A short proof that the $L^p$–diameter of $\text{Diff}_0(S, \text{area})$ is infinite

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We give a short proof that the $L^p$–diameter of the group of area preserving diffeomorphisms isotopic to the identity of a compact surface is infinite.

37E30, 57K10, 58D05

1 Introduction

Let $(M, g)$ be a Riemannian manifold and let $\mu$ be the measure induced by the metric $g$. We denote the group of all diffeomorphisms of $M$ that preserve $\mu$ and are isotopic to the identity by $\text{Diff}_0(M, \mu)$.

In [12] Shnirelman showed that the $L^2$–diameter of $\text{Diff}_0(M, \mu)$ is finite if $M$ is the $n$–dimensional ball for $n > 2$ see also Shnirelman [13]. Conjecturally, the same is true for any compact simply connected Riemannian manifold of dimension greater than 2 (it is stated in Eliashberg and Ratiu [8] without proof).

The situation is different for 2–dimensional manifolds. In this case it is customary to denote the measure induced by $g$ by area. For simplicity, let us restrict the discussion to orientable compact connected Riemannian surfaces $(S, g)$. Eliashberg and Ratiu [8] proved that the $L^p$–diameter ($p \geq 1$) of $\text{Diff}_0(S, \text{area})$ is infinite if $S$ is a surface with boundary. They show that the Calabi homomorphism is Lipschitz with respect to the $L^p$–norm. Later Gambaudo and Lagrange [9] obtained a similar result for a huge class of quasimorphisms on $\text{Diff}_0(S, \text{area})$ if $S$ is the closed disc (see as well Brandenbursky [3], Brandenbursky and Shelukhin [6] and Shelukhin [11] for more results concerning quasimorphisms and the $L^p$ geometry). Their proof makes use of the braid group of the disc and inequalities relating the geometric intersection number of a braid and its word-length.

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If $S$ has negative Euler characteristic it is relatively easy to show that the $L^p$–diameter for $p \geq 1$ of $\text{Diff}_0(S, \text{area})$ is infinite; see Proposition 3.2 or Brandenbursky and Kędra [4, Theorem 1.2]. In the case of the torus one needs to know in addition that the group of Hamiltonian diffeomorphisms of the torus is simply connected, which is a nontrivial result from symplectic topology; see Brandenbursky and Shelukhin [7, Appendix A].

The last unsolved case was the sphere. Recently Brandenbursky and Shelukhin [7] showed that in this case the diameter is also infinite. Moreover, for each $p \geq 1$, $\text{Diff}_0(S^2, \text{area})$ contains quasi-isometrically embedded right-angled Artin groups (see Kim and Koberda [10]) and $\mathbb{R}^m$ for each natural $m$. Their arguments use some new tools along with the ideas from [9]. However, using intersection numbers in the case of the sphere requires considerably more work.

Our aim is to give a short and elementary proof of the following theorem:

**Theorem 1** Let $(S, g)$ be a compact surface (with or without boundary). Then for every $p \geq 1$ the $L^p$–diameter of $\text{Diff}_0(S, \text{area})$ is infinite.

Our method gives a unified proof for every compact surface $S$. It is partially inspired by [9]; in particular Lemma 5.2 can be seen as a generalization of an inequality obtained in [9] for the disk. The main simplification comes from the fact that instead of using the braid group and intersection numbers, we directly look at the geometry of the configuration space $C_n(S)$ with a certain complete metric described in Section 4. In Section 5 we relate the $L^1$–norm of $f \in \text{Diff}_0(S, \text{area})$ to an $L^1$–norm, defined by this complete metric, of the diffeomorphism on $C_n(S)$ induced by $f$. This allows us to apply the simple technique, described in Section 3, of showing the unboundedness of the $L^p$–norm in the case where the fundamental group of the manifold is complicated enough.

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## 2 The $L^p$–norm

Let $(M, g)$ be a Riemannian manifold and let $\mu$ be a finite measure on $M$. Usually one assumes that $\mu$ is induced by $g$, even though the definition of an $L^p$–norm works as well if $\mu$ is any finite measure (then the $L^p$–norm could be a pseudonorm). We introduce here a more general definition as it is useful for stating results in Section 5.
Suppose $f \in \text{Diff}_0(M, \mu)$ and let $X : M \to TM$ be a map to a tangent space of $M$ such that $X(x) \in T_{f(x)}M$. One can think of $X$ as a tangent vector to $\text{Diff}_0(M, \mu)$ at the point $f$. The $L^p$–norm of $X$ is defined by the formula

$$
\|X\|_p = \left( \int_M |X(x)|^p \, dx \right)^{\frac{1}{p}}.
$$

Let $f_t \in \text{Diff}_0(M, \mu)$ for $t \in [0, 1]$ be a smooth isotopy, ie it defines a smooth map $M \times [0, 1] \to M$. We always assume that isotopies are smooth. The $L^p$–length of $\{f_t\}$ is defined by

$$
l_p(\{f_t\}) = \int_0^1 \|\dot{f}_t\|_p \, dt,
$$

where $\dot{f}_t(x) = (d/ds)f_s(x)|_{s=t} \in T_{f_t(x)}M$. Note that if $p = 1$, then $\int_0^1 |\dot{f}_t(x)| \, dt$ is the length of the path $f_t(x)$, thus $l_1(\{f_t\})$ can be interpreted as the $\mu$–average of the lengths of all paths $f_t(x)$.

Letting $f \in \text{Diff}_0(M, \mu)$, we define the $L^p$–norm of $f$ by

$$
l_p(f) = \inf_{\{f_t\}} l_p(\{f_t\}),
$$

where the infimum is taken over all smooth isotopies $f_t \in \text{Diff}_0(M, \mu)$ connecting the identity on $M$ with $f$. The assumption that $f$ is $\mu$–preserving was not used in the definition, but it is needed to show that $l_p$ satisfies the triangle inequality.

The $L^p$–diameter of $\text{Diff}_0(M, \mu)$ equals

$$
\sup\{l_p(f) : f \in \text{Diff}_0(M, \mu)\}.
$$

It is worth noting that geodesics in $\text{Diff}_0(M, \mu)$ with the $L^2$–metric are solutions of the Euler equations of an incompressible fluid. For more on the connection between the $L^2$–metric and hydrodynamics see [1].

### 3 The base case

In this section we present the basic method which can be used to show that, for $p \geq 1$, the $L^p$–diameter of $\text{Diff}_0(M, \mu)$ is infinite if $\pi_1(M)$ is complicated enough.

**Lemma 3.1** Let $X$ be a topological space and let $f_t \in \text{Homeo}(X)$ for $t \in [0, 1]$ be a loop in $\text{Homeo}(X)$ based at $\text{Id}_X$, ie $f_0 = f_1 = \text{Id}_X$. Then for every $x \in X$, the loop $f_t(x)$ for $t \in [0, 1]$ is in the center of $\pi_1(X, x)$.
Proof Let \( x \in X \) and let \( \gamma_s \) for \( s \in [0, 1] \) be a loop in \( X \) based at \( x \). Consider the map \( \phi : S^1 \times S^1 \to X \) given by \( (t, s) \mapsto f_t(\gamma_s) \), where \( S^1 = [0, 1]/0 \sim 1 \). We have that \( \phi(t, 0) = f_t(x) \) and \( \phi(0, s) = \gamma_s \). Thus loops \( f_t(x) \) and \( \gamma_s \) are in the image of the torus \( S^1 \times S^1 \), therefore they commute.

Let \((M, g)\) be a Riemannian manifold. Suppose \( h \in \pi_1(M) \). Let \( l(h) \) denote the infimum over lengths of based loops in \( M \) that represent \( h \). We denote by \( Z(\pi_1(M)) \) the center of \( \pi_1(M) \).

**Proposition 3.2** Let \((M, g)\) be a Riemannian manifold and \( \mu \) the measure induced by \( g \). Assume that for every \( r \) the set \( \{ h \in \pi_1(M) : l(h) < r \} \) is finite (it holds eg if \( M \) is compact) and \( \pi_1(M)/Z(\pi_1(M)) \) is infinite. Then for every \( p \geq 1 \) the \( L^p \)–diameter of \( \text{Diff}_0(M, \mu) \) is infinite.

**Proof** By the Hölder inequality we can assume \( p = 1 \). Let \( z \in M \) be a basepoint and let \( h \in \pi_1(M, z) \). We represent \( h \) as a loop \( \gamma \) based at \( z \).

Let \( U \) be a contractible neighborhood of \( z \) and let \( f_t \in \text{Diff}_0(M, \mu) \) for \( t \in [0, 1] \) be a finger-pushing isotopy that moves \( U \) all the way along \( \gamma \). For a detailed construction see [5, proof of Lemma 3.1].

For every \( x \in U \) we choose a path \( \phi_x \) contained in \( U \) connecting \( z \) with \( x \). We can assume that \( l(\phi_x) < \text{diam}(U) \), where \( l(\phi_x) \) is the length of \( \phi_x \). We denote by \( \phi_x^* \) the reverse of \( \phi_x \).

The isotopy \( f_t \) is defined so that it satisfies:

1. For every \( x \in U \), \( f_1(x) = x \).
2. For every \( x \in U \), the concatenation of \( \phi_x, f_t(x) \) and \( \phi_x^* \) is a loop based at \( z \) and its homotopy class equals \( h \).

Let \( f_h = f_1 \) and define \( L_h = \min \{ l(hc) : c \in Z(\pi_1(M, z)) \} \). We shall show that

\[
\mu(U)(L_h - 2 \text{ diam}(U)) \leq l_1(f_h).
\]

Let \( g_t \) for \( t \in [0, 1] \) be any isotopy connecting the identity on \( M \) with \( f_h \). Due to Lemma 3.1, for every \( x \in U \) the paths \( g_t(x) \) and \( f_t(x) \) represent elements of \( \pi_1(M, x) \) that differ by an element of the center. Thus the concatenation of \( \phi_x, g_t(x) \) and \( \phi_x^* \) represents an element of the form \( hc \in \pi_1(M, z) \) where \( c \in Z(\pi_1(M, z)) \). Since \( l(\phi_x) < \text{diam}(U) \), we have that \( l(g_t(x)) \geq L_h - 2 \text{ diam}(U) \). Indeed, otherwise the
concatenation of \( \phi_x, g_t(x) \) and \( \phi_x^* \) would be a loop of length less then \( L_h \leq l(hc) \), which is impossible.

Since \( l(g_t(x)) = \int_0^1 |\dot{g}_t(x)| \, dt \), we have

\[
\mu(U)(L_h - 2 \text{diam}(U)) \leq \int_U \int_0^1 |\dot{g}_t(x)| \, dt \, dx \leq \int_M \int_0^1 |\dot{g}_t(x)| \, dt \, dx = l_1(\{g_t\}).
\]

The isotopy \( g_t \) was arbitrary, therefore \( \mu(U)(L_h - 2 \text{diam}(U)) \leq l_1(f_h) \).

By assumption, for every \( r \) the set \( S_h = \{h \in \pi_1(M) : l(h) < r\} \) is finite. Therefore, since \( \pi_1(M)/Z(\pi_1(M)) \) is infinite, there exists \( h \) such that the coset \( hZ(\pi_1(M)) \) does not intersect \( S_h \). For such \( h \) we have \( L_h \geq r \). Since the set \( U \) does not depend on the choice of \( h \), and \( L_h \) can be arbitrary large, we conclude that the \( L^1 \)–diameter of \( \text{Diff}_0(M, \mu) \) is infinite. \( \square \)

In particular, Proposition 3.2 can be applied when \((S, g)\) is a compact surface of negative Euler characteristic (then \( \pi_1(S) \) is infinite and has trivial center). Unfortunately, it says nothing about the \( L^p \)–diameter of \( \text{Diff}_0(S, \text{area}) \) for the remaining surfaces. Our main goal is to find an argument which is still based on the proof of Proposition 3.2, but works for any compact surface \( S \).

To this end, one could pass to the configuration space of \( n \) ordered points in \( S \), denoted by \( C_n(S) \subset S^n \), with the product Riemannian metric \( g^n \). Its fundamental group is the pure braid group \( P_n(S) \), and \( P_n(S)/Z(P_n(S)) \) is infinite for every \( S \) if \( n > 3 \). However, the problem with this space is that every braid \( P_n(S) \) can be represented as a based loop in \((C_n(S), g^n)\) of length at most \( 2n \text{diam}(S) + 1 \), thus one cannot apply Proposition 3.2.

We solve this problem by changing the metric on \( C_n(S) \). We describe it, in a slightly more general setting, in the next section.

## 4 A complete metric on a manifold with removed submanifolds

Let \((M, g)\) be a compact Riemannian manifold and let \( D = \bigcup_{i=1}^k D_i \), where the \( D_i \) are submanifolds of \( M \). The aim of this paragraph is to construct a metric on \( M - D \) satisfying the following property: for every \( L \) the number of elements in \( \pi_1(M - D) \)
that can be represented by a based loop of length less then \( L \) is finite. For \( x \in M \) denote by \( d(x) \) the distance of \( x \) to \( D \), that is
\[
d(x) = d_g(x, D) = \min\{d_g(x, D_i) : i = 1, \ldots, k\},
\]
where \( d_g \) is the metric on \( M \) induced by \( g \).

Rescaling \( g \) by \( 1/d \) we define a new quadratic form \( g_b \) on the tangent space of \( M - D \) by
\[
|v|_{g_b} = \frac{|v|_g}{d(x)},
\]
where \( v \in T_x(M - D) \) is a vector tangent to a point \( x \in M - D \).

Note that \( d(x) \), and consequently \( g_b \), are not differentiable. They are only continuous. In this case \( g_b \) is called a \( C^0 \)-Riemannian metric and a smooth manifold with such a quadratic form is called a \( C^0 \)-Riemannian manifold. A \( C^0 \)-Riemannian structure allows us to define lengths of paths and a metric \( d \) on the underlying manifold. The topology induced by \( d \) is equal to the manifold topology.

**Lemma 4.1** \( M - D \) with the metric \( g_b \) is a complete \( C^0 \)-Riemannian manifold.

**Proof** Let \( N = (M - D, g_b) \) and let \( B_N(x, r) \) denote the closed ball in \( N \) of radius \( r \) and center \( x \in N \). To show completeness we must show that for every \( x \in N \) the ball \( B_N(x, \frac{1}{2}) \) is compact.

Let \( x \in N \). We shall show that the distance from \( B_N(x, \frac{1}{2}) \) to \( D \) is at least \( \frac{1}{2} d(x) \):
\[
B_N(x, \frac{1}{2}) \subset L := \{ y \in N : d(y) \geq \frac{1}{2} d(x) \}.
\]
Since \( L \) is compact, it follows that \( B_N(x, \frac{1}{2}) \) is compact.

Suppose \( y \in B_N(x, \frac{1}{2}) \) and \( d(y) < d(x) \) (otherwise obviously \( y \in L \)). Let \( \epsilon > 0 \) and let \( \gamma : [0, l] \to N \) be a path connecting \( x \) with \( y \) such that \( |\dot{\gamma}(t)|_{g_b} = 1 \) for \( t \in [0, l] \) and \( l < \frac{1}{2} + \epsilon \).

Let
\[
t_0 = \sup\{t \in [0, l] : d(\gamma(t)) \geq d(x)\},
\]
ie \( t_0 \) is the last time when \( d(\gamma(t_0)) = d(x) \). For \( t \geq t_0 \), we have
\[
|\dot{\gamma}(t)|_g = |\dot{\gamma}(t)|_{g_b} d(\gamma(t)) = d(\gamma(t)) \leq d(x).
\]
Let $\gamma'$ be the restriction of $\gamma$ to the interval $[t_0, l]$. Let $l_g(\gamma')$ be the length of $\gamma'$ in the metric $g$. Since $|\dot{\gamma}(t)|_g \leq d(x)$, we have

$$l_g(\gamma') \leq (l - t_0)d(x) \leq \left(\frac{1}{2} + \epsilon\right)d(x).$$

Therefore the distance of $y$ to $D$ in $g$ is at least

$$d(y) \geq d(\gamma(t_0)) - l_g(\gamma') \geq d(x) - \left(\frac{1}{2} + \epsilon\right)d(x) = \frac{1}{2}d(x) - \epsilon d(x).$$

Since $\epsilon$ is arbitrarily small, $y \in L$ and therefore $B_N(x, \frac{1}{2}) \subset L$. \hfill $\square$

Before we proceed we need the following simple lemma. Note that this lemma would be standard if $(M - D, g_b)$ were a complete Riemannian manifold.

**Lemma 4.2** Let $N = (M - D, g_b)$ and let $\tilde{N}$ be the universal cover of $N$ with the pulled-back $C^0$–Riemannian metric. Then every closed ball in $\tilde{N}$ is compact.

**Proof** By the Weierstrass approximation theorem there exists $C \in \mathbb{R}$ and a smooth function $f : N \to \mathbb{R}$ such that $C^{-1}f(x) < 1/d(x) < Cf(x)$ for every $x \in N$. Let $g_s$ be a Riemannian metric defined by $|v|_{g_s} = f(x)|v|_g$, where $v \in T_xN$. Then $C^{-1}|v|_{g_s} < |v|_{g_b} < C|v|_{g_s}$, thus the metrics induced by $g_b$ and $g_s$ are equivalent. By Lemma 4.1, $(N, g_s)$ is a complete Riemannian manifold and it is a standard fact that closed balls in the universal cover of $(N, g_s)$ are compact. Clearly it holds as well for $(N, g_b)$, since the metrics defined by pullbacks of $g_s$ and $g_b$ to the universal cover are equivalent. \hfill $\square$

Let $h \in \pi_1(M - D)$. Denote by $l(h)$ the infimum of lengths (with respect to $g_b$) of based loops representing $h \in \pi_1(M - D)$.

**Lemma 4.3** For every $r$, the set $\{h \in \pi_1(M - D) : l(h) < r\}$ is finite.

**Proof** Let $N = (M - D, g_b)$, let $x \in N$ be a basepoint and let $p : \tilde{N} \to N$ be the universal cover of $N$. Choose $y \in p^{-1}(x)$. The preimage $p^{-1}(x)$ is discrete and $B_{\tilde{N}}(y, r) \subset \tilde{N}$ is compact by Lemma 4.2. Thus $p^{-1}(x) \cap B_{\tilde{N}}(y, r)$ is finite for every $r$ and therefore $\{h \in \pi_1(N) : l(h) < r\}$ is finite. \hfill $\square$

## 5 A Lipschitz embedding

In this section we focus on the particular case where $M - D$ is a configuration space. Let $(S, g)$ be a compact Riemannian surface and $g^n$ be the product metric on $S^n$. 

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Let $D_{ij} = \{(x_1, \ldots, x_n) \in S^n : x_i = x_j\}$. Denote by $C_n(S) = S^n - \bigcup_{i,j} D_{ij}$ the configuration space of $n$ ordered points in $S$. On $S^n$ and $C_n(S)$ we consider the measure induced by the product metric $g^n$.

We shall now find a formula for $d_{g^n}(x, D_{ij})$ in terms of the metric on $S$. Let $x = (x_1, \ldots, x_n) \in S^n$ and let $m$ be the midpoint of a geodesic connecting $x_i$ with $x_j$. If we start moving points $x_i$ and $x_j$ towards $m$ with constant speed, we get a geodesic in $S^n$ connecting $x$ with the closest point in $D_{ij}$. Since $d_{g}(m, x_i) = d_{g}(m, x_j) = \frac{1}{2} d_{g}(x_i, x_j)$ and we are in the product metric,

$$d_{g^n}(x, D_{ij}) = \sqrt{d_{g}(m, x_i)^2 + d_{g}(m, x_j)^2} = \frac{1}{\sqrt{2}} d_{g}(x_i, x_j).$$

The distance function $d$ has the form

$$d(x) = \frac{1}{\sqrt{2}} \min\{d_{g}(x_i, x_j) : 1 \leq i < j \leq n\}.$$

Let $g_b = (g^n)_b$ be the metric on $C_n(S)$ defined in the previous section, namely $|v|_{g_b} = |v|_{g^n}/d(x)$, where $v \in T_x(C_n(S))$.

Let us fix a point $p \in S$ and let $x = (x_1, \ldots, x_{n-1}) \in S^{n-1}$. Then $(p, x) \in S^n$ and $d((p, x))$ is the minimum over $(1/\sqrt{2})d_{g}(p, x_i)$ for $1 \leq i \leq n-1$ and $(1/\sqrt{2})d_{g}(x_i, x_j)$ for $1 \leq i < j \leq n-1$.

We need the following technical lemma.

**Lemma 5.1** There exists $C \in \mathbb{R}$ such that for every $p \in S$ we have

$$\int_{S^{n-1}} \frac{1}{d((p, x))} \, dx \leq C.$$

**Proof** It can be easily seen using polar coordinates that there exists $C'$ such that for every $p \in D^2$, where $D^2$ is the euclidean disc,

$$\int_D \frac{1}{|p-x|} \, dx < C'.$$

Since such $C'$ exists for a disc, we have a similar bound for every compact surface $S$: for every $p \in S$

$$\int_S \frac{1}{d_{g}(p, x)} \, dx < C'.$$

After integrating over all possible $p \in S$ (we assume area$(S) = 1$),

$$\int_{S^2} \frac{1}{d_{g}(p, x)} \, dp \, dx < C'.$$
Let \( x = (x_1, \ldots, x_{n-1}) \). Since \( d((p, x)) \) is the minimum over \((1/\sqrt{2})d_g(p, x_i)\) for \( i = 1, \ldots, n-1 \) and \((1/\sqrt{2})d_g(x_i, x_j)\) for \( 1 \leq i < j \leq n-1 \),

\[
\frac{1}{d((p, x))} \leq \sum_i \frac{\sqrt{2}}{d_g(p, x_i)} + \sum_{i \neq j} \frac{\sqrt{2}}{d_g(x_i, x_j)}.
\]

Thus

\[
\int_{S^{n-1}} \frac{1}{d((p, x))} \, dx \leq \sum_i \int_{S^{n-1}} \frac{\sqrt{2}}{d_g(p, x_i)} \, dx + \sum_{i \neq j} \int_{S^{n-1}} \frac{\sqrt{2}}{d_g(x_i, x_j)} \, dx
\]

\[
= (n-1) \int_S \frac{\sqrt{2}}{d_g(p, x)} \, dx + \frac{1}{2} n(n-1) \int_{S^2} \frac{\sqrt{2}}{d_g(x_1, x_2)} \, dx_1 \, dx_2
\]

\[
\leq \sqrt{2}(n-1)C' + \frac{n(n-1)}{\sqrt{2}} C' =: C.
\]

Let \( \mu \) be the measure on \( C_n(S) \) induced by the product metric \( g^n \). A diffeomorphism \( f \in \text{Diff}_0(S, \text{area}) \) defines a product diffeomorphism \( f_* \in \text{Diff}_0(C_n(S), \mu) \). Namely, for \( x = (x_1, \ldots, x_n) \in S^n \) we have \( f_*(x) = (f(x_1), \ldots, f(x_n)) \). Thus we have a product embedding \( \text{Diff}_0(S, \text{area}) \hookrightarrow \text{Diff}_0(C_n(S), \mu) \).

On \( \text{Diff}_0(C_n(S), \mu) \) we consider the \( L^1 \)–norm defined by the metric \( g_b \) and the measure \( \mu \). Note that here we are in the case where \( g_b \) and \( \mu \) are not compatible, that is, the measure induced by \( g_b \) and the measure \( \mu \) are different.

The following lemma provides a link between the \( L^1 \)–norm on \( \text{Diff}_0(S, \text{area}) \) and the \( L^1 \)–norm on \( \text{Diff}_0(C_n(S), \mu) \) defined above. Note that in the proof it is essential that \( f \) preserves the area on \( S \).

**Lemma 5.2** The product embedding \( \text{Diff}_0(S, \text{area}) \hookrightarrow \text{Diff}_0(C_n(S), \mu) \) is Lipschitz, ie there exists \( C \) such that \( l_1(f_*) \leq C l_1(f) \).

**Proof** Let \( f \in \text{Diff}_0(S, \text{area}) \) and let \( X : S \to TS \) such that \( X(x) \in T_f(x)S \). For \( x = (x_1, \ldots, x_n) \in C_n(S) \) we define \( X_*(x) = (X(x_1), \ldots, X(x_n)) \in T_{f_*(x)}C_n(S) \).

The set \( \bigcup_{i,j} D_{i,j} \subset S^n \) is of measure zero. This means that we can regard \( |X_*(x)|_{g_b} \) as a measurable function defined on \( S^n \). Thus in what follows, we integrate \( |X_*(x)|_{g_b} \) over \( S^n \) with the product measure rather than over \( C_n(S) \).

To prove the lemma it is enough to show that there exists \( C \) such that for every \( f \in \text{Diff}_0(S, \text{area}) \) and every map \( X : S \to TS \) such that \( X(x) \in T_{f(x)}S \) the following inequality holds:

\[
\|X_*\|_1 \leq C \|X\|_1.
\]
Recall that by definition $\|X_*\|_1 = \int_{S^n} |X_*(x)|_{g_b} \, dx$. We have

$$\int_{S^n} |X_*(x)|_{g_b} \, dx = \int_{S^n} \frac{|X_*(x)|_{g^n}}{d(f_*(x))} \, dx = \int_{S^n} \frac{\sqrt{|X(x_1)|_g^2 + \cdots + |X(x_n)|_g^2}}{d(f_*(x))} \, dx$$

$$\leq \int_{S^n} \frac{|X(x_1)|_g + \cdots + |X(x_n)|_g}{d(f_*(x))} \, dx = n \int_{S^n} \frac{|X(x_1)|_g}{d(f_*(x))} \, dx.$$  

Since $f_*$ preserves the measure on $S^n$,

$$\int_{S^n} \frac{|X(x_1)|_g}{d(f_*(x))} \, dx = \int_{S^n} \frac{|X \circ f^{-1}(x_1)|_g}{d(x)} \, dx$$

$$= \int_S |X \circ f^{-1}(x_1)|_g \left( \int_{S^{n-1}} \frac{1}{d(x_1, x)} \, dx \right) \, dx_1$$

$$\leq C \int_S |X \circ f^{-1}(x_1)|_g \, dx_1 \quad \text{(by Lemma 5.1)}$$

$$= C \int_S |X(x_1)|_g \, dx_1 = C \|X\|_1. \quad \square$$

### 6 Proof of the theorem

**Theorem 1** Let $(S, g)$ be a compact surface (with or without boundary). Then for every $p \geq 1$ the $L^p$–diameter of $\text{Diff}_0(S, \text{area})$ is infinite.

**Proof** By the Hölder inequality we can assume $p = 1$. Fix $n > 3$.

Let $z = (z_1, \ldots, z_n) \in C_n(S)$ and let $P_n(S) = \pi_1(C_n(S), z)$ denote the pure braid group on $n$ strings. Suppose $U_i \subset S$ are disjoint discs such that $z_i \in U_i$, then let $U = U_1 \times U_2 \times \cdots \times U_n \subset C_n(S)$.

Choose $h \in P_n(S)$ and $\gamma$ a loop in $C_n(S)$ representing $h$. Let $f_t \in \text{Diff}_0(S, \text{area})$ for $t \in [0, 1]$ be an isotopy such that $(f_t)_* \in \text{Diff}_0(C_n(S), \mu)$ moves $U$ all the way along $\gamma$ and has properties (1) and (2) from the proof of Proposition 3.2. Let $f_h = f_1$.

It is convenient to imagine that $f_t$ moves $U_i$ along the trajectory of $z_i$ given by $\gamma$. In fact, to construct $f_t$ satisfying the above properties for a general $h \in P_n(S)$, it is enough to do it for a given finite set of generators of $P_n(S)$ (or generators of the full braid group $B_n(S)$). In [2] one can find a set of generators of $B_n(S)$ for which the construction of $f_t$ is straightforward.

Recall that on $C_n(S)$ we consider the complete metric $g_b$. By Lemma 4.3, the set $\{h \in \pi_1(C_n(S)) : l(h) < r\}$ is finite for every $r$ and $P_n(S)/Z(P_n(S))$ is infinite. It follows from the proof of Proposition 3.2 that $l_1((f_h)_*)$ can be arbitrarily large.
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Therefore, due to Lemma 5.2, $l_1(f_h)$ can be arbitrarily large. Thus the $L^1$–diameter of $\text{Diff}_0(S, \text{area})$ is infinite. □

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