, and $c_t = c_{t+T} \in C^h(S^1)$ denoting some time-dependent exterior potential.[1] A specific example of this would be a nonlocal sine-Gordon equation with exterior potential:

$$\ddot{\varphi} - \varphi_{xx} - a_t \sin(\varphi * \psi) * \psi - c_t = 0.$$

[1] Some authors write the NLW with an extra term $+mu$ with $m \geq 0$ on the left-hand side. We choose to set $m = 0$.

This appears, for example, in a nonlocal Frenkel–Kontorova model with time- and space-periodic coefficient a_t and exterior potential c_t.

The nonlinear wave equation is a Hamiltonian PDE on the Hilbert space

$$\mathbb{H} = L^2(S^1,\mathbb{R}) \times L^2(S^1,\mathbb{R}).$$

It can be written as a Hamiltonian PDE as

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\pi} \end{pmatrix} = \begin{pmatrix} \pi \\ \varphi_{xx} + \partial_1 g_t(\varphi * \psi, x) * \psi + c_t \end{pmatrix} = J\nabla_{L^2} H_t(\varphi,\pi)$$

with $J(\varphi,\pi) = (-\pi,\varphi)$ and with Hamiltonian

$$H_t(\varphi,\pi) = \frac{1}{2}\int_0^X (-\varphi_x^2 - \pi^2 + 2g_t(\varphi * \psi, x) + 2c_t\varphi)\,dx.$$

However, this Hilbert space on which the NLW is modeled does not admit a complete Darboux basis which is compatible with the differential operator A. We will study a different structure in Example 2.2.

Example 1.2 (nonlinear Schrödinger equation (NLS)) Consider the nonlinear Schrödinger equation with regularizing nonlinearity

$$i\dot{u} + u_{xx} + \partial_1 f_t(|u * \psi|^2, x)(u * \psi) * \psi = 0, \quad u = u(t,x) = u(t, x + X),$$

for $x \in S^1 = \mathbb{R}/X\mathbb{Z}$, with $\psi \in C^h(S^1)$ for $h > 0$ and with $f_{t+T} = f_t$ smooth in both components. We also require that the map $\tilde{f}_t\colon (s,x) \mapsto f_t(|s|^2, x)$ is bounded and has bounded derivatives in the first component. The Hilbert space is $L^2(S^1,\mathbb{C})$. The Hamiltonian is given by

$$H_t(u) = \frac{1}{2}\int_0^X (-|u_x|^2 + f_t(|u * \psi|^2, x))\,dx.$$

This Hamiltonian PDE descends to an infinite-dimensional Hamiltonian system on projective Hilbert space.

See [11] for more details on this last example. In contrast to [11], we will not focus on specific examples but rather consider nonlinear Hamiltonian PDEs with general nonlinearities on linear space. Note that even though the nonlinearity is not local, it can be quasilocal in the sense that the smoothing kernel ψ can have arbitrarily small support.

2 Diophantineness condition

Let us start with the free term of the Hamiltonian. Since A is self-adjoint, we can diagonalize it. Here we have to make an assumption:

Definition 2.1 The differential operator A of the free term H_A is called *admissible* when it is pure of degree $d \geq 1$ and there exists a complete Darboux basis $(e_n^{\pm})$ of eigenvectors of the operator A in the sense that $Ae_n^{\pm} = \lambda_n e_n^{\pm}$ with real eigenvalues of the form $\lambda_n = an^d$ for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_{>0}$.

From now on we assume that the operator A is admissible. Note that the condition that e_n^+ and e_n^- have the same eigenvalue and form a Darboux basis is equal to the statement that the commutator $[J, A]$ equals zero. Our two main examples, see below, satisfy this condition and have eigenvalues of the form $\lambda_n = (2\pi/X)^d n^d$ with $X > 0$ denoting the space periodicity. *Despite the fact that our results apply to general symplectic Hilbert spaces* $\mathbb{H}$, *we will assume in what follows that* $a = (2\pi/X)^d$. So, let us choose such a complete Darboux basis consisting of eigenvectors of A, so that $Ae_n^{\pm} = \lambda_n e_n^{\pm}$. Then the operator JA is diagonal in the complex basis spanned by $z_n = 2^{-\frac{1}{2}}(e_n^+ + ie_n^-)$ with eigenvalues $i\lambda_n$ for $n \in \mathbb{Z}$. Following [21], we note that the flow maps of the free Hamiltonian $\phi_t^A = e^{tJA}$ define a family of linear symplectomorphisms on $\mathbb{H}$ which restrict to linear symplectomorphisms on the finite-dimensional subspaces $\mathbb{C}^{2k+1} = \mathrm{Span}_{\mathbb{C}}\{z_n\}_{n=-k}^k$. The eigenvalues of the time-T flow are e^{ian^dT}. Let us now consider the examples of the NLW and NLS from before; see [21].

Example 2.2 (NLW) The nonlinear wave equation in one space dimension was modeled in Example 1.1 on $\mathbb{H} = L^2(S^1, \mathbb{R}) \times L^2(S^1, \mathbb{R})$. However, we want the Hilbert basis to be a complete Darboux basis of eigenvectors of the operator A. This forces us to choose $\mathbb{H} = W^{\frac{1}{2},2}(S^1, \mathbb{R}) \times W^{\frac{1}{2},2}(S^1, \mathbb{R})$ with $S^1 = \mathbb{R}/X\mathbb{Z}$. In this setting, we study (3) with the same nonlinearity. Now we write the operator A as $A = \mathrm{Diag}(B, B)$ with $B = \sqrt{-\partial_x^2}$ and we write the nonlinear wave equation as

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\pi} \end{pmatrix} = \begin{pmatrix} -B\pi \\ B\varphi - B^{-1}\partial_1 g_t(\varphi * \psi) * \psi - B^{-1}c_t \end{pmatrix}.$$

The inner product on $W^{\frac{1}{2},2}(S^1, \mathbb{R})$ is

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^X Bf(x)g(x)\,dx,$$

and the Hamiltonian on $\mathbb{H}$ is given by

$$H_t(\varphi,\pi) := \frac{1}{2}\left\langle A\begin{pmatrix}\varphi\\ \pi\end{pmatrix}, \begin{pmatrix}\varphi\\ \pi\end{pmatrix}\right\rangle + \frac{1}{2\pi}\int_0^X g_t(\varphi * \psi, x) + c_t\varphi\, dx,$$

where we extend the inner product componentwise to $W^{\frac{1}{2},2}(S^1,\mathbb{R}) \times W^{\frac{1}{2},2}(S^1,\mathbb{R})$. The complete Darboux basis is then given by

$$e_n^+ = \frac{1}{\sqrt{|n|}}(\xi_n(x), 0), \quad e_n^- = \frac{1}{\sqrt{|n|}}(0, \xi_n(x)),$$

where

$$\xi_n(x) = \begin{cases} \sqrt{2}\cos\left(\frac{2\pi n x}{X}\right) & \text{if } n \le 0,\\ \sqrt{2}\sin\left(\frac{2\pi n x}{X}\right) & \text{if } n > 0,\end{cases}$$

and the eigenvalues of A are

$$Ae_n^\pm = \frac{2\pi n}{X} e_n^\pm$$

so $\lambda_n = an^d$ with $a = 2\pi/X$ and $d = 1$. The Hilbert space can be identified with the subspace $\mathrm{Span}_{\mathbb{C}}\{z_n\}_{n\in\mathbb{Z}}$ of the complexified Hilbert space $\mathbb{H}\otimes\mathbb{C}$, with

$$z_n = \frac{1}{\sqrt{2|n|}}(\xi_n(x), i\,\xi_n(x)),$$

and the flow maps are

$$\phi_T^A z_n = e^{iT2\pi n/X} z_n.$$

Example 2.3 (NLS) The Hilbert space for this PDE is $L^2(S^1,\mathbb{C})$ with inner product which, when viewed as a real vector space with inner product

$$\langle f, g\rangle = \mathrm{Re}\,\frac{1}{X}\int_0^X f\bar{g}\,dx,$$

has complete Darboux basis given by

$$e_n^+ = e_n, \quad e_n^- = -i\,e_n,$$

where $\{e_n\}_{n\in\mathbb{Z}}$ is the complete system of eigenfunctions of $-\partial_x^2$ given by

$$e_n(x) = e^{i2\pi n x/X}.$$

These have eigenvalues $\lambda_n = (2\pi n/X)^2$. We can identify this real Hilbert space $(\mathbb{H}, J = i)$ with the complex Hilbert space spanned by $z_n : x \mapsto \sqrt{2}e^{i2\pi n x/X}$ with $n\in\mathbb{Z}$. The time-T flow of the free part of the Hamiltonian is

$$\phi_T^A z_n = e^{iT(2\pi n/X)^2} z_n.$$

Writing the eigenvalues as suggested above, we get $\lambda_n = an^d$ where $a = (2\pi/X)^2$ and $d = 2$.

Here we already catch a glimpse of what will be a problem we need to address, which does not appear in the finite-dimensional case: in order to apply the machinery of Floer theory in our infinite-dimensional situation, we would like to ensure that the system is nondegenerate, ie that the time-T flow map has no eigenvalue equal to one. This in turn means that $aT/2\pi$ must not be rational. Compare this condition for Example 2.2 with [25], which proved existence of forced time-periodic solutions when the number $aT/(2\pi) = T/X$ *is* rational. For general eigenvalues $\lambda_n = an^d$ we need $aT/2\pi = T/X^d \cdot (2\pi)^{d-1}$ to be irrational, where we recall that $a = (2\pi)^d/X^d$ implicitly depends on the space period X. However, even if these numbers are irrational, we are faced with the problem that a subsequence of the eigenvalues of the flow will always converge to one. Let us illustrate this problem somewhat. To prove the existence of a solution to the nonlinear PDE, we will have to assume that the time-T flow of the free Hamiltonian ϕ_T^A has only one fixed point, or, alternatively, that the only solution to the free Hamiltonian equation

$$\dot{u} = JAu, \quad u(0) = u(T) \tag{4}$$

is $u \equiv 0$. When $aT/2\pi$ is irrational, this forces the only solution to (4) to be $u \equiv 0$; if u_0 is a fixed point of the time-T free flow, then expanding u_0 as $\sum \hat{u}_0(n) z_n$ shows that

$$\phi_T^A u_0 = \phi_T^A \sum_n \hat{u}_0(n) z_n = \sum_n e^{iaTn^d} \hat{u}_0(n) z_n.$$

So as long as $aT/2\pi$ is irrational, for any n there are no eigenvalues equal to one. In the limit, however, this is not guaranteed; there could be a subsequence of $(e^{iT\lambda_n})_n$ converging to one. This is an instance of the small divisor problem. To solve this problem, we need to control the way in which the eigenvalues (or a subsequence of them) converge to one. The essence of our approach is that we should not be able to approximate the irrational number $aT/2\pi$ too well by rational ones. More formally:

Definition 2.4 We call the pair of time and space periods $(T, X) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ *admissible* when the number $aT/2\pi = (2\pi)^{d-1} T/X^{d-1}$ is Diophantine.

In particular, the Diophantine number $aT/2\pi$ has finite *irrationality measure*. Let us explain this statement: Every real number can be approximated by a continued fraction and this gives a measure of how well a real number can be approximated by rationals. For all $\sigma \in \mathbb{R}$ there exists $p/q \in \mathbb{Q}$ such that

$$\left|\sigma - \frac{p}{q}\right| < \frac{1}{q^2}.$$

The irrationality measure is a measure of how good this approximation can be. It is defined as the infimum of the set of real numbers ρ for which

$$\frac{c}{q^\rho} < \left|\sigma - \frac{p}{q}\right| < \frac{1}{q^2}$$

holds for all $p/q \in \mathbb{Q}$ with some fixed $c > 0$, and is usually denoted by r. In particular, it is at least two. It turns out that the set of numbers of irrationality measure two (and hence of Diophantine numbers) has full Lebesgue measure [4, Theorem E.3]. For π, it is shown in [27] that $r < 8$. This is used, for example, in [11] to prove a statement similar our main theorem for the NLS on projective Hilbert space. In what follows, by *generic time period* T we will mean T for which $aT/2\pi$ has irrationality measure $r = 2$.

Before turning to the nonlinearity, let us first give an example which shows that the Diophantineness condition (and subsequent regularity conditions for the nonlinearity) are really necessary.

Example 2.5 (counterexample) Consider the linear wave equation with exterior potential

$$\ddot{\varphi} = \varphi_{xx} + c_t, \quad c_{t+T} = c_t. \tag{5}$$

Let us write φ and c as Fourier series

$$\varphi = \sum_{p,n\in\mathbb{Z}} \hat{\varphi}(p,n) e^{2\pi i p t/T} e^{2\pi i n x/X},$$
$$c = \sum_{p,n\in\mathbb{Z}} \hat{c}(p,n) e^{2\pi i p t/T} e^{2\pi i n x/X},$$

with $\overline{\hat{\varphi}(p,n)} = \hat{\varphi}(-p,-n)$ and $\overline{\hat{c}(p,n)} = \hat{c}(-p,-n)$ such that the functions are real-valued. Termwise, (5) becomes

$$\hat{\varphi}(p,n)\Big(\Big(\frac{2\pi n}{X}\Big)^2 - \Big(\frac{2\pi p}{T}\Big)^2\Big) e^{2\pi i p t/T} e^{2\pi i n x/X} = \hat{c}(p,n) e^{2\pi i p t/T} e^{2\pi i n x/X},$$

or

$$\hat{\varphi}(p,n) = \frac{\hat{c}(p,n)\frac{T^2}{(2\pi n)^2}}{\big(\frac{T}{X} - \frac{p}{n}\big)\big(\frac{T}{X} + \frac{p}{n}\big)}.$$

When T/X is not Diophantine, there is a subsequence $(p',n') \subset (p,n)_{p,n\in\mathbb{Z}}$ for which the denominator in the expression for $\hat{\varphi}(p',n')$ goes to zero exponentially fast. If we define the exterior potential c_t by

$$\hat{c}(p,n) := \begin{cases} \frac{(2\pi n)^2}{T^2}\left(\frac{T}{X}-\frac{p}{n}\right)\left(\frac{T}{X}+\frac{p}{n}\right) & \text{if } (p,n)=(p',n'), \\ 0 & \text{otherwise,} \end{cases}$$

then c_t is smooth, but $\hat{\varphi}(p',n')$ is constantly one, so there is no solution. Writing the exterior potential in the Hamiltonian for the NLW in Example 2.2 as $\langle (c_t,0),\cdot\rangle$, we see that c_t should have some minimal regularity, depending on the irrationality measure of T/X, to ensure the existence of a solution.

3 A–admissible and weakly A–admissible nonlinearities

In order to deal with the asymptotic degeneracy caused by the small divisor problem, our key idea is as follows: in order for Gromov–Floer compactness to hold, we want to assume that the nonlinearity can be approximated by finite-dimensional nonlinearities in the sense of Section 5 better than the eigenvalues of the time-T flow of the free Hamiltonian approach one. This puts restrictions on the regularity of the nonlinearity. To explain this, consider expanding $u \in \mathbb{H}$ as $u = \sum_n \hat{u}(n) z_n$ and letting u^k denote the restriction to $\mathbb{C}^{2k+1} = \mathrm{Span}_{\mathbb{C}}\{z_n\}_{n=-k}^{k} \subset \mathbb{H}$ given by

$$u^k := \sum_{n=-k}^{k} \hat{u}(n) z_n.$$

The finite-dimensional restriction F_t^k of the nonlinearity is then

$$F_t^k(u) := F_t(u^k),$$

and we write $X_t^{F,k}$ for the symplectic gradient of this finite-dimensional restriction. Then the flow $\phi_t^{F,k}$ of the restricted Hamiltonian F_t^k restricts to the finite-dimensional subspace $\mathbb{C}^{2k+1} \subset \mathbb{H}$.

To formalize the idea that we need some minimal regularity for the nonlinearity to make our methods work, we start by introducing Hilbert scales in the sense of [22]. Working in the complex Hilbert space spanned by z_n for $n \in \mathbb{Z}$, where we can identify J with i, our separable symplectic Hilbert space $\mathbb{H}$ is isometrically isomorphic to $\ell^2(\mathbb{Z},\mathbb{C})$ via the complete basis $\{z_n\}_{n\in\mathbb{Z}}$ by

$$\mathbb{H} \ni u = \sum_n \hat{u}(n) z_n \mapsto (\hat{u}(n))_{n\in\mathbb{Z}},$$

where the sum is understood to be over $n \in \mathbb{Z}$. We will write $\mathbb{H}_0 = \mathbb{H}$ and define $\mathbb{H}_1$ to be the (dense) subspace of $\mathbb{H}_0$ consisting of those $u = \sum_n \hat{u}(n) z_n$ for which $\sum_n \hat{u}(n) n z_n$ is in $\mathbb{H}_0$. We define $\ell^{2,1}$ to be the image of $\mathbb{H}_1$ under the isomorphism between $\mathbb{H}_0$ and ℓ^2 described above. More generally, we define

$$\mathbb{H}_h := \left\{ u \in \mathbb{H}_0 \,\middle|\, \sum_n \hat{u}(n) n^h z_n \in \mathbb{H}_0 \right\}, \qquad \ell^{2,h} := \{(\hat{u}(n))_{n\in\mathbb{Z}} \in \ell^2 \mid (\hat{u}(n) n^h)_{n\in\mathbb{Z}} \in \ell^2\}$$

for $h \geq 0$. Similarly, we define the sequence space

$$\ell^{2,-h} := \{(\hat{u}(n))_{n\in\mathbb{Z}} \mid (u(n) n^{-h}) \in \ell^2\}$$

for $h > 0$ as the space of possibly diverging sequences and, using the Darboux basis, we identify this with the subspace $\mathbb{H}_{-h}$ of the space of tempered distributions. The totality $(\mathbb{H}_h)_{h\in\mathbb{R}}$ is also known as a Hilbert scale. We let $\mathbb{H}_\infty = \bigcap \mathbb{H}_h$ and $\mathbb{H}_{-\infty} = \bigcup \mathbb{H}_h$. Note that $\mathbb{H}_h$ is dense and embeds compactly in $\mathbb{H}_i$ when $h > i$.

Definition 3.1 A map $F_t \colon \mathbb{H}_0 = \mathbb{H} \to \mathbb{R}$ is called *h–regularizing* if it extends to a smooth map

$$F_t \colon \mathbb{H}_{-h} \to \mathbb{R},$$

and it is called ∞–regularizing when it is h–regularizing for all $h \in \mathbb{N}$.

Note that when F_t is h–regularizing the differential defines a map

$$dF_t \colon \mathbb{H}_{-h} \to (\mathbb{H}_{-h})^* \cong \mathbb{H}_h,$$

and so, in particular, for the gradient (with respect to the inner product on $\mathbb{H}$) we have

$$\nabla F_t \colon \mathbb{H}_0 \subset \mathbb{H}_{-h} \to \mathbb{H}_h.$$

Note that this latter property can also be stated in terms of Kuksin's Hilbert scale theory by saying that ∇F_t is a scale morphism of the Hilbert scale $(\mathbb{H}_k)_k$ of order $-h$.

Lemma 3.2 *Assume the nonlinearity is h–regularizing with $h > 0$ such that the extended map $F_t \colon \mathbb{H}_{-h} \to \mathbb{R}$ has bounded C^α–norms for all $\alpha \in \mathbb{N}$. Then ∇F^k converges to ∇F uniformly with all derivatives when viewed as maps into $\mathbb{H}$. Furthermore, when expanding ∇F_t into a Fourier series*

$$\nabla F_t(u) = \sum_{n\in\mathbb{Z}} \widehat{\nabla F_t(u)}(n) z_n$$

we have $\widehat{\nabla F_t(u)}(n) = o(|n|^{-h})$.

Proof For the first statement note first that the boundedness of the C^1–norm of $F_t : \mathbb{H}_{-h} \to \mathbb{R}$ implies that the C^0–norm of $\nabla F_t : \mathbb{H}(\subset \mathbb{H}_{-h}) \to \mathbb{H}_h$ is bounded, which, together with the compact embedding $\mathbb{H}_h \subset \mathbb{H}$, yields that $\nabla F^k(u) \to \nabla F(u)$ with respect to the $\mathbb{H}$–norm uniformly in $u \in \mathbb{H}$ as $k \to \infty$. Further, note that the boundedness of the C^α–norm of $F_t : \mathbb{H}_{-h} \to \mathbb{R}$ yields a uniform bound for the higher derivatives $\nabla^\alpha F_t(u) \in \mathbb{H}_h^{\otimes\alpha}$ for all $u \in \mathbb{H}$, which again implies that $\nabla^\alpha F_t^k(u) \to \nabla^\alpha F_t(u)$ with respect to the $\mathbb{H}^{\otimes\alpha}$–norm uniformly in $u \in \mathbb{H}$ as $k \to \infty$. Because F_t satisfies the regularity assumption $\nabla F_t : \mathbb{H} \to \mathbb{H}_h$, the coefficients in the expansion of $\nabla F_t(u)$ satisfy $|n|^h \widehat{\nabla F_t(u)}(n) = o(1)$, and hence $\widehat{\nabla F_t(u)}(n) = o(|n|^{-h})$. □

In order to be able to use the results from Floer and symplectic homology for open sets in finite dimensions as in [14; 24; 31], we need a sequence of finite-dimensional Hamiltonians which converges in the proper sense to our infinite-dimensional one as above. To ensure that such an approximating sequence exists and that our methods work, we impose the following restrictions on the nonlinearity:

Definition 3.3 A nonlinearity $F_t : \mathbb{H} \to \mathbb{R}$ is called *A–admissible* if A is admissible and $F_t : \mathbb{H} \to \mathbb{R}$ satisfies:

(i) F_t is T–periodic with (T, X) admissible.

(ii) The nonlinearity is h–regularizing with $h > dr$. Here r is the irrationality measure of $aT/2\pi$ and $d \geq 1$ the order of the differential operator A.

(iii) The extended map $F_t : \mathbb{H}_{-h} \to \mathbb{R}$ has bounded C^α–norms for all α.

(iv) F_t has bounded support, in the sense that for every $k \in \mathbb{N}$ there exists $R_k > 0$ such that $F_t(u) = 0$ for all $u \in \mathbb{H}$ with $|u^k| > R_k$.

F_t is called *weakly A–admissible* when there exists t–dependent $c_t = c_{t+T} \in \mathbb{H}_h$ such that $u \mapsto F_t(u) - \langle c_t, u\rangle$ satisfies (i), (ii) and (iii).

We stress that the notion of (weakly) A–admissibility depends on the operator A because the irrationality measure of the number $aT/2\pi$ associated to the eigenvalues of A, as well as the order of the PDE, dictate what regularity we need for the nonlinearity. Observe that the Diophantineness condition is generic in the sense that Diophantine numbers have full Lebesgue measure. Note, though, that the Diophantineness condition should rather be thought of as a condition on the time period, rather than on the eigenvalues of A; we start with a Hamiltonian PDE and this condition restricts what time periods the solutions can have.

In order to explain the relation between A–admissible and weakly A–admissible nonlinearities, we prove:

Proposition 3.4 *Let $\widetilde{F}_t$ be a weakly A–admissible nonlinearity. Then*

$$F_t(u) := \chi(|u|^2_{-h})\widetilde{F}_t(u)$$

with h as in Definition 3.3 condition (ii), and where χ a smooth cutoff function with $\operatorname{supp}(\chi) \subseteq [0, R]$ *for some $R > 0$, is A–admissible.*

Proof The first condition is immediate. In order to see that F_t satisfies conditions (ii) and (iii), note that for every $c_t \in \mathbb{H}_h$ the map $u \mapsto \chi(|u|^2_{-h})\langle c_t, u\rangle$ satisfies conditions (ii) and (iii) as $\langle c_t, u\rangle \leq |c_t|_h |u|_{-h}$. To establish condition (iv), let $R_k := k^h R^{\frac{1}{2}}$ so that when the finite-dimensional restriction u^k of u satisfies $|u^k|_0 > R_k$, then

$$|u|^2_{-h} \geq |u^k|^2_{-h} = \sum_{n=0}^{k} |\hat{u}(\pm n)|^2 n^{-2h} > k^{-2h} \sum_{n=0}^{k} |\hat{u}(\pm n)|^2 > R$$

so that $F_t(u) = 0$. □

By the above proposition it hence suffices to find examples of weakly A–admissible nonlinearities.

Example 3.5 The nonlinearities from Examples 2.2 and 1.2 are weakly A-admissible as long as (X, T) is admissible and $h > dr$. When $u, \varphi \in \mathbb{H}_{-h}$ and $\psi \in C^h$, then $u * \psi, \varphi * \psi \in C^0$ for both examples. Together with the smoothness of $\tilde{f}_t$ and g_t it follows that the maps $x \mapsto \partial_1^\alpha \tilde{f}_t((u * \psi)(x), x)$ and $x \mapsto \partial_1^\alpha g_t((\varphi * \psi)(x), x)$ are continuous and hence (square-) integrable over $\mathbb{R}/X\mathbb{Z}$ for all $\alpha \in \mathbb{N}$. Altogether this is sufficient to prove that F_t is smooth as a map from $\mathbb{H}_{-h}$ to $\mathbb{R}$ in both examples. Since $\tilde{f}_t$ and g_t have uniformly bounded derivatives, the maps $x \mapsto \partial_1^\alpha \tilde{f}_t((u * \psi)(x), x)$ and $x \mapsto \partial_1^\alpha g_t((\varphi * \psi)(x), x)$ are uniformly bounded with respect to u and φ. But this implies that $\nabla^\alpha F_t$ is uniformly bounded for $\alpha = 1, 2, \ldots$; for $\alpha = 0$ in Example 2.2 $\nabla^\alpha F_t$ is only uniformly bounded after subtracting a linear term as allowed in Definition 3.3.

In Main Theorem 4.1, we prove the existence of a Floer curve together with the existence of a periodic solution only for a Hamiltonian with A–admissible nonlinearity. In order to employ the maximum principle for proving compactness of the relevant moduli space of pseudoholomorphic curves, we do have to make the technical assumption Definition 3.3(iv) concerning the support of the nonlinearity. In Section 9, however, we prove that the existence of a forced time-periodic solution is still guaranteed when the nonlinearity is weakly A–admissible instead of A–admissible.

4 Main theorem

Before stating the main theorem, let us rewrite the setting a little. When the Hamiltonian H_t is a sum of two terms H_A and F_t, then the flows of H_t and of H_A and F_t are related via

$$\phi^{H_t} = \phi^{H_A + F_t} = \phi^{H_A \# G_t} = \phi^A \circ \phi^{G_t},$$

where $G_t := F_t \circ \phi_t^A$ and where $(H_A \# f)_t := H_A + f_t \circ \phi_{-t}^A$ for any function f_t. We will work with ϕ_t^A and G_t rather than with $H_t = H_A + F_t$ because H_A (and hence H_t) is only densely defined, whereas the flow ϕ_t^A is a symplectomorphism which is defined on the whole of $\mathbb{H}$. Also G_t turns out to have sufficiently nice properties; see Lemma 6.1 where we show that even though ϕ_t^A is only differentiable on dense subspaces, G_t is at least four times continuously differentiable in t.

Going back, we see that T–periodic solutions of (1) are in one-to-one correspondence with u satisfying

$$\dot{u} = X_t^G(u), \quad u(t+T) = \phi_{-T}^A(u(t)). \tag{6}$$

We call such solutions ϕ_T^A–periodic. We will prove existence of ϕ_T^A–periodic solutions, which by the above correspondence implies existence of a true T–periodic solution. From now on we will use G_t as in (6) instead of F_t and say that G_t is A–admissible when F_t is. Note that the norms of F_t and G_t coincide for fixed t because the free flow is unitary. Recall that when $G_t \equiv 0$, the only solution to the PDE is $u \equiv 0$.

Let $\varphi \in C^\infty(\mathbb{R})$ be a cutoff function specified by

$$\varphi(s) = 0 \quad \text{for } s \le -1, \qquad \varphi(s) = 1 \quad \text{for } s \ge 0, \qquad 0 \le \varphi'(s) \le 2.$$

Main Theorem 4.1 *For a Hamiltonian PDE with A–admissible nonlinearity G_t there exists a $(\lfloor h/d \rfloor - 1)$–times differentiable map $\tilde{u}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{H}_{h-d(r-1)-\frac{1}{2}} \subset \mathbb{H}$ for $h > dr$, called a* ***Floer curve****, which satisfies the Floer equation and ϕ_T^A–periodicity condition*

$$\bar{\partial}\tilde{u} + \varphi(s)\nabla G_t(\tilde{u}) = 0, \quad \tilde{u}(s, t+T) = \phi_{-T}^A \tilde{u}(s,t), \tag{7}$$

where $\bar{\partial} = \partial_s + i\partial_t$. The Floer curve $\tilde{u}$ connects $u_0 \equiv 0$ with a (weak) solution $u_1(t)$ of the nonlinear Hamiltonian PDE (6) and hence of (1), in the sense that there exist sequences $s_n^\pm \in \mathbb{R}$ with $s_n^\pm \to \pm\infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \tilde{u}(s_n^-, t) = 0, \qquad \lim_{n\to\infty} \tilde{u}(s_n^+, t) = u_1(t)$$

in the $C^{\lfloor h/d \rfloor -1}$–sense. In particular, when the nonlinearity is ∞–regularizing, then both the Floer curve and the periodic orbit are smooth in all variables s, t and x.

Note that we call u_1 a weak solution, since $h-d(r-1)-\frac{1}{2} > d-\frac{1}{2}$ might not be large enough to guarantee that u_1 is also a solution in the classical sense. Here and after we continue to identify $\mathbb{H}$ with a subspace of the complexified Hilbert space spanned by z_n for $n \in \mathbb{Z}$, and write i instead of J. We emphasize that we are using the setup of Floer homology for general symplectomorphisms from [7] because even though the Hamiltonian H_A is only densely defined, its flow ϕ_t^A is an everywhere defined symplectomorphism. To use this setup, we use the fact that $(\phi_{-T}^A)_* i = i$.

To go back from G_t to F_t and obtain a true T–periodic Floer curve for the Hamiltonian $H_t = H_A + F_t$, we define $\tilde{\tilde{u}}(s,t) := \phi_t^A \tilde{u}(s,t)$ for $(s,t) \in \mathbb{R} \times \mathbb{R}$. It immediately follows that (7) is equivalent to

$$\bar{\partial}\tilde{\tilde{u}} + A\tilde{\tilde{u}} + \varphi(s)\nabla F_t(\tilde{\tilde{u}}) = 0, \quad \tilde{\tilde{u}}(s,t+T) = \tilde{\tilde{u}}(s,t). \tag{8}$$

Note that the flow ϕ_t^A preserves Hilbert scales so that a solution to (7) indeed gives us a solution to (8) of the same regularity. Note as well that the asymptotics $\lim_{s\to\pm\infty} \tilde{\tilde{u}}(s,t)$ of the solution $\tilde{\tilde{u}}$ to (8) are T–periodic solutions to (1).

A result similar to our main theorem is proven in [11] for the nonlinear Schrödinger equation on projective Hilbert space (see also Example 1.2). Because of the extra topology on projective Hilbert space, the author can prove the existence of infinitely many solutions rather than just one. *We stress that our paper is self-contained, as in contrast to* [11] *we study the case of general Hamiltonian PDEs.*

Suppose F_t is any A–admissible nonlinearity with finite-dimensional restrictions $F_t^k : \mathbb{C}^{2k+1} \to \mathbb{R}$ given by $F_t^k(u) := F_t(u^k)$ with u^k denoting the projection of $u \in \mathbb{H}$ onto the finite-dimensional subspace $\mathbb{C}^{2k+1}$. Analogously, for $G_t := F_t \circ \phi_t^A$ let G_t^k be its finite-dimensional restriction given by $G_t^k := F_t^k \circ \phi_t^A$ with symplectic gradient $X_t^{G,k}$. In order to prove the main theorem for the infinite-dimensional nonlinearity F_t, we show that, after passing to a subsequence, finite-dimensional Floer curves $\tilde{u}^k$ for the restricted nonlinearity F_t^k converge as $k \to \infty$ to a Floer curve on the infinite-dimensional Hilbert space, as in the main theorem. This can be done because even though the time-T free flow map is asymptotically degenerate, our assumptions on the nonlinearity assure that this is no problem.

Here $\tilde{u}^k$ satisfies the Floer equation

$$\bar{\partial}\tilde{u}^k + \varphi_k(s)\nabla G_t(\tilde{u}^k) = 0, \quad \tilde{u}^k(s,t+T) = \phi_{-T}^A \tilde{u}^k(s,t),$$

with $\varphi_k(s)$ for $k \geq 1$ meeting the requirements

$$\varphi_k(s) = 0 \quad \text{for } s \leq -1 \text{ and } s \geq 2k+1, \qquad \varphi_k(s) = 1 \quad \text{for } s \in [0, 2k],$$
$$0 \leq \varphi_k'(s) \leq 2 \quad \text{for } s < 0, \qquad -2 \leq \varphi_k'(s) \leq 0 \quad \text{for } s > 0,$$

such that $\varphi_k(s) \to \varphi(s)$ as $k \to \infty$ for every $s \in \mathbb{R}$. Furthermore, we impose the asymptotic condition $\lim_{s\to\pm\infty} \tilde{u}^k(s,t) = 0$.

This said, the main ingredient for the proof of Main Theorem 4.1 is the following infinite-dimensional generalization of the Gromov–Floer compactness theorem; see Theorem 8.1.

Theorem 4.2 *There is a subsequence of the sequence $(\tilde{u}^k)_k$ of Floer curves*

$$\tilde{u}^k : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{2k+1}$$

which $C^{\lfloor h/d \rfloor -1}_{\mathrm{loc}}$–converges to a solution $\tilde{u} : \mathbb{R} \times \mathbb{R} \to \mathbb{H}$ of the Floer equation

$$(\partial_s + i\,\partial_t)\tilde{u} + \varphi(s)\nabla G_t(\tilde{u}) = 0, \quad \tilde{u}(s, t+T) = \phi^A_{-T}\tilde{u}(s,t)$$

as in Main Theorem 4.1.

After establishing the existence of a Floer curve, we can directly deduce the existence of a periodic orbit:

Theorem 4.3 *Using finiteness of energy, the limit Floer curve $\tilde{u} : \mathbb{R} \times \mathbb{R} \to \mathbb{H}$ satisfies the following asymptotic conditions: there exists sequences $s_n^\pm \in \mathbb{R}$ with $s_n^\pm \to \pm\infty$ as $n \to \infty$ such that*

$$\lim_{n\to\infty} \tilde{u}(s_n^-, t) = u_0 = 0, \qquad \lim_{n\to\infty} \tilde{u}(s_n^+, t) = u_1(t),$$

in the $C^{\lfloor h/d \rfloor -1}$–sense. Here $u_0 = 0$ is the trivial and only fixed point of the free flow and u_1 is a ϕ^A_T–periodic orbit of G_t.

We finish by discussing the regularity of the solution:

Theorem 4.4 *The Floer curve $\tilde{u}$, and in particular the T–periodic solution*

$$u(t) = \phi^A_t u_1(t)$$

we obtain from the ϕ^A_T–periodic solution $u_1(t)$, is of regularity $h - d(r-1) - \frac{1}{2}$ for any $h > dr$, ie $\tilde{u} : \mathbb{R} \times \mathbb{R} \to \mathbb{H}_{h-d(r-1)-\frac{1}{2}} \subset \mathbb{H}$.

In particular, when $h=\infty$ we obtain a smooth solution to the Floer equation and associated Hamiltonian PDE.

These results will play an important role in the construction of a symplectic cohomology theory for Hamiltonian PDEs with regularizing nonlinearities, which is an ongoing project of the authors. In an upcoming paper we will prove a Lagrangian version of the results above.

5 Finite-dimensional case

As mentioned above, the approach that we take is to start with the case of *finite-dimensional nonlinearities*, that is, we consider nonlinearities which are given by the composition of any smooth T–periodic time-dependent map $F_t\colon \mathbb{C}^{2k+1}\to\mathbb{R}$ with bounded support with the orthogonal projection from $\mathbb{H}$ onto the finite-dimensional subspace

$$\mathbb{C}^{2k+1}=\mathrm{Span}_{\mathbb{C}}\{z_n\}_{n=-k}^{k}\subset\mathbb{H}.$$

Note that any nonlinearity of this form is automatically A–admissible for any admissible A and any admissible periods (T,X). Since the linear symplectomorphism ϕ_t^A restricts to any $\mathbb{C}^{2k+1}$, it turns out that, in order to prove Main Theorem 4.1 for these finite-dimensional nonlinearities, it suffices to replace the infinite-dimensional symplectic Hilbert space $\mathbb{H}$ by the finite-dimensional symplectic space $\mathbb{C}^{2k+1}$ and employ well-established results of Floer theory in finite dimensions. To prove Main Theorem 4.1 for general infinite-dimensional A–admissible nonlinearities, we will prove in the upcoming sections a generalized Gromov–Floer compactness result for the Floer curves introduced in this section. More precisely, we will consider the case when the dimension k is allowed to vary, in particular, allowed to approach infinity.

Let $F_t\colon\mathbb{C}^{2k+1}\to\mathbb{R}$ be any smooth T–periodic time-dependent map with bounded support in the ball $B_{R_k}(0)$ of radius $R_k>0$ and define again $G_t:=F_t\circ\phi_t^A$. Consider now the τ–dependent Floer equation in $\mathbb{C}^{2k+1}$

$$\bar\partial\tilde u+\varphi_\tau(s)\nabla G_t(\tilde u)=0,\quad \tilde u(s,t+T)=\phi_{-T}^A\tilde u(s,t) \tag{9}$$

using the family of cutoff functions $\varphi_\tau\colon\mathbb{R}\to[0,1]$ for $\tau\geq 0$, with $\varphi_0(s)=0$ and $\varphi_\tau(s)$ for $\tau\geq 1$ meeting the requirements

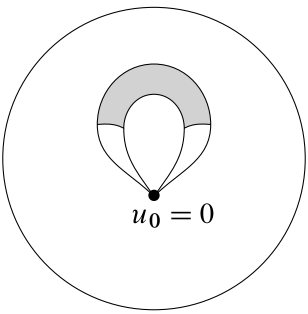
$$\varphi_\tau(s)=0\quad\text{for } s\leq -1 \text{ and } s\geq 2\tau+1,\qquad \varphi_\tau(s)=1\quad\text{for } s\in[0,2\tau],$$
$$0\leq\varphi_\tau'(s)\leq 2\quad\text{for } s<0,\qquad -2\leq\varphi_\tau'(s)\leq 0\quad\text{for } s>0.$$

such as

Our results stem from a careful analysis of the moduli space of curves satisfying this Floer equation (9). We define the moduli space $\mathcal{M}^k$ for the finite-dimensional problem by

$$\{\tilde{u}^\tau := (\tilde{u}, \tau) \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{C}^{2k+1}) \times \mathbb{R}_{\geq 0} \mid \tilde{u} \text{ satisfies (9) and } \lim_{s \to \pm\infty} \tilde{u}(s,t) = 0\}.$$

After restricting to $\mathbb{R} \times [0, T]$, pictorially such Floer curves look like



where the gray area depicts the part where $\varphi_\tau(s) = 1$. In order to show that we can compactify $\mathcal{M}^k$, we crucially use the bounded support condition in Definition 3.3 and:

Proposition 5.1 (maximum principle) *If (Σ, j) is a Riemann surface and the map $\tilde{u}: (\Sigma, j) \to (\mathbb{C}^{2k+1}, i)$ is holomorphic, then*

$$\Sigma \to [0, \infty) \quad \textit{given by } z \mapsto |\tilde{u}(z)|^2$$

has no local maximum.

This implies that Floer curves $\tilde{u}$ cannot escape the ball $B_{R_k}(0)$. If they could, they would be holomorphic outside the ball, where $G_t = 0$, and so by the above they could not have a maximum, which is impossible. So even though the target space of the Floer curve is not compact, the image is contained in a compact set.

Proposition 5.2 *For every $\tau \in \mathbb{N}$ there is a Floer curve $\tilde{u}^\tau$ in $\mathcal{M}^k$.*

Proof For the proof we use well-known results from Floer theory [26], and from Floer theory for general symplectomorphisms [7]. Since all these results are well-established in the literature, we freely use established terminology without giving definitions. Note that since H_A is smooth on finite-dimensional subspaces of $\mathbb{H}$, one can either use a solution $\tilde{u}$ to (9) or $\tilde{\tilde{u}}$ solving

$$\bar{\partial}\tilde{\tilde{u}} + A\tilde{\tilde{u}} + \varphi_\tau(s)\nabla F_t(\tilde{\tilde{u}}) = 0, \quad \tilde{\tilde{u}}(s,t+T) = \tilde{\tilde{u}}(s,t).$$

To start, note that the energy $E(\tilde{u}) = \|\partial_s\tilde{u}\|^2_{L^2}$ of the Floer curves is uniformly bounded by $4T\|F\|_{C^0}$ (see [23, Chapter 8]) which is finite by Definition 3.3 condition (iii). Assuming transversality for the nonlinear Cauchy–Riemann operator for the moment, the moduli space of such pairs $(\tilde{u},\tau)$ is a 1–dimensional manifold. Since for $\tau = 0$ the unique Floer curve $(\tilde{u},0)$ is the constant curve $\tilde{u} \equiv 0$, the moduli space is not empty. Indeed, Floer curves $(\tilde{u},\tau)$ exist for all $\tau > 0$ by Gromov–Floer compactness, as we can exclude bubbling-off of holomorphic spheres as well as breaking-off of cylinders for finite τ. Note that existence of holomorphic spheres is excluded due to the fact that the symplectic form is exact. Note that the assumption that the Hamiltonian PDE with $F_t = 0$ only has the trivial periodic solution $u_0 = 0$ is essential here to conclude that breaking of Floer curves cannot happen for finite $\tau > 0$.

It remains to discuss the problem with transversality of the perturbed Cauchy–Riemann operator $\bar{\partial} + \varphi_\tau(s)\nabla G^k_t$. Since we cannot expect transversality to hold, we first need to approximate i by a family of time-dependent almost-complex structures J^ν_t satisfying $(\phi^A_{-T})_* J^\nu_t = J^\nu_{t+T}$, in the sense that $J^\nu_t \to J^0_t = i$ as $\nu \to \infty$. We assume that the perturbed almost-complex structure J^ν_t agrees with i outside the ball $B_{R_k}(0)$ so that the maximum principle still holds. The existence of Floer curves as claimed above then holds for all $\nu \neq 0$, and by applying Gromov–Floer compactness as $\nu \to 0$ this implies the existence of Floer curves for $\nu = 0$. □

6 Bubbling-off analysis

After settling the case of finite-dimensional nonlinearities in the previous section, we start by recalling the detailed strategy for the case of general infinite-dimensional A–admissible nonlinearities. Let F_t be any A–admissible nonlinearity with finite-dimensional restrictions $F^k_t : \mathbb{C}^{2k+1} \to \mathbb{R}$ given by $F^k_t(u) := F_t(u^k)$, with u^k denoting the projection of $u \in \mathbb{H}$ onto the finite-dimensional subspace $\mathbb{C}^{2k+1}$. In analogy, for $G_t := F_t \circ \phi^A_t$ let G^k_t be its finite-dimensional restriction given by $G^k_t := F^k_t \circ \phi^A_t$

with symplectic gradient $X_t^{G,k}$. In order to prove the main theorem for the infinite-dimensional nonlinearity F_t, we choose for every $k \in \mathbb{N}$ a Floer curve $\tilde{u}^k$ for the restricted nonlinearity F_t^k such that $(\tilde{u}^k, k) \in \mathcal{M}^k$. We then show that, after passing to a subsequence, these finite-dimensional Floer curves converge as $k \to \infty$ to a Floer curve on the infinite-dimensional Hilbert space, as in the main theorem. This can be done because even though the time-T free flow map is asymptotically degenerate, our assumptions on the nonlinearity assure that this is no problem. Note that $\tilde{u}^k$ satisfies a τ–dependent Floer equation with $\tau = k$ and $\varphi_k(s) \to \varphi(s)$ as $k \to \infty$ for every $s \in \mathbb{R}$.

As a first step we would like to bound the Floer curves $\tilde{u}^k$, for all k, in the C^m–norm, where $m = \lfloor h/d \rfloor$. We will do this by showing the first derivatives are bounded and then using an elliptic bootstrapping argument. We use ideas similar to those in [11]. We stress, however, that contrary to [11] for our problem we work on linear space and with general Hamiltonians with minimal regularity.

We start by proving the analogue of Lemma 3.2 on the convergence of $G^k : \mathbb{R} \times \mathbb{H} \to \mathbb{R}$ given by $G^k(t,u) = G_t^k(u)$ to $G : \mathbb{R} \times \mathbb{H} \to \mathbb{R}$ given by $G(t,u) = G_t(u)$. Note that we explicitly want to include into our discussion not only the derivatives with respect to $u \in \mathbb{H}$, but also the derivatives with respect to the time $t \in \mathbb{R}$.

Lemma 6.1 *$\nabla G^k : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ converges to $\nabla G : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ uniformly with all u–derivatives, and with all t–derivatives up to order $m = \lfloor h/d \rfloor$ (which is at least two). Furthermore, the Fourier coefficients of ∇G in the expansion*

$$\nabla G(u)(t) = \sum_{p,n} \widehat{\nabla G(u)}(p,n) e^{i2\pi pt/T} z_n$$

with respect to $n \in \mathbb{Z}$ and $p \in \mathbb{Z} - an^d T/(2\pi)$ satisfy

$$\widehat{\nabla G(u)}(p,n) |n|^h |p|^m \to 0 \quad \text{as } |n|, |p| \to \infty.$$

Proof The statement about the u–derivatives directly follows from Lemma 3.2, as the u–derivatives of G^k are obtained from the u–derivatives of F^k by composition with the linear unitary map ϕ_t^A. In order to compute the t–derivatives of G one does not only have to take the t–derivatives of F into account, but also the t–derivatives of ϕ^A. While F is assumed to be smooth in $t \in \mathbb{R}$, the t–derivatives of ϕ^A are given by $\partial_t^\alpha \phi^A = (JA)^\alpha \cdot \phi^A : \mathbb{H}_0 \to \mathbb{H}_{-\alpha d}$ and hence have decreasing regularity. But since F is assumed to be h–regularizing and hence F_t extends to a smooth map $\mathbb{H}_{-h} \to \mathbb{R}$ with $h > dr$, it follows that derivatives up to order $m = \lfloor h/d \rfloor \geq 2$ are no problem.

Moreover, since, in analogy with Lemma 3.2, we already know that ∇F^k converges to ∇F uniformly with all derivatives when viewed as maps from $\mathbb{H}_{-h}$ into $\mathbb{H}_h \subset \mathbb{H}$, it follows that $\partial_t^\alpha G^k$ converges to $\partial_t^\alpha G$ for $k \to \infty$ as long as $\alpha \leq m$. When we expand

$$\nabla G(u)(t) = \sum_{p,n} \widehat{\nabla G(u)}(p,n) e^{i 2\pi p t/T} z_n$$

the statement as $|p| \to \infty$ follows. The statement for the n–variable follows from the fact that $\nabla G(u)$ is in $\mathbb{H}_h$. □

Lemma 6.2 *The first derivatives of the Floer curves $\tilde{u}^k$ are bounded uniformly in k, ie $\sup_k \|T\tilde{u}^k\|_{C^0} < \infty$.*

Proof Showing that the first derivatives are bounded is done by assuming that

$$\sup_k \|T\tilde{u}^k\|_{C^0} = \infty \tag{10}$$

and showing that this assumption leads to the formation of a sphere. We will not argue, as in the finite-dimensional case, that because ω is exact no holomorphic spheres can exist, this would require Gromov–Floer compactness in infinite dimensions. Rather, assuming the first derivative is unbounded, we show that a sphere is formed as an image of the disc where the length of the image of the boundary of the disc converges to zero. We then bound the derivative of the Floer curve by the symplectic area of these discs, which by exactness of ω is given by an integral over the boundary, thereby deriving a contradiction. This implies boundedness in the C^1–norm. Although the proof is very similar to the proof of the well-established finite-dimensional result, we include it with all details as our infinite-dimensional result does *not* follow from the finite-dimensional bubbling-off result.

Henceforth assume that the first derivative is unbounded in the sense that for

$$C_k := \max_{z=(s,t)\in\mathbb{R}\times\mathbb{R}} \{|\partial_s \tilde{u}^k(z)|\} =: |\partial_s \tilde{u}^k(z_k)|$$

the sequence $(C_k)_{k\in I}$ converges to ∞ for some index-set I. We can assume that the Floer curve $\tilde{u}^k$ attains this maximum at some point z_k because of the asymptotic conditions. Now we reparametrize as

$$\tilde{v}^k : B_{\sqrt{C_k}}(0) \to \mathbb{C}^{2k+1} \quad \text{given by } z \mapsto \tilde{u}^k\Big(\frac{z}{C_k} + z_k\Big),$$

so that $|\partial_s \tilde{v}^k(0)| = 1$ and $|\partial_s \tilde{v}^k(z)| \leq 1$ for $|z| \leq \sqrt{C_k}$. Then we define a family of maps γ_r^k for $0 \leq r \leq \sqrt{C_k}$:

$$\gamma_r^k : S^1 \to \mathbb{C}^{2k+1} \quad \text{given by } \theta \mapsto \tilde{v}^k(re^{i\theta}).$$

Let $L\colon C^\infty(S^1,\mathbb{C}^{2k+1})\to\mathbb{R}$ be the map which assigns to a loop its length with respect to the metric $\omega(\cdot\,,i\,\cdot)$ restricted to $\mathbb{C}^{2k+1}$. Let $A\colon C^\infty(B_R(0),\mathbb{C}^{2k+1})\to\mathbb{R}$ be the area functional $A(v):=\int v^*\omega$, where again we restrict the symplectic form ω to $\mathbb{C}^{2k+1}$. Now we show that for increasing dimension k, the length of the image of the boundary circle decreases. More precisely, we show that for all k, there exists an r_k with $\frac{1}{2}\sqrt{C_k}\le r_k\le\sqrt{C_k}$ such that $L(\gamma_{r_k})\to 0$. By the exactness of ω the area of $\tilde{v}^k_{r_k}$, which is the restriction of $\tilde{v}^k$ to the disk of radius r_k, goes to zero.

As a first step, we show that A is bounded by the energy of the solution $\tilde{u}^k$ as $k\to\infty$, which will show that the area is bounded.

$$\begin{aligned}A(\tilde{v}^k)&=\int_{B_{\sqrt{C_k}}(0)}\tilde{v}^{k*}\omega=\int_{B_{\sqrt{C_k}}(0)}\omega(\partial_s\tilde{v}^k,\partial_t\tilde{v}^k)\,ds\wedge dt\\&\le E(\tilde{u}^k)+\int_{B_{1/\sqrt{C_k}}(z^k)}\varphi_k(s)dG^k_t(\partial_s\tilde{u}^k)\,ds\wedge dt\\&\le E(\tilde{u}^k)+\int_{B_{1/\sqrt{C_k}}(z^k)}\|G^k\|_{C^1}\,ds\wedge dt.\end{aligned}$$

Since $\sqrt{C_k}\to\infty$ by our assumption (10), the second term vanishes. Now we write $\tilde{v}^k(z)=\tilde{v}^k(re^{i\theta})$ and, assuming k is sufficiently large, compute

$$\begin{aligned}\int_{\sqrt{C_k}/2}^{\sqrt{C_k}}rL(\gamma^k_r)^2dr&=\int_{\sqrt{C_k}/2}^{\sqrt{C_k}}r\Big(\int_0^{2\pi}|\partial_\theta\tilde{v}^k(re^{i\theta})|\,d\theta\Big)^2\,dr\\&\le 2\pi\int_{\sqrt{C_k}/2}^{\sqrt{C_k}}\int_0^{2\pi}r|\partial_\theta\tilde{v}^k|^2\,d\theta\,dr\\&\le 10\pi T\|F\|_{C^0}\end{aligned}$$

using Cauchy–Schwarz, the previous inequality and the fact that $E(\tilde{u}^k)<5T\|F\|_{C^0}$; see [23]. By setting L^k_0 to be the minimum of $L(\gamma^k_r)$ for $\sqrt{C_k}/2\le r\le\sqrt{C_k}$, we get

$$10\pi T\|F\|_{C^0}\ge\int_{\sqrt{C_k}/2}^{\sqrt{C_k}}r(L^k_0)^2dr=\tfrac{1}{8}3(L^k_0)^2C_k,$$

so

$$L^k_0\le\sqrt{\frac{80\pi T\|F\|_{C^0}}{3C_k}},$$

which tends to zero as $k\to\infty$. Since $\omega=d\lambda$, for any disc $v\colon B_R(0)\to\mathbb{C}^{2k+1}$ we have

$$A(v)=\int_{B_R(0)}v^*\omega=\int_{\partial B_R(0)}v^*\lambda,$$

and so the area $A(\tilde{v}^k_{r_k}) \to 0$ as $L^k_0 \to 0$. Now there are two ways to prove the desired result. First, it follows from the a priori estimate

$$|\partial_s \tilde{v}^k(0)|^2 < c\,\frac{A(\tilde{v}^k_{r_k})}{r_k^2}$$

in [23, Chapter 4] by observing that the Floer curve can be realized as an actual $\phi^{H,k}_T$–periodic J–holomorphic curve when we set $J^k_t := (\phi^{G,k}_{-t})_* i$. Note that contrary to [23, Chapter 4] we don't work with a single almost-complex structure J, but with a sequence J^k_t which converges to $J_t = (\phi^G_{-t})_* i$. Since we have $|\partial_s \tilde{v}^k(0)| = 1$, the contradiction then follows by letting $r_k \to \infty$.

Alternatively, consider the following. We first observe that

$$\bar{\partial}\tilde{v}^k = -C_k^{-1}\varphi_k(s)\nabla G^k_t(\tilde{v}^k) \to 0 \quad \text{as } k \to \infty,$$

and so

$$\Delta\partial_s \tilde{v}^k = -(\partial \circ \partial_s) C_k^{-1}\varphi_k(s)\nabla G^k_t(\tilde{v}^k) \to 0 \quad \text{as } k \to \infty.$$

Writing $v := \partial_s \tilde{v}^k$ and using the divergence theorem, we get

$$\partial_\rho\left(\frac{1}{\rho}\int_{\partial B_\rho(0)} v\right) = \frac{1}{\rho}\int_{B_\rho(0)} \Delta v \to 0 \quad \text{as } k \to \infty$$

uniformly in ρ for $\rho \le \epsilon$ for some $\epsilon > 0$. Using the fact that $(2\pi\rho)^{-1}\int_{\partial B_\rho(0)} v \to v(0)$ as $\rho \to 0$ as well as the above convergence to zero as $k \to \infty$, we get

$$v(0) - \frac{1}{\pi\epsilon^2}\int_{B_\epsilon(0)} v(z)\,dz \to 0 \quad \text{as } k \to \infty.$$

Now

$$\frac{1}{\pi\epsilon^2}\left|\int_{B_\epsilon(0)} v(z)\,dz\right| \le \frac{1}{\pi^{\frac{1}{2}}\epsilon}\left(\int_{B_\epsilon(0)} |v(z)|^2 dz\right)^{\frac{1}{2}} \le \frac{1}{\pi^{\frac{1}{2}}\epsilon}\|v\|_{L^2},$$

so that indeed

$$|\partial_s \tilde{v}^k(0)|^2 < c\,\frac{A(\tilde{v}^k_\epsilon)}{\epsilon^2}$$

for k sufficiently large and some positive constant c which is independent of the dimension. Since $A(\tilde{v}^k_\epsilon) \to 0$ as $k \to \infty$ we obtain a contradiction to the equation $|\partial_s \tilde{v}^k(0)| = 1$. □

We can now apply the aforementioned bootstrapping argument, to show boundedness of the Floer curves in the C^m–norm. Recall that $m = \lfloor h/d \rfloor \ge 2$.

Proposition 6.3 *The Floer curves $\tilde{u}^k$ are C^m–bounded uniformly in k; that is,*

$$\sup_k \|\tilde{u}^k\|_{C^m} < \infty.$$

Proof For the proof we choose the bounded open subset

$$B = (s + \Delta s, s - \Delta s) \times (0, 1) \subset \mathbb{R}^2$$

for some fixed Δs and all norms are understood after restricting the maps $\tilde{u}^k$ to this bounded open subset. By the result above, and the discussion following the maximum principle, we know that $\|\tilde{u}^k\|_{C^1}$ is bounded. We will use the fact that our sequence of finite-dimensional nonlinearities approximates the original one and an elliptic bootstrapping argument to show boundedness in all C^α–norms up to $\alpha = m$. By the Sobolev embedding theorem (see [2]), the inequality

$$\|\tilde{u}^k\|_{C^\beta} \leq c_0 \|\tilde{u}^k\|_{W^{\alpha,p}}$$

holds for $p > 2$, with $\alpha, \beta \in \mathbb{N}$ and for all $\beta \leq \alpha - 2/p$, and a constant $c_0 > 0$ which is independent of the dimension of the codomain. It follows that it suffices to show boundedness of $\tilde{u}^k$ in the $W^{\alpha,p}$–norms up to $\alpha = m + 1$.

We first observe that the boundedness in C^1 implies boundedness in $W^{1,p}$; note that this is the point where it is crucial that we first restrict $\tilde{u}^k$ to a bounded open subset. Assume now that $\|\tilde{u}^k\|_{W^{\alpha,p}}$ is bounded, for some $\alpha > 1$, uniformly in k. We have that $\tilde{u}^k$ satisfies

$$\bar{\partial}\tilde{u}^k = -\varphi_k(s)\nabla G_t^k(\tilde{u}^k) =: \eta^k$$

and η^k is bounded in the $W^{\alpha,p}$–norm if and only if the $W^{\alpha,p}$–norm of $\nabla G_t^k(\tilde{u}^k)$ is bounded with

$$\nabla G^k(\tilde{u}^k)(s,t) = \nabla G_t^k(\tilde{u}^k(s,t)).$$

On the other hand, viewing $\nabla G^k(\tilde{u}^k)\colon B \to \mathbb{C}^{2k+1}$ as a composition of the maps $\check{u}^k : B \to B \times \mathbb{C}^{2k+1}$ given by $(s,t) \mapsto (s,t,\tilde{u}^k(s,t))$ and $\nabla G^k : B \times \mathbb{C}^{2k+1} \to \mathbb{C}^{2k+1}$ given by $(s,t,u) \mapsto \nabla G_t^k(u)$, by [23, Appendix B] it holds true that

$$\|\nabla G^k(\tilde{u}^k)\|_{W^{\alpha,p}} \leq c_1 \|\nabla G^k\|_{C^\alpha}(\|\check{u}^k\|_{C^0}^{\alpha-1} + 1)(\|\check{u}^k\|_{W^{\alpha,p}} + 1)$$

with a constant $c_1 > 0$ which is independent of the dimension of the target space. Note that the C^α–norm of ∇G^k also contains t–derivatives of $t \mapsto \nabla G_t^k$. Since by Lemma 6.1 we have, for all $\alpha \leq m$, that $\|\nabla G^k\|_{C^\alpha} \to \|\nabla G\|_{C^\alpha}$ as $k \to \infty$, it follows that $\|\nabla G^k\|_{C^\alpha}$ is bounded for $\alpha \leq m$. Together with the induction hypothesis, we get boundedness of $\nabla G_t^k(\tilde{u}^k)$ in the $W^{\alpha,p}$–norm as long as $\alpha \leq m$. Now, local regularity of the Cauchy–Riemann operator $\bar{\partial}$ together with boundedness of η in the $W^{\alpha,p}$–norm implies

$$\|\tilde{u}^k\|_{W^{\alpha+1,p}} \leq c_2(\|\bar{\partial}\tilde{u}^k\|_{W^{\alpha,p}} + \|\tilde{u}^k\|_{L^p})$$

is finite for $\alpha \leq m$. Note that, again, $c_2 > 0$ is independent of the dimension of the codomain. Finally we remark that all constants depend on the bounded open subset B but not on s, so that we obtain a bound which is uniform in s. □

7 Small divisor problem

We have chosen the setting so that the nonlinearity can be approximated by finite-dimensional ones better than the eigenvalues of the time-T flow of the free Hamiltonian approach one. In this section, we will make this statement precise by giving bounds on the norms of the tail of $\tilde{u}^k$, and invoke a result from number theory to overcome the small divisor problem which arises as we increase the dimension k.

Let us write a finite-dimensional solution of the Floer equation (9) as

$$\tilde{u}^k = (\tilde{u}^{k,\ell}, \tilde{u}_\perp^{k,\ell}) \colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{2\ell+1} \oplus \mathbb{C}^{2k-2\ell} = \mathbb{C}^{2k+1} \subset \mathbb{H},$$

and call the tail $\tilde{u}_\perp^{k,\ell}$ of $\tilde{u}^k$ the *normal component*. The statement (Proposition 7.2) needed for the proof in Section 8 of the main theorem, is then that we have

$$\sup_{k \geq \ell} \|\tilde{u}_\perp^{k,\ell}\|_{C^{m-1}} \to 0 \quad \text{as } \ell \to \infty \tag{11}$$

for $m = \lfloor h/d \rfloor$. We prove this by observing that the Fourier coefficients of the Floer curve, which depend on the s–coordinate, satisfy an ODE involving the Fourier coefficients of the Hamiltonian vector field of the nonlinearity and which satisfy a decay property as $s \to \pm\infty$. The following elementary lemma then allows us to show that the coefficients themselves decay to zero with some rate which we compute.

Lemma 7.1 *Let $w = w_R + i w_I \colon \mathbb{R} \to \mathbb{C}$ be a continuously differentiable solution to the ODE with asymptotic condition*

$$w'(s) = \lambda w(s) + f(s), \qquad w(s) \to 0 \quad \text{as } s \to \pm\infty, \tag{12}$$

where $\lambda \in \mathbb{R}$. If $f = f_R + i f_I \colon \mathbb{R} \to \mathbb{C}$ satisfies $\|f\|_{C^0} < \infty$, then

$$\|w\|_{C^0} \leq \sqrt{2}\frac{\|f\|_{C^0}}{|\lambda|}.$$

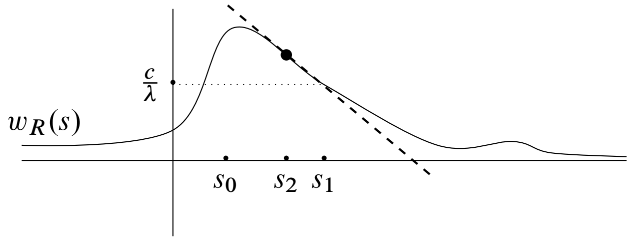


Figure 1

Proof The proof is by contradiction. Assume $|w(s_0)| > \sqrt{2}\|f\|_{C^0}/|\lambda|$ for some $s_0 \in \mathbb{R}$ and, without loss of generality, that $|w_R(s_0)| \geq |w_I(s_0)|$ so that $|w_R(s_0)| > \|f\|_{C^0}/|\lambda|$ by the Pythagorean theorem. Assume that $w_R(s_0) > 0$ and $\lambda > 0$ (different signs lead to obvious changes in the proof). Since $w(s) \to 0$ as $s \to +\infty$, by the intermediate value theorem we know that there is some $s_1 > s_0$ such that $w_R(s_1) = \|f\|_{C^0}/\lambda$ and $w_R(s) > \|f\|_{C^0}/\lambda$ for all $s \in (s_0, s_1)$. By the mean value theorem, there exists some $s_2 \in (s_0, s_1)$ such that $w'_R(s_2) < 0$. Since $|f_R(s)| \leq |f(s)| \leq \|f\|_{C^0}$, we have $w'_R(s_2) < 0$ but $\lambda w_R(s_2) + f_R(s_2) > 0$, which contradicts the assumption that w satisfies (12); see Figure 1. □

In order to prove (11), we essentially expand the Floer curve into a Fourier series and show that the coefficients, viewed as functions of the variable s, satisfy (12), and use this bound to show that the C^{m-1}–norms of $\tilde{u}_\perp^{k,\ell}$ go to zero uniformly in k.

Proposition 7.2 *The C^{m-1}–norm of the normal component $\tilde{u}_\perp^{k,\ell}$ converges to zero as $\ell \to \infty$; that is,*

$$\sup_{k \geq \ell} \|\tilde{u}_\perp^{k,\ell}\|_{C^{m-1}} \to 0 \quad \text{as } \ell \to \infty.$$

Proof Consider the space $L^2_{\phi_T^A}(\mathbb{R}, \mathbb{H})$ of ϕ_T^A–periodic maps

$$L^2_{\phi_T^A}(\mathbb{R}, \mathbb{H}) := \{u \in L^2(\mathbb{R}, \mathbb{H}) \mid u(t+T) = \phi_{-T}^A u(t)\},$$

and, acting on it, the densely defined operator $-i\partial_t$. Using the fact that the maps z_n are a complete eigenbasis of ϕ_T^A with eigenvalues e^{iaTn^d}, we observe that the space $L^2_{\phi_T^A}$ has a complete basis of eigenfunctions $u_{p,n}$ of $-i\partial_t$ with eigenvalues $\lambda_{p,n}$ given by

$$u_{p,n}(t) = e^{i((2\pi/T)p - an^d)t} z_n, \quad \lambda_{p,n} = \frac{2\pi}{T}p - an^d,$$

for $p,n\in\mathbb{Z}$. Even though $\lambda_{p,n}\neq 0$ for all $p,n\in\mathbb{Z}$, there exist sequences for which $(\lambda_{p',n'})\to 0$, or

$$\inf_{p,n\in\mathbb{Z}}\left|\frac{2\pi}{T}p-an^d\right|=0.$$

We overcome this *small divisor problem* by using the assumption that the number $aT/(2\pi)$ is Diophantine with irrationality measure $r<\infty$: for fixed $n\in\mathbb{N}$, we have the bound

$$\inf_{p\in\mathbb{Z}}\left|\frac{2\pi}{T}p-an^d\right|=\frac{2\pi n^d}{T}\inf_{p\in\mathbb{Z}}\left|\frac{aT}{2\pi}-\frac{p}{n^d}\right|\geq\frac{c}{n^{d(r-1)}}\tag{13}$$

for some $c>0$.

We now view a Floer curve $\tilde{u}^k$ as a map

$$\tilde{u}^k:\mathbb{R}\to L^2_{\phi^A_T}(\mathbb{R},\mathbb{C}^{2k+1})\subset L^2_{\phi^A_T}(\mathbb{R},\mathbb{H})$$

satisfying

$$\partial_s\tilde{u}^k=-i\,\partial_t\tilde{u}^k-\varphi_k(s)\nabla G^k_t(\tilde{u}^k)$$

and the asymptotic conditions $\tilde{u}^k(s,\cdot)\to 0$ as $s\to\pm\infty$. Expanding the s–evaluation as a Fourier series with respect to $n\in\mathbb{Z}$ and $p\in\mathbb{Z}-an^dT/(2\pi)$,

$$\tilde{u}^k(s,t)=\tilde{u}^k(s)(t)=\sum_{n=-k}^{k}\sum_{p}\widehat{\tilde{u}^k(s)}(p,n)e^{i2\pi pt/T}z_n$$

with $\widehat{\tilde{u}^k(s)}:\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$. We thus obtain s–dependent sequences $w^k_{p,n}(s)=\widehat{\tilde{u}^k(s)}(p,n)$ which satisfy

$$(w^k_{p,n})'(s)=\lambda_{p,n}w^k_{p,n}(s)+f^k_{p,n}(s),\qquad w^k_{p,n}(s)\to 0\quad\text{as } s\to\pm\infty,$$

where $f^k_{p,n}(s):=-\widehat{\nabla G^k_t(\tilde{u}^k)(s)}(p,n)\in\mathbb{C}$ are the Fourier coefficients of $\nabla G^k(\tilde{u}^k)(s)$. Here we view $\nabla G^k(\tilde{u}^k)$ as a map from $\mathbb{R}$ to $L^2_{\phi^A_T}(\mathbb{R},\mathbb{H})$ by

$$\nabla G^k(\tilde{u}^k)(s)(t):=\varphi_k(s)\nabla G^k_t(\tilde{u}^k(s,t))$$

so that

$$\nabla G^k(\tilde{u}^k)(s)(t)=\varphi_k(s)\sum_{n=-k}^{k}\sum_{p}\widehat{\nabla G^k(\tilde{u}^k)(s)}(p,n)e^{i2\pi pt/T}z_n.$$

Since $\nabla G^k(\tilde{u}^k)(s)$ is $m=\lfloor h/d\rfloor$–times continuously differentiable with respect to time and has uniformly bounded derivatives, and by the decay property of the h–regularizing nonlinearity, we know that

$$\|f^k_{p,n}\|_{C^0}|p|^m|n|^h=\|\widehat{\nabla G^k(\tilde{u}^k)}(\cdot)(p,n)\|_{C^0}|p|^m|n|^h\to 0\quad\text{as } |p|,|n|\to\infty,\tag{14}$$

where the C^0–norm is with respect to $s \in \mathbb{R}$. Note that here and below it is implicitly assumed that the limit is uniform with respect to $k \in \mathbb{Z}$, and since the argument $\tilde{u}^k$ of ∇G^k also depends on t, we additionally have to use the result in Proposition 6.3 that $\tilde{u}^k$ is $\lfloor h/d \rfloor$–times continuously differentiable and its derivatives are uniformly bounded in s and t. Combining this with (13) and Lemma 7.1, we obtain

$$\text{(15)} \qquad \|w^k_{p,n}\|_{C^0} |p|^m |n|^{h-d(r-1)} = \|\widehat{\tilde{u}^k}(\cdot)(p,n)\|_{C^0} |p|^m |n|^{h-d(r-1)} \to 0$$

as $|p|, |n| \to \infty$, where again the C^0–norm is with respect to $s \in \mathbb{R}$.

We now bound the time derivative which, together with the bound on the gradient of the nonlinearity, also leads to a bound of the s–derivative which concludes the proof. From

$$|\partial_t^{m-1} \tilde{u}_\perp^{k,\ell}(s,t)|^2 \le \sum_{|n|=\ell+1}^{k} \biggl(\sum_p |\widehat{\tilde{u}^k(s)}(p,n)||p|^{m-1} \biggr)^2$$

and the above, it follows that the tail $\tilde{u}_\perp^{k,\ell}$ for $k \ge \ell$ satisfies

$$\|\partial_t^{m-1} \tilde{u}_\perp^{k,\ell}\|_{C^0} = o(\ell^{-h+d(r-1)+\frac{1}{2}}).$$

Let $\nabla_\perp^\ell G_t^k(u)$ denote the component of the gradient of $G_t^k(u)$ which is normal to the finite-dimensional subspace $\mathbb{C}^{2\ell+1} \subset \mathbb{H}$. Since $\|\nabla_\perp^\ell G_t^k(\tilde{u}^k)\|_{C^{m-1}}$ goes to zero uniformly in $k \ge \ell$ as $\ell \to \infty$ by (14), and since $\tilde{u}^k$ satisfies the Floer equation, we obtain that the s–derivatives also go to zero uniformly in k, so that $\|\tilde{u}_\perp^{k,\ell}\|_{C^{m-1}}$ goes to zero uniformly in k as long as $h > dr > d(r-1) + \frac{1}{2}$. □

Since almost all numbers have $r = 2$, generically this bound comes down to $h > 2d$. In the case of the Schrödinger equation this means we need $h > 4$, and for the wave equation $h > 2$.

8 Completing the proof

We now complete the proof of Main Theorem 4.1. This consists of three parts. First, we prove convergence of the sequence (or a subsequence) of Floer curves $(\tilde{u}^k)_k$ to a solution $\tilde{u}$ of the Floer equation on the full Hilbert space. This is not immediate, since $\mathbb{H}$, or even the support of the nonlinearity in $\mathbb{H}$, is not compact, so we cannot use Gromov–Floer compactness. We will prove this convergence in the C^{m-1}_{loc}–topology, where $m = \lfloor h/d \rfloor \ge 2$.

Second, we establish the asymptotic properties to conclude that this Floer curve connects the single (trivial) solution of the free Hamiltonian equation, to a (nontrivial) solution of the full Hamiltonian equation.

Finally, we discuss the regularity of the solution. The regularity of the solution we find will, of course, depend on the regularity of the nonlinearity. We stress here that the existence of the finite-dimensional Floer curves $\tilde{u}^k$ for the finite-dimensional nonlinearities G_t^k, which make up the sequence $(\tilde{u}^k)_k$, is proven in Section 5.

Theorem 8.1 *There exists a subsequence of the sequence $(\tilde{u}^k)_k$ of Floer curves $\tilde{u}^k: \mathbb{R}\times\mathbb{R}\to\mathbb{C}^{2k+1}$ which C^{m-1}_{loc}–converges to a solution $\tilde{u}: \mathbb{R}\times\mathbb{R}\to\mathbb{H}$ of the Floer equation*

$$(\partial_s + i\,\partial_t)\tilde{u} + \varphi(s)\nabla G_t(\tilde{u}) = 0$$

satisfying $\tilde{u}(s,t+T) = \phi^A_{-T}\tilde{u}(s,t)$.

Proof By Proposition 7.2 we know that the C^{m-1}–norms of $\tilde{u}^{k,\ell}_\perp$ converge to zero as k increases. To show that the limit of $(\tilde{u}^k)_k$ exists, we start with the observation that there is a subsequence of $(\tilde{u}^{k,\ell})_k$ of maps from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{C}^{2\ell+1}$ which C^{m-1}_{loc}–converges to a smooth map $\tilde{u}^\ell: \mathbb{R}\times\mathbb{R}\to\mathbb{C}^{2\ell+1}$ as $k\to\infty$ for all ℓ. We stress that the maps $\tilde{u}^{k,\ell}$ take values in $\mathbb{C}^{2\ell+1}$, so compactness holds by analogous reasons as for finite-dimensional nonlinearities. In particular, by the bounded support condition in Definition 3.3, the maximum principle ensures that the image is in a ball of radius $R_\ell\subset\mathbb{C}^{2\ell+1}$. Because we have locally bounded $W^{m+1,p}$–norms and hence, by elliptic bootstrapping and passing to a diagonal subsequence, local $W^{m,p}$–convergence, by Sobolev embedding we also have local convergence in the C^{m-1}–norm. Passing to a diagonal subsequence yet again, we obtain C^{m-1}_{loc}–convergence for all ℓ simultaneously.

After restricting to any bounded open subset, we now show that the sequence of maps $(\tilde{u}^k)_k$ thus obtained is Cauchy in the C^{m-1}–norm, which is sufficient to prove C^{m-1}_{loc}–convergence. Let $\epsilon>0$. Then there is an ℓ such that $\sup_{k\geq\ell}\|\tilde{u}^{k,\ell}_\perp\|_{C^{m-1}} < \frac{1}{3}\epsilon$. For this ℓ, the sequence $(\tilde{u}^{k,\ell})_k$ converges to $\tilde{u}^\ell$, so there is $k_0\geq\ell$ such that for $k,k'\geq k_0$ we have $\|\tilde{u}^{k,\ell}-\tilde{u}^{k',\ell}\|_{C^{m-1}} < \frac{1}{3}\epsilon$. Hence

$$\|\tilde{u}^k-\tilde{u}^{k'}\|_{C^{m-1}} \leq \|\tilde{u}^{k,\ell}_\perp\|_{C^{m-1}} + \|\tilde{u}^{k,\ell}-\tilde{u}^{k',\ell}\|_{C^{m-1}} + \|\tilde{u}^{k',\ell}_\perp\|_{C^{m-1}} < \epsilon. \qquad \square$$

Let us now establish the asymptotic behavior of the Floer curve:

Theorem 8.2 *Using finiteness of energy, the limit Floer curve $\tilde{u}: \mathbb{R} \times \mathbb{R} \to \mathbb{H}$ satisfies the following asymptotic conditions: there exist sequences $s_n^{\pm} \in \mathbb{R}$ with $s_n^{\pm} \to \pm\infty$ as $n \to \infty$ such that*

$$\lim_{n\to\infty} \tilde{u}(s_n^-, t) = u_0(t) \quad \textit{and} \quad \lim_{n\to\infty} \tilde{u}(s_n^+, t) = u_1(t)$$

in the C^{m-1}–sense. Here $u_0 = 0$ is the trivial and only fixed point of the free flow and u_1 is a ϕ_T^A–periodic orbit of G_t.

Proof Because the energy is bounded in terms of the C^0–norm of G (see Theorem 8.1), we get

$$E(\tilde{u}) = \int_{-\infty}^{\infty}\int_0^T |\partial_t \tilde{u}(s,t) - \varphi(s) X_t^G(\tilde{u}(s,t))|^2 \, dt \, ds \le 4T \|F\|_{C^0}.$$

Choose sequences $s_\gamma^{\pm} \in \mathbb{R}$ with $\gamma \le s_\gamma^+ \le 2\gamma$ and $\gamma \le -s_\gamma^- \le 2\gamma$ such that

$$\gamma \int_0^T |\partial_t \tilde{u}(s_\gamma^{\pm}, t) - \varphi(s_\gamma^{\pm}) X_t^G(\tilde{u}(s_\gamma^{\pm}, t))|^2 \, dt$$

is bounded by

$$\int_\gamma^{2\gamma}\int_0^T |\partial_t \tilde{u}(s,t) - \varphi(s) X_t^G(\tilde{u}(s,t))|^2 \, dt \, ds$$

or

$$\int_{-2\gamma}^{-\gamma}\int_0^T |\partial_t \tilde{u}(s,t) - \varphi(s) X_t^G(\tilde{u}(s,t))|^2 \, dt \, ds,$$

respectively. This implies

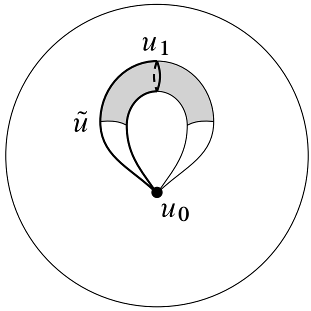
$$\int_0^T |\partial_t \tilde{u}(s_\gamma^{\pm}, t) - \varphi(s_\gamma^{\pm}) X_t^G(\tilde{u}(s_\gamma^{\pm}, t))|^2 \, dt \le \frac{4T\|F\|_{C^0}}{\gamma} \to 0 \quad \text{as } \gamma \to \infty.$$

Now we write $\tilde{u} = (\tilde{u}^\ell, \tilde{u}_\perp^\ell): \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{2\ell+1} \oplus \mathbb{H}/\mathbb{C}^{2\ell+1}$ for $\ell \in \mathbb{N}$. Since, by the maximum principle, $\tilde{u}^\ell$ takes values in $B_{R_\ell}(0) \subset \mathbb{C}^{2\ell+1}$, after passing to a subsequence we can assume that $\tilde{u}^\ell(s_\gamma^{\pm}, \cdot)$ C^{m-1}–converges as $\gamma \to \infty$. After passing to a diagonal subsequence we can assume $\tilde{u}^\ell(s_\gamma^{\pm}, \cdot)$ C^{m-1}–converges for all ℓ simultaneously. Since $\|\tilde{u}_\perp^\ell\|_{C^{m-1}} \to 0$ by Proposition 7.2 and Theorem 8.1, we have that $\tilde{u}(s_\gamma^{\pm}, \cdot)$ C^{m-1}–converges, that is

$$\lim_{\gamma\to\infty} \tilde{u}(s_\gamma^-, t) = u_0(t) \quad \text{and} \quad \lim_{\gamma\to\infty} \tilde{u}(s_\gamma^+, t) = u_1(t)$$

which both satisfy the Hamiltonian equation (6). Because $\varphi(s_\gamma^-) = 0$, the solution $u_0(t)$ is the trivial solution. Because $\varphi(s_\gamma^+) = 1$ we have indeed found a solution u_1 to (6). □

Pictorially, the limit looks like the breaking



Since there is no other fixed point of the free flow than the trivial solution, we indeed find a nontrivial fixed point of the full flow, provided that $\nabla F_t(0) \neq 0$.

We finish by discussing the regularity of the solution:

Theorem 8.3 *The Floer curve $\tilde{u}$, and in particular the T–periodic solution*

$$u(t) = \phi_t^A u_1(t)$$

we obtain from the ϕ_T^A–periodic solution $u_1(t)$ found in Theorem 8.2, is of regularity $h - d(r-1) - \frac{1}{2} > 0$ for every $h > dr$, ie $\tilde{u}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{H}_{h-d(r-1)-\frac{1}{2}} \subset \mathbb{H}$.

Proof Let F_t be A–admissible. From (15) in the proof of Proposition 7.2 we know that the coefficients in the Fourier expansion of the Floer curve $\tilde{u}$ satisfy

$$|\widehat{\tilde{u}(s)}(p,n)||n|^{h-d(r-1)}|p|^m \to 0 \quad \text{as } |n|, |p| \to \infty$$

with $m = \lfloor h/d \rfloor \geq 2$ uniformly for all s, which implies that

$$|\tilde{u}(s,t)|^2_{h-d(r-1)-\frac{1}{2}} \leq \sum_n \biggl(|n|^{h-d(r-1)-\frac{1}{2}} \sum_p |\widehat{\tilde{u}(s)}(p,n)| \biggr)^2,$$

where we sum over $n \in \mathbb{Z}$ and $p \in \mathbb{Z} - an^d T/(2\pi)$, is uniformly bounded for all $(s,t) \in \mathbb{R} \times \mathbb{R}$. In particular, this holds as we let s go to infinity, so we obtain the same regularity for the nontrivial solution u_1. Subsequently, the solution $u(t) = \phi_t^A u_1(t)$ also has the same regularity since ϕ_t^A preserves Hilbert scales. □

We again stress that for generic time period T, the irrationality measure is $r = 2$, so $h > 2d$. The regularity of the solution depends on h but also on the specific Hilbert space on which the Hamiltonian PDE is modeled. Let us apply our results to our two examples:

Proposition 8.4 *Viewing it as a PDE in the variables s, t and x with asymptotic conditions, the Floer equation*

$$\bar{\partial}\tilde{\tilde{u}}(s,t,x) + A\tilde{\tilde{u}}(s,t,x) + \varphi(s)\nabla F_t(\tilde{\tilde{u}}(s,t,x)) = 0$$

with A–admissible nonlinearities admits a strong (T, X)–periodic solution

$$\tilde{\tilde{u}}(s,t+T,x) = \tilde{\tilde{u}}(s,t,x) = \tilde{\tilde{u}}(s,t,x+X) \quad \text{for } (s,t,x) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R},$$

for generic T when $h > 2\frac{1}{2}$ for the nonlinear wave equation and $h > 5$ for the nonlinear Schrödinger equation.

Proof We define $\tilde{\tilde{u}}(s,t) := \phi_t^A \tilde{u}(s,t)$ for $(s,t) \in \mathbb{R}\times\mathbb{R}$ and subsequently view $\tilde{\tilde{u}}$ as a function of s, t and x. Recall that the Hilbert space for the nonlinear wave equation is $\mathbb{H} = W^{\frac{1}{2},2} \times W^{\frac{1}{2},2}$. In Hilbert scale notation we have $(\mathbb{H})_k = W^{\frac{1}{2}+k,2} \times W^{\frac{1}{2}+k,2}$. By the Sobolev embedding theorem we have $W^{\frac{1}{2}+k,2} \subset C^k$. Since A is of order 1, we need our solution $\tilde{\tilde{u}}$ to be an element of $W^{\frac{1}{2}+1,2} \times W^{\frac{1}{2}+1,2}$ in order for it to be in $C^1 \times C^1$. For generic time period T the irrationality measure of $aT/2\pi$ is $r = 2$, so for the solution to land in $\mathbb{H}_1 = W^{\frac{1}{2}+1,2} \times W^{\frac{1}{2}+1,2} \subset C^1 \times C^1$ and be a strong solution to the Floer equation, we need $h > 2\frac{1}{2}$. Then $\tilde{\tilde{u}} = (\tilde{\tilde{\varphi}}, \tilde{\tilde{\pi}})\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{pmatrix} \partial_s\tilde{\tilde{\varphi}} - \partial_t\tilde{\tilde{\pi}} \\ \partial_s\tilde{\tilde{\pi}} + \partial_t\tilde{\tilde{\varphi}} \end{pmatrix} = \chi(|\tilde{\tilde{u}}|_{-h})\varphi(s)\begin{pmatrix} B\tilde{\tilde{\varphi}} - B^{-1}\partial_1 g_t(\tilde{\tilde{\varphi}} * \psi) * \psi - B^{-1}c_t \\ B\tilde{\tilde{\pi}} \end{pmatrix},$$

with cutoff function $\chi\colon [0, R] \to [0, 1]$ to make the nonlinearity A–admissible.

The Hilbert space for the nonlinear Schrödinger equation is L^2. We have $(L^2)_k = W^{k,2}$. In order to get a strong solution $\tilde{\tilde{u}}$ to the Floer equation with nonlinear Schrödinger type Hamiltonian, we need $\tilde{\tilde{u}}$ to be of class C^2 in the spatial variable. For generic time period T the irrationality measure of $aT/2\pi$ is $r = 2$ and so when the A–admissible nonlinearity F_t is h–regularizing with $h > 5$, we have $\tilde{\tilde{u}}(s,t) \in \mathbb{H}_{2+\frac{1}{2}} = W^{2+\frac{1}{2},2} \subset C^2$ so that we get a strong solution of

$$\partial_s\tilde{\tilde{u}} + i\partial_t\tilde{\tilde{u}} = -\partial_x^2\tilde{\tilde{u}} + \chi(|\tilde{\tilde{u}}|_{-h})\varphi(s)\partial_1 f(|\tilde{\tilde{u}} * \psi|^2, x, t)(\tilde{\tilde{u}} * \psi) * \psi.$$

Since the Floer curve $\tilde{u}$ is of class C^{m-1} in the s– and t–variables, all s– and t–derivatives of $\tilde{\tilde{u}}(s,t) := \phi_t^A \tilde{u}(s,t)$ up to order $m-1 \geq 1$ exist. Note that differentiability in the time coordinate of the ϕ_T^A–periodic solution itself does not immediately imply differentiability in t for the corresponding T–periodic solution of the Floer equation. This is because the t–derivative of ϕ_t^A involves JA, which decreases regularity. So our results do *not* follow from elliptic regularity. More specifically, the time derivative of $\tilde{\tilde{u}}$ is given by

$$\frac{d}{dt}\tilde{\tilde{u}} = \phi_t^A\Big(\frac{d}{dt}\tilde{u}\Big) + \Big(\frac{d}{dt}\phi_t^A\Big)\tilde{u}.$$

Since $\tilde{u}$ is of class C^{m-1} in the time variable, the first term is sufficiently regular. For the second term, recall that $(d/dt)\phi_t^A = JA\phi_t^A$ and so the second term only changes the regularity of $\tilde{u}$ with respect to the space variable by decreasing it by d. In particular, the regularity in the time coordinate depends on the regularity in the space coordinate. However, since above we gave conditions to ensure that we have enough regularity in the space variable, that is, $\tilde{\tilde{u}}(s,t) \in \mathbb{H}_d = \mathrm{Dom}(A)$, the time derivatives in the strong sense exist as well. Observe that the regularity requirements stated above ensure that the single s–derivative also exists. Finally we remark that by Theorem 8.2 and Theorem 8.3 the asymptotics of the Floer curve have the same t– and x–regularity as the Floer curve itself. □

9 Periodic solutions for Hamiltonian PDEs

As a corollary to the existence of a fixed point of ϕ_T^H for a Hamiltonian with A–admissible nonlinearity with, in particular, bounded support in our weaker sense of Definition 3.3, we can now prove the existence of a fixed point when the nonlinearity is only weakly A–admissible. We want to stress here that we do not claim that these result could not be obtained using different methods, and we rather include this as an application of our compactness result. We remark that there has been a significant amount of research on the problem of finding time-periodic solutions of Hamiltonian PDEs, eg [3; 5; 20; 30; 25] to mention just a few; we refer to the comprehensive book [1] for an overview of the current state in the field. In particular, a KAM result was proven in [8; 9] for the Schrödinger equation with regularizing nonlinearity that we consider. Note that the small divisor problem and regularization also play a key role in their considerations. The existence of time-periodic solutions was proven when the nonlinearity is time-independent or when it has a prescribed time dependence, for example, in [15; 16]. We want to stress that we are studying general nonautonomous Hamiltonian PDEs without any predefined time-behavior of the nonlinearity.

The idea, now, is that given a weakly A–admissible nonlinearity $\widetilde{F}_t$ we compose it with a cutoff function χ to get an A–admissible nonlinearity F_t. We then show that when the support of χ is sufficiently large, the region where a possible T–periodic solution could exist stays away from the cutoff region. Main Theorem 4.1 then implies that there exists a periodic solution for the Hamiltonian with this A–admissible nonlinearity F_t. Since this solution remains in the region where $\chi = 1$, that is, where $F_t = \widetilde{F}_t$, we find that the solution is also a solution for the Hamiltonian with weakly A–admissible nonlinearity $\widetilde{F}_t$.

Lemma 9.1 *Let A be admissible and of degree d, let (T, X) be admissible and let $h > dr$. Then there exists a positive $c \in \mathbb{R}$ such that*

$$|\phi_T^A u - u|^2 \geq c|u|_{-h}^2.$$

Proof This is a similar occurrence of the small divisor problem as we have already seen:

$$|\phi_T^A u - u|^2 = \sum_{n=0}^{\infty} |e^{ia(\pm n)^d T} - 1|^2 |\hat{u}(\pm n)|^2 \geq c \sum_{n=0}^{\infty} n^{-2d(r-1)} |\hat{u}(\pm n)|^2 = c|u|_{-d(r-1)}^2,$$

where for the inequality we use the small angle approximation and Diophantineness condition to write

$$|e^{ian^d T} - 1| \approx \inf_{p \in \mathbb{Z}} |an^d T - 2\pi p| \geq 2\pi \frac{c'}{n^{d(r-1)}}$$

similar to the proof of Proposition 7.2. Since $h > dr$ and $|u|_h < |u|_i$ whenever $h < i$, the result follows. □

Theorem 9.2 *For a Hamiltonian PDE with weakly A–admissible nonlinearity there exists a forced time-periodic solution which is of regularity $h - d(r-1) - \frac{1}{2}$ for $h > dr$, that is, $u\colon \mathbb{R} \to \mathbb{H}_{h-d(r-1)-\frac{1}{2}} \subset \mathbb{H}$ with*

$$\partial_t u = JAu + J\nabla F_t(u), \quad u(t+T) = u(t).$$

Proof Choose a cutoff function $\chi^R\colon \mathbb{R}_{\geq 0} \to [0, 1]$ which equals one on $[0, R]$, is zero on $[R+1, \infty)$ and has slope $-2 \leq (\chi^R)'(r) \leq 0$ for $r \in [R, R+1]$. Defining $F_t = F_t^R$ as in Proposition 3.4 using χ^R, it follows that $F_t\colon \mathbb{H}_{-h} \to \mathbb{R}$, and hence also when viewed as a map $F_t\colon \mathbb{H} \to \mathbb{R}$, has bounded first derivatives, independent of R. Here we use that $\widetilde{F}_t$ has bounded first derivatives even when $c_t \neq 0$ in Definition 3.3. Since therefore G_t has bounded first derivatives with respect to u, we have $|X_t^G(u)| \leq c'$ for some $c' > 0$ which is independent of R, and hence $|\phi_T^G(u) - u| \leq c'T$. Since $|\phi_T^A u - u| \geq \sqrt{c}|u|_{-h}$, u cannot be a fixed point whenever $|u|_{-h} > c'T/\sqrt{c}$. This is because $\phi_T^H = \phi_T^A \circ \phi_T^G$, and ϕ_T^A moves any u a distance at least $\sqrt{c}|u|_{-h}$ away while preserving the $\mathbb{H}_{-h}$–norm. However, ϕ_T^G only moves the point $\phi_{-T}^A u$ a distance at most $c'T < \sqrt{c}|u|_{-h}$, so u cannot be a fixed point. In fact, this shows that the entire ϕ_T^A–periodic solution u_1 stays inside the $\mathbb{H}_{-h}$–ball of radius $c'T/\sqrt{c}+\epsilon$ for any $\epsilon > 0$. This continues to hold for the T–periodic solution u because ϕ_t^A preserves the $\mathbb{H}_{-h}$–norm. Now we choose $R = cT'/\sqrt{c}+\epsilon$. For this A–admissible nonlinearity, the existence of a fixed point follows from the main theorem. By the above argument, this fixed point is also a fixed point of the time-T flow of the Hamiltonian with weakly A–admissible nonlinearity $\widetilde{F}_t$ we started with, thus proving the theorem. □

We now show that when the nonlinearity is h–regularizing for all $h \in \mathbb{N}$ we find a periodic solution for almost all time periods, since Diophantine numbers have full measure, which is of class C^∞ in both the time and spatial variable.

Corollary 9.3 *Consider a Hamiltonian PDE with ∞–regularizing T–periodic nonlinearity $\widetilde{F}_t$ with bounded C^α–norms as in condition* (iii) *of Definition 3.3, and with admissible A. Then for admissible (X,T) there exists a strong forced T–periodic solution which is smooth in both the time and space coordinates.*

Proof This does not follow immediately from Theorem 9.2, since there is no complete norm on $\mathbb{H}_{-\infty}$. In order to prove that we still find a periodic orbit for the Hamiltonian PDE with ∞–regularizing weakly A–admissible nonlinearity $\widetilde{G}_t$, which is even smooth in both the time and space variable, compose $\widetilde{G}_t$ as above with a cutoff function $\chi(|\cdot|_{-h})$ for any finite $h > dr$ to obtain an h–regularizing A–admissible nonlinearity. Applying the above result we find a periodic solution $u(t) \in \mathbb{H}_{h-d(r-1)-\frac{1}{2}}$. Since it is a solution to the PDE with ∞–regularizing weakly A–admissible nonlinearity we started with, we can a posteriori show that $u(t)$ has image in $\mathbb{H}_\infty$ and that it is smooth with respect to t. First, since $\widetilde{G}_t$ is ∞–regularizing, by definition its gradient takes values in $\mathbb{H}_\infty$, so that

$$\partial_t u = J\nabla\widetilde{G}_t(u) \in \mathbb{H}_\infty, \quad u(t+T) = \phi_{-T}^A u(t),$$

that is, the Fourier coefficients of $\partial_t u(t)$ decay exponentially fast, which in turn shows that the Fourier coefficients of $\phi_T^A u(t) - u(t)$ have the same decay rate. Now the n^{th} component $u_n(t) = \widehat{u(t)}(n) z_n$ of $u(t) = \sum_n \widehat{u(t)}(n) z_n$ satisfies

$$|\phi_T^A u_n(t) - u_n(t)| = |e^{iaTn^d} - 1||u_n(t)| \approx \inf_{p\in\mathbb{Z}} |aTn^d - 2\pi p||u_n(t)| \geq \frac{c}{n^{d(r-1)}}|u_n(t)|,$$

so that $|u_n(t)|$ still decays exponentially fast with $|n| \in \mathbb{N}$, that is, $u(t) \in \mathbb{H}_\infty$ for all t. It remains to show that $t \mapsto u(t)$ is of class C^∞, for which we again use the fact that it satisfies the Hamiltonian equation. Applying ∂_t to both sides of $\partial_t u = X_t^G(u)$ and observing that G is smooth in t, and both $u(t)$ and $\partial_t u(t)$ are in $\mathbb{H}_\infty$, we see that $\partial_t^2 u(t) \in \mathbb{H}_\infty$. By repeatedly applying ∂_t to both sides of the equation, it follows that $\partial_t^\alpha u(t) \in \mathbb{H}_\infty$ for all $\alpha \in \mathbb{N}$. □

Recalling the fact that not only Diophantine numbers have full measure, but even just those numbers with irrationality measure $r = 2$, for generic T we need $h > 2d$ and so:

Corollary 9.4 *Consider a Hamiltonian PDE with h–regularizing time-periodic nonlinearity with bounded C^α–norms as in condition (iii) of Definition 3.3, and with admissible A. Then for generic time period T, there exists a (weak) forced T–periodic solution which is of regularity $h - d - \frac{1}{2}$ for $h > 2d$.*

In particular, for our examples the main theorem provides us with the following results. Here we use the result from Proposition 8.4 combined with Theorem 9.2.

Corollary 9.5 *The nonlinear wave equation*

$$\ddot{\varphi} - \varphi_{xx} - \partial_1 g_t(\varphi * \psi, x) * \psi - c_t = 0, \quad \varphi = \varphi(t, x) = \varphi(t, x + X)$$

for $x \in S^1 = \mathbb{R}/X\mathbb{Z}$, with $\psi, c_t = c_{t+T} \in C^h$ and $g_{t+T} = g_t$ being bounded and having bounded derivatives, admits a strong T–periodic solution for generic T, provided that $h > 3\frac{1}{2}$. When $\psi, c_t = c_{t+T} \in C^\infty$, the solution is smooth in both time and space coordinates.

The fact that $h > 3$ suffices follows from Theorem 8.3 and the proof of Proposition 8.4, together with the observation that we need two spatial derivatives.

In order to see that one can only expect to find a periodic solution for generic T for $h > 0$ large enough, we emphasize that this can even be seen from a direct computation using Fourier series in the case when $g_t = 0$.

Remark Expanding $\varphi(t, x)$ and $c(t, x) = c_t(x)$ in terms of a Fourier series as in Example 2.5, it follows that the resulting Fourier coefficients satisfy the equation

$$\Big(\frac{2\pi n}{X} - \frac{2\pi p}{T}\Big)\Big(\frac{2\pi n}{X} + \frac{2\pi p}{T}\Big)\hat{\varphi}(p, n) = \hat{c}(p, n).$$

Since for any subsequence $(p',n') \subset (p,n)_{p,n\in\mathbb{Z}}$ each of the two factors can only converge to zero like n^{1-r} (and only one of the two factors is close to zero), it follows that for $c \in W^{h+\frac{1}{2},2}$, that is, $c = (c,0) \in \mathbb{H}_h$, we find a solution $\varphi \in W^{h+\frac{1}{2}-d(r-1),2}$, that is, $u = (\varphi,\pi) \in \mathbb{H}_{h-d(r-1)} \subset \mathbb{H}$ with $d = 1$, provided that $h > dr$. On the other hand, it also follows that we cannot expect to find a solution of higher regularity.

Corollary 9.6 *The nonlinear Schrödinger equation*

$$i\dot{u} + u_{xx} + \partial_1 f_t(|u*\psi|^2, x)(u*\psi)*\psi = 0, \quad u = u(t,x) = u(t,x+X)$$

for $x \in S^1 = \mathbb{R}/X\mathbb{Z}$, *with* $\psi \in C^h$ *and* $\tilde{f}_{t+T} = \tilde{f}_t : (s,x) \mapsto f_t(|s|^2,x)$ *being bounded and having bounded derivatives, admits a strong* T*–periodic solution for generic* T, *provided that* $h > 5$. *When* $\psi \in C^\infty$, *the solution is smooth in both time and space coordinates.*

Remark One could alternatively think about the admissibility condition for the periods (X,T) as a condition on X: one could fix a time period T, so that for generic X the number $aT/2\pi$ is Diophantine (with $r = 2$). Since $a = (2\pi/X)^d$ in the two main examples, this means that for fixed T, the space period X should be such that $(2\pi)^{d-1}TX^{-d}$ is Diophantine (with $r = 2$). We stress the Diophantineness condition (with $r = 2$) can explicitly be checked for any chosen pair (X,T).

10 A cup-length estimate

While our ultimate goal is to develop a full Floer homology theory for Hamiltonian PDEs with regularizing nonlinearities, we will now give an example of a result which definitely needs pseudoholomorphic curve techniques and cannot be proven using more classical techniques such as in [25]. We consider the classical result by Schwarz [28] and use our results to prove a cup-length estimate for a Hamiltonian system on a phase space which is the product of linear symplectic Hilbert space with a closed symplectic manifold.

Let $M = (M,\omega_M)$ be a closed (finite-dimensional) symplectic manifold that has vanishing second homotopy group $\pi_2(M) = \{0\}$. Then $\widetilde{M} := M \times \mathbb{H}$ is an infinite-dimensional symplectic Hilbert manifold equipped with the product symplectic form $\omega = \pi_M^*\omega_M + \pi_{\mathbb{H}}^*\omega_{\mathbb{H}}$ and with a scale structure given by $\widetilde{M}_h := M \times \mathbb{H}_h$ for $h \in \mathbb{R}$. Here $\pi_M : \widetilde{M}_h \to M$ and $\pi_{\mathbb{H}} : \widetilde{M}_h \to \mathbb{H}_h$ denote the projection onto the first or second factor, respectively.

Note that infinite-dimensional phase spaces of this form appear when performing symplectic reduction using a Hamiltonian action on $\mathbb{H}$ which is nontrivial only on finitely many components. Alternatively, they arise in Hamiltonian systems incorporating both Hamiltonian mechanics and Hamiltonian field theory. Indeed, generalizing the class of Hamiltonian particle-field systems that we introduce in [12], consider a symplectic manifold (B,ω_B) with a foliation by Lagrangian submanifolds, which contains (M,ω_M) as a symplectic submanifold, as well as a symplectic vector bundle $E\to B$ over $B=(B,\omega_B)$. Let $\mathbb{H}=(\mathbb{H},\omega_{\mathbb{H}})$ denote a symplectic Hilbert space of sections in this bundle which are constant along leaves, where the symplectic bilinear form $\omega_{\mathbb{H}}$ on $\mathbb{H}$ is defined using the symplectic structures on the fibers. Now consider time-periodic Hamiltonians

$$H_t=H^A+F_t\colon M\times\mathbb{H}\to\mathbb{R}\quad\text{with}\ F_t(u_M,u_{\mathbb{H}})=f_t(u_M,u_{\mathbb{H}}^{\rho}(u_M)),$$

where $u_{\mathbb{H}}\mapsto u_{\mathbb{H}}^{\rho}$ denotes a smoothing operator $\mathbb{H}_{s-h}\to\mathbb{H}_s$ for all $s\in\mathbb{R}$. Note that this indeed generalizes the class of time-periodic particle-field Hamiltonian systems in [12], which model the interaction of a scalar wave field on the d–dimensional torus T^d with a particle constrained to a submanifold $Q\subset T^d$. Here

$$M=T^*Q\subset T^*T^d=B,\quad E=B\times\mathbb{C}\quad\text{and}\quad\mathbb{H}=H^{\frac{1}{2}}(T^d,\mathbb{C})$$

can be viewed as a space of sections in the trivial bundle that are constant along leaves of the canonical Lagrangian foliation on T^*T^d given by the cotangent fibers. Furthermore, the smoothing operator is given by convolution with a C^h–function ρ which models the charge distribution of the particle. By contrast, recall that in this paper we consider the case where (M,ω_M) is closed.

Definition 10.1 A map $F_t\colon\widetilde{M}\to\mathbb{R}$ is called *h–regularizing* if it extends to a smooth map

$$F_t\colon\widetilde{M}_{-h}\to\mathbb{R},$$

and it is called ∞–regularizing when it is h–regularizing for all $h\in\mathbb{N}$.

With this, we again define:

Definition 10.2 A nonlinearity $F_t\colon\widetilde{M}\to\mathbb{R}$ is called *A–admissible* if it satisfies:

(i) F_t is T–periodic with (T,X) admissible.

(ii) The nonlinearity is h–regularizing with $h>dr$. Here r is the irrationality measure of $aT/2\pi$ and d the order of the differential operator A.

(iii) The extended map $F_t\colon \widetilde{M}_{-h} \to \mathbb{R}$ has bounded C^α–norms for all α.

(iv) F_t has bounded support, in the sense that for every $k \in \mathbb{N}$ there exists $R_k > 0$ such that $F_t(u) = 0$ for all $u \in \widetilde{M}$ with $|(\pi_k \circ \pi_{\mathbb{H}})(u)| > R_k$.

F_t is called *weakly A–admissible* when there exists t–dependent $c_t = c_{t+T} \in \mathbb{H}_h$ such that $u \mapsto F_t(u) - \langle c_t, \pi_{\mathbb{H}}(u)\rangle$ satisfies (i), (ii) and (iii).

Again we find:

Proposition 10.3 *Let $\widetilde{F}_t\colon \widetilde{M} \to \mathbb{R}$ be a weakly A–admissible nonlinearity. Then*

$$F_t(u) := \chi(|\pi_{\mathbb{H}}(u)|^2_{-h})\widetilde{F}_t(u)$$

with h as in Definition 3.3 condition (ii), and where χ a smooth cutoff function with $\operatorname{supp}(\chi) \subseteq [0, R]$ *for some $R > 0$, is A–admissible.*

In this final chapter we want to show how our infinite-dimensional Gromov–Floer compactness result can be used to prove the existence of multiple different time-periodic solutions $u\colon \mathbb{R} \to \widetilde{M}$, $u(t+T) = u(t)$ of $\dot{u} = X_H(u)$ for the time-periodic infinite-dimensional Hamiltonian

$$H_t(u) = \tfrac{1}{2}\langle A\pi_{\mathbb{H}}(u), \pi_{\mathbb{H}}(u)\rangle + F_t(u) =: H_A(u) + F_t(u),$$

given as the sum of some weakly A–admissible nonlinearity $F_t\colon M \times \mathbb{H} \to \mathbb{R}$ and the quadratic term H_A defined by a linear, possibly unbounded, self-adjoint (differential) operator $A\colon \mathbb{H} \to \mathbb{H}$, which we again assume to be admissible in the sense of Definition 2.1. We want to emphasize that it is natural to assume that the unbounded free Hamiltonian H_A depends only on the $\mathbb{H}$–component of u, since the restriction of H_A to every finite-dimensional subspace is a smooth Hamiltonian. The flows of H_t and of H_A and F_t are still related via

$$\phi^{H_t} = \phi^{H_A+F_t} = \phi^{H_A\#G_t} = \phi^A \circ \phi^{G_t},$$

where $G_t := F_t \circ \phi^A_t$, and we will work with ϕ^A_t and G_t rather than with $H_t = H_A + F_t$ because H_A (and hence H_t) is only densely defined, whereas the flow ϕ^A_t is a symplectomorphism which is defined on the whole of $\mathbb{H}$, and hence on $\widetilde{M}$. Note that $\phi^A_t \cdot u = (\pi_M(u), e^{itA} \cdot \pi_{\mathbb{H}}(u))$, that is, ϕ^A_t acts trivially on the first factor of $\widetilde{M} = M \times \mathbb{H}$.

Note that in contrast to before, the infinite-dimensional phase space $\widetilde{M} = M \times \mathbb{H}$ inherits nontrivial topology from the finite-dimensional closed symplectic manifold M, which we will use to prove an infinite-dimensional version of the degenerate Arnold conjecture. Let

$$\mathrm{cl}(M) := \max\Big\{ N+1 : \exists \theta_1, \dots, \theta_N \in \bigoplus_{d=1}^{\dim M} H^d(M) \backslash \{0\} \text{ with } \theta_1 \cup \dots \cup \theta_N \neq 0 \Big\}$$

denote the cup-length of M, which is a topological invariant of M and hence of $\widetilde{M}$. After fixing some collection $\theta_1, \dots, \theta_N$ of $N = \mathrm{cl}(M) - 1$ nonzero cohomology classes of M of nonzero degrees with $\theta_1 \cup \dots \cup \theta_N \neq 0$, we choose homology cycles $C_1, \dots, C_N$ representing the chosen cohomology classes via Poincaré duality,

$$\theta_1 = \mathrm{PD}[C_1], \dots, \theta_N = \mathrm{PD}[C_N].$$

More precisely, we consider pseudocycles defined using Morse theory on M; see [28] for details.

As we want to employ pseudoholomorphic curve methods, let J_M denote an arbitrary ω_M–compatible almost-complex structure on M and denote by $J = J_M \times J_{\mathbb{H}}$ the product almost-complex structure on $M \times \mathbb{H}$, where we again assume without loss of generality that the linear complex structure $J_{\mathbb{H}}$ on $\mathbb{H}$ is given by i. The following statement is a generalization of the main result in [28], under the simplifying assumption that $\pi_2(M) = \{0\}$.

Theorem 10.4 *For every Hamiltonian $H_t(u) = H_A(u) + F_t(u)$ with A–admissible nonlinearity F_t, there exist N $(\lfloor h/d \rfloor - 1)$–times differentiable maps*

$$\tilde{u} = \tilde{u}_1, \dots, \tilde{u}_N : \mathbb{R} \times \mathbb{R} \to M \times \mathbb{H}_{h-d(r-1)-\frac{1}{2}} \subset M \times \mathbb{H} \quad \textit{for } h > dr$$

satisfying the Floer equation and ϕ_T^A–periodicity condition

$$\bar{\partial}_J \tilde{u} + \nabla G_t(\tilde{u}) = 0 \quad \textit{and} \quad \tilde{u}(s, t+T) = \phi_{-T}^A \tilde{u}(s,t).$$

For every $\alpha = 1, \dots, N$, the Floer curve $\tilde{u}_\alpha$ connects two different solutions

$$u = u_\alpha^-, u_\alpha^+ : \mathbb{R} \to M \times \mathbb{H}$$

of

$$\dot{u} = X_t^G(u), \quad u(t+T) = \phi_{-T}^A(u(t)) \tag{16}$$

in the sense that there exist sequences $s_{\alpha,n}^\pm \in \mathbb{R}$ with $s_{\alpha,n}^\pm \to \pm\infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \tilde{u}_\alpha(s_{\alpha,n}^-, t) = u_\alpha^-(t) \quad \textit{and} \quad \lim_{n\to\infty} \tilde{u}_\alpha(s_{\alpha,n}^+, t) = u_\alpha^+(t).$$

Furthermore, since for the symplectic actions we have

$$\mathcal{A}(u_1^-) < \mathcal{A}(u_1^+) \leq \mathcal{A}(u_2^-) < \cdots < \mathcal{A}(u_{N-1}^+) \leq \mathcal{A}(u_N^-) < \mathcal{A}(u_N^+),$$

it follows that there are at least $N+1 = \mathrm{cl}(M)$ *mutually different solutions of* (16).

Here the symplectic action $\mathcal{A}(u)$ of a solution $u\colon \mathbb{R} \to M \times \mathbb{H}$ of (16) is defined as

$$\mathcal{A}(u) = \int_{D^2} \bar{u}^*\omega + \int_0^T G_t(u(t))\,dt,$$

where $\bar{u}$ is a filling of u, when viewed as a T–periodic orbit in the symplectic mapping torus $\mathbb{R} \times M \times \mathbb{H}/\{(t,u) \sim (t+T, \phi_{-T}^A(u))\}$; note that since $\pi_2(M) = \{0\}$, this definition is independent of the choice of $\bar{u}$. Following the proof of Theorem 9.2:

Corollary 10.5 *For every Hamiltonian* $H_t(u) = H_A(u) + F_t(u)$ *with weakly* A*–admissible nonlinearity* F_t *there exist* $\mathrm{cl}(M)$ *mutually different* T*–periodic solutions* $u = u_1, \dots, u_{N+1}$ *of regularity* $h - d(r-1) - \frac{1}{2}$ *for* $h > dr$*, that is,*

$$u\colon \mathbb{R} \to \widetilde{M}_{h-d(r-1)-\frac{1}{2}} \subset M \times \mathbb{H}$$

with

$$\partial_t u = JA\pi_{\mathbb{H}}(u) + J\nabla F_t(u) \quad \textit{and} \quad u(t+T) = u(t).$$

For the proof we use the existence of Floer curves in finite dimensions as we did before. More precisely, for every $k \in \mathbb{N}$ let $F_t^k\colon M \times \mathbb{C}^{2k+1} \to \mathbb{R}$ denote the restriction of $F_t\colon \widetilde{M} \to \mathbb{R}$ to the finite-dimensional submanifold $M \times \mathbb{C}^{2k+1} \subset M \times \mathbb{H}$. Note that F_t^k now has bounded support in $M \times B_{R_k}(0)$ and we again define $G_t^k := F_t^k \circ \phi_t^A$. Let $\mathcal{M}^k$ denote the moduli space of tuples $(\tilde{u}, \tau)$, where $\tilde{u}\colon \mathbb{R} \times \mathbb{R} \to M \times \mathbb{C}^{2k+1}$ is again a Floer curve satisfying the asymptotic condition $\lim_{s\to\pm\infty}(\pi_{\mathbb{H}} \circ \tilde{u})(s,t) = 0$, the τ–dependent Floer equation in $M \times \mathbb{C}^{2k+1}$ with periodicity condition

$$\bar{\partial}_J \tilde{u} + \varphi_\tau(s)\nabla G_t(\tilde{u}) = 0, \quad \tilde{u}(s, t+T) = \phi_{-T}^A \tilde{u}(s,t)$$

and the following *intersection property*: every Floer curve $(\tilde{u}, \tau)$ in $\mathcal{M}^k$ is required to intersect all the cycles $C_1, \dots, C_N$ in the sense that

$$(\pi_M \circ \tilde{u})\Big(2\tau\frac{1}{N+1}, 0\Big) \in C_1, \dots, (\pi_M \circ \tilde{u})\Big(2\tau\frac{N}{N+1}, 0\Big) \in C_N.$$

Lemma 10.6 *For every* $\tau \in \mathbb{N}$ *there is a Floer curve* $(\tilde{u}, \tau)$ *in* $\mathcal{M}^k$.

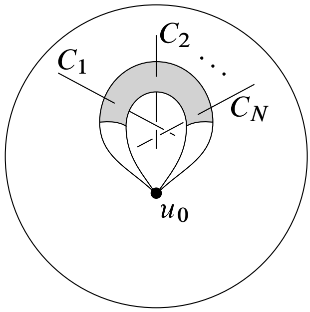


Figure 2

Proof The proof is analogous to the proof in Section 5, so we will only focus on the differences and refer to [28] for further details. Assuming again transversality for the nonlinear Cauchy–Riemann operator for the moment, the moduli space of such pairs $(\tilde{u},\tau)$ is a 1–dimensional manifold. While in the proof of Proposition 5.2 it was readily clear that a Floer curve for $\tau=0$ exists, here we have to additionally take the intersection property into account: since $\mathrm{PD}[C_1]\cup\cdots\cup\mathrm{PD}[C_N]\neq 0$, we may assume without loss of generality that $C_1,\dots,C_N$ intersect transversally in a point, $C_1\cdot\ldots\cdot C_N=\{\text{point}\}$, so that the constant curve with image in $\{\text{point}\}\times\{0\}\subset M\times\mathbb{C}^{2k+1}$ is the unique Floer curve for $\tau=0$. Again, Floer curves $(\tilde{u},\tau)$ exist for all $\tau>0$ by Gromov–Floer compactness (see Figure 2), as we can exclude bubbling-off of holomorphic spheres as well as breaking-off of cylinders for finite τ. Note that in order to exclude existence of holomorphic spheres we additionally use that $\pi_2(M)=\{0\}$.

Since we cannot expect transversality to hold, we again first need to approximate J by a family of time-dependent almost-complex structures J_t^ν satisfying $(\phi^A_{-T})_*J_t^\nu=J_{t+T}^\nu$, in the sense that $J_t^\nu\to J_t^0=i$ as $\nu\to\infty$. We emphasize that transversality now additionally includes that the evaluation map $\mathrm{ev}=(\mathrm{ev}_1,\dots,\mathrm{ev}_N)$ with

$$\mathrm{ev}_\alpha\colon \mathcal{M}^k\to M \quad \text{given by } \tilde{u}\mapsto(\pi_M\circ\tilde{u})\Big(2\tau\frac{\alpha}{N+1},0\Big) \quad \text{for } \alpha=1,\dots,N$$

is transversal to $C_1\times\cdots\times C_N\subset M\times\cdots\times M$. □

For every $k\in\mathbb{N}$ again let $\tilde{u}^k\colon\mathbb{R}\times\mathbb{R}\to M\times\mathbb{C}^{2k+1}$ be a Floer curve in $\mathcal{M}^k$ for $\tau=k$, that is, $(\tilde{u}^k,k)\in\mathcal{M}^k$. As before, the idea is to apply our infinite-dimensional generalization of the Gromov–Floer compactness result to the sequence of Floer curves $\tilde{u}^k$ in order to obtain a Floer curve in $\widetilde{M}=M\times\mathbb{H}$. More precisely, the proof of Theorem 8.1 immediately leads to a proof of the following:

Lemma 10.7 *For every $\alpha = 1, \dots, N = \mathrm{cl}(M) - 1$, a subsequence of the sequence of* ***shifted*** *Floer curves*

$$\tilde{u}^k_\alpha \colon \mathbb{R} \times \mathbb{R} \to M \times \mathbb{C}^{2k+1} \quad \text{given by } \tilde{u}^k_\alpha(s,t) = \tilde{u}^k\Big(s + 2k\frac{\alpha}{N+1}, t\Big)$$

C^{m-1}_{loc}–converges (where $m = \lfloor h/d \rfloor$) to a solution $\tilde{u} = \tilde{u}_\alpha \colon \mathbb{R} \times \mathbb{R} \to M \times \mathbb{H}$ of the Floer equation

$$\bar{\partial}_J \tilde{u} + \nabla G_t(\tilde{u}) = 0, \quad \tilde{u}(s, t+T) = \phi^A_{-T} \tilde{u}(s,t)$$

satisfying the intersection property $(\pi_M \circ \tilde{u}_\alpha)(0,t) \in C_\alpha$.

Proof The key observation is that, while in the unshifted case $\varphi_k(s,t) \to \varphi(s,t)$, in the shifted case we have $\varphi_k(s + 2k\alpha/(N+1)) \to 1$ for every $(s,t) \in \mathbb{R} \times \mathbb{R}$ as $k \to \infty$. We start by observing that we can write the finite-dimensional Floer curve as a tuple

$$\tilde{u}^k = (\tilde{u}^{k,\ell}, \tilde{u}^{k,\ell}_\perp) \colon \mathbb{R} \times \mathbb{R} \to (M \times \mathbb{C}^{2\ell+1}) \times \mathbb{C}^{2k-2\ell} = M \times \mathbb{C}^{2k+1} \subset M \times \mathbb{H},$$

where $\tilde{u}^{k,\ell}_\perp$ again denotes the normal component of $\tilde{u}^k$. Again the extra statement needed for the proof is that we still have, for $m = \lfloor h/d \rfloor$, that

$$\sup_{k \geq \ell} \|\tilde{u}^{k,\ell}_\perp\|_{C^{m-1}} \to 0 \quad \text{as } \ell \to \infty.$$

Note that this relies on the fact that we have bounded derivatives, proven using bubbling-off, where we emphasize that the condition $\pi_2(M) = \{0\}$ ensures that the proof of Lemma 6.2 still goes through. Note that this also proves, using standard elliptic bootstrapping, that there is a subsequence of $(\tilde{u}^{k,\ell})_k$ of maps from $\mathbb{R} \times \mathbb{R}$ to $M \times \mathbb{C}^{2\ell+1}$ which C^{m-1}_{loc}–converges to a map $\tilde{u}^\ell \colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{2\ell+1}$ as $k \to \infty$ for all ℓ. We stress that the maps $\tilde{u}^{k,\ell}$ still take values in finite-dimensional compact manifold $M \times B^{2\ell+1}_{R_\ell}(0)$ by the bounded support condition and the maximum principle. Because we have locally bounded $W^{m+1,p}$–norms and hence, by elliptic bootstrapping and passing to a diagonal subsequence, local $W^{m,p}$–convergence, by Sobolev embedding we also have local convergence in the C^{m-1}–norm. Passing to a diagonal subsequence yet again, we obtain C^{m-1}_{loc}–convergence for all ℓ simultaneously, which, together with our result about the normal component, proves that a subsequence of $\tilde{u}^k \colon \mathbb{R} \times \mathbb{R} \to M \times \mathbb{H}$ is locally Cauchy. □

But this implies that Theorem 8.2 generalizes:

Lemma 10.8 *For every $\alpha = 1, \dots, N = \mathrm{cl}(M)-1$ the limit Floer curve $\tilde{u}_\alpha : \mathbb{R}\times\mathbb{R} \to \mathbb{H}$ satisfies the following asymptotic conditions: there exists sequences $s^{\pm}_{\alpha,n} \in \mathbb{R}$ with $s^{\pm}_{\alpha,n} \to \pm\infty$ as $n \to \infty$ such that*

$$\lim_{n\to\infty} \tilde{u}_\alpha(s^-_{\alpha,n}, t) = u^-_\alpha(t) \quad \text{and} \quad \lim_{n\to\infty} \tilde{u}(s^+_{\alpha,n}, t) = u^+_\alpha(t)$$

in the C^{m-1}–sense ($m = \lfloor h/d \rfloor$) where u^-_α and u^+_α are two different ϕ^A_T–periodic orbits of G_t.

Proof The fact that u^-_α and u^+_α need to be different follows, as in [28], from the fact that

$$\mathcal{A}(u^+_\alpha) - \mathcal{A}(u^-_\alpha) = E(\tilde{u}_\alpha) = \int_{-\infty}^{\infty}\int_0^T |\partial_t \tilde{u}_\alpha(s,t) - X^G_t(\tilde{u}_\alpha(s,t))|^2 \, dt \, ds$$

with $E(\tilde{u}_\alpha) > 0$, since $\tilde{u}_\alpha$ must satisfy the intersection property $(\pi_M \circ \tilde{u}_\alpha)(0,t) \in C_\alpha$. □

Appendix Sc-Hamiltonian flows

Let us address the problem that the Hamiltonian H_t is only densely defined, while the flow is defined on all of $\mathbb{H}$. In particular, we do not have a Hamiltonian flow in the usual sense. Rather, it is an sc–*Hamiltonian flow*:

Definition A.1 A map $H : \mathbb{H}_h \to \mathbb{R}$ is called *strongly* sc^1 when the differential $dH : \mathbb{H}_h \times \mathbb{H}_h \to \mathbb{R}$ extends to a family of maps

$$dH : \mathbb{H}_{h+\ell} \times \mathbb{H}_{h-\ell} \to \mathbb{R}$$

for all $\ell \in \mathbb{R}$.

Let $d \in \mathbb{N}$ be the order of the differential operator A, then note that H_A is a map $H_A : \mathbb{H}_{d/2} \to \mathbb{R}$. It is strongly sc^1 because dH_A is given by

$$dH_A(u) \cdot v = \langle Au, v \rangle,$$

and this defines a family of maps $dH_A : \mathbb{H}_{d/2+\ell} \times \mathbb{H}_{d/2-\ell} \to \mathbb{R}$ with $\ell \in \mathbb{R}$. If we write $k = \ell - \frac{1}{2}d$, then $dH_t : \mathbb{H}_{d+k} \times \mathbb{H}_{-k} \to \mathbb{R}$ for $k \in \mathbb{R}$. Note that ω induces an isomorphism $\omega : \mathbb{H}_k \xrightarrow{\sim} \mathbb{H}^*_{-k}$ and so the (sc-)symplectic gradient X^H_t defined by $\omega(X^H_t, \cdot) = dH_t$ is given by a family of maps $X^H_t : \mathbb{H}_{d+k} \to \mathbb{H}_k$ for all $k \in \mathbb{R}$. That is, X^H_t is a scale morphism of order d for all k.

Definition A.2 We say $\phi\colon \mathbb{R}\times\mathbb{H}\to\mathbb{H}$ is an sc–*Hamiltonian flow of degree* d when:

(i) ϕ is sc^∞ in the sense of [19] for the Hilbert scale $(\mathbb{H}_{dn})_{n\in\mathbb{N}}$. In particular, the time-derivative defines a family of maps $\partial_t\phi\colon \mathbb{H}_{d(n+1)}\to\mathbb{H}_{dn}$ for all $n\in\mathbb{N}$.

(ii) There exists a strongly sc^1 map $H_t\colon \mathbb{H}_{d/2}\to\mathbb{R}$ such that $\partial_t\phi = X_t^H$.

The free flow ϕ_t^A is an sc–Hamiltonian flow. To show that we still get an sc–Hamiltonian flow after we have added the nonlinearity, it is sufficient to show that the flow of F_t is smooth on $\mathbb{H}_k$ for all k. Then it is immediately sc–Hamiltonian. This follows from the fact that $J\nabla F_t$ is smooth as a map from $\mathbb{H}_k$ to $\mathbb{H}_{k+h}$ for $h>0$ with uniform bounds, as the compact inclusion $\mathbb{H}_{k+h}\subset\mathbb{H}_k$ guarantees that the flow on $\mathbb{H}_k$ exists by Picard–Lindelöf. The nonlinearities in our examples satisfy this.

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