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Algebraic & Geometric Topology

Realization of graded monomial ideal rings modulo torsion

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Let A be the quotient of a graded polynomial ring $\mathbb{Z}[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_n]$ by an ideal generated by monomials with leading coefficients 1. We construct a space X_A such that A is isomorphic to $H^*(X_A)$ modulo torsion elements.

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1 Introduction

A classical problem in algebraic topology asks: which commutative graded R–algebras A are isomorphic to $H^*(X_A; R)$ for some space X_A? The space X_A, if it exists, is called a realization of A. According to Aguadé [1] the problem goes back to at least Hopf, and was later explicitly stated by Steenrod [14]. To solve the problem in general is probably too ambitious, but many special cases have been proven.

One of Quillen's motivations for his seminal work on rational homotopy theory [13] was to solve this problem over $\mathbb{Q}$. He showed that all simply connected graded $\mathbb{Q}$–algebras have a realization. The problem of which polynomial algebras over $\mathbb{Z}$ have realizations has a long history, and a complete solution was given by Anderson and Grodal [2]; see also Notbohm [12]. More recently Trevisan [15] and later Bahri, Bendersky, Cohen and Gitler [4] constructed realizations of $\mathbb{Z}[x_1, \ldots, x_m]/I$, where $|x_i| = 2$ and I is an ideal generated by monomials with leading coefficient 1.

We want to consider a related problem that lies between the solved realization problem over $\mathbb{Q}$ and the very difficult realization problem over $\mathbb{Z}$. We do this by modding out torsion.

Problem 1.1 *Which commutative graded R–algebras A are isomorphic to*

$$H^*(X_A; R)/\text{torsion}$$

for some space X_A?

Such an X_A is called a realization modulo torsion of A. For example, a polynomial ring $\mathbb{Z}[x]$ has a realization modulo torsion given by the Eilenberg–Mac Lane space $K(\mathbb{Z}, |x|)$ if $|x|$ is even, while $\mathbb{Z}[x]$ has a realization (before modding out torsion) if and only if $|x| = 2$ or 4 [14]. Here we ask: do all finite type connected commutative graded $\mathbb{Z}$–algebras have a realization modulo torsion?

Notice that modding out by torsion is different from taking rational coefficients. For example, both $H^*(\Omega S^{2n+1}; \mathbb{Q})$ and $H^*(K(\mathbb{Z}, 2n); \mathbb{Q})$ are $\mathbb{Q}[x]$ generated by x of degree $2n$. But $H^*(K(\mathbb{Z}, 2n))/\text{torsion}$ is $\mathbb{Z}[x]$, while $H^*(\Omega S^{2n+1}) \cong \Gamma[x]$ is free as a $\mathbb{Z}$–module and is the divided polynomial algebra generated by x.

In this paper, we construct realizations modulo torsion of graded monomial ideal rings A which are tensors of polynomial algebras and exterior algebras modulo monomial ideals. More precisely, let $P = \mathbb{Z}[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_n]$ be a graded polynomial ring where the x_i's have arbitrary positive even degrees and the y_j's have arbitrary positive odd degrees, and let $I = (M_1, \ldots, M_r)$ be an ideal generated by r minimal monomials

$$M_j = x_1^{a_{1j}} x_2^{a_{2j}} \cdots x_m^{a_{mj}} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}}, \quad 1 \le j \le r,$$

where the indices a_{ij} are nonnegative integers and b_{ij} are either 0 or 1. Then the quotient algebra $A = P/I$ is called a *graded monomial ideal ring*.

Theorem 1.2 (main theorem) *Let A be a graded monomial ideal ring. Then there exists a space X_A such that $H^*(X_A)/T$ is isomorphic to A, where T is the ideal consisting of torsion elements in $H^*(X_A)$. Moreover, there is a ring morphism $A \to H^*(X_A)$ that is right inverse to the quotient map $H^*(X_A) \to H^*(X_A)/T \cong A$.*

If all of the even degree generators are in degree 2, then we do not need to mod out by torsion and so we get a generalization (Theorem 4.6) of the results of Bahri, Bendersky, Cohen and Gitler [4, Theorem 2.2] and Trevisan [15, Theorem 3.6].

The structure of the paper is as follows. Section 2 contains preliminaries, algebraic tools and lemmas that are used in later sections. In Section 3 we recall the definition of polyhedral products and modify a result of Bahri, Bendersky, Cohen and Gitler [3] to compute $H^*((\underline{X}, *)^K)/T$. In Sections 4 and 5 we prove Theorem 1.2 in several steps. First, we prove it in the special case where the ideal I is square-free. Then for the general case, we construct a fibration sequence inspired by algebraic polarization method and show that the fiber X_A is a realization modulo torsion of A. In Section 6 we illustrate how to construct X_A for an easy example of A.

2 Preliminaries

2.1 Quotients of algebras by torsion elements

It is natural to study an algebra A by factoring out the torsion elements since the quotient algebra is torsion-free and has a simpler structure. Driven by this, we start investigating the quotients of cohomology rings of spaces by their torsion elements. Since we cannot find related references in the literature, here we fix the notation and develop lemmas for our purpose.

A graded module $A = \{A_i\}_{i \in S}$ is a family of indexed modules A_i. Since we are interested in cochain complexes and cohomology rings of connected, finite type CW–complexes, we assume A to be a connected, finite type graded module with nonpositive degrees. That is, $S = \mathbb{N}_{\leq 0}$, $A_0 = \mathbb{Z}$ and each component A_i is finitely generated. We follow the convention and denote A_i by A^{-i}.

Remark 2.1 Equivalently we can define a graded module to be a module with a grading structure, that is the direct sum $A = \bigoplus_{i \in S} A_i$ of a family of indexed modules. This definition is slightly different from the definition above. We will use both definitions interchangeably.

An element $x \in A$ is *torsion* if $cx = 0$ for some nonzero integer c, and is *torsion-free* otherwise. The torsion submodule A_t of A is the graded submodule consisting of torsion elements and the torsion-free quotient module $A_f = A/A_t$ is their quotient. If B is another graded module and $g \colon A \to B$ is a morphism, then it induces a morphism $g_f \colon A_f \to B_f$ sending $a + A_t \in A_f$ to $g(a) + B_t \in B_f$. This kind of structure is important in abelian categories and was formalized with Dixon's notion of a torsion theory [6], but in this paper we only use the structure in a naive way.

Lemma 2.2 *If $0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0$ is a short exact sequence of graded modules, then $C_f \cong (B_f/A_f)_f$. Furthermore, if the sequence is split exact, then so is*

$$0 \to A_f \xrightarrow{g_f} B_f \xrightarrow{h_f} C_f \to 0.$$

Proof Consider a commutative diagram as in Figure 1, where g_t is the restriction of g to A_t, p and q are the quotient maps, and u and v are the induced maps. By construction all rows and columns are exact sequences except for the right column. A diagram chase implies that u is injective and v is surjective. We claim that the column is exact at C.

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & A_t & \xrightarrow{g_t} & B_t & \xrightarrow{p} & B_t/A_t & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow u & \\
0 \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{h} & C & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow v & \\
0 \longrightarrow & A_f & \xrightarrow{g_f} & B_f & \xrightarrow{q} & B_f/A_f & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& & 0 & & 0 & & 0
\end{array}
$$

<div align="center">Figure 1</div>

Obviously $v \circ u$ is trivial. Take an element $c \in \ker(v)$ and its preimage $b \in B$. A diagram chase implies $b = g(a) + b'$ for some $a \in A$ and $b' \in B_t$. So $c = h(b') = u \circ p(b')$ is in $\mathrm{Im}(u)$ and the right column $0 \to B_t/A_t \xrightarrow{u} C \xrightarrow{v} B_f/A_f \to 0$ is exact.

For the first part of the lemma, we show that $v_f : C_f \to (B_f/A_f)_f$ is an isomorphism. Since v is surjective, so is v_f. Take $c' \in \ker(v_f)$ and its preimage $\tilde{c}' \in C$. Then $v(\tilde{c}')$ is a torsion element in B_f/A_f and $m v(\tilde{c}') = 0$ for some nonzero integer m. So $m\tilde{c}' \in \ker(v)$. As $\ker(v) = \mathrm{Im}(u)$ consists of torsion elements, $m\tilde{c}'$ is torsion and so is $\tilde{c}'$. Therefore $c' = 0$ in C_f and v_f is injective.

Notice that an exact sequence being split is equivalent to $B \cong A \oplus C$. So $B_f \cong A_f \oplus C_f$ and $0 \to A_f \xrightarrow{g_f} B_f \xrightarrow{h_f} C_f \to 0$ is a split exact sequence. $\hfill\square$

A graded algebra (A, m) consists of a graded module A and an associative bilinear multiplication $m = \{m^{i,j} : A^i \otimes A^j \to A^{i+j}\}$ such that $1 \in A^0$ is the multiplicative identity. A pair (M, μ) is a left (resp. right) A–module if M is a graded module and μ is an associative bilinear multiplication $\mu = \{\mu^{i,j} : A^i \otimes M^j \to M^{i+j}\}$ such that $\mu(1 \otimes x) = x$ (resp. $\mu = \{\mu^{i,j} : M^i \otimes A^j \to M^{i+j}\}$ such that $\mu(1, x) = x$) for all $x \in M$. We check that modding out torsion and multiplications are compatible.

Lemma 2.3 *If A and M are graded modules (not necessarily of finite type), then there is a unique isomorphism $\theta : (A \otimes M)_f \to A_f \otimes M_f$ of graded modules making the diagram*

$$
\begin{array}{ccc}
A \otimes M & \longrightarrow & A_f \otimes M_f \\
\downarrow & \nearrow{\scriptstyle\theta} & \\
(A \otimes M)_f & &
\end{array}
$$

commute, where the vertical and the horizontal maps are quotient maps.

Proof It suffices to show that $(A^i \otimes M^j)_f \cong A^i_f \otimes M^j_f$ for any positive integers i and j. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^i_t \otimes M^j \oplus A^i \otimes M^j_t & \xrightarrow{\iota_1} & A^i \otimes M^j & \xrightarrow{\pi_1} & A^i_f \otimes M^j_f & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \| & & \downarrow{\scriptstyle b} & & \\
0 & \longrightarrow & (A^i \otimes M^j)_t & \xrightarrow{\iota_2} & A^i \otimes M^j & \xrightarrow{\pi_2} & (A^i \otimes M^j)_f & \longrightarrow & 0
\end{array}
$$

where a, ι_1 and ι_2 are inclusions, π_1 and π_2 are quotient maps, and b is the induced map. We want to show that b is an isomorphism, which is equivalent to showing that a is an isomorphism. If A and M are of finite type, then a is an isomorphism since A^i and M^j are finitely generated abelian groups. In the general case, a is an isomorphism by [9, Theorem 61.5]. $\qquad\square$

Corollary 2.4 *Let (A, m) be a graded algebra and let m'_f be the composition*

$$m'_f : A_f \otimes A_f \cong (A \otimes A)_f \xrightarrow{m_f} A_f.$$

Then (A_f, m'_f) is a graded algebra and there is a commutative diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{m} & A \\
\downarrow & & \downarrow \\
A_f \otimes A_f & \xrightarrow{m'_f} & A_f
\end{array}
$$

where the vertical maps are quotient maps.

Let (M, μ) be a left or right A–module and let μ'_f be the composition

$$\mu'_f : A_f \otimes M_f \cong (A \otimes M)_f \xrightarrow{\mu_f} M_f \quad \text{or} \quad \mu'_f : M_f \otimes A_f \cong (M \otimes A)_f \xrightarrow{\mu_f} M_f),$$

respectively. Then (M_f, μ'_f) is respectively a left or right A_f–module and there is a commutative diagram

$$
\begin{array}{ccccc}
A \otimes M & \xrightarrow{\mu} & M & & \\
\downarrow & & \downarrow & \text{or} & \\
A_f \otimes M_f & \xrightarrow{\mu'_f} & M_f & &
\end{array}
\qquad
\begin{array}{ccc}
M \otimes A & \xrightarrow{\mu} & M \\
\downarrow & & \downarrow \\
M_f \otimes A_f & \xrightarrow{\mu'_f} & M_f
\end{array}
$$

respectively, where the vertical maps are quotient maps.

A *cochain complex* (A, d) consists of a graded module A and a differential

$$d = \{d^i : A^i \to A^{i+1}\}$$

such that $d \circ d = 0$. Let $d_f = \{d_f^i : A_f^i \to A_f^{i+1}\}$ be the induced differential on A_f. Then (A_f, d_f) forms a cochain complex and its cohomology

$$H^*(A_f, d_f) = \{H^i(A_f, d_f)\}_{i \geq 0}$$

is a graded module.

A *differential graded algebra* (A, m, d) is a cochain complex (A, d) such that (A, m) is a graded algebra and d and m satisfy the Leibniz rule. Let d_t be the restriction of d to A_t. Then (A_t, d_t) is a differential ideal and (A_f, d_f) is a differential graded algebra, so $H^*(A_f, d_f)$ is a graded algebra.

A left (resp. right) *dg-algebra module* (M, μ, δ) over (A, m, d) if (M, μ) is a left (resp. right) (A, m)–module, (M, δ) is a cochain complex and δ and μ satisfy the Leibniz rule. Then $H^*(M_f, \delta_f)$ is a left (resp. right) $H^*(A_f)$–module.

Even if (A_f, d_f) is torsion-free, $H^*(A_f, d_f)$ is not necessarily torsion-free. Denote $(H^*(A, d))_f$ by $H_f^*(A, d)$. The following lemma compares $H_f^*(A, d)$ and $H_f^*(A_f, d_f)$.

Lemma 2.5 *Let (A, d) be a cochain complex. Then there is a monomorphism of modules*

$$\psi : H_f^*(A, d) \to H_f^*(A_f, d_f).$$

If $H^{i+1}(A_t, d_t) = 0$, then $\psi : H_f^i(A, d) \to H_f^i(A_f, d_f)$ is an isomorphism. Moreover, suppose (A, m, d) is a differential graded algebra. Then ψ is a morphism of algebras.

Proof Assume (A, d) is a cochain complex. Let $\iota : (A_t, d_t) \to (A, d)$ be the inclusion and let $\pi : (A, d) \to (A_f, d_f)$ be the quotient map. Then the short exact sequence of cochain complexes $0 \to (A_t, d_t) \xrightarrow{\iota} (A, d) \xrightarrow{\pi} (A_f, d_f) \to 0$ induces a long exact sequence

$$\cdots \to H^{i-1}(A_f, d_f) \to H^i(A_t, d_t) \xrightarrow{\iota^*} H^i(A, d) \xrightarrow{\pi^*} H^i(A_f, d_f) \to H^{i+1}(A_t, d_t) \to \cdots .$$

Take $\psi : H_f^*(A, d) \to H_f^*(A_f, d_f)$ to be the morphism induced by

$$\pi^* : H^*(A, d) \to H^*(A_f, d_f).$$

We show that it has the asserted properties.

To show the injectivity of ψ, take an equivalence class $[a] \in H_f^*(A, d)$ such that $\psi[a] = 0$. Represent it by a cocycle class $a \in H^i(A, d)$. Then $\pi^*(a)$ is torsion and $\pi^*(ca) = 0$ for some nonzero number c. By exactness, $ca \in \text{Im}(\iota^*)$. Since $H^i(A_t, d_t)$ is torsion, so is $\text{Im}(\iota^*)$ and ca is a torsion. Therefore $a \in H^i(A, d)$ is a torsion. By definition, $[a] \in H_f^i(A, d)$ is zero. So ψ is injective.

Suppose A^{i+1} has no torsion elements. Then $A_t^{i+1} = 0$ and $H^{i+1}(A_t, d_t) = 0$. So π^* is surjective. By definition we have commutative diagram

$$
\begin{array}{ccc}
H^i(A, d) & \xrightarrow{\pi^*} & H^i(A_f, d_f) \\
\downarrow & & \downarrow \\
H_f^i(A, d) & \xrightarrow{\psi} & H_f^i(A_f, d_f)
\end{array}
$$

where vertical arrows are quotient maps and are surjective. So

$$
\psi : H_f^i(A, d) \to H_f^i(A_f, d_f)
$$

is surjective and hence isomorphic.

If A is a differential graded algebra, then $\pi^* : H^*(A, d) \to H^*(A_f, d_f)$ is a morphism of graded algebras. By Corollary 2.4, the induced morphism ψ is multiplicative. $\quad\square$

Example The surjectivity of $\psi : H_f^i(A, d) \to H_f^i(A_f, d_f)$ may fail if A^{i+1} contains torsion elements. Let (A, d) be a cochain complex where

$$
A^i = \begin{cases}
\mathbb{Z} & \text{if } i = 0, \\
\mathbb{Z}/2\mathbb{Z} & \text{if } i = 1, \\
0 & \text{otherwise,}
\end{cases}
$$

and d^i are trivial for all i except for $d^0 \colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ being the quotient map. Then $H^0(A)$ and $H^0(A_f)$ are $\mathbb{Z}$ while $\psi \colon H^0(A) \to H_f^0(A)$ is multiplication $2 \colon \mathbb{Z} \to \mathbb{Z}$.

2.2 Eilenberg–Moore spectral sequence

Given a differential graded algebra (A, d) and a right A–module (M, d_M), we first define the bar bicomplex $B^{*,*}(M, A)$ as follows. For any positive integer i, let $B^{-i}(M, A) = M \otimes (\bar{A})^{\otimes i}$ where $\bar{A} = \{A^n\}_{n>0}$. Denote an element in $B^{-i}(M, A)$ by $x[a_1|\cdots|a_i]$ for $x \in M$ and $a_i \in \bar{A}$. Let $B^{-i,j}(M, A)$ be the submodule of $B^{-i}(M, A)$

consisting elements $x[a_1|\cdots|a_i]$ such that $|x| + \sum_{k=1}^{i} |a_k| = j$. The internal and external differentials

$$d_I : \mathrm{B}^{-i,j}(M, A) \to \mathrm{B}^{-i,j+1}(M, A) \quad \text{and} \quad d_E : \mathrm{B}^{-i,j}(M, A) \to \mathrm{B}^{-i+1,j}(M, A)$$

are given by

$$d_I(x[a_1|\cdots|a_i]) = (d_M x)[a_1|\cdots|a_i] + \sum_{j=1}^{i} (-1)^{\epsilon_{j-1}} x[a_1|\cdots|a_{j-1}|d_A a_j|a_{j+1}|\cdots|a_i],$$

$$d_E(x[a_1|\cdots|a_i]) = (-1)^{|x|}(xa_1)[a_2|\cdots|a_i] + \sum_{j=1}^{i-1} (-1)^{\epsilon_j} x[a_1|\cdots|a_{j-1}|a_j \cdot a_{j+1}|\cdots|a_i],$$

where $\epsilon_k = k + |x| + \sum_{j=1}^{k} |a_j|$. Then we define the bar construction $(\mathcal{B}(M, A), d_{\mathcal{B}})$ to be a graded module where

$$\mathcal{B}(M, A)^n = \bigoplus_{-i+j=n} \mathrm{B}^{-i,j}(M, A) \quad \text{and} \quad d_{\mathcal{B}} = \bigoplus_{-i+j=n} (d_I + d_E)$$

for $n \geq 0$.

Take the filtration $\mathcal{F}^{-p} = \bigoplus_{0 \leq i \leq p} \mathrm{B}^{-i}(M, A)$. The associated spectral sequence $\{E_r^{*,*}\}_{r=0}^{\infty}$ is the Eilenberg–Moore spectral sequence converging to $H^*(\mathcal{B}(M, A))$; see [7, Remark 2.3] and [11, Corollary 7.9].

Lemma 2.6 *Let A be a simply connected differential graded algebra and M be a right A–module such that A and M are free as $\mathbb{Z}$–modules. Then there is a monomorphism of modules*

$$\psi : (E_2^{-p,q})_f \to \left(\mathrm{Tor}_{H_f(A)}^{-p,q}(H_f(M), \mathbb{Z}) \right)_f$$

which is an isomorphism for $p = 0$. Moreover, if $H(A)$ and $H(M)$ are free modules, then $E_2^{-p,q} \cong \mathrm{Tor}_{H(A)}^{-p,q}(H(M), \mathbb{Z})$.

Proof The E_0–page is given by

$$E_0^{-p,*} = \mathcal{F}^{-p}/\mathcal{F}^{-p+1} = M \otimes (\bar{A}^{\otimes p})$$

and $d_0 = d_I$. By the Künneth theorem, the E_1–page is given by

$$E_1^{-p,*} \cong H(M) \otimes (\tilde{H}(A)^{\otimes p}) \oplus T \cong \mathrm{B}^{-p}(H(M), H(A)) \oplus T,$$

where T is a torsion term and d_1 is induced by d_E. Denote $H(M)$ by M' and $H(A)$ by A' for short. By Lemma 2.3, there is an isomorphism of graded modules

$$\theta : (E_1^{-p,*})_f \cong (\mathrm{B}^{-p}(M', A'))_f \to \mathrm{B}^{-p}(M'_f, A'_f)$$

such that

$$
\begin{array}{ccc}
\mathrm{B}^{-p}(M', A') & & \\
\downarrow & \searrow & \\
(\mathrm{B}^{-p}(M', A'))_f & \xrightarrow{\ \theta\ } & \mathrm{B}^{-p}(M'_f, A'_f)
\end{array}
$$

where the downward maps are quotient maps. Let d' be the external differential of $\mathrm{B}^*(M'_f, A'_f)$. Then $\theta\colon ((\mathrm{B}^{-p}(M', A'))_f, (d_1)_f) \to (\mathrm{B}^*(M'_f, A'_f), d')$ is an isomorphism of cochain complexes. By Lemma 2.5, there is a monomorphism of graded modules

$$
\psi\colon (E_2^{-p,q})_f = H_f^{-p}(E_1^{*,q}, d_1) \to H_f^{-p}((\mathrm{B}^{*,q}(M', A'))_f, (d_1)_f) \cong H_f^{-p}(\mathrm{B}^{*,q}(M'_f, A'_f), d').
$$

Notice that $\mathrm{B}^*(M'_f, A'_f) \cong M'_f \otimes_{A'_f} \mathrm{B}^*(A'_f, A'_f)$ and $d' = \mathbb{1} \otimes_{A'_f} d''$, where d'' is the external differential of $\mathrm{B}^*(A'_f, A'_f)$. Since, by [11, Proposition 7.8],

$$
\cdots \to \mathrm{B}^{-1}(A'_f, A'_f) \xrightarrow{d''} \mathrm{B}^0(A'_f, A'_f) \xrightarrow{\epsilon} \mathbb{Z} \to 0
$$

is a projective resolution of $\mathbb{Z}$ over A'_f–modules where $\epsilon\colon \mathrm{B}^0(A'_f, A'_f) \cong A'_f \to \mathbb{Z}$ is the augmentation, the monomorphism becomes

$$
\psi\colon (E_2^{-p,q})_f \to (\mathrm{Tor}_{A'_f}^{-p,q}(M'_f, \mathbb{Z}))_f.
$$

Since $\mathrm{B}^1(M', A') = 0$, ψ is isomorphic for $p = 0$ by Lemma 2.5.

Suppose $H(A)$ and $H(M)$ are free $\mathbb{Z}$–modules. By the Künneth theorem,

$$
E_1^{*,*} \cong \mathrm{B}^{*,*}(H(M), H(A))
$$

and d_1 is the external differential. So $E_2^{-p,q} \cong \mathrm{Tor}_{H(A)}^{-p,q}(H(M), \mathbb{Z})$. □

Let $F \to E \xrightarrow{\pi} X$ be a fibration sequence where all spaces are connected, finite type CW–complexes, and X is simply connected. In [7, Theorem III] there is a quasi-isomorphism

$$
\Theta\colon \Omega(C_*^\pi(E), C_*(X)) \to CN_*(F)
$$

of dg-algebra modules, which is natural in π. Here $\Omega(-, -)$ is the cobar construction, $C_*^\pi(E)$ is a nonnegative chain complex, $C_*(X)$ is a simply connected chain complex, $CN_*(F)$ is a chain complex, and $C_*^\pi(E)$, $C_*(X)$ and $CN_*(F)$ are quasi-isomorphic to the singular chain complexes of E, X and F, respectively.

Denote the dual of a (co)chain complex C by $C^\vee = \mathrm{Hom}(C, \mathbb{Z})$. Since X is simply connected, $H^1(X) = 0$ and $H^2(X)$ is free. By [7, Propositions 4.2 and 4.6] there are

finite type graded free modules $V = \{V^i\}_{i \geq 2}$ and $W = \{W^j\}_{j \geq 0}$, a quasi-isomorphism of dg-algebras

$$\phi: T(V) \to (C_*(X))^\vee$$

and a quasi-isomorphism of dg-algebra modules

$$\varphi: T(V) \otimes W \to (C_*^\pi(E))^\vee,$$

where $T(V)$ is the tensor algebra on V. Write $\widetilde{X} = T(V)$ and $\widetilde{E} = T(V) \otimes W$ for short. Then the compositions

$$C_*(X) \xrightarrow{\text{incl}} (C^*(X))^\vee \xrightarrow{\phi^\vee} \widetilde{X}^\vee \quad \text{and} \quad C_*^\pi(E) \xrightarrow{\text{incl}} (C^*(E))^\vee \xrightarrow{\varphi^\vee} \widetilde{E}^\vee$$

are quasi-isomorphisms of dg-coalgebras and of dg-coalgebra modules. Since $C_*(X)$ and $\widetilde{X}^\vee$ are simply connected free chain complexes, and $C_*^\pi(E)$ and $\widetilde{E}^\vee$ are nonnegative chain complexes, we have a zig-zag of quasi-isomorphisms

$$\Omega(\widetilde{E}^\vee, \widetilde{X}^\vee) \xleftarrow{\cong} \Omega(C_*^\pi(E), C_*(X)) \xrightarrow{\Theta} CN_*(F).$$

Since $\widetilde{E}$ and $\widetilde{X}$ are of finite type, dualize the zig-zag and take cohomology to get an isomorphism

$$H^*(\mathcal{B}(\widetilde{E}, \widetilde{X})) \xrightarrow{\cong} H^*(F).$$

The Eilenberg–Moore spectral sequence $\{E_r^{*,*}\}_{r=0}^\infty$ on $F \to E \xrightarrow{\pi} X$ is the Eilenberg–Moore spectral sequence given by $A = \widetilde{X}$ and $M = \widetilde{E}$. Note that this definition depends on the choice of the pair $(\widetilde{X}, \widetilde{E}, \phi, \varphi)$. Any two choices may give spectral sequences with different E_0–pages, but their E_r–pages are isomorphic for $r \geq 1$.

Lemma 2.7 Let $F \to E \xrightarrow{\pi} X$ be a fibration sequence such that all spaces are finite type spaces and X is simply connected, and let $\{E_2^{-p,q}\}$ be the E_2–page of Eilenberg–Moore spectral sequence on this fibration. Then there is a monomorphism

$$\psi: (E_2^{-p,q})_f \to \left(\text{Tor}_{H_f^*(X)}^{-p,q}(H_f^*(E), \mathbb{Z})\right)_f$$

as modules such that ψ is an isomorphism for $p = 0$.

Proof Since $H(\widetilde{E}) \cong H^*(E)$ and $H(\widetilde{X}) \cong H^*(X)$, Lemma 2.6 implies that there is a monomorphism $\psi: (E_2^{-p,q})_f \to \left(\text{Tor}_{H_f^*(X)}^{-p,q}(H_f^*(E), \mathbb{Z})\right)_f$ such that ψ is an isomorphism at $p = 0$.   □

Recall that the E_0–page is given by $E_0^{p,*} = \mathcal{F}^{-p}/\mathcal{F}^{-p+1} \cong \widetilde{E} \otimes (\overline{\widetilde{X}})^{\otimes p}$. In particular, if $p = 0$, then $E_0^{0,*} \cong \widetilde{E}$. On the other hand, $\{E_r^{*,*}\}_{r=0}^\infty$ is a second quadrant spectral sequence. So $E_r^{0,*}$ is the kernel of the differential map and $E_{r+1}^{0,*}$ is a quotient group of $E_r^{0,*}$. For $r \in \mathbb{N} \cup \{\infty\}$, define the edge homomorphism e_r to be the composition

$$e_r \colon H(E) \cong H(\widetilde{E}) \cong E_1^{0,*} \to E_r^{0,*}$$

where the unnamed arrow is the quotient map. The following lemma tells how the edge homomorphisms relate the E_r–page to $H^*(E)$ and $H^*(F)$.

Lemma 2.8 *Under the hypotheses of [Lemma 2.7](), the edge homomorphisms make the diagram*

$$
\begin{array}{ccccccc}
H^*(E) & = & H^*(E) & = \cdots \longrightarrow & H^*(E) & = & H^*(E) \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_\infty} & & \downarrow{\scriptstyle \iota^*} \\
E_1^{0,*} & \xrightarrow{J_1} & E_2^{0,*} & \xrightarrow{J_2} \cdots \longrightarrow & E_\infty^{0,*} & \xrightarrow{J} & H^*(F)
\end{array}
$$

commute, where ι^* is induced by $\iota \colon F \to E$, J is the inclusion and the J_r's are the quotient maps.

Proof We use the notation above. Consider the commutative diagram

$$
\begin{array}{ccccc}
F & \xrightarrow{\iota} & E & \xrightarrow{\pi} & X \\
\downarrow{\scriptstyle \iota} & & \| & & \downarrow \\
E & = & E & \xrightarrow{c} & \mathrm{pt}
\end{array}
$$

where c is the constant map. We have

$$
\begin{array}{ccc}
\Omega(C_*^\pi(E), C_*(X)) & \xrightarrow{\Theta} & CN_*(F) \\
\downarrow & & \downarrow{\scriptstyle \iota_*} \\
\Omega(C_*^c(E), C_*(\mathrm{pt})) & \xrightarrow{\Theta} & CN_*(E)
\end{array}
$$

since the quasi-isomorphism Θ is natural. The supplement $\mathbb{Z} \to (C_*(\mathrm{pt}))^\vee$ is a quasi-isomorphism of dg-algebras and $\varphi \colon \widetilde{E} \to (C_*^\pi(E))^\vee$ is a quasi-isomorphism of dg-algebra modules. Using this replacement and taking dual and cohomology of the diagram, we obtain

(1)
$$
\begin{array}{ccc}
H^*(\widetilde{E}) & \xrightarrow{\cong} & H^*(E) \\
\downarrow{\scriptstyle e^*} & & \downarrow{\scriptstyle \iota^*} \\
H^*(\mathcal{B}(\widetilde{E}, \widetilde{X})) & \xrightarrow{\cong} & H^*(F)
\end{array}
$$

where e^* is the composition

$$e^* \colon H^*(\widetilde{E}) \cong H^*(\mathcal{B}(\widetilde{E}, \mathbb{Z})) \xrightarrow{e'} H^*(\mathcal{B}(\widetilde{E}, \widetilde{X}))$$

and e' is induced by the inclusion $e \colon \mathrm{B}^{*,*}(\widetilde{E}, \mathbb{Z}) \to \mathrm{B}^{*,*}(\widetilde{E}, \widetilde{X})$. Let $\{\widehat{E}_r^{*,*}\}_{r=0}^{\infty}$ be the Eilenberg–Moore spectral sequence on $E \xRightarrow{} E \xrightarrow{c} \mathrm{pt}$. Then $\widehat{E}_0^{*,*} \cong \mathrm{B}^{*,*}(\widetilde{E}, \mathbb{Z})$ and the $\widehat{E}_1$–page collapses to $H^*(\widetilde{E})$. The inclusion $e \colon \mathrm{B}^{*,*}(\widetilde{E}, \mathbb{Z}) \to \mathrm{B}^{*,*}(\widetilde{E}, \widetilde{X})$ gives the commutative diagram

$$
\begin{array}{ccccccc}
H^*(\widetilde{E}) & = & H^*(\widetilde{E}) & = \cdots \longrightarrow & H^*(\widetilde{E}) & = & H^*(\widetilde{E}) \\
\downarrow{\scriptstyle \tilde{e}_1} & & \downarrow{\scriptstyle \tilde{e}_2} & & \downarrow{\scriptstyle \tilde{e}_\infty} & & \downarrow{\scriptstyle e^*} \\
E_1^{0,*} & \xrightarrow{J_1} & E_2^{0,*} & \xrightarrow{J_2} \cdots \longrightarrow & E_\infty^{0,*} & \xrightarrow{\tilde{j}} & H^*(\mathcal{B}(\widetilde{E}, \widetilde{X}))
\end{array}
$$

where $\tilde{e}_r \colon H^*(\widetilde{E}) \cong H^*(E) \xrightarrow{e_r} E_r^{0,*}$ and $\tilde{j} \colon E_\infty^{0,*} \xrightarrow{J} H^*(F) \cong H^*(\mathcal{B}(\widetilde{E}, \widetilde{X}))$. Combine this with (1) and obtain the asserted commutative diagram.   □

2.3 Regular sequences and freeness

Here we use the alternative description of graded objects. A commutative graded algebra $A = \bigoplus_{i \geq 0} A_i$ is an algebra with a grading such that $ab = (-1)^{ij} ba$ for $a \in A^i$ and $b \in A^j$, and a graded A–module $M = \bigoplus_{j \geq 0} M_j$ is the direct sum of a family of A–modules. A set $\{r_1, \ldots, r_n\}$ of elements in M is called an M–*regular sequence* if the ideal $(r_1, \ldots, r_n)M$ is not equal to M and the multiplication

$$r_i \colon M/(r_1, \ldots, r_{i-1})M \to M/(r_1, \ldots, r_{i-1})M$$

is injective for $1 \leq i \leq n$. In the special case where M is a $\mathbb{K}[x_1, \ldots, x_n]$–module for some field $\mathbb{K}$ and the grading of M has a lower bound, M is a free $\mathbb{K}[x_1, \ldots, x_n]$–module if $\{x_i\}_{i=1}^n$ is a regular sequence in M. We want to extend this fact to the case where M is a $\mathbb{Z}[x_1, \ldots, x_n]$–module. Recall a corollary of the graded Nakayama lemma.

Lemma 2.9 *Let A be a graded ring and let M be an A–module. Suppose A and M are nonnegatively graded, and $I = (r_1, \ldots, r_n) \subset A$ is an ideal generated by homogeneous elements r_i of positive degrees. If $\{m_\alpha\}_{\alpha \in S}$ is a set of homogeneous elements in M whose images generate M/IM, then $\{m_\alpha\}_{\alpha \in S}$ generates M.*

Lemma 2.10 *Let M be a $\mathbb{Z}[x_1, \ldots, x_n]$–module with nonnegative degrees. If*

$$M/(x_1, \ldots, x_n)M$$

is a free $\mathbb{Z}$–module and $\{x_1, \ldots, x_n\}$ is an M–regular sequence, then M is a free $\mathbb{Z}[x_1, \ldots, x_n]$–module.

Proof Let $I = (x_1, \ldots, x_n)$. By assumption there is a set $\{m_\alpha\}_{\alpha \in S}$ of homogeneous elements in M such that their quotient images form a basis in M/IM. By Lemma 2.9, $\{m_\alpha\}_{\alpha \in S}$ generates M. We need to show that $\{m_\alpha\}_{\alpha \in S}$ is linear independent over $\mathbb{Z}[x_1, \ldots, x_n]$.

For $0 \le i \le n$, let $M_i = M/(x_1, \ldots, x_{n-i})M$, $A_i = \mathbb{Z}[x_{n+1-i}, \ldots, x_n]$ and $m_{\alpha,i}$ be the quotient image of m_α in M_i. We prove that M_i is a free A_i–module with a basis $\{m_{\alpha,i}\}_{\alpha \in S}$ by induction on i. For $i = 0$, $M_0 = M/IM$ and $A_0 = \mathbb{Z}$. The statement is true since $\{m_{\alpha,0}\}_{\alpha \in S}$ is a basis by construction. Assume the statement holds for $i \le k$. For $i = k + 1$, if there is a collection $\{f_\alpha\}_{\alpha \in S}$ of polynomials satisfying

$$(2) \qquad \sum_{\alpha \in S} f_\alpha \cdot m_{\alpha, k+1} = 0,$$

we show that all f_α's are zero.

If not, then there are finitely many nonzero polynomials $f_{j_1}, \ldots, f_{j_r}$. Quotient M_{k+1} and A_{k+1} by the ideal (x_{n-k}) and let $\bar{f}_{j_i}$ be the image of f_{j_i} in A_k. Then (2) becomes

$$\sum_{i=1}^{r} \bar{f}_{j_i} \cdot m_{j_i, k} = 0.$$

By our inductive assumption, $\{m_{\alpha,k}\}$ is a basis in M_k. So $\bar{f}_{j_i} = 0$ and $f_{j_i} = x_{n-k} g_{j_i}$ for some polynomial $g_{j_i} \in A_{k+1}$. Since x_{n-k} is not a zero-divisor, putting $f_{j_i} = x_{n-k} g_{j_i}$ in (2) gives

$$\sum_{i=1}^{r} g_{j_i} \cdot m_{j_i, k+1} = 0.$$

So $g_{j_1}, \ldots, g_{j_r}$ are nonzero polynomials satisfying (2) and $|g_{j_i}| < |f_{j_i}|$ for $1 \le i \le r$. Iterating this argument implies that the $|f_{j_i}|$'s are arbitrarily large, but this is impossible. So the f_{j_i}'s must be zero and $\{m_{\alpha, k+1}\}$ is linearly independent. It follows that M_{k+1} is a free A_{k+1}–module. $\qquad \square$

3   Cohomology rings of polyhedral products

Let $[m] = \{1, \ldots, m\}$, K be a simplicial complex on $[m]$ and $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be a sequence of pairs of relative CW–complexes. For any simplex $\sigma \in K$, define

$$(\underline{X}, \underline{A})^\sigma = \left\{ (x_1, \ldots, x_m) \in \prod_{i=1}^m X_i \,\middle|\, x_i \in A_i \text{ for } i \notin \sigma \right\}$$

as a subspace of $\prod_{i=1}^m X_i$, and define the *polyhedral product*

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, A)^\sigma$$

to be the union of $(\underline{X}, \underline{A})^\sigma$ over $\sigma \in K$.

If $X_i = \mathbb{CP}^\infty$ and $A_i = *$ for all i, then $(\mathbb{CP}^\infty, *)^K$ is homotopy equivalent to Davis–Januszkiewicz space [5, Theorem 4.3.2]. For any principal ideal domain R, $H^*((\mathbb{CP}^\infty, *)^K; R)$ is isomorphic to the Stanley–Reisner ring $R[x_1, \ldots, x_m]/I_K$. Here I_K is the ideal generated by $x_{j_1} \cdots x_{j_k}$ for $x_{j_i} \in \tilde{H}^*(X_{j_i}; R)$ and $\{j_1, \ldots, j_k\} \notin K$, and is called the *Stanley–Reisner ideal of K*. In general, a similar formula holds for $H^*((\underline{X}, *)^K)$ whenever the X_i's are any spaces with free cohomology.

Theorem 3.1   [3]   *Let R be a principal ideal domain, K be a simplicial complex on $[m]$ and $\underline{X} = \{X_i\}_{i=1}^m$ be a sequence of CW–complexes. If $H^*(X_i; R)$ is a free R–module for all i, then*

$$H^*((\underline{X}, *)^K; R) \cong \bigotimes_{i=1}^m H^*(X_i; R)/I_K,$$

where I_K is generated by $x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $x_{j_i} \in \tilde{H}^*(X_{j_i}; R)$ and $\{j_1, \ldots, j_k\} \notin K$ and is called the generalized Stanley–Reisner ideal of K.

The proof of Theorem 3.1 uses the strong form of the Künneth theorem, which says that

$$\mu \colon \bigotimes_{i=1}^m H^*(X_i; R) \to H^*\left(\prod_{i=1}^m X_i; R \right), \quad x_1 \otimes \cdots \otimes x_m \mapsto \pi_1^*(x_1) \cup \cdots \cup \pi_m^*(x_m),$$

where π_j^* is induced by the projection $\pi_j \colon \prod_{i=1}^m X_i \to X_j$, is an isomorphism if all $H^*(X_i; R)$'s are free. In the reduced version of the Künneth theorem,

$$\bar{\mu} \colon \bigotimes_{i=1}^m \tilde{H}^*(X_i) \to \tilde{H}^*\left(\bigwedge_{i=1}^m X_i \right)$$

is also an isomorphism if all $\tilde{H}^*(X_i; R)$'s are free. The goal of this section is to modify Theorem 3.1 by removing the freeness assumption on $H^*(X_i)$. As a trade-off, we need to mod out the torsion elements of $H^*(X_i)$. First let us refine the Künneth theorem.

Lemma 3.2 *Let $\underline{X} = \{X_i\}_{i=1}^m$ be a sequence of spaces X_i. Then the induced morphisms*

$$\mu_f : \bigotimes_{i=1}^m H_f^*(X_i) \to H_f^*\left(\prod_{i=1}^m X_i\right) \quad \text{and} \quad \bar{\mu}_f : \bigotimes_{i=1}^m \tilde{H}_f^*(X_i) \to \tilde{H}_f^*\left(\bigwedge_{i=1}^m X_i\right).$$

are isomorphisms as algebras, and there is a commutative diagram

$$
\begin{array}{ccc}
\bigotimes_{i=1}^m \tilde{H}_f^*(X_i) & \xrightarrow{\bar{\mu}_f} & \tilde{H}_f^*\left(\bigwedge_{i=1}^m X_i\right) \\
\downarrow & & \downarrow{q_f^*} \\
\bigotimes_{i=1}^m H_f^*(X_i) & \xrightarrow{\mu_f} & H_f^*\left(\prod_{i=1}^m X_i\right)
\end{array}
$$

where q_f^* is induced by the quotient map $q : \prod_{i=1}^m X_i \to \bigwedge_{i=1}^m X_i$.

Proof It suffices to show the $m = 2$ case. Let (X, A) and (Y, B) be pairs of relative CW–complexes and let $\pi_X : (X \times Y, A \times Y) \to (X, A)$ and $\pi_Y : (X \times Y, X \times B) \to (Y, B)$ be projections. By the generalized version of Künneth theorem [10, Chapter XIII, Theorem 11.2], the sequence

$$0 \to \bigoplus_{i+j=n} H^i(X, A) \otimes H^j(Y, B) \xrightarrow{\mu'} H^n(X \times Y, X \times B \cup A \times Y) \to T \to 0,$$

where T is a torsion term and μ' sends $u \otimes v \in H^i(X, A) \otimes H^j(Y, B)$ to $\pi_X^*(u) \cup \pi_Y^*(v)$, is split exact. By Lemma 2.3 $(H^*(X, A) \otimes H^*(Y, B))_f \cong H_f^*(X, A) \otimes H_f^*(Y, B)$ and by Lemma 2.2

$$\mu'_f : H_f^*(X, A) \otimes H_f^*(Y, B) \to H_f^*(X \times Y, X \times B \cup A \times Y)$$

is an isomorphism. Since μ' is multiplicative, so is μ'_f. Letting A and B be the basepoints of X and Y, or be the empty set, gives the isomorphisms

$$\mu_f : H_f^*(X) \otimes H_f^*(Y) \cong H_f^*(X \times Y) \quad \text{and} \quad \bar{\mu}_f : \tilde{H}_f^*(X) \otimes \tilde{H}_f^*(Y) \cong \tilde{H}_f^*(X \wedge Y).$$

The commutative diagram

$$
\begin{array}{ccc}
\bigotimes_{i=1}^{m} \tilde{H}^*(X_i) & \xrightarrow{\ \bar{\mu}\ } & \tilde{H}^*\big(\bigwedge_{i=1}^{m} X_i\big) \\
\downarrow & & \downarrow{\scriptstyle q^*} \\
\bigotimes_{i=1}^{m} H^*(X_i) & \xrightarrow{\ \mu\ } & H^*\big(\prod_{i=1}^{m} X_i\big)
\end{array}
$$

leads to the asserted commutative diagram. $\hspace{2cm}\square$

Proposition 3.3   *Let $\underline{X} = \{X_i\}_{i=1}^{m}$ be a sequence of spaces X_i, and let K be a simplicial complex on $[m]$. Then the inclusion $\iota : (\underline{X}, *)^K \to \prod_{i=1}^{m} X_i$ induces a ring isomorphism*

$$
H_f^*\big((\underline{X}, *)^K\big) \cong \bigg(\bigotimes_{i=1}^{m} H_f^*(X_i) \bigg) / I_K
$$

where I_K is generated by $x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $x_{j_i} \in \tilde{H}_f^*(X_{j_i})$ and $\{j_1, \ldots, j_k\} \notin K$.

Proof   This proof modifies the proofs in [3; 5]. Consider the homotopy cofibration sequence

$$
(\underline{X}, *)^K \xrightarrow{\ \iota\ } \prod_{i=1}^{m} X_i \xrightarrow{\ J\ } C,
$$

where C is the mapping cone of ι and J is the inclusion. Suspend it and obtain a diagram of homotopy cofibration sequences

(3)
$$
\begin{array}{ccccc}
\Sigma(\underline{X}, *)^K & \xrightarrow{\ \Sigma\iota\ } & \Sigma\big(\prod_{i=1}^{m} X_i\big) & \xrightarrow{\ \Sigma J\ } & \Sigma C \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
\bigvee_{J \in K} \Sigma \underline{X}^{\wedge J} & \xrightarrow{\ \bar{\iota}\ } & \bigvee_{J \in [m]} \Sigma \underline{X}^{\wedge J} & \xrightarrow{\ \bar{J}\ } & \bigvee_{J \notin K} \Sigma \underline{X}^{\wedge J}
\end{array}
$$

where $\underline{X}^{\wedge J} = X_{j_1} \wedge \cdots \wedge X_{j_k}$ for $J = \{j_1, \ldots, j_k\}$, $\bar{\iota}$ is the inclusion, $\bar{J}$ is the pinch map, a is a homotopy equivalence by [3, Theorem 2.21], b is a homotopy equivalence, and c is an induced homotopy equivalence. Take cohomology and get the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{J \notin K} \tilde{H}^*(\underline{X}^{\wedge J}) & \xrightarrow{\ \bar{J}^*\ } & \bigoplus_{J \in [m]} \tilde{H}^*(\underline{X}^{\wedge J}) & \xrightarrow{\ \bar{\iota}^*\ } & \bigoplus_{J \in K} \tilde{H}^*(\underline{X}^{\wedge J}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle c^*} & & \downarrow{\scriptstyle b^*} & & \downarrow{\scriptstyle a^*} & & \\
0 & \longrightarrow & \tilde{H}^*(C) & \xrightarrow{\ J^*\ } & \tilde{H}^*\big(\prod_{i=1}^{m} X_i\big) & \xrightarrow{\ \iota^*\ } & \tilde{H}^*\big((\underline{X}, *)^K\big) & \longrightarrow & 0
\end{array}
$$

where the rows are split exact sequences, all vertical maps are isomorphisms, and all maps are additive while ι^* is multiplicative. Apply Lemma 2.2 to the diagram and get:

(4)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{J \notin K} \tilde{H}_f^*(\underline{X}^{\wedge J}) & \xrightarrow{\bar{J}_f^*} & \bigoplus_{J \in [m]} \tilde{H}_f^*(\underline{X}^{\wedge J}) & \xrightarrow{\bar{\iota}_f^*} & \bigoplus_{J \in K} \tilde{H}_f^*(\underline{X}^{\wedge J}) & \longrightarrow & 0 \\
& & \downarrow{c_f^*} & & \downarrow{b_f^*} & & \downarrow{a_f^*} & & \\
0 & \longrightarrow & \tilde{H}_f^*(C) & \xrightarrow{J_f^*} & \tilde{H}_f^*(\prod_{i=1}^m X_i) & \xrightarrow{\iota_f^*} & \tilde{H}_f^*((\underline{X}, *)^K) & \longrightarrow & 0
\end{array}
$$

By Lemma 3.2, $H_f^*\left(\prod_{i=1}^m X_i\right) \cong \bigotimes_{i=1}^m H_f^*(X_i)$ so there is a ring isomorphism

$$
H_f^*((\underline{X}, *)^K) \cong \left(\bigotimes_{i=1}^m H_f^*(X_i)\right)/\ker(\iota_f^*).
$$

Since the rows are split exact and the vertical maps are isomorphic in (4), $\ker(\iota_f^*)$ is generated by $x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $x_{j_i} \in \tilde{H}_f^*(X_{j_i})$ and $\{j_1, \ldots, j_k\} \notin K$. Therefore $\ker(\iota_f^*) = I_K$ and $H_f^*((\underline{X}, *)^K) \cong \left(\bigotimes_{i=1}^m H_f^*(X_i)\right)/I_K$. $\square$

Proposition 3.3 can be refined as follows. If the quotient map $H^*(X_i) \to H_f^*(X_i)$ has right inverse for all i, then so does $H^*((\underline{X}, *)^K) \to H_f^*((\underline{X}, *)^K)$. To formulate this, we introduce new definition.

Definition 3.3.1 A graded algebra $\mathcal{A}$ is *free split* if the quotient map $\pi: \mathcal{A} \to \mathcal{A}_f$ has a section as algebras. In other words, there is a ring morphism $s: \mathcal{A}_f \to \mathcal{A}$ making the diagrams

$$
\begin{array}{ccc}
\mathcal{A}_f & \xrightarrow{s} & \mathcal{A} \\
& \searrow & \downarrow{\pi} \\
& & \mathcal{A}_f
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathcal{A}_f \otimes \mathcal{A}_f & \xrightarrow{m_f} & \mathcal{A}_f \\
\downarrow{s \otimes s} & & \downarrow{s} \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{m} & \mathcal{A}
\end{array}
$$

commute, where m and m_f are multiplications in $\mathcal{A}$ and $\mathcal{A}_f$. We call s a *free splitting* of $\mathcal{A}$.

In general, a free splitting of $\mathcal{A}$ is not unique. Any two free splittings s_1 and s_2 differ by a torsion element.

Remark 3.4 Not all cohomology rings of spaces are free split. Let C be the mapping cone of the composite

$$
P^3(2) \xrightarrow{\rho} S^3 \xrightarrow{[\iota_1, \iota_2]} S^2 \vee S^2,
$$

where $P^3(2)$ is the mapping cone of degree map $2: S^2 \to S^2$, ρ is the quotient map and $[\iota_1, \iota_2]$ is the Whitehead product. Then $H^*(C) \cong \mathbb{Z}[a, b]/(a^2 = b^2 = 2ab = 0)$ where $|a| = |b| = 2$, and it is not free split.

Lemma 3.5 *Under the conditions of Proposition 3.3, if $H^*(X_i)$ is free split for all i, then $H^*((\underline{X}, *)^K)$ is free split.*

Proof Use the notations in the proof of Proposition 3.3. Let $s_i : H_f^*(X_i) \to H^*(X_i)$, for $1 \leq i \leq m$, be a free splitting and let s be the composite

$$s : \bigotimes_{i=1}^m H_f^*(X_i) \xrightarrow{\otimes_{i=1}^m s_i} \bigotimes_{i=1}^m H^*(X_i) \xrightarrow{\mu} H^*\left(\prod_{i=1}^m X_i\right).$$

Then s is a free splitting of $H^*\left(\prod_{i=1}^m X_i\right)$. As $\iota_f^* : H_f^*\left(\prod_{i=1}^m X_i\right) \to H_f^*((\underline{X}, *)^K)$ is surjective, define $s' : H_f^*((\underline{X}, *)^K) \to H^*((\underline{X}, *)^K)$ by the diagram

$$
\begin{array}{ccc}
\bigotimes_{i=1}^m H_f^*(X_i) & \xrightarrow{\ s\ } & H^*\left(\prod_{i=1}^m X_i\right) \\
\downarrow{\scriptstyle \iota_f^*} & & \downarrow{\scriptstyle \iota^*} \\
H_f^*((\underline{X}, *)^K) & \xrightarrow{\ s'\ } & H^*((\underline{X}, *)^K)
\end{array}
$$

We need to show that s' is well defined. For $x \in H_f^*(\underline{X}, *)^K$, let $y, y' \in \bigotimes_{i=1}^m H_f^*(X_i)$ be two preimages of x. Then $y - y' \in \ker(\iota_f^*) = I_K$. For $J \notin K$, s sends $\tilde{H}_f^*(\underline{X})^{\otimes J}$ to $\mu(\tilde{H}^*(\underline{X})^{\otimes J})$ which is contained in $\ker(\iota^*)$. So $\iota^* \circ s(y - y') = 0$ and s' is well defined. Since s, ι^* and ι_f^* are multiplicative, so is s'. Furthermore, s' is right inverse to the quotient map $H^*((\underline{X}, *)^K) \to H_f^*((\underline{X}, *)^K)$. So s' is a free splitting. $\square$

4 Realization of graded monomial ideal rings

We follow the idea of [3] and prove Theorem 1.2 in several steps. In Section 4.1 we use Proposition 3.3 to prove the special case where the ideal I of A is square-free. In Sections 4.2 and 4.3 we construct a fibration sequence inspired by algebraic polarization method and show that the fiber X_A is a realization modulo torsion of A. More precisely, we apply the Eilenberg–Moore spectral sequence defined in Section 2.2 to calculate $H_f^*(X_A)$ and give the E_∞–page by the end of this section. The extension problem is long and complicated and will be discussed in Section 5.

4.1 Quotient rings of square-free ideals

Let $P = \mathbb{Z}[x_1, \ldots, x_m] \otimes \Lambda[y_1, \ldots, y_n]$ be a graded polynomial ring where the x_i's have arbitrary positive even degrees and the y_j's have arbitrary positive odd degrees, and let $I = (M_1, \ldots, M_r)$ be an ideal generated by monomials

$$M_j = x_1^{a_{1j}} \cdots x_m^{a_{mj}} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}},$$

where the a_{ij}'s are nonnegative integers and the b_{ij}'s are either 0 or 1. Then $A = P/I$ is a *graded monomial ideal ring*. We say that I is *square-free* if the M_j's are square-free monomials, that is all a_{ij}'s are either 0 or 1.

In the following let

- $\{i_1, \ldots, i_k\} + \{j_1, \ldots, j_l\} = \{i_1, \ldots, i_k, j_1+m, \ldots, j_l+m\}$ for $\{i_1, \ldots, i_k\} \subset [m]$ and $\{j_1, \ldots, j_l\} \subset [n]$, and

- $\underline{X} + \underline{Y} = \{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$ for sequences of spaces $\underline{X} = \{X_i\}_{i=1}^m$ and $\underline{Y} = \{Y_j\}_{j=1}^n$.

Given a square-free ideal I of A, take K to be a simplicial complex on $[m+n]$ by removing faces $\{i_1, \ldots, i_k\} + \{j_1, \ldots, j_l\}$ whenever $x_{i_1} \cdots x_{i_k} \otimes y_{j_1} \cdots y_{j_l} \in I$. Then I is the generalized Stanley–Reisner ideal of K.

Lemma 4.1 *Let $\underline{X} = \{K(\mathbb{Z}, |x_i|)\}_{i=1}^m$ and $\underline{Y} = \{S^{|y_j|}\}_{j=1}^n$ and let K be the simplicial complex defined as above. Then there is a ring isomorphism $H_f^*((\underline{X} + \underline{Y}, *)^K) \cong A$. Furthermore, $H^*((\underline{X} + \underline{Y}, *)^K)$ is free split.*

Proof Since $H_f^*(X_i) \cong \mathbb{Z}[x_i]$ and $H^*(Y_j) \cong \Lambda[y_j]$, the first part follows from [Proposition 3.3](#).

For the second part, it suffices to show that $H^*(X_i)$ and $H^*(Y_j)$ are free split by [Lemma 3.5](#). For $1 \leq j \leq n$, $H^*(Y_j)$ is free and hence free split. For $1 \leq i \leq m$, let x_i' be a generator of $H^{|x_i|}(X_i) \cong \mathbb{Z}$. Then inclusion $\iota \colon \mathbb{Z}\langle x_i' \rangle \to H^*(X_i)$ extends to a ring morphism

$$s \colon \mathbb{Z}[x_i'] \cong H_f^*(X_i) \to H^*(X_i).$$

Let $\pi \colon H^*(X_i) \to H_f^*(X_i)$ be the quotient map. Since $\pi \circ \iota$ sends x_i' to itself, by the universal property $\pi \circ s$ is the identity map. So s is a free splitting of $H^*(X_i)$. $\qquad\square$

4.2 Polarization of graded monomial ideal rings

Now drop the square-free assumption on $I = (x_1^{a_{1j}} \cdots x_m^{a_{mj}} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}} \mid 1 \leq j \leq r)$ and suppose some a_{ij}'s are greater than 1. Following ideas from [3] and [15], we use polarization to reduce the realization problem of A to the special case when I is square-free.

For $1 \leq i \leq m$, let $a_i = \max\{a_{i1}, \ldots, a_{ir}\}$ be the largest index of x_i among the M_j's, and let

$$\Omega = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m, 1 \leq j \leq a_i\}$$

where $(i, j) \in \Omega$ are ordered in left lexicographical order, that is $(i, j) < (i', j')$ if $i < i'$, or if $i = i'$ and $j < j'$. Let

$$P' = \mathbb{Z}[x_{ij} \mid (i, j) \in \Omega] \otimes \Lambda[y_1, \ldots, y_n]$$
$$= \mathbb{Z}[x_{11}, \ldots, x_{1a_1}, x_{21}, \ldots, x_{2a_2}, \ldots, x_{m1}, \ldots, x_{ma_m}] \otimes \Lambda[y_1, \ldots, y_n],$$

be a graded polynomial ring where $|x_{ij}| = |x_i|$, let

$$M_j' = (x_{11}x_{12} \cdots x_{1a_{1j}})(x_{21}x_{22} \cdots x_{2a_{2j}}) \cdots (x_{m1}x_{m2} \cdots x_{ma_{mj}}) \otimes (y_1^{b_{1j}} \cdots y_n^{b_{nj}})$$

and let $I' = (M_1', \ldots, M_r')$. Then I' is square-free and $A' = P'/I'$ is called the *polarization* of A.

Conversely, we can reverse the polarization process and obtain A back from A'. Let

$$\overline{\Omega} = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m, 2 \leq j \leq a_i\}$$

where $(i, j) \in \overline{\Omega}$ are ordered in left lexicographical order, and let W be a graded polynomial ring

$$W = \mathbb{Z}[w_{ij} \mid (i, j) \in \overline{\Omega}] = \mathbb{Z}[w_{12}, \ldots, w_{1a_1}, w_{22}, \ldots, w_{2a_2}, \ldots, w_{m2}, \ldots, w_{ma_m}],$$

where $|w_{ij}| = |x_i|$. Define a ring morphism $\delta \colon W \to P'$ by $\delta(w_{ij}) = x_{ij} - x_{i1}$ and make P' a W–module via δ. Then A' is a W–module and $A \cong A'/\overline{W} A'$, where $\overline{W} = \{W^i\}_{i>0}$.

Lemma 4.2 *Let A' be a square-free graded monomial ideal ring and let W and δ be defined as above. Then A' is a free W–module.*

Proof Since $A'/\overline{W} A'$ is a free $\mathbb{Z}$–module, by Lemma 2.10 it suffices to show that $\{w_{ij}\}_{(i,j)\in\overline{\Omega}}$ is a A'–regular sequence. Set $N = |\overline{\Omega}| = \sum_{i=1}^{m} a_i - m$. For $1 \leq k \leq N$, let $(i_k, j_k) \in \overline{\Omega}$ be the kth pair under lexicographical order and let

$$I_k = (w_{12}, w_{13}, \ldots, w_{i_k j_k}).$$

We need to show that multiplication $w_{i_{k+1} j_{k+1}} \colon A'/I_k A' \to A'/I_k A'$ is injective. Observe that $A'/I_k A' = \widetilde{P}/\widetilde{I}$, where

$$\widetilde{P} = \mathbb{Z}[x_{11}, x_{21}, \ldots, x_{m1}, x_{i_{k+1} j_{k+1}}, x_{i_{k+2} j_{k+2}}, \ldots x_{i_N j_N}] \otimes \Lambda[y_1, \ldots, y_n]$$

and $\widetilde{I} = (\widetilde{M}_1, \ldots, \widetilde{M}_r)$ is generated by monomials $\widetilde{M}_j$ obtained by identifying x_{ij} with x_{i1} in M'_j for $(i, j) \leq (i_k, j_k)$. Suppose there is a polynomial $p \in \widetilde{P}$ such that

$$(x_{i_{k+1} j_{k+1}} - x_{i_{k+1} 1}) \cdot p \in \widetilde{I}.$$

We can use the combinatorial argument of [8, page 31] to show $p \in \widetilde{I}$. Here is an outline of the argument. Write $p = \sum_\alpha p_\alpha$ as a sum of monomials p_α. For each monomial p_α, it can be shown that $x_{i_{k+1} j_{k+1}} p_\alpha$ and $x_{i_{k+1} 1} p_\alpha$ are in $\widetilde{I}$. Counting the indices of variables implies $p_\alpha \in \widetilde{I}$. So p is in $\widetilde{I}$ and multiplication $w_{i_{k+1} j_{k+1}} \colon A'/I_k A' \to A'/I_k A'$ is injective. Therefore $\{w_{ij}\}_{(i,j) \in \overline{\Omega}}$ is a regular sequence and A' is a free W–module. $\square$

4.3 Constructing a realization modulo torsion X_A

Let $A' = P'/I'$ be the polarization of A and let K be a simplicial complex on $\left(\sum_{i=1}^m a_i + n\right)$ vertices such that I' is the generalized Stanley–Reisner ideal of K. Construct a polyhedral product to realize A'. Take

$$\underline{X} = \{X_{ij} = K(\mathbb{Z}, |x_i|) \mid (i, j) \in \Omega\}$$
$$= \{\underbrace{K(\mathbb{Z}, |x_1|), \ldots, K(\mathbb{Z}, |x_1|)}_{a_1}, \underbrace{K(\mathbb{Z}, |x_2|), \ldots, K(\mathbb{Z}, |x_2|)}_{a_2},$$
$$\ldots, \underbrace{K(\mathbb{Z}, |x_m|), \ldots, K(\mathbb{Z}, |x_m|)}_{a_m}\}$$

and

$$\underline{Y} = \{Y_k = S^{|y_k|} \mid 1 \leq k \leq n\} = \{S^{|y_1|}, S^{|y_2|}, \ldots, S^{|y_n|}\}.$$

By Lemma 4.1, $H_f^*((\underline{X} + \underline{Y}, *)^K)$ is isomorphic to A'.

For $1 \leq i \leq m$, define $\delta_i \colon \prod_{j=1}^{a_i} X_{ij} \to \prod_{j=2}^{a_i} X_{ij}$ by

$$\delta_i(u_1, \ldots, u_{a_i}) = (u_2 \cdot u_1^{-1}, \ldots, u_{a_i} \cdot u_1^{-1}),$$

and define $\delta \colon (\underline{X} + \underline{Y}, *)^K \to \prod_{(i,j) \in \overline{\Omega}} X_{ij}$ to be the composite

$$\delta \colon (\underline{X} + \underline{Y}, *)^K \hookrightarrow \prod_{(i,j) \in \Omega} X_{ij} \times \prod_{k=1}^n Y_k \xrightarrow{\text{proj}} \prod_{(i,j) \in \Omega} X_{ij} \xrightarrow{\prod_{i=1}^m \delta_i} \prod_{(i,j) \in \overline{\Omega}} X_{ij}.$$

As δ is a fibration, take X_A to be its fiber. We claim that $H_f^*(X_A) \cong A$.

Notation 4.3 Let $\{E_r^{*,*}\}_{r=0}^{\infty}$ be the Eilenberg–Moore spectral sequence defined in Section 2.2 on the fibration sequence

$$(5) \qquad X_A \to (\underline{X} + \underline{Y}, *)^K \xrightarrow{\delta} \prod_{(ij) \in \overline{\Omega}} X_{ij},$$

where $H^*((\underline{X} + \underline{Y}, *)^K)$ is an $H^*\big(\prod_{(ij) \in \overline{\Omega}} X_{ij}\big)$–module via δ^*.

Lemma 4.4 For the E_∞–page, $(E_\infty^{0,q})_f \cong A^q$ as modules and $(E_\infty^{-p,q})_f = 0$ for $p \neq 0$.

Proof The E_2–page is given by $E_2^{-p,*} = \mathrm{Tor}_{H^*(\prod_{(ij) \in \overline{\Omega}} X_{ij})}^{-p,*}(H^*((\underline{X} + \underline{Y}, *)^K), \mathbb{Z})$. By Lemma 2.6, there is a monomorphism

$$\pi' \colon (E_2^{-p,*})_f \to \big(\mathrm{Tor}_{H_f^*(\prod_{(ij) \in \overline{\Omega}} X_{ij})}^{-p,*}(H_f^*((\underline{X} + \underline{Y}, *)^K), \mathbb{Z})\big)_f,$$

which is an isomorphism for $p = 0$. By Lemmas 3.2 and 3.3, $H_f^*((\underline{X} + \underline{Y}, *)^K) \cong A'$ and

$$H_f^*\bigg(\prod_{(ij) \in \overline{\Omega}} X_{ij} \bigg) \cong \mathbb{Z}[w_{12}, \ldots, w_{1a_1}, w_{22}, \ldots, w_{2a_2}, \ldots, w_{m2}, \ldots, w_{ma_m}].$$

Denote $H_f^*\big(\prod_{(ij) \in \overline{\Omega}} X_{ij}\big)$ by W. So A' is a W–module via δ^*. By Lemma 4.2, A' is a free W–module, so

$$\mathrm{Tor}_W^{-p,q}(A', \mathbb{Z}) \cong \begin{cases} A^q & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $(E_2^{-p,q})_f$ is A^q for $p = 0$ and is zero otherwise.

Suppose $(E_r^{-p,q})_f$ is A^q for $p = 0$ and is zero otherwise. Since $(E_r^{-p,*})_f$ is concentrated in the column $p = 0$, any differentials d_r in and out of torsion-free elements are trivial. So we have $\ker(d_r)_f = (E_r^{-p,q})_f$ and $\mathrm{Im}(d_r)_f = 0$. By Lemma 2.2, $(E_{r+1}^{-p,q})_f \cong (E_r^{-p,q})_f$. Therefore $(E_\infty^{-p,q})_f$ is isomorphic to A^q for $p = 0$ and is zero otherwise. $\square$

Lemma 4.5 There is an additive isomorphism $H_f^q(X_A) \cong A^q$.

Proof Since the Eilenberg–Moore spectral sequence strongly converges to $H^*(X_A)$, for any fixed q there is a decreasing filtration $\{\mathscr{F}^{-p}\}$ of $H^q(X_A)$ such that

$$\mathscr{F}^{-\infty} = H^q(X_A), \quad \mathscr{F}^1 = 0, \quad E_\infty^{-p,p+q} \cong \mathscr{F}^{-p}/\mathscr{F}^{-p+1}.$$

By Lemma 2.2, $(E_\infty^{-p,p+q})_f \cong \big((\mathscr{F}^{-p})_f/(\mathscr{F}^{-p+1})_f\big)_f$. By Lemma 4.4, $(E_\infty^{-p,p+q})_f$ is zero unless $p = 0$, so $H_f^q(X_A) \cong (E_\infty^{0,q})_f \cong A^q$ as modules. $\square$

Before going to the extension problem of the E_∞–page, we consider the special case where all of the even degree generators of A are in degree 2. The following theorem refines Lemma 4.5 and shows that $H^*(X_A) \cong A$ as algebras without modding out the cohomology by torsion. This generalizes the results of Bahri, Bendersky, Cohen and Gitler [4, Theorem 2.2] and Trevisan [15, Theorem 3.6].

Theorem 4.6 *Let A be a graded monomial ideal ring where its generators have either degree 2 or arbitrary positive odd degrees. Then $H^*(X_A) \cong A$ as algebras.*

Proof The E_2–page is given by

$$E_2^{-p,*} = \mathrm{Tor}^{-p,*}_{H^*(\prod_{(ij)\in\overline{\Omega}} X_{ij})}(H^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z}).$$

By hypothesis, $X_{ij} = \mathbb{CP}^\infty$ for $(i,j) \in \Omega$, and $H^*\big(\prod_{(ij)\in\overline{\Omega}} X_{ij}\big)$ and $H^*((\underline{X}+\underline{Y},*)^K$ are free. Following the argument in the proof of Lemma 4.4, $E_2^{-p,q}$ is A^q for $p = 0$ and is zero otherwise. Since the E_2–page is concentrated in the column $p = 0$, the spectral sequence collapses and $H^*(X_A) \cong A$ as modules.

Let $\phi\colon X_A \to (\underline{X}+\underline{Y},*)^K$ be the fiber inclusion. Lemma 2.8 implies the commutative diagram

$$
\begin{array}{ccc}
A' & \xrightarrow{\;\cong\;} & H^*((\underline{X}+\underline{Y},*)^K) \\[2pt]
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \phi^*} \\[2pt]
E_2^{*,*} & \xrightarrow{\;\cong\;} & H^*(X_A)
\end{array}
$$

where e is surjective. Since ϕ^* is surjective and multiplicative and its kernel is W, $H^*(X_A) \cong A'/W \cong A$ as algebras. $\qquad\square$

5 The extension problem

In this section we continue using Notation 4.3. Lemma 4.5 shows that $H^*_f(X_A)$ and A are free $\mathbb{Z}$–modules of same rank at each degree. We claim that they are isomorphic as algebras. The idea is to construct a space Z_A related to X_A such that $H^*(Z_A)$ is free and computable. Then we define a map $g_A\colon Z_A \to X_A$ and compare $H^*(X_A)$ with $H^*(Z_A)$ via g_A^*.

Construction of Z_A   For $1 \leq i \leq m$ let $|x_i| = 2c_i$, and let

$$
\begin{aligned}
\underline{Z} &= \{Z_{ij} = (\mathbb{CP}^\infty)^{c_i} \mid (i,j) \in \Omega\} \\
&= \{\underbrace{(\mathbb{CP}^\infty)^{c_1}, \ldots, (\mathbb{CP}^\infty)^{c_1}}_{a_1}, \underbrace{(\mathbb{CP}^\infty)^{c_2}, \ldots, (\mathbb{CP}^\infty)^{c_2}}_{a_2}, \\
&\qquad\qquad\qquad\qquad\qquad \ldots, \underbrace{(\mathbb{CP}^\infty)^{c_m}, \ldots, (\mathbb{CP}^\infty)^{c_m}}_{a_m}\}
\end{aligned}
$$

and construct the polyhedral product $(\underline{Z} + \underline{Y}, *)^K$. Fix a generator z of $H^2(\mathbb{CP}^\infty)$. For $(i,j) \in \Omega$ and $1 \leq k \leq c_i$, let $\pi_{ijk} \colon Z_{ij} \to \mathbb{CP}^\infty$ be the projection onto the kth copy of $\mathbb{CP}^\infty$ and let $z_{ijk} = \pi_{ijk}^*(z)$. By Theorem 3.1,

$$
H^*((\underline{Z} + \underline{Y}, *)^K) \cong Q'/L',
$$

where $Q' = \mathbb{Z}[z_{ijk} \mid (i,j) \in \Omega, 1 \leq k \leq c_i] \otimes \Lambda[y_1, \ldots, y_n]$ and L' is the ideal generated by monomials

$$
z_{i_1 j_1 k_1} \cdots z_{i_t j_t k_t} \otimes y_{l_1} \cdots y_{l_\tau}
$$

for $\{j_1 + \sum_{s=1}^{i_1-1} a_s, \ldots, j_t + \sum_{s=1}^{i_t-1} a_s\} + \{l_1, \ldots, l_\tau\} \notin K$. For $1 \leq i \leq m$, define

$$
\tilde{\delta}_i \colon \prod_{j=1}^{a_i} Z_{ij} \to \prod_{j=2}^{a_i} Z_{ij}, \quad \tilde{\delta}_i(u_1, \ldots, u_{a_i}) = (u_2 \cdot u_1^{-1}, \ldots, u_{a_i} \cdot u_1^{-1}),
$$

and define $\tilde{\delta} \colon (\underline{Z} + \underline{Y}, *)^K \to \prod_{(i,j) \in \overline{\Omega}} Z_{ij}$ to be the composite

$$
\tilde{\delta} \colon (\underline{Z} + \underline{Y}, *)^K \hookrightarrow \prod_{(i,j) \in \Omega} Z_{ij} \times \prod_{k=1}^n Y_k \xrightarrow{\text{proj}} \prod_{(i,j) \in \Omega} Z_{ij} \xrightarrow{\prod_{i=1}^m \tilde{\delta}_i} \prod_{(i,j) \in \overline{\Omega}} Z_{ij}.
$$

Lemma 5.1   *Let Z_A be the fiber of δ'. Then $H^*(Z_A) \cong Q/L$, where*

$$
Q = \mathbb{Z}[z_{ik} \mid 1 \leq i \leq m, 1 \leq k \leq c_i] \otimes \Lambda[y_1, \ldots, y_n]
$$

with $|z_{ik}| = 2$ and L is generated by monomials $z_{i_1 k_1} \cdots z_{i_N k_N} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}}$ satisfying

$$
1 \leq j \leq r, \quad 1 \leq k_l \leq c_{i_l} \quad \text{and} \quad (i_1, \ldots, i_N) = (\underbrace{1, \ldots, 1}_{a_{1j}}, \underbrace{2, \ldots, 2}_{a_{2j}}, \ldots, \underbrace{m, \ldots, m}_{a_{mj}}).
$$

Proof   Apply the Eilenberg–Moore spectral sequence to the fibration sequence

$$
Z_A \to (\underline{Z} + \underline{Y}, *)^K \xrightarrow{\tilde{\delta}} \prod_{(i,j) \in \overline{\Omega}} Z_{ij}.
$$

The E_2–page is given by $\widetilde{E}_2^{-p,*} = \mathrm{Tor}_{H^*(\prod_{(i,j)\in\overline{\Omega}} Z_{ij})}^{-p,*}(\mathbb{Z}, H^*((\underline{Z}+\underline{Y},*)^K))$. By the Künneth theorem,

$$H^*\left(\prod_{(i,j)\in\overline{\Omega}} Z_{ij}\right) \cong \mathbb{Z}[v_{ijk} \mid (i,j)\in\overline{\Omega}, 1\le k\le c_i],$$

where $|v_{ijk}| = 2$. Denote $H^*(\prod_{(i,j)\in\overline{\Omega}} Z_{ij})$ by V. By definition $\widetilde{\delta}^*(v_{ijk}) = z_{ijk} - z_{i1k}$. This gives an action of V on Q'. By [Lemma 4.2](), Q'/L' is a free V–module, so

$$\mathrm{Tor}_V^{-p,*}(Q'/L',\mathbb{Z}) = \begin{cases} (Q'/L')/(z_{ijk}-z_{ilk}) & \text{if } p=0, \\ 0 & \text{otherwise.} \end{cases}$$

Modding out $(z_{ijk}-z_{ilk})$ identifies z_{ijk} with z_{ilk} in Q'/L', so

$$(Q'/L')/(z_{ijk}-z_{ilk}) \cong Q/L.$$

Since the E_2–page is concentrated in the column $p=0$, $H^*(Z_A) \cong Q/L$.

[Lemma 2.8]() implies a commutative diagram

$$
\begin{array}{ccc}
Q' & \xrightarrow{\ \cong\ } & H^*((\underline{Z}+\underline{Y})^K) \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \phi^*} \\
E_2^{*,*} & \xrightarrow{\ \cong\ } & H^*(Z_A)
\end{array}
$$

where e is surjective and ϕ^* is induced by the fiber inclusion $\phi\colon Z_A \to (\underline{Z}+\underline{Y})^K$. This implies ϕ^* is surjective. Since ϕ^* is multiplicative, $H^*(Z_A) \cong Q/L$ as algebras. $\qquad\square$

Construction of g_A Fix a generator $z\in H^2(\mathbb{CP}^\infty)$. Let $\pi_j\colon (\mathbb{CP}^\infty)^{c_i} \to \mathbb{CP}^\infty$, for $1\le j\le c_i$, be the projection onto the jth copy of $\mathbb{CP}^\infty$ and let $z_j = \pi_j^*(z)$. For $1\le i\le m$, take a map $g_i\colon (\mathbb{CP}^\infty)^{c_i} \to K(\mathbb{Z}, 2c_i)$ that represents the cocycle class $z_1\cdots z_{c_i} \in H^{2c_i}((\mathbb{CP}^\infty)^{c_i})$. For $(i,j)\in\Omega$, let $g_{ij}\colon Z_{ij}\to X_{ij}$ be g_i, and for $1\le k\le n$, let $h_k\colon Y_k\to Y_k$ be the identity map. Then $\{g_{ij}, h_k \mid (i,j)\in\Omega, 1\le k\le n\}$ induces a map $g_K\colon (\underline{Z}+\underline{Y},*)^K \to (\underline{X}+\underline{Y},*)^K$ by the functoriality of polyhedral products.

Lemma 5.2 *Let $\{x_{ij}, y_k \mid (i,j)\in\Omega, 1\le k\le n\}$ be generators of*

$$H_f^*((\underline{X}+\underline{Y},*)^K) \cong P'/I'$$

and $\{z_{ijl}, y_k' \mid (i,j)\in\Omega, 1\le l\le c_i, 1\le k\le n\}$ be generators of

$$H^*((\underline{Z}+\underline{Y},*)^K) \cong Q'/L'.$$

Then $(g_K^*)_f(x_{ij}) = \prod_{l=1}^{c_i} z_{ijl}$ and $(g_K^*)_f(y_k) = y_k'$.

Proof　There is a commutative diagram

$$
\begin{array}{ccc}
(\underline{Z}+\underline{Y},*)^K & \xrightarrow{\;J\;} & \prod_{(i,j)\in\Omega} Z_{ij} \times \prod_{k=1}^n Y_k \\
\Big\downarrow{\scriptstyle g_K} & & \Big\downarrow{\scriptstyle g} \\
(\underline{X}+\underline{Y},*)^K & \xrightarrow{\;\iota\;} & \prod_{(i,j)\in\Omega} X_{ij} \times \prod_{k=1}^n Y_k
\end{array}
$$

where ι and J are inclusions, and $g = \prod_{(i,j)\in\Omega} g_{ij} \times \prod_{k=1}^n h_k$. Taking cohomology and modding out torsion elements, we obtain the commutative diagram

$$
\begin{array}{ccc}
P' & \xrightarrow{\;\iota_f^*\;} & P'/I' \\
{\scriptstyle g_f^*}\Big\downarrow & & \Big\downarrow{\scriptstyle (g_K^*)_f} \\
Q' & \xrightarrow{\;J^*\;} & Q'/L'
\end{array}
$$

where ι_f^* and J^* are the quotient maps. Let $\tilde{x}_{ij}, \tilde{y}_k \in P'$ and $\tilde{z}_{ijl}, \tilde{y}_k' \in Q'$ be generators such that $\iota_f^*(\tilde{x}_{ij}) = x_{ij}, \iota_f^*(\tilde{y}_k) = y_k, J_f^*(\tilde{y}_k') = y_k'$ and $J_f^*(\tilde{z}_{ijl}) = z_{ijl}$. By construction $g_f^*(\tilde{x}_{ij}) = \prod_{l=1}^{c_i} \tilde{z}_{ijl}$ and $g_f^*(\tilde{y}_k) = \tilde{y}_k'$, so we have $(g_K^*)_f(x_{ij}) = \prod_{l=1}^{c_i} z_{ijl}$ and $(g_K^*)_f(y_k) = y_k'$.　□

Lemma 5.3　*There is a map $g_A \colon Z_A \to X_A$ making the diagram*

$$
\begin{array}{ccc}
Z_A & \longrightarrow & (\underline{Z}+\underline{Y},*)^K \\
{\scriptstyle g_A}\Big\downarrow & & \Big\downarrow{\scriptstyle g_K} \\
X_A & \longrightarrow & (\underline{X}+\underline{Y},*)^K
\end{array}
$$

commute, where the horizontal maps are the inclusion maps.

Proof　One may want to construct g_A by showing the diagram

$$
\begin{array}{ccc}
(\underline{Z}+\underline{Y},*)^K & \xrightarrow{\;\tilde{\delta}\;} & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\
{\scriptstyle g_K}\Big\downarrow & & \Big\downarrow{\scriptstyle \prod_{(i,j)\in\overline{\Omega}} g_{ij}} \\
(\underline{X}+\underline{Y},*)^K & \xrightarrow{\;\delta\;} & \prod_{(i,j)\in\overline{\Omega}} X_{ij}
\end{array}
$$

commutes. However, as $\left(\prod_{(i,j)\in\overline{\Omega}} g_{ij}\right) \circ \tilde{\delta}$ and $\delta \circ g_K$ induce different morphisms on cohomology, the diagram cannot commute. Instead, we show that the composite

$$
Z_A \to (\underline{Z}+\underline{Y},*)^K \xrightarrow{\;g_K\;} (\underline{X}+\underline{Y},*)^K \xrightarrow{\;\delta\;} \prod_{(i,j)\in\overline{\Omega}} X_{ij}
$$

is trivial. If so, there will exist a map $g_A \colon Z_A \to X_A$ as asserted since X_A is the fiber of δ.

By definition of $\bar{\bar{\delta}}$ there is a commutative diagram

$$
\begin{array}{ccc}
(\underline{Z}+\underline{Y},*)^K & \xrightarrow{\ \tilde{\delta}\ } & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\
\downarrow{\scriptstyle J} & & \uparrow{\scriptstyle \prod_{i=1}^m \tilde{\delta}_i} \\
\prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k & \xrightarrow{\text{proj}} & \prod_{(i,j)\in\Omega} Z_{ij}
\end{array}
$$

where J is the inclusion. Denote $\left(\prod_{i=1}^m \tilde{\delta}_i\right)\circ\text{proj}$ by $\tilde{\delta}'$ and extend the diagram to

$$
\begin{array}{ccccc}
Z_A & \longrightarrow & (\underline{Z}+\underline{Y},*)^K & \xrightarrow{\ \tilde{\delta}\ } & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle J} & & \| \\
\prod_{i=1}^m (\mathbb{CP}^\infty)^{c_i}\times\prod_{k=1}^n Y_k & \xrightarrow{\triangle'\times h} & \prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k & \xrightarrow{\tilde{\delta}'} & \prod_{(i,j)\in\overline{\Omega}} Z_{ij}
\end{array}
$$

where $\triangle'\colon\prod_{i=1}^m(\mathbb{CP}^\infty)^{c_i}\to\prod_{j=1}^{a_i} Z_{ij}$ is the diagonal map, $h\colon\prod_{k=1}^n Y_k\to\prod_{k=1}^n Y_k$ is the identity map, and e is an induced map. The top and the bottom row are fibration sequences. The left square fits into the commutative diagram

where ι is the inclusion, $\triangle\colon\prod_{i=1}^m K(\mathbb{Z},|x_i|)\to\prod_{j=1}^{a_i} X_{ij}$ is the diagonal map, and δ' is the composite

$$
\delta'\colon\prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k\xrightarrow{\text{proj}}\prod_{(i,j)\in\Omega} X_{ij}\xrightarrow{\prod_{i=1}^m \delta_i}\prod_{(i,j)\in\overline{\Omega}} X_{ij}.
$$

The middle square is due to the functoriality of polyhedral products, the right square is due to the definition of δ and the bottom triangle is due to the naturality of diagonal maps.

The composite of maps from Z_A to $\prod_{(i,j)\in\overline{\Omega}} X_{ij}$ round the bottom triangle is trivial since

$$
\prod_{i=1}^m K(\mathbb{Z},|x_i|)\times\prod_{k=1}^n Y_k\xrightarrow{\triangle\times h}\prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k\xrightarrow{\delta'}\prod_{(i,j)\in\overline{\Omega}} X_{ij}
$$

is a fibration sequence. So the composite in the top row is trivial and this induces a map $g_A \colon Z_A \to X_A$ as asserted.   □

Since $g_K^* \colon H^*((\underline{X} + \underline{Y})^K) \to H^*((\underline{Z} + \underline{Y})^K)$ is multiplicative and $H^*(Z_A)$ is a quotient algebra of $H^*((\underline{Z} + \underline{Y})^K)$, we use g_A to compare $H^*(X_A)$ and $H^*(Z_A)$ and show that $H_f^*(X_A)$ is a quotient algebra of $H_f^*((\underline{X} + \underline{Y})^K)$.

Lemma 5.4   *Let $\phi \colon X_A \to ((\underline{X} + \underline{Y}, *)^K)$ be the inclusion. Then the induced morphism*

$$\phi_f^* \colon H_f^*((\underline{X} + \underline{Y}, *)^K) \to H_f^*(X_A)$$

is surjective and $\ker(\phi_f^*)$ is generated by $x_{ij} - x_{i1}$ for $(i, j) \in \overline{\Omega}$.

Proof   Fix a positive integer q and let $\psi \colon Z_A \to (\underline{Z} + \underline{Y}, *)^K$ be the inclusion. Consider the commutative diagram

$$
\begin{array}{ccc}
H^q((\underline{X} + \underline{Y}, *)^K) & \xrightarrow{\;g_K^*\;} & H^q((\underline{Z} + \underline{Y}, *)^K) \\
{\scriptstyle e}\;\swarrow\;\;\;\downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \psi^*} \\
E_\infty^{0,q} \;\xrightarrow{\;h\;}\; H^q(X_A) & \xrightarrow{\;g_A^*\;} & H^q(Z_A)
\end{array}
$$

where e is surjective and h is injective. The left triangle commutes due to Lemma 2.8 and the right square commutes due to Lemma 5.3. Mod out torsion elements and take a generator

$$x_{i_1 j_1} \cdots x_{i_s j_s} \otimes y_{l_1} \cdots y_{l_t} \in H_f^q((\underline{X} + \underline{Y}, *)^K).$$

By Lemma 5.2 and the above diagram,

$$(g_A^* \circ h \circ e)_f (x_{i_1 j_1} \cdots x_{i_s j_s} \otimes y_{l_1} \cdots y_{l_t}) = (\psi^* \circ g_K^*)_f (x_{i_1 j_1} \cdots x_{i_s j_s} \otimes y_{l_1} \cdots y_{l_t}),$$

$$(g_A^* \circ h)_f (x_{i_1} \cdots x_{i_s} \otimes y_{l_1} \cdots y_{l_t}) = \left(\prod_{u=1}^{s} \prod_{k=1}^{c_{i_u}} z_{i_u j_u k} \right) \otimes y_{l_1} \cdots y_{l_t}.$$

Since $x_{i_1} \cdots x_{i_s} \otimes y_{l_1} \cdots y_{l_t}$ and $\left(\prod_{u=1}^{s} \prod_{k=1}^{c_{i_u}} z_{i_u j_u k} \right) \otimes y_{l_1} \cdots y_{l_t}$ are generators, $(g_A \circ h)_f^*$ is the inclusion of a direct summand into $H_f^q(Z_A)$. By Lemma 4.4, $(E_\infty^{0,q})_f$ and $H_f^q(X_A)$ are free modules of same rank, so h_f is an isomorphism. Since e_f is a surjection, so is ϕ_f^*.

For the second part of the lemma, suppose there is a polynomial $p \in \ker(\phi_f^*)$ not contained in $(x_{ij} - x_{i1})_{(i,j) \in \overline{\Omega}}$. Since ϕ_f^* is a degree 0 morphism, we assume $p = \sum_\alpha p_\alpha$ is a sum of monomials p_α of some fixed degree q. Then the p_α's are linearly dependent. So the rank of $H_f^q(X_A)$ is less than the rank of A^q, contradicting to Lemma 4.4. Thus $\ker(\phi_f^*) = (x_{ij} - x_{i1})_{(i,j) \in \overline{\Omega}}$.   □

Next we restate our main theorem (Theorem 1.2) and prove it.

Theorem 5.5 *Let A be a graded monomial ideal ring. Then there exists a space X_A such that $H_f^*(X_A)$ is ring isomorphic to A. Moreover, $H^*(X_A)$ is free split.*

Proof For the first part of the statement, the ring isomorphism $H_f^*(X_A) \cong A$ follows from Lemma 5.4.

In Lemma 4.1 we construct a free splitting $s_K \colon H_f^*(\underline{X} + \underline{Y}, *)^K \to H^*(\underline{X} + \underline{Y}, *)^K$ out of free splittings $s_{ij} \colon H_f^*(X_{ij}) \to H^*(X_{ij})$ and the identity maps on $H^*(Y_k)$. Define a map $s \colon H_f^*(X_A) \to H^*(X_A)$ by

$$
\begin{array}{ccc}
H_f^*((\underline{X} + \underline{Y}, *)^K) & \xrightarrow{\ s_K\ } & H^*((\underline{X} + \underline{Y}, *)^K) \\
\downarrow{\scriptstyle \phi_f^*} & & \downarrow{\scriptstyle \phi^*} \\
H_f^*(X_A) & \xrightarrow{\ \ \ s\ \ \ } & H^*(X_A)
\end{array}
$$

We need to show that s is well defined. By Lemma 5.4, ϕ_f^* is a surjection and $\ker(\phi_f^*)$ is generated by polynomials $x_{ij} - x_{i1}$ for $(i, j) \in \overline{\Omega}$. It suffices to show that $\phi^* \circ s_K(x_{ij} - x_{i1}) = 0$. Let $\tilde{x}_{ij} \in H^{2c_i}(X_{ij})$ and $\tilde{x}_{ij}' \in H_f^{2c_i}(X_{ij})$ be generators such that $s_{ij}(\tilde{x}_{ij}') = \tilde{x}_{ij}$. There is a string of equations

$$
\begin{aligned}
\phi^* \circ s_K(x_{ij} - x_{i1}) &= \phi^* \circ \mu(s_{ij}(\tilde{x}_{ij}') - s_{i1}(\tilde{x}_{i1}')) \\
&= \phi^* \circ \mu(\tilde{x}_{ij} - \tilde{x}_{i1}) \\
&= \phi^* \circ \delta^* \circ \mu(1 \otimes \cdots \otimes \tilde{x}_{ij} \otimes \cdots \otimes 1) \\
&= 0,
\end{aligned}
$$

where the first line is due to the definition of s_K, the third line is due to the naturality of μ, and the last line is due to the fact that δ and ϕ are two consecutive maps in the fibration sequence $X_A \xrightarrow{\phi} (\underline{X} + \underline{Y}, *)^K \xrightarrow{\delta} \prod_{(i,j) \in \overline{\Omega}} X_{ij}$. So s is well defined.

Obviously s is right inverse to the quotient map $H^*(X_A) \to H_f^*(X_A)$. Since ϕ_f^*, ϕ^* and s_K are multiplicative, so is s. Therefore s is a free splitting. $\square$

6 An example

Now we illustrate how to construct X_A for $A = \mathbb{Z}[x] \otimes \Lambda[y]/(x^2 y)$, where $|x| = 4$ and $|y| = 1$. First, polarize A by introducing two new variables x_1 and x_2 of degree 4

and let

$$A' = \mathbb{Z}[x_1, x_2] \otimes \Lambda[y]/(x_1 x_2 y).$$

Let K be the boundary of a 2–simplex. Then $(x_1 x_2 y)$ is the Stanley–Reisner ideal of K. Take

$$\underline{X} = \{K(\mathbb{Z}, 4), K(\mathbb{Z}, 4)\}, \quad \underline{Y} = \{S^1\}$$

and construct polyhedral product $(\underline{X} + \underline{Y}, *)^K$. By Proposition 3.3,

$$H_f^*((\underline{X} + \underline{Y}, *)^K) \cong \mathbb{Z}[x_1, x_2] \otimes \Lambda[y]/(x_1 x_2 y).$$

Define $\delta \colon (\underline{X} + \underline{Y}, *)^K \to K(\mathbb{Z}, 4)$ by $\delta_1(u_1, u_2, t) = u_2 \cdot u_1^{-1}$, and define X_A to be the fiber of δ. By Theorem 5.5, $H_f^*(X_A) \cong A$.

Next, we construct Z_A and g_A to illustrate the proof of the extension problem. In this case, take $\underline{Z} = \{(\mathbb{CP}^\infty)^2, (\mathbb{CP}^\infty)^2\}$. Denote the first $(\mathbb{CP}^\infty)^2$ by Z_1 and the second $(\mathbb{CP}^\infty)^2$ by Z_2. Then $H^*(Z_1) = \mathbb{Z}[z_{11}, z_{12}]$ and $H^*(Z_2) = \mathbb{Z}[z_{21}, z_{22}]$, where $|z_{ij}| = 2$ for $i, j \in \{1, 2\}$, and

$$H^*((\underline{Z} + \underline{Y}, *)^K) \cong \mathbb{Z}[z_{11}, z_{12}, z_{21}, z_{22}] \otimes \Lambda[y]/L',$$

where $L' = (z_{11} z_{21} y, z_{11} z_{22} y, z_{12} z_{21} y, z_{12} z_{22} y)$. Define

$$\tilde{\delta} \colon (\underline{Z} + \underline{Y}, *)^K \to (\mathbb{CP}^\infty)^2, \quad \tilde{\delta}(v_1, v_2, t) = v_2 \cdot v_1^{-1},$$

and define Z_A to be the fiber of $\tilde{\delta}$. Then $H_f^*(Z_A) \cong \mathbb{Z}[z_1, z_2] \otimes \Lambda[y]/L$, where $|z_1| = |z_2| = 2$ and $L = (z_1^2 y, z_2^2 y, z_1 z_2 y)$.

For $i = \{1, 2\}$, let $g_i \colon Z_i \to K(\mathbb{Z}, 4)$ be a map representing $z_{i1} z_{i2} \in H^4(Z_i)$, and let $h \colon S^1 \to S^1$ be the identity map. Then g_1, g_2 and h induce

$$g_K \colon (\underline{Z} + \underline{Y}, *)^K \to (\underline{X} + \underline{Y}, *)^K$$

such that $g_K^*(x_i) = z_{i1} z_{i2}$ and $g_K^*(y) = y$. Lemma 5.3 gives a map $g_A \colon Z_A \to X_A$ making the diagram

$$\begin{array}{ccc} Z_A & \longrightarrow & (\underline{Z} + \underline{Y}, *)^K \\ \downarrow{\scriptstyle g_A} & & \downarrow{\scriptstyle g_K} \\ X_A & \longrightarrow & (\underline{X} + \underline{Y}, *)^K \end{array}$$

commute.

Acknowledgements

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