AGT

Algebraic & Geometric Topology

Volume 23 (2023)

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*Algebraic & Geometric Topology* **23**:2 (2023) 733–764
DOI: 10.2140/agt.2023.23.733
Published: 9 May 2023

# Realization of graded monomial ideal rings modulo torsion

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Let $A$ be the quotient of a graded polynomial ring $\mathbb{Z}[x_1, \dots, x_m] \otimes \Lambda[y_1, \dots, y_n]$ by an ideal generated by monomials with leading coefficients 1. We construct a space $X_A$ such that $A$ is isomorphic to $H^*(X_A)$ modulo torsion elements.

55N10; 13F55, 55P99, 55T20

## 1 Introduction

A classical problem in algebraic topology asks: which commutative graded $R$–algebras $A$ are isomorphic to $H^*(X_A; R)$ for some space $X_A$? The space $X_A$, if it exists, is called a realization of $A$. According to Aguadé [1] the problem goes back to at least Hopf, and was later explicitly stated by Steenrod [14]. To solve the problem in general is probably too ambitious, but many special cases have been proven.

One of Quillen's motivations for his seminal work on rational homotopy theory [13] was to solve this problem over $\mathbb{Q}$. He showed that all simply connected graded $\mathbb{Q}$–algebras have a realization. The problem of which polynomial algebras over $\mathbb{Z}$ have realizations has a long history, and a complete solution was given by Anderson and Grodal [2]; see also Notbohm [12]. More recently Trevisan [15] and later Bahri, Bendersky, Cohen and Gitler [4] constructed realizations of $\mathbb{Z}[x_1, \dots, x_m]/I$, where $|x_i| = 2$ and $I$ is an ideal generated by monomials with leading coefficient 1.

We want to consider a related problem that lies between the solved realization problem over $\mathbb{Q}$ and the very difficult realization problem over $\mathbb{Z}$. We do this by modding out torsion.

**Problem 1.1** *Which commutative graded $R$–algebras $A$ are isomorphic to*

$$H^*(X_A; R)/\mathit{torsion}$$

*for some space $X_A$?*

Such an $X_A$ is called a realization modulo torsion of $A$. For example, a polynomial ring $\mathbb{Z}[x]$ has a realization modulo torsion given by the Eilenberg–Mac Lane space $K(\mathbb{Z},|x|)$ if $|x|$ is even, while $\mathbb{Z}[x]$ has a realization (before modding out torsion) if and only if $|x|=2$ or 4 [14]. Here we ask: do all finite type connected commutative graded $\mathbb{Z}$–algebras have a realization modulo torsion?

Notice that modding out by torsion is different from taking rational coefficients. For example, both $H^*(\Omega S^{2n+1};\mathbb{Q})$ and $H^*(K(\mathbb{Z},2n);\mathbb{Q})$ are $\mathbb{Q}[x]$ generated by $x$ of degree $2n$. But $H^*(K(\mathbb{Z},2n))/\text{torsion}$ is $\mathbb{Z}[x]$, while $H^*(\Omega S^{2n+1})\cong\Gamma[x]$ is free as a $\mathbb{Z}$–module and is the divided polynomial algebra generated by $x$.

In this paper, we construct realizations modulo torsion of graded monomial ideal rings $A$ which are tensors of polynomial algebras and exterior algebras modulo monomial ideals. More precisely, let $P=\mathbb{Z}[x_1,\dots,x_m]\otimes\Lambda[y_1,\dots,y_n]$ be a graded polynomial ring where the $x_i$'s have arbitrary positive even degrees and the $y_j$'s have arbitrary positive odd degrees, and let $I=(M_1,\dots,M_r)$ be an ideal generated by $r$ minimal monomials

$$M_j=x_1^{a_{1j}}x_2^{a_{2j}}\cdots x_m^{a_{mj}}\otimes y_1^{b_{1j}}\cdots y_n^{b_{nj}},\quad 1\le j\le r,$$

where the indices $a_{ij}$ are nonnegative integers and $b_{ij}$ are either 0 or 1. Then the quotient algebra $A=P/I$ is called a *graded monomial ideal ring*.

**Theorem 1.2** (main theorem) *Let $A$ be a graded monomial ideal ring. Then there exists a space $X_A$ such that $H^*(X_A)/T$ is isomorphic to $A$, where $T$ is the ideal consisting of torsion elements in $H^*(X_A)$. Moreover, there is a ring morphism $A\to H^*(X_A)$ that is right inverse to the quotient map $H^*(X_A)\to H^*(X_A)/T\cong A$.*

If all of the even degree generators are in degree 2, then we do not need to mod out by torsion and so we get a generalization (Theorem 4.6) of the results of Bahri, Bendersky, Cohen and Gitler [4, Theorem 2.2] and Trevisan [15, Theorem 3.6].

The structure of the paper is as follows. Section 2 contains preliminaries, algebraic tools and lemmas that are used in later sections. In Section 3 we recall the definition of polyhedral products and modify a result of Bahri, Bendersky, Cohen and Gitler [3] to compute $H^*((\underline{X},*)^K)/T$. In Sections 4 and 5 we prove Theorem 1.2 in several steps. First, we prove it in the special case where the ideal $I$ is square-free. Then for the general case, we construct a fibration sequence inspired by algebraic polarization method and show that the fiber $X_A$ is a realization modulo torsion of $A$. In Section 6 we illustrate how to construct $X_A$ for an easy example of $A$.

# 2 Preliminaries

## 2.1 Quotients of algebras by torsion elements

It is natural to study an algebra $A$ by factoring out the torsion elements since the quotient algebra is torsion-free and has a simpler structure. Driven by this, we start investigating the quotients of cohomology rings of spaces by their torsion elements. Since we cannot find related references in the literature, here we fix the notation and develop lemmas for our purpose.

A graded module $A = \{A_i\}_{i \in S}$ is a family of indexed modules $A_i$. Since we are interested in cochain complexes and cohomology rings of connected, finite type CW–complexes, we assume $A$ to be a connected, finite type graded module with nonpositive degrees. That is, $S = \mathbb{N}_{\leq 0}$, $A_0 = \mathbb{Z}$ and each component $A_i$ is finitely generated. We follow the convention and denote $A_i$ by $A^{-i}$.

**Remark 2.1** Equivalently we can define a graded module to be a module with a grading structure, that is the direct sum $A = \bigoplus_{i \in S} A_i$ of a family of indexed modules. This definition is slightly different from the definition above. We will use both definitions interchangeably.

An element $x \in A$ is *torsion* if $cx = 0$ for some nonzero integer $c$, and is *torsion-free* otherwise. The torsion submodule $A_t$ of $A$ is the graded submodule consisting of torsion elements and the torsion-free quotient module $A_f = A/A_t$ is their quotient. If $B$ is another graded module and $g\colon A \to B$ is a morphism, then it induces a morphism $g_f\colon A_f \to B_f$ sending $a + A_t \in A_f$ to $g(a) + B_t \in B_f$. This kind of structure is important in abelian categories and was formalized with Dixon's notion of a torsion theory [6], but in this paper we only use the structure in a naive way.

**Lemma 2.2** *If* $0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0$ *is a short exact sequence of graded modules, then* $C_f \cong (B_f/A_f)_f$. *Furthermore, if the sequence is split exact, then so is*

$$0 \to A_f \xrightarrow{g_f} B_f \xrightarrow{h_f} C_f \to 0.$$

**Proof** Consider a commutative diagram as in Figure 1, where $g_t$ is the restriction of $g$ to $A_t$, $p$ and $q$ are the quotient maps, and $u$ and $v$ are the induced maps. By construction all rows and columns are exact sequences except for the right column. A diagram chase implies that $u$ is injective and $v$ is surjective. We claim that the column is exact at $C$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_t & \xrightarrow{g_t} & B_t & \xrightarrow{p} & B_t/A_t & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow u & & \\
0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{h} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow v & & \\
0 & \longrightarrow & A_f & \xrightarrow{g_f} & B_f & \xrightarrow{q} & B_f/A_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 1

Obviously $v \circ u$ is trivial. Take an element $c \in \ker(v)$ and its preimage $b \in B$. A diagram chase implies $b = g(a) + b'$ for some $a \in A$ and $b' \in B_t$. So $c = h(b') = u \circ p(b')$ is in $\mathrm{Im}(u)$ and the right column $0 \to B_t/A_t \xrightarrow{u} C \xrightarrow{v} B_f/A_f \to 0$ is exact.

For the first part of the lemma, we show that $v_f : C_f \to (B_f/A_f)_f$ is an isomorphism. Since $v$ is surjective, so is $v_f$. Take $c' \in \ker(v_f)$ and its preimage $\tilde{c}' \in C$. Then $v(\tilde{c}')$ is a torsion element in $B_f/A_f$ and $mv(\tilde{c}') = 0$ for some nonzero integer $m$. So $m\tilde{c}' \in \ker(v)$. As $\ker(v) = \mathrm{Im}(u)$ consists of torsion elements, $m\tilde{c}'$ is torsion and so is $\tilde{c}'$. Therefore $c' = 0$ in $C_f$ and $v_f$ is injective.

Notice that an exact sequence being split is equivalent to $B \cong A \oplus C$. So $B_f \cong A_f \oplus C_f$ and $0 \to A_f \xrightarrow{g_f} B_f \xrightarrow{h_f} C_f \to 0$ is a split exact sequence. □

A graded algebra $(A, m)$ consists of a graded module $A$ and an associative bilinear multiplication $m = \{m^{i,j} : A^i \otimes A^j \to A^{i+j}\}$ such that $1 \in A^0$ is the multiplicative identity. A pair $(M, \mu)$ is a left (resp. right) $A$–module if $M$ is a graded module and $\mu$ is an associative bilinear multiplication $\mu = \{\mu^{i,j} : A^i \otimes M^j \to M^{i+j}\}$ such that $\mu(1 \otimes x) = x$ (resp. $\mu = \{\mu^{i,j} : M^i \otimes A^j \to M^{i+j}\}$ such that $\mu(1, x) = x$) for all $x \in M$. We check that modding out torsion and multiplications are compatible.

**Lemma 2.3** *If $A$ and $M$ are graded modules (not necessarily of finite type), then there is a unique isomorphism $\theta : (A \otimes M)_f \to A_f \otimes M_f$ of graded modules making the diagram*

$$\begin{tikzcd}
A \otimes M \arrow[r] \arrow[d] & A_f \otimes M_f \\
(A \otimes M)_f \arrow[ur, "\theta"'] &
\end{tikzcd}$$

*commute, where the vertical and the horizontal maps are quotient maps.*

**Proof** It suffices to show that $(A^i \otimes M^j)_f \cong A^i_f \otimes M^j_f$ for any positive integers $i$ and $j$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^i_t \otimes M^j \oplus A^i \otimes M^j_t \arrow[r, "\iota_1"] \arrow[d, "a"] & A^i \otimes M^j \arrow[r, "\pi_1"] \arrow[d, equal] & A^i_f \otimes M^j_f \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & (A^i \otimes M^j)_t \arrow[r, "\iota_2"] & A^i \otimes M^j \arrow[r, "\pi_2"] & (A^i \otimes M^j)_f \arrow[r] & 0
\end{tikzcd}$$

where $a$, $\iota_1$ and $\iota_2$ are inclusions, $\pi_1$ and $\pi_2$ are quotient maps, and $b$ is the induced map. We want to show that $b$ is an isomorphism, which is equivalent to showing that $a$ is an isomorphism. If $A$ and $M$ are of finite type, then $a$ is an isomorphism since $A^i$ and $M^j$ are finitely generated abelian groups. In the general case, $a$ is an isomorphism by [9, Theorem 61.5]. □

**Corollary 2.4** *Let $(A, m)$ be a graded algebra and let $m'_f$ be the composition*

$$m'_f : A_f \otimes A_f \cong (A \otimes A)_f \xrightarrow{m_f} A_f .$$

*Then $(A_f, m'_f)$ is a graded algebra and there is a commutative diagram*

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d] & A \arrow[d] \\
A_f \otimes A_f \arrow[r, "m'_f"] & A_f
\end{tikzcd}$$

*where the vertical maps are quotient maps.*

*Let $(M, \mu)$ be a left or right $A$–module and let $\mu'_f$ be the composition*

$$\mu'_f : A_f \otimes M_f \cong (A \otimes M)_f \xrightarrow{\mu_f} M_f \quad \text{or} \quad \mu'_f : M_f \otimes A_f \cong (M \otimes A)_f \xrightarrow{\mu_f} M_f),$$

*respectively. Then $(M_f, \mu'_f)$ is respectively a left or right $A_f$–module and there is a commutative diagram*

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mu"] \arrow[d] & M \arrow[d] \\
A_f \otimes M_f \arrow[r, "\mu'_f"] & M_f
\end{tikzcd} \quad \text{or} \quad \begin{tikzcd}
M \otimes A \arrow[r, "\mu"] \arrow[d] & M \arrow[d] \\
M_f \otimes A_f \arrow[r, "\mu'_f"] & M_f
\end{tikzcd}$$

*respectively, where the vertical maps are quotient maps.*

A *cochain complex* $(A, d)$ consists of a graded module $A$ and a differential

$$d = \{d^i : A^i \to A^{i+1}\}$$

such that $d \circ d = 0$. Let $d_f = \{d_f^i : A_f^i \to A_f^{i+1}\}$ be the induced differential on $A_f$. Then $(A_f, d_f)$ forms a cochain complex and its cohomology

$$H^*(A_f, d_f) = \{H^i(A_f, d_f)\}_{i \geq 0}$$

is a graded module.

A *differential graded algebra* $(A, m, d)$ is a cochain complex $(A, d)$ such that $(A, m)$ is a graded algebra and $d$ and $m$ satisfy the Leibniz rule. Let $d_t$ be the restriction of $d$ to $A_t$. Then $(A_t, d_t)$ is a differential ideal and $(A_f, d_f)$ is a differential graded algebra, so $H^*(A_f, d_f)$ is a graded algebra.

A left (resp. right) *dg-algebra module* $(M, \mu, \delta)$ over $(A, m, d)$ if $(M, \mu)$ is a left (resp. right) $(A, m)$–module, $(M, \delta)$ is a cochain complex and $\delta$ and $\mu$ satisfy the Leibniz rule. Then $H^*(M_f, \delta_f)$ is a left (resp. right) $H^*(A_f)$–module.

Even if $(A_f, d_f)$ is torsion-free, $H^*(A_f, d_f)$ is not necessarily torsion-free. Denote $(H^*(A, d))_f$ by $H_f^*(A, d)$. The following lemma compares $H_f^*(A, d)$ and $H_f^*(A_f, d_f)$.

**Lemma 2.5** *Let $(A, d)$ be a cochain complex. Then there is a monomorphism of modules*

$$\psi : H_f^*(A, d) \to H_f^*(A_f, d_f).$$

*If $H^{i+1}(A_t, d_t) = 0$, then $\psi : H_f^i(A, d) \to H_f^i(A_f, d_f)$ is an isomorphism. Moreover, suppose $(A, m, d)$ is a differential graded algebra. Then $\psi$ is a morphism of algebras.*

**Proof** Assume $(A, d)$ is a cochain complex. Let $\iota : (A_t, d_t) \to (A, d)$ be the inclusion and let $\pi : (A, d) \to (A_f, d_f)$ be the quotient map. Then the short exact sequence of cochain complexes $0 \to (A_t, d_t) \xrightarrow{\iota} (A, d) \xrightarrow{\pi} (A_f, d_f) \to 0$ induces a long exact sequence

$$\cdots \to H^{i-1}(A_f, d_f) \to H^i(A_t, d_t) \xrightarrow{\iota^*} H^i(A, d) \xrightarrow{\pi^*} H^i(A_f, d_f) \to H^{i+1}(A_t, d_t) \to \cdots.$$

Take $\psi : H_f^*(A, d) \to H_f^*(A_f, d_f)$ to be the morphism induced by

$$\pi^* : H^*(A, d) \to H^*(A_f, d_f).$$

We show that it has the asserted properties.

To show the injectivity of $\psi$, take an equivalence class $[a] \in H^*_f(A,d)$ such that $\psi[a]=0$. Represent it by a cocycle class $a \in H^i(A,d)$. Then $\pi^*(a)$ is torsion and $\pi^*(ca)=0$ for some nonzero number $c$. By exactness, $ca \in \mathrm{Im}(\imath^*)$. Since $H^i(A_t,d_t)$ is torsion, so is $\mathrm{Im}(\imath^*)$ and $ca$ is a torsion. Therefore $a \in H^i(A,d)$ is a torsion. By definition, $[a] \in H^i_f(A,d)$ is zero. So $\psi$ is injective.

Suppose $A^{i+1}$ has no torsion elements. Then $A_t^{i+1}=0$ and $H^{i+1}(A_t,d_t)=0$. So $\pi^*$ is surjective. By definition we have commutative diagram

$$\begin{array}{ccc} H^i(A,d) & \xrightarrow{\pi^*} & H^i(A_f,d_f) \\ \downarrow & & \downarrow \\ H^i_f(A,d) & \xrightarrow{\psi} & H^i_f(A_f,d_f) \end{array}$$

where vertical arrows are quotient maps and are surjective. So

$$\psi\colon H^i_f(A,d) \to H^i_f(A_f,d_f)$$

is surjective and hence isomorphic.

If $A$ is a differential graded algebra, then $\pi^*\colon H^*(A,d) \to H^*(A_f,d_f)$ is a morphism of graded algebras. By Corollary 2.4, the induced morphism $\psi$ is multiplicative. □

**Example** The surjectivity of $\psi\colon H^i_f(A,d) \to H^i_f(A_f,d_f)$ may fail if $A^{i+1}$ contains torsion elements. Let $(A,d)$ be a cochain complex where

$$A^i = \begin{cases} \mathbb{Z} & \text{if } i=0, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i=1, \\ 0 & \text{otherwise,} \end{cases}$$

and $d^i$ are trivial for all $i$ except for $d^0\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ being the quotient map. Then $H^0(A)$ and $H^0(A_f)$ are $\mathbb{Z}$ while $\psi\colon H^0(A) \to H^0_f(A)$ is multiplication $2\colon \mathbb{Z} \to \mathbb{Z}$.

## 2.2 Eilenberg–Moore spectral sequence

Given a differential graded algebra $(A,d)$ and a right $A$–module $(M,d_M)$, we first define the bar bicomplex $\mathrm{B}^{*,*}(M,A)$ as follows. For any positive integer $i$, let $\mathrm{B}^{-i}(M,A) = M \otimes (\bar{A})^{\otimes i}$ where $\bar{A} = \{A^n\}_{n>0}$. Denote an element in $\mathrm{B}^{-i}(M,A)$ by $x[a_1|\cdots|a_i]$ for $x \in M$ and $a_i \in \bar{A}$. Let $\mathrm{B}^{-i,j}(M,A)$ be the submodule of $\mathrm{B}^{-i}(M,A)$

consisting elements $x[a_1|\cdots|a_i]$ such that $|x|+\sum_{k=1}^{i}|a_k|=j$. The internal and external differentials

$$d_I\colon \mathrm{B}^{-i,j}(M,A)\to \mathrm{B}^{-i,j+1}(M,A)\quad\text{and}\quad d_E\colon \mathrm{B}^{-i,j}(M,A)\to \mathrm{B}^{-i+1,j}(M,A)$$

are given by

$$d_I(x[a_1|\cdots|a_i])=(d_Mx)[a_1|\cdots|a_i]+\sum_{j=1}^{i}(-1)^{\epsilon_{j-1}}x[a_1|\cdots|a_{j-1}|d_Aa_j|a_{j+1}|\cdots|a_i],$$

$$d_E(x[a_1|\cdots|a_i])=(-1)^{|x|}(xa_1)[a_2|\cdots|a_i]+\sum_{j=1}^{i-1}(-1)^{\epsilon_j}x[a_1|\cdots|a_{j-1}|a_j\cdot a_{j+1}|\cdots|a_i],$$

where $\epsilon_k=k+|x|+\sum_{j=1}^{k}|a_j|$. Then we define the bar construction $(\mathcal{B}(M,A),d_{\mathcal{B}})$ to be a graded module where

$$\mathcal{B}(M,A)^n=\bigoplus_{-i+j=n}\mathrm{B}^{-i,j}(M,A)\quad\text{and}\quad d_{\mathcal{B}}=\bigoplus_{-i+j=n}(d_I+d_E)$$

for $n\geq 0$.

Take the filtration $\mathscr{F}^{-p}=\bigoplus_{0\leq i\leq p}\mathrm{B}^{-i}(M,A)$. The associated spectral sequence $\{E_r^{*,*}\}_{r=0}^{\infty}$ is the Eilenberg–Moore spectral sequence converging to $H^*(\mathcal{B}(M,A))$; see [7, Remark 2.3] and [11, Corollary 7.9].

**Lemma 2.6** *Let $A$ be a simply connected differential graded algebra and $M$ be a right $A$–module such that $A$ and $M$ are free as $\mathbb{Z}$–modules. Then there is a monomorphism of modules*

$$\psi\colon (E_2^{-p,q})_f\to \big(\mathrm{Tor}_{H_f(A)}^{-p,q}(H_f(M),\mathbb{Z})\big)_f$$

*which is an isomorphism for $p=0$. Moreover, if $H(A)$ and $H(M)$ are free modules, then $E_2^{-p,q}\cong \mathrm{Tor}_{H(A)}^{-p,q}(H(M),\mathbb{Z})$.*

**Proof** The $E_0$–page is given by

$$E_0^{-p,*}=\mathscr{F}^{-p}/\mathscr{F}^{-p+1}=M\otimes(\bar{A}^{\otimes p})$$

and $d_0=d_I$. By the Künneth theorem, the $E_1$–page is given by

$$E_1^{-p,*}\cong H(M)\otimes(\widetilde{H}(A)^{\otimes p})\oplus T\cong \mathrm{B}^{-p}(H(M),H(A))\oplus T,$$

where $T$ is a torsion term and $d_1$ is induced by $d_E$. Denote $H(M)$ by $M'$ and $H(A)$ by $A'$ for short. By Lemma 2.3, there is an isomorphism of graded modules

$$\theta\colon (E_1^{-p,*})_f\cong(\mathrm{B}^{-p}(M',A'))_f\to \mathrm{B}^{-p}(M'_f,A'_f)$$

such that

$$\begin{tikzcd} \mathrm{B}^{-p}(M',A') \arrow[d] \arrow[dr] & \\ (\mathrm{B}^{-p}(M',A'))_f \arrow[r, "\theta"] & \mathrm{B}^{-p}(M'_f,A'_f) \end{tikzcd}$$

where the downward maps are quotient maps. Let $d'$ be the external differential of $\mathrm{B}^*(M'_f,A'_f)$. Then $\theta\colon ((\mathrm{B}^{-p}(M',A'))_f,(d_1)_f)\to(\mathrm{B}^*(M'_f,A'_f),d')$ is an isomorphism of cochain complexes. By Lemma 2.5, there is a monomorphism of graded modules

$$\psi\colon (E_2^{-p,q})_f=H_f^{-p}(E_1^{*,q},d_1)\to H_f^{-p}((\mathrm{B}^{*,q}(M',A'))_f,(d_1)_f)\cong H_f^{-p}(\mathrm{B}^{*,q}(M'_f,A'_f),d').$$

Notice that $\mathrm{B}^*(M'_f,A'_f)\cong M'_f\otimes_{A'_f}\mathrm{B}^*(A'_f,A'_f)$ and $d'=\mathbb{1}\otimes_{A'_f}d''$, where $d''$ is the external differential of $\mathrm{B}^*(A'_f,A'_f)$. Since, by [11, Proposition 7.8],

$$\cdots\to\mathrm{B}^{-1}(A'_f,A'_f)\xrightarrow{d''}\mathrm{B}^0(A'_f,A'_f)\xrightarrow{\epsilon}\mathbb{Z}\to 0$$

is a projective resolution of $\mathbb{Z}$ over $A'_f$–modules where $\epsilon\colon\mathrm{B}^0(A'_f,A'_f)\cong A'_f\to\mathbb{Z}$ is the augmentation, the monomorphism becomes

$$\psi\colon (E_2^{-p,q})_f\to(\mathrm{Tor}_{A'_f}^{-p,q}(M'_f,\mathbb{Z}))_f.$$

Since $\mathrm{B}^1(M',A')=0$, $\psi$ is isomorphic for $p=0$ by Lemma 2.5.

Suppose $H(A)$ and $H(M)$ are free $\mathbb{Z}$–modules. By the Künneth theorem,

$$E_1^{*,*}\cong\mathrm{B}^{*,*}(H(M),H(A))$$

and $d_1$ is the external differential. So $E_2^{-p,q}\cong\mathrm{Tor}_{H(A)}^{-p,q}(H(M),\mathbb{Z})$. □

Let $F\to E\xrightarrow{\pi}X$ be a fibration sequence where all spaces are connected, finite type CW–complexes, and $X$ is simply connected. In [7, Theorem III] there is a quasi-isomorphism

$$\Theta\colon\Omega(C_*^{\pi}(E),C_*(X))\to CN_*(F)$$

of dg-algebra modules, which is natural in $\pi$. Here $\Omega(-,-)$ is the cobar construction, $C_*^{\pi}(E)$ is a nonnegative chain complex, $C_*(X)$ is a simply connected chain complex, $CN_*(F)$ is a chain complex, and $C_*^{\pi}(E)$, $C_*(X)$ and $CN_*(F)$ are quasi-isomorphic to the singular chain complexes of $E$, $X$ and $F$, respectively.

Denote the dual of a (co)chain complex $C$ by $C^{\vee}=\mathrm{Hom}(C,\mathbb{Z})$. Since $X$ is simply connected, $H^1(X)=0$ and $H^2(X)$ is free. By [7, Propositions 4.2 and 4.6] there are finite type graded free modules $V=\{V^i\}_{i\geq 2}$ and $W=\{W^j\}_{j\geq 0}$, a quasi-isomorphism of dg-algebras

$$\phi\colon T(V)\to (C_*(X))^\vee$$

and a quasi-isomorphism of dg-algebra modules

$$\varphi\colon T(V)\otimes W\to (C_*^\pi(E))^\vee,$$

where $T(V)$ is the tensor algebra on $V$. Write $\widetilde{X}=T(V)$ and $\widetilde{E}=T(V)\otimes W$ for short. Then the compositions

$$C_*(X)\xrightarrow{\text{incl}}(C^*(X))^\vee\xrightarrow{\phi^\vee}\widetilde{X}^\vee \quad\text{and}\quad C_*^\pi(E)\xrightarrow{\text{incl}}(C^*(E))^\vee\xrightarrow{\varphi^\vee}\widetilde{E}^\vee$$

are quasi-isomorphisms of dg-coalgebras and of dg-coalgebra modules. Since $C_*(X)$ and $\widetilde{X}^\vee$ are simply connected free chain complexes, and $C_*^\pi(E)$ and $\widetilde{E}^\vee$ are nonnegative chain complexes, we have a zig-zag of quasi-isomorphisms

$$\Omega(\widetilde{E}^\vee,\widetilde{X}^\vee)\xleftarrow{\simeq}\Omega(C_*^\pi(E),C_*(X))\xrightarrow{\Theta}CN_*(F).$$

Since $\widetilde{E}$ and $\widetilde{X}$ are of finite type, dualize the zig-zag and take cohomology to get an isomorphism

$$H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))\xrightarrow{\cong}H^*(F).$$

The Eilenberg–Moore spectral sequence $\{E_r^{*,*}\}_{r=0}^\infty$ on $F\to E\xrightarrow{\pi}X$ is the Eilenberg–Moore spectral sequence given by $A=\widetilde{X}$ and $M=\widetilde{E}$. Note that this definition depends on the choice of the pair $(\widetilde{X},\widetilde{E},\phi,\varphi)$. Any two choices may give spectral sequences with different $E_0$–pages, but their $E_r$–pages are isomorphic for $r\geq 1$.

**Lemma 2.7** *Let $F\to E\xrightarrow{\pi}X$ be a fibration sequence such that all spaces are finite type spaces and $X$ is simply connected, and let $\{E_2^{-p,q}\}$ be the $E_2$–page of Eilenberg–Moore spectral sequence on this fibration. Then there is a monomorphism*

$$\psi\colon (E_2^{-p,q})_f\to\big(\mathrm{Tor}^{-p,q}_{H_f^*(X)}(H_f^*(E),\mathbb{Z})\big)_f$$

*as modules such that $\psi$ is an isomorphism for $p=0$.*

**Proof** Since $H(\widetilde{E})\cong H^*(E)$ and $H(\widetilde{X})\cong H^*(X)$, Lemma 2.6 implies that there is a monomorphism $\psi\colon (E_2^{-p,q})_f\to\big(\mathrm{Tor}^{-p,q}_{H_f^*(X)}(H_f^*(E),\mathbb{Z})\big)_f$ such that $\psi$ is an isomorphism at $p=0$. □

Recall that the $E_0$–page is given by $E_0^{p,*} = \mathscr{F}^{-p}/\mathscr{F}^{-p+1} \cong \widetilde{E} \otimes (\overline{\widetilde{X}})^{\otimes p}$. In particular, if $p = 0$, then $E_0^{0,*} \cong \widetilde{E}$. On the other hand, $\{E_r^{*,*}\}_{r=0}^{\infty}$ is a second quadrant spectral sequence. So $E_r^{0,*}$ is the kernel of the differential map and $E_{r+1}^{0,*}$ is a quotient group of $E_r^{0,*}$. For $r \in \mathbb{N} \cup \{\infty\}$, define the edge homomorphism $e_r$ to be the composition

$$e_r \colon H(E) \cong H(\widetilde{E}) \cong E_1^{0,*} \to E_r^{0,*}$$

where the unnamed arrow is the quotient map. The following lemma tells how the edge homomorphisms relate the $E_r$–page to $H^*(E)$ and $H^*(F)$.

**Lemma 2.8** *Under the hypotheses of Lemma 2.7, the edge homomorphisms make the diagram*

$$\begin{tikzcd} H^*(E) \arrow[r, equal] \arrow[d, "e_1"] & H^*(E) \arrow[r, equal] \arrow[d, "e_2"] & \cdots \arrow[r] & H^*(E) \arrow[r, equal] \arrow[d, "e_\infty"] & H^*(E) \arrow[d, "\iota^*"] \\ E_1^{0,*} \arrow[r, "\jmath_1"] & E_2^{0,*} \arrow[r, "\jmath_2"] & \cdots \arrow[r] & E_\infty^{0,*} \arrow[r, "\jmath"] & H^*(F) \end{tikzcd}$$

*commute, where $\iota^*$ is induced by $\iota \colon F \to E$, $\jmath$ is the inclusion and the $\jmath_r$'s are the quotient maps.*

**Proof** We use the notation above. Consider the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[d, "\iota"] & E \arrow[r, "\pi"] \arrow[d, equal] & X \arrow[d] \\ E \arrow[r, equal] & E \arrow[r, "c"] & \mathrm{pt} \end{tikzcd}$$

where $c$ is the constant map. We have

$$\begin{tikzcd} \Omega(C_*^{\pi}(E), C_*(X)) \arrow[r, "\Theta"] \arrow[d] & CN_*(F) \arrow[d, "\iota_*"] \\ \Omega(C_*^{c}(E), C_*(\mathrm{pt})) \arrow[r, "\Theta"] & CN_*(E) \end{tikzcd}$$

since the quasi-isomorphism $\Theta$ is natural. The supplement $\mathbb{Z} \to (C_*(\mathrm{pt}))^{\vee}$ is a quasi-isomorphism of dg-algebras and $\varphi \colon \widetilde{E} \to (C_*^{\pi}(E))^{\vee}$ is a quasi-isomorphism of dg-algebra modules. Using this replacement and taking dual and cohomology of the diagram, we obtain

$$\begin{tikzcd} H^*(\widetilde{E}) \arrow[r, "\cong"] \arrow[d, "e^*"] & H^*(E) \arrow[d, "\iota^*"] \\ H^*(\mathcal{B}(\widetilde{E}, \widetilde{X})) \arrow[r, "\cong"] & H^*(F) \end{tikzcd} \tag{1}$$

where $e^*$ is the composition

$$e^*\colon H^*(\widetilde{E}) \cong H^*(\mathcal{B}(\widetilde{E},\mathbb{Z})) \xrightarrow{e'} H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))$$

and $e'$ is induced by the inclusion $e\colon \mathrm{B}^{*,*}(\widetilde{E},\mathbb{Z}) \to \mathrm{B}^{*,*}(\widetilde{E},\widetilde{X})$. Let $\{\widehat{E}_r^{*,*}\}_{r=0}^{\infty}$ be the Eilenberg–Moore spectral sequence on $E \xrightarrow{=} E \xrightarrow{c} \mathrm{pt}$. Then $\widehat{E}_0^{*,*} \cong \mathrm{B}^{*,*}(\widetilde{E},\mathbb{Z})$ and the $\widehat{E}_1$–page collapses to $H^*(\widetilde{E})$. The inclusion $e\colon \mathrm{B}^{*,*}(\widetilde{E},\mathbb{Z}) \to \mathrm{B}^{*,*}(\widetilde{E},\widetilde{X})$ gives the commutative diagram

$$\begin{tikzcd}
H^*(\widetilde{E}) \arrow[r, equal] \arrow[d, "\tilde{e}_1"] & H^*(\widetilde{E}) \arrow[r, equal] \arrow[d, "\tilde{e}_2"] & \cdots \arrow[r] & H^*(\widetilde{E}) \arrow[r, equal] \arrow[d, "\tilde{e}_\infty"] & H^*(\widetilde{E}) \arrow[d, "e^*"] \\
E_1^{0,*} \arrow[r, "j_1"] & E_2^{0,*} \arrow[r, "j_2"] & \cdots \arrow[r] & E_\infty^{0,*} \arrow[r, "\tilde{j}"] & H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))
\end{tikzcd}$$

where $\tilde{e}_r\colon H^*(\widetilde{E}) \cong H^*(E) \xrightarrow{e_r} E_r^{0,*}$ and $\tilde{j}\colon E_\infty^{0,*} \xrightarrow{j} H^*(F) \cong H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))$. Combine this with (1) and obtain the asserted commutative diagram. □

## 2.3 Regular sequences and freeness

Here we use the alternative description of graded objects. A commutative graded algebra $A = \bigoplus_{i\ge 0} A_i$ is an algebra with a grading such that $ab = (-1)^{ij} ba$ for $a \in A^i$ and $b \in A^j$, and a graded $A$–module $M = \bigoplus_{j\ge 0} M_j$ is the direct sum of a family of $A$–modules. A set $\{r_1,\dots,r_n\}$ of elements in $M$ is called an *$M$–regular sequence* if the ideal $(r_1,\dots,r_n)M$ is not equal to $M$ and the multiplication

$$r_i\colon M/(r_1,\dots,r_{i-1})M \to M/(r_1,\dots,r_{i-1})M$$

is injective for $1 \le i \le n$. In the special case where $M$ is a $\mathbb{K}[x_1,\dots,x_n]$–module for some field $\mathbb{K}$ and the grading of $M$ has a lower bound, $M$ is a free $\mathbb{K}[x_1,\dots,x_n]$–module if $\{x_i\}_{i=1}^n$ is a regular sequence in $M$. We want to extend this fact to the case where $M$ is a $\mathbb{Z}[x_1,\dots,x_n]$–module. Recall a corollary of the graded Nakayama lemma.

**Lemma 2.9** *Let $A$ be a graded ring and let $M$ be an $A$–module. Suppose $A$ and $M$ are nonnegatively graded, and $I = (r_1,\dots,r_n) \subset A$ is an ideal generated by homogeneous elements $r_i$ of positive degrees. If $\{m_\alpha\}_{\alpha\in S}$ is a set of homogeneous elements in $M$ whose images generate $M/IM$, then $\{m_\alpha\}_{\alpha\in S}$ generates $M$.*

**Lemma 2.10** *Let $M$ be a $\mathbb{Z}[x_1,\dots,x_n]$–module with nonnegative degrees. If*

$$M/(x_1,\dots,x_n)M$$

*is a free $\mathbb{Z}$–module and $\{x_1,\dots,x_n\}$ is an $M$–regular sequence, then $M$ is a free $\mathbb{Z}[x_1,\dots,x_n]$–module.*

**Proof** Let $I=(x_1,\dots,x_n)$. By assumption there is a set $\{m_\alpha\}_{\alpha\in S}$ of homogeneous elements in $M$ such that their quotient images form a basis in $M/IM$. By Lemma 2.9, $\{m_\alpha\}_{\alpha\in S}$ generates $M$. We need to show that $\{m_\alpha\}_{\alpha\in S}$ is linear independent over $\mathbb{Z}[x_1,\dots,x_n]$.

For $0\le i\le n$, let $M_i=M/(x_1,\dots,x_{n-i})M$, $A_i=\mathbb{Z}[x_{n+1-i},\dots,x_n]$ and $m_{\alpha,i}$ be the quotient image of $m_\alpha$ in $M_i$. We prove that $M_i$ is a free $A_i$–module with a basis $\{m_{\alpha,i}\}_{\alpha\in S}$ by induction on $i$. For $i=0$, $M_0=M/IM$ and $A_0=\mathbb{Z}$. The statement is true since $\{m_{\alpha,0}\}_{\alpha\in S}$ is a basis by construction. Assume the statement holds for $i\le k$. For $i=k+1$, if there is a collection $\{f_\alpha\}_{\alpha\in S}$ of polynomials satisfying

$$\sum_{\alpha\in S} f_\alpha\cdot m_{\alpha,k+1}=0, \tag{2}$$

we show that all $f_\alpha$'s are zero.

If not, then there are finitely many nonzero polynomials $f_{j_1},\dots,f_{j_r}$. Quotient $M_{k+1}$ and $A_{k+1}$ by the ideal $(x_{n-k})$ and let $\bar f_{j_i}$ be the image of $f_{j_i}$ in $A_k$. Then (2) becomes

$$\sum_{i=1}^{r}\bar f_{j_i}\cdot m_{j_i,k}=0.$$

By our inductive assumption, $\{m_{\alpha,k}\}$ is a basis in $M_k$. So $\bar f_{j_i}=0$ and $f_{j_i}=x_{n-k}g_{j_i}$ for some polynomial $g_{j_i}\in A_{k+1}$. Since $x_{n-k}$ is not a zero-divisor, putting $f_{j_i}=x_{n-k}g_{j_i}$ in (2) gives

$$\sum_{i=1}^{r} g_{j_i}\cdot m_{j_i,k+1}=0.$$

So $g_{j_1},\dots,g_{j_r}$ are nonzero polynomials satisfying (2) and $|g_{j_i}|<|f_{j_i}|$ for $1\le i\le r$. Iterating this argument implies that the $|f_{j_i}|$'s are arbitrarily large, but this is impossible. So the $f_{j_i}$'s must be zero and $\{m_{\alpha,k+1}\}$ is linearly independent. It follows that $M_{k+1}$ is a free $A_{k+1}$–module. □

# 3 Cohomology rings of polyhedral products

Let $[m]=\{1,\dots,m\}$, $K$ be a simplicial complex on $[m]$ and $(\underline{X},\underline{A})=\{(X_i,A_i)\}_{i=1}^m$ be a sequence of pairs of relative CW–complexes. For any simplex $\sigma\in K$, define

$$(\underline{X},\underline{A})^\sigma=\left\{(x_1,\dots,x_m)\in\prod_{i=1}^m X_i \;\middle|\; x_i\in A_i \text{ for } i\notin\sigma\right\}$$

as a subspace of $\prod_{i=1}^m X_i$, and define the *polyhedral product*

$$(\underline{X},\underline{A})^K=\bigcup_{\sigma\in K}(\underline{X},A)^\sigma$$

to be the union of $(\underline{X},\underline{A})^\sigma$ over $\sigma\in K$.

If $X_i=\mathbb{CP}^\infty$ and $A_i=*$ for all $i$, then $(\mathbb{CP}^\infty,*)^K$ is homotopy equivalent to Davis–Januszkiewicz space [5, Theorem 4.3.2]. For any principal ideal domain $R$, $H^*((\mathbb{CP}^\infty,*)^K;R)$ is isomorphic to the Stanley–Reisner ring $R[x_1,\dots,x_m]/I_K$. Here $I_K$ is the ideal generated by $x_{j_1}\cdots x_{j_k}$ for $x_{j_i}\in\widetilde{H}^*(X_{j_i};R)$ and $\{j_1,\dots,j_k\}\notin K$, and is called the *Stanley–Reisner ideal of* $K$. In general, a similar formula holds for $H^*((\underline{X},*)^K)$ whenever the $X_i$'s are any spaces with free cohomology.

**Theorem 3.1** [3] *Let $R$ be a principal ideal domain, $K$ be a simplicial complex on $[m]$ and $\underline{X}=\{X_i\}_{i=1}^m$ be a sequence of CW–complexes. If $H^*(X_i;R)$ is a free $R$–module for all $i$, then*

$$H^*((\underline{X},*)^K;R)\cong\bigotimes_{i=1}^m H^*(X_i;R)/I_K,$$

*where $I_K$ is generated by $x_{j_1}\otimes\cdots\otimes x_{j_k}$ for $x_{j_i}\in\widetilde{H}^*(X_{j_i};R)$ and $\{j_1,\dots,j_k\}\notin K$ and is called the generalized Stanley–Reisner ideal of $K$.*

The proof of Theorem 3.1 uses the strong form of the Künneth theorem, which says that

$$\mu\colon\bigotimes_{i=1}^m H^*(X_i;R)\to H^*\left(\prod_{i=1}^m X_i;R\right),\qquad x_1\otimes\cdots\otimes x_m\mapsto\pi_1^*(x_1)\cup\cdots\cup\pi_m^*(x_m),$$

where $\pi_j^*$ is induced by the projection $\pi_j\colon\prod_{i=1}^m X_i\to X_j$, is an isomorphism if all $H^*(X_i;R)$'s are free. In the reduced version of the Künneth theorem,

$$\bar{\mu}\colon\bigotimes_{i=1}^m\widetilde{H}^*(X_i)\to\widetilde{H}^*\left(\bigwedge_{i=1}^m X_i\right)$$

is also an isomorphism if all $\widetilde{H}^*(X_i;R)$'s are free. The goal of this section is to modify Theorem 3.1 by removing the freeness assumption on $H^*(X_i)$. As a trade-off, we need to mod out the torsion elements of $H^*(X_i)$. First let us refine the Künneth theorem.

**Lemma 3.2** *Let $\underline{X} = \{X_i\}_{i=1}^m$ be a sequence of spaces $X_i$. Then the induced morphisms*

$$\mu_f\colon \bigotimes_{i=1}^m H_f^*(X_i) \to H_f^*\Big(\prod_{i=1}^m X_i\Big) \quad \text{and} \quad \bar{\mu}_f\colon \bigotimes_{i=1}^m \widetilde{H}_f^*(X_i) \to \widetilde{H}_f^*\Big(\bigwedge_{i=1}^m X_i\Big).$$

*are isomorphisms as algebras, and there is a commutative diagram*

$$\begin{tikzcd} \bigotimes_{i=1}^m \widetilde{H}_f^*(X_i) \arrow[r, "\bar{\mu}_f"] \arrow[d] & \widetilde{H}_f^*(\bigwedge_{i=1}^m X_i) \arrow[d, "q_f^*"] \\ \bigotimes_{i=1}^m H_f^*(X_i) \arrow[r, "\mu_f"] & H_f^*(\prod_{i=1}^m X_i) \end{tikzcd}$$

*where $q_f^*$ is induced by the quotient map $q\colon \prod_{i=1}^m X_i \to \bigwedge_{i=1}^m X_i$.*

**Proof** It suffices to show the $m=2$ case. Let $(X,A)$ and $(Y,B)$ be pairs of relative CW–complexes and let $\pi_X\colon (X\times Y, A\times Y)\to (X,A)$ and $\pi_Y\colon (X\times Y, X\times B)\to (Y,B)$ be projections. By the generalized version of Künneth theorem [10, Chapter XIII, Theorem 11.2], the sequence

$$0\to \bigoplus_{i+j=n} H^i(X,A)\otimes H^j(Y,B) \xrightarrow{\mu'} H^n(X\times Y, X\times B\cup A\times Y)\to T\to 0,$$

where $T$ is a torsion term and $\mu'$ sends $u\otimes v\in H^i(X,A)\otimes H^j(Y,B)$ to $\pi_X^*(u)\cup\pi_Y^*(v)$, is split exact. By Lemma 2.3 $(H^*(X,A)\otimes H^*(Y,B))_f\cong H_f^*(X,A)\otimes H_f^*(Y,B)$ and by Lemma 2.2

$$\mu_f'\colon H_f^*(X,A)\otimes H_f^*(Y,B)\to H_f^*(X\times Y, X\times B\cup A\times Y)$$

is an isomorphism. Since $\mu'$ is multiplicative, so is $\mu_f'$. Letting $A$ and $B$ be the basepoints of $X$ and $Y$, or be the empty set, gives the isomorphisms

$$\mu_f\colon H_f^*(X)\otimes H_f^*(Y)\cong H_f^*(X\times Y) \quad \text{and} \quad \bar{\mu}_f\colon \widetilde{H}_f^*(X)\otimes \widetilde{H}_f^*(Y)\cong \widetilde{H}_f^*(X\wedge Y).$$

The commutative diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^m \widetilde{H}^*(X_i) \arrow[r, "\bar{\mu}"] \arrow[d] & \widetilde{H}^*\big(\bigwedge_{i=1}^m X_i\big) \arrow[d, "q^*"] \\
\bigotimes_{i=1}^m H^*(X_i) \arrow[r, "\mu"] & H^*\big(\prod_{i=1}^m X_i\big)
\end{tikzcd}$$

leads to the asserted commutative diagram. □

**Proposition 3.3** *Let $\underline{X}=\{X_i\}_{i=1}^m$ be a sequence of spaces $X_i$, and let $K$ be a simplicial complex on $[m]$. Then the inclusion $\imath\colon(\underline{X},*)^K\to\prod_{i=1}^m X_i$ induces a ring isomorphism*

$$H_f^*((\underline{X},*)^K)\cong\Big(\bigotimes_{i=1}^m H_f^*(X_i)\Big)/I_K$$

*where $I_K$ is generated by $x_{j_1}\otimes\cdots\otimes x_{j_k}$ for $x_{j_i}\in\widetilde{H}_f^*(X_{j_i})$ and $\{j_1,\dots,j_k\}\notin K$.*

**Proof** This proof modifies the proofs in [3; 5]. Consider the homotopy cofibration sequence

$$(\underline{X},*)^K\xrightarrow{\imath}\prod_{i=1}^m X_i\xrightarrow{\jmath}C,$$

where $C$ is the mapping cone of $\imath$ and $\jmath$ is the inclusion. Suspend it and obtain a diagram of homotopy cofibration sequences

$$\begin{tikzcd}
\Sigma(\underline{X},*)^K \arrow[r, "\Sigma\imath"] \arrow[d, "a"] & \Sigma\big(\prod_{i=1}^m X_i\big) \arrow[r, "\Sigma\jmath"] \arrow[d, "b"] & \Sigma C \arrow[d, "c"] \\
\bigvee_{J\in K}\Sigma\underline{X}^{\wedge J} \arrow[r, "\bar{\imath}"] & \bigvee_{J\in[m]}\Sigma\underline{X}^{\wedge J} \arrow[r, "\bar{\jmath}"] & \bigvee_{J\notin K}\Sigma\underline{X}^{\wedge J}
\end{tikzcd} \tag{3}$$

where $\underline{X}^{\wedge J}=X_{j_1}\wedge\cdots\wedge X_{j_k}$ for $J=\{j_1,\dots,j_k\}$, $\bar{\imath}$ is the inclusion, $\bar{\jmath}$ is the pinch map, $a$ is a homotopy equivalence by [3, Theorem 2.21], $b$ is a homotopy equivalence, and $c$ is an induced homotopy equivalence. Take cohomology and get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{J\notin K}\widetilde{H}^*(\underline{X}^{\wedge J}) \arrow[r, "\bar{\jmath}^*"] \arrow[d, "c^*"] & \bigoplus_{J\in[m]}\widetilde{H}^*(\underline{X}^{\wedge J}) \arrow[r, "\bar{\imath}^*"] \arrow[d, "b^*"] & \bigoplus_{J\in K}\widetilde{H}^*(\underline{X}^{\wedge J}) \arrow[r] \arrow[d, "a^*"] & 0 \\
0 \arrow[r] & \widetilde{H}^*(C) \arrow[r, "\jmath^*"] & \widetilde{H}^*\big(\prod_{i=1}^m X_i\big) \arrow[r, "\imath^*"] & \widetilde{H}^*((\underline{X},*)^K) \arrow[r] & 0
\end{tikzcd}$$

where the rows are split exact sequences, all vertical maps are isomorphisms, and all maps are additive while $\imath^*$ is multiplicative. Apply Lemma 2.2 to the diagram and get:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{J\notin K} \widetilde{H}_f^*(\underline{X}^{\wedge J}) \arrow[r, "\bar{\jmath}_f^*"] \arrow[d, "c_f^*"] & \bigoplus_{J\in[m]} \widetilde{H}_f^*(\underline{X}^{\wedge J}) \arrow[r, "\bar{\imath}_f^*"] \arrow[d, "b_f^*"] & \bigoplus_{J\in K} \widetilde{H}_f^*(\underline{X}^{\wedge J}) \arrow[r] \arrow[d, "a_f^*"] & 0 \\
0 \arrow[r] & \widetilde{H}_f^*(C) \arrow[r, "\jmath_f^*"] & \widetilde{H}_f^*\bigl(\prod_{i=1}^m X_i\bigr) \arrow[r, "\imath_f^*"] & \widetilde{H}_f^*((\underline{X},*)^K) \arrow[r] & 0
\end{tikzcd} \tag{4}$$

By Lemma 3.2, $H_f^*\bigl(\prod_{i=1}^m X_i\bigr) \cong \bigotimes_{i=1}^m H_f^*(X_i)$ so there is a ring isomorphism

$$H_f^*((\underline{X},*)^K) \cong \left(\bigotimes_{i=1}^m H_f^*(X_i)\right)/\ker(\imath_f^*).$$

Since the rows are split exact and the vertical maps are isomorphic in (4), $\ker(\imath_f^*)$ is generated by $x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $x_{j_i} \in \widetilde{H}_f^*(X_{j_i})$ and $\{j_1, \dots, j_k\} \notin K$. Therefore $\ker(\imath_f^*) = I_K$ and $H_f^*((\underline{X},*)^K) \cong \bigl(\bigotimes_{i=1}^m H_f^*(X_i)\bigr)/I_K$. □

Proposition 3.3 can be refined as follows. If the quotient map $H^*(X_i) \to H_f^*(X_i)$ has right inverse for all $i$, then so does $H^*((\underline{X},*)^K) \to H_f^*((\underline{X},*)^K)$. To formulate this, we introduce new definition.

**Definition 3.3.1** A graded algebra $\mathcal{A}$ is *free split* if the quotient map $\pi\colon \mathcal{A} \to \mathcal{A}_f$ has a section as algebras. In other words, there is a ring morphism $s\colon \mathcal{A}_f \to \mathcal{A}$ making the diagrams

$$\begin{tikzcd}
\mathcal{A}_f \arrow[r, "s"] \arrow[dr, equal] & \mathcal{A} \arrow[d, "\pi"] \\
& \mathcal{A}_f
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathcal{A}_f \otimes \mathcal{A}_f \arrow[r, "m_f"] \arrow[d, "s\otimes s"] & \mathcal{A}_f \arrow[d, "s"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}$$

commute, where $m$ and $m_f$ are multiplications in $\mathcal{A}$ and $\mathcal{A}_f$. We call $s$ a *free splitting* of $\mathcal{A}$.

In general, a free splitting of $\mathcal{A}$ is not unique. Any two free splittings $s_1$ and $s_2$ differ by a torsion element.

**Remark 3.4** Not all cohomology rings of spaces are free split. Let $C$ be the mapping cone of the composite

$$P^3(2) \xrightarrow{\rho} S^3 \xrightarrow{[\imath_1,\imath_2]} S^2 \vee S^2,$$

where $P^3(2)$ is the mapping cone of degree map $2\colon S^2 \to S^2$, $\rho$ is the quotient map and $[\iota_1, \iota_2]$ is the Whitehead product. Then $H^*(C) \cong \mathbb{Z}[a,b]/(a^2 = b^2 = 2ab = 0)$ where $|a| = |b| = 2$, and it is not free split.

**Lemma 3.5** *Under the conditions of Proposition 3.3, if $H^*(X_i)$ is free split for all $i$, then $H^*((\underline{X}, *)^K)$ is free split.*

**Proof** Use the notations in the proof of Proposition 3.3. Let $s_i : H^*_f(X_i) \to H^*(X_i)$, for $1 \le i \le m$, be a free splitting and let $s$ be the composite

$$s\colon \bigotimes_{i=1}^m H^*_f(X_i) \xrightarrow{\bigotimes_{i=1}^m s_i} \bigotimes_{i=1}^m H^*(X_i) \xrightarrow{\mu} H^*\Big(\prod_{i=1}^m X_i\Big).$$

Then $s$ is a free splitting of $H^*\big(\prod_{i=1}^m X_i\big)$. As $\iota_f^*\colon H^*_f\big(\prod_{i=1}^m X_i\big) \to H^*_f((\underline{X}, *)^K)$ is surjective, define $s'\colon H^*_f((\underline{X}, *)^K) \to H^*((\underline{X}, *)^K)$ by the diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^m H^*_f(X_i) \arrow[r, "s"] \arrow[d, "\iota_f^*"] & H^*\big(\prod_{i=1}^m X_i\big) \arrow[d, "\iota^*"] \\
H^*_f((\underline{X}, *)^K) \arrow[r, "s'"] & H^*((\underline{X}, *)^K)
\end{tikzcd}$$

We need to show that $s'$ is well defined. For $x \in H^*_f(\underline{X}, *)^K$, let $y, y' \in \bigotimes_{i=1}^m H^*_f(X_i)$ be two preimages of $x$. Then $y - y' \in \ker(\iota_f^*) = I_K$. For $J \notin K$, $s$ sends $\widetilde{H}^*_f(\underline{X})^{\otimes J}$ to $\mu(\widetilde{H}^*(\underline{X})^{\otimes J})$ which is contained in $\ker(\iota^*)$. So $\iota^* \circ s(y - y') = 0$ and $s'$ is well defined. Since $s$, $\iota^*$ and $\iota_f^*$ are multiplicative, so is $s'$. Furthermore, $s'$ is right inverse to the quotient map $H^*((\underline{X}, *)^K) \to H^*_f((\underline{X}, *)^K)$. So $s'$ is a free splitting. □

# 4 Realization of graded monomial ideal rings

We follow the idea of [3] and prove Theorem 1.2 in several steps. In Section 4.1 we use Proposition 3.3 to prove the special case where the ideal $I$ of $A$ is square-free. In Sections 4.2 and 4.3 we construct a fibration sequence inspired by algebraic polarization method and show that the fiber $X_A$ is a realization modulo torsion of $A$. More precisely, we apply the Eilenberg–Moore spectral sequence defined in Section 2.2 to calculate $H^*_f(X_A)$ and give the $E_\infty$–page by the end of this section. The extension problem is long and complicated and will be discussed in Section 5.

## 4.1 Quotient rings of square-free ideals

Let $P = \mathbb{Z}[x_1, \dots, x_m] \otimes \Lambda[y_1, \dots, y_n]$ be a graded polynomial ring where the $x_i$'s have arbitrary positive even degrees and the $y_j$'s have arbitrary positive odd degrees, and let $I = (M_1, \dots, M_r)$ be an ideal generated by monomials

$$M_j = x_1^{a_{1j}} \cdots x_m^{a_{mj}} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}},$$

where the $a_{ij}$'s are nonnegative integers and the $b_{ij}$'s are either 0 or 1. Then $A = P/I$ is a *graded monomial ideal ring*. We say that $I$ is *square-free* if the $M_j$'s are square-free monomials, that is all $a_{ij}$'s are either 0 or 1.

In the following let

- $\{i_1, \dots, i_k\} + \{j_1, \dots, j_l\} = \{i_1, \dots, i_k, j_1 + m, \dots, j_l + m\}$ for $\{i_1, \dots, i_k\} \subset [m]$ and $\{j_1, \dots, j_l\} \subset [n]$, and
- $\underline{X} + \underline{Y} = \{X_1, \dots, X_m, Y_1 \dots, Y_n\}$ for sequences of spaces $\underline{X} = \{X_i\}_{i=1}^m$ and $\underline{Y} = \{Y_j\}_{j=1}^n$.

Given a square-free ideal $I$ of $A$, take $K$ to be a simplicial complex on $[m+n]$ by removing faces $\{i_1, \dots, i_k\} + \{j_1, \dots, j_l\}$ whenever $x_{i_1} \cdots x_{i_k} \otimes y_{j_1} \cdots y_{j_l} \in I$. Then $I$ is the generalized Stanley–Reisner ideal of $K$.

**Lemma 4.1** *Let $\underline{X} = \{K(\mathbb{Z}, |x_i|)\}_{i=1}^m$ and $\underline{Y} = \{S^{|y_j|}\}_{j=1}^n$ and let $K$ be the simplicial complex defined as above. Then there is a ring isomorphism $H_f^*((\underline{X} + \underline{Y}, *)^K) \cong A$. Furthermore, $H^*((\underline{X} + \underline{Y}, *)^K)$ is free split.*

**Proof** Since $H_f^*(X_i) \cong \mathbb{Z}[x_i]$ and $H^*(Y_j) \cong \Lambda[y_j]$, the first part follows from Proposition 3.3.

For the second part, it suffices to show that $H^*(X_i)$ and $H^*(Y_j)$ are free split by Lemma 3.5. For $1 \le j \le n$, $H^*(Y_j)$ is free and hence free split. For $1 \le i \le m$, let $x_i'$ be a generator of $H^{|x_i|}(X_i) \cong \mathbb{Z}$. Then inclusion $\iota\colon \mathbb{Z}\langle x_i' \rangle \to H^*(X_i)$ extends to a ring morphism

$$s\colon \mathbb{Z}[x_i'] \cong H_f^*(X_i) \to H^*(X_i).$$

Let $\pi\colon H^*(X_i) \to H_f^*(X_i)$ be the quotient map. Since $\pi \circ \iota$ sends $x_i'$ to itself, by the universal property $\pi \circ s$ is the identity map. So $s$ is a free splitting of $H^*(X_i)$. □

## 4.2 Polarization of graded monomial ideal rings

Now drop the square-free assumption on $I=(x_1^{a_{1j}}\cdots x_m^{a_{mj}}\otimes y_1^{b_{1j}}\cdots y_n^{b_{nj}}\mid 1\le j\le r)$ and suppose some $a_{ij}$'s are greater than 1. Following ideas from [3] and [15], we use polarization to reduce the realization problem of $A$ to the special case when $I$ is square-free.

For $1\le i\le m$, let $a_i=\max\{a_{i1},\dots,a_{ir}\}$ be the largest index of $x_i$ among the $M_j$'s, and let

$$\Omega=\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid 1\le i\le m, 1\le j\le a_i\}$$

where $(i,j)\in\Omega$ are ordered in left lexicographical order, that is $(i,j)<(i',j')$ if $i<i'$, or if $i=i'$ and $j<j'$. Let

$$\begin{aligned}P'&=\mathbb{Z}[x_{ij}\mid (i,j)\in\Omega]\otimes\Lambda[y_1,\dots,y_n]\\&=\mathbb{Z}[x_{11},\dots,x_{1a_1},x_{21},\dots,x_{2a_2},\dots,x_{m1},\dots,x_{ma_m}]\otimes\Lambda[y_1,\dots,y_n],\end{aligned}$$

be a graded polynomial ring where $|x_{ij}|=|x_i|$, let

$$M'_j=(x_{11}x_{12}\cdots x_{1a_{1j}})(x_{21}x_{22}\cdots x_{2a_{2j}})\cdots(x_{m1}x_{m2}\cdots x_{ma_{mj}})\otimes(y_1^{b_{1j}}\cdots y_n^{b_{nj}})$$

and let $I'=(M'_1,\dots,M'_r)$. Then $I'$ is square-free and $A'=P'/I'$ is called the *polarization* of $A$.

Conversely, we can reverse the polarization process and obtain $A$ back from $A'$. Let

$$\overline{\Omega}=\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid 1\le i\le m, 2\le j\le a_i\}$$

where $(i,j)\in\overline{\Omega}$ are ordered in left lexicographical order, and let $W$ be a graded polynomial ring

$$W=\mathbb{Z}[w_{ij}|(i,j)\in\overline{\Omega}]=\mathbb{Z}[w_{12},\dots,w_{1a_1},w_{22},\dots,w_{2a_2},\dots,w_{m2},\dots,w_{ma_m}],$$

where $|w_{ij}|=|x_i|$. Define a ring morphism $\delta\colon W\to P'$ by $\delta(w_{ij})=x_{ij}-x_{i1}$ and make $P'$ a $W$–module via $\delta$. Then $A'$ is a $W$–module and $A\cong A'/\overline{W}A'$, where $\overline{W}=\{W^i\}_{i>0}$.

**Lemma 4.2** *Let $A'$ be a square-free graded monomial ideal ring and let $W$ and $\delta$ be defined as above. Then $A'$ is a free $W$–module.*

**Proof** Since $A'/\overline{W}A'$ is a free $\mathbb{Z}$–module, by Lemma 2.10 it suffices to show that $\{w_{ij}\}_{(i,j)\in\overline{\Omega}}$ is a $A'$–regular sequence. Set $N=|\overline{\Omega}|=\sum_{i=1}^m a_i-m$. For $1\le k\le N$, let $(i_k,j_k)\in\overline{\Omega}$ be the $k^{\text{th}}$ pair under lexicographical order and let

$$I_k=(w_{12},w_{13},\dots,w_{i_kj_k}).$$

We need to show that multiplication $w_{i_{k+1}j_{k+1}}\colon A'/I_kA' \to A'/I_kA'$ is injective.

Observe that $A'/I_kA' = \widetilde{P}/\tilde{I}$, where

$$\widetilde{P} = \mathbb{Z}[x_{11}, x_{21}, \dots, x_{m1}, x_{i_{k+1}j_{k+1}}, x_{i_{k+2}j_{k+2}}, \dots x_{i_N j_N}] \otimes \Lambda[y_1, \dots, y_n]$$

and $\tilde{I} = (\widetilde{M}_1, \dots, \widetilde{M}_r)$ is generated by monomials $\widetilde{M}_j$ obtained by identifying $x_{ij}$ with $x_{i1}$ in $M'_j$ for $(i,j) \le (i_k, j_k)$. Suppose there is a polynomial $p \in \widetilde{P}$ such that

$$(x_{i_{k+1}j_{k+1}} - x_{i_{k+1}1}) \cdot p \in \tilde{I}.$$

We can use the combinatorial argument of [8, page 31] to show $p \in \tilde{I}$. Here is an outline of the argument. Write $p = \sum_\alpha p_\alpha$ as a sum of monomials $p_\alpha$. For each monomial $p_\alpha$, it can be shown that $x_{i_{k+1}j_{k+1}} p_\alpha$ and $x_{i_{k+1}1} p_\alpha$ are in $\tilde{I}$. Counting the indices of variables implies $p_\alpha \in \tilde{I}$. So $p$ is in $\tilde{I}$ and multiplication $w_{i_{k+1}j_{k+1}}\colon A'/I_kA' \to A'/I_kA'$ is injective. Therefore $\{w_{ij}\}_{(i,j)\in\overline{\Omega}}$ is a regular sequence and $A'$ is a free $W$–module. $\square$

## 4.3 Constructing a realization modulo torsion $X_A$

Let $A' = P'/I'$ be the polarization of $A$ and let $K$ be a simplicial complex on $\left(\sum_{i=1}^m a_i + n\right)$ vertices such that $I'$ is the generalized Stanley–Reisner ideal of $K$. Construct a polyhedral product to realize $A'$. Take

$$\begin{aligned}\underline{X} &= \{X_{ij} = K(\mathbb{Z}, |x_i|) \mid (i,j) \in \Omega\} \\ &= \{\underbrace{K(\mathbb{Z}, |x_1|), \dots, K(\mathbb{Z}, |x_1|)}_{a_1}, \underbrace{K(\mathbb{Z}, |x_2|), \dots, K(\mathbb{Z}, |x_2|)}_{a_2}, \\ &\qquad\qquad \dots, \underbrace{K(\mathbb{Z}, |x_m|), \dots, K(\mathbb{Z}, |x_m|)}_{a_m}\}\end{aligned}$$

and

$$\underline{Y} = \{Y_k = S^{|y_k|} \mid 1 \le k \le n\} = \{S^{|y_1|}, S^{|y_2|}, \dots, S^{|y_n|}\}.$$

By Lemma 4.1, $H^*_f((\underline{X} + \underline{Y}, *)^K)$ is isomorphic to $A'$.

For $1 \le i \le m$, define $\delta_i\colon \prod_{j=1}^{a_i} X_{ij} \to \prod_{j=2}^{a_i} X_{ij}$ by

$$\delta_i(u_1, \dots, u_{a_i}) = (u_2 \cdot u_1^{-1}, \dots, u_{a_i} \cdot u_1^{-1}),$$

and define $\delta\colon (\underline{X} + \underline{Y}, *)^K \to \prod_{(i,j)\in\overline{\Omega}} X_{ij}$ to be the composite

$$\delta\colon (\underline{X} + \underline{Y}, *)^K \hookrightarrow \prod_{(i,j)\in\Omega} X_{ij} \times \prod_{k=1}^n Y_k \xrightarrow{\text{proj}} \prod_{(i,j)\in\Omega} X_{ij} \xrightarrow{\prod_{i=1}^m \delta_i} \prod_{(i,j)\in\overline{\Omega}} X_{ij}.$$

As $\delta$ is a fibration, take $X_A$ to be its fiber. We claim that $H^*_f(X_A) \cong A$.

**Notation 4.3** Let $\{E_r^{*,*}\}_{r=0}^{\infty}$ be the Eilenberg–Moore spectral sequence defined in Section 2.2 on the fibration sequence

$$X_A \to (\underline{X}+\underline{Y},*)^K \xrightarrow{\delta} \prod_{(ij)\in\overline{\Omega}} X_{ij}, \tag{5}$$

where $H^*((\underline{X}+\underline{Y},*)^K)$ is an $H^*\big(\prod_{(ij)\in\overline{\Omega}} X_{ij}\big)$–module via $\delta^*$.

**Lemma 4.4** *For the $E_\infty$–page, $(E_\infty^{0,q})_f \cong A^q$ as modules and $(E_\infty^{-p,q})_f = 0$ for $p \neq 0$.*

**Proof** The $E_2$–page is given by $E_2^{-p,*} = \mathrm{Tor}^{-p,*}_{H^*(\prod_{(ij)\in\overline{\Omega}} X_{ij})}(H^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z})$. By Lemma 2.6, there is a monomorphism

$$\pi' \colon (E_2^{-p,*})_f \to \big(\mathrm{Tor}^{-p,*}_{H_f^*(\prod_{(ij)\in\overline{\Omega}} X_{ij})}(H_f^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z})\big)_f,$$

which is an isomorphism for $p=0$. By Lemmas 3.2 and 3.3, $H_f^*((\underline{X}+\underline{Y},*)^K) \cong A'$ and

$$H_f^*\bigg(\prod_{(ij)\in\overline{\Omega}} X_{ij}\bigg) \cong \mathbb{Z}[w_{12},\dots,w_{1a_1},w_{22},\dots,w_{2a_2},\dots,w_{m2},\dots,w_{ma_m}].$$

Denote $H_f^*\big(\prod_{(ij)\in\overline{\Omega}} X_{ij}\big)$ by $W$. So $A'$ is a $W$–module via $\delta^*$. By Lemma 4.2, $A'$ is a free $W$–module, so

$$\mathrm{Tor}_W^{-p,q}(A',\mathbb{Z}) \cong \begin{cases} A^q & \text{if } p=0,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $(E_2^{-p,q})_f$ is $A^q$ for $p=0$ and is zero otherwise.

Suppose $(E_r^{-p,q})_f$ is $A^q$ for $p=0$ and is zero otherwise. Since $(E_r^{-p,*})_f$ is concentrated in the column $p=0$, any differentials $d_r$ in and out of torsion-free elements are trivial. So we have $\ker(d_r)_f = (E_r^{-p,q})_f$ and $\mathrm{Im}(d_r)_f = 0$. By Lemma 2.2, $(E_{r+1}^{-p,q})_f \cong (E_r^{-p,q})_f$. Therefore $(E_\infty^{-p,q})_f$ is isomorphic to $A^q$ for $p=0$ and is zero otherwise. □

**Lemma 4.5** *There is an additive isomorphism $H_f^q(X_A) \cong A^q$.*

**Proof** Since the Eilenberg–Moore spectral sequence strongly converges to $H^*(X_A)$, for any fixed $q$ there is a decreasing filtration $\{\mathscr{F}^{-p}\}$ of $H^q(X_A)$ such that

$$\mathscr{F}^{-\infty} = H^q(X_A), \quad \mathscr{F}^1 = 0, \quad E_\infty^{-p,p+q} \cong \mathscr{F}^{-p}/\mathscr{F}^{-p+1}.$$

By Lemma 2.2, $(E_\infty^{-p,p+q})_f \cong \big((\mathscr{F}^{-p})_f/(\mathscr{F}^{-p+1})_f\big)_f$. By Lemma 4.4, $(E_\infty^{-p,p+q})_f$ is zero unless $p=0$, so $H_f^q(X_A) \cong (E_\infty^{0,q})_f \cong A^q$ as modules. □

Before going to the extension problem of the $E_\infty$–page, we consider the special case where all of the even degree generators of $A$ are in degree 2. The following theorem refines Lemma 4.5 and shows that $H^*(X_A) \cong A$ as algebras without modding out the cohomology by torsion. This generalizes the results of Bahri, Bendersky, Cohen and Gitler [4, Theorem 2.2] and Trevisan [15, Theorem 3.6].

**Theorem 4.6** *Let $A$ be a graded monomial ideal ring where its generators have either degree 2 or arbitrary positive odd degrees. Then $H^*(X_A) \cong A$ as algebras.*

**Proof** The $E_2$–page is given by

$$E_2^{-p,*} = \mathrm{Tor}^{-p,*}_{H^*(\prod_{(ij)\in\bar{\Omega}} X_{ij})}(H^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z}).$$

By hypothesis, $X_{ij} = \mathbb{CP}^\infty$ for $(i,j)\in\Omega$, and $H^*\bigl(\prod_{(ij)\in\bar{\Omega}} X_{ij}\bigr)$ and $H^*((\underline{X}+\underline{Y},*)^K)$ are free. Following the argument in the proof of Lemma 4.4, $E_2^{-p,q}$ is $A^q$ for $p=0$ and is zero otherwise. Since the $E_2$–page is concentrated in the column $p=0$, the spectral sequence collapses and $H^*(X_A)\cong A$ as modules.

Let $\phi\colon X_A \to (\underline{X}+\underline{Y},*)^K$ be the fiber inclusion. Lemma 2.8 implies the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\cong"] \arrow[d, "e"] & H^*((\underline{X}+\underline{Y},*)^K) \arrow[d, "\phi^*"] \\ E_2^{*,*} \arrow[r, "\cong"] & H^*(X_A) \end{tikzcd}$$

where $e$ is surjective. Since $\phi^*$ is surjective and multiplicative and its kernel is $W$, $H^*(X_A)\cong A'/W \cong A$ as algebras. □

## 5 The extension problem

In this section we continue using Notation 4.3. Lemma 4.5 shows that $H^*_f(X_A)$ and $A$ are free $\mathbb{Z}$–modules of same rank at each degree. We claim that they are isomorphic as algebras. The idea is to construct a space $Z_A$ related to $X_A$ such that $H^*(Z_A)$ is free and computable. Then we define a map $g_A\colon Z_A \to X_A$ and compare $H^*(X_A)$ with $H^*(Z_A)$ via $g_A^*$.

**Construction of $Z_A$** For $1 \le i \le m$ let $|x_i| = 2c_i$, and let

$$\begin{aligned}\underline{Z} &= \{Z_{ij} = (\mathbb{CP}^\infty)^{c_i} \mid (i,j) \in \Omega\} \\ &= \{\underbrace{(\mathbb{CP}^\infty)^{c_1}, \dots, (\mathbb{CP}^\infty)^{c_1}}_{a_1}, \underbrace{(\mathbb{CP}^\infty)^{c_2}, \dots, (\mathbb{CP}^\infty)^{c_2}}_{a_2}, \\ &\qquad \dots, \underbrace{(\mathbb{CP}^\infty)^{c_m}, \dots, (\mathbb{CP}^\infty)^{c_m}}_{a_m}\}\end{aligned}$$

and construct the polyhedral product $(\underline{Z} + \underline{Y}, *)^K$. Fix a generator $z$ of $H^2(\mathbb{CP}^\infty)$. For $(i,j) \in \Omega$ and $1 \le k \le c_i$, let $\pi_{ijk} \colon Z_{ij} \to \mathbb{CP}^\infty$ be the projection onto the $k^{\text{th}}$ copy of $\mathbb{CP}^\infty$ and let $z_{ijk} = \pi_{ijk}^*(z)$. By Theorem 3.1,

$$H^*((\underline{Z} + \underline{Y}, *)^K) \cong Q'/L',$$

where $Q' = \mathbb{Z}[z_{ijk} | (i,j) \in \Omega, 1 \le k \le c_i] \otimes \Lambda[y_1, \dots, y_n]$ and $L'$ is the ideal generated by monomials

$$z_{i_1 j_1 k_1} \cdots z_{i_t j_t k_t} \otimes y_{l_1} \cdots y_{l_\tau}$$

for $\bigl\{j_1 + \sum_{s=1}^{i_1 - 1} a_s, \dots, j_t + \sum_{s=1}^{i_t - 1} a_s\bigr\} + \{l_1, \dots, l_\tau\} \notin K$. For $1 \le i \le m$, define

$$\tilde{\delta}_i \colon \prod_{j=1}^{a_i} Z_{ij} \to \prod_{j=2}^{a_i} Z_{ij}, \quad \tilde{\delta}_i(u_1, \dots, u_{a_i}) = (u_2 \cdot u_1^{-1}, \dots, u_{a_i} \cdot u_1^{-1}),$$

and define $\tilde{\delta} \colon (\underline{Z} + \underline{Y}, *)^K \to \prod_{(i,j) \in \overline{\Omega}} Z_{ij}$ to be the composite

$$\tilde{\delta} \colon (\underline{Z} + \underline{Y}, *)^K \hookrightarrow \prod_{(i,j) \in \Omega} Z_{ij} \times \prod_{k=1}^{n} Y_k \xrightarrow{\text{proj}} \prod_{(i,j) \in \Omega} Z_{ij} \xrightarrow{\prod_{i=1}^m \tilde{\delta}_i} \prod_{(i,j) \in \overline{\Omega}} Z_{ij}.$$

**Lemma 5.1** *Let $Z_A$ be the fiber of $\delta'$. Then $H^*(Z_A) \cong Q/L$, where*

$$Q = \mathbb{Z}[z_{ik} \mid 1 \le i \le m, 1 \le k \le c_i] \otimes \Lambda[y_1, \dots, y_n]$$

*with $|z_{ik}| = 2$ and $L$ is generated by monomials $z_{i_1 k_1} \cdots z_{i_N k_N} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}}$ satisfying*

$$1 \le j \le r, \quad 1 \le k_l \le c_{i_l} \quad \text{and} \quad (i_1, \dots, i_N) = (\underbrace{1, \dots, 1}_{a_{1j}}, \underbrace{2, \dots, 2}_{a_{2j}}, \dots, \underbrace{m, \dots, m}_{a_{mj}}).$$

**Proof** Apply the Eilenberg–Moore spectral sequence to the fibration sequence

$$Z_A \to (\underline{Z} + \underline{Y}, *)^K \xrightarrow{\tilde{\delta}} \prod_{(i,j) \in \overline{\Omega}} Z_{ij}.$$

The $E_2$–page is given by $\widetilde{E}_2^{-p,*} = \mathrm{Tor}^{-p,*}_{H^*(\prod_{(i,j)\in\overline{\Omega}} Z_{ij})}(\mathbb{Z}, H^*((\underline{Z}+\underline{Y},*)^K))$. By the Künneth theorem,

$$H^*\Bigl(\prod_{(i,j)\in\overline{\Omega}} Z_{ij}\Bigr) \cong \mathbb{Z}[v_{ijk} \mid (i,j)\in\overline{\Omega}, 1\le k\le c_i],$$

where $|v_{ijk}|=2$. Denote $H^*(\prod_{(i,j)\in\overline{\Omega}} Z_{ij})$ by $V$. By definition $\tilde{\delta}^*(v_{ijk}) = z_{ijk} - z_{i1k}$. This gives an action of $V$ on $Q'$. By Lemma 4.2, $Q'/L'$ is a free $V$–module, so

$$\mathrm{Tor}_V^{-p,*}(Q'/L',\mathbb{Z}) = \begin{cases} (Q'/L')/(z_{ijk}-z_{ilk}) & \text{if } p=0,\\ 0 & \text{otherwise.}\end{cases}$$

Modding out $(z_{ijk}-z_{ilk})$ identifies $z_{ijk}$ with $z_{ilk}$ in $Q'/L'$, so

$$(Q'/L')/(z_{ijk}-z_{ilk}) \cong Q/L.$$

Since the $E_2$–page is concentrated in the column $p=0$, $H^*(Z_A)\cong Q/L$.

Lemma 2.8 implies a commutative diagram

$$\begin{tikzcd} Q' \arrow[r, "\cong"] \arrow[d, "e"] & H^*((\underline{Z}+\underline{Y})^K) \arrow[d, "\phi^*"] \\ E_2^{*,*} \arrow[r, "\cong"] & H^*(Z_A) \end{tikzcd}$$

where $e$ is surjective and $\phi^*$ is induced by the fiber inclusion $\phi\colon Z_A\to(\underline{Z}+\underline{Y})^K$. This implies $\phi^*$ is surjective. Since $\phi^*$ is multiplicative, $H^*(Z_A)\cong Q/L$ as algebras. □

**Construction of $g_A$** Fix a generator $z\in H^2(\mathbb{CP}^\infty)$. Let $\pi_j\colon(\mathbb{CP}^\infty)^{c_i}\to\mathbb{CP}^\infty$, for $1\le j\le c_i$, be the projection onto the $j^{\text{th}}$ copy of $\mathbb{CP}^\infty$ and let $z_j=\pi_j^*(z)$. For $1\le i\le m$, take a map $g_i\colon(\mathbb{CP}^\infty)^{c_i}\to K(\mathbb{Z},2c_i)$ that represents the cocycle class $z_1\cdots z_{c_i}\in H^{2c_i}((\mathbb{CP}^\infty)^{c_i})$. For $(i,j)\in\Omega$, let $g_{ij}\colon Z_{ij}\to X_{ij}$ be $g_i$, and for $1\le k\le n$, let $h_k\colon Y_k\to Y_k$ be the identity map. Then $\{g_{ij},h_k\mid(i,j)\in\Omega,1\le k\le n\}$ induces a map $g_K\colon(\underline{Z}+\underline{Y},*)^K\to(\underline{X}+\underline{Y},*)^K$ by the functoriality of polyhedral products.

**Lemma 5.2** *Let $\{x_{ij},y_k\mid(i,j)\in\Omega,1\le k\le n\}$ be generators of*

$$H_f^*((\underline{X}+\underline{Y},*)^K)\cong P'/I'$$

*and $\{z_{ijl},y'_k\mid(i,j)\in\Omega,1\le l\le c_i,1\le k\le n\}$ be generators of*

$$H^*((\underline{Z}+\underline{Y},*)^K)\cong Q'/L'.$$

*Then $(g_K^*)_f(x_{ij})=\prod_{l=1}^{c_i} z_{ijl}$ and $(g_K^*)_f(y_k)=y'_k$.*

**Proof** There is a commutative diagram

$$\begin{array}{ccc} (\underline{Z}+\underline{Y},*)^K & \xrightarrow{\jmath} & \prod_{(i,j)\in\Omega} Z_{ij} \times \prod_{k=1}^n Y_k \\ \big\downarrow g_K & & \big\downarrow g \\ (\underline{X}+\underline{Y},*)^K & \xrightarrow{\imath} & \prod_{(i,j)\in\Omega} X_{ij} \times \prod_{k=1}^n Y_k \end{array}$$

where $\imath$ and $\jmath$ are inclusions, and $g=\prod_{(i,j)\in\Omega} g_{ij} \times \prod_{k=1}^n h_k$. Taking cohomology and modding out torsion elements, we obtain the commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{\imath_f^*} & P'/I' \\ g_f^* \big\downarrow & & \big\downarrow (g_K^*)_f \\ Q' & \xrightarrow{\jmath^*} & Q'/L' \end{array}$$

where $\imath_f^*$ and $\jmath^*$ are the quotient maps. Let $\tilde{x}_{ij}, \tilde{y}_k \in P'$ and $\tilde{z}_{ijl}, \tilde{y}'_k \in Q'$ be generators such that $\imath_f^*(\tilde{x}_{ij}) = x_{ij}$, $\imath_f^*(\tilde{y}_k) = y_k$, $\jmath_f^*(\tilde{y}'_k) = y'_k$ and $\jmath_f^*(\tilde{z}_{ijl}) = z_{ijl}$. By construction $g_f^*(\tilde{x}_{ij}) = \prod_{l=1}^{c_i} \tilde{z}_{ijl}$ and $g_f^*(\tilde{y}_k) = \tilde{y}'_k$, so we have $(g_K^*)_f(x_{ij}) = \prod_{l=1}^{c_i} z_{ijl}$ and $(g_K^*)_f(y_k) = y'_k$. □

**Lemma 5.3** *There is a map $g_A\colon Z_A \to X_A$ making the diagram*

$$\begin{array}{ccc} Z_A & \longrightarrow & (\underline{Z}+\underline{Y},*)^K \\ \big\downarrow g_A & & \big\downarrow g_K \\ X_A & \longrightarrow & (\underline{X}+\underline{Y},*)^K \end{array}$$

*commute, where the horizontal maps are the inclusion maps.*

**Proof** One may want to construct $g_A$ by showing the diagram

$$\begin{array}{ccc} (\underline{Z}+\underline{Y},*)^K & \xrightarrow{\tilde{\delta}} & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\ \big\downarrow g_K & & \big\downarrow \prod_{(i,j)\in\overline{\Omega}} g_{ij} \\ (\underline{X}+\underline{Y},*)^K & \xrightarrow{\delta} & \prod_{(i,j)\in\overline{\Omega}} X_{ij} \end{array}$$

commutes. However, as $\big(\prod_{(i,j)\in\overline{\Omega}} g_{ij}\big) \circ \tilde{\delta}$ and $\delta \circ g_K$ induce different morphisms on cohomology, the diagram cannot commute. Instead, we show that the composite

$$Z_A \to (\underline{Z}+\underline{Y},*)^K \xrightarrow{g_K} (\underline{X}+\underline{Y},*)^K \xrightarrow{\delta} \prod_{(i,j)\in\overline{\Omega}} X_{ij}$$

is trivial. If so, there will exist a map $g_A\colon Z_A \to X_A$ as asserted since $X_A$ is the fiber of $\delta$.

By definition of $\bar{\delta}$ there is a commutative diagram

$$\begin{tikzcd}
(\underline{Z}+\underline{Y},*)^K \arrow[r, "\tilde{\delta}"] \arrow[d, "\jmath"] & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\
\prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k \arrow[r, "\mathrm{proj}"] & \prod_{(i,j)\in\Omega} Z_{ij} \arrow[u, "\prod_{i=1}^m \tilde{\delta}_i"']
\end{tikzcd}$$

where $\jmath$ is the inclusion. Denote $\left(\prod_{i=1}^m \tilde{\delta}_i\right)\circ\mathrm{proj}$ by $\tilde{\delta}'$ and extend the diagram to

$$\begin{tikzcd}
Z_A \arrow[r] \arrow[d, "e"] & (\underline{Z}+\underline{Y},*)^K \arrow[r, "\tilde{\delta}"] \arrow[d, "\jmath"] & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \arrow[d, equal] \\
\prod_{i=1}^m (\mathbb{C}\mathbb{P}^\infty)^{c_i}\times\prod_{k=1}^n Y_k \arrow[r, "\triangle'\times h"] & \prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k \arrow[r, "\tilde{\delta}'"] & \prod_{(i,j)\in\overline{\Omega}} Z_{ij}
\end{tikzcd}$$

where $\triangle'\colon\prod_{i=1}^m(\mathbb{C}\mathbb{P}^\infty)^{c_i}\to\prod_{j=1}^{a_i} Z_{ij}$ is the diagonal map, $h\colon\prod_{k=1}^n Y_k\to\prod_{k=1}^n Y_k$ is the identity map, and $e$ is an induced map. The top and the bottom row are fibration sequences. The left square fits into the commutative diagram

$$\begin{tikzcd}
Z_A \arrow[r] \arrow[d, "e"] & (\underline{Z}+\underline{Y},*)^K \arrow[r, "g_K"] \arrow[d, "\jmath"] & (\underline{X}+\underline{Y},*)^K \arrow[r, "\delta"] \arrow[d, "\imath"] & \prod_{\overline{\Omega}} X_{ij} \arrow[d, equal] \\
\prod_{i=1}^m(\mathbb{C}\mathbb{P}^\infty)^{c_i}\times\prod_{j=1}^n Y_j \arrow[r, "\triangle'\times h"] \arrow[dr, "\prod g_i\times h"'] & \prod_\Omega Z_{ij}\times\prod_{j=1}^n Y_j \arrow[r, "\prod g_{ij}\times h"] & \prod_\Omega X_{ij}\times\prod_{k=1}^n Y_k \arrow[r, "\delta'"] & \prod_{\overline{\Omega}} X_{ij} \\
& \prod_{i=1}^m K(\mathbb{Z},|x_i|)\times\prod_{j=1}^n Y_j \arrow[ur, "\triangle\times h"'] & &
\end{tikzcd}$$

where $\imath$ is the inclusion, $\triangle\colon\prod_{i=1}^m K(\mathbb{Z},|x_i|)\to\prod_{j=1}^{a_i} X_{ij}$ is the diagonal map, and $\delta'$ is the composite

$$\delta'\colon \prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k \xrightarrow{\mathrm{proj}} \prod_{(i,j)\in\Omega} X_{ij} \xrightarrow{\prod_{i=1}^m \delta_i} \prod_{(i,j)\in\overline{\Omega}} X_{ij}.$$

The middle square is due to the functoriality of polyhedral products, the right square is due to the definition of $\delta$ and the bottom triangle is due to the naturality of diagonal maps.

The composite of maps from $Z_A$ to $\prod_{(i,j)\in\overline{\Omega}} X_{ij}$ round the bottom triangle is trivial since

$$\prod_{i=1}^m K(\mathbb{Z},|x_i|)\times\prod_{k=1}^n Y_k \xrightarrow{\triangle\times h} \prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k \xrightarrow{\delta'} \prod_{(i,j)\in\overline{\Omega}} X_{ij}$$

is a fibration sequence. So the composite in the top row is trivial and this induces a map $g_A\colon Z_A \to X_A$ as asserted. $\square$

Since $g_K^*\colon H^*((\underline{X}+\underline{Y})^K) \to H^*((\underline{Z}+\underline{Y})^K)$ is multiplicative and $H^*(Z_A)$ is a quotient algebra of $H^*((\underline{Z}+\underline{Y})^K)$, we use $g_A$ to compare $H^*(X_A)$ and $H^*(Z_A)$ and show that $H_f^*(X_A)$ is a quotient algebra of $H_f^*((\underline{X}+\underline{Y})^K)$.

**Lemma 5.4** *Let $\phi\colon X_A \to ((\underline{X}+\underline{Y},*)^K)$ be the inclusion. Then the induced morphism*

$$\phi_f^*\colon H_f^*((\underline{X}+\underline{Y},*)^K) \to H_f^*(X_A)$$

*is surjective and* $\ker(\phi_f^*)$ *is generated by $x_{ij}-x_{i1}$ for $(i,j)\in\overline{\Omega}$.*

**Proof** Fix a positive integer $q$ and let $\psi\colon Z_A \to (\underline{Z}+\underline{Y},*)^K$ be the inclusion. Consider the commutative diagram

$$\begin{tikzcd}
 & H^q((\underline{X}+\underline{Y},*)^K) \arrow[r, "g_K^*"] \arrow[dl, "e"'] \arrow[d, "\phi^*"] & H^q((\underline{Z}+\underline{Y},*)^K) \arrow[d, "\psi^*"] \\
E_\infty^{0,q} \arrow[r, "h"] & H^q(X_A) \arrow[r, "g_A^*"] & H^q(Z_A)
\end{tikzcd}$$

where $e$ is surjective and $h$ is injective. The left triangle commutes due to Lemma 2.8 and the right square commutes due to Lemma 5.3. Mod out torsion elements and take a generator

$$x_{i_1j_1}\cdots x_{i_sj_s}\otimes y_{l_1}\cdots y_{l_t} \in H_f^q((\underline{X}+\underline{Y},*)^K).$$

By Lemma 5.2 and the above diagram,

$$(g_A^*\circ h\circ e)_f(x_{i_1j_1}\cdots x_{i_sj_s}\otimes y_{l_1}\cdots y_{l_t}) = (\psi^*\circ g_K^*)_f(x_{i_1j_1}\cdots x_{i_sj_s}\otimes y_{l_1}\cdots y_{l_t}),$$

$$(g_A^*\circ h)_f(x_{i_1}\cdots x_{i_s}\otimes y_{l_1}\cdots y_{l_t}) = \left(\prod_{u=1}^{s}\prod_{k=1}^{c_{i_u}} z_{i_uj_uk}\right)\otimes y_{l_1}\cdots y_{l_t}.$$

Since $x_{i_1}\cdots x_{i_s}\otimes y_{l_1}\cdots y_{l_t}$ and $\left(\prod_{u=1}^{s}\prod_{k=1}^{c_{i_u}} z_{i_uj_uk}\right)\otimes y_{l_1}\cdots y_{l_t}$ are generators, $(g_A\circ h)_f^*$ is the inclusion of a direct summand into $H_f^q(Z_A)$. By Lemma 4.4, $(E_\infty^{0,q})_f$ and $H_f^q(X_A)$ are free modules of same rank, so $h_f$ is an isomorphism. Since $e_f$ is a surjection, so is $\phi_f^*$.

For the second part of the lemma, suppose there is a polynomial $p\in\ker(\phi_f^*)$ not contained in $(x_{ij}-x_{i1})_{(i,j)\in\overline{\Omega}}$. Since $\phi_f^*$ is a degree 0 morphism, we assume $p=\sum_\alpha p_\alpha$ is a sum of monomials $p_\alpha$ of some fixed degree $q$. Then the $p_\alpha$'s are linearly dependent. So the rank of $H_f^q(X_A)$ is less than the rank of $A^q$, contradicting to Lemma 4.4. Thus $\ker(\phi_f^*)=(x_{ij}-x_{i1})_{(i,j)\in\overline{\Omega}}$. $\square$

Next we restate our main theorem (Theorem 1.2) and prove it.

**Theorem 5.5** *Let $A$ be a graded monomial ideal ring. Then there exists a space $X_A$ such that $H_f^*(X_A)$ is ring isomorphic to $A$. Moreover, $H^*(X_A)$ is free split.*

**Proof** For the first part of the statement, the ring isomorphism $H_f^*(X_A) \cong A$ follows from Lemma 5.4.

In Lemma 4.1 we construct a free splitting $s_K\colon H_f^*(\underline{X}+\underline{Y},*)^K \to H^*(\underline{X}+\underline{Y},*)^K$ out of free splittings $s_{ij}\colon H_f^*(X_{ij}) \to H^*(X_{ij})$ and the identity maps on $H^*(Y_k)$. Define a map $s\colon H_f^*(X_A) \to H^*(X_A)$ by

$$\begin{array}{ccc} H_f^*((\underline{X}+\underline{Y},*)^K) & \xrightarrow{s_K} & H^*((\underline{X}+\underline{Y},*)^K) \\ \Big\downarrow \phi_f^* & & \Big\downarrow \phi^* \\ H_f^*(X_A) & \xrightarrow{\qquad s \qquad} & H^*(X_A) \end{array}$$

We need to show that $s$ is well defined. By Lemma 5.4, $\phi_f^*$ is a surjection and $\ker(\phi_f^*)$ is generated by polynomials $x_{ij}-x_{i1}$ for $(i,j)\in\overline{\Omega}$. It suffices to show that $\phi^*\circ s_K(x_{ij}-x_{i1})=0$. Let $\tilde{x}_{ij}\in H^{2c_i}(X_{ij})$ and $\tilde{x}'_{ij}\in H_f^{2c_i}(X_{ij})$ be generators such that $s_{ij}(\tilde{x}'_{ij})=\tilde{x}_{ij}$. There is a string of equations

$$\begin{aligned} \phi^*\circ s_K(x_{ij}-x_{i1}) &= \phi^*\circ\mu(s_{ij}(\tilde{x}'_{ij})-s_{i1}(\tilde{x}'_{i1})) \\ &= \phi^*\circ\mu(\tilde{x}_{ij}-\tilde{x}_{i1}) \\ &= \phi^*\circ\delta^*\circ\mu(1\otimes\cdots\otimes\tilde{x}_{ij}\otimes\cdots\otimes 1) \\ &= 0, \end{aligned}$$

where the first line is due to the definition of $s_K$, the third line is due to the naturality of $\mu$, and the last line is due to the fact that $\delta$ and $\phi$ are two consecutive maps in the fibration sequence $X_A \xrightarrow{\phi} (\underline{X}+\underline{Y},*)^K \xrightarrow{\delta} \prod_{(i,j)\in\overline{\Omega}} X_{ij}$. So $s$ is well defined.

Obviously $s$ is right inverse to the quotient map $H^*(X_A)\to H_f^*(X_A)$. Since $\phi_f^*$, $\phi^*$ and $s_K$ are multiplicative, so is $s$. Therefore $s$ is a free splitting. □

# 6 An example

Now we illustrate how to construct $X_A$ for $A=\mathbb{Z}[x]\otimes\Lambda[y]/(x^2y)$, where $|x|=4$ and $|y|=1$. First, polarize $A$ by introducing two new variables $x_1$ and $x_2$ of degree 4 and let

$$A' = \mathbb{Z}[x_1, x_2] \otimes \Lambda[y]/(x_1x_2y).$$

Let $K$ be the boundary of a 2–simplex. Then $(x_1x_2y)$ is the Stanley–Reisner ideal of $K$. Take

$$\underline{X} = \{K(\mathbb{Z}, 4), K(\mathbb{Z}, 4)\}, \quad \underline{Y} = \{S^1\}$$

and construct polyhedral product $(\underline{X} + \underline{Y}, *)^K$. By Proposition 3.3,

$$H_f^*((\underline{X} + \underline{Y}, *)^K) \cong \mathbb{Z}[x_1, x_2] \otimes \Lambda[y]/(x_1x_2y).$$

Define $\delta\colon (\underline{X} + \underline{Y}, *)^K \to K(\mathbb{Z}, 4)$ by $\delta_1(u_1, u_2, t) = u_2 \cdot u_1^{-1}$, and define $X_A$ to be the fiber of $\delta$. By Theorem 5.5, $H_f^*(X_A) \cong A$.

Next, we construct $Z_A$ and $g_A$ to illustrate the proof of the extension problem. In this case, take $\underline{Z} = \{(\mathbb{CP}^\infty)^2, (\mathbb{CP}^\infty)^2\}$. Denote the first $(\mathbb{CP}^\infty)^2$ by $Z_1$ and the second $(\mathbb{CP}^\infty)^2$ by $Z_2$. Then $H^*(Z_1) = \mathbb{Z}[z_{11}, z_{12}]$ and $H^*(Z_2) = \mathbb{Z}[z_{21}, z_{22}]$, where $|z_{ij}| = 2$ for $i, j \in \{1, 2\}$, and

$$H^*((\underline{Z} + \underline{Y}, *)^K) \cong \mathbb{Z}[z_{11}, z_{12}, z_{21}, z_{22}] \otimes \Lambda[y]/L',$$

where $L' = (z_{11}z_{21}y, z_{11}z_{22}y, z_{12}z_{21}y, z_{12}z_{22}y)$. Define

$$\tilde{\delta}\colon (\underline{Z} + \underline{Y}, *)^K \to (\mathbb{CP}^\infty)^2, \quad \tilde{\delta}(v_1, v_2, t) = v_2 \cdot v_1^{-1},$$

and define $Z_A$ to be the fiber of $\tilde{\delta}$. Then $H_f^*(Z_A) \cong \mathbb{Z}[z_1, z_2] \otimes \Lambda[y]/L$, where $|z_1| = |z_2| = 2$ and $L = (z_1^2y, z_2^2y, z_1z_2y)$.

For $i = \{1, 2\}$, let $g_i : Z_i \to K(\mathbb{Z}, 4)$ be a map representing $z_{i1}z_{i2} \in H^4(Z_i)$, and let $h\colon S^1 \to S^1$ be the identity map. Then $g_1$, $g_2$ and $h$ induce

$$g_K\colon (\underline{Z} + \underline{Y}, *)^K \to (\underline{X} + \underline{Y}, *)^K$$

such that $g_K^*(x_i) = z_{i1}z_{i2}$ and $g_K^*(y) = y$. Lemma 5.3 gives a map $g_A\colon Z_A \to X_A$ making the diagram

$$\begin{tikzcd}
Z_A \arrow[r] \arrow[d, "g_A"] & (\underline{Z} + \underline{Y}, *)^K \arrow[d, "g_K"] \\
X_A \arrow[r] & (\underline{X} + \underline{Y}, *)^K
\end{tikzcd}$$

commute.

## Acknowledgements

This work was done during So's PIMS postdoctoral fellowship. Both authors gratefully acknowledge the support of the Fields Institute and NSERC. So also thanks Hector Durham for looking for references.

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Received: 14 October 2020 Revised: 22 September 2021

**ALGEBRAIC & GEOMETRIC TOPOLOGY**

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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