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Michał Marcinkowski

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A short proof that the L^p–diameter of $\mathrm{Diff}_0(S, \text{area})$ is infinite

MICHAŁ MARCINKOWSKI

We give a short proof that the L^p–diameter of the group of area preserving diffeomorphisms isotopic to the identity of a compact surface is infinite.

37E30, 57K10, 58D05

1 Introduction

Let (M, g) be a Riemannian manifold and let μ be the measure induced by the metric g. We denote the group of all diffeomorphisms of M that preserve μ and are isotopic to the identity by $\mathrm{Diff}_0(M, \mu)$.

In [12] Shnirelman showed that the L^2–diameter of $\mathrm{Diff}_0(M, \mu)$ is finite if M is the n–dimensional ball for $n > 2$ see also Shnirelman [13]. Conjecturally, the same is true for any compact simply connected Riemannian manifold of dimension greater than 2 (it is stated in Eliashberg and Ratiu [8] without proof).

The situation is different for 2–dimensional manifolds. In this case it is customary to denote the measure induced by g by area. For simplicity, let us restrict the discussion to orientable compact connected Riemannian surfaces (S, g). Eliashberg and Ratiu [8] proved that the L^p–diameter ($p \geq 1$) of $\mathrm{Diff}_0(S, \text{area})$ is infinite if S is a surface with boundary. They show that the Calabi homomorphism is Lipschitz with respect to the L^p–norm. Later Gambaudo and Lagrange [9] obtained a similar result for a huge class of quasimorphisms on $\mathrm{Diff}_0(S, \text{area})$ if S is the closed disc (see as well Brandenbursky [3], Brandenbursky and Shelukhin [6] and Shelukhin [11] for more results concerning quasimorphisms and the L^p geometry). Their proof makes use of the braid group of the disc and inequalities relating the geometric intersection number of a braid and its word-length.

If S has negative Euler characteristic it is relatively easy to show that the L^p–diameter for $p \geq 1$ of $\mathrm{Diff}_0(S, \mathrm{area})$ is infinite; see Proposition 3.2 or Brandenbursky and Kędra [4, Theorem 1.2]. In the case of the torus one needs to know in addition that the group of Hamiltonian diffeomorphisms of the torus is simply connected, which is a nontrivial result from symplectic topology; see Brandenbursky and Shelukhin [7, Appendix A].

The last unsolved case was the sphere. Recently Brandenbursky and Shelukhin [7] showed that in this case the diameter is also infinite. Moreover, for each $p \geq 1$, $\mathrm{Diff}_0(S^2, \mathrm{area})$ contains quasi-isometrically embedded right-angled Artin groups (see Kim and Koberda [10]) and $\mathbb{R}^m$ for each natural m. Their arguments use some new tools along with the ideas from [9]. However, using intersection numbers in the case of the sphere requires considerably more work.

Our aim is to give a short and elementary proof of the following theorem:

Theorem 1 *Let* (S, g) *be a compact surface* (*with or without boundary*). *Then for every* $p \geq 1$ *the* L^p*–diameter of* $\mathrm{Diff}_0(S, \mathrm{area})$ *is infinite.*

Our method gives a unified proof for every compact surface S. It is partially inspired by [9]; in particular Lemma 5.2 can be seen as a generalization of an inequality obtained in [9] for the disk. The main simplification comes from the fact that instead of using the braid group and intersection numbers, we directly look at the geometry of the configuration space $C_n(S)$ with a certain complete metric described in Section 4. In Section 5 we relate the L^1–norm of $f \in \mathrm{Diff}_0(S, \mathrm{area})$ to an L^1–norm, defined by this complete metric, of the diffeomorphism on $C_n(S)$ induced by f. This allows us to apply the simple technique, described in Section 3, of showing the unboundedness of the L^p–norm in the case where the fundamental group of the manifold is complicated enough.

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2 The L^p–norm

Let (M, g) be a Riemannian manifold and let μ be a finite measure on M. Usually one assumes that μ is induced by g, even though the definition of an L^p–norm works as well if μ is any finite measure (then the L^p–norm could be a pseudonorm). We introduce here a more general definition as it is useful for stating results in Section 5.

Suppose $f \in \mathrm{Diff}_0(M, \mu)$ and let $X\colon M \to TM$ be a map to a tangent space of M such that $X(x) \in T_{f(x)}M$. One can think of X as a tangent vector to $\mathrm{Diff}_0(M, \mu)$ at the point f. The L^p–norm of X is defined by the formula

$$\|X\|_p = \left(\int_M |X(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Let $f_t \in \mathrm{Diff}_0(M, \mu)$ for $t \in [0, 1]$ be a smooth isotopy, ie it defines a smooth map $M \times [0, 1] \to M$. We always assume that isotopies are smooth. The L^p–length of $\{f_t\}$ is defined by

$$l_p(\{f_t\}) = \int_0^1 \|\dot{f}_t\|_p \, dt,$$

where $\dot{f}_t(x) = (d/ds) f_s(x)|_{s=t} \in T_{f_t(x)}M$. Note that if $p = 1$, then $\int_0^1 |\dot{f}_t(x)| \, dt$ is the length of the path $f_t(x)$, thus $l_1(\{f_t\})$ can be interpreted as the μ–average of the lengths of all paths $f_t(x)$.

Letting $f \in \mathrm{Diff}_0(M, \mu)$, we define the L^p–norm of f by

$$l_p(f) = \inf l_p(\{f_t\}),$$

where the infimum is taken over all smooth isotopies $f_t \in \mathrm{Diff}_0(M, \mu)$ connecting the identity on M with f. The assumption that f is μ–preserving was not used in the definition, but it is needed to show that l_p satisfies the triangle inequality.

The L^p–diameter of $\mathrm{Diff}_0(M, \mu)$ equals

$$\sup\{l_p(f) : f \in \mathrm{Diff}_0(M, \mu)\}.$$

It is worth noting that geodesics in $\mathrm{Diff}_0(M, \mu)$ with the L^2–metric are solutions of the Euler equations of an incompressible fluid. For more on the connection between the L^2–metric and hydrodynamics see [1].

3 The base case

In this section we present the basic method which can be used to show that, for $p \geq 1$, the L^p–diameter of $\mathrm{Diff}_0(M, \mu)$ is infinite if $\pi_1(M)$ is complicated enough.

Lemma 3.1 *Let X be a topological space and let $f_t \in \mathrm{Homeo}(X)$ for $t \in [0, 1]$ be a loop in $\mathrm{Homeo}(X)$ based at Id_X, ie $f_0 = f_1 = \mathrm{Id}_X$. Then for every $x \in X$, the loop $f_t(x)$ for $t \in [0, 1]$ is in the center of $\pi_1(X, x)$.*

Proof Let $x \in X$ and let γ_s for $s \in [0,1]$ be a loop in X based at x. Consider the map $\phi\colon S^1 \times S^1 \to X$ given by $(t,s) \mapsto f_t(\gamma_s)$, where $S^1 = [0,1]/0{\sim}1$. We have that $\phi(t,0) = f_t(x)$ and $\phi(0,s) = \gamma_s$. Thus loops $f_t(x)$ and γ_s are in the image of the torus $S^1 \times S^1$, therefore they commute. □

Let (M,g) be a Riemannian manifold. Suppose $h \in \pi_1(M)$. Let $l(h)$ denote the infimum over lengths of based loops in M that represent h. We denote by $Z(\pi_1(M))$ the center of $\pi_1(M)$.

Proposition 3.2 *Let (M,g) be a Riemannian manifold and μ the measure induced by g. Assume that for every r the set $\{h \in \pi_1(M) : l(h) < r\}$ is finite (it holds eg if M is compact) and $\pi_1(M)/Z(\pi_1(M))$ is infinite. Then for every $p \geq 1$ the L^p–diameter of $\mathrm{Diff}_0(M,\mu)$ is infinite.*

Proof By the Hölder inequality we can assume $p = 1$. Let $z \in M$ be a basepoint and let $h \in \pi_1(M,z)$. We represent h as a loop γ based at z.

Let U be a contractible neighborhood of z and let $f_t \in \mathrm{Diff}_0(M,\mu)$ for $t \in [0,1]$ be a finger-pushing isotopy that moves U all the way along γ. For a detailed construction see [5, proof of Lemma 3.1].

For every $x \in U$ we choose a path ϕ_x contained in U connecting z with x. We can assume that $l(\phi_x) < \mathrm{diam}(U)$, where $l(\phi_x)$ is the length of ϕ_x. We denote by ϕ_x^* the reverse of ϕ_x.

The isotopy f_t is defined so that it satisfies:

(1) For every $x \in U$, $f_1(x) = x$.

(2) For every $x \in U$, the concatenation of ϕ_x, $f_t(x)$ and ϕ_x^* is a loop based at z and its homotopy class equals h.

Let $f_h = f_1$ and define $L_h = \min\{l(hc) : c \in Z(\pi_1(M,z))\}$. We shall show that

$$\mu(U)(L_h - 2\,\mathrm{diam}(U)) \leq l_1(f_h).$$

Let g_t for $t \in [0,1]$ be any isotopy connecting the identity on M with f_h. Due to Lemma 3.1, for every $x \in U$ the paths $g_t(x)$ and $f_t(x)$ represent elements of $\pi_1(M,x)$ that differ by an element of the center. Thus the concatenation of ϕ_x, $g_t(x)$ and ϕ_x^* represents an element of the form $hc \in \pi_1(M,z)$ where $c \in Z(\pi_1(M,z))$. Since $l(\phi_x) < \mathrm{diam}(U)$, we have that $l(g_t(x)) \geq L_h - 2\,\mathrm{diam}(U)$. Indeed, otherwise the concatenation of ϕ_x, $g_t(x)$ and ϕ_x^* would be a loop of length less then $L_h \le l(hc)$, which is impossible.

Since $l(g_t(x)) = \int_0^1 |\dot{g}_t(x)|\,dt$, we have

$$\mu(U)(L_h - 2\,\mathrm{diam}(U)) \le \int_U \int_0^1 |\dot{g}_t(x)|\,dt\,dx \le \int_M \int_0^1 |\dot{g}_t(x)|\,dt\,dx = l_1(\{g_t\}).$$

The isotopy g_t was arbitrary, therefore $\mu(U)(L_h - 2\,\mathrm{diam}(U)) \le l_1(f_h)$.

By assumption, for every r the set $S_h = \{h \in \pi_1(M) : l(h) < r\}$ is finite. Therefore, since $\pi_1(M)/Z(\pi_1(M))$ is infinite, there exists h such that the coset $hZ(\pi_1(M))$ does not intersect S_h. For such h we have $L_h \ge r$. Since the set U does not depend on the choice of h, and L_h can be arbitrary large, we conclude that the L^1–diameter of $\mathrm{Diff}_0(M,\mu)$ is infinite. □

In particular, Proposition 3.2 can be applied when (S,g) is a compact surface of negative Euler characteristic (then $\pi_1(S)$ is infinite and has trivial center). Unfortunately, it says nothing about the L^p–diameter of $\mathrm{Diff}_0(S,\mathrm{area})$ for the remaining surfaces. Our main goal is to find an argument which is still based on the proof of Proposition 3.2, but works for any compact surface S.

To this end, one could pass to the configuration space of n ordered points in S, denoted by $C_n(S) \subset S^n$, with the product Riemannian metric g^n. Its fundamental group is the pure braid group $P_n(S)$, and $P_n(S)/Z(P_n(S))$ is infinite for every S if $n > 3$. However, the problem with this space is that every braid $P_n(S)$ can be represented as a based loop in $(C_n(S), g^n)$ of length at most $2n\,\mathrm{diam}(S) + 1$, thus one cannot apply Proposition 3.2.

We solve this problem by changing the metric on $C_n(S)$. We describe it, in a slightly more general setting, in the next section.

4 A complete metric on a manifold with removed submanifolds

Let (M,g) be a compact Riemannian manifold and let $D = \bigcup_{i=1}^k D_i$, where the D_i are submanifolds of M. The aim of this paragraph is to construct a metric on $M - D$ satisfying the following property: for every L the number of elements in $\pi_1(M - D)$

that can be represented by a based loop of length less then L is finite. For $x \in M$ denote by $d(x)$ the distance of x to D, that is

$$d(x) = d_g(x, D) = \min\{d_g(x, D_i) : i = 1, \dots, k\},$$

where d_g is the metric on M induced by g.

Rescaling g by $1/d$ we define a new quadratic form g_b on the tangent space of $M - D$ by

$$|v|_{g_b} = \frac{|v|_g}{d(x)},$$

where $v \in T_x(M - D)$ is a vector tangent to a point $x \in M - D$.

Note that $d(x)$, and consequently g_b, are not differentiable. They are only continuous. In this case g_b is called a C^0–Riemannian metric and a smooth manifold with such a quadratic form is called a C^0–Riemannian manifold. A C^0–Riemannian structure allows us to define lengths of paths and a metric d on the underlying manifold. The topology induced by d is equal to the manifold topology.

Lemma 4.1 *$M - D$ with the metric g_b is a complete C^0–Riemannian manifold.*

Proof Let $N = (M - D, g_b)$ and let $B_N(x, r)$ denote the closed ball in N of radius r and center $x \in N$. To show completeness we must show that for every $x \in N$ the ball $B_N\left(x, \frac{1}{2}\right)$ is compact.

Let $x \in N$. We shall show that the distance from $B_N\left(x, \frac{1}{2}\right)$ to D is at least $\frac{1}{2}d(x)$:

$$B_N\left(x, \tfrac{1}{2}\right) \subset L := \left\{y \in N : d(y) \geq \tfrac{1}{2}d(x)\right\}.$$

Since L is compact, it follows that $B_N\left(x, \frac{1}{2}\right)$ is compact.

Suppose $y \in B_N\left(x, \frac{1}{2}\right)$ and $d(y) < d(x)$ (otherwise obviously $y \in L$). Let $\epsilon > 0$ and let $\gamma\colon [0, l] \to N$ be a path connecting x with y such that $|\dot{\gamma}(t)|_{g_b} = 1$ for $t \in [0, l]$ and $l < \frac{1}{2} + \epsilon$.

Let

$$t_0 = \sup\{t \in [0, l] : d(\gamma(t)) \geq d(x)\},$$

ie t_0 is the last time when $d(\gamma(t_0)) = d(x)$. For $t \geq t_0$, we have

$$|\dot{\gamma}(t)|_g = |\dot{\gamma}(t)|_{g_b} d(\gamma(t)) = d(\gamma(t)) \leq d(x).$$

Let γ' be the restriction of γ to the interval $[t_0, l]$. Let $l_g(\gamma')$ be the length of γ' in the metric g. Since $|\dot{\gamma}(t)|_g \le d(x)$, we have

$$l_g(\gamma') \le (l - t_0)d(x) \le \left(\tfrac{1}{2} + \epsilon\right)d(x).$$

Therefore the distance of y to D in g is at least

$$d(y) \ge d(\gamma(t_0)) - l_g(\gamma') \ge d(x) - \left(\tfrac{1}{2} + \epsilon\right)d(x) = \tfrac{1}{2}d(x) - \epsilon d(x).$$

Since ϵ is arbitrarily small, $y \in L$ and therefore $B_N\left(x, \frac{1}{2}\right) \subset L$. □

Before we proceed we need the following simple lemma. Note that this lemma would be standard if $(M - D, g_b)$ were a complete Riemannian manifold.

Lemma 4.2 *Let $N = (M - D, g_b)$ and let $\widetilde{N}$ be the universal cover of N with the pulled-back C^0–Riemannian metric. Then every closed ball in $\widetilde{N}$ is compact.*

Proof By the Weierstrass approximation theorem there exists $C \in \mathbb{R}$ and a smooth function $f : N \to \mathbb{R}$ such that $C^{-1}f(x) < 1/d(x) < Cf(x)$ for every $x \in N$. Let g_s be a Riemannian metric defined by $|v|_{g_s} = f(x)|v|_g$, where $v \in T_xN$. Then $C^{-1}|v|_{g_s} < |v|_{g_b} < C|v|_{g_s}$, thus the metrics induced by g_b and g_s are equivalent. By Lemma 4.1, (N, g_s) is a complete Riemannian manifold and it is a standard fact that closed balls in the universal cover of (N, g_s) are compact. Clearly it holds as well for (N, g_b), since the metrics defined by pullbacks of g_s and g_b to the universal cover are equivalent. □

Let $h \in \pi_1(M - D)$. Denote by $l(h)$ the infimum of lengths (with respect to g_b) of based loops representing $h \in \pi_1(M - D)$.

Lemma 4.3 *For every r, the set $\{h \in \pi_1(M - D) : l(h) < r\}$ is finite.*

Proof Let $N = (M - D, g_b)$, let $x \in N$ be a basepoint and let $p \colon \widetilde{N} \to N$ be the universal cover of N. Choose $y \in p^{-1}(x)$. The preimage $p^{-1}(x)$ is discrete and $B_{\widetilde{N}}(y, r) \subset \widetilde{N}$ is compact by Lemma 4.2. Thus $p^{-1}(x) \cap B_{\widetilde{N}}(y, r)$ is finite for every r and therefore $\{h \in \pi_1(N) : l(h) < r\}$ is finite. □

5 A Lipschitz embedding

In this section we focus on the particular case where $M - D$ is a configuration space. Let (S, g) be a compact Riemannian surface and g^n be the product metric on S^n.

Let $D_{ij} = \{(x_1, \dots, x_n) \in S^n : x_i = x_j\}$. Denote by $C_n(S) = S^n - \bigcup_{i,j} D_{ij}$ the configuration space of n ordered points in S. On S^n and $C_n(S)$ we consider the measure induced by the product metric g^n.

We shall now find a formula for $d_{g^n}(x, D_{ij})$ in terms of the metric on S. Let $x = (x_1, \dots, x_n) \in S^n$ and let m be the midpoint of a geodesic connecting x_i with x_j. If we start moving points x_i and x_j towards m with constant speed, we get a geodesic in S^n connecting x with the closest point in D_{ij}. Since $d_g(m, x_i) = d_g(m, x_j) = \frac{1}{2} d_g(x_i, x_j)$ and we are in the product metric,

$$d_{g^n}(x, D_{ij}) = \sqrt{d_g(m, x_i)^2 + d_g(m, x_j)^2} = \frac{1}{\sqrt{2}} d_g(x_i, x_j).$$

The distance function d has the form

$$d(x) = \frac{1}{\sqrt{2}} \min\{d_g(x_i, x_j) : 1 \le i < j \le n\}.$$

Let $g_b = (g^n)_b$ be the metric on $C_n(S)$ defined in the previous section, namely $|v|_{g_b} = |v|_{g^n}/d(x)$, where $v \in T_x(C_n(S))$.

Let us fix a point $p \in S$ and let $x = (x_1, \dots, x_{n-1}) \in S^{n-1}$. Then $(p, x) \in S^n$ and $d((p, x))$ is the minimum over $(1/\sqrt{2}) d_g(p, x_i)$ for $1 \le i \le n-1$ and $(1/\sqrt{2}) d_g(x_i, x_j)$ for $1 \le i < j \le n-1$.

We need the following technical lemma.

Lemma 5.1 *There exists $C \in \mathbb{R}$ such that for every $p \in S$ we have*

$$\int_{S^{n-1}} \frac{1}{d((p, x))} \, dx \le C.$$

Proof It can be easily seen using polar coordinates that there exists C' such that for every $p \in D^2$, where D^2 is the euclidean disc,

$$\int_D \frac{1}{|p-x|} \, dx < C'.$$

Since such C' exists for a disc, we have a similar bound for every compact surface S: for every $p \in S$

$$\int_S \frac{1}{d_g(p, x)} \, dx < C'.$$

After integrating over all possible $p \in S$ (we assume area$(S) = 1$),

$$\int_{S^2} \frac{1}{d_g(p, x)} \, dp \, dx < C'.$$

Let $x=(x_1,\dots,x_{n-1})$. Since $d((p,x))$ is the minimum over $(1/\sqrt{2})d_g(p,x_i)$ for $i=1,\dots,n-1$ and $(1/\sqrt{2})d_g(x_i,x_j)$ for $1\le i<j\le n-1$,

$$\frac{1}{d((p,x))}\le\sum_i\frac{\sqrt{2}}{d_g(p,x_i)}+\sum_{i\neq j}\frac{\sqrt{2}}{d_g(x_i,x_j)}.$$

Thus

$$\begin{aligned}\int_{S^{n-1}}\frac{1}{d((p,x))}\,dx&\le\sum_i\int_{S^{n-1}}\frac{\sqrt{2}}{d_g(p,x_i)}\,dx+\sum_{i\neq j}\int_{S^{n-1}}\frac{\sqrt{2}}{d_g(x_i,x_j)}\,dx\\&=(n-1)\int_S\frac{\sqrt{2}}{d_g(p,x)}\,dx+\tfrac{1}{2}n(n-1)\int_{S^2}\frac{\sqrt{2}}{d_g(x_1,x_2)}\,dx_1\,dx_2\\&\le\sqrt{2}(n-1)C'+\frac{n(n-1)}{\sqrt{2}}C'=:C.\end{aligned}$$

□

Let μ be the measure on $C_n(S)$ induced by the product metric g^n. A diffeomorphism $f\in\mathrm{Diff}_0(S,\text{area})$ defines a product diffeomorphism $f_*\in\mathrm{Diff}_0(C_n(S),\mu)$. Namely, for $x=(x_1,\dots,x_n)\in S^n$ we have $f_*(x)=(f(x_1),\dots,f(x_n))$. Thus we have a product embedding $\mathrm{Diff}_0(S,\text{area})\hookrightarrow\mathrm{Diff}_0(C_n(S),\mu)$.

On $\mathrm{Diff}_0(C_n(S),\mu)$ we consider the L^1–norm defined by the metric g_b and the measure μ. Note that here we are in the case where g_b and μ are not compatible, that is, the measure induced by g_b and the measure μ are different.

The following lemma provides a link between the L^1–norm on $\mathrm{Diff}_0(S,\text{area})$ and the L^1–norm on $\mathrm{Diff}_0(C_n(S),\mu)$ defined above. Note that in the proof it is essential that f preserves the area on S.

Lemma 5.2 *The product embedding* $\mathrm{Diff}_0(S,\text{area})\hookrightarrow\mathrm{Diff}_0(C_n(S),\mu)$ *is Lipschitz, ie there exists C such that $l_1(f_*)\le C\,l_1(f)$.*

Proof Let $f\in\mathrm{Diff}_0(S,\text{area})$ and let $X\colon S\to TS$ such that $X(x)\in T_{f(x)}S$. For $x=(x_1,\dots,x_n)\in C_n(S)$ we define $X_*(x)=(X(x_1),\dots,X(x_n))\in T_{f_*(x)}C_n(S)$.

The set $\bigcup_{i,j}D_{ij}\subset S^n$ is of measure zero. This means that we can regard $|X_*(x)|_{g_b}$ as a measurable function defined on S^n. Thus in what follows, we integrate $|X_*(x)|_{g_b}$ over S^n with the product measure rather then over $C_n(S)$.

To prove the lemma it is enough to show that there exists C such that for every $f\in\mathrm{Diff}_0(S,\text{area})$ and every map $X\colon S\to TS$ such that $X(x)\in T_{f(x)}S$ the following inequality holds:

$$\|X_*\|_1\le C\|X\|_1.$$

Recall that by definition $\|X_*\|_1 = \int_{S^n} |X_*(x)|_{g_b}\, dx$. We have

$$\begin{aligned}\int_{S^n} |X_*(x)|_{g_b}\, dx = \int_{S^n} \frac{|X_*(x)|_{g^n}}{d(f_*(x))}\, dx = \int_{S^n} \frac{\sqrt{|X(x_1)|_g^2 + \cdots + |X(x_n)|_g^2}}{d(f_*(x))}\, dx \\ \le \int_{S^n} \frac{|X(x_1)|_g + \cdots + |X(x_n)|_g}{d(f_*(x))}\, dx = n \int_{S^n} \frac{|X(x_1)|_g}{d(f_*(x))}\, dx.\end{aligned}$$

Since f_* preserves the measure on S^n,

$$\begin{aligned}\int_{S^n} \frac{|X(x_1)|_g}{d(f_*(x))}\, dx &= \int_{S^n} \frac{|X \circ f^{-1}(x_1)|_g}{d(x)}\, dx \\ &= \int_S |X \circ f^{-1}(x_1)|_g \left(\int_{S^{n-1}} \frac{1}{d(x_1, x)}\, dx \right) dx_1 \\ &\le C \int_S |X \circ f^{-1}(x_1)|_g\, dx_1 \qquad \text{(by Lemma 5.1)} \\ &= C \int_S |X(x_1)|_g dx_1 = C\|X\|_1. \qquad \square\end{aligned}$$

6 Proof of the theorem

Theorem 1 *Let (S, g) be a compact surface (with or without boundary). Then for every $p \ge 1$ the L^p–diameter of $\mathrm{Diff}_0(S, \text{area})$ is infinite.*

Proof By the Hölder inequality we can assume $p = 1$. Fix $n > 3$.

Let $z = (z_1, \dots, z_n) \in C_n(S)$ and let $P_n(S) = \pi_1(C_n(S), z)$ denote the pure braid group on n strings. Suppose $U_i \subset S$ are disjoint discs such that $z_i \in U_i$, then let $U = U_1 \times U_2 \times \cdots \times U_n \subset C_n(S)$.

Choose $h \in P_n(S)$ and γ a loop in $C_n(S)$ representing h. Let $f_t \in \mathrm{Diff}_0(S, \text{area})$ for $t \in [0, 1]$ be an isotopy such that $(f_t)_* \in \mathrm{Diff}_0(C_n(S), \mu)$ moves U all the way along γ and has properties (1) and (2) from the proof of Proposition 3.2. Let $f_h = f_1$.

It is convenient to imagine that f_t moves U_i along the trajectory of z_i given by γ. In fact, to construct f_t satisfying the above properties for a general $h \in P_n(S)$, it is enough to do it for a given finite set of generators of $P_n(S)$ (or generators of the full braid group $B_n(S)$). In [2] one can find a set of generators of $B_n(S)$ for which the construction of f_t is straightforward.

Recall that on $C_n(S)$ we consider the complete metric g_b. By Lemma 4.3, the set $\{h \in \pi_1(C_n(S)) : l(h) < r\}$ is finite for every r and $P_n(S)/Z(P_n(S))$ is infinite. It follows from the proof of Proposition 3.2 that $l_1((f_h)_*)$ can be arbitrarily large.

Therefore, due to Lemma 5.2, $l_1(f_h)$ can be arbitrarily large. Thus the L^1–diameter of $\mathrm{Diff}_0(S, \text{area})$ is infinite. □

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Institute of Mathematics, University of Wrocław
Wrocław, Poland

marcinkow@math.uni.wroc.pl

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