Bounded cohomology of classifying spaces for
families of subgroups

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We introduce a bounded version of Bredon cohomology for groups relative to a family of subgroups. Our theory generalizes bounded cohomology and differs from Mineyev and Yaman's relative bounded cohomology for pairs. We obtain cohomological characterizations of relative amenability and relative hyperbolicity, analogous to the results of Johnson and Mineyev for bounded cohomology.

1 Introduction

Bounded cohomology is a homotopy invariant of topological spaces with deep connections to Riemannian geometry via the simplicial volume of manifolds; see Gromov [7]. An astonishing phenomenon known as Gromov's mapping theorem is that for every CW–complex X, the classifying map $X \to B\pi_1(X)$ induces an isometric isomorphism on bounded cohomology. This emphasizes the importance of the corresponding theory of bounded cohomology for groups, which is also of independent interest due to its plentiful applications in geometric group theory; see Frigerio [6] and Monod [20; 21]. The bounded cohomology $H_b^n(G; V)$ of a (discrete) group G with coefficients in a normed G–module V is the cohomology of the cochain complex of bounded G–maps $G^{n+1} \to V$. The inclusion of bounded G–maps into (not necessarily bounded) G–maps

induces the so-called *comparison map* $H_b^n(G; V) \to H^n(G; V)$. On the one hand, the bounded cohomology groups are very difficult to compute in general. On the other hand, they characterize interesting group-theoretic properties such as amenability — Johnson [11] — and hyperbolicity — Mineyev [17; 18].

Theorem 1.1 (Johnson) *Let G be a group. The following are equivalent*:

(i) *G is amenable.*

(ii) *$H_b^n(G; V^\#) = 0$ for all dual normed $\mathbb{R}G$–modules $V^\#$ and all $n \geq 1$.*

(iii) *$H_b^1(G; V^\#) = 0$ for all dual normed $\mathbb{R}G$–modules $V^\#$.*

Theorem 1.2 (Mineyev) *Let G be a finitely presented group. The following are equivalent*:

(i) *G is hyperbolic.*

(ii) *The comparison map $H_b^n(G; V) \to H^n(G; V)$ is surjective for all normed $\mathbb{Q}G$–modules V and all $n \geq 2$.*

(iii) *The comparison map $H_b^2(G; V) \to H^2(G; V)$ is surjective for all normed $\mathbb{R}G$–modules V.*

There are well-studied notions of relative amenability and relative hyperbolicity in the literature; see Hruska [8] and Ji, Ogle and Ramsey [10]. In the present article we introduce a new "relative bounded cohomology theory" characterizing these relative group-theoretic properties as a bounded version of Bredon cohomology. For a group G, a family of subgroups $\mathcal{F}$ is a nonempty set of subgroups which is closed under conjugation and taking subgroups. For a set of subgroups $\mathcal{H}$ of G, we denote by $\mathcal{F}\langle\mathcal{H}\rangle$ the smallest family containing $\mathcal{H}$. The Bredon cohomology $H_{\mathcal{F}}^n(G; V)$ with coefficients in a G–module V (or more general coefficient systems) is a generalization of group cohomology, which is recovered when $\mathcal{F}$ consists only of the trivial subgroup. A fundamental feature of Bredon cohomology is that for a normal subgroup N of G there is an isomorphism $H_{\mathcal{F}\langle N\rangle}^n(G; V) \cong H^n(G/N; V^N)$. From a topological point of view, the Bredon cohomology of G can be identified with the equivariant cohomology of the classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$, which is a terminal object in the G–homotopy category of G–CW–complexes with stabilizers in $\mathcal{F}$. Especially the classifying spaces $E_{\mathcal{FIN}}G$ and $E_{\mathcal{VCY}}G$ for the family of finite groups and virtually cyclic groups have received a lot of attention in recent years due to their prominent role in the isomorphism conjectures of Baum–Connes and Farrell–Jones, respectively.

We introduce the *bounded Bredon cohomology* $H^n_{\mathcal{F},b}(G; V)$ of G with coefficients in a normed G–module V, which generalizes bounded cohomology (Definition 3.1). Our theory still is well-behaved with respect to normal subgroups (Corollary 3.11) and admits a topological interpretation in terms of classifying spaces for families (Theorem 3.10). We obtain the following generalizations of Theorems 1.1 and 1.2. A group G is called amenable relative to a set of subgroups $\mathcal{H}$ if there exists a G–invariant mean on the G–set $\bigsqcup_{H \in \mathcal{H}} G/H$.

Theorem 1.3 *Let G be a group and $\mathcal{H}$ be a set of subgroups. The following are equivalent*:

(i) *G is amenable relative to $\mathcal{H}$.*

(ii) $H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G; V^{\#}) = 0$ *for all dual normed $\mathbb{R}G$–modules $V^{\#}$ and all $n \geq 1$.*

(iii) $H^1_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G; V^{\#}) = 0$ *for all dual normed $\mathbb{R}G$–modules $V^{\#}$.*

Theorem 1.3 is a special case of the more general Theorem 4.5. We also provide a characterization of relative amenability in terms of relatively injective modules (Proposition 4.8). Recall that a finite set of subgroups $\mathcal{H}$ is called a malnormal (resp. almost malnormal) collection if for all $H_i, H_j \in \mathcal{H}$ and $g \in G$ we have that $H_i \cap g H_j g^{-1}$ is trivial (resp. finite), unless $i = j$ and $g \in H_i$. A group G is said to be of type $F_{n,\mathcal{F}}$ for a family of subgroups $\mathcal{F}$, if there exists a model for the classifying space $E_{\mathcal{F}}G$ with cocompact n–skeleton.

Theorem 1.4 (Theorem 5.4) *Let G be a finitely generated torsionfree group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Suppose that G is of type $F_{2,\mathcal{F}\langle\mathcal{H}\rangle}$ (eg G and all subgroups in $\mathcal{H}$ are finitely presented). Then the following are equivalent*:

(i) *G is hyperbolic relative to $\mathcal{H}$.*

(ii) *The comparison map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G; V) \to H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G; V)$ is surjective for all normed $\mathbb{Q}G$–modules V and all $n \geq 2$.*

(iii) *The comparison map $H^2_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G; V) \to H^2_{\mathcal{F}\langle\mathcal{H}\rangle}(G; V)$ is surjective for all normed $\mathbb{R}G$–modules V.*

In Theorem 1.4 the equivalence of (i) and (iii) still holds if the group G contains torsion and $\mathcal{H}$ is almost malnormal, see Remark 5.5. Note that condition (iii) is trivially satisfied for groups of Bredon cohomological dimension $\mathrm{cd}_{\mathcal{F}\langle\mathcal{H}\rangle}$ equal to 1.

The topological interpretation of bounded Bredon cohomology via classifying spaces for families was used by Löh and Sauer [12] to give a new proof of the nerve theorem and vanishing theorem for amenable covers. We prove a converse of [12, Proposition 5.2], generalizing a recent result of Moraschini and Raptis [23, Theorem 3.1.3], where the case of a normal subgroup is treated.

Theorem 1.5 *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent*:

  (i) *All subgroups in $\mathcal{F}$ are amenable.*

 (ii) *The canonical map $H^n_{\mathcal{F},b}(G; V^{\#}) \to H^n_b(G; V^{\#})$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^{\#}$ and all $n \geq 0$.*

(iii) *The canonical map $H^1_{\mathcal{F},b}(G; V^{\#}) \to H^1_b(G; V^{\#})$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^{\#}$.*

Theorem 1.5 is a special case of the more general Theorem 4.5. As an application of Theorem 1.5, the comparison map vanishes for groups which admit a "small" model for $E_{\mathcal{F}}G$, where $\mathcal{F}$ is any family consisting of amenable subgroups (Corollary 4.6). Examples are graph products of amenable groups (eg right-angled Artin groups) and fundamental groups of graphs of amenable groups.

There is another natural relative cohomology theory given by the relative cohomology of a pair of spaces. For a set of subgroups $\mathcal{H}$, it gives rise to the cohomology $H^n(G, \mathcal{H}; V)$ of the group pair $(G, \mathcal{H})$ introduced by Bieri and Eckmann [2]. A bounded version $H^n_b(G, \mathcal{H}; V)$ was defined by Mineyev and Yaman [19] to give a characterization of relative hyperbolicity; see also Franceschini [5]. A characterization of relative amenability in terms of this relative theory was obtained in [10]. There is a canonical map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G; V) \to H^n(G, \mathcal{H}; V)$ for $n \geq 2$ which is an isomorphism if $\mathcal{H}$ is malnormal; see Remark 2.1. Similarly, there is a map for the bounded versions but we do not know when it is an isomorphism due to the failure of the excision axiom for bounded cohomology; see Remark 3.12. We also mention that Mineyev and Yaman's relative bounded cohomology was extended to pairs of groupoids by Blank in [3].

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2 Preliminaries on Bredon cohomology and classifying spaces

In this section we briefly recall the notion of Bredon cohomology for groups and its topological interpretation as the equivariant cohomology of classifying spaces for families of subgroups. For an introduction to Bredon cohomology we refer to [4] and for a survey on classifying spaces to [13].

Let G be a group, which shall always mean a discrete group. A *family of subgroups $\mathcal{F}$* is a nonempty set of subgroups of G that is closed under conjugation by elements of G and under taking subgroups. Typical examples are

$$\mathcal{TR} = \{1\},$$
$$\mathcal{FIN} = \{\text{finite subgroups}\},$$
$$\mathcal{VCY} = \{\text{virtually cyclic subgroups}\},$$
$$\mathcal{ALL} = \{\text{all subgroups}\}.$$

We will moreover be interested in $\mathcal{AME} = \{\text{amenable subgroups}\}$. For a subgroup H of G, we denote by $\mathcal{F}|_H$ the family $\{L \cap H \mid L \in \mathcal{F}\}$ of subgroups of H. (In the literature this family is sometimes denoted by $\mathcal{F} \cap H$ instead.) For a set of subgroups $\mathcal{H}$, one can consider the smallest family containing $\mathcal{H}$ which is defined by $\mathcal{F}\langle\mathcal{H}\rangle = \{\text{conjugates of elements in } \mathcal{H} \text{ and their subgroups}\}$ and called the *family generated by $\mathcal{H}$*. When $\mathcal{H}$ consists of a single subgroup H, we denote $\mathcal{F}\langle\mathcal{H}\rangle$ instead by $\mathcal{F}\langle H\rangle$ and call it the *family generated by H*. We denote by $G/\mathcal{H}$ the G–set $\bigsqcup_{H \in \mathcal{H}} G/H$.

Let R be a ring and Mod_R denote the category of R–modules. We will often suppress the ring R, so that G–modules are understood to be RG–modules. The ($\mathcal{F}$–*restricted*) *orbit category $\mathcal{O}_\mathcal{F}G$* has as objects G–sets of the form G/H with $H \in \mathcal{F}$ and as morphisms G–maps. An $\mathcal{O}_\mathcal{F}G$–*module* is a contravariant functor $M : \mathcal{O}_\mathcal{F}G \to \mathrm{Mod}_R$, the category of which is denoted by $\mathcal{O}_\mathcal{F}G$–Mod_R. Note that $\mathcal{O}_{\mathcal{TR}}G$–$\mathrm{Mod}_R$ can be identified with the category of G–modules (see eg [4, Chapter 1, Section 4]). For a G–module V, there is a coinduced $\mathcal{O}_\mathcal{F}G$–module $V^?$ given by $V^?(G/H) = V^H$. (In the literature this is sometimes called a *fixed-point functor*.) Observe that $(\,\cdot\,)^?$ is

right-adjoint to the restriction $\mathcal{O}_{\mathcal{F}}G\text{–Mod}_R \to \mathcal{O}_{\mathcal{TR}}G\text{–Mod}_R$, $M \mapsto M(G/1)$; see eg [4, Proposition 1.31]. That is, for every $\mathcal{O}_{\mathcal{F}}G$–module M and G–module V there is a natural isomorphism

$$(2\text{-}1) \qquad \operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{–Mod}_R}(M, V^?) \cong \operatorname{Hom}_{RG}(M(G/1), V).$$

For a G–space X and a G–CW–complex Y with stabilizers in $\mathcal{F}$, there are singular and cellular $\mathcal{O}_{\mathcal{F}}G$–chain complexes

$$C_*(X^?)(G/H) = C_*(X^H) \quad \text{and} \quad C_*^{\operatorname{cell}}(Y^?)(G/H) = C_*^{\operatorname{cell}}(Y^H),$$

where $C_*(X^H)$ and $C_*^{\operatorname{cell}}(Y^H)$ denote the usual singular and cellular chain complexes, respectively.

The *Bredon cohomology* of G with coefficients in an $\mathcal{O}_{\mathcal{F}}G$–module M is defined as the R–module

$$H_{\mathcal{F}}^n(G; M) := \operatorname{Ext}_{\mathcal{O}_{\mathcal{F}}G\text{–Mod}_R}^n(R, M)$$

for $n \geq 0$, where R is regarded as a constant $\mathcal{O}_{\mathcal{F}}G$–module. It can be computed as the cohomology of the cochain complex $\operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{–Mod}_R}(R[((G/\mathcal{F})^{*+1})^?], M)$; see eg [4, Proposition 3.5]. We define the G–chain complex $C_*^{\mathcal{F}}(G)$ given by G–modules

$$C_n^{\mathcal{F}}(G) := R[(G/\mathcal{F})^{n+1}]$$

with the diagonal G–action and differentials $\partial_n \colon C_n^{\mathcal{F}}(G) \to C_{n-1}^{\mathcal{F}}(G)$,

$$\partial_n(g_0 H_0, \ldots, g_n H_n) = \sum_{i=0}^n (-1)^i (g_0 H_0, \ldots, \widehat{g_i H_i}, \ldots, g_n H_n).$$

For a G–module V, the G–cochain complex $C_{\mathcal{F}}^*(G; V)$ is given by

$$C_{\mathcal{F}}^n(G; V) := \operatorname{Hom}_R(C_n^{\mathcal{F}}(G), V)$$

so that by adjunction (2-1),

$$H_{\mathcal{F}}^n(G; V) := H_{\mathcal{F}}^n(G; V^?) \cong H^n(C_{\mathcal{F}}^*(G; V)^G).$$

For a G–space X with stabilizers in $\mathcal{F}$, the *Bredon cohomology* of X with coefficients in an $\mathcal{O}_{\mathcal{F}}G$–module M is defined as

$$H_G^n(X; M) := H^n(\operatorname{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{–Mod}_R}(C_*(X^?), M))$$

for $n \geq 0$. If X is a G–CW–complex, then $H_G^n(X; M)$ can be computed using $C_*^{\operatorname{cell}}(X^?)$ instead of $C_*(X^?)$.

A *classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$* is a terminal object in the G–homotopy category of G–CW–complexes with stabilizers in $\mathcal{F}$. It can be shown that a G–CW–complex X is a model for $E_{\mathcal{F}}G$ if and only if the fixed-point set X^H is contractible for $H \in \mathcal{F}$ and empty otherwise; see eg [13, Theorem 1.9]. An explicit model is given by the geometric realization Y of the semisimplicial set $\{(G/\mathcal{F})^{n+1} \mid n \geq 0\}$ with the usual face maps. Then Y has (nonequivariant) n–cells corresponding to $(G/\mathcal{F})^{n+1}$ and we refer to Y as the *simplicial model* for $E_{\mathcal{F}}G$. Note that a model for $E_{\mathcal{TR}}G$ is given by EG and a model for $E_{\mathcal{ALL}}G$ is the point G/G. The cellular $\mathcal{O}_{\mathcal{F}}G$–chain complex of any model for $E_{\mathcal{F}}G$ is a projective resolution of the constant $\mathcal{O}_{\mathcal{F}}G$–module R (see eg [4, Proposition 2.9]) and thus we have

$$(2\text{-}2) \qquad\qquad H^n_{\mathcal{F}}(G; M) \cong H^n_G(E_{\mathcal{F}}G; M)$$

for all $\mathcal{O}_{\mathcal{F}}G$–modules M. If N is a normal subgroup of G, then a model for $E_{\mathcal{F}\langle N \rangle}G$ is given by $E(G/N)$ regarded as a G–CW–complex and we find

$$(2\text{-}3) \qquad\qquad H^n_{\mathcal{F}\langle N \rangle}(G; M) \cong H^n(G/N; M(G/N))$$

(see eg [1, Corollary 4.11]).

For a subgroup H of G, when viewed as an H–space $E_{\mathcal{F}}G$ is a model for $E_{\mathcal{F}|_H}H$ which induces the restriction map

$$(2\text{-}4) \qquad\qquad \mathrm{res}^n_{H \subset G} : H^n_{\mathcal{F}}(G; M) \to H^n_{\mathcal{F}|_H}(H; M)$$

for all $\mathcal{O}_{\mathcal{F}}G$–modules M. For two families of subgroups $\mathcal{F} \subset \mathcal{G}$, the up to G–homotopy unique G–map $E_{\mathcal{F}}G \to E_{\mathcal{G}}G$ induces the canonical map

$$(2\text{-}5) \qquad\qquad \mathrm{can}^n_{\mathcal{F} \subset \mathcal{G}} : H^n_{\mathcal{G}}(G; M) \to H^n_{\mathcal{F}}(G; M)$$

for all $\mathcal{O}_{\mathcal{G}}G$–modules M.

Remark 2.1 (Bieri and Eckmann's relative cohomology) For a group G and a set of subgroups $\mathcal{H}$, Bieri and Eckmann [2] have introduced the *relative cohomology $H^n(G, \mathcal{H}; V)$ of the pair $(G, \mathcal{H})$* with coefficients in a G–module V. It can be identified with the relative cohomology $H^n_G\big(EG, \coprod_{H \in \mathcal{H}} G \times_H EH; V\big)$ of the pair of G–spaces $\big(EG, \coprod_{H \in \mathcal{H}} G \times_H EH\big)$. Here a model for EG is chosen that contains $\coprod_{H \in \mathcal{H}} G \times_H EH$ as a subcomplex by taking mapping cylinders. Hence there is a long exact sequence

$$\cdots \to H^n(G, \mathcal{H}; V) \to H^n(G; V) \to \prod_{H \in \mathcal{H}} H^n(H; V) \to \cdots,$$

which is one of the main features of the relative cohomology groups.

There is a relation between Bredon cohomology and Bieri and Eckmann's relative cohomology as follows. Consider the G–space X obtained as the G–pushout

$$\begin{array}{ccc} \coprod_{H\in\mathcal{H}} G\times_H EH & \longrightarrow & EG \\ \downarrow & & \downarrow \\ \coprod_{H\in\mathcal{H}} G/H & \longrightarrow & X \end{array}$$

where the left vertical map is induced by collapsing each EH to a point. Then the G–space X has stabilizers in $\mathcal{F}\langle\mathcal{H}\rangle$ and hence admits a G–map $X \to E_{\mathcal{F}\langle\mathcal{H}\rangle}G$. For an $\mathcal{O}_{\mathcal{F}}G$–module M, we have maps

$$\begin{array}{ccc} H_G^n(X; M) & \longleftarrow & H_G^n(X, \coprod_{H\in\mathcal{H}} G/H; M) \\ \uparrow & & \downarrow{\scriptstyle\cong} \\ H_G^n(E_{\mathcal{F}\langle\mathcal{H}\rangle}G; M) & & H_G^n(EG, \coprod_{H\in\mathcal{H}} G\times_H EH; M) \end{array}$$

where the right vertical map is an isomorphism by excision. Now, if $\mathcal{H}$ is a malnormal collection, then X is a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ and

$$H_{\mathcal{F}\langle\mathcal{H}\rangle}^n(G; M) \cong H^n(G, \mathcal{H}; M(G/1))$$

for $n \geq 2$. This was shown in [1, Theorem 4.16] for the special case when $\mathcal{H}$ consists of a single subgroup.

3 Bounded Bredon cohomology

In this section we introduce a bounded version of Bredon cohomology and develop some of its basic properties. We follow the exposition in [6] for bounded cohomology. Throughout, let G be a group and $\mathcal{F}$ be a family of subgroups.

From now on, let the ring R be one of $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. A *normed G–module* V is a G–module equipped with a G–invariant norm $\|\cdot\| \colon V \to \mathbb{R}$. (That is, for all $v, u \in V$, $r \in R$, and $g \in G$ we have $\|v\| = 0$ if and only if $v = 0$, $\|rv\| \leq |r|\cdot\|v\|$, $\|v+u\| \leq \|v\| + \|u\|$, and $\|g\cdot v\| = \|v\|$.) A morphism $f \colon V \to W$ of normed G–modules is a morphism of G–modules with finite operator-norm $\|f\|_\infty$. We denote by $\mathrm{bHom}_R(V, W)$ the G–module of R–linear maps $f \colon V \to W$ with finite operator-norm, where the G–action is given by $(g\cdot f)(v) = g\cdot f(g^{-1}v)$. We denote the topological dual $\mathrm{bHom}_R(V, \mathbb{R})$ of V by $V^{\#}$. For a set S and a normed module V, we denote by $\mathrm{bMap}(S, V)$ the module of functions $S \to V$ with bounded image. Instead of $\mathrm{bMap}(S, \mathbb{R})$ we also write $\ell^\infty(S)$.

The following is our key definition. Recall the notation $G/\mathcal{F} = \coprod_{H \in \mathcal{F}} G/H$ and consider $C_n^{\mathcal{F}}(G) = R[(G/\mathcal{F})^{n+1}]$ as a normed G–module equipped with the ℓ^1–norm with respect to the R–basis $(G/\mathcal{F})^{n+1}$. For a normed G–module V, we define the cochain complex $C_{\mathcal{F},b}^*(G; V)$ of normed G–modules by

$$C_{\mathcal{F},b}^n(G; V) := \mathrm{bHom}_R(C_n^{\mathcal{F}}(G), V)$$

together with the differentials $\delta^n \colon C_{\mathcal{F},b}^n(G; V) \to C_{\mathcal{F},b}^{n+1}(G; V)$,

$$\delta^n(f)(g_0 H_0, \dots, g_{n+1} H_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0 H_0, \dots, \widehat{g_i H_i}, \dots, g_{n+1} H_{n+1}).$$

Definition 3.1 (bounded Bredon cohomology of groups) The *bounded Bredon cohomology* of G with coefficients in a normed G–module V is defined as

$$H_{\mathcal{F},b}^n(G; V) := H^n(C_{\mathcal{F},b}^*(G; V)^G)$$

for $n \geq 0$. The inclusion $C_{\mathcal{F},b}^n(G; V) \subset C_{\mathcal{F}}^n(G; V)$ induces a map

$$c_{\mathcal{F}}^n \colon H_{\mathcal{F},b}^n(G; V) \to H_{\mathcal{F}}^n(G; V),$$

called the *comparison map*.

Note that for $\mathcal{F} = \mathcal{TR}$, Definition 3.1 recovers the usual definition of bounded cohomology.

Remark 3.2 (coefficient modules) We only consider normed G–modules as coefficients, rather than more general $\mathcal{O}_{\mathcal{F}}G$–modules equipped with a "compatible norm". Hence strictly speaking our theory is a bounded version of Nucinkis' cohomology relative to the G–set $G/\mathcal{F}$ [24], rather than a bounded version of Bredon cohomology.

Remark 3.3 (canonical seminorm) The ℓ^∞–norm on $C_{\mathcal{F},b}^n(G; V)$ descends to a *canonical seminorm* on $H_{\mathcal{F},b}^n(G; V)$. However, we do not consider seminorms anywhere in this article and regard $H_{\mathcal{F},b}^n(G; V)$ merely as an R–module.

Bounded Bredon cohomology satisfies the following basic properties.

Lemma 3.4 *The following hold*:

(i) *Let $0 \to V_0 \to V_1 \to V_2 \to 0$ be a short exact sequence of normed G–modules such that $0 \to V_0^H \to V_1^H \to V_2^H \to 0$ is exact for each $H \in \mathcal{F}$. Then there exists a long exact sequence*

$$0 \to H_{\mathcal{F},b}^0(G; V_0) \to H_{\mathcal{F},b}^0(G; V_1) \to H_{\mathcal{F},b}^0(G; V_2) \to H_{\mathcal{F},b}^1(G; V_0) \to \cdots.$$

(ii) $H^0_{\mathcal{F},b}(G;V) \cong V^G$ for all normed G–modules V.

(iii) $H^1_{\mathcal{F},b}(G;\mathbb{R}) = 0$.

Proof (i) For a G–set $S = \coprod_{i \in I} G/H_i$ and a normed G–module V, we can identify the module $\mathrm{bMap}_G(S,V)$ with the submodule of $\prod_{i \in I} V^{H_i}$ consisting of the elements $(v_i)_{i \in I}$ satisfying $\sup_{i \in I} \|v_i\| < \infty$. It follows that for a G–set S with stabilizers in $\mathcal{F}$, the sequence of modules

$$0 \to \mathrm{bMap}_G(S,V_0) \to \mathrm{bMap}_G(S,V_1) \to \mathrm{bMap}_G(S,V_2) \to 0$$

is exact. Applying the above to the G–sets $(G/\mathcal{F})^{n+1}$ for $n \geq 0$, we obtain that the sequence of cochain complexes

$$0 \to C^*_{\mathcal{F},b}(G;V_0)^G \to C^*_{\mathcal{F},b}(G;V_1)^G \to C^*_{\mathcal{F},b}(G;V_2)^G \to 0$$

is exact. Then the associated long exact sequence on cohomology is as desired.

(ii) We have $H^0_{\mathcal{F},b}(G;V) = \ker(\delta^0)$, where

$$\delta^0 : \mathrm{bHom}_{RG}(R[G/\mathcal{F}],V) \to \mathrm{bHom}_{RG}(R[(G/\mathcal{F})^2],V)$$

is given by $\delta^0(f)(g_0 H_0, g_1 H_1) = f(g_1 H_1) - f(g_0 H_0)$. Hence $\ker(\delta^0)$ consists precisely of the constant G–maps $G/\mathcal{F} \to V$, which are in correspondence to V^G.

(iii) We identify

$$C^n_{\mathcal{F},b}(G;\mathbb{R})^G \cong \mathrm{bMap}\left(\coprod_{H_0,\dots,H_n \in \mathcal{F}} H_0 \backslash (G/H_1 \times \cdots \times G/H_n), \mathbb{R} \right)$$

for $n \geq 1$ and $C^0_{\mathcal{F},b}(G;\mathbb{R})^G \cong \mathrm{bMap}(\coprod_{H_0 \in \mathcal{F}} *_{H_0}, \mathbb{R})$. The differentials of this "inhomogeneous" complex in low degrees are given by

$$\delta^0(f)(H_0 g_1 H_1) = f(*_{H_1}) - f(*_{H_0}),$$
$$\delta^1(\varphi)(H_0(g_1 H_1, g_2 H_2)) = \varphi(H_1 g_1^{-1} g_2 H_2) - \varphi(H_0 g_2 H_2) + \varphi(H_0 g_1 H_1).$$

Then it is not difficult to check that $\ker(\delta^1) = \mathrm{im}(\delta^0)$. $\qquad\square$

We also define the bounded cohomology of a G–space X as follows. Denote by $S_n(X)$ the set of singular n–simplices in X and consider $C_n(X) = R[S_n(X)]$ equipped with the ℓ^1–norm as a normed G–module. For a normed G–module V, we define the cochain complex $C^*_b(X;V)$ of normed G–modules by

$$C^n_b(X;V) := \mathrm{bHom}_R(C_n(X),V)$$

together with the usual differentials.

Definition 3.5 (bounded cohomology of G–spaces) The (G–*equivariant*) *bounded cohomology* of a G–space X with coefficients in a normed G–module V is defined as

$$H_{G,b}^n(X;V) := H^n(C_b^*(X;V)^G)$$

for $n \geq 0$. The inclusion $C_b^n(X;V) \subset C^n(X;V)$ induces a map

$$c_X^n : H_{G,b}^n(X;V) \to H_G^n(X;V)$$

called the *comparison map*.

Note that the functors $H_{G,b}^*$ are G–homotopy invariant and that $H_{G,b}^n(G/H;V)$ is isomorphic to V^H for $n = 0$ and trivial otherwise. However, beware that $H_{G,b}^*$ is neither a G–cohomology theory, nor can it be computed cellularly for G–CW–complexes, as is the case already when G is the trivial group; see eg [6, Remark 5.6].

Relative homological algebra We develop the relative homological algebra that will allow us to compute bounded Bredon cohomology via resolutions, analogous to Ivanov's approach for bounded cohomology [9].

A map $p: A \to B$ of G–modules is called $\mathcal{F}$–*strongly surjective* if for each $H \in \mathcal{F}$ there exists a map $\tau_H : B \to A$ of H–modules such that $p \circ \tau_H = \mathrm{id}_B$. A G–module P is called *relatively $\mathcal{F}$–projective* if for every $\mathcal{F}$–strongly surjective G–map $p: A \to B$ and every G–map $\phi: P \to B$, there exists a G–map $\Phi: P \to A$ such that $p \circ \Phi = \phi$. A chain complex of G–modules is called *relatively $\mathcal{F}$–projective* if each chain module is relatively $\mathcal{F}$–projective. A resolution (C_*, ∂_*) of G–modules is called $\mathcal{F}$–*strong* if it is contractible as a resolution of H–modules for each $H \in \mathcal{F}$. (That is, there exist H–maps $k_*^H : C_* \to C_{*+1}$ such that $\partial_{n+1} \circ k_n^H + k_{n-1}^H \circ \partial_n = \mathrm{id}_{C_n}$.)

Lemma 3.6 *The following hold*:

  (i) *If S is a G–set with stabilizers in $\mathcal{F}$, then the G–module $R[S]$ is relatively $\mathcal{F}$–projective.*

  (ii) *If S is a G–set with $S^H \neq \varnothing$ for all $H \in \mathcal{F}$, then the resolution $R[S^{*+1}] \to R$ of G–modules is $\mathcal{F}$–strong.*

  (iii) *If X is a G–space with contractible fixed-point set X^H for each $H \in \mathcal{F}$, then the resolution $C_*(X) \to R$ of G–modules is $\mathcal{F}$–strong.*

Proof (i) Given a lifting problem as in the definition of relative $\mathcal{F}$–projectivity,

$$
\begin{array}{ccc}
 & R[S] & \\
\Phi \nearrow & \downarrow \phi & \\
A \xrightarrow{p} & B \longrightarrow & 0 \\
\underset{\tau_H}{\longleftarrow} & &
\end{array}
$$

we construct a lift Φ as follows. Let T be a set of representatives of $G\backslash S$ and denote the stabilizer of an element $t \in T$ by G_t. Then for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1}s = t_s$. Define $\Phi: R[S] \to A$ on generators by

$$\Phi(s) = g_s \cdot \tau_{G_{t_s}}(\phi(g_s^{-1}s))$$

which is independent of the choice of g_s, since the map $\tau_{G_{t_s}}$ is G_{t_s}–equivariant. Then Φ is a G–equivariant lift of ϕ.

(ii) For $H \in \mathcal{F}$, fix an element $s_H \in S^H$ and define $k_*^H: R[S^{*+1}] \to R[S^{*+2}]$ on generators by

$$k_n^H(s_0, \ldots, s_n) = (s_H, s_0, \ldots, s_n).$$

Then k_*^H is an H–equivariant contraction.

(iii) For $H \in \mathcal{F}$, fix a point $x_H \in X^H$ and define a contraction $k_*^H: C_*(X) \to C_{*+1}(X)$ of H–chain complexes inductively as follows. Starting with $k_{-1}^H: R \to C_0(X)$, given by $r \mapsto r \cdot x_H$, we may assume that k_{n-1}^H has been constructed. Let s be a singular n–simplex in X and denote its stabilizer by H_s. Then there exists a singular $(n+1)$–simplex s' with 0th vertex x_H and opposite face s satisfying $\partial_{n+1}(s')+k_{n-1}^H(\partial_n(s)) = s$. Moreover, since X^{H_s} is contractible we may choose s' such that its image is contained in X^{H_s}. Now, for each H–orbit of singular n–simplices in X choose a representative s, define $k_n^H(s)$ to be s', and then extend H–equivariantly. $\square$

The proof of the following proposition is standard and omitted.

Proposition 3.7 *Let $f: V \to W$ be a map of G–modules, $P_* \to V$ be a G–chain complex with P_n relatively $\mathcal{F}$–projective for all $n \geq 0$, and $C_* \to W$ be an $\mathcal{F}$–strong resolution of G–modules. Then there exists a G–chain map $f_*: P_* \to C_*$ extending f, which is unique up to G–chain homotopy.*

While relatively $\mathcal{F}$–projective $\mathcal{F}$–strong resolutions are useful to compute Bredon homology, the following dual approach will compute bounded Bredon cohomology.

A map $i\colon A \to B$ of normed G–modules is called $\mathcal{F}$–*strongly injective* if for each $H \in \mathcal{F}$ there exists a map $\sigma_H\colon B \to A$ of normed H–modules with $\|\sigma_H\|_\infty \leq K$ such that $\sigma_H \circ i = \mathrm{id}_A$, for a uniform constant $K \geq 0$. A normed G–module I is called *relatively $\mathcal{F}$–injective* if for every $\mathcal{F}$–strongly injective G–map $i\colon A \to B$ and every map $\psi\colon A \to I$ of normed G–modules, there exists a map $\Psi\colon B \to I$ of normed G–modules such that $\Psi \circ i = \psi$. A chain complex of normed G–modules is called *relatively $\mathcal{F}$–injective* if each chain module is relatively $\mathcal{F}$–injective. A resolution of normed G–modules is called $\mathcal{F}$–*strong* if it is contractible as a resolution of normed H–modules for each $H \in \mathcal{F}$.

Dually to Lemma 3.6 and Proposition 3.7 we obtain the following.

Lemma 3.8 *Let V be a normed G–module. The following hold:*

(i) *If S is a G–set with stabilizers in $\mathcal{F}$, then $\mathrm{bHom}_R(R[S], V)$ is a relatively $\mathcal{F}$–injective normed G–module.*

(ii) *If S is a G–set with $S^H \neq \varnothing$ for all $H \in \mathcal{F}$, then the resolution*
$$V \to \mathrm{bHom}_R(R[S^{*+1}], V)$$
of normed G–modules is $\mathcal{F}$–strong.

(iii) *If X is a G–space with contractible fixed-point set X^H for each $H \in \mathcal{F}$, then the resolution $V \to C_b^*(X; V)$ of normed G–modules is $\mathcal{F}$–strong.*

Proof Given an extension problem as in the definition of relative $\mathcal{F}$–injectivity,

we construct an extension Ψ as follows. Let T be a set of representatives of $G\backslash S$ and denote the stabilizer of an element $t \in T$ by G_t. Then for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1} s = t_s$. Define
$$\Psi\colon B \to \mathrm{bHom}_R(R[S], V)$$
for $b \in B$ and $s \in S$ by
$$\Psi(b)(s) = \psi(g_s \cdot \sigma_{G_{t_s}}(g_s^{-1}b))(s).$$
One checks that Ψ is a well-defined map of normed RG–modules extending ψ.

The proofs of (ii) and (iii) are dual to those of Lemma 3.6(ii) and (iii), respectively, and are left to the reader. □

Proposition 3.9 Let $f\colon V \to W$ be a map of normed G–modules, $V \to C^*$ be an $\mathcal{F}$–strong resolution of normed G–modules, and $W \to I^*$ be a G–chain complex with I^n relatively $\mathcal{F}$–injective for all $n \geq 0$. Then there exists a G–chain map $f^*\colon C^* \to I^*$ extending f, which is unique up to G–chain homotopy.

As a consequence of Proposition 3.9, we may use any relatively $\mathcal{F}$–injective $\mathcal{F}$–strong resolution to compute bounded Bredon cohomology. We obtain the isomorphisms analogous to (2-2) and (2-3) for Bredon cohomology.

Theorem 3.10 Let G be a group, $\mathcal{F}$ be a family of subgroups, and V be a normed G–module. For all $n \geq 0$ there is an isomorphism

$$H^n_{\mathcal{F},b}(G;V) \cong H^n_{G,b}(E_{\mathcal{F}}G;V).$$

Proof Both $C^*_{\mathcal{F},b}(G;V)$ and $C^*_b(E_{\mathcal{F}}G;V)$ are relatively $\mathcal{F}$–injective $\mathcal{F}$–strong resolutions of V by Lemma 3.8; hence G–chain homotopy equivalent by Proposition 3.9. □

Corollary 3.11 Let G be a group, N be a normal subgroup of G, and V be a normed G–module. For all $n \geq 0$ there is an isomorphism

$$H^n_{\mathcal{F}\langle N\rangle,b}(G;V) \cong H^n_b(G/N;V^N).$$

Proof As a model for $E_{\mathcal{F}\langle N\rangle}G$ we take $E(G/N)$ regarded as a G–space. Then it suffices to observe that

$$\mathrm{bHom}_{RG}(R[S_n(E(G/N))], V) \cong \mathrm{bHom}_{R[G/N]}(R[S_n(E(G/N))], V^N)$$

and to apply Theorem 3.10 twice. □

Analogous to (2-4) and (2-5) for Bredon cohomology, for a subgroup H of G and two families of subgroups $\mathcal{F} \subset \mathcal{G}$, we have the maps

$$\mathrm{res}^n_{H\subset G,b}\colon H^n_{\mathcal{F},b}(G;V) \to H^n_{\mathcal{F}|_H,b}(H;V), \quad \mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}\colon H^n_{\mathcal{G},b}(G;V) \to H^n_{\mathcal{F},b}(G;V)$$

for all normed G–modules V.

Remark 3.12 (Mineyev and Yaman's relative bounded cohomology) Mineyev and Yaman have introduced the bounded analogue of Bieri and Eckmann's relative cohomology for pairs (Remark 2.1) in [19]. For a group G, a finite set of subgroups $\mathcal{H}$, and a normed G–module V, their relative bounded cohomology groups $H_b^n(G, \mathcal{H}; V)$ can be identified with $H_{G,b}^n\big(EG, \bigsqcup_{H \in \mathcal{H}} G \times_H EH; V\big)$ and therefore fit in a long exact sequence

$$\cdots \to H_b^n(G, \mathcal{H}; V) \to H_b^n(G; V) \to \prod_{H \in \mathcal{H}} H_b^n(H; V) \to \cdots .$$

As in Remark 2.1, we denote by X the G–space obtained as a G–pushout from EG by collapsing $G \times_H EH$ to G/H for each $H \in \mathcal{H}$. Then we have maps

$$
\begin{array}{ccc}
H_{G,b}^n(X; V) & \longleftarrow & H_{G,b}^n(X, \bigsqcup_{H \in \mathcal{H}} G/H; V) \\
\uparrow & & \downarrow \\
H_{G,b}^n(E_{\mathcal{F}\langle\mathcal{H}\rangle}G; V) & & H_{G,b}^n(EG, \bigsqcup_{H \in \mathcal{H}} G \times_H EH; V)
\end{array}
$$

where, for $n \geq 2$, the horizontal map is an isomorphism by the long exact sequence of a pair, using the fact that $H_{G,b}^*(G/H; V) = 0$ for $* \geq 1$. Hence for $n \geq 2$ we obtain a map

$$H_{\mathcal{F}\langle\mathcal{H}\rangle,b}^n(G; V) \to H_b^n(G, \mathcal{H}; V).$$

However, even if $\mathcal{H}$ is a malnormal collection in which case X is a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$, this map need not be an isomorphism due to the failure of the excision axiom for bounded cohomology.

4 Characterization of relative amenability

In this section we prove a characterization of relatively amenable groups in terms of bounded Bredon cohomology analogous to Theorem 1.1.

Recall that a G–*invariant mean* on a G–set S is an $\mathbb{R}$–linear map $m \colon \ell^\infty(S) \to \mathbb{R}$ which is normalized, nonnegative, and G–invariant. (That is, for the constant function $1 \in \ell^\infty(S)$, $f \in \ell^\infty(S)$, and $g \in G$ we have $m(1) = 1$, $m(f) \geq 0$ if $f \geq 0$, and $m(g \cdot f) = m(f)$.) Note that for a G–map $S_1 \to S_2$ of G–sets, a G–invariant mean on S_1 is pushed forward to a G–invariant mean on S_2.

Definition 4.1 (relative amenability) A group G is *amenable relative* to a set of subgroups $\mathcal{H}$ if the G–set $G/\mathcal{H}$ admits a G–invariant mean. When G is amenable relative to $\mathcal{H}$ consisting of a single subgroup H, we say that H is *coamenable* in G.

When $\mathcal{H}$ is a finite set of subgroups, we recover the notion of relative amenability studied in [10]; see also [22].

Example 4.2 Let G be a group, H be a subgroup, and $\mathcal{H}$ be a set of subgroups.

  (i) If G is amenable, then G is amenable relative to $\mathcal{H}$.

 (ii) If H is a normal subgroup, then H is coamenable in G if and only if the quotient group G/H is amenable.

(iii) If H has finite index in G or contains the commutator subgroup $[G, G]$, then H is coamenable in G.

 (iv) If $\mathcal{H}$ is finite and G is amenable relative to $\mathcal{H}$, then $\mathcal{H}$ contains an element that is coamenable in G.

  (v) G is amenable relative to $\mathcal{H}$ if and only if G is amenable relative to $\mathcal{F}\langle\mathcal{H}\rangle$.

The following lemma is proved analogously to [6, Lemma 3.2]; see also [20, Corollary 5.3.8].

Lemma 4.3 *Let G be a group and $\mathcal{H}$ be a set of subgroups. Then G is amenable relative to $\mathcal{H}$ if and only if there exists a nontrivial G–invariant element in $\ell^\infty(G/\mathcal{H})^{\#}$.*

By Proposition 3.9 bounded Bredon cohomology can be computed using relatively $\mathcal{F}$–injective $\mathcal{F}$–strong resolutions. If one considers coefficients in dual normed $\mathbb{R}G$–modules, then such resolutions can be obtained from G–sets whose stabilizers are amenable relative to $\mathcal{F}$.

Lemma 4.4 *Let G be a group, $\mathcal{F}$ be a family of subgroups, and $V^{\#}$ be a dual normed $\mathbb{R}G$–module. If S is a G–set such that every stabilizer G_s is amenable relative to $\mathcal{F}|_{G_s}$, then the normed $\mathbb{R}G$–module $\mathrm{bHom}_{\mathbb{R}}(\mathbb{R}[S^{n+1}], V^{\#})$ is relatively $\mathcal{F}$–injective for all $n \geq 0$.*

Proof Since the stabilizers of S^{n+1} are intersections of stabilizers of S, and relative amenability passes to subgroups, it is enough to consider the case $n = 0$. Let an extension problem as in the definition of relative $\mathcal{F}$–injectivity

$$
\begin{array}{ccc}
& & \overset{\sigma_H}{\overbrace{}} \\
0 \longrightarrow A & \underset{i}{\overset{}{\rightleftarrows}} & B \\
\psi \downarrow & \swarrow \Psi & \\
\mathrm{bHom}_{\mathbb{R}}(\mathbb{R}[S], V^{\#}) & &
\end{array}
$$

be given. Let T be a set of representatives of $G \backslash S$. We denote the stabilizer of an element $t \in T$ by G_t and by assumption there exists a G_t–invariant mean m_t on $G_t / \mathcal{F}|_{G_t}$. Note that any subgroup $L \in \mathcal{F}|_{G_t}$ can also be viewed as an element in $\mathcal{F}$. Now, for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1} s = t_s$. Define $\Psi \colon B \to \mathrm{bHom}_{\mathbb{R}}(\mathbb{R}[S], V^\#)$ for $b \in B$, $s \in S$, and $v \in V$ by

$$\Psi(b)(s)(v) = m_{t_s}\big(gL \mapsto (g_s g \cdot \psi(\sigma_L(g^{-1} g_s^{-1} b)))(s)(v)\big).$$

One checks that Ψ is a well-defined map of normed $\mathbb{R}G$–modules extending ψ. $\qquad\square$

For a family of subgroups $\mathcal{F}$, consider the short exact sequence of normed $\mathbb{R}G$–modules

$$0 \to \mathbb{R} \to \ell^\infty(G/\mathcal{F}) \to \ell^\infty(G/\mathcal{F})/\mathbb{R} \to 0,$$

where $\mathbb{R}$ is regarded as the constant functions. Then the sequence of topological duals

$$0 \to (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\# \to \ell^\infty(G/\mathcal{F})^\# \to \mathbb{R} \to 0$$

is exact, since an $\mathbb{R}$–linear split $\mathbb{R} \to \ell^\infty(G/\mathcal{F})^\#$ is given by evaluation at the trivial coset of the trivial subgroup in $G/\mathcal{F}$. We define the *relative Johnson class*

$$[J_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\#)$$

as the cohomology class of the 1–cocycle $J_\mathcal{F} \in C^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\#)$ given by

$$J_\mathcal{F}(g_0 H_0, g_1 H_1) = \epsilon_{g_1 H_1} - \epsilon_{g_0 H_0},$$

where $\epsilon_{g_i H_i}$ is the evaluation map at $g_i H_i$ for $i = 0, 1$.

Theorem 4.5 *Let G be a group and $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups. The following are equivalent:*

(i) *Every subgroup $H \in \mathcal{G}$ is amenable relative to $\mathcal{F}|_H$.*

(ii) *The canonical map $H^n_{\mathcal{G},b}(G; V^\#) \to H^n_{\mathcal{F},b}(G; V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^\#$ and all $n \geq 0$.*

(iii) *The canonical map $H^1_{\mathcal{G},b}(G; V^\#) \to H^1_{\mathcal{F},b}(G; V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^\#$.*

(iv) *The relative Johnson class $[J_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^\#)$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F} \subset \mathcal{G},b}$.*

Proof Suppose that every subgroup $H \in \mathcal{G}$ is amenable relative to $\mathcal{F}|_H$. Then the resolution of normed $\mathbb{R}G$–modules $V^\# \to C_\mathcal{G}^*(G; V^\#)$ is $\mathcal{F}$–strong and relatively $\mathcal{F}$–injective by Lemmas 3.8(ii) and 4.4 applied to the G–set $G/\mathcal{G}$. Hence the canonical map $\mathrm{can}_{\mathcal{F} \subset \mathcal{G}, b}^n$ is an isomorphism for all $n \geq 0$ by Proposition 3.9.

The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious. Suppose that the relative Johnson class $[J_\mathcal{F}]$ lies in the image of the canonical map $\mathrm{can}_{\mathcal{F} \subset \mathcal{G}, b}^1$ and let $V := \ell^\infty(G/\mathcal{F})/\mathbb{R}$. We claim that for every subgroup $H \in \mathcal{G}$, the image of $[J_\mathcal{F}]$ under the restriction map

$$\mathrm{res}_{H \subset G, b}^1 \colon H_{\mathcal{F}, b}^1(G; V^\#) \to H_{\mathcal{F}|_H, b}^1(H; V^\#)$$

is trivial. Indeed, there is a commutative diagram

$$
\begin{array}{ccc}
H_{G,b}^1(E_\mathcal{G}G; V^\#) & \xrightarrow{\mathrm{can}_{\mathcal{F} \subset \mathcal{G}, b}^1} & H_{G,b}^1(E_\mathcal{F}G; V^\#) \\
\downarrow & & \downarrow \qquad \searrow^{\mathrm{res}_{H \subset G, b}^1} \\
H_{H,b}^1(E_\mathcal{G}G; V^\#) & \longrightarrow & H_{H,b}^1(E_\mathcal{F}G; V^\#) \xrightarrow{\cong} H_{H,b}^1(E_{\mathcal{F}|_H}H; V^\#)
\end{array}
$$

where the vertical maps are induced by viewing a G–space as an H–space. Observe that the lower left corner $H_{H,b}^1(E_\mathcal{G}G; V^\#)$ is trivial, since when viewed as an H–space $E_\mathcal{G}G$ is a model for $E_{\mathcal{ALL}|_H}H$ and hence H–equivariantly contractible. This proves the claim.

Now, fix a subgroup $H \in \mathcal{G}$ and denote $W := \ell^\infty(H/\mathcal{F}|_H)/\mathbb{R}$. Consider the commutative diagram of normed $\mathbb{R}H$–modules

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V^\# & \longrightarrow & \ell^\infty(G/\mathcal{F})^\# & \longrightarrow & \mathbb{R} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & W^\# & \longrightarrow & \ell^\infty(H/\mathcal{F}|_H)^\# & \longrightarrow & \mathbb{R} & \longrightarrow & 0
\end{array}
$$

where the rows are exact, and remain exact when restricted to L–fixed-points for every $L \in \mathcal{F}|_H$. By Lemma 3.4 there are associated long exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (V^\#)^H & \longrightarrow & (\ell^\infty(G/\mathcal{F})^\#)^H & \longrightarrow & \mathbb{R} & \xrightarrow{\partial_{V^\#}^0} & H_{\mathcal{F}|_H, b}^1(H; V^\#) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & (W^\#)^H & \longrightarrow & (\ell^\infty(H/\mathcal{F}|_H)^\#)^H & \longrightarrow & \mathbb{R} & \xrightarrow{\partial_{W^\#}^0} & H_{\mathcal{F}|_H, b}^1(H; W^\#) & \longrightarrow & \cdots
\end{array}
$$

on bounded cohomology. Observe that the image of $\partial_{V^\#}^0$ is precisely $\mathbb{R} \cdot \mathrm{res}_{H \subset G, b}^1[J_\mathcal{F}]$ and hence trivial by the claim above. This implies that the map $\partial_{W^\#}^0$ is trivial and hence

there exists a nontrivial H–invariant element in $\ell^\infty(H/\mathcal{F}|_H)^\#$. Thus H is amenable relative to $\mathcal{F}|_H$ by Lemma 4.3. This finishes the proof. $\qquad\square$

As special cases of Theorem 4.5 we obtain Theorem 1.3 by taking $\mathcal{G} = \mathcal{ALL}$ and Theorem 1.5 by taking $\mathcal{F} = \mathcal{TR}$. The case when $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G} = \mathcal{ALL}$ recovers Theorem 1.1.

Corollary 4.6 *Let X be a CW–complex with fundamental group G and $\mathcal{F}$ be a family consisting of amenable subgroups of G. Suppose that there exists a model for $E_\mathcal{F}G$ whose orbit space $G \backslash E_\mathcal{F}G$ is homotopy equivalent to a k–dimensional CW–complex. Then the comparison map $c_X^n \colon H_b^n(X;\mathbb{R}) \to H^n(X;\mathbb{R})$ vanishes for all $n > k$.*

Proof By Gromov's mapping theorem [7, page 40] — see also [6, Theorem 5.9] — the comparison map c_X^n vanishes if the comparison map

$$c_{EG}^n \colon H_{G,b}^n(EG;\mathbb{R}) \to H_G^n(EG;\mathbb{R})$$

vanishes. The G–map $EG \to E_\mathcal{F}G$ induces a commutative square

$$
\begin{array}{ccc}
H_{G,b}^n(EG;\mathbb{R}) & \xrightarrow{\ c_{EG}^n\ } & H_G^n(EG;\mathbb{R}) \\[4pt]
{\scriptstyle \mathrm{can}_{\mathcal{TR}\subset\mathcal{F},b}^n}\Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle \mathrm{can}_{\mathcal{TR}\subset\mathcal{F}}^n} \\[4pt]
H_{G,b}^n(E_\mathcal{F}G;\mathbb{R}) & \xrightarrow{\ c_{E_\mathcal{F}G}^n\ } & H_G^n(E_\mathcal{F}G;\mathbb{R})
\end{array}
$$

where the canonical map $\mathrm{can}_{\mathcal{TR}\subset\mathcal{F},b}^n$ is an isomorphism by Theorem 1.5. Since we are considering trivial coefficients, the lower right corner can be identified with the (nonequivariant) cohomology of the orbit space

$$H_G^n(E_\mathcal{F}G;\mathbb{R}) \cong H^n(G\backslash E_\mathcal{F}G;\mathbb{R})$$

(see eg [4, Theorem 4.2]). $\qquad\square$

As an application of Corollary 4.6 we obtain the following well-known examples.

Example 4.7 The comparison map vanishes in all positive degrees for CW–complexes whose fundamental groups are

  (i)   graph products of amenable groups (eg right-angled Artin groups);

  (ii)   fundamental groups of graphs of groups with amenable vertex groups.

Indeed, if G_Γ is a graph product of amenable groups, we consider the family $\mathcal{F}$ generated by the vertex groups and direct products of vertex groups whenever the corresponding vertices form a clique in the underlying graph Γ. We claim that there exists a model for $E_\mathcal{F}(G_\Gamma)$ with contractible orbit space. If Γ is a complete graph, then a model for $E_\mathcal{F}(G_\Gamma)$ is given by the point. Otherwise, Γ can be written as $\Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where Γ_i is a proper full subgraph of Γ for $i = 0, 1, 2$, and we have $G_\Gamma \cong G_{\Gamma_1} *_{G_{\Gamma_0}} G_{\Gamma_2}$. Let $\mathcal{F}_i$ be the corresponding family of subgroups of G_{Γ_i} for $i = 0, 1, 2$. Then a model for the classifying space $E_\mathcal{F}(G_\Gamma)$ can be constructed as the following G–pushout:

$$
\begin{array}{ccc}
G_\Gamma \times_{G_{\Gamma_0}} E_{\mathcal{F}_0}(G_{\Gamma_0}) \times S^0 & \longrightarrow & G_\Gamma \times_{G_{\Gamma_1}} E_{\mathcal{F}_1}(G_{\Gamma_1}) \coprod G_\Gamma \times_{G_{\Gamma_2}} E_{\mathcal{F}_2}(G_{\Gamma_2}) \\
\downarrow & & \downarrow \\
G_\Gamma \times_{G_{\Gamma_0}} E_{\mathcal{F}_0}(G_{\Gamma_0}) \times D^1 & \longrightarrow & E_\mathcal{F}(G_\Gamma)
\end{array}
$$

By induction on the number of vertices of Γ, the classifying spaces $E_{\mathcal{F}_i}(G_{\Gamma_i})$ have contractible orbit spaces for $i = 0, 1, 2$, and hence so does $E_\mathcal{F}(G_\Gamma)$.

If G is the fundamental group of a graph of groups with amenable vertex groups, we consider the family $\mathcal{F}$ generated by the vertex groups. Then the Bass–Serre tree is a 1–dimensional model for $E_\mathcal{F}G$. Recall that the comparison map always vanishes in degree 1, since $H_b^1(G; \mathbb{R})$ is trivial for every group G.

We also obtain a characterization of relative amenability via relatively $\mathcal{F}$–injective modules, analogous to [6, Proposition 4.18]; see also [20, Theorem 5.7.1].

Proposition 4.8 *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent*:

(i)   *G is amenable relative to $\mathcal{F}$.*

(ii) *Every dual normed $\mathbb{R}G$–module $V^\#$ is relatively $\mathcal{F}$–injective.*

(iii) *The trivial normed $\mathbb{R}G$–module $\mathbb{R}$ is relatively $\mathcal{F}$–injective.*

Proof   Suppose that G is amenable relative to $\mathcal{F}$ and let $m_\mathcal{F}$ be a G–invariant mean on $G/\mathcal{F}$. The inclusion $V^\# \to C_{\mathcal{F},b}^0(G; V^\#)$ of normed G–modules admits a right inverse r given by

$$
r(f)(v) = m_\mathcal{F}(gH \mapsto f(gH)(v))
$$

for $f \in C_{\mathcal{F},b}^0(G; V^\#)$ and $v \in V$. Then the relative $\mathcal{F}$–injectivity of $V^\#$ follows from the relative $\mathcal{F}$–injectivity of $C_{\mathcal{F},b}^0(G; V^\#)$.

Clearly, condition (ii) implies (iii). Suppose that $\mathbb{R}$ is relatively $\mathcal{F}$–injective. Consider the strongly $\mathcal{F}$–injective map $i : \mathbb{R} \to \ell^\infty(G/\mathcal{F})$ of normed G–modules that has an H–section τ_H given by $\tau_H(f) = f(eH)$ for each $H \in \mathcal{F}$. Then the identity $\mathrm{id}_\mathbb{R}$ admits an extension along i which yields a nontrivial G–invariant element in $\ell^\infty(G/\mathcal{F})^\#$. By Lemma 4.3 this finishes the proof. $\qquad\square$

Characterization of relative finiteness Analogously to Theorem 4.5, when instead considering all (not necessarily dual) normed $\mathbb{R}G$–modules, one obtains the theorem below. Let G be a group and $\mathcal{F}$ be a family of subgroups.

Let $\ell^1(G/\mathcal{F})$ denote the normed $\mathbb{R}G$–module of summable functions $f : G/\mathcal{F} \to \mathbb{R}$ with $\|f\|_1 = \sum_{gH \in G/\mathcal{F}} |f(gH)|$. Let $\ell^1_0(G/\mathcal{F})$ be the kernel of the map $\ell^1(G/\mathcal{F}) \to \mathbb{R}$, given by $f \mapsto \sum_{gH \in G/\mathcal{F}} f(gH)$. We define the class $[K_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; \ell^1_0(G/\mathcal{F}))$ as the cohomology class of the 1–cocycle $K_\mathcal{F} \in C^1_{\mathcal{F},b}(G; \ell^1_0(G/\mathcal{F}))$ given by

$$K_\mathcal{F}(g_0 H_0, g_1 H_1) = \chi_{g_1 H_1} - \chi_{g_0 H_0},$$

where $\chi_{g_i H_i}$ is the characteristic function supported at $g_i H_i$ for $i = 0, 1$.

We say that G is *finite relative* to $\mathcal{F}$, if $\mathcal{F}$ contains a finite index subgroup of G.

Theorem 4.9 *Let G be a group and $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups. The following are equivalent:*

(i) *Every subgroup $H \in \mathcal{G}$ is finite relative to $\mathcal{F}|_H$.*

(ii) *The canonical map $H^n_{\mathcal{G},b}(G; V) \to H^n_{\mathcal{F},b}(G; V)$ is an isomorphism for all normed $\mathbb{R}G$–modules V and all $n \geq 0$.*

(iii) *The canonical map $H^1_{\mathcal{G},b}(G; V) \to H^1_{\mathcal{F},b}(G; V)$ is an isomorphism for all normed $\mathbb{R}G$–modules V.*

(iv) *The class $[K_\mathcal{F}] \in H^1_{\mathcal{F},b}(G; \ell^1_0(G/\mathcal{F}))$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F} \subset \mathcal{G},b}$.*

Proof We only give a sketch of the proof which is entirely analogous to that of Theorem 4.5. Suppose that every subgroup $H \in \mathcal{G}$ is finite relative to $\mathcal{F}|_H$. One shows that the resolution of normed $\mathbb{R}G$–modules $V \to C^*_\mathcal{G}(G; V)$ is relatively $\mathcal{F}$–injective by taking averages over finite sets of cosets. Moreover, the resolution is $\mathcal{F}$–strong by Lemma 3.8(ii) and hence the canonical map $\mathrm{can}^n_{\mathcal{F} \subset \mathcal{G},b}$ is an isomorphism for all $n \geq 0$ by Proposition 3.9.

The implications (ii) $\implies$ (iii) $\implies$ (iv) are obvious. Suppose that the class $[K_{\mathcal{F}}]$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F} \subset \mathcal{G}, b}$. Fix a subgroup $H \in \mathcal{G}$ and consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ell^1_0(G/\mathcal{F}) & \longrightarrow & \ell^1(G/\mathcal{F}) & \longrightarrow & \mathbb{R} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \ell^1_0(H/\mathcal{F}|_H) & \longrightarrow & \ell^1(H/\mathcal{F}|_H) & \longrightarrow & \mathbb{R} & \longrightarrow & 0
\end{array}
$$

of normed $\mathbb{R}H$–modules. Following the proof of Theorem 4.5, one obtains a nontrivial H–invariant element $f \in \ell^1(H/\mathcal{F}|_H)$. Since f is constant on H–orbits, nontrivial, and summable, there exists a finite H–orbit in $H/\mathcal{F}|_H$. Thus H is finite relative to $\mathcal{F}|_H$. $\qquad\square$

Theorem 4.9 has the following interesting special cases. If $\mathcal{F}$ is arbitrary and $\mathcal{G} = \mathcal{ALL}$, we characterize that $\mathcal{F}$ contains a finite index subgroup of G. If $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G}$ is arbitrary, we characterize that all subgroups in $\mathcal{G}$ are finite, generalizing [23, Theorem B]. We recover the characterization of finite groups [6, Theorem 3.12] for $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G} = \mathcal{ALL}$.

5 Characterization of relative hyperbolicity

In this section we prove a characterization of relatively hyperbolic groups in terms of bounded Bredon cohomology analogous to Theorem 1.2.

Let G be a finitely generated group and $\mathcal{H}$ be a finite set of subgroups. Recall that G is *hyperbolic relative to* $\mathcal{H}$ if the coned–off Cayley graph is hyperbolic and fine; see eg [8]. For example, hyperbolic groups are hyperbolic relative to the trivial subgroup, free products $G_1 * G_2$ are hyperbolic relative to $\{G_1, G_2\}$, and fundamental groups of finite volume hyperbolic manifolds are hyperbolic relative to the cusp subgroups. If G is hyperbolic relative to $\mathcal{H}$, then $\mathcal{H}$ is almost malnormal and hence malnormal if G is torsionfree.

From now on, let the ring R be either $\mathbb{Q}$ or $\mathbb{R}$. A map $f : C \to B$ of normed RG–modules is called *undistorted* if there exists a constant $K \geq 0$ such that for all $b \in \mathrm{im}(f)$ there exists $c \in C$ with $f(c) = b$ such that $\|c\|_C \leq K \cdot \|b\|_B$. A normed RG–module P is called *boundedly projective* if for every undistorted epimorphism $f : C \to B$ and every map $\phi : P \to B$ of normed RG–modules, there exists a map $\Phi : P \to C$ of normed RG–modules such that $f \circ \Phi = \phi$.

The following lemma [19, Lemma 52] is useful to construct G–equivariant maps.

Lemma 5.1 (Mineyev and Yaman) *Let G be a group and S be a G–set with finite stabilizers. Then $\mathbb{Q}[S]$ is projective as a $\mathbb{Q}G$–module and boundedly projective as a normed $\mathbb{Q}G$–module when equipped with the ℓ^1–norm.*

Let X be a G–CW–complex with cocompact $(n+1)$–skeleton and consider for $k \geq 0$ the cellular chains $C_k^{\mathrm{cell}}(X; R)$ as a normed RG–module equipped with the ℓ^1–norm. We say that X satisfies a *linear homological isoperimetric inequality over R in degree n* if the boundary map

$$\partial_{n+1} \colon C_{n+1}^{\mathrm{cell}}(X; R) \to C_n^{\mathrm{cell}}(X; R)$$

is undistorted. Equivalently, there exists a constant $K \geq 0$ such that for every cellular n–boundary $b \in B_n^{\mathrm{cell}}(X; R)$ we have $\|b\|_\partial \leq K \cdot \|b\|_1$, where

$$\|b\|_\partial := \inf\{\|c\|_1 \mid c \in C_{n+1}^{\mathrm{cell}}(X; R), \partial_{n+1}(c) = b\}$$

(which is sometimes called the *filling norm*). (In [16], the terminology of the *uniform boundary condition* is used for a linear homological isoperimetric inequality.)

If G is hyperbolic relative to $\mathcal{H}$, Mineyev and Yaman [19, Theorem 41] have constructed the so-called "ideal complex" X. It is in particular a cocompact G–CW–complex with precisely one equivariant 0–cell G/H for each $H \in \mathcal{H}$ and finite edge-stabilizers. Moreover, X is (nonequivariantly) contractible and hence a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ provided that G is torsionfree. We summarize some of its properties [19, Theorems 47 and 51].

Theorem 5.2 (Mineyev and Yaman) *Let G be a finitely generated torsionfree group and $\mathcal{H}$ be a finite set of subgroups. If G is hyperbolic relative to $\mathcal{H}$, then there exists a cocompact model X for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ such that*

(i) *X satisfies linear homological isoperimetric inequalities over $\mathbb{Q}$ in degree n for all $n \geq 1$;*

(ii) *there exists a map $q \colon X^{(0)} \times X^{(0)} \to C_1^{\mathrm{cell}}(X; \mathbb{Q})$ with $\partial_1(q(a, b)) = b - a$, called a homological $\mathbb{Q}$–bicombing, that is G–equivariant and satisfies*

$$\|q(a, b) + q(b, c) - q(a, c)\|_1 \leq K$$

for all $a, b, c \in X^{(0)}$ and a uniform constant $K \geq 0$.

The following criterion for relative hyperbolicity is a combination of [5, Proposition 8.3 and Theorem 8.5]; see also [14, Theorems 1.6 and 1.10].

Theorem 5.3 (Franceschini [5] and Martínez-Pedroza [14]) *Let G be a group and $\mathcal{H}$ be a finite set of subgroups. Then G is hyperbolic relative to $\mathcal{H}$ if there exists a G–CW–complex Z such that*

(i) *Z is simply connected;*

(ii) *the 2–skeleton $Z^{(2)}$ is cocompact;*

(iii) *$\mathcal{H}$ is a set of representatives of distinct conjugacy classes of vertex-stabilizers such that each infinite stabilizer is represented;*

(iv) *the edge-stabilizers of Z are finite;*

(v) *Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1.*

We prove the following characterization of relative hyperbolicity closely following Mineyev's original proof of Theorem 1.2 — [17, Theorem 11] and [18, Theorem 9].

Theorem 5.4 *Let G be a finitely generated torsionfree group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Let $\mathcal{F}$ be the family $\mathcal{F}\langle\mathcal{H}\rangle$ and suppose that G is of type $F_{2,\mathcal{F}}$. Then the following are equivalent:*

(i) *G is hyperbolic relative to $\mathcal{H}$.*

(ii) *The comparison map $H_{\mathcal{F},b}^n(G;V) \to H_{\mathcal{F}}^n(G;V)$ is surjective for all normed $\mathbb{Q}G$–modules V and all $n \geq 2$.*

(iii) *The comparison map $H_{\mathcal{F},b}^2(G;V) \to H_{\mathcal{F}}^2(G;V)$ is surjective for all normed $\mathbb{R}G$–modules V.*

Proof Suppose that G is hyperbolic relative to $\mathcal{H}$. Let X be the model for $E_{\mathcal{F}}G$ that is given by Mineyev and Yaman's ideal complex (Theorem 5.2) and Y be the simplicial model for $E_{\mathcal{F}}G$ with (nonequivariant) n–cells corresponding to $(G/\mathcal{F})^{n+1}$ for all $n \geq 0$. We claim that there is a G–chain map

$$\varphi_* : C_*^{\text{cell}}(Y;\mathbb{Q}) \to C_*^{\text{cell}}(X;\mathbb{Q})$$

with φ_n bounded for all $n \geq 2$, admitting a G–homotopy left inverse. We construct φ_* inductively as follows. In degree 0, we define

$$\varphi_0 : C_0^{\text{cell}}(Y;\mathbb{Q}) = \mathbb{Q}[G/\mathcal{F}] \to C_0^{\text{cell}}(X;\mathbb{Q})$$

to map a generator of the form eH to the vertex of X with stabilizer containing H. Then extend G–equivariantly and $\mathbb{Q}$–linearly to all of $\mathbb{Q}[G/\mathcal{F}]$. In degree 1, we define $\varphi_1 \colon C_1^{\text{cell}}(Y;\mathbb{Q}) \to C_1^{\text{cell}}(X;\mathbb{Q})$ on generators by

$$\varphi_1(g_0 H_0, g_1 H_1) = q(\varphi_0(g_0 H_0), \varphi_0(g_1 H_1)),$$

where q is the homological $\mathbb{Q}$–bicombing on X from Theorem 5.2(ii). Since both q and φ_0 are G–equivariant, so is φ_1. In degree 2, we consider the maps

(5-1)
$$
\begin{array}{ccc}
C_2^{\text{cell}}(Y;\mathbb{Q}) & \xrightarrow{\partial_2^Y} & C_1^{\text{cell}}(Y;\mathbb{Q}) \\
& & \downarrow{\varphi_1} \\
C_2^{\text{cell}}(X;\mathbb{Q}) & \xrightarrow{\partial_2^X} & C_1^{\text{cell}}(X;\mathbb{Q})
\end{array}
$$

and observe that the composition $\varphi_1 \circ \partial_2^Y$ is bounded by properties of q and that ∂_2^X is undistorted by Theorem 5.2(i). There is a G–invariant decomposition

$$C_2^{\text{cell}}(Y;\mathbb{Q}) \cong \mathbb{Q}[S_1] \oplus \mathbb{Q}[S_2],$$

where S_1 and S_2 denote the sets of 2–cells of Y with trivial and nontrivial stabilizers, respectively. We obtain a bounded G–map $\varphi_2 \colon C_2^{\text{cell}}(Y;\mathbb{Q}) \to C_2^{\text{cell}}(X;\mathbb{Q})$ by using the bounded projectivity of $\mathbb{Q}[S_1]$ (Lemma 5.1) and by setting φ_2 to be zero on $\mathbb{Q}[S_2]$. This renders the square (5-1) commutative because the edge-stabilizers of X are trivial.

Assuming that φ_n has been constructed, one analogously defines a bounded G–map φ_{n+1} using that ∂_{n+1}^X is undistorted by Theorem 5.2(i). Thus one obtains a G–chain map φ_* with φ_n bounded for $n \geq 2$. To conclude the claim, we note that $C_*^{\text{cell}}(Y;\mathbb{Q})$ is a relatively $\mathcal{F}$–projective $\mathcal{F}$–strong resolution of $\mathbb{Q}$ by Lemma 3.6. Hence by Proposition 3.7 any G–chain map $\psi_* \colon C_*^{\text{cell}}(X;\mathbb{Q}) \to C_*^{\text{cell}}(Y;\mathbb{Q})$ extending $\mathrm{id}_{\mathbb{Q}}$ is a G–homotopy left inverse of φ_*.

Now, let V be a normed $\mathbb{Q}G$–module. Applying $\mathrm{Hom}_{\mathbb{Q}G}(\,\cdot\,, V)$ yields a cochain map

$$\varphi^* \colon C_{\text{cell}}^*(X;V)^G \to C_{\text{cell}}^*(Y;V)^G$$

with homotopy right inverse ψ^*. In particular, the composition $\varphi^* \circ \psi^*$ induces the identity on $H^*(C_{\text{cell}}^*(Y;V)^G) \cong H_{\mathcal{F}}^*(G;V)$. Finally, for $n \geq 2$ let $c \in C_{\text{cell}}^n(Y;V)^G$ be a cocycle. Then $\varphi^n(\psi^n(c))$ and c represent the same cohomology class in $H_{\mathcal{F}}^n(G;V)$. We have

$$\|\varphi^n(\psi^n(c))\|_\infty = \|\psi^n(c) \circ \varphi_n\|_\infty \leq \|\psi^n(c)\|_\infty \cdot \|\varphi_n\|_\infty,$$

where φ_n is bounded by construction and so is $\psi^n(c) \in C^n_{\mathrm{cell}}(X; V)^G$ because X has only finitely many orbits of n–cells. Thus we have shown that for $n \geq 2$ every cohomology class in $H^n_{\mathcal{F}}(G; V)$ admits a bounded representative.

Obviously condition (ii) implies (iii). Suppose that the comparison map is surjective in degree 2 for coefficients in every normed $\mathbb{R}G$–module. Let Z be a model for $E_{\mathcal{F}}G$ with cocompact 2–skeleton. Since $\mathcal{H}$ is malnormal, by collapsing fixed-point sets of Z we may assume that for every nontrivial subgroup $H \in \mathcal{F}$ the fixed-point set Z^H consists of precisely one point. In other words, Z has one equivariant 0–cell of the form G/H for each $H \in \mathcal{H}$ and all other cells have trivial stabilizers. In order to apply Theorem 5.3 and conclude that G is hyperbolic relative to $\mathcal{H}$, it remains to verify that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1.

We take as coefficients the cellular 1–boundaries $V := B^{\mathrm{cell}}_1(Z; \mathbb{R})$ equipped with the norm $\|\cdot\|_\partial$. Let Y be the simplicial model for $E_{\mathcal{F}}G$. Then there is a G–chain homotopy equivalence

$$\psi_* \colon C^{\mathrm{cell}}_*(Z; \mathbb{R}) \to C^{\mathrm{cell}}_*(Y; \mathbb{R})$$

with G–homotopy inverse φ_*. Applying $\mathrm{Hom}_{\mathbb{R}G}(\,\cdot\,, V)$ yields a cochain homotopy equivalence

$$\psi^* \colon C^*_{\mathrm{cell}}(Y; V)^G \to C^*_{\mathrm{cell}}(Z; V)^G$$

with homotopy inverse φ^*. In particular, the composition $\psi^* \circ \varphi^*$ induces the identity on $H^*(C^*_{\mathrm{cell}}(Z; V)^G) \cong H^*_{\mathcal{F}}(G; V)$. Consider the 2–cocycle $u \in C^2_{\mathrm{cell}}(Z; V)^G$ given by the boundary map

$$u = \partial_2 \colon C^{\mathrm{cell}}_2(Z; \mathbb{R}) \to B^{\mathrm{cell}}_1(Z; \mathbb{R}) = V.$$

Then we can write

$$(5\text{-}2) \qquad\qquad u = (\psi^2 \circ \varphi^2)(u) + \delta^1_Z(v)$$

for some $v \in C^1_{\mathrm{cell}}(Z; V)^G$. Since the comparison map $H^2_{\mathcal{F}, b}(G; V) \to H^2_{\mathcal{F}}(G; V)$ is surjective by hypothesis, we can write

$$(5\text{-}3) \qquad\qquad \varphi^2(u) = u' + \delta^1_Y(v')$$

for a *bounded* 2–cocycle $u' \in C^2_{\mathrm{cell}}(Y; V)^G$ and some $v' \in C^1_{\mathrm{cell}}(Y; V)^G$. For a fixed vertex $y \in Y^{(0)} = G/\mathcal{F}$, let $\mathrm{Cone}_y \colon C^{\mathrm{cell}}_1(Y; \mathbb{R}) \to C^{\mathrm{cell}}_2(Y; \mathbb{R})$ be defined on generators by

$$\mathrm{Cone}_y((g_0 H_0, g_1 H_1)) = (y, g_0 H_0, g_1, H_1).$$

Obviously Cone_y preserves the ℓ^1–norms. For a G–CW–complex W, we denote the evaluation pairing by

$$\langle \cdot, \cdot \rangle_W \colon C^*_{\mathrm{cell}}(W; V)^G \times C^{\mathrm{cell}}_*(W; \mathbb{R}) \to V.$$

Now, for $b \in C^{\mathrm{cell}}_1(Z; \mathbb{R})$ and $c \in C^{\mathrm{cell}}_2(Z; \mathbb{R})$ with $\partial_2(c) = b$, we find by (5-2) that

$$
\begin{aligned}
b = \langle u, c \rangle_Z &= \langle (\psi^2 \circ \varphi^2)(u) + \delta^1_Z(v), c \rangle_Z \\
&= \langle (\psi^2 \circ \varphi^2)(u), c \rangle_Z + \langle v, \partial^Y_2(c) \rangle_Z \\
&= \langle \varphi^2(u), \psi_2(c) \rangle_Y + \langle v, b \rangle_Z.
\end{aligned}
$$

Since $\varphi^2(u)$ is a cocycle and $\psi_2(c) - \mathrm{Cone}_y(\partial^Y_2(\psi_2(c)))$ is a cycle and hence a boundary,

$$
\begin{aligned}
\langle \varphi^2(u), \psi_2(c) \rangle_Y &= \langle \varphi^2(u), \mathrm{Cone}_y(\partial^Y_2(\psi_2(c))) \rangle_Y \\
&= \langle \varphi^2(u), \mathrm{Cone}_y(\psi_1(b)) \rangle_Y \\
&= \langle u' + \delta^1_Y(v'), \mathrm{Cone}_y(\psi_1(b)) \rangle_Y \\
&= \langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y + \langle v', \partial^Y_2(\mathrm{Cone}_y(\psi_1(b))) \rangle_Y \\
&= \langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y + \langle v', \psi_1(b) \rangle_Y \\
&= \langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y + \langle \psi^1(v'), b \rangle_Z,
\end{aligned}
$$

where we used (5-3). Together, we have

$$b = \langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y + \langle \psi^1(v') + v, b \rangle_Z.$$

We claim that $\| \langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y \|_\partial \leq \| u' \|_\infty \cdot \| \mathrm{Cone}_y(\psi_1(b)) \|_1$. Indeed, consider the map induced by ∂_2 on coefficients

$$(\partial_2)_* \colon C^2_{\mathrm{cell}}(Y; C^{\mathrm{cell}}_2(Z; \mathbb{R}))^G \to C^2_{\mathrm{cell}}(Y; V)^G.$$

Since ∂_2 is surjective, there exists a preimage $\tilde{u}' \in C^2_{\mathrm{cell}}(Y; C^{\mathrm{cell}}_2(Z; \mathbb{R}))^G$ of u' under $(\partial_2)_*$ with $\| \tilde{u}' \|_\infty \leq \| u' \|_\infty$. Then $\langle \tilde{u}', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y \in C^{\mathrm{cell}}_2(Z; \mathbb{R})$ is a preimage of $\langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y$ under ∂_2 witnessing the desired inequality. Similarly, one shows that $\| \langle \psi^1(v') + v, b \rangle_Z \|_\partial \leq \| \psi^1(v') + v \|_\infty \cdot \| b \|_1$. It follows that

$$
\begin{aligned}
\| b \|_\partial &\leq \| \langle u', \mathrm{Cone}_y(\psi_1(b)) \rangle_Y \|_\partial + \| \langle \psi^1(v') + v, b \rangle_Z \|_\partial \\
&\leq \| u' \|_\infty \cdot \| \mathrm{Cone}_y(\psi_1(b)) \|_1 + \| \psi^1(v') + v \|_\infty \cdot \| b \|_1 \\
&= \| u' \|_\infty \cdot \| \psi_1(b) \|_1 + \| \psi^1(v') + v \|_\infty \cdot \| b \|_1 \\
&\leq (\| u' \|_\infty \cdot \| \psi_1 \|_\infty + \| \psi^1(v') + v \|_\infty) \cdot \| b \|_1.
\end{aligned}
$$

Finally, u' is bounded by construction and so are ψ_1 and $\psi^1(v') + v$ because they are G–maps with domain $C_1^{\text{cell}}(Z; \mathbb{R})$ and Z has only finitely many orbits of 1–cells. Thus we have shown that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1. This finishes the proof. $\qquad\square$

Remark 5.5 (groups with torsion) In Theorem 5.4, if the group G is not assumed to be torsionfree and $\mathcal{H}$ is instead assumed to be almost malnormal, one can still prove the equivalence of (i) and (iii). However, a few modifications are necessary which we shall only outline.

Assuming that G is hyperbolic relative to $\mathcal{H}$, Mineyev and Yaman's ideal complex has to be replaced by a Rips type construction X due to Martínez-Pedroza and Przytycki that is a model for $E_{\mathcal{F} \cup \mathcal{FIN}} G$. This complex X satisfies a linear homological isoperimetric inequality over $\mathbb{Z}$ in degree 1 [15, Corollary 1.5]. It is part of a hyperbolic tuple in the sense of [19, Definition 38] and hence admits a homological $\mathbb{Q}$–bicombing by [19, Theorem 47]. Then one can construct a G–chain map φ_* with φ_2 bounded similarly as before and conclude surjectivity of the comparison map in degree 2 for the family $\mathcal{F} \cup \mathcal{FIN}$. This implies the same for the family $\mathcal{F}$ over the ring $\mathbb{R}$.

For the converse implication, since $\mathcal{H}$ is almost malnormal, there exists a model Z for $E_{\mathcal{F}} G$ such that for every infinite subgroup $H \in \mathcal{F}$ the fixed-point set Z^H consists of precisely one point. Then one shows as before that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1 and concludes by Theorem 5.3.

We do not know whether condition (ii) is equivalent to (i) and (iii) in this case.

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