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Parametrized higher category theory

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We develop foundations for the theory of ∞–categories parametrized by a base ∞–category. Our main contribution is a theory of indexed homotopy limits and colimits, which specializes to a theory of G–colimits for G a finite group when the base is chosen to be the orbit category of G. We apply this theory to show that the G–∞–category of G–spaces is freely generated under G–colimits by the contractible G–space, thereby affirming a conjecture of Mike Hill.

55U35, 55U40; 55U10

1 Introduction

Motivation from equivariant homotopy theory This paper lays foundations for a theory of ∞–categories parametrized by a base ∞–category S. Our interest in this project originates in attempting to locate the core homotopy theories of interest in equivariant homotopy theory — those of G–spaces and G–spectra — within the appropriate ∞–categorical framework. To explain, let G be a finite group and let us review the definitions of the ∞–categories of G–spaces and G–spectra, with a view towards endowing them with universal properties.

Consider a category $\mathbf{Top}_G$ of (nice) topological spaces equipped with G–action, with morphisms given by the G–equivariant continuous maps. There are various homotopy theories that derive from this category, depending on the class of weak equivalences that one chooses to invert. At one end, we can invert the class $\mathcal{W}_1$ of G–equivariant maps which induce a weak homotopy equivalence of underlying topological spaces, forgetting the G–action. If we let $\mathbf{Spc}$ denote the ∞–category of spaces (ie ∞–groupoids), then inverting $\mathcal{W}_1$ obtains the *∞–category of spaces with G–action*

$$\mathbf{Top}_G[\mathcal{W}_1^{-1}] \simeq \mathrm{Fun}(BG, \mathbf{Spc}).$$

For many purposes, $\mathrm{Fun}(BG, \mathbf{Spc})$ is the homotopy theory that one wishes to contemplate, but here we instead highlight its main deficiency. Namely, passing to this homotopy theory blurs the distinction between homotopy fixed points and actual fixed points, in that the functor $\mathbf{Top}_G \to \mathrm{Fun}(BG, \mathbf{Spc})$ forgets the homotopy types of the various spaces X^H for H a nontrivial subgroup of G. Because many arguments in equivariant homotopy theory involve comparing X^H with the homotopy fixed points X^{hH}, we want to retain this data. To this end, we can instead let $\mathcal{W}$ be the class of G–equivariant maps which induce an equivalence on H–fixed points for every subgroup H of G. Let $\mathbf{Spc}_G := \mathbf{Top}_G[\mathcal{W}^{-1}]$; this is the *$\infty$–category of G–spaces*.

As with $\mathbf{Top}_G[\mathcal{W}_1^{-1}]$, we would like a description of $\mathbf{Spc}_G$ which eliminates any reference to topological spaces with G–action, for the purpose of comprehending its universal property. Elmendorf's theorem grants such a description: we have

$$\mathbf{Spc}_G \simeq \mathrm{Fun}(\boldsymbol{O}_G^{\mathrm{op}}, \mathbf{Spc}),$$

where $\boldsymbol{O}_G$ is the category of orbits of the group G. Thus, as an ∞–category, $\mathbf{Spc}_G$ is the *free cocompletion* of $\boldsymbol{O}_G$.

It is a more subtle matter to define the homotopy theory of G–spectra. There are at least three possibilities:

(1) The ∞–category of *Borel G–spectra*, ie spectra with G–action. This is
$$\mathbf{Sp}^{hG} := \mathrm{Fun}(BG, \mathbf{Sp}),$$
which is the stabilization of $\mathrm{Fun}(BG, \mathbf{Spc})$.

(2) The ∞–category of *"naive" G–spectra*, ie spectral presheaves on $\boldsymbol{O}_G$. This is
$$\mathbf{Sp}_G := \mathrm{Fun}(\boldsymbol{O}_G^{\mathrm{op}}, \mathbf{Sp}),$$
which is the stabilization of $\mathbf{Spc}_G$.[1]

(3) The ∞–category of *"genuine" G–spectra*, ie spectral Mackey functors on the category $\boldsymbol{F}_G$ of finite G–sets: Let $A^{\mathrm{eff}}(\boldsymbol{F}_G)$ be the effective Burnside $(2,1)$–category of G, given by taking as objects finite G–sets, as morphisms spans of finite G–sets, and as 2–morphisms isomorphisms between spans. Then the ∞–category of genuine G–spectra is defined to be
$$\mathbf{Sp}^G := \mathrm{Fun}^{\oplus}(A^{\mathrm{eff}}(\boldsymbol{F}_G), \mathbf{Sp}),$$
the ∞–category of direct-sum preserving functors from $A^{\mathrm{eff}}(\boldsymbol{F}_G)$ to $\mathbf{Sp}$.[2]

The third possibility incorporates essential examples of cohomology theories for G–spaces, such as equivariant K–theory, because G–spectra in this sense possess transfers along maps of finite G–sets, encoded by the covariant maps in $A^{\mathrm{eff}}(\boldsymbol{F}_G)$. It is thus what homotopy theorists customarily mean by G–spectra. However, from a categorical perspective it is a more mysterious object than the ∞–category of naive G–spectra, since it is *not* the stabilization of G–spaces. We are led to ask:

Question What is the universal property of $\mathbf{Sp}^G$? More precisely, we have an adjunction
$$\Sigma^\infty_+ : \mathbf{Spc}_G \rightleftarrows \mathbf{Sp}^G : \Omega^\infty$$
with the right adjoint given by taking $\Omega^\infty : \mathbf{Sp} \to \mathbf{Spc}$ objectwise and restricting along the evident map $\boldsymbol{O}_G^{\mathrm{op}} \to A^{\mathrm{eff}}(\boldsymbol{F}_G)$, and we would like a universal property for Σ^∞_+ or Ω^∞.

Put another way, what is the categorical procedure which manufactures $\mathbf{Sp}^G$ from $\mathbf{Spc}_G$?

[1] The usage of a subscript G to indicate presheaves on $\boldsymbol{O}_G$ (whether valued in spaces or spectra) is consistent with our later notation for the S–category of S–objects in an arbitrary ∞–category — see Construction 3.9.

[2] This is not the definition which first appeared in the literature for G–spectra, but it is equivalent to, for example, the homotopy theory of orthogonal G–spectra by the pioneering work of Guillou and May [6]. For an ∞–categorical treatment, see Barwick [1].

The key idea is that for this procedure of "G–stabilization" one needs to enforce "G–additivity" over and above the usual additivity satisfied by a stable ∞–category; that is, one wants the coincidence of coproducts and products indexed not just by finite sets but by finite sets with G–action. Reflecting upon the possible homotopical meaning of such a G–(co)product, we see that for a transitive G–set G/H, $\coprod_{G/H}$ and $\prod_{G/H}$ should be interpreted to mean the left and right adjoints to the restriction functor $\mathbf{Sp}^G \to \mathbf{Sp}^H$, ie the induction and coinduction functors, and G–additivity then becomes the Wirthmüller isomorphism. In particular, we see that G–additivity is not a property that $\mathbf{Sp}^G$ can be said to enjoy in isolation, but rather one satisfied by the *presheaf* $\underline{\mathbf{Sp}}^G$ of ∞–categories indexed by $\boldsymbol{O}_G$; here, for every G–orbit U, a choice of basepoint specifying an isomorphism $U \cong G/H$ yields an equivalence $\underline{\mathbf{Sp}}^G(U) \simeq \mathbf{Sp}^H$, and the functoriality in maps of orbits is that of conjugation and restriction (in particular, recording the residual actions of the Weyl groups on $\mathbf{Sp}^H$). Correspondingly, we must rephrase our question so as to inquire after the universal property of the *morphism of $\boldsymbol{O}_G$–presheaves*, $\underline{\Sigma}^\infty_+ \colon \underline{\mathbf{Spc}}_G \to \underline{\mathbf{Sp}}^G$, where $\underline{\Sigma}^\infty_+$ is objectwise given by genuine H–suspension ranging over all subgroups $H \leq G$.

We now pause to observe that for the purpose of this analysis the group G is of secondary importance as compared to its associated category of orbits $\boldsymbol{O}_G$. Indeed, we focused on G–additivity as the distinguishing feature of genuine vs naive G–spectra, as opposed to the invertibility of representation spheres, in order to evade representation theoretic aspects of equivariant stable homotopy theory. In order to frame our situation in its proper generality, let us now dispense with the group G and replace $\boldsymbol{O}_G$ by an arbitrary ∞–category T. Call a presheaf of ∞–categories on T a *T–category*. The *T–category of T–spaces* $\underline{\mathbf{Spc}}_T$ is given by the functor $T^{\mathrm{op}} \to \mathbf{Cat}_\infty$, $t \mapsto \mathrm{Fun}((T^{/t})^{\mathrm{op}}, \mathbf{Spc})$. Note that this specializes to $\underline{\mathbf{Spc}}_G$ when $T = \boldsymbol{O}_G$ because $\boldsymbol{O}_H \simeq (\boldsymbol{O}_G)^{/(G/H)}$; slice categories stand in for subgroups in our theory. With the theory of T–colimits advanced in this paper, we can then supply a universal property for $\underline{\mathbf{Spc}}_T$ *as a T–category*. Write $\underline{\mathrm{Fun}}_T$ for the internal hom in the ∞–category of T–categories, which is cartesian closed.

1.1 Theorem *Suppose T is any ∞–category. Then $\underline{\mathbf{Spc}}_T$ is T–cocomplete, and for any T–category E which is T–cocomplete, the T–functor of evaluation at the T–final object*[3]

$$\underline{\mathrm{Fun}}^L_T(\underline{\mathbf{Spc}}_T, E) \to \underline{\mathrm{Fun}}_T(*_T, E) \simeq E$$

[3] We define $*_T$ to be the constant T–presheaf valued at $*$, which is the final object in the ∞–category of T–categories.

induces an equivalence from the T–category of T–functors $\underline{\mathbf{Spc}}_T \to E$ which strongly preserve T–colimits to E. In other words, $\underline{\mathbf{Spc}}_T$ is freely generated under T–colimits by the final T–category.

1.2 Remark The notion of T–cocompleteness needed for the theorem is slightly more elaborate than one might naively expect. Namely, we say that a T–category C is *T–cocomplete* if for all $t \in T$, the pullback of C to a $T^{/t}$–category $C_{\underline{t}}$ (Notation 2.29) admits all (small) $T^{/t}$–colimits (Definition 5.13). Correspondingly, we say that a T–functor $F\colon C \to D$ strongly preserves T–colimits if for all $t \in T$, the pulled-back $T^{/t}$–functor $F_{\underline{t}}\colon C_{\underline{t}} \to D_{\underline{t}}$ preserves all $T^{/t}$–colimits (Definition 11.2).

When $T = \boldsymbol{O}_G$, this result was originally conjectured by Mike Hill.

To go further and define T–spectra, we need a condition on T so that it supports a theory of spectral Mackey functors. We say that T is *orbital* if T admits multipullbacks, by which we mean that its finite coproduct completion $\boldsymbol{F}_T$ admits pullbacks. The purpose of the orbitality assumption is to ensure that the effective Burnside category $A^{\mathrm{eff}}(\boldsymbol{F}_T)$ is well defined. Note that the slice categories $T_{/t}$ are orbital if T is. We define the *T–category of T–spectra* $\underline{\mathbf{Sp}}^T$ to be the functor $T^{\mathrm{op}} \to \mathbf{Cat}_\infty$ given by $t \mapsto \mathrm{Fun}^{\oplus}(A^{\mathrm{eff}}(\boldsymbol{F}_{T_{/t}}), \mathbf{Sp})$. We then have the following theorem of Denis Nardin concerning $\underline{\mathbf{Sp}}^T$ from [15], which resolves our question:

1.3 Theorem [15, Theorem 7.4] *Suppose T is an atomic[4] orbital ∞–category. Then $\underline{\mathbf{Sp}}^T$ is T–stable, and for any pointed T–category C which has all finite T–colimits, the functor of postcomposition by Ω^∞*

$$(\Omega^\infty)_*\colon \mathrm{Fun}_T^{T-\mathrm{rex}}(C, \underline{\mathbf{Sp}}^T) \to \mathrm{Lin}^T(C, \underline{\mathbf{Spc}}_T)$$

induces an equivalence from the ∞–category of T–functors $C \to \underline{\mathbf{Sp}}^T$ which preserve finite T–colimits to the ∞–category of T–linear functors $C \to \underline{\mathbf{Spc}}_T$, ie those T–functors which are fiberwise linear and send finite T–coproducts to T–products.

We hope that the two aforementioned theorems will serve to impress upon the reader the utility of the purely ∞–categorical work that we undertake in this paper.

1.4 Warning In contrast to this introduction thus far and the conventions adopted elsewhere — eg in [15] — we will henceforth speak of S–categories, S–colimits, etc for $S = T^{\mathrm{op}}$.

[4] This is an additional technical hypothesis which we do not explain here. It will not concern us in the body of the paper.

What is parametrized ∞–category theory?

Roughly speaking, parametrized ∞–category theory is an interpretation of the familiar notions of ordinary or "absolute" ∞–category theory within the $(\infty, 2)$–category of functors $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$, done relative to a fixed "base" ∞–category S. By "interpretation", we mean something along the lines of the program of Emily Riehl and Dominic Verity [16], which axiomatizes the essential properties of an $(\infty, 2)$–category that one needs to do formal category theory into the notion of an *∞–cosmos*, of which $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is an example. In an ∞–cosmos, one can write down in a formal way notions of limits and colimits, adjunctions, Kan extensions, and so forth. Working out what this means in the example of $\mathbf{Cat}_\infty$–valued functors is the goal of this paper. In the classical 2–categorical setting, such limits and colimits are referred to as "indexed" limits and colimits, so another perspective on this paper is that it extends indexed category theory to the ∞–categorical setting.

In contrast to Riehl and Verity, we will work within the model of *quasicategories* and not hesitate to use special aspects of our model (eg combinatorial arguments involving simplicial sets). We are motivated in this respect by the existence of a highly developed theory of *cocartesian fibrations* due to Jacob Lurie, which we review in Section 2. Cocartesian fibrations are our preferred way to model $\mathbf{Cat}_\infty$–valued functors, for two reasons:

(1) The data of a functor $F\colon S \to \mathbf{Cat}_\infty$ is overdetermined compared to that of a cocartesian fibration over S, in the sense that to define F one must prescribe an infinite hierarchy of coherence data, which under the functor-fibration correspondence amounts to prescribing an infinite sequence of compatible horn fillings.[5] Because of this, specifying a cocartesian fibration (which one ultimately needs to do in order to connect our theory to applications) is typically an easier task than specifying the corresponding functor to $\mathbf{Cat}_\infty$.

(2) The Grothendieck construction on a functor $S \to \mathbf{Cat}_\infty$ is made visible in the cocartesian fibration setup, as the total category of the cocartesian fibration. Many of our arguments involve direct manipulation of the Grothendieck construction, in order to relate or reduce notions of parametrized ∞–category theory to absolute ∞–category theory.

We have therefore tailored our exposition to the reader familiar with the first five chapters of Lurie [9]; the only additional major prerequisite is the part of Lurie [11,

[5] It is for this reason that one speaks of *straightening* a cocartesian fibration to a functor.

Appendix B] dealing with variants of the cocartesian model structure of [9, Section 3] and functoriality in the base.

Linear overview

Let us now give a section-by-section summary of the contents of this paper.

In Section 2 we define an *S–category* as a cocartesian fibration over S, and then collect some necessary preliminaries on cocartesian fibrations and model structures on categories of marked simplicial sets. In particular, we recapitulate Lurie's theorem that establishes conditions under which change-of-base adjunctions are Quillen (Theorem 2.24). This theorem will allow us to efficiently verify the fibrancy of many of the simplicial set constructions introduced in this paper.

In Section 3 we first define and study the internal hom $\underline{\mathrm{Fun}}_S(-,-)$ of S–categories (Definition 3.2). We then recall the *S–category of S–objects* $\underline{E}_S$ in an ∞–category E from Barwick, Dotto, Glasman, Nardin and Shah [2] (Construction 3.9), which computes the right adjoint to the forgetful functor $[C \to S] \mapsto C$. When $S = \boldsymbol{O}_G^{\mathrm{op}}$ and $E = \mathbf{Spc}$, this recovers the G–category of G–spaces $\underline{\mathbf{Spc}}_G$.

In Section 4 we first introduce the S–join $(- \star_S -)$ (Definition 4.1), which in terms of presheaves computes the fiberwise join. We then define and study two (canonically equivalent) S–slice constructions: for an S–functor $p\colon K \to C$, we have S–undercategories $C_{(p,S)/}$ and $C^{(p,S)/}$ and S–overcategories $C_{/(p,S)}$ and $C^{/(p,S)}$. The "lower" construction (Definition 4.17) is a direct generalization of Joyal's slice construction — cf [9, Proposition 1.2.9.2] — and participates in a Quillen adjunction with the S–join. The "upper" construction (Definition 4.26) proceeds by taking an S–fiber of the relevant map of S–functor categories. In practice, the upper S–slice is far easier to work with as its definition is less bound up with the intricate combinatorics of the S–join (which need to be thoroughly understood to even establish the fibrancy of the lower S–slice; see Proposition 4.11). However, it is easier to establish the universal mapping property of the S–slice using its lower incarnation (Proposition 4.25).

In Section 5 we initiate our study of S–colimits and S–limits by giving the basic Definition 5.2, and then discuss a few special cases: S–(co)limits in an S–category of S–objects, S–colimits indexed by constant S–diagrams, and S–colimits indexed by S–points (ie S–coproducts). We then explain how to deduce results about S–limits from S–colimits (or vice versa) by means of the vertical opposite construction (Corollary 5.25).

In Section 6 our main goal is to establish an S–analogue of Joyal's cofinality theorem [9, Theorem 4.1.3.1]: an S–functor $C \to D$ is S*–final* if and only if it is fiberwise final[6] (Theorem 6.7). Our strategy is to control the functoriality encoded by the S–slice category in terms of a construction, the *twisted slice* (Definition 6.5), fibered over the twisted arrow category $\widetilde{\mathbb{O}}(S)$; the right Kan extension of the latter will then obtain the former (Theorem 6.6). In fact, we first do the same for the internal hom $\underline{\mathrm{Fun}}_S$ itself (equation (6.3.1)). This may be thought of as a refinement of the end formula for an ∞–category of natural transformations (see Remark 6.4).

In Section 7 we introduce the notions of S–fibration, S–(co)cartesian fibration and S–bifibration (Definitions 7.1 and 7.9). We also introduce the free S–(co)cartesian fibration as an example (Definition 7.6).

In Section 8 we recall Lurie's definition of a relative adjunction and specialize it to the notion of an S–adjunction (Definition 8.3). We then prove a number of fundamental results about S–adjunctions — most notably, the fact that a left S–adjoint preserve S–colimits (Corollary 8.9).

In Section 9, given an S–cocartesian fibration $\phi\colon C \to D$ and an S–functor $F\colon C \to E$, we construct the left S–Kan extension $\phi_! F\colon D \to E$, which will also call the D–*parametrized* S*–colimit* of F. With our assumption on ϕ, we have that for every object $x \in D_s$, $(\phi_! F)(x)$ is computed as the $S^{s/}$–colimit of the restriction of F to the $S^{s/}$–fiber $C_{\underline{x}}$. This is precisely analogous to the situation where the left Kan extension along a cocartesian fibration is computed by taking colimits fiberwise. In order to construct $\phi_! F$, we need to solve the coherence problem of assembling the individual $S^{s/}$–colimits of $F_{\underline{s}}\colon C_{\underline{x}} \to E_{\underline{s}}$ (ranging over all $x \in D_s$) into a single S–functor out of D. We introduce the S–pairing (Construction 9.1), and subsequently the D–parametrized slice (Construction 9.8), to facilitate this. The problem of constructing $\phi_! F$ then ultimately reduces to choosing a section of a certain trivial Kan fibration defined in terms of the D–parametrized slice (Theorem 9.15).

In Section 10 we define left S–Kan extensions in general (Definition 10.1) and prove the basic existence and uniqueness result about them (Theorem 10.3). In contrast to the brutal simplex-by-simplex approach taken in [9, Section 4.3.2] to the construction of Kan extensions (cf [9, Lemma 4.3.2.13]), we instead reduce to the solved coherence problem for D–parametrized S–colimits via factoring the S–functor $\phi\colon C \to D$ to be extended along through the free S–cocartesian fibration on it. We remark that, to

[6]We write final and initial for what Lurie calls (left) cofinal and right cofinal, respectively.

our knowledge, the approach of Sections 9 and 10 give a novel[7] and more conceptual construction of Kan extensions even in the context of ordinary ∞–category theory. Lurie has since independently written up a treatment of (relative) Kan extensions along these lines in Kerodon [12, Section 7.3].

In Section 11 we recall the S–category of presheaves $\boldsymbol{P}_S(-)$, prove the S–Yoneda Lemma 11.1, discuss S–mapping spaces, and establish the universal property of $\boldsymbol{P}_S(-)$ as free S–cocompletion (Theorem 11.5), thereby proving Theorem 1.1.

In Section 12 we prove two Bousfield–Kan-style[8] decomposition results that express an arbitrary S–colimit as a geometric realization of either S–coproducts or S–space-indexed S–colimits (Theorems 12.13 and 12.6). The essential content behind such formulas lies in replacing a given diagram C with one fibered over $\Delta^{\mathrm{op}} \times S$ that possesses an S–final map to C. As a warmup, we first explain how this goes when S is a point (Corollaries 12.3 and 12.5); the resulting formula appears to be new in the case of coproducts, whereas the case of spaces was first obtained by Aaron Mazel-Gee in [14]. We then apply the S–Bousfield–Kan formula to show that, supposing S^{op} admits multipullbacks, an S–category is S–cocomplete if and only if it admits all S–(co)products and geometric realizations (Corollary 12.15).

Notation and conventions

Let C be an ∞–category. We write

$$\mathbb{O}(C) := \mathrm{Fun}(\Delta^1, C)$$

for the ∞–category of arrows in C. In this paper, we will frequently encounter fiber products of the form

$$A \times_{F,C,\mathrm{ev}_0} \mathbb{O}(C) \times_{\mathrm{ev}_1,C,G} B$$

where $F \colon A \to C$ and $G \colon B \to C$ are functors. To avoid notational clutter, we adopt the global convention that, unless otherwise decorated, fiber products with the source functor ev_0 are to be written on the left, and fiber products with the target functor ev_1 are to written on the right. Moreover, we will drop F and G from the notation if they are understood from context. For instance, we would write the preceding expression as $A \times_C \mathbb{O}(C) \times_C B$.

[7] All these results date to 2017.

[8] By this, we mean to refer to generalizations of the classical formula for writing a colimit as a coequalizer of coproducts, which were studied by Bousfield and Kan in the context of homotopy colimits with coequalizers replaced by geometric realization.

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2 Cocartesian fibrations and model categories of marked simplicial sets

Let S be an ∞–category. In this section, we give a rapid review of the theory of cocartesian fibrations and the surrounding apparatus of marked simplicial sets. This primarily serves to fix some of our notation and conventions for the remainder of the paper; for a more detailed exposition of these concepts, we refer the reader to [4]. In particular, the reader should be aware of our special notation (Notation 2.29) for the S–fibers of an S–functor.

Cocartesian fibrations

We begin with the basic definitions:

2.1 Definition Let $\pi: X \to S$ be a map of simplicial sets. Then π is a *cocartesian fibration* if:

(1) It is an *inner fibration*; for every $n > 1$, $0 < k < n$ and commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

the dotted lift exists.

(2) For every edge $\alpha: s_0 \to s_1$ in S and $x_0 \in X$ with $\pi(x_0) = s_0$, there exists an edge $e: x_0 \to x_1$ in X with $\pi(e) = \alpha$, such that e is π–*cocartesian*; for every

$n > 1$ and commutative square

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "f"] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

with $f|_{\Delta^{\{0,1\}}} = e$, the dotted lift exists.

Dually, π is a *cartesian fibration* if π^{op} is a cocartesian fibration.

A cocartesian (resp. cartesian) fibration $\pi \colon X \to S$ is said to be a *left* (resp. *right*) fibration if for every object $s \in S$ the fiber X_s is a Kan complex.

Now suppose $\pi \colon X \to S$ and $\rho \colon Y \to S$ are (co)cartesian fibrations. Then a *map of (co)cartesian fibrations* $f \colon X \to Y$ is a map of simplicial sets such that $\rho \circ f = \pi$ and f carries π–(co)cartesian edges to ρ–(co)cartesian edges. The collection of cocartesian fibrations over S and maps thereof organize into a subcategory $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/S}$ of the overcategory $\mathbf{Cat}_{\infty/S}$.

In this paper, owing to the importance of these notions we see fit to introduce more concise and suggestive terminology for cocartesian fibrations and left fibrations over S.

2.2 Definition An S*–category* (resp. S*–space*) C is a cocartesian (resp. left) fibration $\pi \colon C \to S$. An S*–functor* $F \colon C \to D$ between S–categories C and D is a map of cocartesian fibrations over S.

Given an S–category $\pi \colon C \to S$, an S*–subcategory* $D \subset C$ is a subcategory such that the restriction $\pi|_D$ is a cocartesian fibration and an edge in D is $\pi|_D$–cocartesian if and only if it is π–cocartesian. The inclusion functor then necessarily preserves cocartesian edges, so is an S–functor. We further say that D is a *full* S*–subcategory* if $D \subset C$ is in addition a full subcategory, or equivalently, for every $s \in S$, $D_s \subset C_s$ is a full subcategory.

2.3 Example (arrow ∞–categories) The arrow ∞–category $\mathbb{O}(S)$ of S is cocartesian over S via the target morphism ev_1, and cartesian over S via the source morphism ev_0. An edge

$$e \colon [s_0 \to t_0] \to [s_1 \to t_1]$$

in $\mathbb{O}(S)$ is ev_1–cocartesian (resp. ev_0–cartesian) if and only if $\mathrm{ev}_0(e)$ (resp. $\mathrm{ev}_1(e)$) is an equivalence in S.

The fiber of $\mathrm{ev}_0\colon \mathcal{O}(S) \to S$ over s is isomorphic to Lurie's "alternative" slice ∞–category $S^{s/}$. Using our knowledge of the ev_1–cocartesian edges, we see that ev_1 restricts to a left fibration $S^{s/} \to S$. In the terminology of [9, Proposition 4.4.4.5], this is a *corepresentable* left fibration. We will refer to the corepresentable left fibrations as *S–points*. Further emphasizing this viewpoint, we will often let $\underline{s}$ denote $S^{s/}$.

To a beginner, the lifting conditions of Definition 2.1 can seem opaque. Under our standing assumption that S is an ∞–category, we have a reformulation of the definition of cocartesian edge, and hence that of cocartesian fibration, which serves to illuminate its homotopical meaning.

2.4 Proposition *Let $\pi\colon X \to S$ be an inner fibration (so X is an ∞–category). Then an edge $e\colon x_0 \to x_1$ in X is π–cocartesian if and only if for every $x_2 \in X$, the commutative square of mapping spaces*

$$\begin{tikzcd} \mathrm{Map}_X(x_1, x_2) \arrow[r, "e^*"] \arrow[d, "\pi"] & \mathrm{Map}_X(x_0, x_2) \arrow[d, "\pi"] \\ \mathrm{Map}_S(\pi(x_1), \pi(x_2)) \arrow[r, "\pi(e)^*"] & \mathrm{Map}_S(\pi(x_0), \pi(x_2)) \end{tikzcd}$$

is homotopy cartesian.

With some work, Proposition 2.4 can be used to give an alternative, model-independent definition of a cocartesian fibration. We refer to Mazel-Gee's paper [13] for an exposition along these lines.

2.5 Example [9, Section 3.2.2] Let $\mathbf{Cat}_\infty$ denote the (large) ∞–category of (small) ∞–categories. Then there exists a *universal cocartesian fibration* $\mathcal{U} \to \mathbf{Cat}_\infty$, which is characterized up to contractible choice by the requirement that any cocartesian fibration $\pi\colon X \to S$ (with essentially small fibers) fits into a homotopy pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"] & \mathcal{U} \arrow[d] \\ S \arrow[r, "F_\pi"] & \mathbf{Cat}_\infty \end{tikzcd}$$

Concretely, one can take $\mathcal{U}$ to be the subcategory of the arrow category $\mathcal{O}(\mathbf{Cat}_\infty)$ spanned by the representable right fibrations and morphisms thereof.

As suggested by Example 2.5, the functor

$$\mathrm{Fun}(S, \mathbf{Cat}_\infty) \to \mathbf{Cat}_{\infty/S}^{\mathrm{cocart}}$$

given by pulling back $\mathcal{U} \to \mathbf{Cat}_\infty$ is an equivalence. The composition

$$\mathrm{Gr}\colon \mathrm{Fun}(S, \mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Cat}_{\infty/S}^{\mathrm{cocart}} \subset \mathbf{Cat}_{\infty/S}$$

is the *Grothendieck construction* functor. Since equivalences in $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ are detected objectwise, Gr is conservative. Moreover, one can check that Gr preserves limit and colimits, so by the adjoint functor theorem Gr admits both a left and a right adjoint.

2.6 Notation Let

$$\mathrm{Fr} \dashv \mathrm{Gr} \dashv H$$

denote the left and right adjoints of Gr.

We call Fr the *free cocartesian fibration* functor (see also [5]); concretely, one has

$$\mathrm{Fr}(X \to S) = X \times_S \mathcal{O}(S) \xrightarrow{\mathrm{ev}_1} S,$$

or as a functor $s \mapsto X \times_S S_{/s}$ with functoriality obtained from $S_{/(-)}$. The functor H can also be concretely described using its universal mapping property: since

$$\mathrm{Fr}(\{s\} \subset S) = S_{s/},$$

the fiber $H(X)_s$ is given by $\mathrm{Fun}_{/S}(S_{s/}, X)$, and the functoriality in S is obtained from that of $S_{(-)/}$.

A model structure for cocartesian fibrations

We want a model structure which has as its fibrant objects the cocartesian fibrations over a fixed simplicial set. However, it is clear that to define it we need some way to remember the data of the cocartesian edges. This leads us to introduce *marked simplicial sets*.

2.7 Definition A marked simplicial set $(X, \mathcal{E})$ is the data of a simplicial set X and a subset $\mathcal{E} \subset X_1$ of the edges of X, such that $\mathcal{E}$ contains all of the degenerate edges. We call $\mathcal{E}$ the set of *marked edges* of X. A map of marked simplicial sets $f : (X, \mathcal{E}) \to (Y, \mathcal{F})$ is a map of simplicial sets $f : X \to Y$ such that $f(\mathcal{E}) \subset \mathcal{F}$.

2.8 Notation We introduce notation for certain classes of marked simplicial sets. Let X be a simplicial set.

- $X^\flat$ is X with only the degenerate edges marked. To avoid notational clutter, we will sometimes suppress this notation and simply write X for $X^\flat$.
- $X^\sharp$ is X with all of its edges marked.
- Suppose that X is an ∞–category. Then $X^\sim$ is X with its equivalences marked.
- Suppose that $\pi : X \to S$ is an inner fibration. Then ${}_\natural X$ is X with its π–cocartesian edges marked, and $X^\natural$ is X with its π–cartesian edges marked.
- Let $n > 0$. Let ${}_\natural\Delta^n$ and ${}_\natural\Lambda^n_0$ denote Δ^n and Λ^n_0, respectively, with the edge $\{0,1\}$ marked (if it exists, ie excluding Δ^0 and $\Lambda^1_0 = \{0\}$) along with the degenerate edges. Dually, let $\Delta^{n\natural}$ and $\Lambda^{n\natural}_n$ denote Δ^n and Λ^n_n, respectively, with the edge $\{n-1, n\}$ marked.

Note that our choice of notation ${}_\natural\Delta^n$ and ${}_\natural\Lambda^n_0$ is not meant to be interpreted as a special instance of marking cocartesian edges (though the map $\Delta^n \to \Delta^1$ given by $0 \mapsto 0$ and $1, \dots, n \mapsto 1$ renders it as such for the former); rather, we mean to indicate that the relevant lifting problem for a cocartesian fibration as a marked simplicial set is to lift along the marked horn inclusion ${}_\natural\Lambda^n_0 \to {}_\natural\Delta^n$ (cf Definition 2.9 below), and vice versa for cartesian fibrations and $\Lambda^{n\natural}_n \to \Delta^{n\natural}$.

For the rest of this section, fix a marked simplicial set $(Z, \mathcal{E})$ where Z is an ∞–category and $\mathcal{E}$ contains all of the equivalences in Z — in our applications, Z will generally be some type of fibration over S. Let $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ denote the category of marked simplicial sets over $(Z, \mathcal{E})$. Following Lurie [9, Notation 3.1.0.2], we will also denote $s\mathbf{Set}^+_{/Z^\sharp}$ more simply as $s\mathbf{Set}^+_{/Z}$. We will frequently abuse notation by referring an object $\pi : (X, \mathcal{F}) \to (Z, \mathcal{E})$ of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ by its domain $(X, \mathcal{F})$, or even just by X.

2.9 Definition An object $(X, \mathcal{F})$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is *$(Z, \mathcal{E})$–fibered*[9] if:

(1) $\pi : X \to Z$ is an inner fibration.

(2) For every $n > 0$ and commutative square

$$\begin{tikzcd} {}_\natural\Lambda^n_0 \arrow[r] \arrow[d] & (X, \mathcal{F}) \arrow[d] \\ {}_\natural\Delta^n \arrow[r] \arrow[ur, dotted] & (Z, \mathcal{E}) \end{tikzcd}$$

[9]This differs from [11, Definition B.0.19], but nonetheless defines the correct class of anodyne morphisms [11, Definition B.1.1].

a dotted lift exists. In other words, letting $n = 1$, π–cocartesian lifts exist over marked edges in Z, and letting $n > 1$, marked edges in X are π–cocartesian.[10]

(3) For every commutative square

$$\begin{tikzcd} (\Lambda^2_1)^\sharp \cup_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \arrow[r] \arrow[d] & (X, \mathcal{F}) \arrow[d] \\ (\Delta^2)^\sharp \arrow[r] \arrow[ur, dotted] & (Z, \mathcal{E}) \end{tikzcd}$$

a dotted lift exists. In other words, marked edges are closed under composition.[11]

(4) Let $Q = \Delta^0 \amalg_{\Delta^{\{0,2\}}} \Delta^3 \amalg_{\Delta^{\{1,3\}}} \Delta^0$. For every commutative square

$$\begin{tikzcd} Q^\flat \arrow[r] \arrow[d] & (X, \mathcal{F}) \arrow[d] \\ Q^\sharp \arrow[r] \arrow[ur, dotted] & (Z, \mathcal{E}) \end{tikzcd}$$

a dotted lift exists. Since we assumed that $\mathcal{E}$ contains all equivalences in Z, this implies that all equivalences in X are marked.

2.10 Example Let $\pi\colon X \to Z$ be an inner fibration. Comparing with Definition 2.1, it is clear that $(X, \mathcal{F})$ is $Z^\sharp$–fibered if and only if π is a cocartesian fibration and $(X, \mathcal{F}) = {}_\natural X$. At the other extreme, $(X, \mathcal{F})$ is $Z^\sim$–fibered if and only if π is a categorical fibration and $(X, \mathcal{F}) = X^\sim$.

Recall that a model structure, if it exists, is determined by its cofibrations and fibrant objects. Collecting results of Lurie from [11, Appendix B], we now define a model structure on $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ with cofibrations the monomorphisms and fibrant objects given by the $(Z, \mathcal{E})$–fibered objects.

2.11 Definition Define functors[12]

$$\mathrm{Map}_Z(-,-)\colon {s\mathbf{Set}^+_{/(Z,\mathcal{E})}}^{\mathrm{op}} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \to s\mathbf{Set},$$
$$\mathrm{Fun}_Z(-,-)\colon {s\mathbf{Set}^+_{/(Z,\mathcal{E})}}^{\mathrm{op}} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \to s\mathbf{Set}$$

[10]Condition (2) already guarantees that $X \to Z$ is a cocartesian fibration if $\mathscr{E} = Z_1$; however, one additionally needs condition (4) to ensure that *all* of the π–cocartesian edges are marked in X.

[11]Strictly speaking, condition (3) by itself only guarantees that for any pair of composable marked edges, there exists a composite that is again marked. One additionally needs condition (4) to ensure that *all* compositions of marked edges are again marked.

[12]In [11, Appendix B], these functors are denoted as $\mathrm{Map}^\sharp_Z$ and $\mathrm{Map}^\flat_Z$ respectively.

by

$$\mathrm{Hom}(A, \mathrm{Map}_Z(X,Y)) = \mathrm{Hom}_{/(Z,\mathcal{E})}(A^\sharp \times X, Y),$$
$$\mathrm{Hom}(A, \mathrm{Fun}_Z(X,Y)) = \mathrm{Hom}_{/(Z,\mathcal{E})}(A^\flat \times X, Y).$$

2.12 Definition A map $f\colon A \to B$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ is a *cocartesian equivalence* (with respect to $(Z,\mathcal{E})$) if one of the following equivalent conditions hold.

(1) For all $(Z,\mathcal{E})$–fibered X, $f^*\colon \mathrm{Map}_Z(B,X) \to \mathrm{Map}_Z(A,X)$ is an equivalence of Kan complexes.

(2) For all $(Z,\mathcal{E})$–fibered X, $f^*\colon \mathrm{Fun}_Z(B,X) \to \mathrm{Fun}_Z(A,X)$ is an equivalence of ∞–categories.

2.13 Theorem [11, Theorem B.0.20] *There exists a left proper combinatorial model structure on the category $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$, which we call the* ***cocartesian model structure****, such that*

(1) *the cofibrations are the monomorphisms*,

(2) *the weak equivalences are the cocartesian equivalences*,

(3) *the fibrant objects are the $(Z,\mathcal{E})$–fibered objects.*

Dually, we define the ***cartesian model structure*** *on $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ to be the cocartesian model structure on $s\mathbf{Set}^+_{/(Z,\mathcal{E})^{\mathrm{op}}}$ under the isomorphism given by taking opposites.*

2.14 Remark The underlying ∞–category of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ identifies as the subcategory of $\mathbf{Cat}_{\infty/Z}$ on those isofibrations[13] $X \to Z$ that admit cocartesian lifts over $\mathcal{E}$, and with morphisms preserving cocartesian edges. In particular, passing to the closure of $\mathcal{E}$ under composition does not change the underlying ∞–category.

We have the following characterization of the cocartesian equivalences between fibrant objects — which is unsurprising, in light of the equivalence $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/Z} \simeq \mathrm{Fun}(Z, \mathbf{Cat}_\infty)$.

2.15 Proposition [11, Lemma B.2.4] *Let X and Y be fibrant objects in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ equipped with the cocartesian model structure, and let $f\colon X \to Y$ be a map in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$. Then the following are equivalent*:

[13] With this choice, the resulting subcategory is not stable under equivalence. One could alternatively appeal to a homotopy-invariant notion of cocartesian fibration and instead replace isofibrations with functors — cf [13], which admits an obvious generalization to this setting.

(1) f *is a cocartesian equivalence.*

(2) f *is a homotopy equivalence, ie f admits a homotopy inverse; there exists a map $g\colon Y \to X$ and homotopies $h\colon (\Delta^1)^\sharp \times X \to X$ and $h'\colon (\Delta^1)^\sharp \times Y \to Y$ in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ connecting $g \circ f$ to id_X and $f \circ g$ to id_Y, respectively.*

(3) f *is a categorical equivalence.*

(4) *For every (not necessarily marked) edge $\alpha\colon \Delta^1 \to Z$, $f_\alpha\colon \Delta^1 \times_Z X \to \Delta^1 \times_Z Y$ is a categorical equivalence.*

If every edge of Z is marked, then (4) can be replaced by the following apparently weaker condition:

(4′) *For every object $z \in Z$, $f_z\colon X_z \to Y_z$ is a categorical equivalence.*

We also have the following characterization of the fibrations between fibrant objects.

2.16 Proposition [11, Proposition B.2.7] *Let $Y = (Y, \mathcal{F})$ be a fibrant object in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ equipped with the cocartesian model structure, and let $f\colon X \to Y$ be a map in $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$. Then the following are equivalent:*

(1) f *is a fibration.*

(2) X *is fibrant, and f is a categorical fibration.*

(3) f *is fibrant in $s\mathbf{Set}^+_{/(Y,\mathcal{F})}$.*

2.17 Corollary *Suppose $Z \to S$ is a cocartesian fibration. Then the cocartesian model structure $s\mathbf{Set}^+_{/_\natural Z}$ coincides with the "slice" model structure on $(s\mathbf{Set}^+_{/S})_{/_\natural Z}$ created by the forgetful functor to $s\mathbf{Set}^+_{/S}$ equipped with its cocartesian model structure.*

Proof This immediately follows from Proposition 2.16. □

2.18 Example Suppose that Z is a Kan complex. Then the cocartesian and cartesian model structures on $s\mathbf{Set}^+_{/Z}$ coincide. In particular, taking $Z = \Delta^0$, we will also refer to the cocartesian model structure on $s\mathbf{Set}^+$ as the *marked model structure*. Since this model structure on $s\mathbf{Set}^+$ is unambiguous, we will always regard $s\mathbf{Set}^+$ as equipped with it. Then the fibrant objects of $s\mathbf{Set}^+$ are precisely the ∞–categories with their equivalences marked.

2.19 Example Suppose that $(Z, \mathcal{E}) = Z^{\sim}$. Then the cocartesian and cartesian model structures on $s\mathbf{Set}^+_{/Z^{\sim}}$ coincide. Moreover, we have a Quillen equivalence

$$(-)^\flat \colon (s\mathbf{Set}_{\mathrm{Joyal}})_{/Z} \rightleftarrows s\mathbf{Set}^+_{/Z^{\sim}} : U$$

where the functor U forgets the marking. In particular, $(-)^\flat$ sends categorical equivalences to marked equivalences.

2.20 Example The inclusion functor $\mathbf{Spc} \subset \mathbf{Cat}_\infty$ admits left and right adjoints B and ι, where B is the classifying space functor that inverts all edges and ι is the "core" functor that takes the maximal sub–∞–groupoid. These two adjunctions are modeled by the two Quillen adjunctions

$$U \colon s\mathbf{Set}^+ \rightleftarrows s\mathbf{Set}_{\mathrm{Quillen}} : (-)^\sharp, \quad (-)^\sharp \colon s\mathbf{Set}_{\mathrm{Quillen}} \rightleftarrows s\mathbf{Set}^+ : M.$$

Here $M(X, \mathscr{E})$ is the maximal subsimplicial set of X such that all of its edges are marked. In particular, $(-)^\sharp$ sends weak homotopy equivalences to marked equivalences.

2.21 Proposition [11, Remark B.2.5] *The bifunctor*

$$- \times - \colon s\mathbf{Set}^+_{/(Z_1, \mathcal{E}_1)} \times s\mathbf{Set}^+_{/(Z_2, \mathcal{E}_2)} \to s\mathbf{Set}^+_{/(Z_1 \times Z_2, \mathcal{E}_1 \times \mathcal{E}_2)}$$

is left Quillen. Consequently, the bifunctors

$$\mathrm{Map}_Z(-,-) \colon {s\mathbf{Set}^+_{/(Z,\mathcal{E})}}^{\mathrm{op}} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \to s\mathbf{Set}_{\mathrm{Quillen}},$$
$$\mathrm{Fun}_Z(-,-) \colon {s\mathbf{Set}^+_{/(Z,\mathcal{E})}}^{\mathrm{op}} \times s\mathbf{Set}^+_{/(Z,\mathcal{E})} \to s\mathbf{Set}_{\mathrm{Joyal}}$$

are right Quillen, so $s\mathbf{Set}^+_{/(Z,E)}$ is both an $s\mathbf{Set}_{\mathrm{Quillen}}$–enriched model category (with respect to Map_Z) and $s\mathbf{Set}_{\mathrm{Joyal}}$–enriched model category (with respect to Fun_Z).

2.22 Remark As explained in [16, Digression 1.2.13], by Proposition 2.21 the full subcategory of $s\mathbf{Set}^+_{/(Z,\mathcal{E})}$ spanned by the fibrant objects is an example of an ∞–cosmos [16, Definition 1.2.1].

Finally, we explain how the formalism of marked simplicial sets can be used to extract the pushforward functors implicitly defined by a cocartesian fibration. First, we need a lemma.

2.23 Lemma *For $n > 0$, the inclusion $i_n \colon \Delta^{n-1} \cong \Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}} \to {}_\natural\Delta^n$ is left marked anodyne. Consequently, for a cocartesian fibration $C \to S$, the map*

$$\mathrm{Fun}({}_\natural\Delta^n, {}_\natural C) \to \mathrm{Fun}(\Delta^{n-1}, C) \times_{\mathrm{Fun}(\Delta^{n-1}, C)} \mathrm{Fun}(\Delta^n, S)$$

induced by i_n is a trivial Kan fibration.

Proof We proceed by induction on n, the base case $n = 1$ being the left marked anodyne map $\Delta^{\{0\}} \to {}_\natural\Delta^1 = (\Delta^1)^\sharp$. Consider the commutative diagram

$$\begin{tikzcd}
\Delta^{\{0\}} \star \partial\Delta^{n-2} \arrow[r] \arrow[d, hook, "i_{n-1}"] & \Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}} \arrow[d] \arrow[dd, bend left, "i_n"] \\
(\Delta^{\{0\}} \star \Lambda^{n-1}_0, \mathscr{E}) \arrow[r] & {}_\natural\Lambda^n_0 \arrow[d] \\
& {}_\natural\Delta^n
\end{tikzcd}$$

where $\mathscr{E}$ is the collection of edges $\{0, i\}$, $0 < i \leq n$ (and the degenerate edges). The square is a pushout, and by the inductive hypothesis, the left-hand vertical map is left marked anodyne. We deduce that i_n is left marked anodyne. The second statement now follows because the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{Fun}({}_\natural\Delta^n, {}_\natural C) \arrow[d] \\
B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(\Delta^{n-1}, C) \times_{\mathrm{Fun}(\Delta^{n-1}, C)} \mathrm{Fun}(\Delta^n, S)
\end{tikzcd}$$

transposes to

$$\begin{tikzcd}
A \times {}_\natural\Delta^n \cup_{A \times \Delta^{n-1}} B \times \Delta^{n-1} \arrow[r] \arrow[d] & {}_\natural C \arrow[d] \\
B \times {}_\natural\Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

and the left-hand vertical map is left marked anodyne for any cofibration $A \to B$ by [9, Proposition 3.1.2.3]. $\square$

The main case of interest in Lemma 2.23 is when $n = 1$, which shows that

$$\mathbb{O}^{\mathrm{cocart}}(C) \to C \times_S \mathbb{O}(S)$$

is a trivial Kan fibration. Let

$$P : C \times_S \mathbb{O}(S) \to \mathbb{O}^{\mathrm{cocart}}(C)$$

be a section that fixes the inclusion $C \subset \mathbb{O}^{\mathrm{cocart}}(C)$ (for this, note that $C \subset C \times_S \mathbb{O}(S)$ is a cofibration as it is a monomorphism of simplicial sets). Then we say that P or the further composite $P' = \mathrm{ev}_1 \circ P$ is a *cocartesian pushforward* for $C \to S$. Given an edge α of S, $P'_\alpha : C_s \to C_t$ is the pushforward functor $\alpha_!$ determined under the equivalence $\mathbf{Cat}^{\mathrm{cocart}}_{\infty/S} \simeq \mathrm{Fun}(S, \mathbf{Cat}_\infty)$.

Functoriality in the base

Let $\pi : X \to Z$ be a map of simplicial sets. Then the pullback functor

$$\pi^* : s\mathbf{Set}_{/Z} \to s\mathbf{Set}_{/X}$$

admits a left adjoint $\pi_!$, given by postcomposing with π. In addition, since $s\mathbf{Set}$ is a topos, π^* also admits a right adjoint π_*, which may be thought of as the functor of relative sections because $\mathrm{Hom}_{/X}(A, \pi_*(B)) \cong \mathrm{Hom}_{/Z}(A \times_X Z, B)$.

Now supposing that π is a map of marked simplicial sets, π^*, $\pi_!$, and π_* extend to functors of marked simplicial sets over X or Y in an evident manner. We then seek conditions under which the adjunctions $\pi_! \dashv \pi^*$ and $\pi^* \dashv \pi_*$ are Quillen with respect to the cocartesian model structures. To this end, we have the following theorem of Lurie.

2.24 Theorem [11, Theorem B.4.2] *Let*

$$(Z, \mathcal{E}) \xleftarrow{\pi} (X, \mathcal{F}) \xrightarrow{\rho} (X', \mathcal{F}')$$

be a span of marked simplicial sets such that Z, X, X' are ∞–categories and the collections of markings contain all the equivalences.

(i) *The adjunction*

$$\rho_! : s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(X',\mathcal{F}')} : \rho^*$$

is Quillen with respect to the cocartesian model structures.

(ii) *Further suppose that*:

(1) *For every object $x \in X$ and marked edge $f : z \to \pi(x)$ in Z, there exists a locally π–cartesian edge $x_0 \to x$ in X lifting f.*

(2) *π is a flat categorical fibration.*

(3) *$\mathcal{E}$ and $\mathcal{F}$ are closed under composition.*

(4) *Suppose given a commutative diagram*

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"] & & x_2 \end{tikzcd}$$

in X where g is locally π–cartesian, $\pi(g)$ is marked, and $\pi(f)$ is an equivalence. Then f is marked if and only if h is marked. (*Note in particular that, taking f to be an identity morphism, every locally π–cartesian edge lying over a marked edge is itself marked.*)

Then the adjunction

$$\pi^* : s\mathbf{Set}^+_{/(X,\mathcal{F})} \rightleftarrows s\mathbf{Set}^+_{/(Z,\mathcal{E})} : \pi_*$$

is Quillen with respect to the cocartesian model structures.

We formulated Theorem 2.24 as a theorem concerning a span $Z \xleftarrow{\pi} X \xrightarrow{\rho} X'$ because in applications we will typically be interested in the composite Quillen adjunction

$$\rho_! \pi^* : s\mathbf{Set}^+_{/(Z,\mathcal{E})} \rightleftarrows s\mathbf{Set}^+_{/(X',\mathcal{F}')} : \pi_* \rho^*.$$

Here are two examples.

2.25 Example (pairing cartesian and cocartesian fibrations) Let $\pi : X \to Z$ be a cartesian fibration. Then the span

$$Z^\sharp \xleftarrow{\pi} X^\natural \xrightarrow{\pi} Z^\sharp$$

satisfies the hypotheses of Theorem 2.24. Now given a cocartesian fibration $Y \to Z$, define

$$\widetilde{\mathrm{Fun}}_Z(X, Y) := (\pi_* \pi^*)({}_\natural Y \to Z^\sharp).$$

Then the fiber of $\widetilde{\mathrm{Fun}}_Z(X, Y)$ over an object $z \in Z$ is $\mathrm{Fun}(X_z, Y_z)$, and given a morphism $\alpha : z_0 \to z_1$, the pushforward functor

$$\alpha_! : \mathrm{Fun}(X_{z_0}, Y_{z_0}) \to \mathrm{Fun}(X_{z_1}, Y_{z_1})$$

is given by precomposition in the source and postcomposition in the target. Note how this example highlights the relevance of condition (1) in Theorem 2.24(ii).

2.26 Example (right Kan extension) Let $f : Y \to Z$ be a functor. We can apply Theorem 2.24 to perform the operation of right Kan extension at the level of cocartesian fibrations. Consider the span

$$Z^\sharp \xleftarrow{\mathrm{ev}_0} (\mathbb{O}(Z) \times_{Z,f} Y)^\sharp \xrightarrow{\mathrm{pr}_Y} Y^\sharp.$$

Then the conditions of Theorem 2.24 are satisfied, so we obtain a Quillen adjunction

$$(\mathrm{pr}_Y)_! (\mathrm{ev}_0)^* : s\mathbf{Set}^+_{/Z} \rightleftarrows s\mathbf{Set}^+_{/Y} : (\mathrm{ev}_0)_* (\mathrm{pr}_Y)^*.$$

In addition, the map $C \times_Z Y^\sharp \to C \times_Z \mathbb{O}(Z)^\sharp \times_Z Y^\sharp$ induced by the identity section $\iota : Z \to \mathbb{O}(Z)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/Y}$ for $C \to Z$ *fibrant* in $s\mathbf{Set}^+_{/Z}$, by [2, Lemma 9.8]. Consequently, the induced adjunction of ∞–categories

$$(\mathrm{pr}_Y)_! (\mathrm{ev}_0)^* : \mathbf{Cat}^{\mathrm{cocart}}_{\infty/Z} \rightleftarrows \mathbf{Cat}^{\mathrm{cocart}}_{\infty/Y} : (\mathrm{ev}_0)_* (\mathrm{pr}_Y)^*$$

is equivalent to

$$f^*\colon \mathrm{Fun}(Z,\mathbf{Cat}_\infty) \rightleftarrows \mathrm{Fun}(Y,\mathbf{Cat}_\infty) : f_*$$

under the straightening/unstraightening equivalence (which is natural with respect to pullback).

Note that as a special case, if $Z = \Delta^0$ we recover the formula $\mathrm{Fun}_Y(Y^\sharp, {}_\natural C) \simeq \varprojlim F_C$ of [9, Corollary 3.3.3.2] (where $C \to Y$ is a cocartesian fibration and $F_C\colon Y \to \mathbf{Cat}_\infty$ the corresponding functor). Indeed, this construction of the right Kan extension of a cocartesian fibration is suggested by that result and the pointwise formula for a right Kan extension.

Finally, we will use the following two observations concerning the interaction of Theorem 2.24 with compositions and homotopy equivalences of spans — which we also recorded in [4].

2.27 Lemma *Suppose we have spans of marked simplicial sets*

$$X_0 \xleftarrow{\pi_0} Z_0 \xrightarrow{\rho_0} X_1 \quad \textit{and} \quad X_1 \xleftarrow{\pi_1} Z_1 \xrightarrow{\rho_1} X_2$$

which each satisfy the hypotheses of Theorem 2.24. Then the span

$$Z_0 \xleftarrow{\mathrm{pr}_0} Z_0 \times_{X_1} Z_1 \xrightarrow{\mathrm{pr}_1} Z_1$$

also satisfies the hypothesis of Theorem 2.24.[14] *Consequently, we obtain a Quillen adjunction*

$$(\rho_1 \circ \mathrm{pr}_1)_!(\pi_0 \circ \mathrm{pr}_0)^*\colon s\mathbf{Set}^+_{/X_0} \rightleftarrows s\mathbf{Set}^+_{X_2} : (\pi_0 \circ \mathrm{pr}_0)_*(\rho_1 \circ \mathrm{pr}_1)^*,$$

which is the composite of the Quillen adjunction from $s\mathbf{Set}^+_{/X_0}$ to $s\mathbf{Set}^+_{/X_1}$ with the one from $s\mathbf{Set}^+_{/X_1}$ to $s\mathbf{Set}^+_{/X_2}$.

Proof The assertion that the span satisfies the hypotheses of Theorem 2.24 is by inspection. The other assertion that the Quillen adjunction factors as a composite follows from the base-change isomorphism $\rho_0^*\pi_{1,*} \cong \mathrm{pr}_{0,*} \circ \mathrm{pr}_1^*$. □

[14] However, one should beware that the "long" span $X_0 \leftarrow Z_0 \times_{X_1} Z_1 \to X_2$ may fail to satisfy the hypotheses of Theorem 2.24, because the composition of locally cartesian fibrations may fail to again be locally cartesian; this explains the roundabout formulation of the statement.

2.28 Lemma *Suppose a morphism of spans of marked simplicial sets*

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[d, "f"] \arrow[dr, "\rho"] & \\ X & Z' \arrow[l, "\pi'"] \arrow[r, "\rho'"'] & X' \end{tikzcd}$$

where $\rho_!\pi^*$ and $(\rho')_!(\pi')^*$ are left Quillen with respect to the cocartesian model structures on X and X'. Suppose moreover that f is a homotopy equivalence in $s\mathbf{Set}^+_{/X'}$, so that there exists a homotopy inverse g and homotopies

$$h\colon \mathrm{id} \simeq g \circ f \quad \text{and} \quad k\colon \mathrm{id} \simeq f \circ g.$$

Then the natural transformation $\rho_!\pi^* \to (\rho')_!(\pi')^*$ induced by f is a cocartesian equivalence on all objects, and, consequently, the adjoint natural transformation $(\pi')_*(\rho')^* \to \pi_*\rho^*$ is a cocartesian equivalence on all fibrant objects.

Proof The homotopies h and k pull back to show that for all $X \to C$, the map

$$\mathrm{id}_X \times_C f\colon X \times_C K \to X \times_C L$$

is a homotopy equivalence with inverse $\mathrm{id}_X \times_C g$. The last statement now follows from [7, Corollary 1.4.4(b)]. □

Parametrized fibers

In this brief subsection, we record notation for the S–fibers of an S–functor.

2.29 Notation Given an S–category $\pi\colon D \to S$ and an object $x \in D$, define

$$\mathbb{O}_{x\to}(D) := \{x\} \times_D \mathbb{O}(D).$$

For the full subcategory of cocartesian edges $\mathbb{O}^{\mathrm{cocart}}(D) \subset \mathbb{O}(D)$, also define

$$\underline{x} := \{x\} \times_D \mathbb{O}^{\mathrm{cocart}}(D).$$

Given an S–functor $\phi\colon C \to D$, define

$$C_{\underline{x}} := \underline{x} \times_{D,\phi} C.$$

Note that by definition, the objects of $\underline{x}$ are π–cocartesian edges in D with source x. Then by the right cancellative property of π–cocartesian edges [9, Lemma 2.4.2.7], the morphisms in $\underline{x}$ are 2–simplices of cocartesian edges with source x; hence $\underline{x}$ is an S–space (via the map $\mathrm{ev}_1 \colon \underline{x} \to S$). In fact, by Lemma 12.10, $\mathrm{ev}_1 \colon \underline{x} \to S^{\pi x/}$ is a trivial fibration, so we may think of $\underline{x}$ as an "S–point" of D.

In view of this, we will also regard $C_{\underline{x}}$ as an $S^{\pi x/}$–category (and we will sometimes be cavalier about the distinction between $\underline{x}$ and $S^{\pi x/}$). Note however, that the functor $\underline{x} \to D$ is canonical in our setup, whereas we need to make a choice of cocartesian pushforward to choose an S–functor $S^{\pi x/} \to D$ that selects $x \in D$.

3 Functor categories

Let S be an ∞–category. Then $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$ is cartesian closed, so it possesses an internal hom. As a basic application of Theorem 2.24, we will define this internal hom at the level of cocartesian fibrations over S.

3.1 Proposition *Let $C \to S$ be a cocartesian fibration. Let $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathcal{O}(S) \times_S C \to S$ denote the source and target maps. Then the functor*

$$(\mathrm{ev}_1)_!(\mathrm{ev}_0)^* \colon s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{/\mathcal{O}(S)^\sharp \times_S {}_\natural C} \to s\mathbf{Set}^+_{/S}$$

is left Quillen with respect to the cocartesian model structures.

Proof We verify the hypotheses of Theorem 2.24 as applied to the span

$$S^\sharp \xleftarrow{\mathrm{ev}_0} \mathcal{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{\mathrm{ev}_1} S^\sharp.$$

By [9, Corollary 2.4.7.12], ev_0 is a cartesian fibration and an edge e in $\mathcal{O}(S) \times_S C$ is ev_0–cartesian if and only if its projection to C is an equivalence. Thus (1) holds. Item (2) holds since cartesian fibrations are flat categorical fibrations. Item (3) is obvious. Item (4) follows from the stability of cocartesian edges under equivalence. □

3.2 Definition In the statement of Proposition 3.1, let

$$\underline{\mathrm{Fun}}_S(C, -) := (\mathrm{ev}_0)_*(\mathrm{ev}_1)^* \colon s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{/S}.$$

We will also write this as $\underline{\mathrm{Fun}}_S({}_\natural C, -)$ if we wish to emphasize the marking.

Proposition 3.1 implies that if $D \to S$ is a cocartesian fibration, then $\underline{\mathrm{Fun}}_S(C, D) \to S$ is a cocartesian fibration. Unwinding the definitions, we see that an object of $\underline{\mathrm{Fun}}_S(C, D)$ over $s \in S$ is an $S^{s/}$–functor

$$S^{s/} \times_S C \to S^{s/} \times_S D,$$

and a cocartesian edge of $\underline{\mathrm{Fun}}_S(C,D)$ over an edge $e\colon \Delta^1 \to S$ is a $\Delta^1 \times_S \mathcal{O}(S)$–functor

$$\Delta^1 \times_S \mathcal{O}(S) \times_S C \to \Delta^1 \times_S \mathcal{O}(S) \times_S D.$$

Our first goal is to prove that the construction $\underline{\mathrm{Fun}}_S(C,-)$ implements the internal hom at the level of underlying ∞–categories. To this end, we have the following lemma and proposition.

3.3 Lemma *Let $\iota\colon S \to \mathcal{O}(S)$ be the identity section and regard $\mathcal{O}(S)^\sharp$ as a marked simplicial set over S via the target map. Then*:

(1) *For every marked simplicial set $X \to S$ and cartesian fibration $C \to S$,*

$$\mathrm{id}_X \times \iota \times \mathrm{id}_C : X \times_S C^\natural \to X \times_S \mathcal{O}(S)^\sharp \times_S C^\natural$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.

(1′) *For every marked simplicial set $X \to S$ and cartesian fibration $C \to S$,*

$$\iota \times \mathrm{id}_C : X \times_S C^\natural \to \mathrm{Fun}((\Delta^1)^\sharp, X) \times_S C^\natural$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, where the marked edges in $\mathrm{Fun}((\Delta^1)^\sharp, X)$ are the marked squares in X.

(2) *For every marked simplicial set $X \to S$ and cocartesian fibration $C \to S$,*

$$\mathrm{id}_C \times \iota \times \mathrm{id}_X : {}_\natural C \times_S X \to {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X$$

is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$.

Proof (1) Because $- \times_S C^\natural$ preserves cocartesian equivalences, we reduce to the case where $C = S$. By definition, $X \to X \times_S \mathcal{O}(S)^\sharp$ is a cocartesian equivalence if and only if for every cocartesian fibration $Z \to S$, $\mathrm{Map}^\sharp_S(X \times_S \mathcal{O}(S)^\sharp, {}_\natural Z) \to \mathrm{Map}^\sharp_S(X, {}_\natural Z)$ is a trivial Kan fibration. In other words, for every monomorphism of simplicial sets $A \to B$ and cocartesian fibration $Z \to S$, we need to provide a lift in the commutative square

$$\begin{tikzcd}
B^\sharp \times X \sqcup_{A^\sharp \times X} (A^\sharp \times X) \times_S \mathcal{O}(S)^\sharp \arrow[r, "\phi"] \arrow[d] & {}_\natural Z \arrow[d] \\
(B^\sharp \times X) \times_S \mathcal{O}(S)^\sharp \arrow[r] \arrow[ur, dotted] & S^\sharp
\end{tikzcd}$$

Define $h_0\colon \mathbb{O}(S)^\sharp \times (\Delta^1)^\sharp \to \mathbb{O}(S)^\sharp$ to be the adjoint to the map $\mathbb{O}(S)^\sharp \to \mathbb{O}(\mathbb{O}(S))^\sharp$ obtained by precomposing by the map of posets $\Delta^1 \times \Delta^1 \to \Delta^1$ which sends $(1,1)$ to 1 and the other vertices to 0. Precomposing ϕ by $\mathrm{id}_{A^\sharp \times X} \times h_0$, define a homotopy

$$h\colon (A^\sharp \times X) \times_S \mathbb{O}(S)^\sharp \times (\Delta^1)^\sharp \to {}_\natural Z$$

from $\phi|_{A^\sharp \times X} \circ \mathrm{pr}_{A^\sharp \times X}$ to $\phi|_{(A^\sharp \times X)\times_S \mathbb{O}(S)^\sharp}$. Using h and $\phi|_{B^\sharp \times X}$, define a map

$$\psi\colon B^\sharp \times X \sqcup_{A^\sharp \times X} (A^\sharp \times X) \times_S \mathbb{O}(S)^\sharp \to \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z)$$

such that $\psi|_{B^\sharp \times X}$ is adjoint to $\phi|_{B^\sharp \times X} \circ \mathrm{pr}_{B^\sharp \times X}$ and $\psi|_{(A^\sharp \times X)\times_S \mathbb{O}(S)^\sharp}$ is adjoint to h. Then we may factor the above square through the trivial fibration

$$\mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z) \to {}_\natural Z \times_S \mathbb{O}(S)^\sharp$$

to obtain the commutative rectangle:

$$\begin{tikzcd}
B^\sharp \times X \sqcup_{A^\sharp \times X} (A^\sharp \times X) \times_S \mathbb{O}(S)^\sharp \arrow[r, "\psi"] \arrow[d] & \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural Z) \arrow[r, "e_1"] \arrow[d, two heads, "\simeq"] & {}_\natural Z \arrow[d] \\
(B^\sharp \times X) \times_S \mathbb{O}(S)^\sharp \arrow[r, "\phi|_{B^\sharp \times X} \times \mathrm{id}"'] \arrow[ur, dotted, "\widetilde{\psi}"] & {}_\natural Z \times_S \mathbb{O}(S)^\sharp \arrow[r, "e_1"] & S^\sharp
\end{tikzcd}$$

The dotted lift $\widetilde{\psi}$ exists, and $e_1 \circ \widetilde{\psi}$ is our desired lift.

(1′) Repeat the argument of (1) with $\mathrm{Fun}((\Delta^1)^\sharp, X)$ in place of $\mathbb{O}(S)^\sharp$.

(2) Let $p\colon C \to S$ denote the structure map and let P be a lift in the commutative square

$$\begin{tikzcd}
{}_\natural C \arrow[r, "\iota_C"] \arrow[d] & \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \arrow[d, two heads, "\simeq"', "{(e_0, \mathbb{O}(p))}"] \\
{}_\natural C \times_S \mathbb{O}(S)^\sharp \arrow[r, equal] \arrow[ur, dotted, "P"] & {}_\natural C \times_S \mathbb{O}(S)^\sharp
\end{tikzcd}$$

Let

$$g = (e_1 \times \mathrm{id}_X) \circ (P \times \mathrm{id}_X)\colon {}_\natural C \times_S \mathbb{O}(S)^\sharp \times_S X \to {}_\natural C \times_S X$$

and note that g is a map over S. We claim that g is a marked homotopy inverse of $f = \mathrm{id}_C \times \iota \times \mathrm{id}_X$. By construction, $g \circ f = \mathrm{id}$. For the other direction, define

$$h_0\colon \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \times (\Delta^1)^\sharp \to \mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C)$$

as the adjoint of the map $\mathrm{Fun}((\Delta^1)^\sharp, {}_\natural C) \to \mathrm{Fun}((\Delta^1 \times \Delta^1)^\sharp, {}_\natural C)$ obtained by precomposing by the map of posets $\Delta^1 \times \Delta^1 \to \Delta^1$ which sends $(0,0)$ to 0 and the other vertices to 1. Define

$$h\colon {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X \times (\Delta^1)^\sharp \to {}_\natural C \times_S \mathcal{O}(S)^\sharp \times_S X$$

as the composite $((e_0, \mathcal{O}(p)) \times X) \circ (h_0 \times X) \circ (P \times \mathrm{id}_{X \times (\Delta^1)^\sharp})$. Then h is a homotopy over S from id to $f \circ g$. □

3.4 Proposition *Let $C, C', D \to S$ be cocartesian fibrations and let $F\colon C \to C'$ be a monomorphism of cocartesian fibrations over S (so preserving cocartesian edges). For all marked simplicial sets Y over S, the map*

$$\mathrm{Fun}_S({}_\natural D, \underline{\mathrm{Fun}}_S({}_\natural C', Y)) \to \mathrm{Fun}_S({}_\natural D \times_S {}_\natural C', Y) \times_{\mathrm{Fun}_S({}_\natural D \times_S {}_\natural C, Y)} \mathrm{Fun}_S({}_\natural D, \underline{\mathrm{Fun}}_S({}_\natural C, Y))$$

which precomposes by F is a trivial Kan fibration.

Proof From the defining adjunction, for all $X, Y \in s\mathbf{Set}^+_{/S}$ we have a natural isomorphism

$$\mathrm{Fun}_S(X, \underline{\mathrm{Fun}}_S({}_\natural C, Y)) \cong \mathrm{Fun}_S(X \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural C, Y)$$

of simplicial sets. Since $\mathrm{Fun}_S(-,-)$ is a right Quillen bifunctor, the assertion reduces to showing that

$${}_\natural D \times_S {}_\natural C' \sqcup_{{}_\natural D \times_S {}_\natural C} {}_\natural D \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural C \to {}_\natural D \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural C'$$

is a trivial cofibration in $s\mathbf{Set}^+_{/S}$, which follows from Lemma 3.3(2). □

In Proposition 3.4, letting $C = \varnothing$ and $Y = {}_\natural E$ for another cocartesian fibration $E \to S$, we deduce that $\underline{\mathrm{Fun}}_S(C', -)$ is right adjoint to $C' \times_S -$ as an endofunctor of $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$. Further setting $D = S$, we deduce that the category of cocartesian sections of $\underline{\mathrm{Fun}}_S({}_\natural C, {}_\natural E)$ is equivalent to $\mathrm{Fun}_S({}_\natural C, {}_\natural E)$. We will employ the following notation to explicitly track objects under this correspondence.

3.5 Notation Given a map $f\colon {}_\natural C \to {}_\natural E$, let σ_f denote the cocartesian section $S^\sharp \to \underline{\mathrm{Fun}}_S({}_\natural C, {}_\natural E)$ given by adjointing the map $\mathcal{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{\mathrm{pr}_C} {}_\natural C \xrightarrow{f} {}_\natural E$.

We next study varying the second variable in the construction $\underline{\mathrm{Fun}}_S(-,-)$.

3.6 Lemma *Let $C \to D$ be a fibration of marked simplicial sets over S.*

(1) *Let $K \to S$ be a cocartesian fibration. Then*

$$\underline{\mathrm{Fun}}_S({}_\natural K, C) \to \underline{\mathrm{Fun}}_S({}_\natural K, D) \times_D C$$

is a fibration in $s\mathbf{Set}^+_{/S}$.

(2) *The map*

$$\underline{\mathrm{Fun}}_S(S^\sharp, C) \to \underline{\mathrm{Fun}}_S(S^\sharp, D) \times_D C$$

is a trivial fibration in $s\mathbf{Set}^+_{/S}$.

Proof Let $i\colon A \to B$ be a map of marked simplicial sets. For (1), we use that if i is a trivial cofibration, then

$$B \sqcup_A A \times_S \mathbb{O}(S)^\sharp \times_S {}_\natural K \to B \times_S \mathbb{O}(S) \times_S {}_\natural K$$

is a trivial cofibration, which follows from Proposition 3.1. For (2), we use that if i is a cofibration, then

$$B \sqcup_A A \times_S \mathbb{O}(S)^\sharp \to B \times_S \mathbb{O}(S)$$

is a trivial cofibration, which follows from Lemma 3.3(1). □

The following proposition indicates that we can promote the conclusion $\underline{\mathrm{Fun}}_S(S, -) \simeq \mathrm{id}$ (as an endofunctor of $\mathrm{Fun}(S, \mathbf{Cat}_\infty)$) of Proposition 3.4 to the level of cocartesian model structures. It will not be used in the sequel and is included only for illustrative purposes.

3.7 Proposition *The Quillen adjunction*

$$- \times_S \mathbb{O}(S)^\sharp : s\mathbf{Set}^+_{/S} \rightleftarrows s\mathbf{Set}^+_{/S} : \underline{\mathrm{Fun}}_S(S^\sharp, -)$$

is a Quillen equivalence.

Proof We first check that for every cocartesian fibration $C \to S$, the counit map

$$\underline{\mathrm{Fun}}_S(S^\sharp, {}_\natural C) \times_S \mathbb{O}(S)^\sharp \to {}_\natural C$$

is a cocartesian equivalence. By Lemma 3.3(1), it suffices to show that

$$\underline{\mathrm{Fun}}_S(S^\sharp, {}_\natural C) \to {}_\natural C$$

is a trivial marked fibration, which follows from Lemma 3.6(2) (taking $D = S$). We now complete the proof by checking that $- \times_S \mathbb{O}(S)^\sharp$ reflects cocartesian equivalences; ie given the commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A \times_S \mathbb{O}(S)^\sharp \arrow[r] & B \times_S \mathbb{O}(S)^\sharp \end{tikzcd}$$

if the lower horizontal map is a cocartesian equivalence over S (with respect to the target map) then the upper horizontal map is a cocartesian equivalence over S. But the vertical maps are cocartesian equivalences by Lemma 3.3(1). ◻

The construction $\underline{\mathrm{Fun}}_S(-,-)$ does not make homotopical sense when the first variable is not fibrant, so it does not yield a Quillen bifunctor. Nevertheless, we can say the following about varying the first variable.

3.8 Proposition *Let K, L, and C be fibrant marked simplicial sets over S, let $f : K \to L$ be a map and let*

$$f^* : \underline{\mathrm{Fun}}_S(L, C) \to \underline{\mathrm{Fun}}_S(K, C)$$

denote the induced map.

(1) *Suppose that f is a cocartesian equivalence over S. Then f^* is a cocartesian equivalence over S.*

(2) *Suppose that f is a cofibration. Then f^* is a fibration in $s\mathbf{Set}^+_{/S}$.*

Proof (1) It suffices to check that for all $s \in S$, f^* induces a categorical equivalence between the fibers over s, ie that

$$\mathrm{Fun}_S((S^{s/})^\sharp \times_S L, C) \to \mathrm{Fun}_S((S^{s/})^\sharp \times_S K, C)$$

is a categorical equivalence. Our assumption implies that $(S^{s/})^\sharp \times_S K \to (S^{s/})^\sharp \times_S L$ is a cocartesian equivalence over S, so this holds.

(2) For any trivial cofibration $A \to B$ in $s\mathbf{Set}^+_S$, we need to check that

$$A \times_S \mathbb{O}(S) \times_S L \sqcup_{A \times_S \mathbb{O}(S) \times_S K} B \times_S \mathbb{O}(S) \times_S K \to B \times_S \mathbb{O}(S) \times_S L$$

is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. By Proposition 3.1, $- \times_S \mathbb{O}(S) \times_S K$ preserves trivial cofibrations and ditto for L. The result then follows. ◻

A final word on notation: since $\underline{\mathrm{Fun}}_S(-,-)$ is only homotopically meaningful (and fibrant) when both variables are fibrant, we will henceforth cease to denote the markings on the variables.

S–categories of S–objects

For the convenience of the reader, we briefly review the construction and basic properties of the S–category of S–objects in an ∞–category C. This is a construction, at the level of marked simplicial sets, of the right adjoint to the Grothendieck construction functor[15]

$$\mathrm{Gr}_U : \mathbf{Cat}_{\infty/S}^{\mathrm{cocart}} \to \mathbf{Cat}_\infty, \quad (C \to S) \mapsto C.$$

This material is originally due to Denis Nardin in [2, Section 7].

3.9 Construction [2, Definition 7.4] The span

$$S^\sharp \xleftarrow{\mathrm{ev}_0} \mathbb{O}(S)^\natural \xrightarrow{\rho} \Delta^0$$

defines a right Quillen functor $(\mathrm{ev}_0)_* \rho^* : s\mathbf{Set}^+ \to s\mathbf{Set}^+_{/S}$, which sends an ∞–category E to $\widetilde{\mathrm{Fun}}_S(\mathbb{O}(S), E \times S)$ (cf Example 2.25). This is the *S–category of objects in E*, which we will denote by $\underline{E}_S$.

The next proposition shows that the functor $E \mapsto \underline{E}_S$ indeed implements the right adjoint to Gr_U.

3.10 Proposition *Suppose C an S–category and E an ∞–category. Then we have an equivalence*

$$\psi : \mathrm{Fun}_S(C, \underline{E}_S) \xrightarrow{\simeq} \mathrm{Fun}(C, E).$$

Proof Consider the commutative diagram

$$\begin{tikzcd} C^\sim \arrow[r] \arrow[d] & \mathbb{O}(S)^\natural \arrow[r] \arrow[d] & \Delta^0 \\ {}_\natural C \arrow[r] \arrow[d] & S^\sharp & \\ \Delta^0 & & \end{tikzcd}$$

[15] We write Gr_U to distinguish from Notation 2.6.

Given an ∞–category E, traveling along the outer span (ie pulling back and then pushing forward) yields $\mathrm{Fun}(C, E)$, traveling along the two inner spans yields $\mathrm{Fun}_S(C, \underline{E}_S)$, and the comparison functor ψ is induced by the map $\iota\colon C^{\sim} \to {}_{\natural}C \times_S \mathbb{O}(S)^{\natural}$. By [2, Proposition 6.2], ι is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$. Therefore, combining Lemma 2.27 and Lemma 2.28, we deduce the claim. □

3.11 Notation Given an S–functor $p\colon C \to \underline{E}_S$, let $p^{\dagger}\colon C \to E$ denote the corresponding functor under the equivalence of Proposition 3.10.

3.12 Example Let $E = \mathbf{Spc}$ or $\mathbf{Cat}_\infty$. Then $\underline{\mathbf{Spc}}_S$ (resp. $\underline{\mathbf{Cat}}_{\infty,S}$) is the S–category of S–spaces (resp. S–categories). In particular, suppose $E = \mathbf{Spc}$ and $S = \boldsymbol{O}_G^{\mathrm{op}}$. Then we also call $\underline{\mathbf{Spc}}_{\boldsymbol{O}_G^{\mathrm{op}}}$ the G–∞–category of G–spaces. Note that the fiber of this cocartesian fibration over a transitive G–set G/H is equivalent to the ∞–category of H–spaces $\mathrm{Fun}(\boldsymbol{O}_H^{\mathrm{op}}, \mathbf{Spc})$, and the pushforward functors are given by restriction along a subgroup and conjugation.

3.13 Remark Let C be an S–category and $\pi\colon X \to C$ a left fibration. Then π straightens to a functor $F\colon C \to \mathbf{Spc}$, which under the equivalence of Proposition 3.10 corresponds to an S–functor $F'\colon C \to \underline{\mathbf{Spc}}_S$. We will say that π *S–straightens* to F'. Similarly, if π is a cocartesian fibration, then π S–straightens to an S–functor valued in $\underline{\mathbf{Cat}}_{\infty,S}$.

4 Join and slice

The join and slice constructions are at the heart of the ∞–categorical approach to limits and colimits. In this section, we introduce relative join and slice constructions and explore their homotopical properties.

The S–join

4.1 Definition Let $\iota\colon S \times \partial\Delta^1 \hookrightarrow S \times \Delta^1$ be the inclusion. Define the *S–join* to be the functor

$$(- \star_S -) := \iota_*\colon s\mathbf{Set}_{/S \times \partial\Delta^1} \to s\mathbf{Set}_{/S \times \Delta^1}.$$

Define the *marked S–join* to be the functor

$$(- \star_S -) := \iota_*\colon s\mathbf{Set}^+_{/S^{\sharp} \times (\partial\Delta^1)^{\flat}} \to s\mathbf{Set}^+_{/S^{\sharp} \times (\Delta^1)^{\flat}}.$$

4.2 Notation Given X and Y, marked simplicial sets over S, we will usually refer to the structure maps to S by $\pi_1\colon X \to S$, $\pi_2\colon Y \to S$, and $\pi\colon X \star_S Y \to S$. Explicitly, an $(i+j+1)$–simplex λ of $X \star_S Y$ is the data of simplices $\sigma\colon \Delta^i \to X$, $\tau\colon \Delta^j \to Y$, and $\lambda'\colon \Delta^i \star \Delta^j \to S$ such that the diagram

$$\begin{tikzcd} \Delta^i \arrow[r] \arrow[d, "\sigma"] & \Delta^i \star \Delta^j \arrow[d, "\lambda'"] & \Delta^j \arrow[l] \arrow[d, "\tau"] \\ X \arrow[r, "\pi_1"] & S & Y \arrow[l, "\pi_2"'] \end{tikzcd}$$

commutes; we then have that $\lambda' = \pi \circ \lambda$. We will sometimes write $\lambda = (\sigma, \tau)$ so as to remember the data of the i–simplex of X and the j–simplex of Y in the notation. If given an n–simplex of $X \star_S Y$, we will indicate the decomposition of Δ^n given by the structure map to Δ^1 as $\Delta^{n_0} \star \Delta^{n_1}$ (with either side possibly empty).

4.3 Proposition *Let $\iota\colon S \times \partial\Delta^1 \hookrightarrow S \times \Delta^1$ be the inclusion. Then*

(a) $\iota_*\colon s\mathbf{Set}_{/S\times\partial\Delta^1} \to s\mathbf{Set}_{/S\times\Delta^1}$ *is a right Quillen functor.*

(b) $\iota_*\colon s\mathbf{Set}^+_{/S^\sharp\times(\partial\Delta^1)^\flat} \to s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}$ *is a right Quillen functor.*

Consequently, if X and Y are categorical (*resp. cocartesian*) *fibrations over S, then $X \star_S Y$ is a categorical* (*resp. cocartesian*) *fibration over S, with the cocartesian edges given by those in X and Y.*

Proof For (b), we verify the hypotheses of Theorem 2.24(ii). All of the requirements are immediate except for (1) and (2).

(1) Let (s,i) be a vertex of $S^\sharp \times (\partial\Delta^1)^\flat$, $i = 0$ or 1. Let $f\colon (s',i') \to (s,i)$ be a marked edge in $S^\sharp \times (\Delta^1)^\flat$. Then $i' = i$ and f viewed as an edge in $S^\sharp \times (\partial\Delta^1)^\flat$ is locally ι–cartesian.

(2) It is obvious that $\partial\Delta^1 \hookrightarrow \Delta^1$ is a flat categorical fibration, so by stability of flat categorical fibrations under base change, $S \times \partial\Delta^1 \hookrightarrow S \times \Delta^1$ is a flat categorical fibration.

Part (a) also follows from (2) by [11, Proposition B.4.5]. By (a), if X and Y are categorical fibrations over S, $X \star_S Y$ is a categorical fibration over $S \times \Delta^1$. The projection map $S \times \Delta^1 \to S$ is a categorical fibration, so $X \star_S Y$ is also a categorical fibration over S. By (b), if X and Y are cocartesian fibrations over S, ${}_\natural X \star_S {}_\natural Y$ is fibrant in $s\mathbf{Set}^+_{/S^\sharp\times(\Delta^1)^\flat}$. Since $S^\sharp \times (\Delta^1)^\flat$ is marked as a cocartesian fibration over S, ${}_\natural X \star_S {}_\natural Y$ is marked as a cocartesian fibration over S. □

We have the compatibility of the relative join with base change.

4.4 Lemma *Let $f: T \to S$ be a functor and let X and Y be (marked) simplicial sets over S. Then we have a canonical isomorphism*

$$(X \star_S Y) \times_S T \cong (X \times_S T) \star_T (Y \times_S T).$$

Proof From the pullback square

$$\begin{tikzcd} T \times \partial\Delta^1 \arrow[r, "\iota_T"] \arrow[d, "f \times \mathrm{id}"] & T \times \Delta^1 \arrow[d, "f \times \mathrm{id}"] \\ S \times \partial\Delta^1 \arrow[r, "\iota_S"] & S \times \Delta^1 \end{tikzcd}$$

we obtain the base-change isomorphism $f^*(\iota_S)_* \cong (\iota_T)_* f^*$. □

In [9, Section 4.2.2], Lurie introduces the relative "diamond" join operation $\diamond_S$, which we now recall. Given X and Y marked simplicial sets over S, define

$$X \diamond_S Y = X \sqcup_{X \times_S Y \times \{0\}} X \times_S Y \times (\Delta^1)^\flat \sqcup_{X \times_S Y \times \{1\}} Y.$$

There is a comparison map

$$\psi_{(X,Y)}\colon X \diamond_S Y \to X \star_S Y = \iota_*(X, Y),$$

adjoint to the isomorphism $\iota^*(X \star_S Y) \cong (X, Y)$.

4.5 Lemma *Let X be a marked simplicial set. Then $\psi_{(X,S)}\colon X \diamond_S S^\sharp \to X \star_S S^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Dually, if X is in addition fibrant, then $\psi_{(S,X)}\colon S^\sharp \diamond_S X \to S^\sharp \star_S X$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof We first address the map $\psi_{(X,S)}$. By left properness of the cocartesian model structure, the defining pushout for $X \diamond_S S^\sharp$ is a homotopy pushout. By Theorem 4.16,[16] $- \star_S S^\sharp$ preserves cocartesian equivalences. Therefore, choosing a fibrant replacement for X and using naturality of the comparison map $\psi_{(X,S)}$, we may reduce to the case that X is fibrant. Then we have to check that

$$\begin{tikzcd} X \times \{1\} \arrow[r] \arrow[d] & X \times (\Delta^1)^\flat \arrow[d] \\ S^\sharp \arrow[r] & X \star_S S^\sharp \end{tikzcd}$$

[16] There is no circularity since Lemma 4.5 is only later referenced in this paper at the beginning of Section 9.

is a homotopy pushout square. Since this is a square of fibrant objects, this assertion can be checked fiberwise, in which case it reduces to the equivalence $X_s \diamond \Delta^0 \xrightarrow{\simeq} X^{\rhd}$ of [9, Proposition 4.2.1.2].

The second statement concerning $\psi_{(S,X)}$ follows by the same type of argument, but without the reduction step. □

4.6 Warning In general, $\psi_{(X,Y)}$ is not a cocartesian equivalence. As a counterexample, consider $S = \Delta^1$, $X = \{0\}$, and $Y = \{1\}$. Then $\psi_{(X,Y)}$ is the inclusion of

$$X \diamond_S Y \cong \Delta^{\{0\}} \sqcup \Delta^{\{1\}}$$

into $X \star_S Y \cong \Delta^1$, which is not a cocartesian equivalence over Δ^1.

We will later need the following strengthening of the conclusion of Proposition 4.3.

4.7 Proposition (1) *Let $C, C', D \to S$ be inner fibrations and let $C, C' \to D$ be functors over S. Then $C \star_D C' \to S$ is an inner fibration.*

(2) *Let $C, C', D \to S$ be S–categories and let $C, C' \to D$ be S–functors. Then $C \star_D C' \to S$ is an S–category with cocartesian edges given by those in C or C', and $C \star_D C' \to D$ is an S–functor.*

Proof (1) Let $0 < k < n$. We need to solve the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\lambda_0"] \arrow[d] & C \star_D C' \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\lambda"] & S \end{tikzcd}$$

Let $\bar{\lambda}\colon \Delta^n \to D$ be a lift in the commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & D \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, "\bar{\lambda}"] & S \end{tikzcd}$$

Define λ using the data $(\lambda_0|_{\Delta^{n_0}}, \lambda_0|_{\Delta^{n_1}}, \bar{\lambda})$. Then λ is a valid lift.

(2) Consider $C \star_D C'$ as a marked simplicial set with marked edges those in ${}_\natural C$ or in ${}_\natural C'$. We need to solve the lifting problem

$$\begin{tikzcd} {}_\natural\Lambda^n_0 \arrow[r, "\lambda_0"] \arrow[d] & C \star_D C' \arrow[d] \\ {}_\natural\Delta^n \arrow[r] \arrow[ur, dotted, "\lambda"] & S \end{tikzcd}$$

Let $\bar\lambda \colon \Delta^n \to D$ be a lift in the commutative square

$$\begin{tikzcd} {}_\natural\Lambda^n_0 \arrow[r] \arrow[d] & {}_\natural D \arrow[d] \\ {}_\natural\Delta^n \arrow[r] \arrow[ur, "\bar\lambda"] & S \end{tikzcd}$$

Define λ using the data $(\lambda_0|_{\Delta^{n_0}}, \lambda_0|_{\Delta^{n_1}}, \bar\lambda)$. Then λ is a valid lift. Finally, note that we may obviously lift against classes (3) and (4) of [9, Definition 3.1.1.1]. We conclude that $C \star_D C' \to S$ is fibrant in $s\mathbf{Set}^+_{/S}$, hence an S–category with cocartesian edges as marked. □

Since the S–join is defined as a right Kan extension, it is simple to map into. In the other direction, we can offer the following lemma.

4.8 Lemma *Let C, C', D, and E be S–categories and let $C, C' \to D$ be S–functors. Then*

$$\mathrm{Fun}_S(C \star_D C', E) \to \mathrm{Fun}_S(C, E) \times \mathrm{Fun}_S(C', E)$$

is a bifibration [9, Definition 2.4.7.2]. *Consequently*,

$$\mathrm{Fun}_S(C \star_D C', E) \to \mathrm{Fun}_S(C, E)$$

is a cartesian fibration with cartesian edges those sent to equivalences in $\mathrm{Fun}_S(C', E)$*, and*

$$\mathrm{Fun}_S(C \star_D C', E) \to \mathrm{Fun}_S(C', E)$$

is a cocartesian fibration with cocartesian edges those sent to equivalences in $\mathrm{Fun}_S(C', E)$.

Proof By inspection, the span

$$(\Delta^1)^\flat \xleftarrow{\pi} {}_\natural(C \star_D C') \xrightarrow{\pi'} S^\sharp$$

satisfies the hypotheses of Theorem 2.24. Therefore, $\pi_*\pi'^*({}_\natural E \to S)$ is a categorical fibration over Δ^1. The claim now follows from [9, Proposition 2.4.7.10], and the consequence from [9, Lemma 2.4.7.5] and its opposite. □

The Quillen adjunction between S–join and S–slice

Our next goal is to obtain a relative join and slice Quillen adjunction. To this end, we need a good understanding of the combinatorics of the relative join (Proposition 4.11). We prepare for the proof of that proposition with a few lemmas.

4.9 Lemma *Let $i, l \geq -1$ and $j, k \geq 0$. Then the map*

$$\Delta^i \star \Delta^j \star \partial\Delta^k \star \Delta^l \sqcup_{\Delta^j \star \partial\Delta^k \star \Delta^l} \Delta^{j+k+l+2} \hookrightarrow \Delta^{i+j+k+l+3}$$

is inner anodyne.

Proof Let $f\colon \Delta^{j-1} \hookrightarrow \Delta^i \star \Delta^{j-1}$ and $g\colon \Lambda_0^{k+1} \hookrightarrow \Delta^{k+1}$. The map in question is $f \star g \star \Delta^l$, so is inner anodyne by [9, Lemma 2.1.2.3]. □

By [9, Lemma 2.1.2.4], the join of a left anodyne map and an inclusion is left anodyne. We need a slight refinement of this result:

4.10 Lemma *Let $f : A_0 \hookrightarrow A$ be a cofibration of simplicial sets.*

(1) *Let $g\colon B_0 \hookrightarrow B$ be a right marked anodyne map between marked simplicial sets. Then*

$$f^\flat \star g\colon A_0^\flat \star B \sqcup_{A_0^\flat \star B_0} A^\flat \star B_0 \hookrightarrow A^\flat \star B$$

is a right marked anodyne map.

(2) *Let $g\colon B_0 \hookrightarrow B$ be a left marked anodyne map between marked simplicial sets. Then*

$$g \star f^\flat\colon B \star A_0^\flat \sqcup_{B_0 \star A_0^\flat} B_0 \star A^\flat \hookrightarrow B \star A^\flat$$

is a left marked anodyne map.

Proof We prove (1); the dual assertion (2) is proven by a similar argument. As f lies in the weakly saturated closure of the inclusions $i_m\colon \partial\Delta^m \hookrightarrow \Delta^m$, it suffices to check that $i_m^\flat \star g$ is right marked anodyne for the four classes of morphisms enumerated in [9, Definition 3.1.1.1]. For $g\colon (\Lambda_i^n)^\flat \hookrightarrow (\Delta^n)^\flat$, $0 < i < n$, $i_m^\flat \star g$ obtained from an inner anodyne map by marking common edges, so is marked right anodyne. For $g\colon \Lambda_n^{n\natural} \hookrightarrow \Delta^{n\natural}$, $i_m^\flat \star g$ is $\Lambda_{n+m+1}^{n+m+1\natural} \hookrightarrow \Delta^{n+m+1\natural}$, so $i_m^\flat \star g$ is marked right anodyne. For the remaining two classes, $i_m^\flat \star g$ is the identity because no markings are introduced when joining two marked simplicial sets. □

The following proposition reveals a basic asymmetry of the relative join, which is related to our choice of *cocartesian* fibrations to model functors.

4.11 Proposition *Let K be a marked simplicial set over S.*

(1) *For every marked left horn inclusion ${}_\natural\Lambda^n_0 \hookrightarrow {}_\natural\Delta^n$ over S, the induced map*

$$K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural) \hookrightarrow K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural)$$

is left marked anodyne, where the pullbacks ${}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural$ and ${}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural$ are formed with respect to the source map e_0 and are regarded as marked simplicial sets over S via the target map e_1.

(1′) *For every left horn inclusion $\Lambda^n_0 \hookrightarrow \Delta^n$ over S, the induced map*

$$\Delta^n \times_S \mathcal{O}(S) \sqcup_{\Lambda^n_0 \times_S \mathcal{O}(S)} K \star_S (\Lambda^n_0 \times_S \mathcal{O}(S)) \hookrightarrow K \star_S (\Delta^n \times_S \mathcal{O}(S))$$

is an inner anodyne map.

(2) *Let $e_0 \colon C \to S$ be a cartesian fibration over S and let $e_1 \colon C \to S$ be any map of simplicial sets. For every inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, $0 < k < n$ over S, the induced map*

$$K \star_S (\Lambda^n_k \times_S C) \hookrightarrow K \star_S (\Delta^n \times_S C)$$

is inner anodyne, where the pullbacks $\Lambda^n_k \times_S C$ and $\Delta^n \times_S C$ are formed with respect to e_0 and are regarded as simplicial sets over S via e_1.

(3) *For every marked right horn inclusion $\Lambda^{n\natural}_n \hookrightarrow \Delta^{n\natural}$ over S, the induced map*

$$K \star_S \Lambda^{n\natural}_n \hookrightarrow K \star_S \Delta^{n\natural}$$

is right marked anodyne.

Proof Let I be the set of simplices of K endowed with a total order such that $\sigma < \sigma'$ if the dimension of σ is less than that of σ', where we view the empty set as a simplex of dimension -1. Let J be the set of epimorphisms $\chi \colon \Delta^j \twoheadrightarrow \Delta^{n-1}$ endowed with a total order such that $\chi < \chi'$ if the dimension of χ is less than that of χ'. Order $I \times J$ by $(\sigma, \chi) < (\sigma', \chi')$ if $\sigma < \sigma'$ or $\sigma = \sigma'$ and $\chi < \chi'$. For any simplex $\tau \colon \Delta^j \to \Delta^n$, we let $r_k(\tau)$ be the pullback

$$\begin{tikzcd} \Delta^{r_k(\tau)_0} \arrow[r, "r_k(\tau)"] \arrow[d] & \Delta^{n-1} \arrow[d, "d_k"] \\ \Delta^j \arrow[r, "\tau"] & \Delta^n \end{tikzcd}$$

We will let ι denote the map under consideration. We first prove (1). Given $\sigma \in I$ and $\chi \in J$, let $X_{\sigma,\chi}$ be the submarked simplicial set of $K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural)$ on

$K \star_S ({}_\natural\Lambda^n_0 \times_S \mathbb{O}(S)^\natural)$ and simplices $(\sigma', \tau')\colon \Delta^i \star \Delta^j \to K \star_S (\Delta^n \times_S \mathbb{O}(S))$ not in $K \star_S (\Lambda^n_0 \times_S \mathbb{O}(S))$ with $(\sigma', r_0(e_0 \circ \tau')) \leq (\sigma, \chi)$. If $(\sigma, \chi) < (\sigma', \chi')$, then we have an obvious inclusion $X_{\sigma,\chi} \hookrightarrow X_{\sigma',\chi'}$, and we let

$$X_{<(\sigma,\chi)} = ({}_\natural\Lambda^n_0 \times_S \mathbb{O}(S)^\natural) \cup \Bigl(\bigcup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma',\chi'} \Bigr).$$

Since $K \star_S ({}_\natural\Delta^n \times_S \mathbb{O}(S)^\natural) = \operatorname{colim}_{(\sigma,\chi)\in I \times J} X_{\sigma,\chi}$, in order to show that ι is left marked anodyne it suffices to show that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is left marked anodyne for all $(\sigma, \chi) \in I \times J$. We will say that a simplex of $X_{\sigma,\chi}$ is *new* if it does not belong to $X_{<(\sigma,\chi)}$.

Let $\sigma\colon \Delta^i \to K$ be an element of I and $\chi\colon \Delta^j \twoheadrightarrow \Delta^{n-1}$ an element of J. Let $\lambda = (\sigma, \tau)\colon \Delta^i \star \Delta^j \to K \star_S (\Delta^n \times_S \mathbb{O}(S))$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_0(e_0 \circ \tau) = \chi$. Let $\bar\chi\colon \Delta^{j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_0(\bar\chi) = \chi$ and let $e\colon \Delta^1 \to \Delta^n \times_S \mathbb{O}(S)$ be a cartesian edge over $\{0,1\}$ with $e(1) = \tau(0)$. The inclusion $(\Delta^1)^\sharp \sqcup_{\Delta^0} \Delta^j \hookrightarrow {}_\natural\Delta^{j+1}$ is right marked anodyne, so we have a lift $\bar\tau$ in the diagram

$$\begin{tikzcd} \Delta^1 \sqcup_{\Delta^0} \Delta^j \arrow[r, "e\cup\tau"] \arrow[d] & \Delta^n \times_S \mathbb{O}(S) \arrow[d] \\ \Delta^{j+1} \arrow[r, "\bar\chi"] \arrow[ur, dotted, "\bar\tau"] & \Delta^n \end{tikzcd}$$

By Lemma 4.10,

$$\Delta^i \star \Delta^j \sqcup_{\Delta^j} {}_\natural\Delta^{j+1} \hookrightarrow \Delta^i \star {}_\natural\Delta^{j+1}$$

is right marked anodyne. Using that $(e_1 \circ \bar\tau)(e)$ is an equivalence, we obtain a lift

$$\begin{tikzcd} \Delta^i \star \Delta^j \sqcup_{\Delta^j} {}_\natural\Delta^{j+1} \arrow[r, "\pi\lambda\cup e_1\bar\tau"] \arrow[d] & S^\sim \\ \Delta^i \star {}_\natural\Delta^{j+1} \arrow[ur, dotted] & \end{tikzcd}$$

which allows us to define $\bar\lambda\colon \Delta^i \star \Delta^{j+1} \to K \star_S (\Delta^n \times_S \mathbb{O}(S))$ extending λ and $\bar\tau$. Then $\bar\lambda$ is a nondegenerate new simplex of $X_{\sigma,\chi}$ and every face of $\bar\lambda$ except for $\lambda = d_{i+1}(\bar\lambda)$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{tikzcd} \bigsqcup_\lambda (\Lambda^{i+j+2}_{i+1}, \{i+1, i+2\}) \arrow[r] \arrow[d] & X_{<(\sigma,\chi)} \arrow[d] \\ \bigsqcup_\lambda (\Delta^{i+j+2}, \{i+1, i+2\}) \arrow[r] & X_{<(\sigma,\chi),1} \end{tikzcd}$$

which factors the inclusion $X_{<(\sigma,\chi)} \hookrightarrow X_{(\sigma,\chi)}$ as the composition of a left marked anodyne map and an inclusion. (There is one further complication involving markings: in the special case $n = 1$, $\sigma = \varnothing$ and $j = 1$, we may have that $\lambda = \tau$ is a marked edge, ie an equivalence over 1. Then the edges of $\bar{\tau}$ are all marked, so we should form the pushout via maps $(\Lambda^2_0)^\sharp \hookrightarrow (\Delta^2)^\sharp$, which are left marked anodyne by [9, Corollary 3.1.1.7]).

Now for the inductive step suppose that we have defined a sequence of left marked anodyne maps

$$X_{<(\sigma,\chi)} \hookrightarrow \dots \hookrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i+l+j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to an $(i+l+j+1)$–simplex with the edge $\{i+l, i+l+1\}$ marked in $X_{<(\sigma,\chi),l}$, and no new nondegenerate simplices of dimension $> i+l+j+1$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma, \tau)$ be any new nondegenerate $(i+m+j+1)$–simplex not in $X_{<(\sigma,\chi),m}$. For $0 \leq l < m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $(i+m+j+1)$–simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m}(\lambda_l) = d_{i+l+1}(\lambda)$ and edge $\{i+m, i+m+1\}$ marked. τ and $\tau_0, \dots, \tau_{m-1}$ together define a map

$$\tau' \colon \Lambda^{m+1}_{m+1} \star \Delta^{j-1} \to \Delta^n \times_S \mathbb{O}(S)$$

where the domain of τ is the subset $\{0, \dots, m-1, m+1, \dots, m+j+1\}$ and the domain of τ_l is the subset $\{0, \dots, \hat{l}, \dots, m+j+1\}$. Observe that the map

$$\Lambda^{m+1}_{m+1}{}^\natural \star \Delta^{j-1} \hookrightarrow \Delta^{m+1}{}^\natural \star \Delta^{j-1}$$

is right marked anodyne, since it factors as

$$\Lambda^{m+1}_{m+1}{}^\natural \star \Delta^{j-1} \hookrightarrow \Delta^{m+1}{}^\natural \sqcup_{\Lambda^{m+1}_{m+1}{}^\natural} \Lambda^{m+1}_{m+1}{}^\natural \star \Delta^{j-1} \hookrightarrow \Delta^{m+1}{}^\natural \star \Delta^{j-1}$$

where the first map is obtained as the pushout of the right marked anodyne map $\Lambda^{m+1}_{m+1}{}^\natural \hookrightarrow \Delta^{m+1}{}^\natural$ along the inclusion $\Lambda^{m+1}_{m+1}{}^\natural \hookrightarrow \Lambda^{m+1}_{m+1}{}^\natural \star \Delta^{j-1}$ and the second map is obtained by marking a common edge of an inner anodyne map. Let $\bar{\chi} \colon \Delta^{m+j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_0(\bar{\chi}) = \chi$. Then we have a lift $\bar{\tau}$ in the commutative diagram

$$\begin{tikzcd}
\Lambda^{m+1}_{m+1} \star \Delta^{j-1} \arrow[r, "\tau'"] \arrow[d] & \Delta^n \times_S \mathbb{O}(S) \arrow[d] \\
\Delta^{m+1} \star \Delta^{j-1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n
\end{tikzcd}$$

By Lemma 4.10, the map

$$\Delta^i \star \Lambda^{m+1^\natural}_{m+1} \star \Delta^{j-1} \sqcup_{\Lambda^{m+1^\natural}_{m+1} \star \Delta^{j-1}} \Delta^{m+1^\natural} \star \Delta^{j-1} \hookrightarrow \Delta^i \star \Delta^{m+1^\natural} \star \Delta^{j-1}$$

is right marked anodyne. Since $(e_1 \circ \bar{\tau})(\{m, m+1\})$ is an equivalence, we may extend $\left(\bigcup_l \pi\lambda_l\right) \cup \pi\lambda \cup e_1\bar{\tau}$ to a map $\Delta^{i+m+j+2} \to S$, which defines a nondegenerate $(i+m+j+2)$–simplex $\bar{\lambda}$ with λ as its $(i+m+1)^{\text{th}}$ face and which extends $\bar{\tau}$. By construction, every other face of $\bar{\lambda}$ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{tikzcd}
\bigsqcup_\lambda (\Lambda^{i+m+j+2}_{i+m+1}, \{i+m+1, i+m+2\}) \arrow[r] \arrow[d] & X_{<(\sigma,\chi),m} \arrow[d] \\
\bigsqcup_\lambda (\Delta^{i+m+j+2}, \{i+m+1, i+m+2\}) \arrow[r] & X_{<(\sigma,\chi),m+1}
\end{tikzcd}$$

and complete the inductive step. (Again, there is one further complication involving markings: in the special case $i=-1$, $n=1$, $j=0$ and $m=1$, we may have that λ is marked. Then every edge of $\bar{\lambda}$ is marked since $(\Lambda^2_2)^\sharp \hookrightarrow (\Delta^2)^\sharp$ is right marked anodyne, and we form the pushout along maps $(\Lambda^2_1)^\sharp \hookrightarrow (\Delta^2)^\sharp$). Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is marked left anodyne, which completes the proof.

For $(1')$, simply observe that if $i > -1$ we are attaching along inner horns.

We now modify the above proof to prove (2). Let $X_{\sigma,\chi}$ be the subsimplicial set of $K \star_S (\Delta^n \times_S C)$ on $K \star_S (\Lambda^n_k \times_S C)$ and simplices $(\sigma', \tau')\colon \Delta^i \star \Delta^j \to K \star_S (\Delta^n \times_S C)$ not in $K \star_S (\Lambda^n_k \times_S C)$ with $(\sigma', r_k(e_0 \circ \tau')) \le (\sigma, \chi)$. Let

$$X_{<(\sigma,\chi)} = (K \star (\Lambda^n_k \times_S C)) \cup \left(\bigcup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma',\chi'} \right).$$

We will show that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is inner anodyne for all $(\sigma,\chi) \in I \times J$.

Let $\sigma\colon \Delta^i \to K$ be an element of I, $\chi\colon \Delta^j \twoheadrightarrow \Delta^{n-1}$ an element of J, and let k' be the first vertex of χ with $\chi(k') = k$. Let $\lambda = (\sigma, \tau)\colon \Delta^i \star \Delta^j \to K \star_S (\Delta^n \times_S C)$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_k(e_0 \circ \tau) = \chi$. Let $\bar{\chi}\colon \Delta^{j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_k(\bar{\chi}) = \chi$. Combining [9, Lemma 2.1.2.3] and Lemma 4.10, we see that the inclusion

$$d_{k'}\colon \Delta^j = \Delta^{k'-1} \star \Delta^{j-k'} \hookrightarrow \Delta^{k'-1} \star {}_\natural\Delta^{j-k'+1}$$

is right marked anodyne, so we have a lift $\bar{\tau}$ in

$$\begin{tikzcd} \Delta^j \arrow[r, "\tau"] \arrow[d] & \Delta^n \times_S C \arrow[d] \\ \Delta^{j+1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n \end{tikzcd}$$

where $\bar{\tau}(\{k', k'+1\})$ is a cartesian edge. By Lemma 4.9,

$$\Delta^i \star \Delta^j \sqcup_{\Delta^j} \Delta^{j+1} \hookrightarrow \Delta^i \star \Delta^{j+1}$$

is inner anodyne. We thus obtain an extension

$$\begin{tikzcd} \Delta^i \star \Delta^j \sqcup_{\Delta^j} \Delta^{j+1} \arrow[r, "\pi\lambda \cup e_1 \bar{\tau}"] \arrow[d] & S \\ \Delta^i \star \Delta^{j+1} \arrow[ur, dotted] & \end{tikzcd}$$

which allows us to define $\bar{\lambda}\colon \Delta^i \star \Delta^{j+1} \to K \star_S (\Delta^n \times_S C)$ extending λ and $\bar{\tau}$. Then $\bar{\lambda}$ is nondegenerate and every face of $\bar{\lambda}$ except for $\lambda = d_{i+k'+1}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{tikzcd} \coprod_\lambda \Lambda^{i+j+2}_{i+k'+1} \arrow[r] \arrow[d] & X_{<(\sigma,\chi)} \arrow[d] \\ \coprod_\lambda \Delta^{i+j+2} \arrow[r] & X_{<(\sigma,\chi),1} \end{tikzcd}$$

which factors the inclusion $X_{<(\sigma,\chi)} \hookrightarrow X_{(\sigma,\chi)}$ as the composition of an inner anodyne map and an inclusion.

Now for the inductive step suppose that we have defined a sequence of inner anodyne maps

$$X_{<(\sigma,\chi)} \hookrightarrow \cdots \hookrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i+l+j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to an $(i+l+j+1)$–simplex such that the edge $\{i+k'+l, i+k'+l+1\}$ is sent to a cartesian edge of $\Delta^n \times_S C$, and no new nondegenerate simplices of dimension $> i+l+j+1$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma, \tau)$ be any new nondegenerate $(i+m+j+1)$–simplex not in $X_{<(\sigma,\chi),m}$. For

$0 \leq l < m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $(i+m+j+1)$–simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m+k'}(\lambda_l) = d_{i+l+k'+1}(\lambda)$. Then τ and $\tau_0, \dots, \tau_{m-1}$ together define a map

$$\tau'\colon \Delta^{k'-1} \star \Lambda^{m+1}_{m+1} \star \Delta^{j-k'-1} \to \Delta^n \times_S C,$$

where the domain of τ is the subset $\{0, \dots, k'+m-1, k'+m+1, \dots, m+j+1\}$ and the domain of τ_l is the subset $\{0, \dots, \widehat{k'+l}, \dots, m+j+1\}$. The map

$$\Delta^{k'-1} \star \Lambda^{m+1\natural}_{m+1} \star \Delta^{j-k'-1} \hookrightarrow \Delta^{k'-1} \star \Delta^{m+1\natural} \star \Delta^{j-k'-1}$$

is $\Delta^{k'-1}$ joined with a right marked anodyne map, so is right marked anodyne by Lemma 4.10. Let $\bar{\chi}\colon \Delta^{m+j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_k(\bar{\chi}) = \chi$. Then we have a lift $\bar{\tau}$ in the following commutative diagram

$$\begin{tikzcd}
\Delta^{k'-1} \star \Lambda^{m+1}_{m+1} \star \Delta^{j-k'-1} \arrow[r, "\tau'"] \arrow[d] & \Delta^n \times_S C \arrow[d] \\
\Delta^{m+j+1} \arrow[r, "\bar{\chi}"] \arrow[ur, dotted, "\bar{\tau}"] & \Delta^n
\end{tikzcd}$$

such that $\bar{\tau}(\{k'+m, k'+m+1\})$ is a cartesian edge. By Lemma 4.9, the map

$$\Delta^i \star \Delta^{k'-1} \star \partial\Delta^m \star \Delta^{j-k'} \sqcup_{\Delta^{k'-1} \star \partial\Delta^m \star \Delta^{j-k'}} \Delta^{m+j+1} \hookrightarrow \Delta^{i+m+j+2}$$

is inner anodyne. Thus, we may extend $\left(\bigcup_l \pi\lambda_l\right) \cup \pi\lambda \cup e_1\bar{\tau}$ to a map $\Delta^{i+m+j+2} \to S$, which defines a nondegenerate $(i+m+j+2)$–simplex $\bar{\lambda}$ with λ as its $(i+k'+m+1)^{\text{th}}$ face and which extends $\bar{\tau}$. By construction every other face of $\bar{\lambda}$ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{tikzcd}
\coprod_\lambda \Lambda^{i+m+j+2}_{i+k'+m+1} \arrow[r] \arrow[d] & X_{<(\sigma,\chi),m} \arrow[d] \\
\coprod_\lambda \Delta^{i+m+j+2} \arrow[r] & X_{<(\sigma,\chi),m+1}
\end{tikzcd}$$

and complete the inductive step. Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is inner anodyne, which completes the proof.

We finally modify the above proof to prove (3). Given $\sigma \in I$ and $\chi \in J$, let $X_{\sigma,\chi}$ be the submarked simplicial set of $K \star_S \Delta^{n\natural}$ on $K \star_S \Lambda^{n\natural}_n$ and simplices

$$(\sigma', \tau')\colon \Delta^i \star \Delta^j \to K \star_S \Delta^{n\natural}$$

not in $K \star_S \Lambda^{n\natural}_n$ with $(\sigma', r_n(\tau')) \leq (\sigma, \chi)$. Let

$$X_{<(\sigma,\chi)} = (K \star_S \Lambda^{n\natural}_n) \cup \left(\bigcup_{(\sigma',\chi')<(\sigma,\chi)} X_{\sigma',\chi'} \right).$$

We will show that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is right marked anodyne for all $(\sigma, \chi) \in I \times J$.

Let $\sigma \colon \Delta^i \to K$ be an element of I and $\chi \colon \Delta^j \twoheadrightarrow \Delta^{n-1}$ an element of J. Let $\lambda = (\sigma, \tau) \colon \Delta^i \star \Delta^j \to K \star_S \Delta^{n\natural}$ be any nondegenerate new simplex of $X_{\sigma,\chi}$, so $r_n(\tau) = \chi$. Let $\bar{\chi} \colon \Delta^{j+1} \twoheadrightarrow \Delta^n$ be the unique epimorphism with $r_n(\bar{\chi}) = \chi$. By Lemma 4.9, the inclusion

$$\Delta^i \star \Delta^j \sqcup_{\Delta^j} \Delta^{j+1} \hookrightarrow \Delta^i \star \Delta^{j+1}$$

is inner anodyne, so we have an extension in

$$\begin{tikzcd} \Delta^i \star \Delta^j \sqcup_{\Delta^j} \Delta^{j+1} \arrow[r, "\pi\lambda \cup \pi_2\bar{\chi}"] \arrow[d] & S \\ \Delta^i \star \Delta^{j+1} \arrow[ur, dotted] & \end{tikzcd}$$

which allows us to define $\bar{\lambda} \colon \Delta^i \star \Delta^{j+1} \to K \star_S \Delta^{n\natural}$ extending λ and $\bar{\chi}$. Then $\bar{\lambda}$ is nondegenerate and every face of $\bar{\lambda}$ except for $\lambda = d_{i+j+2}(\bar{\lambda})$ lies in $X_{<(\sigma,\chi)}$. We may thus form the pushout

$$\begin{tikzcd} \coprod_\lambda \Lambda^{i+j+2\natural}_{i+j+2} \arrow[r] \arrow[d] & X_{<(\sigma,\chi)} \arrow[d] \\ \coprod_\lambda \Delta^{i+j+2\natural} \arrow[r] & X_{<(\sigma,\chi),1} \end{tikzcd}$$

which factors the inclusion $X_{<(\sigma,\chi)} \to X_{(\sigma,\chi)}$ as the composition of a right marked anodyne map and an inclusion.

Now for the inductive step suppose that we have defined a sequence of right marked anodyne maps

$$X_{<(\sigma,\chi)} \hookrightarrow \cdots \hookrightarrow X_{<(\sigma,\chi),m} \subset X_{(\sigma,\chi)}$$

such that for all $0 < l \leq m$ all new nondegenerate simplices in $X_{(\sigma,\chi)}$ of dimension $i + l + j$ lie in $X_{<(\sigma,\chi),l}$ and admit an extension to an $(i+l+j+1)$–simplex, and no new nondegenerate simplices of dimension $> i + l + j + 1$ lie in $X_{<(\sigma,\chi),l}$. Let $\lambda = (\sigma, \tau)$ be any new nondegenerate $(i+m+j+1)$–simplex not in $X_{<(\sigma,\chi),m}$. For

$0 < l \leq m$ let $\lambda_l = (\sigma, \tau_l)$ be a nondegenerate $(i+m+j+1)$–simplex in $X_{<(\sigma,\chi),m}$ with $d_{i+m+j+1}(\lambda_l) = d_{i+j+l+1}(\lambda)$ (note that $\tau_l = \tau$). By Lemma 4.9, the map

$$\Delta^i \star \Delta^j \star \partial\Delta^m \sqcup_{\Delta^j \star \partial\Delta^m} \Delta^j \star \Delta^m \hookrightarrow \Delta^i \star \Delta^j \star \Delta^m$$

is inner anodyne. Therefore, we may extend $\pi\lambda \cup \left(\bigcup_l \pi\lambda_l\right)$ to a map $\Delta^{i+j+m+2} \to S$ and define an $(i+j+m+2)$–simplex $\bar{\lambda}$ of $K \star \Delta^{n\natural}$ with

$$d_{i+j+m+2}\bar{\lambda} = \lambda \quad \text{and} \quad d_{i+j+l+1}\bar{\lambda} = \lambda + l.$$

By construction, every face of $\bar{\lambda}$ except for λ lies in $X_{<(\sigma,\chi),m}$. Thus we may form the pushout

$$\begin{tikzcd} \bigsqcup_\lambda \Lambda^{i+j+m+2\natural}_{i+j+m+2} \arrow[r] \arrow[d] & X_{<(\sigma,\chi),m} \arrow[d] \\ \bigsqcup_\lambda \Delta^{i+j+m+2\natural} \arrow[r] & X_{<(\sigma,\chi),m+1} \end{tikzcd}$$

and complete the inductive step. Passing to the colimit, we deduce that $X_{<(\sigma,\chi)} \hookrightarrow X_{\sigma,\chi}$ is right marked anodyne, which completes the proof. $\square$

4.12 Remark The proof of Proposition 4.11 can be adapted to show that for any cartesian fibration $C \to S$, ${}_\natural\Lambda^n_0 \times_S C^\natural \hookrightarrow {}_\natural\Delta^n \times_S C^\natural$ is marked left anodyne (in the $\sigma = \varnothing$ case, we only use that $e_0 \colon \mathcal{O}(S) \to S$ is a cartesian fibration). As well, letting $K = \varnothing$, part (2) of Proposition 4.11 shows that $\Lambda^n_k \times_S C \hookrightarrow \Delta^n \times_S C$ is inner anodyne. This refines the theorem that marked left (resp. inner) anodyne maps pullback to cocartesian (resp. categorical) equivalences along cartesian fibrations.

For later use, we state a criterion for showing that a functor is left Quillen.

4.13 Lemma *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $F \colon \mathcal{M} \to \mathcal{N}$ be a functor which preserves cofibrations. Let I be a weakly saturated* [9, Definition A.1.2.2] *subset of the trivial cofibrations in $\mathcal{M}$ such that for every object $A \in \mathcal{M}$, we have a map $f \colon A \to A'$ where $f \in I$ and A' is fibrant. Then F preserves trivial cofibrations if and only if*

(1) *for every $f \in I$, $F(f)$ is a trivial cofibration*;

(2) *F preserves trivial cofibrations between fibrant objects.*

Proof The "only if" direction is obvious. For the other direction, let $A \to B$ be a trivial cofibration in $\mathcal{M}$. We may form the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dr] & \\ A' \arrow[r] & A' \sqcup_A B \arrow[r] & (A' \sqcup_A B)' \end{tikzcd}$$

where the vertical and lower right horizontal arrows are in I. Then our two assumptions along with the two-out-of-three property of the weak equivalences shows that $F(A) \to F(B)$ is a trivial cofibration. $\square$

4.14 Lemma *Let K be a simplicial set over S. Then*

$$K \star_S -, - \star_S K \colon s\mathbf{Set}_{/S} \to s\mathbf{Set}_{K/\!/S}$$

are left adjoints. Similarly, for K a marked simplicial set over S,

$$K \star_S -, - \star_S K \colon s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{K/\!/S}$$

are left adjoints.

Proof We prove that $K \star_S -$ is a left adjoint in the unmarked case and leave the other cases to the reader. Let F denote $K \star_S -$ and define a functor $G \colon s\mathbf{Set}_{K/\!/S} \to s\mathbf{Set}_{/S}$ by letting $G(K \to C)$ be the simplicial set over S which satisfies

$$\mathrm{Hom}_{/S}(\Delta^n, G(K \to C)) = \mathrm{Hom}_{K/\!/S}(K \star_S \Delta^n, C);$$

this is evidently natural in $K \to C$. Define a unit map $\eta\colon \mathrm{id} \to GF$ on objects X by sending $\sigma\colon \Delta^n \to X$ to $K \star_S \sigma\colon K \star_S \Delta^n \to K \star_S X$, which corresponds to $\Delta^n \to G(K \star_S X)$. Define a counit map $\eta\colon FG \to \mathrm{id}$ on objects $K \to C$ by sending $\lambda = (\sigma, \tau)\colon \Delta^i \star \Delta^j \to K \star_S G(K \to C)$ to $\Delta^i \star \Delta^j \xrightarrow{(\sigma,\mathrm{id})} K \star_S \Delta^j \xrightarrow{\tau'} C$, where τ' corresponds to $\tau\colon \Delta^j \to G(K \to C)$. Then it is straightforward to verify the triangle identities, so F is adjoint to G. $\square$

For the following pair of results, endow $s\mathbf{Set}^+_{/S}$ with the cocartesian model structure and $s\mathbf{Set}^+_{K/\!/S} = (s\mathbf{Set}^+_{/S})_{K/}$ with the model structure created by the forgetful functor to $s\mathbf{Set}^+_{/S}$.

4.15 Theorem *Let K be a marked simplicial set over S. The functor*

$$K \star_S (- \times_S \mathcal{O}(S)^\sharp) \colon s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{K/\!/S}$$

is left Quillen.

Proof We will denote the functor in question by F. First observe that F is the composite of the three left adjoints e_0^*, $e_{1!}$, and $K \star_S -$, so F is a left adjoint. F evidently preserve cofibrations, so it only remains to check that F preserves the trivial cofibrations. We first verify that F preserves the left marked anodyne maps. Since F preserves colimits it suffices to check that F preserves a collection of morphisms which generate the left marked anodyne maps as a weakly saturated class. We verify that F preserves the four classes of maps enumerated in [9, Definition 3.1.1.1].

(1) For $\iota\colon (\Lambda^n_k)^\flat \to (\Delta^n)^\flat$, $0 < k < n$, the underlying map of simplicial sets of $F(\iota)$ is inner anodyne by Proposition 4.11. $F(\iota)$ is obtained by marking common edges of an inner anodyne map, so is left marked anodyne.

(2) For $\iota\colon {}_\natural\Lambda^n_0 \to {}_\natural\Delta^n$, we observe that the map

$$K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\sharp) \sqcup_{K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural)} K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural) \to K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\sharp)$$

in the case $n = 1$ is marked left anodyne, since every marked edge in the codomain factors as a composite of two marked edges in the domain, and is the identity if $n > 1$. It thus suffices to show that $K \star_S ({}_\natural\Lambda^n_0 \times_S \mathcal{O}(S)^\natural) \to K \star_S ({}_\natural\Delta^n \times_S \mathcal{O}(S)^\natural)$ is left marked anodyne, which is the content of part (1) of Proposition 4.11.

(3) and (4) In both of these cases one has a map of marked simplicial sets $A \to B$ whose underlying map is an isomorphism of simplicial sets. Then

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F(A) \arrow[d] \\ B \arrow[r] & F(B) \end{tikzcd}$$

is a pushout square, so $F(A) \to F(B)$ is left marked anodyne if $A \to B$ is.

Next, let $f\colon {}_\natural C \to {}_\natural D$ be a cocartesian equivalence between cocartesian fibrations over S. Let $g\colon {}_\natural D \to {}_\natural C$ be a homotopy inverse of f, so that there exists a homotopy $h\colon {}_\natural C \times (\Delta^1)^\sharp \to {}_\natural C$ over S from id_C to $g \circ f$. Define a map

$$\phi\colon (K \star_S ({}_\natural C \times_S \mathcal{O}(S)^\sharp)) \times (\Delta^1)^\sharp \to K \star_S (({}_\natural C \times_S \mathcal{O}(S)^\sharp) \times (\Delta^1)^\sharp)$$

by sending an $(i+j+1)$–simplex (λ, α) given by the data

$$\sigma\colon \Delta^i \to K, \quad \tau\colon \Delta^j \to {}_\natural C \times_S \mathcal{O}(S)^\sharp, \quad \pi \circ \lambda\colon \Delta^{i+j+1} \to \Delta^1, \quad \alpha\colon \Delta^{i+j+1} \to \Delta^1$$

to an $(i+j+1)$–simplex λ' given by σ, $(\tau, \alpha \circ \iota)$ and $\pi \circ \lambda$, where $\iota\colon \Delta^j \to \Delta^i \star \Delta^j$ is the inclusion. It is easy to see that ϕ restricts to an isomorphism on

$$(K \star_S ({}_\natural C \times_S \mathbb{O}(S)^\sharp)) \times \partial\Delta^1.$$

We deduce that $F(h) \circ \phi$ is a homotopy from $F(g \circ f)$ to the identity. A similar argument concerning a chosen homotopy from $f \circ g$ to id_D shows that $F(f)$ is a cocartesian equivalence.

Finally, invoking Lemma 4.13 completes the proof. $\square$

4.16 Theorem *Let K be a marked simplicial set over S. The functor*

$$- \star_S K\colon s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{K/\!/S}$$

is left Quillen.

Proof As with the proof of Theorem 4.15, the proof will be an application of Lemma 4.13. We first verify that $- \star_S K$ preserves the four classes of left marked anodyne maps enumerated in [9, Definition 3.1.1.1]. Class (1) is handled by the dual of part (2) of Proposition 4.11. Class (2) is handled by the dual of part (3) of Proposition 4.11. Classes (3) and (4) are handled as in the proof of Theorem 4.15. Finally, the case of $A \to B$ a cocartesian equivalence between fibrant objects is also handled as in the proof of Theorem 4.15. $\square$

4.17 Definition Let $K, C \to S$ be marked simplicial sets over S and let $p\colon K \to C$ be a map over S. Define the marked simplicial set $C_{(p,S)/} \to S$ as the value of the right adjoint to $K \star_S (- \times_S \mathbb{O}(S)^\sharp)$ on $K \to C \to S$ in $s\mathbf{Set}^+_{K/\!/S}$. By Theorem 4.15, if $C \to S$ is an S–category, then $C_{(p,S)/} \to S$ is an S–category. We will refer to $C_{(p,S)/}$ as a *S–undercategory* of C.

Dually, define the marked simplicial set $C_{/(p,S)} \to S$ as the value of the right adjoint to $- \star_S (K \times_S \mathbb{O}(S)^\sharp)$ on $K \to C \to S$ in $s\mathbf{Set}^+_{K/\!/S}$. By Theorem 4.16 applied to $K \times_S \mathbb{O}(S)^\sharp$, if $C \to S$ is an S–category, then $C_{/(p,S)} \to S$ is an S–category. We will refer to $C_{/(p,S)}$ as an *S–overcategory* of C.

In the sequel, we will focus our attention on the S–undercategory and leave proofs of the evident dual assertions to the reader.

Functoriality in the diagram

We now study the functoriality of the S–undercategory with respect to the diagram category. Given maps $f\colon K \to L$ and $p\colon L \to X$ of marked simplicial sets over S, we have an induced map $X_{(p,S)/} \to X_{(pf,S)/}$, which in terms of the functors that $X_{(p,S)/}$ and $X_{(pf,S)/}$ represent is given by precomposing $L \star_S (A \times_S \mathcal{O}(S)^\sharp) \to X$ by $f \star_S \mathrm{id}$.

Recall that for a category $\mathcal{M}$ admitting pushouts and a map $f\colon K \to L$, we have an adjunction

$$f_!\colon \mathcal{M}_{K/} \rightleftarrows \mathcal{M}_{L/} : f^*$$

where $f_!(K \to X) = X \sqcup_K L$ and $f^*(L \xrightarrow{p} X) = p \circ f$. If $\mathcal{M}$ is a model category and $\mathcal{M}_{K/}$ and $\mathcal{M}_{L/}$ are provided with the model structures induced from $\mathcal{M}$, then $(f_!, f^*)$ is a Quillen adjunction. Moreover, if $\mathcal{M}$ is a left proper model category and f is a weak equivalence, then $(f_!, f^*)$ is a Quillen equivalence.

4.18 Proposition *Let $f\colon K \to L$ be a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Let C be an S–category and let $p\colon L \to {}_\natural C$ be a map. Then ${}_\natural C_{(p,S)/} \to {}_\natural C_{(pf,S)/}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.*

Proof Let $F = f_! \circ (K \star_S (- \times_S \mathcal{O}(S)^\sharp))$ and let $F' = L \star_S (- \times_S \mathcal{O}(S)^\sharp)$. Let G and G' be the right adjoints to F and F', respectively. Let $\alpha\colon F \to F'$ be the evident natural transformation and let $\beta\colon G' \to G$ be the dual natural transformation, defined by $G' \xrightarrow{\eta_{G'}} GFG' \xrightarrow{G\alpha G'} GF'G' \xrightarrow{G\epsilon'} G$. Then $\beta_C\colon {}_\natural C_{(p,S)/} \to {}_\natural C_{(pf,S)/}$ is the map under consideration. By Theorem 4.16, α_X is a cocartesian equivalence for all $X \in s\mathbf{Set}^+_{/S}$. Therefore, by [7, Corollary 1.4.4(b)], β_C is a cocartesian equivalence. □

4.19 Proposition *Consider a commutative diagram of marked simplicial sets*

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & C \arrow[d, "q"] \\ L \arrow[r] \arrow[ur, "p"] & D \end{tikzcd}$$

where i is a cofibration and q is a fibration.

(1) *The map*

$$C_{(p,S)/} \to C_{(pi,S)/} \times_{D_{(qpi,S)/}} D_{(qp,S)/}$$

is a fibration.

(2) *Let $K = \varnothing$ and $D = S^\sharp$. Then the map*

$$C_{(p,S)/} \to C_{(pi,S)/} \cong \underline{\mathrm{Fun}}_S(S^\sharp, C)$$

is a left fibration (*of the underlying simplicial sets*).

Proof (1) Given a trivial cofibration $A \to B$, we need to solve lifting problems of the form

$$\begin{tikzcd} L \star_S (A \times_S \mathcal{O}(S)^\sharp) \sqcup_{K \star_S (A \times_S \mathcal{O}(S)^\sharp)} K \star_S (B \times_S \mathcal{O}(S)^\sharp) \arrow[r] \arrow[d] & C \arrow[d] \\ L \star_S (B \times_S \mathcal{O}(S)^\sharp) \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

but the left-hand map is a trivial cofibration by Theorem 4.15.

(2) We need to solve lifting problems of the form

$$\begin{tikzcd} (\Delta^n)^\flat \times_S \mathcal{O}(S)^\sharp \sqcup_{(\Lambda^n_i)^\flat} K \star_S ((\Lambda^n_i)^\flat \times_S \mathcal{O}(S)^\sharp) \arrow[r] \arrow[d] & C \arrow[d] \\ K \star_S ((\Delta^n)^\flat \times_S \mathcal{O}(S)^\sharp) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

where $0 \le i < n$, but the left-hand map is a trivial cofibration by Proposition 4.11($1'$) and (2). $\square$

Combining (2) of the above proposition with Lemma 3.6(2) — which supplies a trivial marked fibration $\underline{\mathrm{Fun}}_S(S^\sharp, C) \to C$ — we obtain a map $C_{(p,S)/} \to C$ which is a marked fibration and a left fibration, and such that for any $f : K \to L$, the triangle

$$\begin{tikzcd} C_{(p,S)/} \arrow[rr] \arrow[dr] & & C_{(pf,S)/} \arrow[dl] \\ & C & \end{tikzcd}$$

commutes.

The universal mapping property of the S–slice

Because the S–join and slice Quillen adjunction is not simplicial, we do not immediately obtain a universal mapping property characterizing the S–slice. Our goal in this subsection is to supply such a universal mapping property (Proposition 4.25). We first recall how to slice Quillen bifunctors. Suppose $\mathcal{V}$ is a closed symmetric monoidal category and $\mathcal{M}$ is enriched, tensored, and cotensored over $\mathcal{V}$. Denote the internal hom by

$$\underline{\mathrm{Hom}}(-,-)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}.$$

Define bifunctors

$$\underline{\mathrm{Hom}}_{x/}(-,-)\colon \mathcal{M}^{\mathrm{op}}_{x/} \times \mathcal{M}_{x/} \to \mathcal{V}, \quad \underline{\mathrm{Hom}}_{/x}(-,-)\colon \mathcal{M}^{\mathrm{op}}_{/x} \times \mathcal{M}_{/x} \to \mathcal{V}$$

on objects $f\colon x \to a$, $g\colon x \to b$ and $f'\colon a \to x$, $g'\colon b \to x$ to be pullbacks

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{x/}(f,g) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(a,b) \arrow[d, "f^*"] \\
1 \arrow[r, "g"] & \underline{\mathrm{Hom}}(x,b)
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathrm{Hom}}_{/x}(f',g') \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(a,b) \arrow[d, "g'_*"] \\
1 \arrow[r, "f'"] & \underline{\mathrm{Hom}}(a,x)
\end{tikzcd}$$

and on morphisms in the obvious way (we abusively denote by $g\colon 1 \to \underline{\mathrm{Hom}}(x,b)$ the map corresponding to g under the natural isomorphisms $\underline{\mathrm{Hom}}(1,\underline{\mathrm{Hom}}(x,b)) \cong \underline{\mathrm{Hom}}(1 \otimes x, b) \cong \underline{\mathrm{Hom}}(x,b)$, and likewise for f'). It is easy to see that $\underline{\mathrm{Hom}}_{x/}$ and $\underline{\mathrm{Hom}}_{/x}$ preserve limits separately in each variable.

4.20 Lemma *In the above situation let $\mathcal{M}$ be a model category and $\mathcal{V}$ be a monoidal model category. If $\underline{\mathrm{Hom}}(-,-)$ is a right Quillen bifunctor, then $\underline{\mathrm{Hom}}_{x/}(-,-)$ and $\underline{\mathrm{Hom}}_{/x}(-,-)$ are right Quillen bifunctors, where we endow $\mathcal{M}_{x/}$ and $\mathcal{M}_{/x}$ with the model structures created by the forgetful functor to $\mathcal{M}$.*

Proof We prove the assertion for $\underline{\mathrm{Hom}}_{x/}(-,-)$, the proof for $\underline{\mathrm{Hom}}_{/x}(-,-)$ being identical. Let $i\colon a \to b$ and $f\colon c \to d$ be morphisms in $\mathcal{M}_{x/}$ (so they are compatible with the structure maps $\pi_a, \dots, \pi_d$). In the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{x/}(\pi_b,\pi_c) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(b,c) \arrow[d] \\
\underline{\mathrm{Hom}}_{x/}(\pi_a,\pi_c) \times_{\underline{\mathrm{Hom}}_{x/}(\pi_a,\pi_d)} \underline{\mathrm{Hom}}_{x/}(\pi_b,\pi_d) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(a,c) \times_{\underline{\mathrm{Hom}}(a,d)} \underline{\mathrm{Hom}}(b,d) \arrow[d] \\
1 \arrow[r] & \underline{\mathrm{Hom}}(x,c)
\end{tikzcd}$$

it is easy to see that the lower square and the rectangle are pullback squares, so the upper square is a pullback square. It is now clear that if $\underline{\mathrm{Hom}}(-,-)$ is a right Quillen bifunctor, then $\underline{\mathrm{Hom}}_{x/}(-,-)$ is as well. □

We apply Lemma 4.20 to the bifunctors

$$\mathrm{Map}_{K/\!\!/S}(-,-)\colon s\mathbf{Set}^{+}_{K/\!\!/S}{}^{\mathrm{op}} \times s\mathbf{Set}^{+}_{K/\!\!/S} \to s\mathbf{Set}_{\mathrm{Quillen}},$$
$$\mathrm{Fun}_{K/\!\!/S}(-,-)\colon s\mathbf{Set}^{+}_{K/\!\!/S}{}^{\mathrm{op}} \times s\mathbf{Set}^{+}_{K/\!\!/S} \to s\mathbf{Set}_{\mathrm{Joyal}}$$

induced by $\mathrm{Map}_S(-,-)$ and $\mathrm{Fun}_S(-,-)$.

4.21 Lemma *Let K, A, and B be simplicial sets and define a map*

$$A \times (K \star B) \to K \star (A \times B)$$

by sending the data $(\Delta^n \to A,\ \Delta^k \to K,\ \Delta^{n-k-1} \to B)$ of a n–simplex of $A \times (K \star B)$ to the data $(\Delta^k \to K,\ \Delta^{n-k-1} \to A \times B)$ of a n–simplex of $K \star (A \times B)$. Then

$$\phi\colon A \times (K \star B) \sqcup_{A \times K} K \to K \star (A \times B)$$

is a categorical equivalence.

Proof Recall [9, Proposition 4.2.1.2] that there is a map

$$\eta_{X,Y} : X \diamond Y = X \sqcup_{X \times Y \times \{0\}} X \times Y \times \Delta^1 \sqcup_{X \times Y \times \{1\}} Y \to X \star Y$$

natural in X and Y which is always a categorical equivalence. Thus

$$f = (A \times \eta_{K,B}) \sqcup \mathrm{id}_K : A \times (K \diamond B) \sqcup_{A \times K} K \to A \times (K \star B) \sqcup_{A \times K} K$$

is a categorical equivalence. The domain is isomorphic to $K \diamond (A \times B)$, and it is easy to check that the map $\eta_{K, A \times B}$ is the composite

$$K \diamond (A \times B) \xrightarrow{f} A \times (K \star B) \sqcup_{A \times K} K \xrightarrow{\phi} K \star (A \times B).$$

Using the two-out-of-three property of the categorical equivalences, we deduce that ϕ is a categorical equivalence. □

4.22 Lemma *For all $L \in s\mathbf{Set}^{+}_{/S}$, we have a natural equivalence*

$$\phi\colon \mathrm{Fun}_S(L, {}_\natural C_{(p,S)/}) \xrightarrow{\simeq} \mathrm{Fun}_{K/\!\!/S}(K \star_S (L \times_S \mathbb{O}(S)^\sharp), {}_\natural C).$$

Proof Define bisimplicial sets $X, Y : \Delta^{\mathrm{op}} \to s\mathbf{Set}$ by

$$X_n = \mathrm{Map}_{K/\!\!/S}\big(K \star_S ((\Delta^n)^\flat \times L \times_S \mathbb{O}(S)^\sharp), {}_\natural C\big),$$
$$Y_n = \mathrm{Map}\big(\Delta^n, \mathrm{Fun}_{K/\!\!/S}(K \star_S (L \times_S \mathbb{O}(S)^\sharp), {}_\natural C)\big)$$
$$\cong \mathrm{Map}_{K/\!\!/S}\big((\Delta^n)^\flat \times \big(K \star_S (L \times_S \mathbb{O}(S)^\sharp) \sqcup_{(\Delta^n)^\flat \times K} K, {}_\natural C\big)\big).$$

and define a map of bisimplicial sets $\Phi\colon X \to Y$ by precomposing levelwise by the map

$$g_{L,n}\colon (\Delta^n)^\flat \times (K \star_S (L \times_S \mathbb{O}(S)^\sharp)) \sqcup_{(\Delta^n)^\flat \times K} K \to K \star_S ((\Delta^n)^\flat \times L \times_S \mathbb{O}(S)^\sharp)$$

adjoint as a map over $S \times \Delta^1$ to the identity over $S \times \partial\Delta^1$. Taking levelwise zero simplices then defines the map ϕ, which is clearly natural in L, K, and C. By Theorem 4.16, taking a fibrant replacement of K we may suppose that K is fibrant. We first check that X and Y are complete Segal spaces. By [8, Theorem 4.12], Y is a complete Segal space as it arises from a ∞–category. For X, since $\mathrm{Map}_{K/\!/S}(-,-)$ is a right Quillen bifunctor, we only have to observe that:

- Every monomorphism $A \to B$ of simplicial sets induces a cofibration
$$K \star_S (A^\flat \times L \times_S \mathbb{O}(S)^\sharp) \to K \star_S (B^\flat \times L \times_S \mathbb{O}(S)^\sharp),$$
so X is Reedy fibrant.
- The spine inclusion $\iota_n\colon \mathrm{Sp}(n) \to \Delta^n$ induces a trivial cofibration
$$K \star_S (\mathrm{Sp}(n)^\flat \times L \times_S \mathbb{O}(S)^\sharp) \to K \star_S ((\Delta^n)^\flat \times L \times_S \mathbb{O}(S)^\sharp).$$
Since ι_n is inner anodyne, this follows from Theorem 4.15 and [9, Proposition 3.1.4.2].
- The map $\pi\colon E \to \Delta^0$, where E is the nerve of the contractible groupoid with two elements, induces a cocartesian equivalence
$$K \star_S (E^\flat \times L \times_S \mathbb{O}(S)^\sharp) \to K \star_S (L \times_S \mathbb{O}(S)^\sharp);$$
$\pi^\flat$ is a cocartesian equivalence (as the composite of $E^\flat \to E^\sharp$ and $E^\sharp \to \Delta^0$), so this also follows from Theorem 4.15 and [9, Proposition 3.1.4.2].

We next prove that Φ is an equivalence in the complete Segal model structure. For this, we will prove that each map $g_{L,n}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Both sides preserve colimits as a functor of L (valued in $s\mathbf{Set}^+_{K/\!/S}$), so by left properness and the stability of cocartesian equivalences under filtered colimits we reduce to the case L is an m–simplex with some marking. In particular, $(\Delta^m)^\flat \times_S \mathbb{O}(S)^\sharp \to S$ is fibrant in $s\mathbf{Set}^+_{/S}$. By [9, Theorem 4.2.4.1] we may check that the square of fibrant objects

$$\begin{tikzcd}
(\Delta^n)^\flat \times K \arrow[r] \arrow[d] & K \arrow[d] \\
(\Delta^n)^\flat \times (K \star_S ((\Delta^m)^\flat \star_S \mathbb{O}(S)^\sharp)) \arrow[r] & K \star_S ((\Delta^n)^\flat \times (\Delta^m)^\flat \times_S \mathbb{O}(S)^\sharp)
\end{tikzcd}$$

is a homotopy pushout square in the underlying ∞–category $\mathbf{Cat}^{\mathrm{cocart}}_{\infty,S} \simeq \mathrm{Fun}(S, \mathbf{Cat}_\infty)$, where colimits are computed objectwise. In other words, we may check that for every $s \in S$, the fiber of the square over s is a homotopy pushout square in $s\mathbf{Set}$, which holds by Lemma 4.21. Pushing out along the cofibration $(\Delta^m)^\flat \times_S \mathcal{O}(S)^\sharp \to L \times_S \mathcal{O}(S)^\sharp$ and using left properness, we deduce that $g_{L,m}$ is a cocartesian equivalence. Finally, we invoke [8, Theorem 4.11] to deduce that ϕ is a categorical equivalence. □

4.23 Lemma *Let $L \to S$ be a cocartesian fibration. Then*

$$\mathrm{id}_K \star \iota_L \colon K \star_S {}_\natural L \to K \star_S ({}_\natural L \times_S \mathcal{O}(S)^\sharp)$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$.

Proof By Theorem 4.16, taking a fibrant replacement of K we may suppose that K is fibrant. By Proposition A.4, it suffices to show that for every $s \in S$,

$$K^\sim_s \star L^\sim_s \to K^\sim_s \star ({}_\natural L \times_S (S^{/s})^\sharp)$$

is a marked equivalence in $s\mathbf{Set}^+$. The *cartesian* equivalence $\{s\} \to (S^{/s})^\sharp$ pulls back by the cocartesian fibration ${}_\natural L \to S^\sharp$ to a marked equivalence $L^\sim_s \to {}_\natural L \times_S (S^{/s})^\sharp$. Then, by Theorem 4.15 for $S = \Delta^0$, $K^\sim_s \star -$ preserves marked equivalences, which concludes the proof. □

4.24 Notation Suppose we have a commutative square of S–categories and S–functors:

$$\begin{tikzcd} K \arrow[r, "G"] \arrow[d, "F"] & D \arrow[d, "\pi"] \\ C \arrow[r, "\rho"] & M \end{tikzcd}$$

Define $\underline{\mathrm{Fun}}_{K/\!/M,S}(C, D)$ to be the pullback

$$\begin{tikzcd} \underline{\mathrm{Fun}}_{K/\!/M,S}(C, D) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(C, D) \arrow[d, "(F^*,\pi_*)"] \\ S \arrow[r, "\sigma_{\pi G}"] & \underline{\mathrm{Fun}}_S(K, M) \end{tikzcd}$$

If $K = \varnothing$, we will also denote $\underline{\mathrm{Fun}}_{K/\!/M,S}(C, D)$ by $\underline{\mathrm{Fun}}_{/M,S}(C, D)$. If $M = S$, we will write $\underline{\mathrm{Fun}}_{K/\!/S}(C, D)$ in place of $\underline{\mathrm{Fun}}_{K/\!/S,S}(C, D)$.

Note that by Propositions 3.8 and 2.16, the defining pullback square is a homotopy pullback square if F is a monomorphism and π is a categorical fibration.

4.25 Proposition *Let K, L and C be S–categories and let $p\colon K \to C$ and $q\colon L \to C$ be S–functors.*

(1) *We have an equivalence*

$$\psi\colon \underline{\mathrm{Fun}}_S(L, C_{(p,S)/}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{K/\!/S}(K \star_S L, C).$$

(2) *We have an equivalence*

$$\psi'\colon \underline{\mathrm{Fun}}_S(L, C_{/(q,S)}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_{L/\!/S}(K \star_S L, C)$$

(3) *We have equivalences*

$$\underline{\mathrm{Fun}}_{/C,S}(L, C_{(p,S)/}) \underset{\simeq}{\xrightarrow{\psi_q}} \underline{\mathrm{Fun}}_{K\sqcup L/\!/S}(K \star_S L, C) \underset{\simeq}{\xleftarrow{\psi'_p}} \underline{\mathrm{Fun}}_{/C,S}(K, C_{/(q,S)}).$$

Proof (1) Define the S–functor ψ as follows. Suppose we are given a marked simplicial set A and a map $A \to \underline{\mathrm{Fun}}_S(L, C_{(p,S)/})$ over S. This is equivalently given by the datum of a map

$$f_A\colon {}_\natural K \star_S ((A \times_S \mathcal{O}(S)^\sharp \times_S {}_\natural L) \times_S \mathcal{O}(S)^\sharp) \to {}_\natural C$$

under K and over S. Let

$${}_\natural K \sqcup_{A\times_S \mathcal{O}(S)^\sharp \times_S {}_\natural K} (A\times_S \mathcal{O}(S)^\sharp)\times_S ({}_\natural K \star_S ({}_\natural L \times_S \mathcal{O}(S)^\sharp)) \to K \star_S (A\times_S \mathcal{O}(S)^\sharp \times_S {}_\natural L \times_S \mathcal{O}(S)^\sharp)$$

be the map over $S \times \Delta^1$ adjoint to the identity over $S \times \partial\Delta^1$. Precomposing f_A by this and $\iota_L\colon {}_\natural L \to {}_\natural L \times_S \mathcal{O}(S)^\sharp$ on that factor defines the desired map

$$A \to \underline{\mathrm{Fun}}_{K/\!/S}(K \star_S L, C).$$

Now to check that ψ is an equivalence, we may work fiberwise and combine Lemmas 4.22 and 4.23.

(2) This follows by a parallel argument to the proof of (1).

(3) We prove that ψ_q is an equivalence; a parallel argument will work for ψ'_p. $\underline{\mathrm{Fun}}_{K\sqcup L/\!/S}(K \star_S L, C)$ fits into a diagram

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_{K\sqcup L/\!/S}(K \star_S L, C) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_{K/\!/S}(K \star_S L, C) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(K \star_S L, C) \arrow[d] \\
S \arrow[r] \arrow[rr, bend left, "\sigma_{p\sqcup q}"] & \underline{\mathrm{Fun}}_{K/\!/S}(K \sqcup L, C) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(K \sqcup L, C) \arrow[d] \\
 & S \arrow[r, "\sigma_p"] & \underline{\mathrm{Fun}}_S(K, C)
\end{tikzcd}$$

in which every square is a pullback square. The map ψ_q is then defined to be the pullback of the map of spans

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_S(L, C_{(p,S)/}) \arrow[r] \arrow[d, "\psi"] & \underline{\mathrm{Fun}}_S(L, C) \arrow[d, "p\sqcup-"] & S \arrow[l, "\sigma_q"'] \arrow[d, equal] \\
\underline{\mathrm{Fun}}_{K/\!\!/S}(K \star_S L, C) \arrow[r] & \underline{\mathrm{Fun}}_{K/\!\!/S}(K \sqcup L, C) & S \arrow[l]
\end{tikzcd}$$

in which the vertical arrows are equivalences. By Proposition 4.19 and $\underline{\mathrm{Fun}}_S(L,-)$ being right Quillen, the top left horizontal arrow is an S–fibration, and by Proposition 3.8, the bottom left horizontal arrow is an S–fibration, so ψ_q is an equivalence. □

In light of Proposition 4.25, we have evident "alternative" S–slice S–categories, whose definition more closely adheres to the intuition that a slice category is a category of extensions.

4.26 Definition Let $p\colon K \to C$ be an S–functor. We define the *alternative S–undercategory*

$$C^{(p,S)/} := \underline{\mathrm{Fun}}_{K/\!\!/S}(K \star_S S, C).$$

Similarly, we define the *alternative S–overcategory*

$$C^{/(p,S)} := \underline{\mathrm{Fun}}_{K/\!\!/S}(S \star_S K, C).$$

4.27 Corollary *Let $p\colon K \to C$ and $q\colon L \to C$ be S–functors.*

(1) *We have equivalences $C_{(p,S)/} \xrightarrow{\simeq} C^{(p,S)/}$ and $C_{/(q,S)} \xrightarrow{\simeq} C^{/(q,S)}$.*

(2) *We have an equivalence $\underline{\mathrm{Fun}}_{/C,S}(L, C^{(p,S)/}) \simeq \underline{\mathrm{Fun}}_{/C,S}(K, C^{/(q,S)})$ through a natural zigzag.*

Proof For (1), let $L = S$ and $K = S$ in Proposition 4.25(1) and (2), respectively. For (2), combine the preceding (1) and Proposition 4.25(3). □

4.28 Warning When $S = \Delta^0$, the alternative S–undercategory

$$C^{(p,S)/} \cong \{p\} \times_{\mathrm{Fun}(K,C)} \mathrm{Fun}(K^{\rhd}, C)$$

differs from Lurie's alternative undercategory $C^{p/}$. However, we have a comparison functor

$$\{p\} \times_{\mathrm{Fun}(K,C)} \mathrm{Fun}(K^{\rhd}, C) \to C^{p/}$$

which is a categorical equivalence and which factors through the categorical equivalence $C_{p/} \to C^{p/}$ of [9, Proposition 4.2.1.5].

Slicing over and under S–points

We give a smaller model for slicing over and under S–points in an S–category C.

4.29 Notation Suppose C an S–category. Let

$$\mathbb{O}_S(C) := \widetilde{\operatorname{Fun}}_S(S \times \Delta^1, C) \cong S \times_{\mathbb{O}(S)} \mathbb{O}(C)$$

denote the fiberwise arrow S–category of C. Given an object $x \in C$, let

$$C^{/\underline{x}} := \mathbb{O}_S(C) \times_C \underline{x}, \quad C^{\underline{x}/} := \underline{x} \times_C \mathbb{O}_S(C).$$

4.30 Proposition *Let $x \in C$ be an object and denote by $i_x \colon \underline{x} \to C_{\underline{x}}$ the $\underline{x}$–functor defined by x. We have natural equivalences of $\underline{x}$–categories*

$$C_{\underline{x}}{}^{/(\underline{x}, i_x)} \simeq C^{/\underline{x}}, \quad C_{\underline{x}}{}^{/(i_x, \underline{x})} \simeq C^{\underline{x}/}.$$

Proof For any functor $S' \to S$ and S–category C, $\mathbb{O}_S(C) \times_S S' \cong \mathbb{O}_{S'}(C \times_S S')$. Therefore, $\mathbb{O}_S(C) \times_C \underline{x} \cong \mathbb{O}_{\underline{x}}(C_{\underline{x}}) \times_{C_{\underline{x}}} \underline{x}$ and likewise for $\underline{x} \times_C \mathbb{O}_S(C)$. Changing base to $\underline{x}$, we may suppose $S = \underline{x}$ and $i_x = i \colon S \to C$ is any S–functor. The identity section $S \to \mathbb{O}(S)$ induces a morphism of spans

$$\begin{tikzcd}
S \arrow[r, "\sigma_i"] \arrow[d, equal] & \underline{\operatorname{Fun}}_S(S, C) \arrow[d] & \underline{\operatorname{Fun}}_S(S \times \Delta^1, C) \arrow[l] \arrow[d] \\
S \arrow[r, "i"] & C & \widetilde{\operatorname{Fun}}_S(S \times \Delta^1, C) \arrow[l]
\end{tikzcd}$$

with the vertical maps equivalences. Taking pullbacks now yields the claim (where we use the isomorphism $S \star_S S \cong S \times \Delta^1$ to identify the upper pullback with the S–slice category in question). □

4.31 Proposition *We have a natural equivalence $C^{\underline{x}/} \simeq C^{x/}$ of left fibrations over C.*

Proof Using the marked left anodyne map ${}_\natural\Lambda^2_1 \to {}_\natural\Delta^2$ and the map of Lemma 2.23 for $n = 2$, we obtain a span

$$\begin{tikzcd}
 & \operatorname{Fun}({}_\natural\Delta^2, {}_\natural C) \arrow[dl, two heads, "\simeq"'] \arrow[dr, two heads, "\simeq"] & \\
\operatorname{Fun}((\Delta^{\{0,1\}})^\sharp, {}_\natural C) \times_{C^{\{1\}}} \operatorname{Fun}(\Delta^{\{1,2\}}, C) & & \operatorname{Fun}(\Delta^{\{0,2\}}, C) \times_{S^{\{0,2\}}} \operatorname{Fun}(\Delta^2, S)
\end{tikzcd}$$

Pulling back via $\{x\} \times_{C^{\{0\}}} -$ on the left and $- \times_{S^{\{1,2\}}} S$ on the right, and using that the inclusion $\Delta^{\{0,2\}} \to \Delta^2 \cup_{\Delta^{\{1,2\}}} \Delta^0$ is a categorical equivalence, we get

$$\begin{tikzcd} & \{x\} \times_{C^{\{0\}}} \mathrm{Fun}({}_\natural\Delta^2, {}_\natural C) \times_{S^{\{1,2\}}} S \arrow[dl, two heads, "\simeq"'] \arrow[dr, two heads, "\simeq"] & \\ C^{\underline{x}/} & & C^{x/} \end{tikzcd}$$

which completes the proof. □

5 Limits and colimits

In this section, we introduce S–colimits and study their basic properties. We then study the correspondence between S–colimits and S–limits through the vertical opposite construction of [3].

5.1 Definition Let C be an S–category and $\sigma\colon S \to C$ be a cocartesian section. We say that σ is a *S–initial object* if $\sigma(s)$ is an initial object for all objects $s \in S$. Dually, σ is an *S–final object* if $\sigma(s)$ is a final object for all $s \in S$.

5.2 Definition Let K and C be S–categories. Let $\bar{p}\colon K \star_S S \to C$ be an extension of an S–functor $p\colon K \to C$. From the commutativity of the diagram

$$\begin{tikzcd} S \arrow[r, "\sigma_{\bar{p}}"] \arrow[d, equal] & \underline{\mathrm{Fun}}_S(K \star_S S, C) \arrow[d] \\ S \arrow[r, "\sigma_p"] & \underline{\mathrm{Fun}}_S(K, C) \end{tikzcd}$$

(recall Notation 3.5 for $\sigma_{(-)}$) we see that $\sigma_{\bar{p}}$ defines a cocartesian section of $C^{(p,S)/}$ (Definition 4.26), which we also denote by $\sigma_{\bar{p}}$. We say that $\bar{p}$ is an *S–colimit diagram* if $\sigma_{\bar{p}}$ is an S–initial object. If $\bar{p}$ is an S–colimit diagram, then $\bar{p}|_S\colon S \to C$ is said to be an *S–colimit* of p. If S admits an initial object s, we will also identify the S–colimit with its value on s.

Dually, substituting $S \star_S K$ for $K \star_S S$ leads in a parallel way to the definition of an *S–limit diagram* and an *S–limit*.

5.3 Remark In view of the comparison result Corollary 4.27, we could also use the S–slice category $C_{(p,S)/}$ to make the definition of an S–colimit diagram. This would yield some additional generality, in that $C_{(p,S)/}$ is defined for an arbitrary marked simplicial set K. However, the construction $C^{(p,S)/}$ is easier to relate to functor categories, which we need to do to show that the left adjoint to the restriction along $K \subset K \star_S S$ computes colimits (a special case of Corollary 9.16).

5.4 Remark Suppose K and C are ∞–categories, and write $\pi\colon K \to *$ for the map to a point. One may define the K–indexed colimit "globally" as the (partially defined) left adjoint $\pi_!$ to the restriction functor $\pi^*\colon C \to \mathrm{Fun}(K, C)$. Given a diagram $p\colon K \to C$ that admits an extension to a colimit diagram $\bar{p}\colon K^{\rhd} \to C$ with cone point $\{v\}$, one then has $\bar{p}|_{\{v\}} \simeq \pi_!(p)$.

To establish a parallel picture for S–colimits, we will first need to introduce the concept of S–adjunctions (Definition 8.3). If we now let K and C be S–categories and $\pi\colon K \to S$ denote the structure map, we will show that if for all $s \in S$, $C_{\underline{s}}$ admits $K_{\underline{s}}$–indexed $S^{s/}$–colimits, then the restriction S–functor $\pi^*\colon C \to \underline{\mathrm{Fun}}_S(K, C)$ admits a left S–adjoint $\pi_!$ such that

$$(\pi_!)_s\colon \mathrm{Fun}_{S^{s/}}(K_{\underline{s}}, C_{\underline{s}}) \to C_s$$

computes the $S^{s/}$–colimit (Theorem 10.5 in the special case $\phi = \pi$). Furthermore, taking cocartesian sections of this S–adjunction then yields an adjunction, which we may abusively denote by

$$\pi_!\colon \mathrm{Fun}_S(K, C) \rightleftarrows \mathrm{Fun}_S(S, C) :\pi_*,$$

in which $\pi_!$ computes the S–colimit.

In proving some of the assertions in this subsection (Corollary 5.9 and Propositions 5.11 and 5.12), it will be convenient to have this relationship between S–colimits and S–adjunctions established. We note that there is no danger of circularity here since the proof of Theorem 10.5 (or its simpler predecessor Theorem 9.15) doesn't use any of the remainder of this subsection (which, apart from S–(co)limits in an S–category of S–objects, is only devoted to working out special classes of diagrams in the theory).

There are a couple instances where the notion of S–colimit specializes to a notion of ordinary category theory. For example, we have the following pair of propositions computing S–colimits and S–limits in an S–category of objects $\underline{C}_S$ as left or right Kan extensions in C; the asymmetry in their formulations arises due to working with *cocartesian* fibrations instead of cartesian fibrations to model S–categories. In the statements, recall Notation 3.11 for the meaning of $(-)^\dagger$.

5.5 Proposition *Let $\bar{p}\colon K \star_S S \to \underline{C}_S$ be an S–functor extending $p\colon K \to \underline{C}_S$. Suppose further that a left Kan extension of $p^\dagger\colon K \to C$ to a functor $K \star_S S \to C$ exists. Then the following are equivalent*:

(1) *$\bar{p}$ is an S–colimit diagram.*

(2) *$\bar{p}^\dagger$ is a left Kan extension of $p^\dagger$.*

(3) *$\bar{p}^\dagger|_{K_s^{\rhd}}$ is a colimit diagram for all $s \in S$.*

Proof (2) and (3) are equivalent because left Kan extensions along cocartesian fibrations are computed fiberwise. Suppose (3). To prove (1), we want to show that for every $s \in S$, $\bar{p}_{\underline{s}}$ is an initial object in $((\underline{C}_S)^{(p,S)/})_s$. But $((\underline{C}_S)^{(p,S)/})_s$ is equivalent to the fiber of $\mathrm{Fun}(K_{\underline{s}} \star_{\underline{s}} \underline{s}, C) \to \mathrm{Fun}(K_{\underline{s}}, C)$ over $p^\dagger|_{K_{\underline{s}}}$, so to prove the claim it suffices to show that the functor $\bar{p}^\dagger|_{K_{\underline{s}}}$ is a left Kan extension of $p|_{K_{\underline{s}}}$. This holds by the equivalence of (2) and (3) for $S^{s/}$.

Conversely, suppose (1). Since we supposed that a left Kan extension of $p^\dagger$ exists, left Kan extensions of $p^\dagger|_{K_s}$ all exist and *any* initial object in the fiber of

$$\mathrm{Fun}(K_{\underline{s}} \star_{\underline{s}} \underline{s}, C) \to \mathrm{Fun}(K_{\underline{s}}, C)$$

over $p^\dagger|_{K_{\underline{s}}}$ is a left Kan extension of $p^\dagger|_{K_{\underline{s}}}$, necessarily a fiberwise colimit diagram (we need this hypothesis because Kan extensions as defined in [9, Section 4.3.2] are always *pointwise* Kan extensions). This implies (3). $\square$

5.6 Proposition *Let $\bar{p}\colon S \star_S K \to \underline{C}_S$ be an S–functor extending $p\colon K \to \underline{C}_S$. Suppose further that a right Kan extension of $p^\dagger\colon K \to C$ to a functor $S \star_S K \to C$ exists. Then the following are equivalent*:

(1) *$\bar{p}$ is an S–limit diagram.*

(2) *$\bar{p}^\dagger$ is a right Kan extension of $p^\dagger$.*

(2′) *$\bar{p}^\dagger|_{\underline{s} \star_{\underline{s}} K_{\underline{s}}}$ is a right Kan extension of $p^\dagger|_{K_{\underline{s}}}$ for all $s \in S$.*

(3) *$\bar{p}^\dagger|_{K_{\underline{s}}^{\lhd}}$ is a limit diagram for all $s \in S$.*

Proof We first observe that because the inclusion $S \to S \star_S K$ is left adjoint to the structure map $S \star_S K \to S$ of the cocartesian fibration,

$$(S \star_S K)^{s/} \simeq S^{s/} \times_S (S \star_S K) \cong \underline{s} \star_{\underline{s}} K_{\underline{s}}.$$

The equivalence of (2) and (2′) now follows from the formula for a right Kan extension. Also, if we view $K_{\underline{s}}^{\triangleleft}$ as mapping to $S \star_S K$ via $\{s\} \star K_{\underline{s}} \to \underline{s} \star_{\underline{s}} K_{\underline{s}} \to S \star_S K$, where the first map is adjoint to $(\{s\} \to \underline{s}, \mathrm{id})$, then (2) and (3) are also equivalent by the same argument. Finally, (2′) implies (1) by definition, and (1) implies (2′) under our additional assumption that a right Kan extension of $p^\dagger$ exists (for the same reason as given in the proof of Proposition 5.5). □

If S is a Kan complex, then the notion of S–colimit reduces to the usual notion of colimit.

5.7 Proposition *Let S be a Kan complex. Then an S–functor $\bar{p}\colon K \star_S S \to C$ is an S–colimit diagram if and only if for every object $s \in S$, $\bar{p}|_s\colon (K_s)^{\triangleright} \to C_s$ is a colimit diagram.*

Proof If S is a Kan complex, then for every $s \in S$, $S^{s/}$ is a contractible Kan complex. Therefore, for all $s \in S$ we have $(C^{(p,S)/})_s \simeq \{p_s\} \times_{\mathrm{Fun}(K_s, C_s)} \mathrm{Fun}(K_s^{\triangleright}, C_s)$, which proves the claim. □

We say that K is a *constant* S–category if it is equivalent to $S \times L$ for L an ∞–category. We have an isomorphism $L^{\triangleright} \times S \to (L \times S) \star_S S$ (defined as a map over $S \times \Delta^1$ to be the adjoint to the identity on $(L \times S, S)$).

5.8 Proposition *An S–functor $\bar{p}\colon L^{\triangleright} \times S \to C$ is an S–colimit diagram if and only if for every object $s \in S$, $\bar{p}_s\colon L^{\triangleright} \to C_s$ is a colimit diagram.*

Proof Observe that

$$\begin{aligned}(C^{(p,S)/})_s &= \{p_{\underline{s}}\} \times_{\mathrm{Fun}_{S^{s/}}(L \times S^{s/}, C_{\underline{s}})} \mathrm{Fun}_{S^{s/}}(L^{\triangleright} \times S^{s/}, C_{\underline{s}}) \\ &\simeq \{p_s\} \times_{\mathrm{Fun}(L, C_s)} \mathrm{Fun}(L^{\triangleright}, C_s).\end{aligned}$$

Therefore, $\sigma_{\bar{p}}\colon S \to C^{(p,S)/}$ is S–initial if and only if $\{\bar{p}_s\} \in \{p_s\} \times_{\mathrm{Fun}(L,C_s)} \mathrm{Fun}(L^{\triangleright}, C_s)$ is an initial object for all $s \in S$, which is the claim. □

5.9 Corollary *Suppose C is an S–category such that C_s admits all colimits for every object $s \in S$ and the pushforward functors $\alpha_!\colon C_s \to C_t$ preserve all colimits for every morphism $\alpha\colon s \to t$ in S. Then C admits all S–colimits indexed by constant diagrams.*

Proof First suppose that S has an initial object s. Suppose that $p\colon L\times S\to C$ is an S–functor. Let $\bar{p}_s\colon L^{\rhd}\to C_s$ be a colimit diagram extending p_s. Let $\bar{p}\colon L^{\rhd}\times S\to C$ be an S–functor corresponding to $\bar{p}_s$ under the equivalence $\mathrm{Fun}_S(L^{\rhd}\times S,C)\simeq\mathrm{Fun}(L^{\rhd},C_s)$, which we may suppose extends p. By Proposition 5.8, $\bar{p}$ is an S–colimit diagram.

The general case follows from Theorem 9.15, taking $\phi\colon C\to D$ to be $L\times S\to S$. □

We now turn to the example of corepresentable fibrations.

5.10 Definition Let $s\in S$ be an object and let K be an $S^{s/}$–category which is equivalent to a coproduct of corepresentable fibrations

$$\coprod_{i\in I} S^{\alpha_i/}\simeq\coprod_{i\in I} S^{t_i/}\xrightarrow{\coprod\alpha_i^*} S^{s/}$$

for $\{\alpha_i : s\to t_i\}_{i\in I}$ a collection of morphisms in S. Let $p\colon K\to C\times_S S^{s/}$ be an $S^{s/}$–functor, so p is precisely the data of objects $\{x_i\in C_{t_i}\}_{i\in I}$. Let

$$\bar{p}\colon K\star_{S^{s/}} S^{s/}\to C\times_S S^{s/}$$

be an $S^{s/}$–colimit diagram extending p, and let $y=\bar{p}(v)\in C_s$ for $v=\mathrm{id}_s$ be the cone point. Then we say that y is the *S–coproduct* of $\{x_i\}_{i\in I}$ along $\{\alpha_i\}_{i\in I}$, and we adopt the notation $y=\coprod_{\alpha_i} x_i$.

Our choice of terminology is guided by the following result, which shows that an $S^{s/}$–colimit of an $S^{s/}$–functor $p\colon S^{\alpha/}\simeq S^{t/}\to C$ obtains the value of a left adjoint to the pushforward functor $\alpha_!$ on $p(t)$. In the case of $S=\boldsymbol{O}_G^{\mathrm{op}}$, $C=\underline{\mathbf{Spc}}_G$ or $\underline{\mathbf{Sp}}^G$, and $K=\boldsymbol{O}_H^{\mathrm{op}}$, this is the induction or indexed coproduct functor from H to G.

5.11 Proposition *Let C be an S–category, let $\alpha: s\to t$ be a morphism in C, and let $\pi: M\to\Delta^1$ be a* **cartesian** *fibration classified by the pushforward functor $\alpha_!\colon C_s\to C_t$. Let $p\colon S^{t/}\to C\times_S S^{s/}$ be an $S^{s/}$–functor and let $x=p(\mathrm{id}_t)\in C_t$. Then the data of an $S^{s/}$–colimit diagram extending p yields a π–cocartesian edge e in M with $d_0(e)=x$ and lifting $0\to 1$.*

Proof Let $\bar{p}\colon S^{t/}\star_{S^{s/}} S^{s/}\to C\times_S S^{s/}$ be an $S^{s/}$–colimit diagram extending p. Let $y=\bar{p}(\mathrm{id}_s)$ and let $f'\colon\Delta^1\to S^{t/}\star_{S^{s/}} S^{s/}$ be the edge connecting id_t to α. We may suppose that M is given by the relative nerve of $\alpha_!$, so that edges in M over Δ^1 are given by commutative squares

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & C_s \arrow[d, "\alpha_!"] \\ \Delta^1 \arrow[r] & C_t \end{tikzcd}$$

Then let e be the edge in M determined by y and $f = \bar{p} \circ f' \colon x \to \alpha_! y$. By definition, $d_0(e) = x$.

We claim that e is π–cocartesian. This holds if and only if for every $y' \in C_s$ the map

$$\mathrm{Map}_{C_s}(y, y') \to \mathrm{Map}_{C_t}(x, \alpha_! y')$$

induced by f is an equivalence. But the local variant of the adjunction of Theorem 10.5 implies this (passing to global sections). □

S–coproducts also satisfy a base-change condition. This is awkward to articulate in general, because the pullback of a corepresentable fibration along another need not be corepresentable. However, if we impose the additional hypothesis that $T = S^{\mathrm{op}}$ admits multipullbacks, then a pullback of a corepresentable fibration decomposes as a finite coproduct of corepresentable fibrations. In this case, we have the following useful reformulation of the base-change condition. Recall from the introduction that we let $\boldsymbol{F}_T$ denote the finite coproduct completion of T. Let $X \subset \mathbb{O}(\boldsymbol{F}_T)$ be the full subcategory on those arrows whose source lies in T and consider the span

$$(\boldsymbol{F}_T)^\sharp \xleftarrow{\mathrm{ev}_1} {}_\natural X \xrightarrow{\mathrm{ev}_0} T^\sharp.$$

This satisfies the dual of the hypotheses of Theorem 2.24, so

$$C^\times := (\mathrm{ev}_0)_*(\mathrm{ev}_1)^*((C^\vee)^\natural)$$

is a cartesian fibration over $\boldsymbol{F}_T$ (with the cartesian edges marked), where $C^\vee \to T$ is the dual cartesian fibration of [3]. Unwinding the definitions, given a finite T–set $U = \coprod_i s_i$, we have that the fiber

$$(C^\times)_U \simeq \mathrm{Fun}_T\left(\coprod_i T^{/s_i}, C^\vee\right) \simeq \prod_i C_{s_i}$$

(where $\mathrm{Fun}_T(-,-)$ denotes those functors over T that preserve cartesian edges), and given a morphism of T–sets $\alpha \colon U \to V$, the pullback functor $\alpha^* \colon (C^\times)_U \to (C^\times)_V$ is induced by restriction.

5.12 Proposition *C admits finite S–coproducts if and only if $\pi : C^{\times} \to \boldsymbol{F}_T$ is a* ***Beck–Chevalley fibration****, ie π is both cocartesian and cartesian, and for every pullback square*

$$\begin{tikzcd} W \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & V' \arrow[d, "\beta"] \\ U \arrow[r, "\alpha"] & V \end{tikzcd}$$

in $\boldsymbol{F}_T$, the natural transformation

$$(\alpha')_!(\beta')^* \to \beta^* \alpha_! \tag{$*$}$$

adjoint to the equivalence $(\beta')^*\alpha^* \simeq (\alpha')^*\beta^*$ is itself an equivalence.

Proof By Theorem 10.5, C admits finite S–coproducts if and only if for every finite collection of morphisms $\{\alpha_i : s \to t_i\}$, the restriction functor

$$\Big(\coprod \alpha_i\Big)^* : \underline{\mathrm{Fun}}_S(S^{s/}, C) \to \underline{\mathrm{Fun}}_S\Big(\coprod_i S^{t_i/}, C\Big)$$

admits a left S–adjoint, in which case that left S–adjoint is computed by the S–coproduct along the α_i. This in turn is immediately equivalent to π being additionally cocartesian and $(*)$ being an equivalence for $\alpha = \coprod \alpha_i : \coprod t_i \to s$ and all morphisms $\beta : s' \to s$ in T. Finally, note that the apparently more general case of $(*)$ being an equivalence for any pullback square is actually determined by this, because any map $\alpha : U = \coprod t_i \to V = \coprod s_j$ is the data of $f : I \to J$ and $\{\alpha_{ij} : s_j \to t_i\}_{i \in f^{-1}(j)}$, whence $\alpha^* = (\alpha_{ij})^* : \prod_j C_{s_j} \to \prod_i C_{t_i}$, etc yields a decomposition of the map $(*)$ in terms of the "basic" squares that we already handled. □

We conclude this subsection by introducing a bit of useful terminology.

5.13 Definition Let C be an S–category. We say that C is *S–cocomplete* if, for every object $s \in S$ and $S^{s/}$–diagram $p : K \to C_{\underline{s}}$ (with K fiberwise small), p admits an $S^{s/}$–colimit.

5.14 Remark Suppose that E is S–cocomplete. Then taking $D = S$ in Theorem 9.15, E admits all (small) S–colimits. However, the converse may fail: if we suppose that E admits all S–colimits, then any $S^{s/}$–diagram $K_{\underline{s}} \to E_{\underline{s}}$ pulled back from an S–diagram $K \to E$ admits an $S^{s/}$–colimit; however, not every $S^{s/}$–diagram need be of this form.

Vertical opposites

In this subsection we study the vertical opposite construction of [3], with the goal of justifying our intuition that the theory of S–limits can be recovered from that of S–colimits, and vice versa (Corollary 5.25). We first recall the definition of the twisted arrow ∞–category from [1, Section 2].

5.15 Definition Given a simplicial set X, we define $\widetilde{\mathcal{O}}(X)$ to be the simplicial set whose n–simplices are given by the formula

$$\widetilde{\mathcal{O}}(X)_n := \mathrm{Hom}((\Delta^n)^{\mathrm{op}} \star \Delta^n, X).$$

If X is an ∞–category, then $\widetilde{\mathcal{O}}(X)$ is the twisted arrow ∞–category of X.

5.16 Warning By definition, $\widetilde{\mathcal{O}}(X)$ comes equipped with a source and target functors $\mathrm{ev}_0 \colon \widetilde{\mathcal{O}}(X) \to X^{\mathrm{op}}$ and $\mathrm{ev}_1 \colon \widetilde{\mathcal{O}}(X) \to X$, respectively. In other words, twisted arrows are *contravariant* in the source and *covariant* in the target. This convention is opposite to that in [11], but agrees with [3].

5.17 Recollection Suppose $X \to T$ a cocartesian fibration. Then the simplicial set X^{vop} is defined to have n–simplices

$$\begin{tikzcd} {}_\natural\widetilde{\mathcal{O}}(\Delta^n) \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & {}_\natural X \arrow[d] \\ (\Delta^n)^\sharp \arrow[r] & T^\sharp \end{tikzcd}$$

The forgetful map $X^{\mathrm{vop}} \to T$ is a cocartesian fibration with cocartesian edges given by $\widetilde{\mathcal{O}}(\Delta^1)^\sharp \to {}_\natural X$. For every $t \in T$, we have an equivalence $(X_t)^{\mathrm{op}} \xrightarrow{\simeq} (X^{\mathrm{vop}})_t$ implemented by the map which precomposes by $\mathrm{ev}_0 \colon {}_\natural\widetilde{\mathcal{O}}(\Delta^n) \to ((\Delta^n)^{\mathrm{op}})^\flat$, which is an equivalence in $s\mathbf{Set}^+$.

Dually, suppose $Y \to T$ a cartesian fibration. Then the simplicial set Y^{vop} is defined to have n–simplices

$$\begin{tikzcd} (\widetilde{\mathcal{O}}(\Delta^n)^{\mathrm{op}})^\natural \arrow[r] \arrow[d, "\mathrm{ev}_0^{\mathrm{op}}"'] & Y^\natural \arrow[d] \\ (\Delta^n)^\sharp \arrow[r] & T^\sharp \end{tikzcd}$$

and the forgetful map $Y^{\mathrm{vop}} \to T$ is a cartesian fibration with fibers $(Y^{\mathrm{vop}})_t \xleftarrow{\simeq} (Y_t)^{\mathrm{op}}$. As a warning, note that the definition of the underlying simplicial set of $(-)^{\mathrm{vop}}$ changes depending on whether the input is a cocartesian or cartesian fibration; in particular, the notation is potentially ambiguous for a bicartesian fibration. We will not apply $(-)^{\mathrm{vop}}$ to bicartesian fibrations in this paper.

Define a functor $\widetilde{\mathbb{O}}'(-)\colon s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{/S}$ by

$$\widetilde{\mathbb{O}}'(A \xrightarrow{\pi} S) = (\widetilde{\mathbb{O}}(A), \mathscr{E}_A) \xrightarrow{\pi\circ \mathrm{ev}_1} S$$

where an edge e is in $\mathscr{E}_A$ just in case $\mathrm{ev}_0(e)$ is marked in A^{op}. Note that $\widetilde{\mathbb{O}}(-)$ preserves colimits since it is defined as precomposition by $\Delta^{\mathrm{op}} \xrightarrow{(\mathrm{rev}\star\mathrm{id})^{\mathrm{op}}} \Delta^{\mathrm{op}}$, and from this it easily follows that $\widetilde{\mathbb{O}}'(-)$ also preserves colimits. By the adjoint functor theorem, $\widetilde{\mathbb{O}}'(-)$ admits a right adjoint, which we label $(-)^{\mathrm{vop}}$ — this agrees with the previously defined $(-)^{\mathrm{vop}}$ for cocartesian fibrations ${}_\natural X \to S^\sharp$.

5.18 Proposition *The adjunction*

$$\widetilde{\mathbb{O}}'(-)\colon s\mathbf{Set}^+_{/S} \rightleftarrows s\mathbf{Set}^+_{/S} :(-)^{\mathrm{vop}}$$

is a Quillen equivalence with respect to the cocartesian model structure on $s\mathbf{Set}^+_{/S}$*.*

Proof We first prove the adjunction is Quillen by employing the criteria of Lemma 4.13. Consider the four classes of maps which generate the left marked anodyne maps:

(1) $i\colon \Lambda^n_k \hookrightarrow \Delta^n$, $0<k<n$: By [1, Lemma 12.15], $\widetilde{\mathbb{O}}(\Lambda^n_k) \hookrightarrow \widetilde{\mathbb{O}}(\Delta^n)$ is inner anodyne, so $\widetilde{\mathbb{O}}'(i)$ is left marked anodyne.

(2) $i\colon {}_\natural\Lambda^n_0 \hookrightarrow {}_\natural\Delta^n$: We can adapt the proof of [1, Lemma 12.16] to show that $\widetilde{\mathbb{O}}'(i)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ (even though it fails to be left marked anodyne). The basic fact underlying this is that a *right* marked anodyne map is an equivalence in $s\mathbf{Set}^+$, so in $s\mathbf{Set}^+_{/S}$ if it lies entirely over an object; details are left to the reader.

(3) $i\colon K^\flat \hookrightarrow K^\sharp$ for K a Kan complex: Because $\widetilde{\mathbb{O}}(K) \to K^{\mathrm{op}} \times K$ is a left fibration, $\widetilde{\mathbb{O}}(K)$ is then again a Kan complex. It follows that $\widetilde{\mathbb{O}}'(i)$ is left marked anodyne.

(4) $(\Lambda^2_1)^\sharp \cup_{\Lambda^2_1} (\Delta^2)^\flat \hookrightarrow (\Delta^2)^\sharp$: obvious from the definitions.

It remains to show that for a trivial cofibration $f\colon {}_\natural X \hookrightarrow {}_\natural Y$ between fibrant objects, $\widetilde{\mathbb{O}}'(f)$ is again a trivial cofibration. Since $\widetilde{\mathbb{O}}(X) \to \widetilde{\mathbb{O}}(Y)$ is a map of cocartesian fibrations over S and the marking on $\widetilde{\mathbb{O}}'(-)$ contains these cocartesian edges, by

Proposition A.4 it suffices to show that for every object $s \in S$, $\widetilde{\mathbb{O}}'(X)_s \to \widetilde{\mathbb{O}}'(Y)_s$ is an equivalence in $s\mathbf{Set}^+$. We have a commutative square

$$\begin{tikzcd}
\widetilde{\mathbb{O}}'(X)_s \arrow[r] \arrow[d] & \widetilde{\mathbb{O}}'(Y)_s \arrow[d] \\
X_s^\sharp \arrow[r, "f_s"] & Y_s^\sharp
\end{tikzcd}$$

where the vertical maps are left fibrations and the bottom map is an equivalence in $s\mathbf{Set}^+$. Therefore, the map $X_s^\sharp \times_{Y_s^\sharp} \widetilde{\mathbb{O}}'(Y)_s \to \widetilde{\mathbb{O}}'(Y)_s$ is an equivalence in $s\mathbf{Set}^+$. Applying Proposition A.4 once more, we reduce to showing that for every object $x_1 \in X$, $\widetilde{\mathbb{O}}'(X)_{x_1} \to \widetilde{\mathbb{O}}'(Y)_{f(x_1)}$ is an equivalence in $s\mathbf{Set}^+$.

Now employing the source maps, we have a commutative square

$$\begin{tikzcd}
\widetilde{\mathbb{O}}'(X)_{x_1} \arrow[r] \arrow[d] & \widetilde{\mathbb{O}}'(Y)_{f(x_1)} \arrow[d] \\
X^{\mathrm{op}\natural} \arrow[r, "f^{\mathrm{op}}"] & Y^{\mathrm{op}\natural}
\end{tikzcd}$$

where the vertical maps are left fibrations and the bottom horizontal map is a *cartesian* equivalence in $s\mathbf{Set}^+_{/S^{\mathrm{op}}}$. Therefore, the map $X^{\mathrm{op}} \times_{Y^{\mathrm{op}}} \widetilde{\mathbb{O}}'(Y)_s \to \widetilde{\mathbb{O}}'(Y)_s$ is a cartesian equivalence. By a third application of Proposition A.4, we reduce to showing that for every object $x_0 \in X$, $\widetilde{\mathbb{O}}'(X)_{(x_0,x_1)} \to \widetilde{\mathbb{O}}'(Y)_{(f(x_0),f(x_1))}$ is an equivalence. But now both sides are endowed with the maximal marking and the map is equivalent to $\mathrm{Map}_X(x_0,x_1) \xrightarrow{f_*} \mathrm{Map}_Y(f(x_0),f(x_1))$, which is an equivalence by assumption.

The fact that this Quillen adjunction is an equivalence follows immediately from [3, Theorem 1.4]. □

5.19 Lemma *Let $C \to S$ be a cocartesian fibration.*

(1) *Let $f\colon S' \to S$ be a functor. Then $f^*(C^{\mathrm{vop}}) \cong f^*(C)^{\mathrm{vop}}$.*

(2) *Let $g\colon S \to T$ be a cartesian fibration and let C be an S–category. Then there is a T–functor $\chi\colon g_*(C)^{\mathrm{vop}} \to g_*(C^{\mathrm{vop}})$ natural in C which is an equivalence.*

Proof Part (1) is obvious from the definitions. For (2), the map χ is defined as follows: an n–simplex of $g_*(C)^{\mathrm{vop}}$ over $\sigma \in T_n$ is given by the data of a commutative diagram

$$\begin{tikzcd}
{}_\natural\widetilde{\mathbb{O}}(\Delta^n) \times_{T^\sharp} S^\sharp \arrow[r] \arrow[d] & {}_\natural C \arrow[d] \\
(\Delta^n \times_T S)^\sharp \arrow[r, "g^*\sigma"] & S^\sharp
\end{tikzcd}$$

and precomposition by the obvious map $\widetilde{\mathbb{O}}(\Delta^n \times_T S) \to \widetilde{\mathbb{O}}(\Delta^n) \times_T S$ yields an n–simplex of $g_*(C^{\mathrm{vop}})$.

We now show that for all $t \in T$, χ_t is a categorical equivalence. Because χ_t is obtained by taking levelwise 0–simplices of the map of complete Segal spaces

$$\mathrm{Map}_S({}_\natural\widetilde{\mathbb{O}}(\Delta^\bullet) \times S_t^\sharp, {}_\natural C) \to \mathrm{Map}_S({}_\natural\widetilde{\mathbb{O}}(\Delta^\bullet) \times \widetilde{\mathbb{O}}(S_t)^\sharp, {}_\natural C),$$

it suffices to show that for all n, ${}_\natural\widetilde{\mathbb{O}}(\Delta^n) \times \widetilde{\mathbb{O}}(S_t)^\sharp \to {}_\natural\widetilde{\mathbb{O}}(\Delta^n) \times S_t^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. As a special case of Proposition 6.3, $\widetilde{\mathbb{O}}(S_t)^\sharp \to S_t^\sharp$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S_t}$, so the claim follows. □

5.20 Lemma *The map* $\mathrm{ev}^{\mathrm{op}} \colon (\widetilde{\mathbb{O}}(\Delta^n)^{\mathrm{op}})^\natural \to (\Delta^n)^\sharp \times ((\Delta^n)^{\mathrm{op}})^\flat$ *is left marked anodyne.*

Proof For convenience, we will relabel $\widetilde{\mathbb{O}}(\Delta^n)^{\mathrm{op}}$ as the nerve of the poset I_n with objects ij, $0 \le i \le j \le n$ and maps $ij \to kl$ for $i \le k$ and $j \le l$. Then an edge $ij \to kl$ is marked in I_n just in case $j = l$, and the map $\mathrm{ev}^{\mathrm{op}}$ becomes the projection $\rho_n \colon I_n \to (\Delta^n)^\sharp \times (\Delta^n)^\flat$, $ij \mapsto (i, j)$. Let $f_n \colon (\Delta^n)^\flat \to I_n$ be the map which sends i to $0i$. Then $\rho_n \circ f_n \colon \{0\} \times (\Delta^n)^\flat \to (\Delta^n)^\sharp \times (\Delta^n)^\flat$ is left marked anodyne, so by the right cancellativity of left marked anodyne maps it suffices to show that i_n is left marked anodyne. For this, we factor f_n as the composition

$$(\Delta^n)^\flat = I_{n,-1} \to I_{n,0} \to \cdots \to I_{n,n} = I_n,$$

where $I_{n,k} \subset I_n$ is the subcategory on objects ij with $i = 0$ or $j \le k$ (and inherits the marking from I_n), and argue that each inclusion $g_k \colon I_{n,k} \subset I_{n,k+1}$ is left marked anodyne. For this, note that g_k fits into a pushout square

$$\begin{tikzcd}
\{0\} \times (\Delta^{k+1})^\flat \cup_{\{0\}\times(\Delta^k)^\flat} (\Delta^{n-k-1})^\sharp \times (\Delta^k)^\flat \arrow[r] \arrow[d] & (\Delta^{n-k-1})^\sharp \times (\Delta^{k+1})^\flat \arrow[d] \\
I_{n,k} \arrow[r, "g_k"] & I_{n,k+1}
\end{tikzcd}$$

with the upper horizontal map marked left anodyne. □

5.21 Construction Suppose T is an ∞–category, $X, Z \to T$ are cocartesian fibrations, $Y \to T$ is a cartesian fibration, and $\mu \colon {}_\natural X \times_T Y^\natural \to {}_\natural Z$ is a map of marked simplicial sets over T. We define a map

$$\mu^{\mathrm{vop}} \colon {}_\natural X^{\mathrm{vop}} \times_T Y^{\mathrm{vop}\natural} \to {}_\natural Z^{\mathrm{vop}}$$

by the following process:

Let J_n be the nerve of the poset with objects ij for $0 \le i \le n$, $-n \le j \le n$ and $-j \le i$, and maps $ij \to kl$ if $i \le k$ and $j \le l$. Mark edges $ij \to kl$ if $j = l$. Let $I_n \subset J_n$ be the subcategory on ij with $j \ge 0$ and $I'_n \subset J_n$ be the subcategory on ij with $j \le 0$; also give I_n and I'_n the induced markings. We have an inclusion $(\Delta^n)^\sharp \to J_n$ given by $i \mapsto i0$ which restricts to inclusions $(\Delta^n)^\sharp \to I_n$, $(\Delta^n)^\sharp \to I'_n$ and induces a map $\gamma_n \colon I_n \cup_{(\Delta^n)^\sharp} I'_n \subset J_n$.

Define auxiliary (unmarked) simplicial sets $Z' \to T$ by

$$\mathrm{Hom}_{/T}(\Delta^n, Z') = \mathrm{Hom}_{/T}(J_n, {}_\natural Z)$$

and $Z'' \to T$ by $\mathrm{Hom}_{/T}(\Delta^n, Z'') = \mathrm{Hom}_{/T}(I_n \cup_{(\Delta^n)^\sharp} I'_n, {}_\natural Z)$, where $J_n \to \Delta^n$ via $ij \mapsto i$. We have a map $r \colon Z' \to Z''$ given by restriction along the γ_n, which we claim is a trivial fibration. By a standard reduction, for this it suffices to show that γ_n is left marked anodyne. Indeed, this follows from Lemma 5.20 applied to $I_n \to (\Delta^n)^\sharp \times \Delta^n$ and the observation that the map $\Delta^n \times \Delta^n \cup_{\Delta^n} I'_n \to J_n$ is inner anodyne, whose proof we leave to the reader.

Define also a map $Z' \to Z^{\mathrm{vop}}$ over T by restriction along the map ${}_\natural\widetilde{\mathbb{O}}(\Delta^n) \to J_n$ which sends ij to jn if $i = 0$ and $j(-i)$ otherwise. Finally, define a map $X^{\mathrm{vop}} \times_T Y^{\mathrm{vop}} \to Z''$ over T as follows. A map $\Delta^n \to X^{\mathrm{vop}} \times_T Y^{\mathrm{vop}}$ is given by the data

$$\begin{tikzcd} {}_\natural\widetilde{\mathbb{O}}(\Delta^n) \arrow[r] \arrow[d] & {}_\natural X \arrow[d] \\ (\Delta^n)^\sharp \arrow[r] & T^\sharp \end{tikzcd} \qquad \begin{tikzcd} (\widetilde{\mathbb{O}}(\Delta^n)^{\mathrm{op}})^\natural \arrow[r] \arrow[d] & Y^\natural \arrow[d] \\ (\Delta^n)^\sharp \arrow[r] & T^\sharp \end{tikzcd}$$

We have isomorphisms ${}_\natural\widetilde{\mathbb{O}}(\Delta^n) \cong I'_n$ and $(\widetilde{\mathbb{O}}(\Delta^n)^{\mathrm{op}})^\natural \cong I_n$, and obvious retractions $I_n \cup_{(\Delta^n)^\sharp} I'_n \to I_n, I'_n$ given by collapsing the complementary part onto Δ^n. Using this, we may define

$$I_n \cup_{(\Delta^n)^\sharp} I'_n \to {}_\natural X \times_T Y^\natural \to {}_\natural Z,$$

which is an n–simplex of Z''.

Choosing a section of r, we may compose these maps to define μ^{vop}, which is then easily checked to also preserve the indicated markings. For example, μ^{vop} on edges is given by

$$\left(\begin{tikzcd} & x_{11} \arrow[d] \\ x_{00} \arrow[r] & x_{01} \\ y_{01} \arrow[r] \arrow[d] & y_{11} \\ y_{00} & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} & \mu(x_{11}, y_{11}) \arrow[d] \\ \mu(x_{00}, y_{01}) \arrow[r] \arrow[d] & \mu(x_{01}, y_{11}) \arrow[d] \\ \mu(x_{00}, y_{00}) \arrow[r] & \alpha_! \mu(x_{00}, y_{00}) \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} & \mu(x_{11}, y_{11}) \arrow[d] \\ \mu(x_{00}, y_{00}) \arrow[r] & \alpha_! \mu(x_{00}, y_{00}) \end{tikzcd}\right)$$

where $\alpha_! \mu(x_{00}, y_{00})$ is a choice of pushforward for the edge α in T that the diagrams are vertically over.

5.22 Lemma *Let $C \to T$ be a cartesian fibration and let $D \to T$ be a cocartesian fibration. There exists a T–equivalence $\psi\colon \widetilde{\mathrm{Fun}}_T(C, D)^{\mathrm{vop}} \to \widetilde{\mathrm{Fun}}_T(C^{\mathrm{vop}}, D^{\mathrm{vop}})$.*

Proof We have a map $\mu\colon \widetilde{\mathrm{Fun}}_T(C, D) \times_T C \to D$ adjoint to the identity. Employing Construction 5.21 on μ and then adjointing, we obtain our desired T–functor ψ. A chase of the definitions then shows that for all objects $t \in T$, ψ_t is homotopic to the known equivalence $\mathrm{Fun}(C_t, D_t)^{\mathrm{op}} \simeq \mathrm{Fun}(C_t^{\mathrm{op}}, D_t^{\mathrm{op}})$. $\square$

5.23 Lemma *Let K and L be S–categories. Then there exists an S–equivalence*

$$\psi\colon (K \star_S L)^{\mathrm{vop}} \xrightarrow{\simeq} L^{\mathrm{vop}} \star_S K^{\mathrm{vop}}$$

over $S \times \Delta^1$.

Proof Note that $(S \times \Delta^1)^{\mathrm{vop}} \cong S \times (\Delta^1)^{\mathrm{op}}$. View $(K \star_S L)^{\mathrm{vop}}$ as lying over $S \times \Delta^1$ via the isomorphism $(\Delta^1)^{\mathrm{op}} \cong \Delta^1$. Since $(K \star_S L)_0^{\mathrm{vop}} \cong L^{\mathrm{vop}}$ and $(K \star_S L)_1^{\mathrm{vop}} \cong K^{\mathrm{vop}}$, our S–functor ψ is adjoint to the identity over $S \times \partial\Delta^1$. Fiberwise, ψ_s is homotopic to the known isomorphism $(K_s \star L_s)^{\mathrm{op}} \cong L_s^{\mathrm{op}} \star K_s^{\mathrm{op}}$, so ψ is an equivalence. $\square$

5.24 Proposition *Suppose K and C are S–categories.*

(1) *The adjoint of the vertical opposite of the evaluation map induces an equivalence*

$$\underline{\mathrm{Fun}}_S(K, C)^{\mathrm{vop}} \xrightarrow{\simeq} \underline{\mathrm{Fun}}_S(K^{\mathrm{vop}}, C^{\mathrm{vop}}).$$

(2) *Suppose $p\colon K \to C$ is an S–functor. We have equivalences*

$$(C^{(p,S)/})^{\mathrm{vop}} \simeq (C^{\mathrm{vop}})^{/(p^{\mathrm{vop}},S)}, \quad (C^{/(p,S)})^{\mathrm{vop}} \simeq (C^{\mathrm{vop}})^{(p^{\mathrm{vop}},S)/}.$$

Proof (1) Recall from (6.3.1) the equivalence $\underline{\mathrm{Fun}}_S(K, C) \simeq \pi_* \pi'^* \{K, C\}_S$. By Lemmas 5.22 and 5.19(1),

$$\{K, C\}_S^{\mathrm{vop}} \simeq \{K^{\mathrm{vop}}, C^{\mathrm{vop}}\}_S.$$

By Lemma 5.19(1) and (2),

$$\pi_* \pi'^* \{K, C\}_S^{\mathrm{vop}} \simeq (\pi_* \pi'^* \{K, C\}_S)^{\mathrm{vop}}.$$

Combining these equivalences supplies an equivalence

$$\underline{\mathrm{Fun}}_S(K, C)^{\mathrm{vop}} \simeq \underline{\mathrm{Fun}}_S(K^{\mathrm{vop}}, C^{\mathrm{vop}}).$$

It is straightforward but tedious to verify that the adjoint of the vertical opposite of the evaluation map $\underline{\mathrm{Fun}}_S(K, C)^{\mathrm{vop}} \times_S K^{\mathrm{vop}} \to C^{\mathrm{vop}}$ is homotopic to this equivalence.

(2) Combine (1), Lemma 5.23, Proposition 5.18 (which shows in particular that $(-)^{\mathrm{vop}}$ is right Quillen), and the definition of the S–slice category. $\square$

5.25 Corollary *Let $\bar{p}\colon S \star_S K \to C$ be an S–functor. Then $\bar{p}$ is an S–limit diagram if and only if $\bar{p}^{\mathrm{vop}}\colon K^{\mathrm{vop}} \star_S S \to C^{\mathrm{vop}}$ is an S–colimit diagram.*

This allows us to deduce statements about S–limits from statements about S–colimits, and vice versa. For this reason, we will primarily concentrate our attention on proving statements concerning S–colimits (and eventually, S–left Kan extensions), leaving the formulation of the dual results to the reader.

5.26 Warning Even with Corollary 5.25, it seems difficult to deduce Proposition 5.6 concerning S–limits in an S–category of objects $\underline{C}_S$ directly from Proposition 5.5 on S–colimits in $\underline{C}_S$. This is because the formation of vertical opposites $\underline{C}_S \mapsto (\underline{C}_S)^{\mathrm{vop}}$ doesn't intertwine with any operation at the level of the ∞–category C.

6 Assembling S–slice categories from ordinary slice categories

Suppose $p\colon K \to C$ is an S–functor. For every morphism $\alpha\colon s \to t$ in S, we have a functor $p_\alpha\colon K_s \to C_t$, and we may consider the collection of "absolute" slice categories $C_{p_\alpha/}$ and examine the functoriality that they satisfy. For this, we have the following basic observation: given a morphism $f\colon t \to t'$, *covariant* functoriality of slice categories in the target yields a functor $C_{p\alpha/} \to C_{pf\alpha/}$, and given a morphism $g\colon s' \to s$, *contravariant* functoriality in the source yields a functor $C_{p\alpha/} \to C_{p\alpha g/}$. Elaborating, we will show in this section that there exists a functor

$$F := F(p\colon K \to C)\colon \widetilde{\mathbb{O}}(S) \to \mathbf{Cat}_\infty$$

out of the twisted arrow category $\widetilde{\mathbb{O}}(S)$ such that $F(\alpha) \simeq C_{p\alpha/}$, which encodes all of this functoriality (Definition 6.5). Moreover, the right Kan extension of F along the target functor $\widetilde{\mathbb{O}}(S) \to S$ is $C_{(p,S)/}$ (Theorem 6.6). We will end with some applications of this result to the theory of cofinality and presentability (Theorem 6.7 and Remark 6.11).

We first record a cofinality result which implies that the values of a right Kan extension along $\mathrm{ev}_1\colon \widetilde{\mathbb{O}}(S) \to S$ are computed as ends.

6.1 Lemma *The functor $\widetilde{\mathbb{O}}(S^{s/}) \to \widetilde{\mathbb{O}}(S) \times_S S^{s/}$ is initial.*

Proof Let $(\alpha\colon u \to t, \beta\colon s \to t)$ be an object of $\widetilde{\mathbb{O}}(S) \times_S S^{s/}$. We will prove that

$$C = \widetilde{\mathbb{O}}(S^{s/}) \times_{\widetilde{\mathbb{O}}(S) \times_S S^{s/}} (\widetilde{\mathbb{O}}(S) \times_S S^{s/})_{/(\alpha,\beta)}$$

is weakly contractible. An object of C is the data of an edge

$$\begin{tikzcd} & s \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"] & & y \end{tikzcd}$$

in $S^{s/}$, which we will abbreviate as $f \xrightarrow{h} g$, and an edge

$$\left(\begin{tikzcd} x \arrow[r, "h"] & y \arrow[d, "\gamma"] \\ u \arrow[u, "\delta"] \arrow[r, "\alpha"] & t \end{tikzcd}, \quad \begin{tikzcd} s \arrow[r, "g"] \arrow[dr, "\beta"'] & y \arrow[d, "\gamma"] \\ & t \end{tikzcd} \right)$$

in $\widetilde{\mathbb{O}}(S) \times_S S^{s/}$, which we will abbreviate as $(h, g) \xrightarrow{(\delta,\gamma)} (\alpha, \beta)$.

Let $C_0 \subset C$ be the full subcategory on objects $c = ((f \xrightarrow{h} g), (h, g) \xrightarrow{(\delta,\gamma)} (\alpha, \beta))$ such that γ is a degenerate edge in $S^{s/}$. We will first show that C_0 is a reflective subcategory of C by verifying the first condition of [9, Proposition 5.2.7.8]. Given an object c of C, define c' to be $((f \xrightarrow{\gamma h} \beta), (\gamma h, \beta) \xrightarrow{(\delta,\mathrm{id}_t)} (\alpha, \beta))$ and let $e\colon c \to c'$ be the edge given by

$$\left(\begin{tikzcd} f \arrow[r, "h"] & g \arrow[d, "\gamma"] \\ f \arrow[u, "\mathrm{id}_f"] \arrow[r, "\gamma h"] & \beta \end{tikzcd}, \quad \begin{tikzcd} (h,g) \arrow[rr, "(\mathrm{id}_x,\gamma)"] \arrow[dr, "(\delta,\gamma)"'] & & (\gamma h, \beta) \arrow[dl, "(\delta,\mathrm{id}_t)"] \\ & (\alpha,\beta) & \end{tikzcd} \right).$$

We need to show that for all $d = ((f' \xrightarrow{h'} \beta), (h', \beta) \xrightarrow{(\delta', \mathrm{id})} (\alpha, \beta)) \in C_0$,

$$\mathrm{Map}_C(c', d) \xrightarrow{e^*} \mathrm{Map}_C(c, d)$$

is a homotopy equivalence. The space $\mathrm{Map}_C(c, d)$ lies in a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_C(c,d) \arrow[r] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S^{s/})}(f \xrightarrow{h} g, f' \xrightarrow{h'} \beta) \arrow[d] \\
\mathrm{Map}_{(\widetilde{\mathbb{O}}(S) \times_S S^{s/})_{/(\alpha,\beta)}}((h,g),(h',\beta)) \arrow[r] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S) \times_S S^{s/}}((h,g),(h',\beta)) \arrow[d, "(\delta',\mathrm{id})_*"] \\
\Delta^0 \arrow[r, "(\delta,\gamma)"] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S) \times_S S^{s/}}((h,g),(\alpha,\beta))
\end{tikzcd}$$

where the two squares are homotopy pullback squares. We also have the analogous diagram for $\mathrm{Map}_C(c', d)$, and the map e^* is induced by a natural transformation of these diagrams. The assertion then reduces to checking that the upper square in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\widetilde{\mathbb{O}}(S^{s/})}(f \xrightarrow{\gamma h} \beta, f' \xrightarrow{h'} \beta) \arrow[r, "(\mathrm{id}_f,\gamma)^*"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S^{s/})}(f \xrightarrow{h} g, f' \xrightarrow{h'} \beta) \arrow[d] \\
\mathrm{Map}_{\widetilde{\mathbb{O}}(S) \times_S S^{s/}}((\gamma h,\beta),(\alpha,\beta)) \arrow[r, "(\mathrm{id}_x,\gamma)^*"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S) \times_S S^{s/}}((h,g),(\alpha,\beta)) \arrow[d] \\
\mathrm{Map}_{S^{s/}}(\beta,\beta) \arrow[r, "\gamma^*"] & \mathrm{Map}_{S^{s/}}(g,\beta)
\end{tikzcd}$$

is a homotopy pullback square. Since (id_x, γ) and (id_f, γ) are ev_1–cocartesian edges in $\widetilde{\mathbb{O}}(S)$ and $\widetilde{\mathbb{O}}(S^{s/})$ respectively, the lower and outer squares are homotopy pullback squares (where we implicitly use that the map (δ', id) covers the identity in $S^{s/}$ to identify the long vertical maps with those induced by ev_1), and the claim is proven.

To complete the proof, we will show that $c = (\beta = \beta, (\mathrm{id}_t, \beta) \xrightarrow{(\alpha, \mathrm{id}_t)} (\alpha, \beta))$ is an initial object in C_0. Let $d \in C_0$ be as above. In the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, "(h',\mathrm{id}_\beta)"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S^{s/})}(\beta = \beta, f' \xrightarrow{h'} \beta) \arrow[d] & \\
\Delta^0 \arrow[r, "(\alpha,\mathrm{id}_t)"] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S) \times_S S^{s/}}((\mathrm{id}_t,\beta),(\alpha,\beta)) \arrow[r] \arrow[d] & \mathrm{Map}_{\widetilde{\mathbb{O}}(S)}(\mathrm{id}_t,\alpha) \arrow[d] \\
\Delta^0 \arrow[r, "\mathrm{id}_\beta"] & \mathrm{Map}_{S^{s/}}(\beta,\beta) \arrow[r] & \mathrm{Map}_S(t,t)
\end{tikzcd}$$

we need to show that the upper square is a homotopy pullback square in order to prove that $\mathrm{Map}_C(c,d) \simeq *$. The fiber of $\widetilde{\mathbb{O}}(S)$ over $t \in S$ is equivalent to $(S_{/t})^{\mathrm{op}}$; in particular, id_t is an initial object in the fiber over t. Therefore, the two outer squares are both homotopy pullbacks. Since the lower right square is a homotopy pullback, this shows that all squares in the diagram are homotopy pullbacks, as desired. $\square$

Let K be an S–category. Let J_n be the poset with objects ij for $0 \le i \le j \le 2n+1$ which has a unique morphism $ij \to kl$ if and only if $k \le i \le j \le l$. Let $I_n \subset J_n$ be the full subcategory on objects ij such that $i \le n$. In view of the isomorphisms

$$J_n \cong \widetilde{\mathbb{O}}(\Delta^{2n+1}) \cong \widetilde{\mathbb{O}}((\Delta^n)^{\mathrm{op}} \star \Delta^n),$$

the I_n and J_n extend to functors

$$I_\bullet \subset J_\bullet \cong \widetilde{\mathbb{O}}((\Delta^\bullet)^{\mathrm{op}} \star \Delta^\bullet) \colon \Delta \to s\mathbf{Set}.$$

Viewing I_n and J_n as marked simplicial sets where $ij \to kl$ is marked just in case $k = i$, we moreover have functors to $s\mathbf{Set}^+$. Define the simplicial set $X \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ to be the functor

$$\mathrm{Hom}_{s\mathbf{Set}^+}(I_\bullet, {}_\natural K) \times_{\mathrm{Hom}(I_\bullet, S)} \mathrm{Hom}((\Delta^\bullet)^{\mathrm{op}} \star \Delta^\bullet, S)$$

where $I_\bullet \subset J_\bullet \to (\Delta^\bullet)^{\mathrm{op}} \star \Delta^\bullet$ is given by the target map. An n–simplex of X is thus the data of a diagram

$$\begin{tikzcd}
 & & & & k_{nn} \arrow[r] \arrow[d] & k_{n(n+1)} \arrow[r] \arrow[d] & \cdots \arrow[r] & k_{n(2n+1)} \arrow[d] \\
 & & \iddots & & \vdots & \vdots & \cdots & \vdots \\
 & k_{11} \arrow[r] \arrow[d] & \cdots \arrow[r] & & k_{1n} \arrow[r] \arrow[d] & k_{1(n+1)} \arrow[r] \arrow[d] & \cdots \arrow[r] & k_{1(2n+1)} \arrow[d] \\
k_{00} \arrow[r] & k_{01} \arrow[r] & \cdots \arrow[r] & & k_{0n} \arrow[r] & k_{0(n+1)} \arrow[r] & \cdots \arrow[r] & k_{0(2n+1)}
\end{tikzcd}$$

where the horizontal edges are cocartesian in K and the vertical edges lie over degeneracies in S.

Declare an edge e in X to be marked if the corresponding map $I_1 \to {}_{\natural}K$ sends all edges to marked edges. We have a commutative square of marked simplicial sets

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{\mathcal{O}}(S)^{\sharp} \arrow[d, "\mathrm{ev}_0"] \\ (K^{\vee})^{\natural} \arrow[r] & (S^{\mathrm{op}})^{\sharp} \end{tikzcd}$$

where $K^{\vee} = (K^{\mathrm{vop}})^{\mathrm{op}} \to S^{\mathrm{op}}$ is the dual cartesian fibration and the map $X \to K^{\vee}$ is defined by restricting $I_n \to K$ to $I'_n \to K$ (where I'_n is the full subcategory of I_n on ij with $j \le n$). Let ψ denote the resulting map from X to the pullback.

6.2 Lemma *$\psi\colon X \to (K^{\vee})^{\natural} \times_{(S^{\mathrm{op}})^{\sharp}} \widetilde{\mathcal{O}}(S)^{\sharp}$ is a trivial fibration of marked simplicial sets.*

Proof Since any lift of a marked edge in $(K^{\vee})^{\natural} \times_{(S^{\mathrm{op}})^{\sharp}} \widetilde{\mathcal{O}}(S)^{\sharp}$ to an edge in X is marked, it suffices to prove that the underlying map of simplicial sets is a trivial fibration.

We first show that $I'_n \subset I_n$ is left marked anodyne. Let $I_{n,k} \subset I_n$ be the full subcategory on objects ij with $i \le k$ and similarly for $I'_{n,k}$. For $0 \le k < n$ we have a pushout decomposition

$$\begin{tikzcd} ((\Delta^{n-k})^{\mathrm{op}})^{\flat} \times (\Delta^{k})^{\sharp} \cup_{((\Delta^{n-k-1})^{\mathrm{op}})^{\flat} \times (\Delta^{k})^{\sharp}} ((\Delta^{n-k-1})^{\mathrm{op}})^{\flat} \times (\Delta^{n+k+1})^{\sharp} \arrow[r] \arrow[d] & I'_{n,n-k} \cup_{I'_{n,n-k-1}} I_{n,n-k-1} \arrow[d] \\ ((\Delta^{n-k})^{\mathrm{op}})^{\flat} \times (\Delta^{n+k+1})^{\sharp} \arrow[r] & I_{n,n-k} \end{tikzcd}$$

and the left-hand map is left marked anodyne by [9, Proposition 3.1.2.3]. It thus suffices to show that

$$I'_{n,0} \cong (\Delta^n)^{\sharp} \to I_{n,0} \cong (\Delta^{2n+1})^{\sharp}$$

is left marked anodyne, and this is clear.

We now explain how to solve the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & K^{\vee} \times_{S^{\mathrm{op}}} \widetilde{\mathcal{O}}(S) \end{tikzcd}$$

To supply the dotted arrow we must provide a lift in the commutative square

$$\begin{tikzcd} \partial I_n \cup_{\partial I_n'} I_n' \arrow[r] \arrow[d, "f"] & {}_\natural K \arrow[d] \\ I_n \arrow[r] \arrow[ur, dotted] & S^\sharp \end{tikzcd}$$

where $\partial I_n = \cup_{[n-1]\subset[n]} I_{n-1}$ as a simplicial subset of I_n and likewise for $\partial I_n'$. Then since $I_n' \to \partial I_n \cup_{\partial I_n'} I_n'$ and $I_n' \to I_n$ are left marked anodyne, f is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, and the lift exists. $\square$

For all $s \in S$, we have trivial cofibrations $i_s\colon K_s \xrightarrow{\simeq} (K^\vee)_s$, and thus commutative squares

$$\begin{tikzcd} K_s \arrow[r, "\mathrm{id}_s"] \arrow[d, hook] & \widetilde{\mathbb{O}}(S) \arrow[d, "\mathrm{ev}_0"] \\ K^\vee \arrow[r] & S^{\mathrm{op}} \end{tikzcd}$$

from which we obtain a cofibration

$$\iota\colon \coprod_{s\in S} K_s \hookrightarrow K^\vee \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S).$$

We have an explicit lift ι' of ι to X, where $K_s \to X$ is given by precomposition by $I_n \to \Delta^n$, $ij \mapsto n-i$.

By Lemma 6.2, there exists a lift σ in the commutative square

$$\begin{tikzcd} \coprod_{s\in S} K_s \arrow[r, "\iota'"] \arrow[d, hook, "\iota"] & X \arrow[d, "\psi"] \\ K^\vee \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S) \arrow[r, equal] \arrow[ur, dotted, "\sigma"] & K^\vee \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S) \end{tikzcd}$$

Let $\chi\colon X \to K$ be the functor induced by $\Delta^n \to I_n$, $i \mapsto (n-i)(n+i)$. Define the *twisted pushforward*

$$\widetilde{P}\colon K^\vee \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S) \to K$$

to be the map over S given by the composite $\chi \circ \sigma$. Then for every object $\alpha\colon s \to t$ in $\widetilde{\mathbb{O}}(S)$, $\widetilde{P}_\alpha \circ i_s\colon K_s \to K_t$ is a choice of pushforward functor over α, which is chosen to be the identity if $\alpha = \mathrm{id}_s$.

6.3 Proposition *For all $A \in s\mathbf{Set}_{/S}$,*

$$\widetilde{P} \times_S \mathrm{id}_A \colon (K^\vee)^\natural \times_{(S^{\mathrm{op}})^\sharp} \widetilde{\mathbb{O}}(S)^\sharp \times_S A^\sharp \to {}_\natural K \times_S A^\sharp$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/A}$.

Proof Let (Z, E) denote the marked simplicial set $(K^\vee)^\natural \times_{(S^{\mathrm{op}})^\sharp} \widetilde{\mathbb{O}}(S)^\sharp$. Viewing Z as $\widetilde{\mathbb{O}}(S) \times_{S^{\mathrm{op}} \times S} (K^\vee \times S)$, we see that $Z \to S$ is a cocartesian fibration with the cocartesian edges a subset of E. Moreover, every edge in E factors as a cocartesian edge followed by an edge in E in the fiber over S. By Proposition A.4, it suffices to verify that for all $s \in S$, $\widetilde{P}_s$ is a cocartesian equivalence in $s\mathbf{Set}^+$. Since id_s is an initial object in $\widetilde{\mathbb{O}}(S) \times_S \{s\}$, the inclusion of the fiber $(K^\vee)^\sim_s \subset (Z_s, E_s)$ is a cocartesian equivalence in $s\mathbf{Set}^+$ by [9, Lemma 3.3.4.1]. We chose $\widetilde{P}$ so as to split the inclusion of K_s in Z, so this completes the proof. $\square$

Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{O}(S)^\sharp \times_S {}_\natural K \arrow[drrrr, bend left=10, "\mathrm{ev}_1"] \arrow[dddr, bend right, "\mathrm{ev}_0"'] & & & & \\
& \mathbb{O}(S)^\sharp \times_{\mathrm{ev}_1, S, \mathrm{ev}_1} ((K^\vee)^\natural \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S)^\sharp) \arrow[ul, "\mathrm{id}_{\mathbb{O}(S)} \times_S \widetilde{P}"'] \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}"] & (K^\vee)^\natural \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S)^\sharp \arrow[r, "\mathrm{id} \times \mathrm{ev}_1"] \arrow[d, "\mathrm{pr}"] & (K^\vee)^\natural \times S^\sharp \arrow[r, "\mathrm{pr}_S"'] \arrow[d, "q^\vee \times \mathrm{id}"'] & S^\sharp \\
& \mathbb{O}(S)^\sharp \times_{\mathrm{ev}_1, S, \mathrm{ev}_1} \widetilde{\mathbb{O}}(S)^\sharp \arrow[r, "\pi'"] \arrow[d, "\pi"] & \widetilde{\mathbb{O}}(S)^\sharp \arrow[r, "\mathrm{ev}"] & (S^{\mathrm{op}})^\sharp \times S^\sharp & \\
& S^\sharp & & &
\end{tikzcd}$$

where $\pi = \mathrm{ev}_0 \circ \mathrm{pr}_{\mathbb{O}(S)}$ and $\pi' = \mathrm{pr}_{\widetilde{\mathbb{O}}(S)}$. Since $K^\vee \to S^{\mathrm{op}}$ is a cartesian fibration, by Theorem 2.24 $(q^\vee \times \mathrm{id})_*$ is right Quillen. Therefore, given an S–category C, we obtain a $\widetilde{\mathbb{O}}(S)$–category

$$\{K, C\}_S := (\mathrm{ev}^* \circ (q^\vee \times \mathrm{id})_* \circ \mathrm{pr}_S^*)({}_\natural C).$$

Moreover, we saw in Example 2.26 that $\pi_* \pi'^*$ is right Quillen and computes right Kan extension along $\mathrm{ev}_1 \colon \widetilde{\mathbb{O}}(S) \to S$. Finally, the map $\mathrm{id}_{\mathbb{O}(S)} \times_S \widetilde{P}$ induces an S–functor

$$\theta \colon \underline{\mathrm{Fun}}_S(K, C) \to \pi_* \pi'^* \{K, C\}_S, \tag{6.3.1}$$

natural in K and C. By Proposition 6.3 applied to $A = S^{s/}$ for all $s \in S$, θ is an equivalence.

6.4 Remark As a corollary, we have that the global sections of $\{K,C\}_S$ are equivalent to $\mathrm{Fun}_S(K,C)$. If we knew that under the straightening functor St, $\{K,C\}_S$ was equivalent to the composite

$$\widetilde{\mathbb{O}}(S) \to S^{\mathrm{op}} \times S \xrightarrow{\mathrm{St}_S(K)^{\mathrm{op}} \times \mathrm{St}_S(C)} \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathbf{Cat}_\infty \xrightarrow{\mathrm{Fun}} \mathbf{Cat}_\infty,$$

then this would yield another proof of the end formula for the ∞–category of natural transformations, as proven in [5, Section 6]. As we manage to always stay within the environment of cocartesian fibrations, this identification is not necessary for our purposes.

6.5 Definition Given an S–functor $p\colon K \to C$ and a choice of twisted pushforward $\widetilde{P}$ for K, define the cocartesian section $\omega_p\colon \widetilde{\mathbb{O}}(S) \to \{K,C\}_S$ to be the adjoint to

$$p \circ \widetilde{P}\colon K^{\vee\natural} \times_{S^{\mathrm{op}}} \widetilde{\mathbb{O}}(S)^\sharp \to {}_\natural K \to {}_\natural C.$$

For objects $[\alpha\colon s \to t]$ in $\widetilde{\mathbb{O}}(S)$, $\omega_p(\alpha) \in \mathrm{Fun}((K^\vee)_s, C_t)$ is the functor

$$p_t \circ \widetilde{P}_\alpha\colon (K^\vee)_s \to K_t \to C_t.$$

Define the *twisted slice $\widetilde{\mathbb{O}}(S)$–category* to be[17]

$$C^{\widetilde{(p,S)}/} := \widetilde{\mathbb{O}}(S) \times_{\{K,C\}_S} \{K \star_S S, C\}_S.$$

Note that the fiber of $C^{\widetilde{(p,S)}/}$ over an object $[\alpha\colon s \to t]$ is $C^{p_t \circ \widetilde{P}_\alpha/}$.

We now connect the constructions $C^{\widetilde{(p,S)}/}$ and $C^{(p,S)/}$. A check of the definitions reveals that $\theta \circ \sigma_p = \pi_* \pi'^*(\omega_p)$ for the canonical cocartesian section

$$\sigma_p\colon S \to \underline{\mathrm{Fun}}_S(K,C).$$

We thus have a morphism of spans

$$\begin{tikzcd}
S \arrow[r, "\sigma_p"'] \arrow[d, equal] & \underline{\mathrm{Fun}}_S(K,C) \arrow[d, "\simeq"] & \underline{\mathrm{Fun}}_S(K \star_S S, C) \arrow[l] \arrow[d, "\simeq"] \\
S \arrow[r, "\pi_*\pi'^*(\omega_p)"'] & \pi_*\pi'^*\{K,C\}_S & \pi_*\pi'^*\{K \star_S S, C\}_S \arrow[l]
\end{tikzcd}$$

with all objects fibrant and the right horizontal maps fibrations by a standard argument. Taking pullbacks, we deduce:

[17] We omit the dependence on $\widetilde{P}$ from the notation.

6.6 Theorem *We have an equivalence*

$$\pi_* \pi'^* (C^{\widetilde{(p,S)}/}) \xrightarrow{\simeq} C^{(p,S)/}.$$

In other words, the right Kan extension of $C^{\widetilde{(p,S)}/}$ along the target functor $\mathrm{ev}_1 : \widetilde{\mathbb{O}}(S) \to S$ is equivalent to $C^{(p,S)/}$.

Proof Our interpretation of this equivalence is by Example 2.26. □

Relative cofinality

Let us now apply Theorem 6.6. We have the S–analogue of the basic cofinality result [9, Proposition 4.1.1.8].

6.7 Theorem *Let $f : K \to L$ be an S–functor. The following conditions are equivalent:*

(1) *For every object $s \in S$, $f_s : K_s \to L_s$ is final.*
(2) *For every S–functor $p : L \to C$, the functor $f^* : C^{(p,S)/} \to C^{(pf,S)/}$ is an equivalence.*
(3) *For every S–colimit diagram $\bar{p} : L \star_S S \to C$, $\bar{p} \circ f^{\rhd} : K \star_S S \to C$ is an S–colimit diagram.*

Proof (1) $\Longrightarrow$ (2) Factoring f as the composition of a cofibration and a trivial fibration, we may suppose that f is a cofibration, in which case we may choose compatible twisted pushforward functors $\widetilde{P}_K$ and $\widetilde{P}_L$. Let $p : L \to C$ be an S–functor. Precomposition by f yields a $\widetilde{\mathbb{O}}(S)$–functor $\widetilde{f}^* : C^{\widetilde{(p,S)}/} \to C^{\widetilde{(pf,S)}/}$. Passing to the fiber over an object $\alpha : s \to t$, the compatibility of $\widetilde{P}_K$ and $\widetilde{P}_L$ implies that the diagram

$$\begin{tikzcd}
(K^{\vee})_s \arrow[r, "(\widetilde{P}_K)_\alpha"] \arrow[d, "(f^{\vee})_s"'] & K_t \arrow[d, "f_t"'] \arrow[dr, "(pf)_t"] & \\
(L^{\vee})_s \arrow[r, "(\widetilde{P}_L)_\alpha"] & L_t \arrow[r, "p_t"] & C_t
\end{tikzcd}$$

commutes and that

$$(\widetilde{f}^*)_\alpha = (f^{\vee})_s^* : C^{p_t \circ (\widetilde{P}_L)_\alpha /} \to C^{(pf)_t \circ (\widetilde{P}_K)_\alpha /}.$$

By [9, Corollary 4.1.1.10], $(f^{\vee})_s$ is final, so by [9, Proposition 4.1.1.8], $(f^{\vee})_s^*$ is an equivalence. Consequently, $\widetilde{f}^*$ is an equivalence. Now by Theorem 6.6, f^* is an equivalence.

(2) $\Longrightarrow$ (3) Immediate from the definition.

(3) $\Longrightarrow$ (1) Let $s \in S$ be any object and $\bar{p}_s \colon L_s^{\rhd} \to \mathbf{Spc}$ a colimit diagram. Let $\bar{p} \colon (L \star_S S)_{\underline{s}} \to \mathbf{Spc}$ be a left Kan extension of $\bar{p}_s$ along the full and faithful inclusion $L_s^{\rhd} \subset (L \star_S S)_{\underline{s}}$. By transitivity of left Kan extensions, $\bar{p}$ is a left Kan extension of its restriction to $L_{\underline{s}}$. By Proposition 5.5, under the equivalence

$$\mathrm{Fun}(L, \mathbf{Spc}) \simeq \mathrm{Fun}_S(L, \underline{\mathbf{Spc}}_S),$$

$\bar{p}$ is an $S^{s/}$–colimit diagram. By assumption, $\bar{p} \circ (f^{\rhd})_{\underline{s}}$ is an $S^{s/}$–colimit diagram. By Proposition 5.5 again, $\bar{p}_s \circ f_s$ is a colimit diagram, as desired. $\square$

6.8 Definition Let $f : K \to L$ be an S–functor. We say that f is *S–final* if it satisfies the equivalent conditions of Theorem 6.7. We say that f is *S–initial* if f^{vop} is S–final.

6.9 Example Let $F : C \rightleftarrows D : G$ be an S–adjunction (Definition 8.3). Then F is S–initial and G is S–final.

6.10 Remark Let C and D be S–categories and $F : C \to D$ be an S–functor.

(1) Suppose F is fiberwise a weak homotopy equivalence. Then F is a weak homotopy equivalence by [9, Proposition 4.1.2.15], [9, Proposition 4.1.2.18], and [9, Proposition 3.1.5.7].

(2) Suppose F is S–final. Then F is final. Indeed, for any diagram $p \colon D \to \mathbf{Spc}$, we have that

$$\operatorname*{colim}_{d \in D} p(d) \simeq \operatorname*{colim}_{s \in S} \operatorname*{colim}_{d \in D_s} p(d) \simeq \operatorname*{colim}_{s \in S} \operatorname*{colim}_{c \in C_s} pF(c) \simeq \operatorname*{colim}_{c \in C} pF(c).$$

(3) Suppose F is S–initial. Then F is initial. To show this, by (the dual of) [9, Theorem 4.1.3.1] it suffices to show that for every $d \in D$, $C \times_D D^{/d}$ is weakly contractible. Let s be the image of d in S. By Lemma 10.9, the inclusion $C_s \times_{D_s} (D_s)^{/d} \to C \times_D D^{/d}$ is final, so in particular is a weak homotopy equivalence. Hence the desired conclusion follows by our assumption that F is S–initial and [9, Theorem 4.1.3.1] again.

We conclude by using the twisted slice $\widetilde{\mathbb{O}}(S)$–category to give a criterion for the presentability of the S–slice.

6.11 Remark (presentability of the parametrized slice) Suppose that the functor $S \to \mathbf{Cat}_\infty$ classifying the cocartesian fibration $C \to S$ factors through $\mathbf{Pr}^R$, ie $C \to S$ is a *right presentable fibration*. For any X a presentable ∞–category and diagram $f\colon A \to X$, $X^{f/}$ is again presentable and the forgetful functor $X^{f/} \to X$ creates limits and filtered colimits. Therefore, the twisted slice $\widetilde{\mathbb{O}}(S)$–category $C^{\widetilde{(p,S)}/}$ is a right presentable fibration. Since the forgetful functor $\mathbf{Pr}^R \to \mathbf{Cat}_\infty$ creates limits, by Theorem 6.6 we deduce that $C^{(p,S)/}$ is a right presentable fibration. In particular, in every fiber there exists an initial object. However, these initial objects may fail to be preserved by the pushforward functors. In fact, even if we assume that $C \to S$ is both left and right presentable, C may fail to be S–cocomplete.

7 Types of S–fibrations

In this section we introduce some additional classes of fibrations which are all defined relative to S.

7.1 Definition Let $\phi\colon C \to D$ be an S–functor. We say that ϕ is an *S–fibration* if it is a categorical fibration. We then say that ϕ is an *S–cocartesian fibration* if it is an S–fibration such that for every object $s \in S$, $\phi_s\colon C_s \to D_s$ is a cocartesian fibration, and for every square in C

$$\begin{array}{ccc} x_s & \xrightarrow{h} & x_t \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ y_s & \xrightarrow{k} & y_t \end{array}$$

with h and k ϕ–cocartesian edges over $\phi(h) = \phi(k)\colon s \to t$, if f is a ϕ_s–cocartesian edge then g is a ϕ_t–cocartesian edge.

Dually, we say that ϕ is an *S–cartesian fibration* if it is an S–fibration such that for every object $s \in S$, $\phi_s\colon C_s \to D_s$ is a cartesian fibration, and for every square in C labeled as above, but now with h and k ϕ–cartesian edges over $\phi(h) = \phi(k)\colon s \to t$, if f is a ϕ_s–cartesian edge then g is a ϕ_t–cartesian edge.

Equivalently, $\phi\colon C \to D$ is S–(co)cartesian if it is a categorical fibration, fiberwise a (co)cartesian fibration, and for every edge in S, the cocartesian pushforward along that edge preserves (co)cartesian edges in the fibers. We formulate our definition as above so as to avoid having to make any "straightening" constructions such as choosing pushforward functors.

7.2 Remark Declare a morphism of S–cocartesian fibrations

$$[C \xrightarrow{\phi} D] \to [C' \xrightarrow{\phi'} D']$$

to be a commutative square of S–functors

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\phi"] & C' \arrow[d, "\phi'"] \\ D \arrow[r, "G"] & D' \end{tikzcd}$$

in which for all $s \in S$, F_s sends ϕ_s–cocartesian edges to ϕ'_s cocartesian edges. Let $\mathbb{O}^{\text{cocart.fib}}(\mathbf{Cat}^{\text{cocart}}_{\infty/S})$ be the ∞–category of S–cocartesian fibrations and morphisms thereof. Then one has the straightening equivalence

$$\mathbb{O}^{\text{cocart.fib}}(\mathbf{Cat}^{\text{cocart}}_{\infty/S}) \simeq \mathrm{Fun}(S, \mathbb{O}^{\text{cocart.fib}}(\mathbf{Cat}_\infty)).$$

7.3 Remark $\phi\colon C \to D$ is an S–fibration if and only if $\phi\colon {}_\natural C \to {}_\natural D$ is a marked fibration.

7.4 Remark In view of [9, Proposition 2.4.2.11, Lemma 2.4.2.7 and Proposition 2.4.2.8], $\phi\colon C \to D$ is an S–cocartesian fibration if and only if ϕ is a cocartesian fibration. However, there is no corresponding simplification of the definition of an S–cartesian fibration.

7.5 Lemma *Let $\phi\colon C \to D$ be an S–cartesian fibration and let $f\colon x \to y$ be a ϕ_s–cartesian edge in C_s. Then f is a ϕ–cartesian edge.*

Proof The property of being ϕ–cartesian may be checked after base-change to the 2–simplices of D. Consequently, we may suppose that $S = \Delta^1$ and $s = \{1\}$. We have to verify that for every object $w \in C$ we have a homotopy pullback square

$$\begin{tikzcd} \mathrm{Map}_C(w, x) \arrow[r, "f_*"] \arrow[d, "\phi_*"] & \mathrm{Map}_C(w, y) \arrow[d, "\phi_*"] \\ \mathrm{Map}_D(\phi w, \phi x) \arrow[r, "\phi(f)_*"] & \mathrm{Map}_D(\phi w, \phi y) \end{tikzcd}$$

If $w \in C_0$, for any choice of cocartesian edge $w \to w'$ over $0 \to 1$, the square is equivalent to

$$\begin{tikzcd} \mathrm{Map}_{C_1}(w', x) \arrow[r, "f_*"] \arrow[d, "\phi_*"] & \mathrm{Map}_{C_1}(w', y) \arrow[d, "\phi_*"] \\ \mathrm{Map}_{D_1}(\phi w', \phi x) \arrow[r, "\phi(f)_*"] & \mathrm{Map}_{D_1}(\phi w', \phi y) \end{tikzcd}$$

Hence we may suppose that $w \in C_1$, in which case the square is a homotopy pullback square since f is a ϕ_1–cartesian edge. □

We next discuss an important example of S–(co)cartesian fibrations. Recall the fiberwise arrow S–category $\mathcal{O}_S(D)$ (Notation 4.29). Fix $\phi\colon C \to D$ an S–functor.

7.6 Definition The *free S–cocartesian* and *free S–cartesian* fibrations on ϕ are the S–functors

$$\mathrm{Fr}^{\mathrm{cocart}}(\phi) := \mathrm{ev}_1 \circ \mathrm{pr}_2\colon C \times_D \mathcal{O}_S(D) \to D,$$
$$\mathrm{Fr}^{\mathrm{cart}}(\phi) := \mathrm{ev}_0 \circ \mathrm{pr}_1\colon \mathcal{O}_S(D) \times_D C \to D.$$

7.7 Proposition *$\mathrm{Fr}^{\mathrm{cocart}}(\phi)$ is an S–cocartesian fibration. Dually, $\mathrm{Fr}^{\mathrm{cart}}(\phi)$ is an S–cartesian fibration.*

Proof We prove the second assertion, the proof of the first being similar but easier. First note that $\mathcal{O}_S(D) \times_D C$ is a subcategory of $\mathcal{O}(D) \times_D C$ stable under equivalences. Therefore, since $\mathrm{ev}_0\colon \mathcal{O}(D) \times_D C \to D$ is a cartesian fibration, $\mathrm{Fr}^{\mathrm{cart}}(\phi)$ is a categorical fibration. Moreover, for every object $s \in S$, $\mathrm{Fr}^{\mathrm{cart}}(\phi)_s\colon \mathcal{O}(D_s) \times_{D_s} C_s$ is the free cartesian fibration on $\phi_s\colon C_s \to D_s$. It remains to show that for every square

$$\begin{tikzcd} (a \to \phi x, x) \arrow[r, "h"] \arrow[d, "f"] & (b \to \phi y, y) \arrow[d, "g"] \\ (a' \to \phi x', x') \arrow[r, "k"] & (b' \to \phi y', y') \end{tikzcd}$$

in $\mathcal{O}_S(D) \times_D C$ with the horizontal edges cocartesian over S and the left vertical edge $\mathrm{Fr}^{\mathrm{cart}}(\phi)_s$–cartesian, the right vertical edge is $\mathrm{Fr}^{\mathrm{cart}}(\phi)_t$–cartesian. This amounts to verifying that $y \to y'$ is an equivalence in C_t. The above square yields a square

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"] & y \arrow[d, "g"] \\ x' \arrow[r, "k"] & y' \end{tikzcd}$$

in C with $x \to x'$ an equivalence and the horizontal edges cocartesian over S, from which the claim follows. □

We conclude this section with an observation about the interaction between S–joins and S–cocartesian fibrations which will be used in the sequel.

7.8 Lemma *Let C, C' and D be S–categories and let $\phi, \phi' \colon C, C' \to D$ be S–functors. If ϕ and ϕ' are S–(co)cartesian, then $\phi \star \phi' \colon C \star_D C' \to D$ is S–(co)cartesian.*

Proof This is an easy corollary of Proposition 4.7. □

7.9 Definition We say that an S–functor $F \colon C \to D \times_S E$ is an S*–bifibration* if for all objects $s \in S$, F_s is a bifibration. Observe it is then automatic that $\mathrm{pr}_D\, F$ is S–cartesian and $\mathrm{pr}_E\, F \colon C \to E$ is S–cocartesian.

7.10 Example The S–functor

$$\underline{\mathrm{Fun}}_S(K \star_S L, C) \to \underline{\mathrm{Fun}}_S(K, C) \times_S \underline{\mathrm{Fun}}_S(L, C)$$

is an S–bifibration by Lemma 4.8. In particular, for an S–functor $p \colon K \to C$, the S–functors $C^{(p,S)/} \to C$ and $C^{/(p,S)} \to C$ are S–cocartesian and S–cartesian, respectively.

8 Relative adjunctions

In [11, Section 7.3.2], Lurie introduces the notion of a *relative adjunction*.

8.1 Definition [11, Definition 7.3.2.2] Suppose we are given categorical fibrations $q \colon C \to S$ and $p \colon D \to S$, and functors $F \colon C \to D$ and $G \colon D \to C$ over S. Suppose there exists a natural transformation $u \colon \mathrm{id}_C \to GF$ such that

(1) u exhibits F as a left adjoint to G, and

(2) $q(u)$ is the identity transformation from q to itself.

Then we say that the adjunction $F \dashv G$ is a *relative adjunction* with respect to S.

8.2 Recollection By [11, Proposition 7.3.2.5], relative adjunctions are stable under base-change; in particular, they restrict to adjunctions over every fiber.

8.3 Definition Let C and D be S–categories. We call a relative adjunction (with respect to S)

$$F : C \rightleftarrows D : G$$

an S*–adjunction* if F and G are S–functors.

We prove some basic results about S–adjunctions in this section. Let us first reformulate the definition of a relative adjunction in terms of a correspondence. Let $F: C \to D$ be an S–functor. By the relative nerve construction, F defines a cocartesian fibration $M \to \Delta^1$ by prescribing, for every $\Delta^n \cong \Delta^{n_0} \star \Delta^{n_1} \to \Delta^1$, the set $\mathrm{Hom}_{\Delta^1}(\Delta^n, M)$ to be the collection of commutative squares

$$\begin{tikzcd} \Delta^{n_0} \arrow[r] \arrow[d] & C \arrow[d, "F"] \\ \Delta^n \arrow[r] & D \end{tikzcd}$$

for $n_1 \geq 0$, and setting $\mathrm{Hom}_{\Delta^1}(\Delta^n, M) = \mathrm{Hom}(\Delta^n, C)$ for $n_1 = -1$. Moreover, the structure maps for C and D to S define a functor $M \to S$ by sending $\Delta^n \to M$ to $\Delta^n \to D \to S$ if $n_1 \geq 0$, and $\Delta^n \to C \to S$ if $n_1 < 0$. Then M is an S–category, $M \to S \times \Delta^1$ is an S–cocartesian fibration, and F admits a right S–adjoint if and only if $M \to S \times \Delta^1$ is an S–cartesian fibration.

8.4 Proposition *Let $F: C \rightleftarrows D : G$ be an S–adjunction and let I be an S–category. Then we have adjunctions*

$$F_*: \mathrm{Fun}_S(I, C) \rightleftarrows \mathrm{Fun}_S(I, D) : G_*, \quad G^*: \mathrm{Fun}_S(C, I) \rightleftarrows \mathrm{Fun}_S(D, I) : F^*.$$

Proof Let $M \to S \times \Delta^1$ be the S–functor obtained from F. We first produce the adjunction $F_* \dashv G_*$. Invoking Theorem 2.24 on the span

$$(\Delta^1) \xleftarrow{\pi} {}_\natural I \times (\Delta^1)^\sharp \xrightarrow{\pi'} S^\sharp \times (\Delta^1)^\sharp$$

we find that $\pi_* \pi'^*: s\mathbf{Set}^+_{/(S^\sharp \times (\Delta^1)^\sharp)} \to s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let $N = \pi_* \pi'^*(M)$. Then $N \to \Delta^1$ is a cocartesian fibration classified by the functor

$$F_*: \mathrm{Fun}_S(I, C) \to \mathrm{Fun}_S(I, D).$$

Now invoking Theorem 2.24 on the span

$$((\Delta^1)^\sharp)^{\mathrm{op}} \xleftarrow{\rho} (I^\sim \times (\Delta^1)^\sharp)^{\mathrm{op}} \xrightarrow{\rho'} (S^\sim \times (\Delta^1)^\sharp)^{\mathrm{op}}$$

we deduce that $\rho_* \rho'^*: s\mathbf{Set}^+_{/(S^\sim \times (\Delta^1)^\sharp)} \to s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$, with respect to the cartesian model structures, is right Quillen. Let $N' = \rho_* \rho'^* M$. Since G is right S–adjoint to F, $N' \to \Delta^1$ is a cartesian fibration classified by the functor

$$G_*: \mathrm{Fun}_{/S}(I, D) \to \mathrm{Fun}_{/S}(I, C)$$

where we view I, C and D as categorical fibrations over S. N is a subcategory of N', and the cartesian edges e in N' with $d_0(e) \in N$ are in N. Hence $N \to \Delta^1$ is also a cartesian fibration classified by the functor

$$G_*\colon \mathrm{Fun}_S(I,D) \to \mathrm{Fun}_S(I,C).$$

We now produce the adjunction $G^* \dashv F^*$ by similar methods. Let $\mathscr{E}_0$ be the collection of edges $e\colon x \to y$ in M such that e admits a factorization as a cocartesian edge over S followed by a cartesian edge in the fiber. Note that since $M \to S \times \Delta^1$ is an S–cartesian fibration, $\mathscr{E}_0$ is closed under composition of edges. Invoking Theorem 2.24 on the span

$$(\Delta^1)^\sharp \xleftarrow{\mu} (M,\mathscr{E}_0) \xrightarrow{\mu'} S^\sharp \times (\Delta^1)^\sharp$$

we deduce that $\mu_*\mu'^*\colon s\mathbf{Set}^+_{/(S^\sharp\times(\Delta^1)^\sharp)} \to s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$ is right Quillen. Let

$$P = \mu_*\mu'^*({}_\natural I \times (\Delta^1)^\sharp).$$

Then $P \to \Delta^1$ is a cocartesian fibration classified by the functor

$$G^*\colon \mathrm{Fun}_S(C,I) \to \mathrm{Fun}_S(D,I).$$

Let $\mathscr{E}_1$ be the collection of edges $e\colon x \to y$ in M such that e is a cocartesian edge over an equivalence in S. Now invoking Theorem 2.24 on the span

$$((\Delta^1)^\sharp)^{\mathrm{op}} \xleftarrow{\nu} (M,\mathscr{E}_1)^{\mathrm{op}} \xrightarrow{\nu'} (S^\sim \times (\Delta^1)^\sharp)^{\mathrm{op}}$$

we deduce that $\nu_*\nu'^*\colon s\mathbf{Set}^+_{/(S^\sim\times(\Delta^1)^\sharp)} \to s\mathbf{Set}^+_{/(\Delta^1)^\sharp}$, with respect to the cartesian model structures, is right Quillen. Let $P' = \nu_*\nu'^*(I^\sim \times (\Delta^1)^\sharp)$. $P' \to \Delta^1$ is a cartesian fibration with P as a subcategory. One may check that $P \to \Delta^1$ inherits the property of being a cartesian fibration, which is classified by the functor

$$F^*\colon \mathrm{Fun}_S(D,I) \to \mathrm{Fun}_S(C,I). \qquad \square$$

8.5 Corollary *Let $F\colon C \rightleftarrows D : G$ be an S–adjunction and let I be an S–category. Then we have S–adjunctions*

$$F_*\colon \underline{\mathrm{Fun}}_S(I,C) \rightleftarrows \underline{\mathrm{Fun}}_S(I,D) : G_*, \quad G^*\colon \underline{\mathrm{Fun}}_S(C,I) \rightleftarrows \underline{\mathrm{Fun}}_S(D,I) : F^*.$$

Proof By Proposition 8.4, for every $s \in S$,

$$F_*\colon \mathrm{Fun}_{S^{s/}}(I \times_S S^{s/}, C \times_S S^{s/}) \rightleftarrows \mathrm{Fun}_{S^{s/}}(I \times_S S^{s/}, D \times_S S^{s/}) : G_*$$

is an adjunction, and similarly for the contravariant case. $\square$

To state the next corollary, it is convenient to introduce a definition.

8.6 Definition Suppose $\pi\colon C \to D$ is an S–fibration. Define the ∞–category $\mathbf{Sect}_{D/S}(\pi)$ of S–sections of π to be the pullback

$$\begin{tikzcd} \mathbf{Sect}_{D/S}(\pi) \arrow[r] \arrow[d] & \mathrm{Fun}_S(D, C) \arrow[d, "\pi_*"] \\ \Delta^0 \arrow[r, "\mathrm{id}_D"] & \mathrm{Fun}_S(D, D) \end{tikzcd}$$

Define the S–category $\underline{\mathbf{Sect}}_{D/S}(\pi)$ to be the pullback

$$\begin{tikzcd} \underline{\mathbf{Sect}}_{D/S}(\pi) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_S(D, C) \arrow[d, "\pi_*"] \\ S \arrow[r, "\sigma_{\mathrm{id}_D}"] & \underline{\mathrm{Fun}}_S(D, D) \end{tikzcd}$$

We will often denote $\mathbf{Sect}_{D/S}(\pi)$ by $\mathbf{Sect}_{D/S}(C)$, the S–functor π being left implicit.

Note that for any object $s \in S$, the fiber $\underline{\mathbf{Sect}}_{D/S}(\pi)_s$ is isomorphic to $\mathbf{Sect}_{D_{\underline{s}}/\underline{s}}(\pi_{\underline{s}})$.

8.7 Corollary *Let $p\colon C \to E$ and $q\colon D \to E$ be S–fibrations. Let $F\colon C \rightleftarrows D : G$ be an adjunction relative to E where F and G are S–functors. Then for any S–category I,*

$$F_*\colon \mathrm{Fun}_S(I, C) \rightleftarrows \mathrm{Fun}_S(I, D) : G_*$$

is an adjunction relative to $\mathrm{Fun}_S(I, E)$. In particular, taking $I = E$ and the fiber over the identity, we deduce that

$$F_*\colon \mathbf{Sect}_{E/S}(p) \rightleftarrows \mathbf{Sect}_{E/S}(q) : G_*$$

is an adjunction, and also that

$$F_*\colon \underline{\mathbf{Sect}}_{E/S}(p) \rightleftarrows \underline{\mathbf{Sect}}_{E/S}(q) : G_*$$

is an S–adjunction.

Proof The proof of Proposition 8.4 shows that the unit for the adjunction $F_* \dashv G_*$ is sent by p_* to a natural transformation through equivalences. $\square$

8.8 Lemma *Let $F : C \rightleftarrows D : G$ be an S–adjunction. For every S–functor $p : K \to D$, we have a homotopy pullback square in $s\mathbf{Set}^+_{/S}$*

$$\begin{tikzcd} C^{/(Gp,S)} \arrow[r] \arrow[d, "\mathrm{ev}_0^C"] & D^{/(p,S)} \arrow[d, "\mathrm{ev}_0^D"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

where the upper horizontal map is defined to be the composite

$$C^{/(Gp,S)} \xrightarrow{F} C^{/(FGp,S)} \xrightarrow{\epsilon(p)_!} D^{/(p,S)}.$$

Dually, for every S–functor $p : K \to D$, we have a homotopy pullback square in $s\mathbf{Set}^+_{/S}$

$$\begin{tikzcd} D^{(Fp,S)/} \arrow[r] \arrow[d, "\mathrm{ev}_1^D"] & C^{(p,S)/} \arrow[d, "\mathrm{ev}_1^C"] \\ D \arrow[r, "G"] & C. \end{tikzcd}$$

where the upper horizontal map is defined to be the composite

$$D^{(Fp,S)/} \xrightarrow{G} C^{(GFp,S)/} \xrightarrow{\eta(p)^*} C^{(p,S)/}.$$

Proof We prove the first assertion; the second then follows by taking vertical opposites. We first explain how to define the map $\epsilon(p)_!$. Choose a counit transformation

$$\epsilon : D \times \Delta^1 \to D$$

for $F \dashv G$ such that $\pi_D \circ \epsilon$ is the identity natural transformation from π_D to itself. Then $\epsilon \circ (p \times \mathrm{id})$ is adjoint to an S–functor $\epsilon(p) : S \times \Delta^1 \to \underline{\mathrm{Fun}}_S(K, D)$ with $\epsilon(p)_0 = \sigma_{FGp}$ and $\epsilon(p)_0 = \sigma_p$. Because $\underline{\mathrm{Fun}}_S(S \star_S K, D) \to D \times_S \underline{\mathrm{Fun}}_S(K, D)$ is an S–bifibration, from $\epsilon(p)$ we obtain a pushforward S–functor $\epsilon(p)_! : D^{/(FGp,S)} \to D^{/(p,S)}$ compatible with the source maps to D.

We need to check that for every object $s \in S$, passage to the fiber over s yields a homotopy pullback square of ∞–categories. Because $(D^{/(p,S)})_s \cong (D_{\underline{s}}^{/(p_{\underline{s}},\underline{s})})_s$, we may replace S by $S^{s/}$ and thereby suppose that s is an initial object in S.

Let $r : \{s\} \star S \to S$ be a left Kan extension of the identity $S \to S$. By the formula for a left Kan extension, $r(s)$ is an initial object in S, which without loss of generality we may suppose to be s. Using $r \circ (\mathrm{id} \star \pi_K)$ as the structure map for $\{s\} \star K$ over S, define $\phi' : \{s\} \star {}_\natural K \to \{s\} \star_S {}_\natural K$ as adjoint to the identity over $S \times \partial\Delta^1$. It is easy to show that ϕ' is a trivial cofibration in $s\mathbf{Set}^+_{/S}$. Moreover, since the inclusion $\{s\} \to S^\sharp$

is a trivial cofibration, $\{s\} \star_S {}_\natural K \to S^\sharp \star_S {}_\natural K$ is a trivial cofibration in $s\mathbf{Set}^+_{/S}$ by Theorem 4.16. Let ϕ be the composition of these two maps. Then because $\mathrm{Fun}_S(-,-)$ is a right Quillen bifunctor, $\phi^* \colon \mathrm{Fun}_S(S^\sharp \star_S {}_\natural K, {}_\natural D) \to \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D)$ is a trivial Kan fibration.

We further claim that the inclusion

$$j \colon \mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D) \to D_s \times_D \mathrm{Fun}(\{s\} \star K, D) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}_S({}_\natural K, {}_\natural D)$$

is an equivalence. Indeed, we have the pullback square

$$\begin{tikzcd}
\mathrm{Fun}_S(\{s\} \star {}_\natural K, {}_\natural D) \arrow[r] \arrow[d] & D_s \times_D \mathrm{Fun}(\{s\} \star K, D) \times_{\mathrm{Fun}(K,D)} \mathrm{Fun}_S({}_\natural K, {}_\natural D) \arrow[d] \\
\Delta^0 \arrow[r, "r \circ (\mathrm{id} \star \pi_K)"] & \{s\} \times_S \mathrm{Fun}(\{s\} \star K, S) \times_{\mathrm{Fun}(K,S)} \{\pi_K\}
\end{tikzcd}$$

and the term in the lower right is contractible since it is equivalent to the full subcategory $\mathrm{Fun}'(\{s\} \star K, S) \subset \mathrm{Fun}(\{s\} \star K, S)$ of functors which are left Kan extensions of π_K.

Now taking the pullback of the composition $j \circ \phi^*$ over $\{p\}$, we obtain an equivalence

$$(D^{/(p,S)})_s \to D_s \times_D D^{/p}.$$

Similarly, we have an equivalence

$$(C^{/(Gp,S)})_s \to C_s \times_C C^{/Gp}.$$

Since $F \dashv G$ is in particular an adjunction, by [9, Lemma 5.2.5.5] $C^{/Gp} \to C \times_D D^{/p}$ is an equivalence. Taking the fiber over s, we deduce the claim. □

8.9 Corollary *Let $F \colon C \rightleftarrows D \colon G$ be an S–adjunction. Then F preserves S–colimits and G preserves S–limits.*

Proof Let $\bar{p} \colon K \star_S S \to C$ be an S–colimit diagram. To show that $F\bar{p}$ is an S–colimit diagram, it suffices to prove that the restriction map $D^{(F\bar{p},S)/} \to D^{(Fp,S)/}$ is an equivalence. We have the commutative square

$$\begin{tikzcd}
D^{(F\bar{p},S)/} \arrow[r] \arrow[d] & C^{(\bar{p},S)/} \times_C D \arrow[d] \\
D^{(Fp,S)/} \arrow[r] & C^{(p,S)/} \times_C D
\end{tikzcd}$$

(here we suppress some details about the naturality of $\epsilon(-)_!$). The right-hand vertical map is an equivalence by assumption, and the horizontal maps are equivalences by Lemma 8.8. Thus the left-hand vertical map is an equivalence. □

Free S–(co)cartesian fibrations revisited

With the theory of S–adjunctions, we can now establish a key property of the free S–(co)cartesian fibration (Definition 7.6). Let $\phi\colon C \to D$ be an S–functor and define S–functors

$$\iota_0\colon C \to C \times_D \mathbb{O}_S(D), \quad \iota_1\colon C \to \mathbb{O}_S(D) \times_D C$$

via the commutative square

$$\begin{tikzcd} C \arrow[r] \arrow[d, equal] & \mathbb{O}_S(D) \arrow[d, "\mathrm{ev}_i"] \\ C \arrow[r, "\phi"] & D \end{tikzcd}$$

where the upper horizontal map is the composite $C \xrightarrow{\iota} \mathbb{O}_S(C) \to \mathbb{O}_S(D)$.

8.10 Proposition *ι_0 is left S–adjoint to pr_C. Dually, ι_1 is right S–adjoint to pr_C.*

Proof We prove the first assertion, the proof of the second being similar. To prove that we have a relative S–adjunction $\iota_0 \dashv \mathrm{pr}_C$, we must prove that for each $s \in S$ we have an adjunction $(\iota_0)_s \dashv (\mathrm{pr}_C)_s$. So suppose that $S = \Delta^0$. Since $\mathrm{pr}_C \circ \iota_0 = \mathrm{id}$, it suffices by [9, Proposition 5.2.2.8] to check that the identity is a unit transformation; that is, for every $x \in C$ and $(y, \phi y \to a) \in C \times_D \mathbb{O}(D)$,

$$\mathrm{pr}_C\colon \mathrm{Map}_{C \times_D \mathbb{O}(D)}((x, \mathrm{id}_{\phi x}), (y, \phi y \to a)) \to \mathrm{Map}_C(x, y)$$

is an equivalence. Under the fiber product decomposition

$$\begin{aligned} &\mathrm{Map}_{C \times_D \mathbb{O}(D)}((x, \mathrm{id}_{\phi x}), (y, \phi y \to a)) \\ &\qquad\simeq \mathrm{Map}_C(x, y) \times_{\mathrm{Map}_D(\phi x, \phi y)} \mathrm{Map}_{\mathbb{O}(D)}((\mathrm{id}_{\phi x}), (\phi y \to b)) \end{aligned}$$

the map pr_C is projection onto the first factor. The adjunction $\iota\colon D \rightleftarrows \mathbb{O}(D) : \mathrm{ev}_0$ obtained by exponentiating the adjunction $i_0\colon \{0\} \rightleftarrows \Delta^1 : p$ implies that

$$\mathrm{Map}_{\mathbb{O}(D)}((\mathrm{id}_{\phi x}), (\phi y \to b)) \to \mathrm{Map}_D(\phi x, \phi y)$$

is an equivalence, so the claim follows. $\square$

8.11 Remark (universal property of the free S–cocartesian fibration) Let $\phi\colon C \to D$ be an S–functor and $\psi\colon E \to D$ be an S–cocartesian fibration. Then we would like to show that the restriction functor

$$\mathrm{Fun}^{\mathrm{cocart}}_{/D}(C \times_D \mathbb{O}_S(D), E) \to \mathrm{Fun}_{/D,S}(C, E) = S \times_{\sigma_\phi, \mathrm{Fun}_S(C,D), \psi_*} \mathrm{Fun}_S(C, E)$$

is an equivalence of ∞–categories.[18] We prove this in [17, Example 3.8] as an application of the theory of parametrized factorization systems.

9 Parametrized colimits

In this section, we first introduce a parametrized generalization of Lurie's pairing construction [9, Corollary 3.2.2.13]. We then employ it to study *D–parametrized S–(co)limits*. This material recovers and extends [9, Section 4.2.2] (in view of Lemma 4.5). It is a precursor to our study of Kan extensions.

An S–pairing construction

9.1 Construction Let $p\colon C \to S$ and $q\colon D \to S$ be S–categories and let $\phi\colon C \to D$ be an S–functor. Let $\pi, \pi'\colon \mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to D$ be given by $\pi = \mathrm{ev}_0 \circ \mathrm{pr}_1$ and $\pi' = \mathrm{ev}_1 \circ \mathrm{pr}_1$. Let $\mathscr{E}$ denote the collection of edges e in $\mathbb{O}^{\mathrm{cocart}}(D) \times_D C$ such that $\pi(e)$ is q–cocartesian and $\mathrm{pr}_2(e)$ is p–cocartesian (so $\pi'(e)$ is q–cocartesian). Then the span

$$_\natural D \xleftarrow{\pi} (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}) \xrightarrow{\pi'} {}_\natural D$$

defines a functor

$$\pi_* \pi'^* \colon s\mathbf{Set}^+_{/_\natural D} \to s\mathbf{Set}^+_{/_\natural D}.$$

For an S–category E and an S–functor $\psi\colon E \to D$, define

$$(\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to {}_\natural D) := \pi_* \pi'^* ({}_\natural E \xrightarrow{\psi} {}_\natural D).$$

9.2 Lemma *Let $q\colon D \to S$ be an S–category.*

(1) $\mathrm{ev}_0\colon \mathbb{O}^{\mathrm{cocart}}(D) \to D$ *is a cartesian fibration, and an edge e in $\mathbb{O}^{\mathrm{cocart}}(D)$ is* ev_0*–cartesian if and only if $(\mathrm{ev}_{S,1} \circ q)(e)$ is an equivalence in S. In particular, if $\mathrm{ev}_0(e)$ is q–cocartesian, then e is* ev_0*–cartesian if and only if $\mathrm{ev}_1(e)$ is an equivalence in D.*

(2) *If $f\colon x \to y$ is an edge in D such that $q(f)$ is an equivalence, then there exists a* ev_0*–cocartesian edge e over f. Moreover, an edge e over f is* ev_0*–cocartesian if and only if it is* ev_0*–cartesian.*

[18] We use Remark 7.4 to simplify the appearance of the left-hand side, which would otherwise be denoted by $\mathrm{Fun}^{\mathrm{cocart}}_{/D,S}(C \times_D \mathbb{O}_S(D), E)$.

Proof $\mathrm{ev}_0\colon \mathbb{O}^{\mathrm{cocart}}(D) \to D$ factors as

$$\mathbb{O}^{\mathrm{cocart}}(D) \to D \times_S \mathbb{O}(S) \to D,$$

where the first functor is a trivial fibration and the second is a cartesian fibration, as the pullback of $\mathrm{ev}_{S,0}\colon \mathbb{O}(S) \to S$. Thus ev_0 is a cartesian fibration with cartesian edges as indicated. Moreover, since $\mathrm{ev}_{S,0}\colon \mathbb{O}(S) \to S$ is a categorical fibration, the second claim follows from [11, Proposition B.2.9]. □

We have designed our construction so that for any object $x \in D$ and cocartesian section $S^{qx/} \to D$, the fiber of $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to D$ over x is equivalent to

$$\mathrm{Fun}_{S^{qx/}}(C \times_D S^{qx/}, E \times_D S^{qx/}).$$

For this reason, we think of $\widetilde{\mathrm{Fun}}_{D/S}(-,-)$ as the parametrized generalization of the pairing construction $\widetilde{\mathrm{Fun}}_D(-,-)$, to which it reduces when $S = \Delta^0$.

9.3 Theorem *With notation as in Construction 9.1, $\widetilde{\mathrm{Fun}}_{D/S}(C, E)$ enjoys the following functoriality*:

(1) *If ϕ is either an S–cartesian fibration or an S–cocartesian fibration and ψ is a categorical fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to S$ is an S–category with cocartesian edges marked as indicated in Construction 9.1, and $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to D$ is a categorical fibration.*

(2) *If ϕ is an S–cartesian fibration and ψ is an S–cocartesian fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to D$ is an S–cocartesian fibration.*

(3) *If ϕ is an S–cocartesian fibration and ψ is an S–cartesian fibration, then $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to D$ is an S–cartesian fibration.*

Proof (1) It suffices to check that Theorem 2.24 applies to the span

$$\natural D \xleftarrow{\pi} (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathcal{E}) \xrightarrow{\pi'} \natural D.$$

In the remainder of this proof we will verify that $\mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to D$ is a flat categorical fibration. For condition (4) we appeal to Lemma 9.2. The rest of the conditions are easy verifications.

(2) By Lemmas 9.2 and 7.5, $\pi\colon \mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to D$ is a cartesian fibration (hence flat) with an edge e π–cartesian if and only if $\mathrm{pr}_1(e)$ is ev_0–cartesian and $\mathrm{pr}_2(e)$ is ϕ–cartesian. Let $\mathcal{E}'$ be the collection of edges e in $\mathbb{O}^{\mathrm{cocart}}(D) \times_{\mathrm{ev}_1, D} C$ such that for

any π–cartesian lift e' of $\pi(e)$, the induced edge $d_1(e) \to d_1(e')$ is in $\mathcal{E}$. Note that since ϕ is S–cartesian (and not just fiberwise cartesian), $\mathcal{E}'$ is closed under composition. Invoking Theorem 2.24 on the span

$$D^\sharp \xleftarrow{\pi} (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathcal{E}') \xrightarrow{\pi'} D^\sharp$$

we deduce that

$$\pi_* \pi'^* \colon s\mathbf{Set}^+_{/D} \to s\mathbf{Set}^+_{/D}$$

is right Quillen. Note that there is no conflict of notation with the functor $\pi_*\pi'^*$ defined before on $s\mathbf{Set}^+_{/\natural D}$ since $\mathcal{E} \subset \mathcal{E}'$ and the two restrict to the same collections of marked edges in the fibers of π. Since S–cocartesian fibrations are cocartesian fibrations over D (Remark 7.4), we conclude.

(3) First note that π factors as a cocartesian fibration followed by a cartesian fibration, so is flat. Let $\mathcal{F}$ be the collection of edges f in D such that $q(f)$ is an equivalence. By Lemma 9.2, we have that $\pi \colon \mathbb{O}^{\mathrm{cocart}}(D) \times_{\mathrm{ev}_1, D} C \to D$ admits cocartesian lifts of edges in $\mathcal{F}$. Let $\mathcal{E}''$ be the collection of those π–cocartesian edges. Invoking Theorem 2.24 on the span

$$(D, \mathcal{F})^{\mathrm{op}} \xleftarrow{\rho} (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathcal{E}'')^{\mathrm{op}} \xrightarrow{\rho'} (D, \mathcal{F})^{\mathrm{op}},$$

where $\rho = \pi^{\mathrm{op}}$ and $\rho' = \pi'^{\mathrm{op}}$, we deduce that with respect to the cartesian model structures

$$\rho_* \rho'^* \colon s\mathbf{Set}^+_{/(D,\mathcal{F})} \to s\mathbf{Set}^+_{/(D,\mathcal{F})}$$

is right Quillen. We have that $\widetilde{\mathrm{Fun}}_{D/S}(C, E)$ is a full subcategory of $\rho_*\rho'^*(\psi)$. Moreover, the compatibility condition in the definition of an S–cartesian fibration ensures that $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to D$ inherits the property of being fibrant in $s\mathbf{Set}^+_{/(D,\mathcal{F})}$. Another routine verification shows that $\widetilde{\mathrm{Fun}}_{D/S}(C, E) \to D$ is indeed S–cartesian. □

9.4 Lemma *Let $C \to C'$ be a monomorphism between S–cartesian or S–cocartesian fibrations over D and let $E \to D$ be an S–fibration. Then the induced functor*

$$\widetilde{\mathrm{Fun}}_{D/S}(C', E) \to \widetilde{\mathrm{Fun}}_{D/S}(C, E)$$

is a categorical fibration.

Proof Given a trivial cofibration $A \to B$ in $s\mathbf{Set}_{\mathrm{Joyal}}$, we need to solve the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \widetilde{\mathrm{Fun}}_{D/S}(C', E) \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \widetilde{\mathrm{Fun}}_{D/S}(C, E) \end{tikzcd}$$

This diagram transposes to

$$\begin{tikzcd} A \times_D \mathbb{O}^{\mathrm{cocart}}(D) \times_D C' \cup_{A \times_D \mathbb{O}^{\mathrm{cocart}}(D) \times_D C} B \times_D \mathbb{O}^{\mathrm{cocart}}(D) \times_D C \arrow[r] \arrow[d] & E \arrow[d] \\ B \times_D \mathbb{O}^{\mathrm{cocart}}(D) \times_D C' \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

By the proof of Theorem 9.3, $\mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to D$ is a flat categorical fibration. Therefore, by [11, Proposition B.4.5] the left vertical arrow is a trivial cofibration in $s\mathbf{Set}_{\mathrm{Joyal}}$. □

For later use, we analyze some degenerate instances of the S–pairing construction.

9.5 Lemma *There is a natural equivalence $\widetilde{\mathrm{Fun}}_{D/S}(D, E) \xrightarrow{\simeq} E$ of S–categories over D.*

Proof The map is induced by the identity section $\iota_D\colon D \to \mathbb{O}^{\mathrm{cocart}}(D)$ fitting into a morphism of spans

$$\begin{tikzcd} & {}_\natural D \arrow[dl, equal] \arrow[d, "\iota_D"] \arrow[dr, equal] & \\ {}_\natural D & (\mathbb{O}^{\mathrm{cocart}}(D), \mathscr{E}) \arrow[l] \arrow[r] & {}_\natural D \end{tikzcd}$$

By Lemma 3.3(1′), ι_D is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ via the target map. Since the cocartesian model structure on $s\mathbf{Set}^+_{/{}_\natural D}$ is created by the forgetful functor to $s\mathbf{Set}^+_{/S}$, the assertion follows. □

9.6 Lemma *Let $C' \to D'$ be a cartesian fibration of ∞–categories and let E' be an S–category. For all $s \in S$, there is a natural equivalence*

$$\widetilde{\mathrm{Fun}}_{D' \times S/S}(C' \times S, D' \times E')_s \xrightarrow{\simeq} \widetilde{\mathrm{Fun}}_{D'}(C', D' \times E'_s)$$

of cartesian fibrations over D'.

Proof The left-hand side is defined using the span

$$(D')^\sharp \times \{s\} \leftarrow ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathbb{O}^{\mathrm{cocart}}(D' \times S) \times_{D'} C', \mathscr{E}') \to S^\sharp$$

with $\mathscr{E}'$ as in the proof of Theorem 9.3. Cocartesian edges (over S) in $D' \times S$ are precisely those edges which become equivalences when projected to D', so

$$\mathbb{O}^{\mathrm{cocart}}(D' \times S) \cong \mathrm{Fun}((\Delta^1)^\sharp, (D')^\sim) \times \mathbb{O}(S),$$

and the identity section $\iota_{D'}\colon D' \to \mathrm{Fun}((\Delta^1)^\sharp, (D')^\sim)$ is a categorical equivalence. Therefore, the map

$$(D' \times S^{s/})^\sharp \to ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathbb{O}^{\mathrm{cocart}}(D' \times S), \mathcal{E})$$

induced by $\iota_{D'}$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Since $C' \times S \to D' \times S$ is a cartesian fibration, it follows that

$$(C')^\natural \times (S^{s/})^\sharp \to ((D')^\sharp \times \{s\}) \times_{D' \times S} (\mathbb{O}^{\mathrm{cocart}}(D' \times S) \times_{D'} C', \mathcal{E}')$$

is also a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. Finally, using the inclusion

$$C' \times \{s\} \to C' \times S^{s/}$$

we obtain a morphism from the span

$$(D')^\sharp \leftarrow (C')^\natural \to \{s\} \subset S^\sharp$$

through a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$. This yields the claimed equivalence. □

Directly from the definition, we have that for an object $x \in D$, the fiber $\widetilde{\mathrm{Fun}}_{D/S}(C, E)_x$ is isomorphic to $\mathrm{Fun}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$. We now proceed to identify the S–fiber $\widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}$.

9.7 Proposition *There is an $\underline{x}$–functor*

$$\epsilon^*\colon \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}} \to \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$$

which is a cocartesian equivalence in $s\mathbf{Set}^+_{/\underline{x}}$.

Proof We first define the $\underline{x}$–functor ϵ^*. The data of maps of marked simplicial sets

$$A \to {}_\natural\widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}, \quad A \to {}_\natural\underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, (E \times_S D)_{\underline{x}})$$

over $\underline{x}$ is identical to the data of maps

$$A \times_{\underline{x}} \underline{x}^\sharp \times_D (\mathbb{O}^{\mathrm{cocart}}(D), \mathcal{E}) \times_D {}_\natural C \to {}_\natural E, \quad A \times_{\underline{x}} \mathbb{O}(\underline{x})^\sharp \times_{\mathrm{ev}_1 \circ \mathrm{ev}_1, D} {}_\natural C \to {}_\natural E$$

over ${}_\natural D$ (where $\mathcal{E}$ is the collection of edges e in $\mathbb{O}^{\mathrm{cocart}}(D)$ such that $\mathrm{ev}_0(e)$ and $\mathrm{ev}_1(e)$ are cocartesian). We have a commutative square

$$\begin{tikzcd}
\mathbb{O}(\underline{x})^\sharp \arrow[r, "\mathrm{ev}_0"] \arrow[d, "\mathbb{O}(\mathrm{ev}_1)"] & \underline{x}^\sharp \arrow[d, "\mathrm{ev}_1"] \\
(\mathbb{O}^{\mathrm{cocart}}(D), \mathcal{E}) \arrow[r, "\mathrm{ev}_0"] & {}_\natural D
\end{tikzcd}$$

which defines the functor $\epsilon\colon \mathbb{O}(\underline{x}) \to \underline{x} \times_D \mathbb{O}^{\mathrm{cocart}}(D)$, and this in turn induces the functor ϵ^*. To show that ϵ^* is a cocartesian equivalence, it will suffice to show that ϵ is a trivial fibration, for then a choice of section σ and homotopy $\sigma \circ \epsilon \simeq \mathrm{id}$ will furnish a strong homotopy inverse to ϵ^* in the sense of [9, Proposition 3.1.3.5]. Since we have a pullback diagram

$$\begin{tikzcd} \mathbb{O}(\underline{x}) \arrow[r] \arrow[d, "\epsilon"] & D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \arrow[d, "\epsilon'"] \\ \underline{x} \times_D \mathbb{O}^{\mathrm{cocart}}(D) \arrow[r] & \mathrm{Fun}(\Lambda^2_1, D) \end{tikzcd}$$

it will further suffice to show that ϵ' is a trivial Kan fibration. Observe that ϵ' factors as the composition

$$D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \xrightarrow{\epsilon''} \mathrm{Fun}(\Delta^2, D) \xrightarrow{\epsilon'''} \mathrm{Fun}(\Lambda^2_1, D),$$

where ϵ'' is defined by precomposing by the inclusion $i\colon \Delta^2 \to \Delta^1 \times \Delta^1$ which avoids the degenerate edge for objects in $D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D)$, and ϵ''' is precomposition by $\Lambda^2_1 \to \Delta^2$. Moreover, ϵ''' is a trivial fibration since $\Lambda^2_1 \to \Delta^2$ is inner anodyne. To argue that ϵ'' is a trivial fibration, first note that ϵ'' inherits the property of being a categorical fibration from $i^*\colon \mathrm{Fun}(\Delta^1 \times \Delta^1, D) \to \mathrm{Fun}(\Delta^2, D)$. Define an inverse σ'' by precomposing by the unique retraction $r\colon \Delta^1 \times \Delta^1 \to \Delta^2$ chosen so that $r \circ i = \mathrm{id}$. Then σ'' is a section of ϵ'' and one can write down an explicit homotopy through equivalences of the identity functor on $D \times_{\mathrm{Fun}(\Delta^1, D)} \mathrm{Fun}(\Delta^1 \times \Delta^1, D)$ to $\sigma'' \circ \epsilon''$, so ϵ'' is a trivial fibration. $\square$

D–parametrized slice

We now study another slice construction defined using the S–pairing construction.

9.8 Construction Let $\phi\colon C \to D$ be an S–cocartesian fibration, let $E \to D$ be an S–fibration, and let $F\colon C \to E$ be an S–functor over D. Then F defines a section S–functor

$$\tau_F\colon D \to \widetilde{\mathrm{Fun}}_{D/S}(C, E)$$

as adjoint to the functor $\mathbb{O}^{\mathrm{cocart}}(D) \times_{ev_1, D} C \to C \xrightarrow{F} E$. Define

$$E^{(\phi,F)/S} := D \times_{\tau_F, \widetilde{\mathrm{Fun}}_{D/S}(C,E)} \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)$$

and let $\pi_{(\phi,F)}$ denote the projection $E^{(\phi,F)/S} \to D$.

Given an object $x \in D$, the functor $\tau_F : D \to \widetilde{\mathrm{Fun}}_{D/S}(C, E)$ induces via pullback an $\underline{x}$–functor

$$\tau_{F_{\underline{x}}} : \underline{x} \to \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}}.$$

We also have the $\underline{x}$–functor

$$\sigma_{F_{\underline{x}}} : \underline{x} \to \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})$$

adjoint to

$$\mathbb{O}(\underline{x}) \times_{\underline{x}} C_{\underline{x}} \xrightarrow{\mathrm{pr}_2} C_{\underline{x}} \xrightarrow{F_{\underline{x}}} E_{\underline{x}}.$$

An inspection of the definition of the comparison functor ϵ^* of Proposition 9.7 shows that the triangle

$$\begin{tikzcd} \underline{x} \arrow[r, "\tau_{F_{\underline{x}}}"] \arrow[dr, "\sigma_{F_{\underline{x}}}"'] & \widetilde{\mathrm{Fun}}_{D/S}(C, E)_{\underline{x}} \arrow[d, "\epsilon^*"] \\ & \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}}) \end{tikzcd}$$

commutes. Recalling the definitions

$$(E^{(\phi,F)/S})_{\underline{x}} = \underline{x} \times_{\widetilde{\mathrm{Fun}}_{D/S}(C,E)_{\underline{x}}} \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_{\underline{x}},$$
$$(E_{\underline{x}})^{F_{\underline{x}}/\underline{x}} = \underline{x} \times_{\underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}}, E_{\underline{x}})} \underline{\mathrm{Fun}}_{\underline{x}}(C_{\underline{x}} \star_{\underline{x}} \underline{x}, E_{\underline{x}}),$$

we therefore obtain a comparison $\underline{x}$–functor

$$\psi : (E^{(\phi,F)/S})_{\underline{x}} \to (E_{\underline{x}})^{F_{\underline{x}}/\underline{x}}.$$

9.9 Corollary *The functor ψ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\underline{x}}$.*

Proof By [9, Proposition 3.3.1.5], we have to verify that ψ induces a categorical equivalence on the fibers. But after passage to the fiber over an object $e = [x \to y]$ in $\underline{x}$, by Lemma 4.8 ψ_e is a functor between two pullback squares in which one leg is a cartesian fibration. Therefore, by Proposition 9.7 and [9, Corollary 3.3.1.4], ψ_e is a categorical equivalence. $\square$

9.10 Proposition *With setup as in Construction 9.8, suppose in addition that $E \to D$ is an S–cartesian fibration. Then $\pi_{(\phi,F)} : E^{(\phi,F)/S} \to D$ is an S–cartesian fibration.*

Proof By Lemma 9.4, $\pi_{(\phi,F)}$ is a categorical fibration. By Theorem 9.3 and Lemmas 9.4, and 4.8, the functor

$$(\iota_C^*)_s : \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_s \to \widetilde{\mathrm{Fun}}_{D/S}(C, E)_s$$

over D_s satisfies the hypotheses of [9, Proposition 2.4.2.11]; hence is a locally cartesian fibration. To then show that $(\iota_C^*)_s$ is a cartesian fibration, it suffices to check that for every square

$$\begin{tikzcd} {[G\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to E_{\underline{x}}]} \arrow[r] \arrow[d] & {[G'\colon C_{\underline{y}} \star_{\underline{y}} \underline{y} \to E_{\underline{y}}]} \arrow[d] \\ {[H\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to E_{\underline{x}}]} \arrow[r] & {[H'\colon C_{\underline{y}} \star_{\underline{y}} \underline{y} \to E_{\underline{y}}]} \end{tikzcd}$$

in $\widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E)_s$ lying over an edge $e\colon x \to y$ in D_s, if the horizontal edges are cartesian lifts over e and the right vertical edge is $(\iota_C^*)_{s,y}$–cartesian, then the left vertical edge is $(\iota_C^*)_{s,x}$–cartesian. In other words, if we let $e_!\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to C_{\underline{y}} \star_{\underline{y}} \underline{y}$ and $e^*\colon E_{\underline{y}} \to E_{\underline{x}}$ denote choices of pushforward and pullback functors, then we want to show that given $G \simeq e^* \circ G' \circ e_!$, $H \simeq e^* \circ H' \circ e_!$, and $G'|_{\underline{y}} \simeq H'|_{\underline{y}}$, we have that $G|_{\underline{x}} \simeq H|_{\underline{x}}$. But this is clear. We deduce that $(\pi_{(\phi,F)})_s$, being pulled back from $(\iota_C^*)_s$, is a cartesian fibration.

For the final verification, let us abbreviate objects

$$(x \in D, [G\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to E_{\underline{x}}] : G|_{C_{\underline{x}}} = F_{\underline{x}}) \in E^{(\phi,F)/S}$$

as $[G\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to E_{\underline{x}}]$, the restriction to $C_{\underline{x}}$ equaling $F_{\underline{x}}$ being left implicit. We must check that given a square

$$\begin{tikzcd} x \arrow[r, "\tilde{\alpha}_x"] \arrow[d, "e"] & x' \arrow[d, "e'"] \\ y \arrow[r, "\tilde{\alpha}_y"] & y' \end{tikzcd}$$

in D lying over $\alpha\colon s \to t$ with the vertical edges in the fiber and the horizontal edges cocartesian lifts of α, and given a lift of that square to a square

$$\begin{tikzcd} {[G\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to E_{\underline{x}}]} \arrow[r] \arrow[d] & {[G'\colon C_{\underline{x}'} \star_{\underline{x}'} \underline{x}' \to E_{\underline{x}'}]} \arrow[d] \\ {[H\colon C_{\underline{y}} \star_{\underline{y}} \underline{y} \to E_{\underline{y}}]} \arrow[r] & {[H'\colon C_{\underline{y}'} \star_{\underline{y}'} \underline{y}' \to E_{\underline{y}'}]} \end{tikzcd}$$

in $E^{(\phi,F)/S}$ with the horizontal edges cocartesian lifts of α and the left vertical edge $(\pi_{(\phi,F)})_s$–cartesian, then the right vertical edge is $(\pi_{(\phi,F)})_t$–cartesian. We will once more translate this compatibility statement into a more obvious looking one so as to conclude. Let $e_!$, e^*, $e'_!$ and e'^* be defined as above. Let $\alpha^*\colon \underline{x}' \to \underline{x}$ and $\alpha^*\colon \underline{y}' \to \underline{y}$

be choices of pullback functors (eg the first sends a cocartesian edge $f: x' \to z$ to $f \circ \widetilde{\alpha}_x : x \to z$), and also label related functors by α^*. Then the cocartesianness of the horizontal edges amounts to the equivalences $G' \simeq G \circ \alpha^*$ and $H' \simeq H \circ \alpha^*$, and the cartesianness of the left vertical edge amounts to the equivalence $G|_{\underline{x}} \simeq (e^* \circ H \circ e_!)|_{\underline{x}}$. Our desired assertion now is implied by the homotopy commutativity of the diagram

$$\begin{tikzcd}
\underline{x}' \arrow[r, "\alpha^*"] \arrow[d, "e'_!"] & \underline{x} \arrow[r, "G|_{\underline{x}}"] \arrow[d, "e_!"] & E_{\underline{x}} \\
\underline{y}' \arrow[r, "\alpha^*"] & \underline{y} \arrow[r, "H|_{\underline{y}}"] & E_{\underline{y}} \arrow[u, "e^*"']
\end{tikzcd}$$

(the content being in the commutativity of the first square), for this demonstrates that $G'|_{\underline{x}'} \simeq (e'^* \circ H' \circ e'_!)|_{\underline{x}'}$. □

9.11 Lemma *Let $p: W \to S$ and $q: D \to S$ be S–categories, and let $\pi: W \to D$ be an S–fibration such that for every object $s \in S$, π_s is a cartesian fibration.*

(1) *Suppose that*:

  (a) *For every object $x \in D$, there exists an initial object in W_x.*

  (b) *For every p–cocartesian edge $w \to w'$ in W, if w is an initial object in $W_{\pi(w)}$, then w' is an initial object in $W_{\pi(w')}$.*

  Let $W' \subset W$ be the full simplicial subset of W spanned by those objects $w \in W$ which are initial in $W_{\pi(w)}$ and let $\pi' = \pi|_{W'}$. Then W' is a full S–subcategory of W and π' is a trivial fibration.

(2) *Let $\sigma: D \to W$ be an S–functor which is a section of π. Then σ is a left adjoint of π relative to D if and only if, for every object $x \in D$, $\sigma(x)$ is an initial object of W_x.*

Proof (1) Condition (b) ensures that W' is an S–subcategory of W. By [9, Proposition 2.4.4.9], for every object $s \in S$, π'_s is a trivial fibration. In particular, π' is S–cocartesian fibration (the compatibility condition being vacuous since all edges in W'_s are π'_s–cocartesian). By Remark 7.4, π' is a cocartesian fibration. As a cocartesian fibration with contractible fibers, π' is a trivial fibration.

(2) Since relative adjunctions are stable under base change, if σ is a left adjoint of π relative to D, passage to the fiber over $x \in D$ shows that $\sigma(x)$ is an initial object of W_x. Conversely, if for all $x \in D$, $\sigma(x)$ is an initial object of W_x, then by [9, Proposition 5.2.4.3], σ_s is left adjoint to π_s for all $s \in S$. Since σ is already given as

an S–functor, this implies that σ is S–left adjoint to π; in particular, σ is left adjoint to π. The existence of σ implies the hypotheses of (1), so σ is fully faithful. Now by definition, σ is left adjoint to π relative to D. $\square$

We now connect the construction $\widetilde{\mathrm{Fun}}_{D/S}(-,-)$ with $\underline{\mathrm{Fun}}_S(-,-)$. To this end, consider the commutative diagram

$$\begin{tikzcd}
\mathbb{O}(S)^\sharp \times_S {}_\natural C \arrow[dr, "i"] \arrow[drrr, bend left] \arrow[dddr] & & & \\
& \mathbb{O}(S)^\sharp \times_S (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}) \arrow[r] \arrow[d] & (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}) \arrow[r] \arrow[d] & S^\sharp \\
& \mathbb{O}(S)^\sharp \times_S {}_\natural D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "\mathrm{ev}_0"] & {}_\natural D & \\
& S^\sharp & &
\end{tikzcd}$$

where the map i is induced by the identity section $D \to \mathbb{O}^{\mathrm{cocart}}(D)$.

9.12 Lemma *The map i is a homotopy equivalence in $s\mathbf{Set}^+_{/S}$ (considered over S via $p\colon C \to S$).*

Proof Define a map $h'\colon \mathbb{O}(S) \times_S \mathbb{O}^{\mathrm{cocart}}(D) \to \mathrm{Fun}(\Delta^1, \mathbb{O}(S) \times_S \mathbb{O}^{\mathrm{cocart}}(D))$ to be the product of the following three maps.

(1) Choose a lift σ

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\{0,1\}}, S) \arrow[r, "s_1"] \arrow[d] & \mathrm{Fun}(\Delta^2, S) \arrow[d, two heads, "\sim"'] \\
\mathrm{Fun}(\Lambda^2_1, S) \arrow[r, equal] \arrow[ur, dotted, "\sigma"] & \mathrm{Fun}(\Lambda^2_1, S)
\end{tikzcd}$$

and let $\Delta^1 \times \Delta^1 \to \Delta^2$ be the unique map such that the induced map

$$\mathrm{Fun}(\Delta^2, S) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, S) \cong \mathrm{Fun}(\Delta^1, \mathbb{O}(S))$$

sends $(s \to t \to u)$ to $[s \to t] \to [s \to u]$. Use these two maps to define

$$\mathbb{O}(S) \times_S \mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to \mathbb{O}(S) \times_S \mathbb{O}(S) \cong \mathrm{Fun}(\Lambda^2_1, S) \to \mathrm{Fun}(\Delta^1, \mathbb{O}(S)).$$

(2) Use the unique map $\Delta^1 \times \Delta^1 \to \Delta^1$ which sends $(0,0)$ to 0 and all other vertices to 1 to define

$$\mathbb{O}(S) \times_S \mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to \mathbb{O}^{\mathrm{cocart}}(D) \to \mathrm{Fun}(\Delta^1, \mathbb{O}^{\mathrm{cocart}}(D)).$$

(3) The degeneracy map $s_0 \colon C \to \mathrm{Fun}(\Delta^1, C)$ defines

$$\mathbb{O}(S) \times_S \mathbb{O}^{\mathrm{cocart}}(D) \times_D C \to C \to \mathrm{Fun}(\Delta^1, C).$$

Then h' is adjoint to a map of marked simplicial sets over S,

$$h \colon (\Delta^1)^\sharp \times \mathbb{O}(S)^\sharp \times_S (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}) \to \mathbb{O}(S)^\sharp \times_S (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}),$$

such that $h_0 = \mathrm{id}$ and h_1 factors as a composition

$$\mathbb{O}(S)^\sharp \times_S (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}) \xrightarrow{r} \mathbb{O}(S)^\sharp \times_S {}_\natural C \xrightarrow{i} \mathbb{O}(S)^\sharp \times_S (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}),$$

where r is defined by

$$\mathbb{O}(S)^\sharp \times_S (\mathbb{O}^{\mathrm{cocart}}(D) \times_D C, \mathscr{E}) \to \mathrm{Fun}(\Lambda^2_1, S)^\sharp \times_S {}_\natural C \xrightarrow{d_1 \circ \sigma} \mathbb{O}(S)^\sharp \times_S {}_\natural C.$$

Our choice of σ ensures that $r \circ i = \mathrm{id}$, completing the proof. □

Note that for any S–fibration $\pi \colon X \to D$, the S–category $\underline{\mathbf{Sect}}_{D/S}(\pi)$ defined in Definition 8.6 may be identified with $(\mathrm{ev}_0)_*(\mathrm{pr}_D)^*({}_\natural X \xrightarrow{\pi} {}_\natural D)$. Combining Lemmas 9.12, 2.27 and 2.28, we see that if E is an S–category and $C \to D$ is S–cocartesian or S–cartesian, then the map induced by i

$$i^* \colon \underline{\mathbf{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C, E \times_S D)) \to \underline{\mathrm{Fun}}_S(C, E)$$

is an equivalence of S–categories. Moreover, a chase of the definitions reveals that for every S–functor $F \colon C \to E$, we have an identification

$$i^* \circ \underline{\mathbf{Sect}}_{D/S}(\tau_{F \times \phi}) = \sigma_F \colon S \to \underline{\mathrm{Fun}}_S(C, E).$$

We thus have a morphism of spans

$$\begin{tikzcd} S \arrow[r, "\underline{\mathbf{Sect}}_{D/S}(\tau_{F\times\phi})"] \arrow[d, equal] & \underline{\mathbf{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C, E \times_S D)) \arrow[d, "\simeq"] & \underline{\mathbf{Sect}}_{D/S}(\widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E \times_S D)) \arrow[l] \arrow[d, "\simeq"] \\ S \arrow[r, "\sigma_F"] & \underline{\mathrm{Fun}}_S(C, E) & \underline{\mathrm{Fun}}_S(C \star_D D, E) \arrow[l] \end{tikzcd}$$

The right horizontal maps are S–fibrations by Lemma 9.4 and [2, Proposition 9.11(2)], so taking pullbacks yields an equivalence

(9.12.1) $$\underline{\mathbf{Sect}}_{D/S}((E \times_S D)^{(\phi, F \times \phi)/S}) \xrightarrow{\simeq} S \times_{\sigma_F, \underline{\mathrm{Fun}}_S(C,E)} \underline{\mathrm{Fun}}_S(C \star_D D, E).$$

We are now prepared to introduce the main definition of this section.

9.13 Definition Let $\phi\colon C \to D$ be an S–cocartesian fibration. An S–functor

$$\overline{F}\colon C \star_D D \to E$$

is a *D–parametrized S–colimit diagram* if for every object $x \in D$, the $\underline{x}$–functor $\overline{F}|_{C_{\underline{x}} \star_{\underline{x}} \underline{x}}\colon C_{\underline{x}} \star_{\underline{x}} \underline{x} \to E_{\underline{s}}$ is an $\underline{s}$–colimit diagram.

9.14 Proposition *Let $\phi\colon C \to D$ be an S–cocartesian fibration, let $F\colon C \to E$ be an S–functor, and let $\overline{F}\colon C \star_D D \to E$ be a D–parametrized S–colimit diagram extending F. Then the section*

$$\mathrm{id}_S \times \sigma_{\overline{F}}\colon S \to S \times_{\sigma_F, \underline{\mathrm{Fun}}_S(C,E)} \underline{\mathrm{Fun}}_S(C \star_D D, E)$$

is an S–initial object.

Proof Combine (9.12.1), Lemma 9.11(2), and Corollary 8.7. $\square$

We have the following existence and uniqueness result for D–parametrized S–colimits.

9.15 Theorem *Let $\phi\colon C \to D$ be an S–cocartesian fibration and let $F\colon C \to E$ be an S–functor. Suppose that for every object $x \in D$, the $\underline{s}$–functor $F|_{C_{\underline{x}}}\colon C_{\underline{x}} \to E_{\underline{s}}$ admits an $\underline{s}$–colimit. Then there exists a D–parametrized S–colimit diagram $\overline{F}\colon C \star_D D \to E$ extending F. Moreover, the full subcategory of $\{F\} \times_{\mathrm{Fun}_S(C,E)} \mathrm{Fun}_S(C \star_D D, E)$ spanned by the D–parametrized S–colimit diagrams coincides with that spanned by the initial objects.*

Proof By Proposition 9.10 and Corollary 9.9, the functor

$$\pi_{(\phi, F \times \phi)}\colon (E \times_S D)^{(\phi, F \times \phi)/S} \to D$$

is an S–cartesian fibration with $\underline{x}$–fibers equivalent to $(E_{\underline{s}})^{(F|_{C_{\underline{x}}}, \underline{s})/}$. Our hypothesis ensures that the conditions of Lemma 9.11(1) are satisfied, so $\pi_{(\phi, F \times \phi)}$ admits a section σ which is an S–functor that selects an initial object in each fiber. The resulting S–functor $D \to \widetilde{\mathrm{Fun}}_{D/S}(C \star_D D, E \times_S D)$ covering $\tau_{F \times \phi}$ is adjoint to an S–functor $\overline{F}\colon C \star_D D \to E$ extending F, which is a D–parametrized S–colimit diagram. Having proven existence, the second statement now follows from Proposition 9.14. $\square$

Theorem 9.15 also admits the following "global" consequence.

9.16 Corollary *Let $\phi\colon C \to D$ be an S–cocartesian fibration and E be an S–category. Suppose that for every $s \in S$ and $x \in D_s$, $E_{\underline{s}}$ admits all $S^{s/}$–colimits of shape $C_{\underline{x}}$. Then $U\colon \underline{\mathrm{Fun}}_S(C \star_D D, E) \to \underline{\mathrm{Fun}}_S(C, E)$ admits a left S–adjoint L which is a section of U such that for every object $F\colon C_{\underline{s}} \to E_{\underline{s}}$, $L(F)$ is a $D_{\underline{s}}$–parametrized $S^{s/}$–colimit diagram.*

Proof By Example 7.10, Theorem 9.15 and the stability of parametrized colimit diagrams under base change, the conditions of Lemma 9.11(1) are satisfied for U. Thus U admits a section L which selects an initial object in each fiber, necessarily a parametrized colimit diagram. By Lemma 9.11(2), L is a left adjoint of U relative to $\underline{\mathrm{Fun}}_S(C, E)$; in particular, L is S–left adjoint to U. □

Application: Functor categories

9.17 Proposition *Let K, I, and C be S–categories.*

(1) *Suppose that for all $s \in S$, $C_{\underline{s}}$ admits all $K_{\underline{s}}$–indexed colimits. Then*

$$\bar{p}\colon K \star_S S \to \underline{\mathrm{Fun}}_S(I, C)$$

is an S–colimit diagram if and only if, for every object $x \in I$ over s,

$$K_{\underline{s}} \star_{\underline{s}} \underline{s} \xrightarrow{\bar{p}_{\underline{s}}} \underline{\mathrm{Fun}}_{\underline{s}}(I_{\underline{s}}, C_{\underline{s}}) \xrightarrow{\mathrm{ev}_x} C_{\underline{s}}$$

is an $S^{s/}$–colimit diagram.

(2) *An S–functor $p\colon K \to \underline{\mathrm{Fun}}_S(I, C)$ admits an extension to an S–colimit diagram $\bar{p}$ if for all $x \in I$, $\mathrm{ev}_x \circ p_{\underline{s}}$ admits an extension to an $S^{s/}$–colimit diagram.*

Proof We prove (1), the proof for (2) being similar. Let

$$\bar{p}'\colon (K \times_S I) \star_I I \cong (K \star_S S) \times_S I \to C$$

be a choice of adjoint of p under the equivalence

$$\mathrm{Fun}_S(K \star_S S, \underline{\mathrm{Fun}}_S(I, C)) \simeq \mathrm{Fun}_S((K \star_S S) \times_S I, C).$$

By Theorem 9.15 applied to the S–cocartesian fibration $K \times_S I \to I$ and the hypothesis on C, there exists an I–parametrized S–colimit diagram p'' extending $p' = \bar{p}'|_{K \times_S I}$. By Proposition 9.14, p'' defines an S–initial object in

$$S \times_{\underline{\mathrm{Fun}}_S(K \times_S I, C)} \underline{\mathrm{Fun}}_S((K \times_S I) \star_I I, C) \simeq \underline{\mathrm{Fun}}_S(I, C)^{(p,S)/}$$

so *its* adjoint is an S–colimit diagram. For the "if" direction, supposing that $\bar{p}$ is an S–colimit diagram, then by the uniqueness of S–initial objects, p'' is equivalent to $\bar{p}'$. Then $\mathrm{ev}_x \circ \bar{p}_{\underline{s}}$ is equivalent to $p''_{\underline{x}}$, which is an $S^{s/}$–colimit diagram by definition of I–parametrized S–colimit diagram. For the "only if" direction, supposing that all the $\mathrm{ev}_x\, \bar{p}_{\underline{s}}$ are $S^{s/}$–colimit diagrams, we get that $\bar{p}'$ is an I–parametrized S–colimit diagram, so is equivalent to p''. □

9.18 Corollary *Suppose C is S–cocomplete and I is an S–category. Then $\underline{\mathrm{Fun}}_S(I, C)$ is S–cocomplete.*

10 Kan extensions

We now combine the theory of S–colimits parametrized by a base S–category D and that of free S–cocartesian fibrations to establish the theory of left S–Kan extensions.

10.1 Definition Suppose a diagram of S–categories

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\phi"'] & E \\ D \arrow[ur, "G"'] \arrow[u, phantom, ""] & \end{tikzcd} \quad \eta \Downarrow$$

where by the "2–cell" η we mean exactly the datum of an S–functor $\eta\colon C \times \Delta^1 \to E$ restricting to F on 0 and $G \circ \phi$ on 1. Let

$$G'\colon (C \times_D \mathbb{O}_S(D)) \star_D D \xrightarrow{\pi_D} D \xrightarrow{G} E,$$

let

$$\theta\colon (C \times_D \mathbb{O}_S(D)) \times \Delta^1 \to E$$

be the natural transformation adjoint to $G_*\colon C \times_D \mathbb{O}_S(D) \to \mathbb{O}_S(E)$, let

$$\eta'\colon (C \times_D \mathbb{O}_S(D)) \times \Delta^1 \to C \times \Delta^1 \xrightarrow{\eta} E$$

be the natural transformation obtained from η, and let $\theta' = \theta \circ \eta'$ be a choice of composition in $\mathrm{Fun}_S(C \times_D \mathbb{O}_S(D), E)$. Let

$$r\colon \mathrm{Fun}_S((C \times_D \mathbb{O}_S(D)) \star_D D, E) \to \mathrm{Fun}_S(C \times_D \mathbb{O}_S(D), E)$$

denote the restriction functor. By Lemma 4.8, we may select an r–cartesian edge e in $\mathrm{Fun}_S((C \times_D \mathbb{O}_S(D)) \star_D D, E)$ with $d_0(e) = G'$ covering θ', chosen so that $e|_D$ is degenerate. Let $G'' = d_1(e)$.

We say that G is a *left S–Kan extension* of F along ϕ if G'' is a D–parametrized S–colimit diagram.

10.2 Remark The following are equivalent:

(1) G is a left S–Kan extension of F along ϕ.

(2) For all $s \in S$, $G_{\underline{s}}$ is a left $S^{s/}$–Kan extension of $F_{\underline{s}}$ along $\phi_{\underline{s}}$.

(3) For all $s \in S$ and $x \in D_s$, $G|_{\underline{x}} \colon \underline{x} \to E_{\underline{s}}$ is a left $S^{s/}$–Kan extension of $F|_{C_{\underline{x}}} \colon C_{\underline{x}} \to E_{\underline{s}}$ along $\phi_{\underline{x}} \colon C_{\underline{x}} \to \underline{x}$.

In other words, our notion of S–Kan extension generalizes the concept of *pointwise* Kan extensions.

We can bootstrap Theorem 9.15 to prove existence and uniqueness of left S–Kan extensions.

10.3 Theorem *Let $\phi \colon C \to D$ and $F \colon C \to E$ be S–functors. Suppose that for every object $x \in D$, the $S^{s/}$–functor*

$$C \times_D D^{/\underline{x}} \to C_{\underline{s}} \xrightarrow{F_{\underline{s}}} E_{\underline{s}}$$

admits an $S^{s/}$–colimit. Then there exists a left S–Kan extension $G \colon D \to E$ of F along ϕ, uniquely specified up to contractible choice.

Proof We spell out the details of existence and leave the proof of uniqueness to the reader. By Theorem 9.15, there exists a D–parametrized S–colimit diagram

$$\overline{F} \colon (C \times_D \mathcal{O}_S(D)) \star_D D \to E$$

extending $C \times_D \mathcal{O}_S(D) \to C \xrightarrow{F} E$. Let $G = \overline{F}|_D$. Define a map

$$h \colon C \times \Delta^1 \to (C \times_D \mathcal{O}_S(D)) \star_D D$$

over $D \times \Delta^1$ as adjoint to $(C \xrightarrow{(\mathrm{id}, \iota\phi)} C \times_D \mathcal{O}_S(D), C \xrightarrow{\phi} D)$ and let $\eta = \overline{F} \circ h$, so that η is a natural transformation from F to $G \circ \phi$.

We claim that η exhibits G as a left Kan extension of F along ϕ. To show this, we will exhibit an r–cartesian edge e from $\overline{F}$ to G' such that the restriction $r(e)$ of e to $C \times_D \mathcal{O}_S(D)$ is a choice of composition $\theta \circ \eta'$. Define

$$e' \colon (C \times_D \mathcal{O}_S(D)) \star_D D \times \Delta^1 \to (C \times_D \mathcal{O}_S(D)) \star_D D$$

over $D \times \Delta^1$ as adjoint to (id, π_D), and let $e = \overline{F} \circ e'$, so that e is an edge from $\overline{F}$ to G'. Since $(\pi_D)|_D = \mathrm{id}_D$, $e|_D$ is a degenerate edge in $\mathrm{Fun}_S(D, E)$, so e is r–cartesian.

To finish the proof, we need to introduce a few more maps. Define

$$\alpha = (\mathrm{pr}_C, \alpha') \colon C \times_D \mathbb{O}_S(D) \times \Delta^1 \to C \times_D \mathbb{O}_S(D)$$

where α' is adjoint to

$$C \times_D \mathbb{O}_S(D) \to \mathbb{O}_S(D) = \widetilde{\mathrm{Fun}}_S(S \times \Delta^1, D) \xrightarrow{\min^*} \widetilde{\mathrm{Fun}}_S(S \times \Delta^1 \times \Delta^1, D).$$

Here $\min \colon \Delta^1 \times \Delta^1 \to \Delta^1$ is the functor which takes the minimum. Define

$$\beta \colon C \times_D \mathbb{O}_S(D) \times \Delta^1 \to \mathbb{O}_S(D) \times \Delta^1 \xrightarrow{\mathrm{ev}} D.$$

Use α and β to define

$$\gamma \colon C \times_D \mathbb{O}_S(D) \times \Delta^1 \times \Delta^1 \to (C \times_D \mathbb{O}_S(D)) \star_D D$$

so that on objects $(c, \phi c \xrightarrow{f} d)$, γ sends $\Delta^1 \times \Delta^1$ to the square

$$\begin{tikzcd}
(c, \phi c = \phi c) \arrow[r] \arrow[d, "(\mathrm{id}, f)"] & \phi c \arrow[d, "f"] \\
(c, \phi c \xrightarrow{f} d) \arrow[r] & d
\end{tikzcd}$$

Then $\overline{F} \circ \gamma$ defines a square

$$\begin{tikzcd}
F \circ \mathrm{pr}_C \arrow[r, "\eta'"] \arrow[d, equal] & G \circ \phi \circ \mathrm{pr}_C \arrow[d, "\theta"] \\
F \circ \mathrm{pr}_C \arrow[r, "r(e)"] & G'
\end{tikzcd}$$

in $\mathrm{Fun}_S(C \times_D \mathbb{O}_S(D), E)$, which proves that $r(e) \simeq \theta \circ \eta'$. □

We also have the Kan extension counterpart to Corollary 9.16.

10.4 Definition Let $\phi \colon C \to D$ be an S–functor and E an S–category. We say that *E admits the relevant S–colimits for ϕ* if for every $s \in S$ and $x \in D_s$, $E_{\underline{s}}$ admits all $S^{s/}$–colimits of shape $C \times_D D^{/\underline{x}}$.

10.5 Theorem *Let $\phi \colon C \to D$ be an S–functor and E an S–category. Suppose that E admits the relevant S–colimits for ϕ. Then the S–functor*

$$\phi^* \colon \underline{\mathrm{Fun}}_S(D, E) \to \underline{\mathrm{Fun}}_S(C, E)$$

given by restriction along ϕ admits a left S–adjoint $\phi_!$ such that for every S–functor $F: C \to E$, the unit map $F \to \phi^*\phi_! F$ exhibits $\phi_! F$ as a left S–Kan extension of F along ϕ.

Proof Factor ϕ as the composition

$$C \xrightarrow{\iota_C} C \times_D \mathbb{O}_S(D) \xrightarrow{i} (C \times_D \mathbb{O}_S(D)) \star_D D \xrightarrow{\pi_D} D.$$

Then ϕ^* factors as the composition

$$\underline{\mathrm{Fun}}_S(D, E) \xrightarrow{\pi_D^*} \underline{\mathrm{Fun}}_S((C \times_D \mathbb{O}_S(D)) \star_D D, E) \xrightarrow{i^*} \underline{\mathrm{Fun}}_S(C \times_D \mathbb{O}_S(D), E) \xrightarrow{\iota_C^*} \underline{\mathrm{Fun}}_S(C, E).$$

By Proposition 8.10 and Corollary 8.5, pr_C^* is left S–adjoint to ι_C^*. Since i_D is right S–adjoint to π_D, by Corollary 8.5 again i_D^* is left S–adjoint to π_D^*. By Corollary 9.16, i^* admits a left S–adjoint L which extends functors to D–parametrized S–colimit diagrams. Let $\phi_!$ be the composite of these three functors. The proof of Theorem 10.3 shows that $\phi_!(F)$ is as asserted. □

The next proposition permits us to eliminate the datum of the natural transformation η from the definition of a left S–Kan extension when ϕ is fully faithful.

10.6 Proposition *Suppose $\phi: C \to D$ is the inclusion of a full S–subcategory. Then for any left S–Kan extension G of $F: C \to E$ along ϕ, η is a natural transformation through equivalences. Consequently, G is homotopic to a functor $\overline{F}: D \to E$ which is both an extension of F and a left S–Kan extension (with the natural transformation $F \to \overline{F} \circ \phi = F$ chosen to be the identity).*

Proof Let $G'': (C \times_D \mathbb{O}_S(D)) \star_D D \to E$ be as in the definition of a left S–Kan extension. Because D–parametrized S–colimit diagrams are stable under restriction to S–subcategories,

$$(G'')_C: (C \times_D \mathbb{O}_S(D) \times_D C) \star_C C \to E$$

is a C–parametrized S–colimit diagram. The additional assumption that C is a full S–subcategory has the consequence that $(C \times_D \mathbb{O}_S(D) \times_D C) \cong \mathbb{O}_S(C)$. Also, for any object $x \in C$, the inclusion $\underline{x}$–functor $i_x: \underline{x} \to C^{/\underline{x}}$ is $\underline{x}$–final, using the first criterion of Theorem 6.7. Therefore, $\mathbb{O}_S(C) \star_C C \xrightarrow{\pi_C} C \xrightarrow{F} E$ is a C–parametrized S–colimit diagram extending $\mathbb{O}_S(C) \xrightarrow{\mathrm{ev}_0} C \xrightarrow{F} E$, so $(G'')_C \simeq F \circ \pi_C$.

The map h in the proof of Theorem 10.3 factors as

$$C \times \Delta^1 \xrightarrow{h'} \mathbb{O}_S(C) \star_C C \to (C \times_D \mathbb{O}_S(D)) \star_D D.$$

We have the chain of equivalences

$$\eta \simeq G'' \circ h \simeq F \circ \pi_C \circ h' = F \circ \mathrm{pr}_C,$$

proving the first assertion. For the second assertion, use that

$$({}_\natural D \times \{1\}) \cup_{{}_\natural C \times \{1\}} ({}_\natural C \times (\Delta^1)^\sharp) \to {}_\natural D \times (\Delta^1)^\sharp$$

is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$ to extend (G, η) to a homotopy between G and an extension $\overline{F}$, which is then necessarily a left S–Kan extension. ◻

10.7 Corollary *Suppose $\phi : C \to D$ a fully faithful S–functor and E an S–cocomplete S–category. Then the left S–adjoint $\phi_!$ to the restriction S–functor ϕ^* exists and is fully faithful.*

Proof Combine Theorem 10.5 and Proposition 10.6. ◻

As expected, S–colimit diagrams are examples of S–left Kan extensions.

10.8 Proposition *Suppose $\phi : C \to D$ an S–cocartesian fibration and $\overline{F} : C \star_D D \to E$ an S–functor extending $F : C \to E$. Then $\overline{F}$ is a D–parametrized S–colimit diagram if and only if $\overline{F}$ is an S–left Kan extension of F.*

Proof We may check the assertion objectwise on D, so let $x \in D_s$. Consider the commutative diagram

$$\begin{tikzcd}
C_{\underline{x}} \arrow[r, hook] \arrow[d, hook, "\theta"'] & C_{\underline{s}} \arrow[d, "F_{\underline{s}}"] \\
C \times_{C \star_D D} (C \star_D D)^{/\underline{x}} \arrow[ur, "\mathrm{pr}_C"] \arrow[r] & E_{\underline{s}}
\end{tikzcd}$$

The value of a D–parametrized colimit of F on x is computed as the $S^{s/}$–colimit of $(F_{\underline{s}})|_{C_{\underline{x}}}$, and that of an S–left Kan extension of F as the $S^{s/}$–colimit of $F_{\underline{s}} \circ \mathrm{pr}_C$. Therefore, it suffices to prove that θ is $\underline{x}$–final. Let $f : x \to y$ be an object in $\underline{x}$, ie a cocartesian edge in D, which lies over $s \to t$. Then θ_f is equivalent to the inclusion

$$C_y \cong C_y \times_{(C_y)^\rhd} ((C_y)^\rhd)^{/\{\infty\}} \to C_t \times_{C_t \star_{D_t} D_t} (C_t \star_{D_t} D_t)^{/y}.$$

Applying Lemma 10.9 to the map $C_t \to C_t \star_{D_t} D_t$ of cocartesian fibrations over D_t, we deduce that θ_f is final. ◻

10.9 Lemma *Let $X \to Y$ be a map of cocartesian fibrations over Z and let $y \in Y$ be an object over $z \in S$. Then the inclusion $X_z \times_{Y_z} Y_z^{/y} \to X \times_Y Y^{/y}$ is final.*

Proof By the dual of [11, Lemma 3.4.1.10], $X \times_Y Y^{/y} \to Z^{/z}$ is a cocartesian fibration. We have a pullback square

$$\begin{tikzcd} X_z \times_{Y_z} Y_z^{/y} \arrow[r] \arrow[d] & X \times_Y Y^{/y} \arrow[d] \\ \{z\} \arrow[r, "\mathrm{id}_z"] & Z^{/z} \end{tikzcd}$$

where, since the bottom horizontal map is final and cocartesian fibrations are smooth, the top horizontal map is final. □

As with S–colimits, S–left Kan extensions reduce to the usual notion of left Kan extension when taken in an S–category of objects.

10.10 Proposition *Suppose we have a diagram of S–categories:*

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\phi"'] \arrow[r, phantom, shift right=3, "\eta \Downarrow"] & \underline{E}_S \\ D \arrow[ur, "G"'] & \end{tikzcd}$$

The following are equivalent:

(1) *G is a left S–Kan extension of F along ϕ.*

(2) *$G^\dagger$ is a left Kan extension of $F^\dagger$ along ϕ.*

(3) *For all objects $s \in S$, $G^\dagger|_{D_s}$ is a left Kan extension of $F^\dagger|_{C_s}$ along ϕ_s.*

Proof We first prove that (1) and (2) are equivalent. Factor $\phi\colon C \to D$ through the free S–cocartesian fibration on ϕ,

$$\phi\colon C \xrightarrow{\iota_C} C \times_D \mathbb{O}_S D \xrightarrow{\mathrm{Fr}^{\mathrm{cocart}}(\phi)} D.$$

Since ι_C is S–left adjoint to pr_C, it is also left adjoint. Therefore, the S–left (resp. left) Kan extension of F (resp. $F^\dagger$) along ι_C is computed by $F \circ \mathrm{pr}_C$ (resp. $F^\dagger \circ \mathrm{pr}_C$). By transitivity of Kan extensions, we thereby reduce to the case that ϕ is S–cocartesian. The claim now follows easily by combining Propositions 5.5 and 10.8.

We next prove that (2) and (3) are equivalent. For this, it suffices to observe that for all objects $d \in D$ over some $s \in S$, $C_s \times_{D_s} D_s^{/d} \to C \times_D D^{/d}$ is final by Lemma 10.9 applied to $C \to D$. □

11 Yoneda lemma

By Proposition 5.5, $\underline{\mathbf{Spc}}_S$ is S–cocomplete, so by Corollary 9.18, the S–category of presheaves

$$\boldsymbol{P}_S(C) := \underline{\mathrm{Fun}}_S(C^{\mathrm{vop}}, \underline{\mathbf{Spc}}_S)$$

is S–cocomplete. The S–Yoneda embedding $j : C \to \boldsymbol{P}_S(C)$ was constructed in [2, Section 10] via S–straightening the left fibration $\widetilde{\mathcal{O}}_S(C) \to C^{\mathrm{vop}} \times_S C$ given fiberwise by twisted arrows. It was shown there that j is fully faithful [2, Theorem 10.4]. In this section, we first prove the S–Yoneda lemma and then establish the universal property of $\boldsymbol{P}_S(C)$ as the free S–cocompletion of C.

11.1 Lemma (S–Yoneda lemma) *Let $j : C \to \boldsymbol{P}_S(C)$ denote the S–Yoneda embedding. Then the identity on $\boldsymbol{P}_S(C)$ is an S–left Kan extension of j along itself.*

Proof By Proposition 9.17, it suffices to show that for every $s \in S$ and object $x \in C_s$, $\mathrm{ev}_x : \boldsymbol{P}_{\underline{s}}(C_{\underline{s}}) \to \underline{\mathbf{Spc}}_{\underline{s}}$ is an $S^{s/}$–left Kan extension of $\mathrm{ev}_x\, j_{\underline{s}}$. To ease notation, let us replace $S^{s/}$ by S and suppose that $s \in S$ is an initial object.

We claim that $(\mathrm{ev}_x\, j)^\dagger : C \to \mathbf{Spc}$ is homotopic to $\mathrm{Map}_C(x, -)$. By definition of the S–Yoneda embedding, $(\mathrm{ev}_x\, j)^\dagger$ classifies the left fibration

$$\mathrm{ev}_1 : \widetilde{\mathcal{O}}_S(C)_{x\to} \to C$$

pulled back from $\widetilde{\mathcal{O}}_S(C) \to C^{\mathrm{vop}} \times_S C$ via the cocartesian section $\sigma : S \to C^{\mathrm{vop}}$ defined by $\sigma(s) = x$. By [9, Proposition 4.4.4.5], it suffices to show that id_x is an initial object in $\widetilde{\mathcal{O}}_S(C)_{x\to}$. For this, because $s \in S$ is an initial object, we reduce to checking that for all edges $\alpha : s \to t$, the pushforward of id_x by α is an initial object in the fiber $(\widetilde{\mathcal{O}}_S(C)_{x\to})_t$. But this fiber is equivalent to $\widetilde{\mathcal{O}}(C_t)_{\alpha_! x\to} \simeq (C_t)^{\alpha_! x/}$.

Applying Proposition 10.10, we reduce to showing that for all $t \in S$, $(\mathrm{ev}_x)^\dagger|_{\boldsymbol{P}_S(C)_t}$ is a left Kan extension of $(\mathrm{ev}_x\, j)^\dagger|_{C_t}$. Note that for y any cocartesian pushforward of x over the essentially unique edge $s \to t$, we have both that $(\mathrm{ev}_x\, j)^\dagger|_{C_t}$ is homotopic to $\mathrm{Map}_{C_t}(y, -)$ and $(\mathrm{ev}_x)^\dagger|_{\boldsymbol{P}_S(C)_t}$ is homotopic to ev_y (regarding y as an object in $C_{\underline{t}}^{\mathrm{vop}}$). The inclusion

$$C_t \to \boldsymbol{P}_S(C)_t \simeq \mathrm{Fun}(C_{\underline{t}}^{\mathrm{vop}}, \mathbf{Spc})$$

factors through $\boldsymbol{P}(C_t)$ with $\boldsymbol{P}(C_t) \to \mathrm{Fun}(C_{\underline{t}}^{\mathrm{vop}}, \mathbf{Spc})$ left adjoint to precomposition by the inclusion $i : C_t^{\mathrm{op}} \to C_{\underline{t}}^{\mathrm{vop}}$. By the usual Yoneda lemma for ∞–categories, $\mathrm{ev}_y : \boldsymbol{P}(C_t) \to \mathbf{Spc}$ is the left Kan extension of $\mathrm{Map}_{C_t}(y, -)$. The left Kan extension of ev_y to $\boldsymbol{P}_S(C)_t$ is then given by precomposition by i, so is again ev_y. □

To state the universal property of $\boldsymbol{P}_S(C)$, we need to introduce a bit of terminology.

11.2 Definition Let $F\colon C \to D$ be an S–functor. We say that F *strongly preserves S–(co)limits* if for all $s \in S$, $F_{\underline{s}}$ preserves $S^{s/}$–(co)limits.

11.3 Remark If F strongly preserves S–colimits then F preserves S–colimits. However, the converse is not necessarily true.

11.4 Notation Suppose that C and D are S–cocomplete S–categories. Let $\mathrm{Fun}^L_S(C, D)$ denote the full subcategory of $\mathrm{Fun}_S(C, D)$ on the S–functors F which strongly preserve S–colimits. Let $\underline{\mathrm{Fun}}^L_S(C, D)$ denote the full S–subcategory of $\underline{\mathrm{Fun}}_S(C, D)$ with fibers $\mathrm{Fun}^L_{S^{s/}}(C, D)$ over $s \in S$.

11.5 Theorem *Let E be an S–cocomplete S–category. Then restriction along the S–Yoneda embedding defines equivalences*

$$\mathrm{Fun}^L_S(\boldsymbol{P}_S(C), E) \xrightarrow{\simeq} \mathrm{Fun}_S(C, E), \quad \underline{\mathrm{Fun}}^L_S(\boldsymbol{P}_S(C), E) \xrightarrow{\simeq} \underline{\mathrm{Fun}}_S(C, E)$$

with the inverse given by S–left Kan extension.

We prepare for the proof of Theorem 11.5 with some necessary results concerning S–mapping spaces. Recall that given an ∞–category C, we have a number of equivalent options for describing mapping spaces in C. The relevant ones to consider for us are:

(1) Straightening the left fibration $\widetilde{\mathbb{O}}(C) \to C^{\mathrm{op}} \times C$, we obtain the mapping space functor
$$\mathrm{Map}_C(-,-)\colon C^{\mathrm{op}} \times C \to \mathbf{Spc}.$$

(2) Fixing an object $x \in C$, straightening the left fibration $C^{x/} \to C$ also yields the functor
$$\mathrm{Map}_C(x,-)\colon C \to \mathbf{Spc}.$$

(3) Fixing objects $x, y \in C$, we have that the space $\mathrm{Map}_C(x, y)$ is given by
$$\{x\} \times_C \mathbb{O}(C) \times_C \{y\}.$$

Likewise, given an S–category C, we have these possibilities:

(1) The S–functor
$$\underline{\mathrm{Map}}_C(-,-)\colon C^{\mathrm{vop}} \times_S C \to \underline{\mathbf{Spc}}_S$$
given by the S–straightening of $\widetilde{\mathbb{O}}_S(C) \to C^{\mathrm{vop}} \times_S C$.

(2) Fixing an object $x \in C$, we have the left fibration $C^{\underline{x}/} = \underline{x} \times_C \mathbb{O}_S(C) \to C$, which S–straightens to

$$\underline{\mathrm{Map}}_C(x,-)\colon C \to \underline{\mathbf{Spc}}_S.$$

(3) Fixing an object $x \in C$, we have the left fibration $C^{x/} \to C$, which S–straightens to

$$\underline{\mathrm{Map}}_C(x,-)\colon C \to \underline{\mathbf{Spc}}_S.$$

(4) Fixing objects $x \in C$ and $y \in C_s$, we have the $S^{s/}$–space

$$\underline{\mathrm{Map}}_C(x,y) = \underline{x} \times_C \mathbb{O}_S(C) \times_C \underline{y} \to \underline{y} \xrightarrow{\simeq} S^{s/}.$$

In the proof of Lemma 11.1, we showed that (1) and (3) were equivalent, and by Proposition 4.31, (2) and (3) are equivalent. Finally, (2) specializes to (4) by definition. We are thus justified in our abuse of notation when we interchangeably refer to any of these options by $\underline{\mathrm{Map}}_C(-,-)$.

Our next goal is to prove that $\underline{\mathrm{Map}}_C(-,-)$ preserves S–limits in the second variable, and dually, takes S–colimits in the first variable to S–limits. For this, we need a few lemmas.

11.6 Lemma *Let $F\colon X \to Y$ be a map of S–cocartesian or S–cartesian fibrations over an S–category C. The following are equivalent:*

(1) *F is an equivalence.*

(2) *For all $s \in S$ and $S^{s/}$–functors $Z \to C_{\underline{s}}$,*

$$\underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(Z, X_{\underline{s}}) \to \underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(Z, Y_{\underline{s}})$$

is an equivalence.

(3) *For all $s \in S$ and $c \in C_s$,*

$$\underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(\underline{c}, X_{\underline{s}}) \to \underline{\mathrm{Fun}}_{/C_{\underline{s}},S^{s/}}(\underline{c}, Y_{\underline{s}})$$

is an equivalence.

(4) *For all $c \in C$, $F_c\colon X_c \to Y_c$ is an equivalence.*

If X and Y are S–left or S–right fibrations over C, then all instances of $\underline{\mathrm{Fun}}$ can be replaced by $\underline{\mathrm{Map}}$.[19]

[19] $\underline{\mathrm{Map}}$ refers here to the maximal subleft fibration of $\underline{\mathrm{Fun}}$ and not the S–mapping space functor.

Proof (1) $\Longrightarrow$ (2) If F is an equivalence, so is $F_{\underline{s}}$ for all $s \in S$. The map in question is then induced by a map of pullbacks through equivalences in which two matching legs are S–fibrations, so is an equivalence.

(2) $\Longrightarrow$ (3) This is obvious.

(3) $\Longrightarrow$ (4) Given $c \in C_s$, take fibers over $\{s\} \in \underline{s}$ and note that

$$\underline{\mathrm{Fun}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, X_{\underline{s}})_s \simeq \mathrm{Fun}_{/C_c}(\{c\}, X_s) \simeq X_c.$$

(4) $\Longrightarrow$ (1) We must check that F_s is an equivalence for all $s \in S$, for which it suffices to check fiberwise over C_s by the hypothesis. □

11.7 Lemma *Let $\bar{q}\colon S \star_S K \to \underline{\mathbf{Spc}}_S$ be an S–functor which extends $q\colon K \to \underline{\mathbf{Spc}}_S$. Let $\overline{X} \to S \star_S K$ be a left fibration which is an unstraightening of $\bar{q}^\dagger$, and let $X = \overline{X} \times_{S \star_S K} K$. Then $\bar{q}$ is an S–limit diagram if and only if the restriction S–functor*

$$R\colon \underline{\mathrm{Map}}_{/S \star_S K, S}(S \star_S K, \overline{X}) \to \underline{\mathrm{Map}}_{/S \star_S K, S}(K, \overline{X}) \cong \underline{\mathrm{Map}}_{/K, S}(K, X)$$

is an equivalence.

Proof In view of [9, Corollary 3.3.3.4], R_s is a map from the limit of $\bar{q}^\dagger|_{\underline{s} \star_{\underline{s}} K_{\underline{s}}}$ to the limit of $q^\dagger|_{K_{\underline{s}}}$ induced by precomposition on the diagram. But by Proposition 5.6, $\bar{q}$ is an S–limit diagram if and only if $\bar{q}^\dagger$ is a right Kan extension of $q^\dagger$, in which case both of the limits in question are equivalent to $\bar{q}^\dagger(s)$. The assertion now follows. □

11.8 Proposition *Let $\bar{p}\colon S \star_S K \to C$ be an S–functor. The following are equivalent*:

1. *$\bar{p}$ is an S–limit diagram.*
2. *For all $s \in S$ and $c \in C_s$,*
$$\underline{\mathrm{Map}}_{C_{\underline{s}}}(c, \bar{p}_{\underline{s}}(-))\colon \underline{s} \star_{\underline{s}} K_{\underline{s}} \to \underline{\mathbf{Spc}}_{S^{s/}}$$
is an $S^{s/}$–limit diagram.
3. *For all $s \in S$ and $c \in C_s$,*
$$\underline{\mathrm{Map}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, C_{\underline{s}}^{/(\bar{p}_{\underline{s}}, S^{s/})}) \to \underline{\mathrm{Map}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, C_{\underline{s}}^{/(p_{\underline{s}}, S^{s/})})$$
is an equivalence.

Moreover, *if the above conditions are obtained*, *then*

$$\underline{\mathrm{Map}}_{/C_{\underline{s}}, S^{s/}}(\underline{c}, C_{\underline{s}}^{/(p_{\underline{s}}, S^{s/})}) \simeq \underline{\mathrm{Map}}_{C_{\underline{s}}}(c, \bar{p}_{\underline{s}}(v)),$$

where v is the cone point $\{s\} \in \underline{s} \star_{\underline{s}} K_{\underline{s}}$.

Proof (2) $\iff$ (3) We will show that the statements match after fixing $c \in C_s$. To ease notation, let us replace $S^{s/}$ by S and suppose that $s \in S$ is an initial object. By Lemma 11.7 and using that $C^{\underline{c}/}$ is the S–unstraightening of $\underline{\mathrm{Map}}_C(c,-)$, $\underline{\mathrm{Map}}_C(c, \bar{p}(-))$ is an S–limit diagram if and only if

$$\underline{\mathrm{Map}}_{/C,S}(S \star_S K, C^{\underline{c}/}) \to \underline{\mathrm{Map}}_{/C,S}(K, C^{\underline{c}/})$$

is an equivalence. By Corollary 4.27, this map is equivalent by a zigzag to the map

$$\underline{\mathrm{Map}}_{/C,S}(\underline{c}, C^{/(\bar{p},S)}) \to \underline{\mathrm{Map}}_{/C,S}(\underline{c}, C^{/(p,S)}).$$

The assertion now follows. The last assertion also follows in view of the equivalence $C^{/(\bar{p},S)} \simeq C^{/\underline{\bar{p}(v)}}$ and $\underline{\mathrm{Map}}_{/C,S}(\underline{c}, C^{/\underline{\bar{p}(v)}}) \simeq \underline{c} \times_C C^{/\underline{\bar{p}(v)}} \simeq \underline{\mathrm{Map}}_C(c, \bar{p}(v))$.

(1) $\iff$ (3) This follows from Lemma 11.6 applied to $C^{/(\bar{p},S)} \to C^{/(p,S)}$, which is a map of S–right fibrations over C. □

11.9 Corollary *Let $F : C \to D$ be an S–functor. Then*

(1) *F strongly preserves S–limits if and only if for all $s \in S$ and $d \in D_s$,*

$$\underline{\mathrm{Map}}_{D_{\underline{s}}}(d, F_{\underline{s}}(-)) : C_{\underline{s}} \to \underline{\mathbf{Spc}}_{S^{s/}}$$

preserves $S^{s/}$–limits.

(2) *F strongly preserves S–colimits if and only if for all $s \in S$ and $d \in D_s$,*

$$\underline{\mathrm{Map}}_{D_{\underline{s}}}(F_{\underline{s}}(-), d) = \underline{\mathrm{Map}}_{D_{\underline{s}}^{\mathrm{vop}}}(d, F_{\underline{s}}^{\mathrm{vop}}(-)) : C_{\underline{s}}^{\mathrm{vop}} \to \underline{\mathbf{Spc}}_{S^{s/}}$$

preserves $S^{s/}$–limits.

11.10 Corollary *Let C be an S–category. The Yoneda embedding $j : C \to \boldsymbol{P}_S(C)$ strongly preserves and detects S–limits.*

Proof Combine Propositions 11.8 and 9.17. □

Proof of Theorem 11.5 By Theorem 10.5, we have an S–adjunction

$$j_! : \underline{\mathrm{Fun}}_S(C, E) \rightleftarrows \underline{\mathrm{Fun}}_S(\boldsymbol{P}_S(C), E) : j^*$$

with $j^* j_! \simeq \mathrm{id}$ and the essential image of $j_!$ spanned by the left $S^{s/}$–Kan extensions ranging over all $s \in S$. By Proposition 8.4, taking cocartesian sections yields an adjunction

$$j_! : \mathrm{Fun}_S(C, E) \rightleftarrows \mathrm{Fun}_S(\boldsymbol{P}_S(C), E) : j^*$$

again with $j^* j_! \simeq \mathrm{id}$ and the essential image of $j_!$ spanned by the left S–Kan extensions. Both assertions will therefore follow if we prove that for an S–functor $F\colon \boldsymbol{P}_S(C) \to E$, F strongly preserves S–colimits if and only if F is a left S–Kan extension of its restriction $f = F|_C$.

For the "only if" direction, because $\mathrm{id}_{\boldsymbol{P}_S(C)}$ is an S–left Kan extension of j by the S–Yoneda Lemma 11.1, $F = F \circ \mathrm{id}_{\boldsymbol{P}_S(C)}$ is a left S–Kan extension as it is the postcomposition of $\mathrm{id}_{\boldsymbol{P}_S(C)}$ with a strongly S–colimit preserving functor.

For the "if" direction, we use the criterion of Corollary 11.9. Replacing $S^{s/}$ by S and supposing that $s \in S$ is an initial object, we reduce to showing that for all $x \in E_s$, $\underline{\mathrm{Map}}_E(F(-), x)\colon \boldsymbol{P}_S(C)^{\mathrm{vop}} \to \underline{\mathbf{Spc}}_S$ preserves S–limits. We first observe that F^{vop} is an S–right Kan extension (of f^{vop}), hence so is

$$\underline{\mathrm{Map}}_E(F(-), x) = \underline{\mathrm{Map}}_{E^{\mathrm{vop}}}(x, -) \circ F^{\mathrm{vop}}$$

as the postcomposition of an S–right Kan extension with a strongly S–limit preserving functor. However, by the vertical opposite of the S–Yoneda lemma, for any S–functor $G\colon C^{\mathrm{vop}} \to \underline{\mathbf{Spc}}_S$, the strongly S–limit preserving S–functor $\underline{\mathrm{Map}}_{\boldsymbol{P}_S(C)}(-, G)$ is an S–right Kan extension of G. Applying this for $G = \underline{\mathrm{Map}}_E(f(-), x)$, we conclude. $\square$

12 Bousfield–Kan formula

In this section, we prove two decomposition formulas for S–colimits which resemble the classical Bousfield–Kan formula for computing homotopy colimits. We first study the situation when $S = \Delta^0$.

12.1 Notation Let K be a simplicial set and let $\Delta_{/K}$ be the nerve of the category of simplices of K. We denote the first vertex map by $\upsilon_K\colon \Delta^{\mathrm{op}}_{/K} \to K$ and the last vertex map by $\mu_K\colon \Delta_{/K} \to K$.

By [9, Proposition 4.2.3.14], μ_K is final. Unfortunately, this is the wrong direction for the purposes of obtaining a Bousfield–Kan type formula, since $\Delta_{/K}$ is a *cartesian* fibration over Δ. To rectify this state of affairs, we prove that υ_K is in fact final.

12.2 Proposition *Let K be a simplicial set. Then the first vertex map $\upsilon_K\colon \Delta^{\mathrm{op}}_{/K} \to K$ is final. Equivalently, the last vertex map $\mu_{K^{\mathrm{op}}}$ is initial.*

Proof Note that υ_K is natural in K and that $\Delta^{\mathrm{op}}_{/(-)}\colon s\mathbf{Set} \to s\mathbf{Set}$ preserves colimits. Recall from [9, Proposition 4.1.2.5] that a map $f: X \to Y$ is final if and only if it is a contravariant equivalence in $s\mathbf{Set}_{/Y}$. It follows that the class of final maps is stable under filtered colimits, so we may suppose that K has finitely many nondegenerate simplices. Using left properness of the contravariant model structure, by induction we reduce to the assertion for $K = \Delta^n$. But in this case υ_K is final by the proof of [9, Variant 4.2.3.15] (which proves the result when K is the nerve of a category).

For the second assertion, we note that the reversal isomorphism $\Delta_{/K^{\mathrm{op}}} \cong \Delta_{/K}$ interchanges $\mu_{K^{\mathrm{op}}}$ and $(\upsilon_K)^{\mathrm{op}}$. □

12.3 Corollary (Bousfield–Kan formula) *Suppose that C admits* (*finite*) *coproducts. Then for a* (*finite*) *simplicial set K and a map $p\colon K \to C$, the colimit of p exists if and only if the geometric realization*

$$\left| \coprod_{x \in K_0} p(x) \Longleftarrow \coprod_{\alpha \in K_1} p(\alpha(0)) \Lleftarrow \coprod_{\sigma \in K_2} p(\sigma(0)) \cdots \right|$$

exists, *in which case the colimit of p is computed by the geometric realization.*

Proof The fibers of the cocartesian fibration $\pi_K\colon \Delta^{\mathrm{op}}_{/K} \to \Delta^{\mathrm{op}}$ are the discrete sets K_n. Therefore, the left Kan extension of $p \circ \upsilon_K$ along π_K exists. By Proposition 12.2, $\operatorname{colim} p \simeq \operatorname{colim} p \circ \upsilon_K$, and the latter is computed as the colimit of $(\pi_K)_!(p \circ \upsilon_K)$ by the transitivity of left Kan extensions. □

We also have a variant of Corollary 12.3 where the coproducts over K_n are replaced by colimits indexed by the spaces $\mathrm{Map}(\Delta^n, K)$. To formulate this, we need to introduce some auxiliary constructions. Let $\xi\colon W \to \Delta^{\mathrm{op}}$ be the opposite of the relative nerve of the inclusion $\Delta \to s\mathbf{Set}$; this is a cartesian fibration which is an explicit model for the tautological cartesian fibration over Δ^{op} pulled back from the universal cartesian fibration over $\mathbf{Cat}_\infty^{\mathrm{op}}$. Let $\lambda\colon \Delta^{\mathrm{op}} \to W$ be the "first vertex" section of ξ which sends an n–simplex $\Delta^{a_0} \leftarrow \cdots \leftarrow \Delta^{a_n}$ to the n–simplex

$$\begin{tikzcd}
\Delta^n \arrow[d, "(\lambda a)_0"] & \cdots \arrow[l] & \Delta^{\{n-1,n\}} \arrow[l] \arrow[d, "(\lambda a)_{n-1}"] & \Delta^{\{n\}} \arrow[l] \arrow[d, "(\lambda a)_n"] \\
\Delta^{a_0} & \cdots \arrow[l] & \Delta^{a_{n-1}} \arrow[l] & \Delta^{a_n} \arrow[l]
\end{tikzcd}$$

of W specified by $(\lambda a)_i(0) = 0$ for all $0 \le i \le n$.

For an ∞–category C, let $Z_C = \widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}}}(W, C \times \Delta^{\mathrm{op}})$ and let $Z'_C \subset Z_C$ be the subsimplicial set on the simplices σ such that every edge of σ is cocartesian (with respect to the structure map to Δ^{op}), so that $Z'_C \to \Delta^{\mathrm{op}}$ is the maximal subleft fibration in $Z_C \to \Delta^{\mathrm{op}}$. Define a Δ^{op}–functor $\Delta^{\mathrm{op}}_{/C} \to Z_C$ as adjoint to the map $\Delta^{\mathrm{op}}_C \times_{\Delta^{\mathrm{op}}} W \to C$ which sends an n–simplex

$$\begin{tikzcd}
\Delta^n \arrow[d, "(\lambda a)_0"] & \cdots \arrow[l] & \Delta^{\{n-1,n\}} \arrow[l] \arrow[d, "(\lambda a)_{n-1}"] & \Delta^{\{n\}} \arrow[l] \arrow[d, "(\lambda a)_n"] \\
\Delta^{a_0} \arrow[d, "\tau"] & \cdots \arrow[l] & \Delta^{a_{n-1}} \arrow[l] \arrow[dll] & \Delta^{a_n} \arrow[l] \arrow[dlll] \\
C & & &
\end{tikzcd}$$

to $\tau \circ (\lambda a)_0 \in C_n$. Note that since $\Delta^{\mathrm{op}}_{/C} \to \Delta^{\mathrm{op}}$ is a left fibration, this functor factors through Z'_C.

Define a "first vertex" functor $\Upsilon_C \colon Z_C \to C$ by precomposition with ι (using the isomorphism $\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}, C \times \Delta^{\mathrm{op}}) \cong C \times \Delta^{\mathrm{op}}$). We then have a factorization of the first vertex map as

$$\Delta^{\mathrm{op}}_{/C} \to Z'_C \to Z_C \xrightarrow{\Upsilon_C} C.$$

12.4 Proposition *The functors Υ_C and $\Upsilon'_C = (\Upsilon_C)|_{Z'_C}$ are final.*

Proof We first prove that Υ_C is final by verifying the hypotheses of [9, Theorem 4.1.3.1]. Let $c \in C$. The map $Z_C \to C$ is functorial in C, so we have a map $Z_{C_{c/}} \to Z_C \times_C C_{c/}$. We claim that this map is a trivial Kan fibration. Unwinding the definitions, this amounts to showing that for every cofibration $A \to B$ of simplicial sets over Δ^{op}, we can solve the lifting problem

$$\begin{tikzcd}
B \cup_A A \times_{\Delta^{\mathrm{op}}} W \arrow[r] \arrow[d] & C_{c/} \arrow[d] \\
B \times_{\Delta^{\mathrm{op}}} W \arrow[r] \arrow[ur, dotted] & C
\end{tikzcd}$$

where, since the class of left anodyne morphisms is right cancellative, we may suppose $A = \varnothing$. It thus suffices to prove that $\lambda_B = B \times_{\Delta^{\mathrm{op}}} \lambda \colon B \to B \times_{\Delta^{\mathrm{op}}} W$ is left anodyne for any map of simplicial sets $B \to \Delta^{\mathrm{op}}$. Observe that even though λ is not a cartesian section, it is a left adjoint relative to Δ^{op} to ξ by [11, Proposition 7.3.2.6] and the uniqueness of adjoints, since on the fibers it restricts to the adjunction $\{0\} \rightleftarrows \Delta^n$.

Consequently, for any ∞–category B and functor $B \to \Delta^{\mathrm{op}}$, by [11, Proposition 7.3.2.5] λ_B is a left adjoint, hence left anodyne. From this, we deduce the general case by using the characterization in [9, Proposition 4.1.2.1] of the left anodyne maps $X \to Y$ as the trivial cofibrations in $s\mathbf{Set}_{/Y}$ equipped with the covariant model structure. Indeed, arguing as in the proof of Proposition 12.2, by induction on the nondegenerate simplices of B we reduce to the known case $B = \Delta^n$.

We next prove that Z_C is weakly contractible if C is, which will conclude the proof for Υ_C. For this, another application of (the opposite of) [11, Proposition 7.3.2.6] shows that the Δ^{op}–functor $C \times \Delta^{\mathrm{op}} \to Z_C$ defined by precomposition by ξ is a left adjoint relative to Δ^{op} to the functor $(\Upsilon_C, \mathrm{id}_{\Delta^{\mathrm{op}}})$, because it restricts to the adjunction $\iota\colon C \rightleftarrows \mathrm{Fun}(\Delta^n, C) : \mathrm{ev}_0$ on the fibers. Hence, $|Z_C| \simeq |C \times \Delta^{\mathrm{op}}| \simeq |C|$, and the latter is contractible by hypothesis.

We employ the same strategy to show that Υ'_C is final. Since $C_{c/} \to C$ is conservative, the trivial Kan fibration above restricts to yield a trivial Kan fibration $Z'_{C_{c/}} \to Z'_C \times_C C_{c/}$. Thus it suffices to show that Z'_C is weakly contractible if C is. By (the opposite of) [5, Proposition 7.3], the cocartesian fibration $Z'_C \to \Delta^{\mathrm{op}}$ is classified by the functor

$$\Delta^{\mathrm{op}} \xrightarrow{i^{\mathrm{op}}} \mathbf{Cat}_\infty \xrightarrow{\mathrm{Map}(-,C)} \mathbf{Spc}.$$

Let R denote the right adjoint to the colimit-preserving functor

$$L\colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Spc}) \to \mathbf{Cat}_\infty$$

left Kan extended from the inclusion $i\colon \Delta \subset \mathbf{Cat}_\infty$; R sends an ∞–category to its corresponding complete Segal space. Then $R(C) \simeq \mathrm{Map}(-, C) \circ i^{\mathrm{op}}$. For any $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Spc})$, we have $\operatorname{colim} X \simeq |L(X_\bullet)|$, hence

$$\operatorname{colim} R(C) \simeq |(L \circ R)(C)| \simeq |C|,$$

where $L \circ R \simeq \mathrm{id}$ by [10, Corollary 4.3.16]. By [9, Corollary 3.3.4.6],

$$|Z'_C| \simeq \operatorname{colim} R(C),$$

so we conclude that $|Z'_C|$ is contractible. $\square$

The following corollary was previously proven by Mazel-Gee in [14].

12.5 Corollary (Bousfield–Kan formula, "simplicial" variant) *Suppose that C admits colimits indexed by spaces. Then for any ∞–category K and functor $p\colon K \to C$, the colimit of p exists if and only if the geometric realization*

$$\Bigl|\ \operatorname*{colim}_{x \in \mathrm{Map}(\Delta^0, K)} p(x) \Longleftarrow \operatorname*{colim}_{\alpha \in \mathrm{Map}(\Delta^1, K)} p(\alpha(0)) \Lleftarrow \operatorname*{colim}_{\sigma \in \mathrm{Map}(\Delta^2, K)} p(\sigma(0)) \cdots \Bigr|$$

exists, in which case the colimit of p is computed by the geometric realization.

Proof Using Proposition 12.4, we may repeat the proof of Corollary 12.3, now using the span

$$\Delta^{\mathrm{op}} \leftarrow Z'_K \xrightarrow{\Upsilon'_K} K. \qquad \square$$

We now proceed to relativize the above picture, starting with the map Υ_C. Let $C \to S$ be an S–category. Define the map

$$\Upsilon_{C,S}\colon \widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}} \times S/S}(W \times S, \Delta^{\mathrm{op}} \times C) \to C$$

to be the composition of the map to $\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}} \times S/S}(\Delta^{\mathrm{op}} \times S, \Delta^{\mathrm{op}} \times C)$ given by precomposition by $\lambda \times \mathrm{id}_S$, together with the equivalence of Lemma 9.5 of this to $\Delta^{\mathrm{op}} \times C$ and the projection to C. Define $\Upsilon'_{C,S}$ to be the restriction of $\Upsilon_{C,S}$ to the maximal subleft fibration (with respect to $\Delta^{\mathrm{op}} \times S$).

12.6 Theorem *The S–functors $\Upsilon_{C,S}$ and $\Upsilon'_{C,S}$ are S–final.*

Proof For every object $s \in S$, we have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}} \times S/S}(W \times S, \Delta^{\mathrm{op}} \times C)_s \arrow[r, "(\lambda \times \mathrm{id}_S)^*_s"] \arrow[d, "\simeq"] \arrow[rr, bend left, "(\Upsilon_{C,S})_s"] & \widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}} \times S/S}(\Delta^{\mathrm{op}} \times S, \Delta^{\mathrm{op}} \times C)_s \arrow[r] \arrow[d, "\simeq"] & C_s \arrow[d, equal] \\
\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}}}(W, \Delta^{\mathrm{op}} \times C_s) \arrow[r, "\lambda^*"] \arrow[rr, bend right, "\Upsilon_{C_s}"'] & \widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}}}(\Delta^{\mathrm{op}}, \Delta^{\mathrm{op}} \times C_s) \cong \Delta^{\mathrm{op}} \times C_s \arrow[r, "\mathrm{pr}_{C_s}"] & C_s
\end{tikzcd}$$

where the left two vertical maps are given by the natural categorical equivalences of Lemma 9.6; the only point to note is that the equivalences of Lemmas 9.5 and 9.6 coincide when the first variable is trivial. By Proposition 12.4, Υ_{C_s} is final, so $(\Upsilon_{C,S})_s$ is final. By the S–cofinality Theorem 6.7, $\Upsilon_{C,S}$ is S–final. A similar argument shows that $\Upsilon'_{C,S}$ is S–final. $\square$

The process of relativizing υ_C is considerably more involved. We begin with some preliminaries on the relative nerve construction. Let J be a category.

12.7 Lemma *The adjunctions*

$$\mathfrak{F}_J : s\mathbf{Set}_{/N(J)} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}) : N_J, \quad \mathfrak{F}^+_J : s\mathbf{Set}^+_{/N(J)} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+) : N^+_J$$

of [9, Section 3.2.5] *are simplicial.*

Proof Let $\underline{K}\colon J \to s\mathbf{Set}$ denote the constant functor at a simplicial set K. We have an obvious map $\chi_K\colon N(J) \times K \to N_J(\underline{K})$ natural in K and hence a map

$$(\eta_X, \chi_K \circ \mathrm{pr})\colon X \times K \to N_J(\mathfrak{F}_J X \times \underline{K}) \cong N_J \mathfrak{F}_J X \times N_J(\underline{K})$$

natural in X and K. We want to show the adjoint

$$\theta_{X,K}\colon \mathfrak{F}_J(X \times K) \to \mathfrak{F}_J(X) \times \underline{K}$$

is an isomorphism. Both sides preserve colimits separately in each variable, so we may suppose $X = \Delta^n \to J$ and $K = \Delta^m$. By [9, Example 3.2.5.6], $\mathfrak{F}_I(I)(-) \cong N(I_{/-})$, and by [9, Remark 3.2.5.8], for any functor $f : I \to J$, the square

$$\begin{tikzcd}
s\mathbf{Set}_{/N(I)} \arrow[r, "f_!"] \arrow[d, "\mathfrak{F}_I"] & s\mathbf{Set}_{/N(J)} \arrow[d, "\mathfrak{F}_J"] \\
\mathrm{Fun}(I, s\mathbf{Set}) \arrow[r, "f_!"] & \mathrm{Fun}(J, s\mathbf{Set})
\end{tikzcd}$$

commutes. Letting $I = \Delta^n \times \Delta^m$ and $f : I \to J$ be the structure map, we have

$$\mathfrak{F}_I(\Delta^n \times \Delta^m)(k,l) \cong (\Delta^n)_{/k} \times (\Delta^m)_{/l} \cong \Delta^k \times \Delta^l.$$

Factoring f as $\Delta^n \times \Delta^m \xrightarrow{g} \Delta^n \xrightarrow{h} J$, we then have

$$g_!\mathfrak{F}_I(\Delta^n \times \Delta^m)(k) \cong \Delta^i \times \Delta^m.$$

Let $G = g_!\mathfrak{F}_I(\Delta^n \times \Delta^m)$, so that $\mathfrak{F}_J(\Delta^n \times \Delta^m)(j) \cong (h_!G)(j)$. Then

$$(h_!G)(j) \cong \operatorname*{colim}_{\Delta^n \times_J J_{/j}} ((k, h(k) \to j) \mapsto \Delta^k) \times \Delta^m \cong \mathfrak{F}_J(\Delta^n)(j) \times \Delta^m$$

and one can verify that $\theta_{X,K}$ implements this isomorphism. For $\mathfrak{F}^+_J \dashv N^+_J$, recall that the simplicial tensor $s\mathbf{Set} \times s\mathbf{Set}^+ \to s\mathbf{Set}^+$ is given by $(K, X) \mapsto K^\sharp \times X$. Consequently, in the above argument we may simply replace Δ^m by $(\Delta^m)^\sharp$ to conclude. $\square$

Since $N_J^+(\underline{S}^\sharp) = N(J) \times S^\sharp$, the adjunction $\mathfrak{F}_J^+ \dashv N_J^+$ lifts to an adjunction

$$\mathfrak{F}_{J,S}^+ : s\mathbf{Set}^+_{/N(J)\times S} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) : N_{J,S}^+$$

between the overcategories. Moreover, for any functor $f : T \to S$, the square

$$\begin{tikzcd} \mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) \arrow[r, "N_{J,S}^+"] \arrow[d, "f^*"] & s\mathbf{Set}^+_{/N(J)\times S} \arrow[d, "(\mathrm{id}\times f)^*"] \\ \mathrm{Fun}(J, s\mathbf{Set}^+_{/T}) \arrow[r, "N_{J,T}^+"] & s\mathbf{Set}^+_{/N(J)\times T} \end{tikzcd}$$

commutes.

12.8 Proposition *Equip $s\mathbf{Set}^+_{/N(J)\times S}$ with the cocartesian model structure and $\mathrm{Fun}(J, s\mathbf{Set}^+_{/S})$ with the projective model structure, where $s\mathbf{Set}^+_{/S}$ has the cocartesian model structure. Then the adjunction*

$$\mathfrak{F}_{J,S}^+ : s\mathbf{Set}^+_{/N(J)\times S} \rightleftarrows \mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) : N_{J,S}^+$$

is a Quillen equivalence.

Proof We first prove that the adjunction is Quillen. Because this is a simplicial adjunction between left proper simplicial model categories, it suffices to show that $\mathfrak{F}_{J,S}^+$ preserves cofibrations and $N_{J,S}^+$ preserves fibrant objects. Observe that the slice model structure on

$$s\mathbf{Set}^+_{/N(J)\times S} \cong (s\mathbf{Set}^+_{/N(J)})_{/(N(J)\times S)^\sharp}$$

is a localization of the cocartesian model structure. Similarly, the slice model structure on

$$\mathrm{Fun}(J, s\mathbf{Set}^+_{/S}) \cong \mathrm{Fun}(J, s\mathbf{Set}^+)_{/\underline{S}^\sharp}$$

is a localization of the projective model structure, since the trivial fibrations for the two model structures coincide and postcomposition by $\pi_! : s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+$ gives a Quillen left adjoint between the projective model structures. Since the lift of a Quillen adjunction $L : M \rightleftarrows N : R$ to the adjunction $\widetilde{L} : M_{/R(x)} \rightleftarrows N_{/x} : \widetilde{R}$ is Quillen for the slice model structures, we deduce that $\mathfrak{F}_{J,S}^+$ preserves cofibrations.

Now suppose $F : J \to s\mathbf{Set}^+_{/S}$ is fibrant. Since S is an ∞–category, $F \to \underline{S}$ is a fibration in $\mathrm{Fun}(J, s\mathbf{Set})$. Hence $N_{J,S}(F) \to N(J) \times S$ is a categorical fibration. We verify that it is a cocartesian fibration (with every marked edge cocartesian) by solving the lifting problem (for $n \geq 1$)

$$\begin{tikzcd} {}^\natural\Lambda^n_0 \arrow[r] \arrow[d] & N^+_{J,S}(F) \arrow[d] \\ {}^\natural\Delta^n \arrow[r, "(j_\bullet, s_\bullet)"'] \arrow[ur, dotted] & (N(J) \times S)^\sharp \end{tikzcd}$$

Unwinding the definitions, this amounts to solving the lifting problem

$$\begin{tikzcd} {}^\natural\Lambda^n_0 \arrow[r] \arrow[d] & F(j_n) \arrow[d] \\ {}^\natural\Delta^n \arrow[r, "s_\bullet"'] \arrow[ur, dotted] & S^\sharp \end{tikzcd}$$

and the dotted lift exists because $F(j_n)$ is cocartesian over S with the cocartesian edges marked. Finally, it is easy to see that marked edges compose and are stable under equivalence. We conclude that $N^+_{J,S}(F)$ is fibrant in $s\mathbf{Set}^+_{/N(J)\times S}$.

To prove that the Quillen adjunction is a Quillen equivalence, we will show that the induced adjunction of ∞–categories

$$\mathfrak{F}'^+_{J,S} : N((s\mathbf{Set}^+_{/N(J)\times S})^\circ) \rightleftarrows N(\mathrm{Fun}(J, s\mathbf{Set}^+_{/S})^\circ) : N'^+_{J,S}$$

is an adjoint equivalence, where $N'^+_{J,S}$ is the simplicial nerve of $N^+_{J,S}$ and $\mathfrak{F}'^+_{J,S}$ is any left adjoint to $N'^+_{J,S}$. We first check that $N'^+_{J,S}$ is conservative. Indeed, for this we may work in the model category: for a natural transformation $\alpha\colon F \to G$ in $\mathrm{Fun}(J, s\mathbf{Set}^+_{/S})$, $N^+_{J,S}(F) \to N^+_{J,S}(G)$ on fibers is given by $F(j)_s \to G(j)_s$; hence if F and G are fibrant and $N^+_{J,S}(\alpha)$ is an equivalence then α is as well. It now suffices to show that the unit transformation $\eta\colon \mathrm{id} \to N'^+_{J,S}\mathfrak{F}'^+_{J,S}$ is an equivalence. We have the known equivalence $N((s\mathbf{Set}^+_{/N(J)\times S})^\circ) \simeq \mathrm{Fun}(N(J)\times S, \mathbf{Cat}_\infty)$ so it further suffices to check that the map

$$(\mathrm{id} \times i_s)^* \to (\mathrm{id} \times i_s)^* N'^+_{J,S}\mathfrak{F}'^+_{J,S} \simeq N'^+_J i_s^* \mathfrak{F}'^+_{J,S}$$

is an equivalence for all $s \in S$, where $i_s\colon \{s\} \to S$ the inclusion. Equivalently, since $\mathfrak{F}^+_J \dashv N^+_J$ is a Quillen equivalence by [9, Proposition 3.2.5.18], we must show that the adjoint map

$$\mathfrak{F}'^+_J i_s^* \to (\mathrm{id} \times i_s)^* \mathfrak{F}'^+_{J,S}$$

is an equivalence. This statement is in turn equivalent to the adjoint map

$$\theta\colon N'^+_{J,S}(i_s)_* \to (\mathrm{id} \times i_s)_* N'^+_J$$

being an equivalence. Recall that for a functor $f\colon T \to S$,

$$f_*\colon \mathrm{Fun}(T, \mathbf{Cat}_\infty) \to \mathrm{Fun}(S, \mathbf{Cat}_\infty)$$

is induced by $\pi_* \rho^*\colon s\mathbf{Set}^+_{/T} \to s\mathbf{Set}^+_{/S}$ for the span

$$S^\sharp \xleftarrow{\pi} (\mathbb{O}(S) \times_S T)^\sharp \xrightarrow{\rho} T^\sharp$$

with π given by evaluation at 0 and ρ projection to T. Moreover, for a functor $\mathrm{id} \times f\colon U \times T \to U \times S$, we may elect to use the span

$$(U \times S)^\sharp \xleftarrow{\mathrm{id}\times\pi} (U \times \mathbb{O}(S) \times_S T)^\sharp \xrightarrow{\mathrm{id}\times\rho} (U \times T)^\sharp$$

to model $(\mathrm{id} \times f)_*$. Letting $f = i_s$, we see that θ is induced by the map

$$N^+_{J,S} \pi_* \rho^* \to (\mathrm{id} \times \pi)_* N^+_{J,S^{s/}} \rho^* \cong (\mathrm{id} \times \pi)_* (\mathrm{id} \times \rho)^* N^+_J,$$

where the first map is adjoint to the isomorphism $(\mathrm{id} \times \pi)^* N^+_{J,S} \cong N^+_{J,S^{s/}} \pi^*$. Direct computation reveals that this map is an equivalence on fibrant $F\colon J \to s\mathbf{Set}^+$. $\square$

We now return to the situation of interest. Let C be an S–category with structure map $\pi\colon C \to S$. We first extend our existing notation $\underline{x}$ for objects $x \in C$.

12.9 Notation For an n–simplex σ of C, define

$$\underline{\sigma} = \{\sigma\} \times_{\mathrm{Fun}(\Delta^n \times \{0\}, C)} \mathrm{Fun}((\Delta^n)^\flat \times (\Delta^1)^\sharp, {}_\natural C) \times_{\mathrm{Fun}(\Delta^n \times \{1\}, S)} S.$$

12.10 Lemma *There exists a map $b_\sigma\colon \underline{\sigma} \to \{\pi\sigma(n)\} \times_S \mathbb{O}(S) = S^{\pi\sigma(n)/}$ which is a trivial Kan fibration.*

Proof First define a map $b'_\sigma\colon \underline{\sigma} \to \underline{\pi\sigma}$ to be the pullback of the map

$$(e_0, \mathbb{O}(\pi))_*\colon \mathrm{Fun}(\Delta^n, \mathbb{O}^{\mathrm{cocart}}(C)) \to C^{\Delta^n} \times_{S^{\Delta^n}} \mathrm{Fun}(\Delta^n, \mathbb{O}(S))$$

over $\{\sigma\}$ and S. Since $(e_0, \mathbb{O}(\pi))$ is a trivial Kan fibration, so is b'_σ. Next, let K be the pushout $\Delta^n \times \{0\} \cup_{\{n\}\times\{0\}} \{n\} \times \Delta^1$. We claim that the map

$$\mathrm{Fun}(\Delta^n, \mathbb{O}(S)) \times_{S^{\Delta^n}} S \to \mathrm{Fun}(K, S)$$

induced by $K \subset \Delta^n \times \Delta^1$ is a trivial Kan fibration. For a monomorphism $A \to B$, we need to solve the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, \mathbb{O}(S)) \times_{S^{\Delta^n}} S \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(K, S) \end{tikzcd}$$

which transposes to

$$\begin{tikzcd} A \times \Delta^n \cup_{A\times\{n\}} B \times \{n\} \arrow[r] \arrow[d] & \mathcal{O}(S) \arrow[d, "\mathrm{ev}_0"] \\ B \times \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

and the left-hand map is right anodyne by [9, Corollary 2.1.2.7]; hence the dotted lift exists as ev_0 is a cartesian fibration. Now define b''_σ to be the pullback

$$\underline{\pi\sigma} = \{\pi\sigma\} \times_{S^{\Delta^n}} \operatorname{Fun}(\Delta^n, \mathcal{O}(S)) \times_{S^{\Delta^n}} S \to \{\pi\sigma\} \times_{S^{\Delta^n}} \operatorname{Fun}(K, S) \cong S^{\pi\sigma(n)/}.$$

This is also a trivial Kan fibration. Finally, let $b_\sigma = b''_\sigma \circ b'_\sigma$. □

We will regard $\underline{\sigma}$ as an $S^{\pi\sigma(n)/}$ or S–category via b_σ. We also have a target map $\underline{\sigma} \to C^{\Delta^n}$ induced by $\Delta^n \times \{1\} \subset \Delta^n \times \Delta^1$. This covers the target map $S^{\pi\sigma(n)/} \to S$ and is an S–functor.

Define a functor $F_C : \Delta^{\mathrm{op}} \to s\mathbf{Set}^+_{/S}$ on objects $[n]$ by

$$F_C([n]) = \coprod_{\sigma \in C_n} \underline{\sigma}^\sharp$$

and on morphisms $\alpha: [m] \to [n]$ by the map $\underline{\sigma} \to \underline{\sigma\alpha}$ induced by precomposition by $\alpha: \Delta^m \to \Delta^n$.

12.11 Remark The map $\underline{\sigma} \to \underline{\sigma(n)}$ is compatible with the maps b_σ and $b_{\sigma(n)}$ of Lemma 12.10, hence is a categorical equivalence (in fact, a trivial Kan fibration). Consequently, given a morphism $f: x \to y$ in C, by choosing an inverse to $\underline{f} \xrightarrow{\simeq} \underline{y}$ we obtain a map $f^*: \underline{y} \to \underline{x}$, unique up to contractible choice. Moreover, if f lies over an equivalence, then $\underline{f} \to \underline{x}$ is a trivial Kan fibration, so we also obtain a map $f_!: \underline{x} \to \underline{y}$.

In order to define the S–first vertex map $N^+_{\Delta^{\mathrm{op}},S}(F_C) \to C$, we need to introduce a few preliminary constructions. Let $A_n \subset \mathcal{O}(\Delta^n)$ be the subsimplicial set where a k–simplex $x_0 y_0 \to \cdots \to x_k y_k$ is in A_n if and only if $x_k \le y_0$. For the reader's aid we draw a picture of the inclusion $A_n \subset \mathbb{O}(\Delta^n)$ for $n = 2$, where dashed edges are not in A_2:

$$\begin{tikzcd}
00 \arrow[d] \arrow[dr, dotted] \arrow[ddr, dotted] \arrow[ddrr, dotted, bend left] \arrow[dd, bend right] & & \\
01 \arrow[r] \arrow[d] \arrow[dr] \arrow[drr, dotted] & 11 \arrow[d] \arrow[dr, dotted] & \\
02 \arrow[r] \arrow[rr, bend right] & 12 \arrow[r] & 22
\end{tikzcd}$$

12.12 Lemma *The inclusion $A_n \to \mathbb{O}(\Delta^n)$ is inner anodyne.*

Proof In this proof we adopt the notation $[x_0 y_0, \dots, x_k y_k]$ for a k–simplex of $\mathbb{O}(\Delta^n)$. Let E be the collection of edges $[ab, xy]$ in $\mathbb{O}(\Delta^n)$ where $x > b$, and choose a total ordering $\leq$ on E such that if we have a factorization

$$\begin{tikzcd}
ab \arrow[r] \arrow[d] & xy \arrow[d] \\
a'b' \arrow[r] & x'y'
\end{tikzcd}$$

then $[a'b', x'y'] \leq [ab, xy]$. Index edges in E by $I = \{0, \dots, N\}$. Define simplicial subsets $A_{n,i}$ of $\mathbb{O}(\Delta^n)$ such that $A_{n,i}$ is obtained by expanding A_n to contain every k–simplex $[x_0 y_0, \dots, x_k y_k]$ with $[x_0 y_0, x_k y_k]$ in $E_{<i}$. We will show that each inclusion $A_{n,i} \to A_{n,i+1}$ is inner anodyne. We may divide the nondegenerate k–simplices $[x_0 y_0, x_1 y_1, \dots, x_k y_k]$ in $A_{n,i+1}$ but not in $A_{n,i}$ into six classes:

- $A1$ $x_1 y_1 \neq x_0(y_0 + 1)$ and $y_1 > y_0$.
- $A2$ $x_1 y_1 = x_0(y_0 + 1)$.
- $B1$ $x_1 y_1 = (x_0 + 1) y_0$, $y_2 > y_0$, and $x_2 y_2 \neq (x_0 + 1)(y_0 + 1)$.
- $B2$ $x_1 y_1 = (x_0 + 1) y_0$ and $x_2 y_2 = (x_0 + 1)(y_0 + 1)$.
- $C1$ $x_1 y_1 \neq (x_0 + 1) y_0$ and $y_1 = y_0$.
- $C2$ $x_1 y_1 = (x_0 + 1) y_0$ and $y_2 = y_0$.

We have bijections between classes of form 1 and classes of form 2 given by

- A $[x_0 y_0, x_1 y_1, \dots, x_k y_k] \mapsto [x_0 y_0, x_0(y_0 + 1), x_1 y_1, \dots, x_k y_k]$,
- B $[x_0 y_0, x_0 + 1 y_1, x_2 y_2, \dots, x_k y_k]$
  $\mapsto [x_0 y_0, (x_0 + 1) y_0, (x_0 + 1)(y_0 + 1), x_2 y_2, \dots, x_k y_k]$,
- C $[x_0 y_0, x_1 y_1, \dots, x_k y_k] \mapsto [x_0 y_0, (x_0 + 1) y_0, x_1 y_1, \dots, x_k y_k]$.

Moreover, this identifies simplices in a class of form 1 as inner faces of simplices in the corresponding class of form 2. Let P be the collection of pairs $\tau \subset \tau'$ of nondegenerate $k-1$ and k–simplices matched by this bijection. Choose a total ordering on P where pairs are ordered first by the dimension of the smaller simplex, and then by $A < B < C$, and then randomly. Let $J = \{0, \dots, M\}$ be the indexing set for P. We define a sequence of inner anodyne maps

$$A_{n,i} = A_{n,i,0} \to A_{n,i,1} \to \cdots \to A_{n,i,M+1} = A_{n,i+1}$$

such that $A_{n,i,j+1}$ is obtained from $A_{n,i,j}$ by attaching the j^{th} pair $\tau \subset \tau'$ along an inner horn. For this to be valid, we need the other faces of τ' to already be in $A_{n,i,j}$. The ordering on E was chosen so that the outer faces of τ' are in $A_{n,i}$. The argument for the inner faces proceeds by cases:

- τ' is in class $A2$: The other inner faces are also in class $A2$ since they contain $x_0(y_0+1)$, hence were added at some earlier stage.
- τ' is in class $B2$: The other inner faces of

  $$[x_0y_0, (x_0+1)y_0, (x_0+1)(y_0+1), x_2y_2, \dots, x_ky_k]$$

  are all in class $B2$, except for $[x_0y_0, (x_0+1)(y_0+1), x_2y_2, \dots, x_ky_k]$, which is in class $A1$. Both of these were added at an earlier stage.
- τ' is in class $C2$: The other inner faces are in class $C2$ or $B1$ since they contain $(x_0+1)y_0$, hence were added at some earlier stage. □

Let $E_n \subset (A_n)_1 \subset \mathbb{O}(\Delta^n)_1$ be the subset of edges $x_0y_0 \to x_1y_1$ where $y_0 = y_1$. Define simplicial sets C' and C'' to be the pullbacks

$$\begin{tikzcd} C'_\bullet \arrow[r] \arrow[d] & \mathrm{Hom}((\mathbb{O}(\Delta^\bullet), E_\bullet), \natural C) \arrow[d] \\ \mathrm{Hom}(\Delta^\bullet, S) \arrow[r, "\mathrm{ev}_0^*"] & \mathrm{Hom}(\mathbb{O}(\Delta^\bullet), S) \end{tikzcd} \qquad \begin{tikzcd} C''_\bullet \arrow[r] \arrow[d] & \mathrm{Hom}((A_\bullet, E_\bullet), \natural C) \arrow[d] \\ \mathrm{Hom}(\Delta^\bullet, S) \arrow[r, "\mathrm{ev}_0^*"] & \mathrm{Hom}(A_\bullet, S) \end{tikzcd}$$

We now show that the map $C' \to C''$ induced by precomposition by $A_\bullet \to \mathbb{O}(\Delta^\bullet)$ is a trivial Kan fibration. Indeed, in order to solve the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & C' \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & C'' \end{tikzcd}$$

we must supply a lift

$$\begin{tikzcd}
A_n \cup_{\cup A_{n-1}} \left(\bigcup \mathbb{O}(\Delta^{n-1})\right) \arrow[r] \arrow[d] & C \arrow[d] \\
\mathbb{O}(\Delta^n) \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

and the left vertical map is a trivial cofibration by Lemma 12.12. Let $\sigma\colon C'' \to C'$ be any section. Also let $\delta\colon C' \to C$ be the map induced by precomposition by the identity section $\Delta^\bullet \to \mathbb{O}(\Delta^\bullet)$.

Define a map $\upsilon_{C,S}\colon N^+_{\Delta^{\mathrm{op}},S}(F_C) \to C$ over S as follows: the data of an n–simplex of $N^+_{\Delta^{\mathrm{op}},S}(F_C)$ consists of

- an n–simplex $\Delta^{a_0} \leftarrow \cdots \leftarrow \Delta^{a_n}$ in Δ^{op} (so we have maps $f_{ij}\colon \Delta^{a_j} \to \Delta^{a_i}$ for $i \le j$);
- an n–simplex $s_\bullet\colon \Delta^n \to S$;
- a choice of a_0–simplex $\sigma_0 \in C_{a_0}$;
- for $0 \le i \le n$, a map $\gamma_i\colon \Delta^i \to \underline{\sigma}_i$, where $\sigma_i = \sigma_0 \circ f_{0i}$

such that for all $0 \le i \le j \le n$, the diagram

$$\begin{tikzcd}
\Delta^i \arrow[r, "\gamma_i"] \arrow[d, "\{0,\dots,i\}\subset[j]"'] & \underline{\sigma}_i \arrow[d, "f_{ij}^*"] \\
\Delta^j \arrow[r, "\gamma_j"] \arrow[dr, "(s_\bullet)|_{\{0,\dots,j\}}"'] & \underline{\sigma}_j \arrow[d] \\
& S
\end{tikzcd}$$

commutes. Let $\overline{\gamma}_i\colon \Delta^i \times \Delta^{a_i} \times \Delta^1 \to C$ denote the adjoint map.

We now define a map $A_n \to C$ to be that uniquely specified by sending for all $0 \le k \le n$ the rectangle $\Delta^k \times \Delta^{n-k} \subset A_n$ given by $00 \mapsto 0k$ and $k(n-k) \mapsto kn$ to

$$\Delta^k \times \Delta^{n-k} \xrightarrow{\mathrm{id}\times(\lambda a)_k} \Delta^k \times \Delta^{a_k} \times \{1\} \xrightarrow{\overline{\gamma}_i|_{\{1\}}} C,$$

where the maps $(\lambda a)_k$ are obtained from the first vertex section of $W \to \Delta^{\mathrm{op}}$ restricted to $a_\bullet$ as before. One may check that the composite $A_n \to C \to S$ factors as

$$A_n \to \Delta^n \xrightarrow{s_\bullet} S,$$

so this defines a n–simplex of C''. This procedure is natural in $\Delta^n \in \Delta$, so yields a map $N^+_{\Delta^{\mathrm{op}},S}(F_C) \to C''$. Finally, postcomposition by $\delta \circ \sigma : C'' \to C$ define our desired map $\upsilon_{C,S}$. By Proposition 12.8, $N^+_{\Delta^{\mathrm{op}},S}(F_C) \xrightarrow{\pi'} S$ is an S–category with an edge π'–cocartesian if and only if it is degenerate when projected to Δ^{op}. These edges are evidently sent to π–cocartesian edges in C, so υ_C is an S–functor.

12.13 Theorem *The S–first vertex map $\upsilon_{C,S} : N^+_{\Delta^{\mathrm{op}},S}(F_C) \to C$ is fiberwise a weak homotopy equivalence. Moreover, $\upsilon_{C,S}$ is S–final if either $C \to S$ is a left fibration, or S is equivalent to the nerve of a 1–category.*

Proof Let $t \in S$ be an object and $i_t : \{t\} \to S$ the inclusion. Then

$$N^+_{\Delta^{\mathrm{op}},S}(F_C)_t \cong N^+_{\Delta^{\mathrm{op}}}(i_t^* F_C).$$

We have a map

$$N^+_{\Delta^{\mathrm{op}}}(i_t^* F_C) \to \Delta^{\mathrm{op}}_{/C} \cong N^+_{\Delta^{\mathrm{op}}}(C_\bullet)$$

of left fibrations over Δ^{op} induced by the natural transformation $i_t^* F_C \to C_\bullet$ which collapses each $\underline{\sigma} \times_S \{t\}$ to a point. Moreover, this natural transformation is objectwise a Kan fibration, so the map itself is a left fibration. Also define a map

$$N^+_{\Delta^{\mathrm{op}}}(i_t^* F_C) \to (S^{/t})^{\mathrm{op}}$$

as follows: in the above notation, the γ_0 map in the data of an n–simplex

$$(a_\bullet, \gamma_i : \Delta^i \to \underline{\sigma}_i \times_S \{t\})$$

yields a map $\pi\gamma_0 : \Delta^{a_0} \to \mathbb{O}(S) \times_S \{t\} = S^{/t}$, and we send the n–simplex to

$$\Delta^n \xrightarrow{(\lambda a^{\mathrm{rev}})_0} (\Delta^{a_0})^{\mathrm{op}} \xrightarrow{(\pi\gamma_0)^{\mathrm{op}}} (S^{/t})^{\mathrm{op}},$$

where $a^{\mathrm{rev}}_\bullet$ is $(\Delta^{a_0})^{\mathrm{op}} \leftarrow \cdots \leftarrow (\Delta^{a_n})^{\mathrm{op}}$. Using these maps we obtain a commutative square

$$\begin{tikzcd}
N^+_{\Delta^{\mathrm{op}}}(i_t^* F_C) \arrow[r] \arrow[d] & C^{\mathrm{op}} \times_{S^{\mathrm{op}}} (S^{/t})^{\mathrm{op}} \arrow[d] \\
\Delta^{\mathrm{op}}_{/C} \arrow[r, "\mu_C^{\mathrm{op}}"] & C^{\mathrm{op}}
\end{tikzcd}$$

We claim that the map

$$\theta_{C,t} : N^+_{\Delta^{\mathrm{op}}}(i_t^* F_C) \to (\Delta^{\mathrm{op}}_{/C}) \times_{C^{\mathrm{op}}} (C \times_S S^{/t})^{\mathrm{op}}$$

is a categorical equivalence. Since $\theta_{C,t}$ is a map of left fibrations over $\Delta^{\mathrm{op}}_{/C}$, it suffices to check that for every object $\sigma \in \Delta^{\mathrm{op}}_{/C}$, the map on fibers

$$\underline{\sigma} \times_S \{t\} \to (S^{\mathrm{op}})^{t/} \times_{S^{\mathrm{op}}} \{\pi\sigma(n)\} \simeq \{\pi\sigma(n)\} \times_S S^{/t}$$

is a homotopy equivalence. But this is the pullback of the trivial Kan fibration of Lemma 12.10 over $\{t\}$.

We next define a map $N^{+}_{\Delta^{\mathrm{op}}}(i_t^* F_C) \to S^{/t}$ by sending $(a_\bullet, \gamma_i)$ to $\pi\gamma_0 \circ (\lambda a)_0$. Then the outer rectangle

$$\begin{tikzcd} N^{+}_{\Delta^{\mathrm{op}}}(i_t^* F_C) \arrow[r, dotted, "\upsilon'_{C,t}"'] \arrow[rr, bend left] \arrow[d] & C \times_S S^{/t} \arrow[r] \arrow[d] & S^{/t} \arrow[d] \\ \Delta^{\mathrm{op}}_{/C} \arrow[r, "\upsilon_C"] & C \arrow[r, "\pi"] & S \end{tikzcd}$$

commutes so we obtain the dotted map $\upsilon'_{C,t}$.

Next, we choose a section P of the trivial Kan fibration $\mathbb{O}^{\mathrm{cocart}}(C) \to C \times_S \mathbb{O}(S)$ which restricts to the identity section on C. P restricts to a map

$$P_t : C \times_S S^{/t} \to \mathbb{O}^{\mathrm{cocart}}(C) \times_S \{t\},$$

and it is tedious but straightforward to construct a homotopy between the composition $(\mathrm{ev}_1\, P_t) \circ \upsilon'_{C,t}$ and $(\upsilon_{C,S})_t$. Finally, we define a map $\upsilon''_{C,t} : \Delta^{\mathrm{op}}_{/C \times_S S^{/t}} \to N^{+}_{\Delta^{\mathrm{op}}}(i_t^* F_C)$ as follows: given an n–simplex

$$\begin{tikzcd} \Delta^{a_0} \arrow[d, "\tau_0"] & \cdots \arrow[l] & \Delta^{a_n} \arrow[l] \arrow[dll, "\tau_n"] \\ C \times_S S^{/t} & & \end{tikzcd}$$

let $\sigma_i = \mathrm{pr}_C \circ \tau_i$, and define $\gamma_i : \Delta^i \to \underline{\sigma}_i \times_S \{t\}$ as the composition of the projection to Δ^0 and the adjoint of the map $P_t \circ \tau_i$. Then $(a_\bullet, \gamma_i)$ assembles to yield an n–simplex of $N^{+}_{\Delta^{\mathrm{op}}}(i_t^* F_C)$.

Unwinding the definitions of the various maps, we identify the composition $\upsilon'_{C,t} \circ \upsilon''_{C,t}$ as given by $\upsilon_{C \times_S S^{/t}}$, and the composition $\theta_{C,t} \circ \upsilon''_{C,t}$ as given by the map $\Delta^{\mathrm{op}}_{/\mathrm{pr}_C}$ to the factor $\Delta^{\mathrm{op}}_{/C}$ and the map $(\mu_{C \times_S S^{/t}})^{\mathrm{op}}$ to the factor $(C \times_S S^{/t})^{\mathrm{op}}$. By Proposition 12.2 and the fact that final maps pull back along cocartesian fibrations, we deduce that in

$$\begin{tikzcd} \Delta^{\mathrm{op}}_{/C \times_S S^{/t}} \arrow[r] \arrow[rr, bend left] & \Delta^{\mathrm{op}}_{/C} \times_{C^{\mathrm{op}}} (C \times_S S^{/t})^{\mathrm{op}} \arrow[r] & (C \times_S S^{/t})^{\mathrm{op}} \end{tikzcd}$$

the long composition and the second map are both final. Consequently, $\theta_{C,t} \circ \upsilon''_{C,t}$ is a weak homotopy equivalence. Moreover, if S is equivalent to the nerve of a 1–category then $\theta_{C,t} \circ \upsilon''_{C,t}$ is a categorical equivalence, as may be verified by checking that the map is a fiberwise equivalence over $\Delta^{\mathrm{op}}_{/C}$. Since $\theta_{C,t}$ is a categorical equivalence, $\upsilon''_{C,t}$ is then a weak homotopy equivalence (resp. a categorical equivalence). Since $\upsilon_{C \times_S S^{/t}}$ is final, $\upsilon'_{C,t}$ is then a weak homotopy equivalence (resp. final).

For the last step, let $j_t \colon C_t \to C \times_S S^{/t}$ denote the inclusion. As the inclusion of the fiber over a final object into a cocartesian fibration, j_t is final. $(\mathrm{ev}_1\ P_t) \circ j_t = \mathrm{id}_{C_t}$, so by right cancellativity of final maps, $\mathrm{ev}_1\ P_t$ is final. We conclude that $(\upsilon_{C,S})_t$ is a weak homotopy equivalence (resp. final). In addition, if $C \to S$ is a left fibration, $(\upsilon_{C,S})_t$ has target a Kan complex, so is final by [11, Lemma 2.3.4.6]. Invoking the S–cofinality Theorem 6.7, we conclude the proof. □

12.14 Remark The above proof that the S–first vertex map $\upsilon_{C,S}$ is final in special cases hinges upon the finality of the map $\theta_{C,t} \circ \upsilon''_{C,t}$. We believe, but are currently unable to prove, that this map is always final.

We conclude this section with our main application to decomposing S–colimits.

12.15 Corollary *Suppose that S^{op} admits multipullbacks. Then C is S–cocomplete if and only C admits all S–coproducts and geometric realizations.*

Proof We prove the if direction, the only if direction being obvious. Let K be an $S^{s/}$–category and $p \colon K \to C_{\underline{s}}$ an $S^{s/}$–diagram. First suppose that $K \to S^{s/}$ is a left fibration. Consider the diagram

$$\begin{tikzcd} N^+_{\Delta^{\mathrm{op}}, S^{s/}}(F_K) \arrow[r, "\upsilon_{K,S^{s/}}"] \arrow[d, "\rho"] & K \arrow[r, "p"] & C_{\underline{s}} \\ \Delta^{\mathrm{op}} \times S^{s/} & & \end{tikzcd}$$

By Theorem 12.13, the $S^{s/}$–colimit of p is equivalent to that of $p \circ \upsilon_{K,S^{s/}}$. Since ρ is S–cocartesian, by Theorem 9.15 the $S^{s/}$–left Kan extension of $p \circ \upsilon_{K,S^{s/}}$ along ρ exists provided that for all $n \in \Delta^{\mathrm{op}}$ and $f : s \to t$, the $S^{t/}$–colimit exists for $(p \circ \upsilon_{K,S^{s/}})_{\underline{(n,f)}}$. To understand the domain of this map, note that because the pullback of ρ along $f^* \colon \Delta^{\mathrm{op}} \times S^{t/} \to \Delta^{\mathrm{op}} \times S^{s/}$ is given by $N^+_{\Delta^{\mathrm{op}}, S^{t/}}(f^* F_K)$, the assumption that S^{op} admits multipullbacks ensures that the $\underline{(n, f)}$–fibers of ρ decompose as coproducts of representable left fibrations. Therefore, these colimits exist since C is assumed to admit S–coproducts. Now by transitivity of left $S^{s/}$–Kan extensions, the $S^{s/}$–colimit of $p \circ \upsilon_{K,S^{s/}}$ is equivalent to that of $\rho_!(p \circ \upsilon_{K,S^{s/}})$, and this exists since C is assumed to admit geometric realizations.

Now suppose that $K \to S^{s/}$ is any cocartesian fibration. Consider the diagram

$$\begin{tikzcd} \iota\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{op}} \times S^{s/}}(W \times S^{s/}, \Delta^{\mathrm{op}} \times K) \arrow[r, "\Upsilon'_{K,S^{s/}}"] \arrow[d, "\rho'"] & K \arrow[r, "p"] & C_{\underline{s}} \\ \Delta^{\mathrm{op}} \times S^{s/} \end{tikzcd}$$

By Theorem 12.6, the $S^{s/}$–colimit of p is equivalent to that of $p \circ \Upsilon'_{K,S^{s/}}$. By Proposition 9.7, the $\underline{(n,f)}$–fiber of ρ' is equivalent to $\iota\underline{\mathrm{Fun}}_{S^{t/}}(\Delta^n \times S^{t/}, K \times_{S^{s/}} S^{t/})$, which in any case remains a left fibration. We just showed that for all $t \in S$, $C_{\underline{t}}$ admits $S^{t/}$–colimits indexed by left fibrations. We are thereby able to repeat the above proof in order to show that the $S^{s/}$–colimit of p exists. $\square$

Appendix Fiberwise fibrant replacement

In this appendix, we formulate a result (Proposition A.4) which will allow us to recognize a map as a cocartesian equivalence if it is a marked equivalence on the fibers. We begin by introducing a marked variant of Lurie's mapping simplex construction.

A.1 Definition Suppose we have a functor $\phi\colon [n] \to s\mathbf{Set}^+$, $A_0 \to \cdots \to A_n$. Define $M(\phi)$ to be the simplicial set which is the opposite of the mapping simplex construction of [9, Section 3.2.2], so that a m–simplex of $M(\phi)$ is given by the data of a map $\alpha\colon \Delta^m \to \Delta^n$ together with a map $\beta\colon \Delta^m \to A_{\alpha(0)}$. Endow $M(\phi)$ with a marking by declaring an edge $e = (\alpha, \beta)$ of $M(\phi)$ to be marked if and only if β is a marked edge of $A_{\alpha(0)}$. Note that if each A_i is given the degenerate marking, then the marking on $M(\phi)$ is that of [9, Notation 3.2.2.3].

A.2 Lemma *Suppose $\eta\colon \phi \to \psi$ is a natural transformation between functors $[n] \to s\mathbf{Set}^+$ such that for all $0 \le i \le n$, $\eta_i : A_i \to B_i$ is a cocartesian equivalence. Then $M(\eta)\colon M(\phi) \to M(\psi)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$.*

Proof Using the decomposition of $M(\phi)$ as the pushout

$$M(\phi') \cup_{A_0 \times \Delta^{n-1}} A_0 \times \Delta^n$$

for $\phi' \colon A_1 \to \cdots \to A_n$, this follows by an inductive argument in view of the left properness of $s\mathbf{Set}^+_{/\Delta^n}$. $\square$

A.3 Construction Let $X \to \Delta^n$ be a cocartesian fibration, let σ be a section of the trivial Kan fibration $\mathbb{O}^{\mathrm{cocart}}(X) \to X \times_{\Delta^n} \mathbb{O}(\Delta^n)$ which restricts to the identity section on X, and let $P = \mathrm{ev}_1 \circ \sigma$ be the corresponding choice of pushforward functor. For $0 \le i < n$, define $f_i \colon X_i \times \Delta^1 \to X$ by $P \circ (\mathrm{id}_{X_i} \times f_i')$ where $f_i' \colon \Delta^1 \to \mathbb{O}(\Delta^n)$ is the edge $(i = i) \to (i \to i+1)$, and let $\phi \colon X_0^{\sim} \to \cdots \to X_n^{\sim}$ be the sequence obtained from the $f_i \times \{1\}$. We will explain how to produce a map $M(\phi) \to X$ over Δ^n via an inductive procedure. Begin by defining the map $M(\phi)_n = X_n \to X_n$ to be the identity. Proceeding, observe that $M(\phi)$ is the pushout

$$\begin{tikzcd}
X_0 \times \Delta^{\{1,\dots,n\}} \arrow[r] \arrow[d, "\gamma"] & X_0 \times \Delta^n \arrow[d] \\
M(\phi') \arrow[r] & M(\phi)
\end{tikzcd}$$

with ϕ' the composable sequence $X_1 \to \cdots \to X_n$ and the map γ given by

$$X_0 \times \Delta^{n-1} \to X_1 \times \Delta^{n-1} \to M(\phi').$$

Given a map $g' \colon M(\phi') \to X$ over Δ^{n-1}, we have a commutative square

$$\begin{tikzcd}
X_0 \times \Delta^1 \cup_{X_0 \times \Delta^{\{1\}}} X_0 \times \Delta^{\{1,\dots,n\}} \arrow[r, "(f_0, g' \circ \gamma)"] \arrow[d] & X \arrow[d] \\
X_0 \times \Delta^n \arrow[r] \arrow[ur, dotted] & \Delta^n
\end{tikzcd}$$

and the left vertical map is inner anodyne by [9, Lemma 2.1.2.3] and [9, Corollary 2.3.2.4]. Thus a dotted lift exists and we may extend g' to $g \colon M(\phi) \to X$.

Note that g_i is the identity for all $0 \le i \le n$. Therefore, if we instead take the marking on $M(\phi)$ which arises from the degenerate marking on the X_i, then g is (the opposite of) a quasiequivalence in the terminology of [9, Definition 3.2.2.6], hence a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$ by [9, Proposition 3.2.2.14]. Now by Lemma A.2, g with the given marking is a cocartesian equivalence.

This construction of $M(\phi) \to X$ enjoys a convenient functoriality property: given a cofibration $F \colon X \to Y$ between cocartesian fibrations over Δ^n, we may first choose

σ_X as above, and then define σ_Y to be a lift in the diagram

$$\begin{tikzcd}
(X \times_{\Delta^n} \mathbb{O}(\Delta^n)) \cup_X Y \arrow[r, "{(F\circ\sigma_X, \iota)}"] \arrow[d] & \mathbb{O}^{\mathrm{cocart}}(Y) \arrow[d, "\sim"] \\
Y \times_{\Delta^n} \mathbb{O}(\Delta^n) \arrow[r, equal] \arrow[ur, dotted, "\sigma_Y"] & Y \times_{\Delta^n} \mathbb{O}(\Delta^n)
\end{tikzcd}$$

Consequently, we obtain compatible pushforward functors and a natural transformation $\eta\colon \phi_X \to \phi_Y$, which yields, by a similar argument, a commutative square

$$\begin{tikzcd}
M(\phi_X) \arrow[r, "M(\eta)"] \arrow[d] & M(\phi_Y) \arrow[d] \\
X \arrow[r, "F"] & Y
\end{tikzcd}$$

where the vertical maps are cocartesian equivalences in $s\mathbf{Set}^+_{/\Delta^n}$.

A.4 Proposition *Let $p\colon X \to S$ and $q\colon Y \to S$ be cocartesian fibrations over S and let $F\colon X \to Y$ be an S–functor. Suppose collections of edges $\mathcal{E}_X$ and $\mathcal{E}_Y$ of X and Y such that*

(1) *$\mathcal{E}_X$ and $\mathcal{E}_Y$ contain the p– and q–cocartesian edges, respectively;*

(2) *for $\mathcal{E}^0_X \subset \mathcal{E}_X$ the subset of edges which are either p–cocartesian or lie in a fiber, we have that $(X, \mathcal{E}^0_X) \subset (X, \mathcal{E}_X)$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/S}$, and ditto for Y;*

(3) *$F(\mathcal{E}_X) \subset \mathcal{E}_Y$;*

(4) *for all $s \in S$, $F_s\colon (X_s, (\mathcal{E}_X)_s) \to (Y_s, (\mathcal{E}_Y)_s)$ is a cocartesian equivalence in $s\mathbf{Set}^+$.*

Let $X' = (X, \mathcal{E}_X)$, $Y' = (Y, \mathcal{E}_Y)$, and $F'\colon X' \to Y'$ be the map given on underlying simplicial sets by F. Then for all simplicial sets U and maps $U \to S$, F'_U is a cocartesian equivalence in $s\mathbf{Set}^+_{/U}$.

Proof Without loss of generality, we may assume that an edge e is in $\mathcal{E}_X$ if and only if either e is p–cocartesian or $p(e)$ is degenerate, and ditto for $\mathcal{E}_Y$. First suppose that F is a trivial fibration in $s\mathbf{Set}^+_{/S}$ and for all $s \in S$, F'_s reflects marked edges. Then F' is again a trivial fibration because F' has the right lifting property against all cofibrations. For the general case, factor F as $X \xrightarrow{G} Z \xrightarrow{H} Y$ where G is a cofibration and H is a trivial fibration, and let $Z' = (Z, \mathcal{E}_Z)$ for $\mathcal{E}_Z$ the collection of edges e where e is in $\mathcal{E}_Z$ if and only if $H(e)$ is in $\mathcal{E}_Y$. Then for all $s \in S$, $Z'_s \to Y'_s$ is a trivial fibration in

$s\mathbf{Set}^+$, so as we just showed $H'\colon Z' \to Y'$ is a trivial fibration. We thereby reduce to the case that F is a cofibration.

Let $\mathcal{U}$ denote the collection of simplicial sets U such that for every map $U \to S$, F'_U is a cocartesian equivalence in $s\mathbf{Set}^+_{/U}$. We need to prove that every simplicial set belongs to $\mathcal{U}$. For this, we will verify the hypotheses of [9, Lemma 2.2.3.5]. Conditions (i) and (ii) are obvious, condition (iv) follows from left properness of the cocartesian model structure and [11, Proposition B.2.9], and condition (v) follows from the stability of cocartesian equivalences under filtered colimits and [11, Proposition B.2.9]. It remains to check that every n–simplex belongs to $\mathcal{U}$, so suppose $S = \Delta^n$. Let

$$\begin{tikzcd} M(\phi_X) \arrow[r, "M(\eta)"] \arrow[d] & M(\phi_Y) \arrow[d] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

be as in Construction A.3. Let ϕ'_X be the sequence $X'_0 \to \cdots \to X'_n$, where the maps are the same as in ϕ_X, and similarly define ϕ'_Y and η'. Then we have pushout squares

$$\begin{tikzcd} M(\phi_X) \arrow[r] \arrow[d] & M(\phi'_X) \arrow[d] \\ X \arrow[r] & X'' \end{tikzcd} \qquad \begin{tikzcd} M(\phi_Y) \arrow[r] \arrow[d] & M(\phi'_Y) \arrow[d] \\ Y \arrow[r] & Y'' \end{tikzcd}$$

with all four vertical maps cocartesian equivalences in $s\mathbf{Set}^+_{/\Delta^n}$. Here we replace X' by X'', which has the same underlying simplicial set X but more edges marked with $X' \subset X''$ left marked anodyne, so that the vertical maps $M(\phi'_X) \to X''$ are defined and the squares are pushout squares (again, ditto for Y''). Note that F defines a map $F''\colon X'' \to Y''$.

Finally, we have the commutative square

$$\begin{tikzcd} M(\phi'_X) \arrow[r, "M(\eta')"] \arrow[d] & M(\phi'_Y) \arrow[d] \\ X'' \arrow[r, "F''"] & Y'' \end{tikzcd}$$

By assumption, $\eta'\colon \phi'_X \to \phi'_Y$ is a natural transformation through cocartesian equivalences in $s\mathbf{Set}^+$. By Lemma A.2, $M(\eta')$ is a cocartesian equivalence in $s\mathbf{Set}^+_{/\Delta^n}$. We deduce that F'', hence F', is as well. $\square$

A.5 Remark By a simple modification of the above arguments, we may further prove that for any marked simplicial set $A \to S$, F'_A is a cocartesian equivalence in $s\mathbf{Set}^+_{/A}$. We leave the details of this to the reader.

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Floer theory of disjointly supported Hamiltonians on symplectically aspherical manifolds

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We study the Floer-theoretic interaction between disjointly supported Hamiltonians by comparing Floer-theoretic invariants of these Hamiltonians with the ones of their sum. These invariants include spectral invariants, boundary depth and Abbondandolo, Haug and Schlenk's action selector. Additionally, our method shows that in certain situations, the spectral invariants of a Hamiltonian supported in an open subset of a symplectic manifold are independent of the ambient manifold.

53D40, 57R17, 57R58

1 Introduction and results

The paper deals with Hamiltonian diffeomorphisms of symplectic manifolds, which model the Hamiltonian dynamics on phase spaces in classical mechanics. A central tool for studying Hamiltonian diffeomorphisms is Floer theory, which is an infinite-dimensional version of Morse theory applied to the action functional on the space of contractible loops. As such, Floer theory associates a chain complex to each Hamiltonian, which is generated by the critical points of the action functional and whose differential counts certain negative gradient flow lines, called *Floer trajectories*.

Our main object of interest is Floer theory for Hamiltonians supported in pairwise disjoint open sets, namely $F = F_1 + \cdots + F_N$ where F_i is supported in U_i and $U_1, \ldots, U_N$ are pairwise disjoint. On the level of dynamics, the Hamiltonian diffeomorphisms φ_i corresponding to F_i do not interact. The Hamiltonian diffeomorphism corresponding to F is the composition $\varphi = \varphi_1 \circ \cdots \circ \varphi_N$, and the diffeomorphisms φ_i commute. However, it is unclear a priori whether in Floer theory there is any communication between the disjointly supported Hamiltonians F_i. The Floer-theoretic interaction between disjointly supported Hamiltonians was studied by Polterovich [15], Seyfaddini [19], Ishikawa [13] and Humilière–Le Roux–Seyfaddini [12], mostly through the relation between invariants of the sum of Hamiltonians and invariants of each one. These works suggest that such an interaction should be quite limited. The main finding of this paper is a construction, on symplectically aspherical manifolds and under some conditions on the domains U_i, of what we call a "barricade" — a specific perturbation of the Hamiltonians F_i near the boundaries of U_i, which prevents Floer trajectories from entering or exiting these domains. The presence of barricades limits the communication between disjointly supported Hamiltonians as expected. The construction is motivated by the following simple idea in Morse theory. Given a smooth function F on a Riemannian manifold, which is supported inside an open subset U, one can perturb it into a Morse function f that has a "bump" in a neighborhood of the boundary, as illustrated in Figure 1. The negative gradient flow-lines of f cannot cross the bump, and therefore a flow-line starting inside U, and away from the boundary, remains there. On the other hand, flow-lines that start on the bump can flow both in and out of U. Since the Morse differential counts negative gradient flow-lines, such constraints can be used to gain information about it.

This idea can be adapted to Floer theory on symplectically aspherical manifolds (that is, when the symplectic form ω and the first Chern class c_1 vanish on $\pi_2(M)$), and under certain assumptions on the domain U. The resulting construction can be used to study Floer-theoretic invariants, such as *spectral invariants* and the *boundary depth*, of Hamiltonians supported in such domains. Spectral invariants measure the minimal action required to represent a given homology class in Floer homology. These invariants

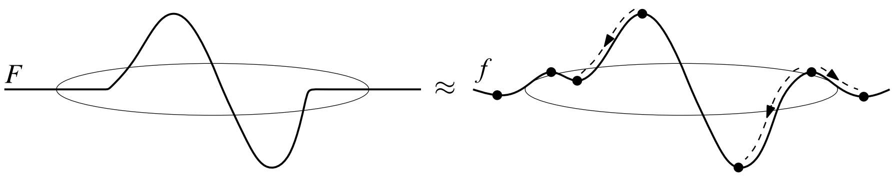

Figure 1: We perturb the function F to create a small "bump" along a neighborhood of ∂U. The dashed lines represent (some of the) flow-lines of $-\operatorname{grad} f$.

play a central role in the study of symplectic topology and Hamiltonian dynamics. Using the barricades construction, we prove that the spectral invariants with respect to the fundamental and the point classes of Hamiltonians supported in certain domains do not depend on the ambient manifold. This result is stated formally in Section 1.1.1. Another application of the barricades construction concerns spectral invariants of Hamiltonians with disjoint supports. This problem was studied in Ishikawa [13], Polterovich [15] and Seyfaddini [19] and lastly in Humilière, Le Roux and Seyfaddini [12]. Humilière, Le Roux and Seyfaddini proved that the spectral invariant with respect to the fundamental class satisfies a "max formula", namely, the invariant of a sum of disjointly supported Hamiltonians is equal to the maximum over the invariants of the summands. This property does not hold for a general homology class. However, using barricades we show that an inequality holds in general; see Section 1.1.2. A third application of this method concerns the boundary depth, which was defined by Usher in [21] and measures the maximal action gap between a boundary term and its smallest primitive in the Floer chain complex; see Section 1.1.3. We prove a relation between the boundary depths of disjointly supported Hamiltonians and that of their sum. The last application concerns a new invariant that was constructed by Abbondandolo, Haug and Schlenk in [1]. We give a partial answer to a question they posed, asking whether a version of Humilière, Le Roux and Seyfaddini's max formula holds for the new invariant; see Section 1.1.4.

1.1 Results

The limitation in Floer-theoretic interaction between disjointly supported Hamiltonians is reflected through Floer-theoretic invariants of these Hamiltonians and their sum. In order to define these invariants, we briefly describe *filtered Floer homology*. For more details, see Section 2 and the references therein. Throughout the paper, (M,ω) denotes a closed symplectically aspherical manifold, namely, $\omega|_{\pi_2(M)}=0$ and $c_1|_{\pi_2(M)}=0$, where c_1 is the first Chern class of M. Given a Hamiltonian $F\colon M\times S^1\to\mathbb{R}$, its symplectic gradient X_F is the time-dependent vector field given by the equation

$$\omega(X_{F_t},\cdot) = -dF_t(\cdot), \quad \text{where } F_t(\cdot) := F(\cdot,t).$$

The 1–periodic orbits of the flow of X_F, whose set is denoted by $\mathcal{P}(F)$, correspond to critical points of the action functional associated to F, and generate the Floer complex $CF_*(F)$. The differential of this chain complex is defined by counting certain negative-gradient flow lines of the action functional, and therefore decreases the value of the action. Note that the gradient of the action functional is taken with respect to a metric induced by an almost complex structure J on M. The homology of this chain complex, denoted by $HF_*(F)$, is known to be isomorphic to the singular homology of M up to a degree-shift, $HF_*(F) \cong H_{*+n}(M;\mathbb{Z}_2)$. The complex $CF_*(F)$ is filtered by the action value, namely, for every $a \in \mathbb{R}$, we denote by $CF_*^a(F)$ the subcomplex generated by 1–periodic orbits whose action is not greater than a. The homology of this subcomplex is denoted by $HF_*^a(F)$.

In what follows we present four applications of the barricades construction, which is an adaptation to Floer theory of the idea presented in Figure 1, and is described in Section 1.2. The barricade construction applies for Hamiltonians supported[1] in certain admissible domains, which include images of symplectic embeddings of nice star-shaped[2] domains in $\mathbb{R}^{2n}$ into M. In order to present this class in full generality we need to recall a few standard notions. Let $U \subset M$ be a domain with a smooth boundary. We say that U has a *contact type boundary* if there exists a vector field Y, called the *Liouville vector field*, that is defined on a neighborhood of ∂U, is transverse to ∂U, points outwards from U and satisfies $\mathcal{L}_Y\omega = \omega$. If the Liouville vector field Y extends to U, the closure of U is called a *Liouville domain*. Finally, a subset $X \subset M$ is called *incompressible* if the map $\iota_*\colon \pi_1(X) \to \pi_1(M)$ induced by the inclusion $X \hookrightarrow M$ is injective. In particular, every simply connected subset is incompressible.

Definition 1.1 An open subset $U \subset M$ with a smooth boundary is called a *CIB* (*Contact Incompressible Boundary*) *domain* if for each connected component U_i of U, one of the following assertions holds:

(i) ∂U_i is of contact type and is incompressible.

(ii) The closure of U_i is an incompressible Liouville domain.

[1] When we say that a Hamiltonian F is supported in a subset U of M, we actually mean that the function $F\colon M \times S^1 \to \mathbb{R}$ is supported in $U \times S^1$.

[2] A nice star-shaped domain is a bounded star-shaped domain in $\mathbb{R}^{2n}$ with a smooth boundary, such that the radial vector field is transverse to the boundary.

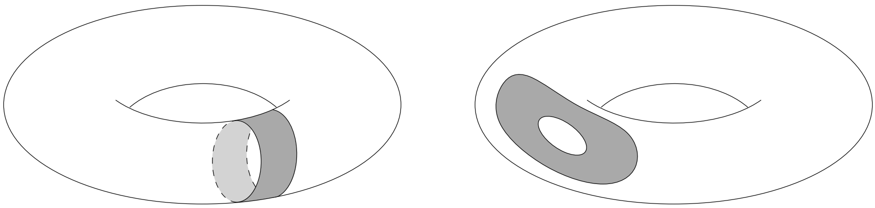

Figure 2: Two embeddings of the annulus into $\mathbb{T}^2$. The embedding on the left is incompressible (as well as its boundary) and hence is a CIB domain. The embedding on the right is contractible in $\mathbb{T}^2$ and therefore not incompressible.

Example 1.2
- The image under a symplectic embedding of a nice star-shaped domain in $\mathbb{R}^{2n}$ into M is a CIB domain.
- A noncontractible annulus in $M = \mathbb{T}^2$ is a CIB domain. More generally, if $M = \mathbb{T}^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$, then certain tubular neighborhoods of $L = \mathbb{R}^n/\mathbb{Z}^n$ in M are CIB domains.

Remark 1.3
- A disjoint union of CIB domains is again a CIB domain.
- The interior of every incompressible Liouville domain is a CIB domain.
- Every CIB domain is incompressible, as the fact that ∂U is incompressible implies that U is incompressible; see the appendix.

1.1.1 Locality of spectral invariants and Schwarz's capacities For a homology class $\alpha \in H_*(M;\mathbb{Z}_2)$ and a Hamiltonian F, the spectral invariant $c(F;\alpha)$ is the smallest action value a for which α appears in $HF_*^a(F,J)$, namely,

$$c(F;\alpha) := \inf\{a \mid \alpha \in \operatorname{im}(\iota_*^a)\},$$

where $\iota_*^a\colon HF_*^a(F) \to HF_*(F)$ is induced by the inclusion $\iota^a\colon CF_*^a(F) \hookrightarrow CF_*(F)$. The following result states that the spectral invariants with respect to the fundamental and the point classes, of a Hamiltonian F supported in a CIB domain, do not depend on the ambient manifold M. More formally, let $U \subset M$ be a CIB domain and assume that there exists a symplectic embedding $\Psi\colon U \hookrightarrow N$ of U into another closed symplectically aspherical manifold (N,Ω), such that $\Psi(U)$ is a CIB domain in N. Denote by $c_M(\cdot\,;\cdot)$ and $c_N(\cdot\,;\cdot)$ the spectral invariants in the manifolds M and N, respectively.

Theorem 1 *Let $F\colon M \times S^1 \to \mathbb{R}$ be a Hamiltonian supported in U. Then*

$$c_M(F;[M]) = c_N(\Psi_*F;[N]) \quad \text{and} \quad c_M(F;[\mathrm{pt}]) = c_N(\Psi_*F;[\mathrm{pt}]), \tag{1}$$

where $\Psi_*F\colon N \times S^1 \to \mathbb{R}$ is the extension by zero of $F \circ \Psi^{-1}$.

The assertion of Theorem 1 does not hold when M is not symplectically aspherical, or when U is not incompressible in M. This is shown in Example 4.6. Theorem 1 also holds for the spectral invariants defined by Frauenfelder and Schlenk [9] on open manifolds obtained as completions of compact manifolds with contact-type boundaries; see Remark 5.1. Moreover, Theorem 1 can be extended to certain other homology classes, as stated in Claim 5.3. One corollary of Theorem 1 concerns Schwarz's relative capacities.[3]

Definition 1.4 (Schwarz [18]) Let (M,ω) be a symplectically aspherical manifold. For a subset $A \subset M$ define the *spectral capacity* by

$$c_\gamma(A;M) := \sup\{c(F;[M]) - c(F;[\mathrm{pt}]) \mid \operatorname{supp} X_F \subset A \times S^1\}. \tag{2}$$

In [18], Schwarz shows that if the spectral capacity of the support of X_F is finite and $\varphi_F^1 \neq \mathbb{1}$, then the Hamiltonian flow of F has infinitely many geometrically distinct nonconstant periodic points corresponding to contractible solutions. In Section 4, we use Theorem 1 to show that when A is a contractible domain with a contact-type boundary, its spectral capacity does not depend on the ambient manifold.

Corollary 1.5 *Let $\mathcal{S}$ be the set of contractible compact symplectic manifolds with contact-type boundaries that can be embedded into symplectically aspherical manifolds, eg nice star-shaped domains in $\mathbb{R}^{2n}$. Then:*

- *Schwarz's spectral capacities $\{c_\gamma(\,\cdot\,;M)\}$ induce a capacity c_γ on the class of symplectic manifolds X which are exhaustible by elements from $\mathcal{S}$, namely there exist $A_i \in \mathcal{S}$ such that*

$$A_1 \subset A_2 \subset \cdots \subset X \quad \textit{and} \quad X = \bigcup_i A_i.$$

- *$c_\gamma(A;M)$ is finite for every $A \subset M$ such that $A \in \mathcal{S}$ and can be symplectically embedded into $(\mathbb{R}^{2n},\omega)$, that is,*

$$c_\gamma(A;M) = c_\gamma(A) \le 2e(A;\mathbb{R}^{2n}) < \infty, \tag{3}$$

where $e(A;\mathbb{R}^{2n})$ is the displacement energy[4] of A in $\mathbb{R}^{2n}$.

[3] We recall the definition of a capacity in Section 4.

[4] We recall the definition of the displacement energy in Section 2, equation (20).

Here we used the fact that every bounded subset of $\mathbb{R}^{2n}$ is displaceable with finite energy. The proof of Corollary 1.5, as well as the definition of c_γ, appears in Section 4.

Another corollary of Theorem 1 concerns the notions of *heavy* and *superheavy* sets, which were introduced by Entov and Polterovich in [7]: A closed subset $X \subset M$ is called heavy if

$$\zeta(F) \geq \inf_{X\times S^1} F \quad \text{for all } F \in \mathcal{C}^\infty(M \times S^1),$$

and is called superheavy if

$$\zeta(F) \leq \sup_{X\times S^1} F \quad \text{for all } F \in \mathcal{C}^\infty(M \times S^1),$$

where

$$\zeta(F) := \lim_{k\to\infty} \frac{c(kF;[M])}{k}$$

is the *partial symplectic quasistate* associated to the spectral invariant c and the fundamental class. The following corollary was suggested to us by Polterovich.

Corollary 1.6 *Let M be a symplectically aspherical manifold and let $A \subset M$ be a contractible domain with a contact-type boundary that can be symplectically embedded in $(\mathbb{R}^{2n}, \omega_0)$. Then $M \setminus A$ is superheavy. In particular, A does not contain a heavy set.*

Corollary 1.6 can be viewed as an extension of the results of [13] to a wider class of domains, when restricting to symplectically aspherical manifolds. Theorem 1 and Corollaries 1.5 and 1.6 are proved in Section 4.

1.1.2 Max-inequality for spectral invariants In [12], Humilière, Le Roux and Seyfaddini proved a max formula for the spectral invariants, with respect to the fundamental class, of Hamiltonians supported in the interiors of disjoint incompressible Liouville domains in symplectically aspherical manifolds.

Theorem (Humilière–Le Roux–Seyfaddini [12, Theorem 45]) *Let $F_1, \dots, F_N$ be Hamiltonians whose supports are contained, respectively, in the interiors of pairwise disjoint incompressible Liouville domains $U_1, \dots, U_N$. Then*

$$c(F_1 + \cdots + F_N;[M]) = \max\{c(F_1;[M]), \dots, c(F_N;[M])\}.$$

The existence of barricades can be used to give an alternative proof for this theorem, as well as to prove a version of it for other homology classes. Clearly, other homology classes do not satisfy such a max formula — for example, by Poincaré duality the class of a point satisfies a min formula. However, an inequality does hold for a general homology class.

Theorem 2 *Let F and G be Hamiltonians supported in disjoint CIB domains and let $\alpha \in H_*(M)$. Then*

$$c(F+G;\alpha) \le \max\{c(F;\alpha), c(G;\alpha)\}. \tag{4}$$

Moreover, when $\alpha = [M]$, we have an equality.

Notice that, by definition, the interior of every incompressible Liouville domain is a CIB domain. Moreover, a disjoint union of CIB domains is again a CIB domain. Hence, the inequality for N Hamiltonians follows by induction. We also mention that a "min inequality" does not hold in general, namely, $c(F+G;\alpha)$ might be strictly smaller than $\min\{c(F,\alpha), c(G,\alpha)\}$, as shown in Example 6.4. Theorem 2 is proved in Section 6.

1.1.3 The boundary depth of disjointly supported Hamiltonians In [21], Usher defined the *boundary depth* of a Hamiltonian F to be the largest action gap between a boundary term in $CF_*(F)$ and its smallest primitive, namely

$$\beta(F) := \inf\{b \in \mathbb{R} \mid CF_*^a(F) \cap \partial_{F,J}(CF_*(F)) \subset \partial_{F,J}(CF_*^{a+b}(F)) \text{ for all } a \in \mathbb{R}\}.$$

The following result relates the boundary depths of disjointly supported Hamiltonians to that of their sum, and is proved in Section 7.

Theorem 3 *Let F and G be Hamiltonians supported in disjoint CIB domains. Then*

$$\beta(F+G) \ge \max\{\beta(F), \beta(G)\}. \tag{5}$$

Note that equality does not hold in (5) in general, as shown in Example 7.2.

1.1.4 Min-inequality for the AHS action selector In a recent paper [1], Abbondandolo, Haug and Schlenk presented a new construction of an action selector, denoted here by c_{AHS}, that does not rely on Floer homology. Roughly speaking, given a Hamiltonian F, the invariant $c_{\mathrm{AHS}}(F)$ is the minimal action value that "survives" under all homotopies starting at F. In Section 8, we review the definition of this selector and a few relevant properties. An open problem, stated in [1, Open Problem 7.5], is whether c_{AHS} coincides with the spectral invariant of the point class. As a starting point, Abbondandolo, Haug and Schlenk ask whether c_{AHS} satisfies a *min formula* like the one proved by Humilière, Le Roux and Seyfaddini in [12] for the spectral invariant with respect to the point class.[5] Due to a result from [12], this will imply that c_{AHS} coincides with the spectral invariant with respect to the point class in dimension 2 on autonomous Hamiltonians. In Section 8, we use barricades in order to prove an inequality for the AHS action selector.

Theorem 4 *Let F and G be Hamiltonians supported in the interiors of disjoint incompressible Liouville domains. Then*

$$c_{\mathrm{AHS}}(F+G) \leq \min\{c_{\mathrm{AHS}}(F), c_{\mathrm{AHS}}(G)\}. \tag{6}$$

1.2 The main tool: barricades

The central construction in this paper is an adaptation of the idea presented in Figure 1 to Floer theory, which is an infinite-dimensional version of Morse theory, applied to the action functional associated to a given Hamiltonian $F\colon M \times S^1 \to \mathbb{R}$. As in Morse theory, the Floer differential counts certain negative-gradient flow lines of the action functional. These flow lines are called "Floer trajectories" and correspond to solutions $u\colon \mathbb{R} \times S^1 \to M$ of a certain partial differential equation, called the "Floer equation" (FE), which converge to 1–periodic orbits of the Hamiltonian flow at the ends,

$$\lim_{s\to\pm\infty} u(s,t) = x_\pm(t) \quad \text{for } x_\pm \in \mathcal{P}(F).$$

In this case we say that u *connects* $x_\pm$; see Section 2 for more details. Following the idea from Morse theory, given a Hamiltonian F supported in a subset $U \subset M$, we wish to construct a perturbation for which Floer trajectories cannot enter or exit the domain. Moreover, we extend this construction to homotopies of Hamiltonians, namely, smooth functions $H\colon M \times S^1 \times \mathbb{R} \to \mathbb{R}$, for the following reason: most of the results presented above compare Floer-theoretic invariants of different Hamiltonians. Such a comparison is usually done using a morphism between the different chain complexes, which is defined by counting solutions of the Floer equation with respect to a homotopy between the two Hamiltonians. We consider only homotopies that are constant outside of a compact set, namely there exists $R > 0$ such that $\partial_s H(\cdot,\cdot,s)$ is supported in $M \times S^1 \times [-R,R]$. We denote by $H_\pm := H(\cdot,\cdot,\pm R)$ the ends of the homotopy H. Note that we think of single Hamiltonians as a special case of this setting, by identifying them with constant homotopies, $H(x,t,s) = F(x,t)$. Given an almost complex structure J on M, we consider solutions of the Floer equation (FE)

[5] As mentioned above, they proved a max formula for the spectral invariant of the fundamental class. By Poincaré duality for spectral invariants, this is equivalent to a min formula for the point class.

with respect to the pair (H, J). The property of having a barricade is defined through constraints on these solutions.

Definition 1.7 Let U and $U_\circ$ be open subsets of M such that $U_\circ \Subset U$. We say that a pair (H, J) of a homotopy and an almost complex structure *has a barricade in U around* $U_\circ$ if the periodic orbits of $H_\pm$ do not intersect the boundaries ∂U and $\partial U_\circ$, and for every $x_\pm \in \mathcal{P}(H_\pm)$ and every solution $u\colon \mathbb{R}\times S^1 \to M$ of the corresponding Floer equation connecting $x_\pm$, we have:

(i) If $x_- \subset U_\circ$, then $\mathrm{im}(u) \subset U_\circ$.

(ii) If $x_+ \subset U$, then $\mathrm{im}(u) \subset U$.

See Figure 3 for an illustration of solutions satisfying and not satisfying these constraints. When H is a constant homotopy, corresponding to a Hamiltonian F, the presence of a barricade yields a decomposition of the Floer complex, in which the differential admits a triangular block form. To describe this decomposition, let us fix some notation: for a subset $X \subset M$, denote by $C_X(F) \subset CF_*(F)$ the subspace generated by orbits contained in X, and by $\partial|_X$ the map obtained by counting only solutions that are contained in X. Then, for a Floer-regular pair (F, J) with a barricade in U around $U_\circ$,

$$CF_*(F) := C_{U_\circ}(F) \oplus C_{U^c}(F) \oplus C_{U\setminus U_\circ}(F), \quad \partial_{F,J} = \begin{pmatrix} \partial|_{U_\circ} & 0 & \partial|_U \\ 0 & * & * \\ 0 & 0 & \partial|_U \end{pmatrix}. \tag{7}$$

The block form (7) implies that the differential restricts to the subspace $C_{U_\circ}(F)$. We study the homology of the resulting subcomplex $(C_{U_\circ}(F), \partial|_{U_\circ})$ in Section 5.1.

Given a homotopy H that is compactly supported in a CIB domain, we construct a small perturbation h of H and an almost complex structure J, so that (h, J) has a barricade.

Theorem 5 *Let U be a CIB domain and let $H\colon M\times S^1\times\mathbb{R} \to \mathbb{R}$ be a homotopy of Hamiltonians, supported in $U\times S^1\times\mathbb{R}$, such that $\partial_s H$ is compactly supported. Then there exist a $\mathcal{C}^\infty$–small perturbation h of H and an almost complex structure J such that the pairs (h, J) and $(h_\pm, J)$ are Floer-regular and have a barricade in U around $U_\circ$. In particular, when H is independent of the $\mathbb{R}$–coordinate (namely, it is a single Hamiltonian), h can be chosen to be independent of the $\mathbb{R}$–coordinate as well.*

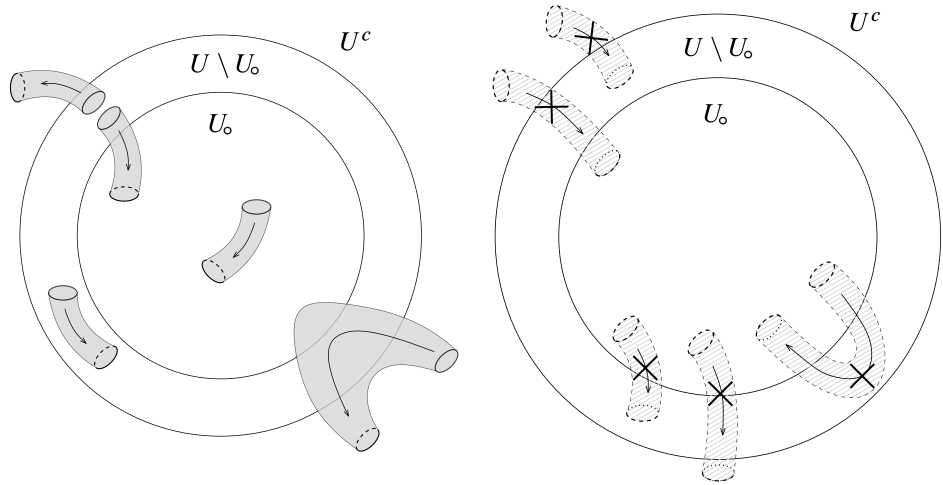

Figure 3: An illustration of allowed solutions, left, and forbidden solutions, right, for a pair (H, J) with a barricade.

This result is proved in Section 3, by an explicit construction of the perturbation h and the almost complex structure J. We remark that the assumptions on (M, ω) being symplectically aspherical and U having either incompressible boundary or being the interior of an incompressible Liouville domain are crucial for this construction. See the proofs of Lemmas 3.4–3.5 for details.

1.3 Related works

There have been several works studying the Floer-theoretic interaction between disjointly supported Hamiltonians, mainly through the spectral invariants of these Hamiltonians and their sum. Early works in this direction, mainly by Polterovich [15], Seyfaddini [19] and Ishikawa [13], established upper bounds for the invariant of the sum of Hamiltonians, which depend on the supports. Later, Humilière, Le Roux and Seyfaddini [12] proved that in certain cases the invariant of the sum is equal to the maximum over the invariants of each individual summand. The method was also conceptually different. While previous works relied solely on the properties of spectral invariants, Humilière, Le Roux and Seyfaddini studied the Floer complex itself. We also take this approach and study the interaction between disjointly supported Hamiltonians on the level of the Floer complex, but our methods are substantially different.

In a broader sense, it is worth mentioning two works which regard *symplectic homology*. Symplectic homology is an umbrella term for a type of homological invariant of symplectic manifolds, or of subsets of symplectic manifolds, constructed via a limiting process from the Floer complexes of properly chosen Hamiltonians. In this setting, questions regarding disjointly supported Hamiltonians correspond to local-to-global relations, such as a Mayer–Vietoris sequence. In [5], Cieliebak and Oancea defined symplectic homology for Liouville domains and Liouville cobordisms and proved a Mayer–Vietoris relation. Their method includes ruling out the existence of certain Floer trajectories, and partially relies on work by Abouzaid and Seidel [2]. Versions of some of these arguments are being used in Section 3 below. Another work concerning the Mayer–Vietoris property is by Varolgunes [22], in which he defines an invariant of compact subsets of closed symplectic manifolds, which is called relative symplectic homology, and finds a condition under which the Mayer–Vietoris property holds. In particular, for a union of disjoint compact sets, the relative symplectic homology splits into a direct sum.

Structure of the paper

In Section 2 we review the necessary preliminaries from Floer theory and contact geometry. In Section 3 we construct barricades and prove Theorem 5. We then use it to prove Theorem 1 in Section 4. In Section 5, we discuss the relation to Floer homology on certain open manifolds and two extensions of Theorem 1. Sections 6–8 are respectively dedicated to the proofs of Theorems 2–4. Finally, in Section 9 we prove several transversality and compactness claims that are required for establishing the main results. The appendix contains a claim about incompressibility, whose proof we include for the sake of completeness.

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2 Preliminaries from Floer theory

In this section we briefly review some preliminaries from Floer theory and contact geometry on closed symplectically aspherical manifolds — namely, when $\omega|_{\pi_2(M)} = 0$ and $c_1|_{\pi_2(M)} = 0$, where c_1 is the first Chern class of M. For more details see, for example, [3; 14; 16]. We also fix some notation that will be used later on.

2.1 Floer homology, regularity and notation

Let $F\colon M \times S^1 \to \mathbb{R}$ be a Hamiltonian on M. The corresponding action functional $\mathcal{A}_F$ is defined on the space of contractible loops in M by

$$\mathcal{A}_F(x) := \int_0^1 F(x(t), t)\, dt - \int \overline{x}^*\omega,$$

where $x\colon S^1 \to M$ and $\overline{x}\colon D^2 \to M$ satisfies $\overline{x}(e^{2\pi i t}) = x(t)$. The critical points of the action functional are the contractible 1–periodic orbits of the flow of X_F, and their set is denoted by $\mathcal{P}(F)$. The Hamiltonian $F\colon M \times S^1 \to \mathbb{R}$ is said to be *nondegenerate* if the graph of the linearized flow of X_F at time 1 intersects the diagonal in $TM \times TM$ transversely. In this case, the flow of X_F has finitely many 1–periodic orbits. The Floer complex $CF_*(F)$ is spanned by these critical points, over $\mathbb{Z}_2$.[6] A time-dependent ω–compatible[7] almost complex structure J induces a metric on the space of contractible loops, in which negative-gradient flow lines of $\mathcal{A}_F$ are maps $u\colon \mathbb{R} \times S^1 \to M$ that solve the Floer equation

$$\text{(FE)} \qquad \partial_s u(s,t) + J \circ u(s,t) \cdot (\partial_t u(s,t) - X_F \circ u(s,t)) = 0.$$

The *energy* of such a solution is defined to be $E(u) := \int_{\mathbb{R}\times S^1} \|\partial_s u\|_J^2\, ds\, dt$, where $\|\cdot\|_J$ is the norm induced by the inner product associated to J, $\langle \cdot, \cdot \rangle_J := \omega(\cdot, J\,\cdot)$. When the Hamiltonian F is nondegenerate, for every solution u with finite energy, there exist $x_\pm \in \mathcal{P}(F)$ such that $\lim_{s\to\pm\infty} u(s,t) = x_\pm(t)$, and we say that u *connects* $x_\pm$. The well-known energy identity for such solutions is a consequence of Stokes' theorem:

$$\text{(8)} \qquad E(u) := \int_{\mathbb{R}\times S^1} \|\partial_s u\|_J^2\, ds\, dt = \mathcal{A}_{F_-}(x_-) - \mathcal{A}_{F_+}(x_+).$$

For two 1–periodic orbits $x_\pm \in \mathcal{P}(F)$ of F, let $\mathcal{M}_{(F,J)}(x_-, x_+)$ denote the set of all solutions $u\colon \mathbb{R}\times S^1 \to M$ of the Floer equation (FE) satisfying $\lim_{s\to\pm\infty} u(s,t) = x_\pm(t)$.

[6] The Floer complex can be defined over other coefficient rings; we chose to work in the simplest setting.
[7] An almost complex structure J is called *ω–compatible* if $\omega(\cdot, J\,\cdot)$ is an inner product on TM. All almost complex structures considered in this paper are assumed to be ω–compatible.

Notice that $\mathbb{R}$ acts on this set by translation in the s variable. We denote by $\mathcal{M}_{(F,J)}$ the set of all finite-energy solutions. It is well known — see eg [3, Theorem 6.5.6] — that when F is nondegenerate, $\mathcal{M}_{(F,J)} := \bigcup_{x_\pm \in \mathcal{P}(H)} \mathcal{M}_{(F,J)}(x_-, x_+)$. Moreover, for nondegenerate Hamiltonians one can define an index $\mu\colon \mathcal{P}(F) \to \mathbb{Z}$, called *the Conley–Zehnder index*, which assigns an integer to each orbit; see eg [3, Chapter 7]. The Floer complex is graded by the index μ; namely, for $k \in \mathbb{Z}$, $CF_k(F)$ is the $\mathbb{Z}_2$–vector space spanned by the periodic orbits $x \in \mathcal{P}(F)$ for which $\mu(x) = k$.

In order to define the Floer differential for the graded complex $CF_*(F)$, one needs an almost complex structure J such that the pair (F, J) is *Floer-regular*. The definition of Floer regularity concerns the surjectivity of a certain linear operator and is given in Section 9.1. When the pair (F, J) is Floer-regular, the space of solutions $\mathcal{M}_{(F,J)}(x_-, x_+)$ is a smooth manifold of dimension $\mu(x_-) - \mu(x_+)$ for all $x_\pm \in \mathcal{P}(F)$. Dividing $\mathcal{M}_{(F,J)}(x_-, x_+)$ by the $\mathbb{R}$ action, we obtain a manifold of dimension $\mu(x_-) - \mu(x_+) - 1$.

Recall that an element $a \in CF_*(F)$ is a formal linear combination $a = \sum_x a_x \cdot x$, where $x \in \mathcal{P}(F)$ and $a_x \in \mathbb{Z}_2$. For a Floer-regular pair (F, J), the Floer differential $\partial_{(F,J)}\colon CF_*(F) \to CF_{*-1}(F)$ is defined by

$$\partial_{(F,J)}(a) := \sum_{x_- \in \mathcal{P}(F)} \sum_{\substack{x_+ \in \mathcal{P}(F) \\ \mu(x_+) = \mu(x_-) - 1}} a_{x_-} \cdot \#_2\left(\frac{\mathcal{M}_{(F,J)}(x_-, x_+)}{\mathbb{R}}\right) \cdot x_+, \tag{9}$$

where $\#_2$ is the number of elements modulo 2. The homology of the complex $(CF_*(F), \partial_{(F,J)})$ is denoted by $HF_*(F, J)$ or $HF_*(F)$. A fundamental result in Floer theory states that Floer homology is isomorphic to the singular homology with a degree shift, $HF_*(F, J) \cong H_{*-n}(M; \mathbb{Z}_2)$. The Floer complex admits a natural filtration by the action value. We denote by $CF_*^a(F)$ the subcomplex spanned by critical points with value not greater than a. Since the differential is action decreasing, it can be restricted to the subcomplex $CF_*^a(F)$. The homology of this subcomplex is denoted by $HF_*^a(F, J)$.

It is well known that when F is a $\mathcal{C}^2$–small Morse function, its 1–periodic orbits are its critical points, $\mathcal{P}(F) \cong \mathrm{Crit}(F)$, and their actions are the values of F, $\mathcal{A}_F(p) = F(p)$. In this case, the Floer complex with respect to a time-independent almost complex structure J coincides with the Morse complex when the degree is shifted by n (which is half the dimension of M), since Morse-ind$(p) = \mu(p) + n$ for every $p \in \mathrm{Crit}(F) \cong \mathcal{P}(F)$:

$$(CF_*(F), \partial^{\mathrm{Floer}}_{(F,J)}) = (CM_{*+n}(F), \partial^{\mathrm{Morse}}_{(F,\langle\cdot,\cdot\rangle_J)}).$$

For a proof, see, for example, [3, Chapter 10]. We conclude this section by fixing notation that will be used later on.

Notation 2.1 Let $a = \sum_x a_x \cdot x$ be an element of $CF_*(H)$.

- We say that $x \in a$ if $a_x \neq 0$.
- We denote the maximal action of an orbit from a by
$$\lambda_H(a) := \max\{\mathcal{A}_H(x) \mid a_x \neq 0\}.$$
- For a subset $X \subset M$, let $C_X(H) \subset CF_*(H)$ be the subspace spanned by the 1–periodic orbits of H that are contained in X. Let $\pi_X: CF_*(H) \to C_X(H)$ be the projection onto this subspace. Note that $C_X(H)$ is not necessarily a subcomplex, and π_X is not a chain map in general.

2.2 Communication between Floer complexes using homotopies

Now let $H: M \times S^1 \times \mathbb{R} \to \mathbb{R}$ denote a homotopy of Hamiltonians, rather than a single Hamiltonian. Throughout the paper, we consider only homotopies that are constant outside of a compact set. Namely, there exists $R > 0$ such that $\partial_s H|_{|s|>R} = 0$, and we denote by $H_\pm(x,t) := \lim_{s\to\pm\infty} H(x,t,s)$ the ends of the homotopy H. Given an almost complex structure J, we consider the Floer equation (FE) with respect to the pair (H, J),

$$\partial_s u(s,t) + J \circ u(s,t) \cdot (\partial_t u(s,t) - X_{H_s} \circ u(s,t)) = 0,$$

where $H_s(\cdot\,,\cdot) := H(\cdot\,,\cdot\,,s)$. We sometimes refer to this equation as *the s–dependent Floer equation*, to stress that it is defined with respect to a homotopy of Hamiltonians. For 1–periodic orbits $x_\pm \in \mathcal{P}(H_\pm)$, we denote by $\mathcal{M}_{(H,J)}(x_-, x_+)$ the set of all solutions $u: \mathbb{R} \times S^1 \to M$ of the s–dependent Floer equation (FE) that satisfy $\lim_{s\to\pm\infty} u(s,t) = x_\pm(t)$. As before, $\mathcal{M}_{(H,J)}$ denotes the set of all finite-energy solutions and when the ends, $H_\pm$, are nondegenerate,

$$\mathcal{M}_{(H,J)} = \bigcup_{x_\pm \in \mathcal{P}(H_\pm)} \mathcal{M}_{(H,J)}(x_-, x_+).$$

(See, for example, [3, Theorem 11.1.1].) The energy identity for homotopies is

$$E(u) := \int_{\mathbb{R}\times S^1} \|\partial_s u\|_J^2 \, ds\, dt = \mathcal{A}_{H_-}(x_-) - \mathcal{A}_{H_+}(x_+) + \int_{\mathbb{R}\times S^1} \partial_s H \circ u \, ds\, dt. \tag{10}$$

As in the case of Hamiltonians, the definition of Floer-regularity concerns the surjectivity of a certain linear operator and is given in Section 9.1. For a Floer-regular pair (H, J), the space $\mathcal{M}_{(H,J)}(x_-, x_+)$ is a smooth manifold of dimension $\mu(x_-) - \mu(x_+)$. In this case, one can define a degree-preserving chain map $\Phi\colon CF_*(H_-) \to CF_*(H_+)$, called *the continuation map*, between the Floer complexes of the ends, by

$$\Phi(a) = \sum_{x_- \in a} \sum_{\substack{x_+ \in \mathcal{P}(H_+) \\ \mu(x_+) = \mu(x_-)}} a_{x_-} \cdot \#_2 \mathcal{M}(x_-, x_+) \cdot x_+. \tag{11}$$

The regularity of the pair guarantees that the map Φ is a well-defined chain map that induces an isomorphism on homologies; see eg [3, Chapter 11].

2.3 Contact-type boundaries

In order to construct barricades for Floer solutions around a given domain, we need the boundary to have a *contact structure*. Let $U \subset M$ be a domain with a smooth boundary. We say that U has a *contact type boundary* if there exists a vector field Y, called the *Liouville vector field*, which is defined on a neighborhood of ∂U, is transverse to ∂U, points outwards from U and satisfies $\mathcal{L}_Y \omega = \omega$. The differential form $\lambda := \iota_Y \omega$ is a primitive of ω, namely $d\lambda = \omega$; it is called the Liouville form and is defined wherever Y is defined. The flow ψ^r of Y is called *the Liouville flow*, and is defined for short times. The Reeb vector field R is then defined by the equations

$$R \in \ker d\lambda|_{T\psi^r \partial U}, \quad \lambda(R)|_{\psi^r \partial U} = e^r. \tag{12}$$

We stress that the vector field R is defined wherever the Liouville vector field Y is defined and is nonvanishing. If the Liouville vector field Y extends to U, the closure of U is called a *Liouville domain*.

3 Barricades for solutions of the (s–dependent) Floer equation

In what follows, $H\colon M \times S^1 \times \mathbb{R} \to \mathbb{R}$ denotes a homotopy of (time-dependent) Hamiltonians and J denotes a (time-dependent) almost complex structure. We assume that $\partial_s H$ is compactly supported and write $H_\pm := \lim_{s\to\pm\infty} H(\cdot, \cdot, s)$. Note that we consider the case where H is a single Hamiltonian as a particular case, by identifying it with a constant homotopy. Fix a CIB domain $U \subset M$, denote by Y and R the Liouville and Reeb vector fields, respectively, and by $\lambda = \iota_Y \omega$ the Liouville form. In order to prove Theorem 5, namely, that there exist a perturbation h of H and an almost complex structure J such that (H, J) has a barricade, we construct h and J explicitly. Let us sketch the idea of this construction before giving the details.

- To construct h, we first add to H a nonnegative bump function in the radial coordinate, which is defined on a neighborhood of ∂U using the Liouville flow. Then we take h to be a small nondegenerate perturbation of it.
- The almost complex structure J is taken to be *cylindrical* near ∂U; see Definition 3.1 below.

We want to rule out the existence of solutions violating the constraints of Definition 1.7. Suppose there exists a solution u connecting $x_- \subset U_\circ$ with $x_+ \subset U_\circ^c$. Then the image of u intersects $\partial U_\circ$, say along a loop Γ. We first bound the action of Γ (Lemma 3.2), and then conclude a negative upper bound for the action of x_+ (Lemma 3.4). Since $h \approx 0$ on $U_\circ^c \supset x_+$, the action of x_+ can be taken to be arbitrarily close to zero, a contradiction.

3.1 Preliminary computations

Some of the arguments and results in this section were carried out by Cieliebak and Oancea in [5] for the setting of completed Liouville domains, instead of closed symplectically aspherical manifolds. Specifically, some of the computations appearing in the proofs of Lemmas 3.2 and 3.5 can be found in the proof of [5, Lemma 2.2], which follows Abouzaid and Seidel's work in [2, Lemma 7.2].

Definition 3.1 We say that a pair (H, J) of a homotopy and an almost complex structure is *δ–cylindrical near ∂U* for $\delta \in \mathbb{R} \setminus \{0\}$, if

(i) J is cylindrical near ∂U, namely, $JY = R$ on an open neighborhood of ∂U,

(ii) $\partial U \times S^1 \times \mathbb{R} = \{H = c\}$ is a regular level set of H,

(iii) the gradient of H with respect to J satisfies $\nabla_J H = \delta e^{-r} Y$ on $\psi^r \partial U$ and H has no 1–periodic orbits near ∂U.

We remark that conditions (ii) and (iii) in the above definition imply that, near ∂U, H does not depend on the $\mathbb{R}$–coordinate. Suppose that (H, J) is δ–cylindrical near ∂U and let $u\colon \mathbb{R} \times S^1 \to M$ be a solution of the (s–dependent) Floer equation (FE) with finite energy $E(u) < \infty$. The following lemma gives an upper bound for the integral of λ along the curve $\Gamma := \mathrm{im}(u) \cap \partial U$ oriented as a connected component of the boundary of $\mathrm{im}(u) \cap U^c$; see Figure 4.

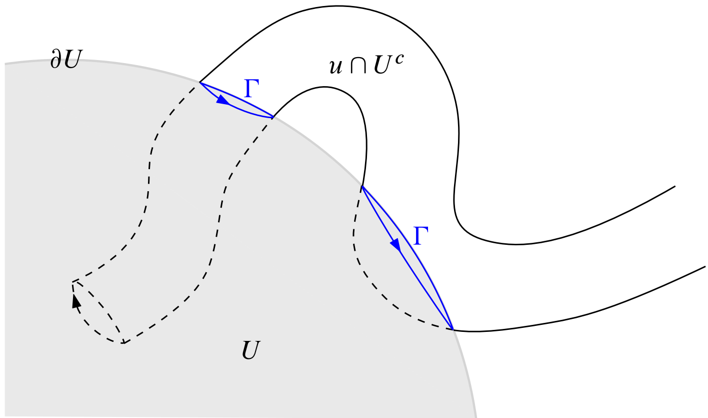

Figure 4: An example of the setting described in Lemma 3.2. The gray region is the set $U \subset M$ and the loops Γ are given by $\operatorname{im}(u) \cap \partial U = \Gamma$, oriented as the boundary of $\operatorname{im}(u) \cap U^c$.

Lemma 3.2 *Let (H, J) be a pair that is δ–cylindrical near ∂U and let $u\colon \mathbb{R} \times S^1 \to M$ be a finite-energy solution of the s–dependent Floer equation connecting $x_\pm \in \mathcal{P}(H_\pm)$. Suppose that u intersects ∂U transversely and write $\Gamma := \operatorname{im}(u) \cap \partial U$ for the intersection, oriented as the boundary of $\operatorname{im}(u) \cap U^c$. Then*

$$\int_\Gamma \lambda \le \begin{cases} -\delta & \text{if } x_- \subset U,\ x_+ \subset U^c, \\ \delta & \text{if } x_- \subset U^c,\ x_+ \subset U, \\ 0 & \text{if } x_\pm \subset U \text{ or } x_\pm \subset U^c. \end{cases} \tag{13}$$

Proof Set $\Sigma := u^{-1}(U^c) \subset \mathbb{R} \times S^1$ and denote its boundary by γ. Then $u(\gamma) = \Gamma$, since the $x_\pm$ do not intersect ∂U. The orientation on Σ is given by the positive frame (∂_s, ∂_t). Let γ_i be a connected component of γ. Then $\Gamma_i := u(\gamma_i)$ is connected. Let $\tau \in [0, T_i]$ be a unit-speed parametrization of γ_i, and notice that this induces a parametrization on Γ_i. Denoting by $\nu(\tau)$ the outer normal to Σ at $\gamma_i(\tau)$, then $\dot\gamma_i(\tau) = j\nu(\tau)$, where j is the standard complex structure on $\mathbb{R} \times S^1$, ie $j\partial_s = \partial_t$. Pushing $(\nu(\tau), \dot\gamma_i(\tau))$ to TM, we obtain

$$N(\tau) := Du(\nu(\tau)), \quad \dot\Gamma_i(\tau) = Du(\dot\gamma_i(\tau)).$$

We remark that $N(\tau)$ is not necessarily normal to ∂U (with respect to the inner product induced by J), but is always pointing inwards (or tangent to the boundary); see Figure 5. The relation between $N(\tau)$ and $\dot\Gamma_i(\tau)$ goes through the Floer equation (FE), which can be written in the form

$$J \circ Du = Du \circ j - X_H \circ u \cdot \langle \cdot, \partial_s \rangle_j + J X_H \circ u \cdot \langle \cdot, \partial_t \rangle_j.$$

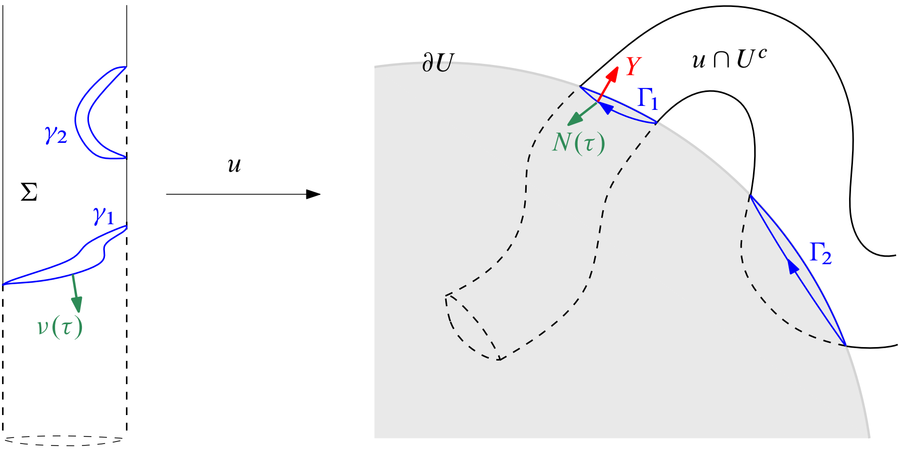

Figure 5: The normal $\nu(\tau)$ to the component γ_1 of $\partial\Sigma$ and its image, $N(\tau)$, under Du.

It follows that $\dot{\Gamma}_i(\tau)$ can be written as a linear combination of $JN(\tau)$, the gradient of H and the symplectic gradient of H:

$$\begin{aligned}\dot{\Gamma}_i(\tau) &= Du(\dot{\gamma}_i(\tau)) = Du(j\nu(\tau))\\ &= JDu(\nu(\tau)) + X_H \circ u \cdot \langle \nu(\tau), \partial_s\rangle_j - JX_H \circ u \cdot \langle \nu(\tau), \partial_t\rangle_j\\ &= JN(\tau) + X_H \circ u \cdot \langle \nu(\tau), \partial_s\rangle_j - JX_H \circ u \cdot \langle \nu(\tau), \partial_t\rangle_j .\end{aligned}$$

Using this to compute the integral of λ along Γ_i, we obtain

$$\begin{aligned}\int_{\Gamma_i} \lambda &= \int \lambda(\dot{\Gamma}_i(\tau))\,d\tau = \int \omega(Y\circ\Gamma_i(\tau), \dot{\Gamma}_i(\tau))\,d\tau\\ &= \int \omega(Y\circ\Gamma_i(\tau), JN(\tau))\,d\tau + \int [\omega(Y, X_H)\cdot\langle \nu, \partial_s\rangle_j - \omega(Y, JX_H)\cdot\langle \nu, \partial_t\rangle_j]\,d\tau\\ &= \int \langle Y\circ\Gamma_i(\tau), N(\tau)\rangle_J\,d\tau\\ &\qquad\qquad + \int [\omega(Y, J\nabla_J H)\cdot\langle \nu, \partial_s\rangle_j - \omega(Y, -\nabla_J H)\cdot\langle \nu, \partial_t\rangle_j]\,d\tau.\end{aligned}$$

Recalling our assumptions that $\nabla_J H = \delta Y$ on ∂U and that JY is the Reeb vector field, we obtain

$$\begin{aligned}\int_{\Gamma_i} \lambda &= \int \langle Y \circ \Gamma_i(\tau), N(\tau)\rangle_J\,d\tau\\ &\qquad + \delta \int [\omega(Y, JY)\cdot\langle \nu, \partial_s\rangle_j - \omega(Y, -Y)\cdot\langle \nu, \partial_t\rangle_j]\,d\tau\\ &= \int \langle Y \circ \Gamma_i(\tau), N(\tau)\rangle_J\,d\tau + \delta\cdot 1\cdot\int \langle \nu, \partial_s\rangle_j\,d\tau.\end{aligned} \tag{14}$$

Let us estimate separately each term in the sum (14), starting with the first: Since $JY = R$, the vector field Y is perpendicular to the hyperplane $T(\partial U)$ at each point and is pointing outwards from U. By our construction, $N(\tau)$ points inwards to U — as it is tangent to $\mathrm{im}(u)$ and points out of $\mathrm{im}(u) \cap U^c$ — and therefore $\langle Y \circ \Gamma_i, N \rangle \le 0$ for all τ. We conclude that

$$\int \langle Y \circ \Gamma_i(\tau), N(\tau) \rangle_J \, d\tau \le 0. \tag{15}$$

We turn to estimate the second summand in (14): Noticing that $\langle \nu, \partial_s \rangle_j = \langle j\nu, j\partial_s \rangle_j = \langle \dot\gamma_i, \partial_t \rangle_j = dt(\dot\gamma_i)$, we have

$$\int \langle \nu, \partial_s \rangle_j \, d\tau = \int dt(\dot\gamma_i)\, d\tau = \int_{\gamma_i} dt.$$

Let $\hat\Sigma$ be the closure of Σ in the compactification $(\mathbb{R} \cup \{\pm\infty\}) \times S^1$ of the cylinder. Then $\partial\hat\Sigma \subset \partial\Sigma \cup \{\pm\infty\} \times S^1$. Notice that $\partial\hat\Sigma$ contains $\{-\infty\} \times S^1$ (resp. $\{+\infty\} \times S^1$) if and only if $x_- \subset U^c$ (resp. $x_+ \subset U^c$). As $\int_{\{\pm\infty\}\times S^1} dt = \pm 1$ and, by Stokes' theorem, $\int_{\partial\hat\Sigma} dt = 0$, we conclude that

$$\sum_i \int_{\gamma_i} dt = \int_\gamma dt = \int_{\partial\hat\Sigma} dt - \begin{cases} 1 & \text{if } x_- \subset U,\ x_+ \subset U^c, \\ -1 & \text{if } x_- \subset U^c,\ x_+ \subset U, \\ 0 & \text{if } x_-, x_+ \subset U \text{ or } x_-, x_+ \subset U^c. \end{cases} \tag{16}$$

Combining (14), (15) and (16) we obtain

$$\int_\Gamma \lambda = \sum_i \int_{\Gamma_i} \lambda \le 0 + \delta \cdot \begin{cases} -1 & \text{if } x_- \subset U,\ x_+ \subset U^c, \\ 1 & \text{if } x_- \subset U^c,\ x_+ \subset U, \\ 0 & \text{if } x_-, x_+ \subset U \text{ or } x_-, x_+ \subset U^c. \end{cases}$$

This completes the proof. □

Remark 3.3 The assertion of Lemma 3.2 continues to hold if we take Γ to be $\mathrm{im}(u) \cap \psi^r(\partial U)$ for some r for which the Liouville flow is defined. The proof of the lemma goes through in this case without any significant changes, under the observation that $\omega(Y, J\nabla_J H)$ is independent of r:

$$\omega(Y, J\nabla_J H) = \omega(Y, \delta e^{-r} JY) = e^{-r}\delta\omega(Y, JY) = \delta e^{-r}\lambda(R) = \delta e^{-r} e^r = \delta.$$

When the homotopy H is nonincreasing in U^c, Lemma 3.2 can be used to bound the action of the ends of solutions that cross the boundary of U. Lemma 3.4 below is similar to a result obtained by Cieliebak and Oancea [5, Lemma 2.2] for the setting of completed Liouville domains, using *neck stretching*. The proof of Lemma 3.4 uses a different approach and is an application of Lemma 3.2 above.

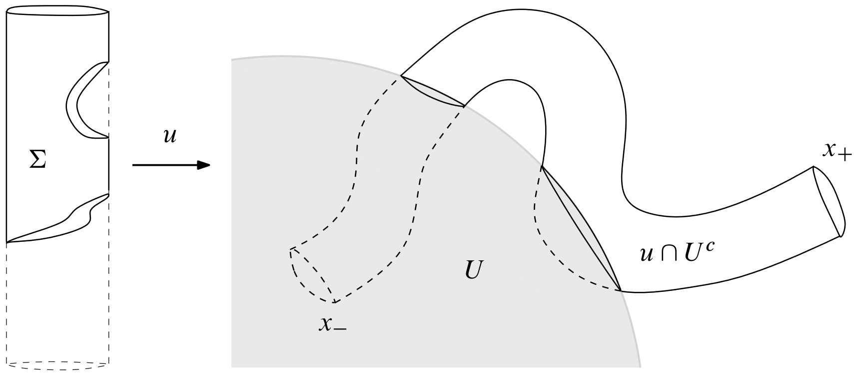

Figure 6: An example for the setting described in Lemma 3.4. The gray region is the set $U \subset M$ and $\Sigma := u^{-1}(U^c) \subset \mathbb{R} \times S^1$.

Lemma 3.4 *Suppose that (H, J) is δ–cylindrical near ∂U and assume in addition that $\partial_s H \le 0$ on U^c. For every finite-energy solution u connecting $x_\pm \in \mathcal{P}(H_\pm)$,*

(i) *if $x_- \subset U$ and $x_+ \subset U^c$, then $\mathcal{A}_{H_+}(x_+) < c - \delta$,*

(ii) *if $x_- \subset U^c$ and $x_+ \subset U$, then $\mathcal{A}_{H_-}(x_-) > c - \delta$,*

where c is the value of H on ∂U.

Proof We prove the first statement, where $x_- \subset U$ and $x_+ \subset U^c$. The second statement is proved similarly. As in [5, Lemma 2.2], after replacing U by its image, $\psi^r U$, under the Liouville flow[8] for small time r, we may assume that u is transverse to ∂U.[9] As explained in Remark 3.3, Lemma 3.2 still applies after such a replacement. Note that, since $\nabla_J H$ is positively proportional to Y on a neighborhood of ∂U, H is constant on $\partial(\psi^r U) = \psi^r(\partial U)$. Moreover, choosing the sign of r to be opposite to the sign of δ, the value of H on $\psi^r(\partial U)$ is smaller than c — in order to prove the second statement, choose r to be of the same sign as δ, and then the value of H on $\psi^r(\partial U)$ will be greater than c. Write $\Sigma := u^{-1}(U^c) \subset \mathbb{R} \times S^1$ and let us compute an energy identity for the restriction $u|_\Sigma$:

$$\begin{aligned}(17)\qquad \int_{u|_\Sigma} \omega &= \int_\Sigma \omega(\partial_s u, \partial_t u)\, ds \wedge dt \overset{\text{(FE)}}{=} \int_\Sigma \omega(\partial_s u, J\partial_s u + X_H \circ u)\, ds \wedge dt \\ &= \int_\Sigma \|\partial_s u\|_J^2\, ds \wedge dt + \int_\Sigma dH(\partial_s u)\, ds \wedge dt\end{aligned}$$

[8]This is an abuse of notation, as the Liouville vector field, and hence its flow, is not necessarily defined on all of U. We define $\psi^r U$ to be $U \cup \bigl(\bigcup_{r'<r} \psi^{r'} \partial U\bigr)$ if $r \ge 0$, and to be $U \setminus \bigl(\bigcup_{r' \in [r,0]} \psi^{r'} \partial U\bigr)$ if $r < 0$.

[9]The proof of this statement is similar to that of Thom's transversality theorem.

$$
\begin{aligned}
&= E(u|_\Sigma) + \int_\Sigma \frac{\partial}{\partial s}(H \circ u)\, ds \wedge dt - \int_\Sigma (\partial_s H) \circ u\, ds \wedge dt \\
&= E(u|_\Sigma) + \int_\Sigma d((H \circ u)\, dt) - \int_\Sigma (\partial_s H) \circ u\, ds \wedge dt \\
&\geq E(u|_\Sigma) + \int_\Sigma d(H \circ u\, dt) > \int_\Sigma d(H \circ u\, dt),
\end{aligned}
$$

where, in the last two inequalities, we used our assumption that $\partial_s H \leq 0$, and the positivity of the energy, respectively. As before, denoting by $\hat{\Sigma}$ the closure of Σ in the compactification $(\mathbb{R} \cup \{\pm\infty\}) \times S^1$, then $\partial\hat{\Sigma} = \gamma \cup \{+\infty\} \times S^1$. Since H is constant on ∂U, $\int_\gamma H \circ u\, dt = H(\partial U) \cdot \int_\gamma dt = -H(\partial U)$, where the last equality follows from (16) for $\gamma = \partial\Sigma$. Therefore, using Stokes' theorem, we obtain

$$
\int_\Sigma d(H \circ u\, dt) = \int_{\partial\hat{\Sigma}} H \circ u\, dt = -H(\partial U) + \int_0^1 H \circ x_+. \tag{18}
$$

Let $\bar{x}_\pm$ be capping disks of $x_\pm$, respectively, and let $v \subset \bar{U}$ be a union of disks capping the connected components of $\Gamma := u(\gamma)$ such that the contact form λ is defined on v. The existence of such disks follows from our definition of a CIB domain: If the relevant connected component of U is the interior of an incompressible Liouville domain, then we can take a capping disk that is contained in that component. Otherwise, the boundary of the relevant connected component of U is incompressible and we can take the capping disk to lie in the boundary. Since M is symplectically aspherical and $\omega = d\lambda$ where λ is defined, we have

$$
\int_{u|_\Sigma} \omega = \int_{\bar{x}_+} \omega + \int_v \omega = \int_{\bar{x}_+} \omega + \int_\Gamma \lambda. \tag{19}
$$

Combining (18) and (19) yields

$$
\begin{aligned}
\mathcal{A}_{H_+}(x_+) = \int_0^1 H \circ x_+ - \int_{\bar{x}_+} \omega &= \int_\Sigma d(H \circ u\, dt) + H(\partial U) - \int_{u|_\Sigma} \omega + \int_\Gamma \lambda \\
&< c + \int_\Gamma \lambda,
\end{aligned}
$$

where the last inequality is due to (17). Using Lemma 3.2 we conclude that

$$
\mathcal{A}_{H_+}(x_+) < c - \delta. \qquad \square
$$

The following lemma is essentially a version of [5, Lemma 2.2] for closed symplectically aspherical manifolds instead of completed Liouville domains.

Lemma 3.5 *Suppose that (H, J) is δ–cylindrical near ∂U and that $\partial_s H \leq 0$ on U^c. Given any pair $x_\pm \in \mathcal{P}(H_\pm) \subset U$, every solution u connecting $x_\pm$ is contained in U.*

Proof As before, after replacing U by its image, $\psi^r U$, under the Liouville flow for a small time r, we may assume that u is transverse to ∂U. Again setting $\Sigma := u^{-1}(U^c) \subset \mathbb{R} \times S^1$ and computing an energy identity, as in (17), for the restriction of u to Σ, we have

$$\int_{u|_\Sigma} \omega \geq E(u|_\Sigma) + \int_{\partial\hat{\Sigma}} H \circ u \, dt,$$

where, as before, $\hat{\Sigma}$ is the closure of Σ in the compactification of the cylinder. This time, both ends $x_\pm$ are contained in U and hence $\partial\hat{\Sigma} = \gamma$. Since H is constant on ∂U, it follows from (16) that

$$\int_{\partial\hat{\Sigma}} H \circ u \, dt = \int_\gamma H \circ u \, dt = H(\partial U) \cdot \int_\gamma dt = 0.$$

On the other hand, taking $v \subset \bar{U}$ to be a union of disks capping the connected components of $\Gamma = u(\gamma)$ (which is oriented as the boundary of $\operatorname{im}(u) \cap U^c$) such that λ is defined on v, the fact that M is symplectically aspherical implies that

$$\int_{u|_\Sigma} \omega = \int_v \omega = \int_\Gamma \lambda \leq 0,$$

where the last inequality follows from Lemma 3.2 (and Remark 3.3). Combining the above two inequalities we find

$$E(u|_\Sigma) \leq \int_{u|_\Sigma} \omega \leq 0.$$

Since we assumed that $H_\pm$ are nondegenerate and have no 1–periodic orbits intersecting ∂U, this implies $\operatorname{im}(u) \cap \operatorname{int}(U^c) = \varnothing$ and hence $\operatorname{im}(u) \subset \bar{U}$. Noticing that we may argue similarly for the image $\psi^r U$ of U under the Liouville flow for small negative time $r < 0$, we conclude that $\operatorname{im}(u) \subset \overline{\psi^r U} \subset U$. $\square$

3.2 Constructing the barricade

As before, U denotes a CIB domain and ψ^r is the flow of the Liouville vector field Y, which is defined in a neighborhood of the boundary ∂U. Consider a pair (H, J) of a homotopy (or, in particular, a Hamiltonian) and an almost complex structure. The following definition is an adaptation of Figure 1 to Floer theory.

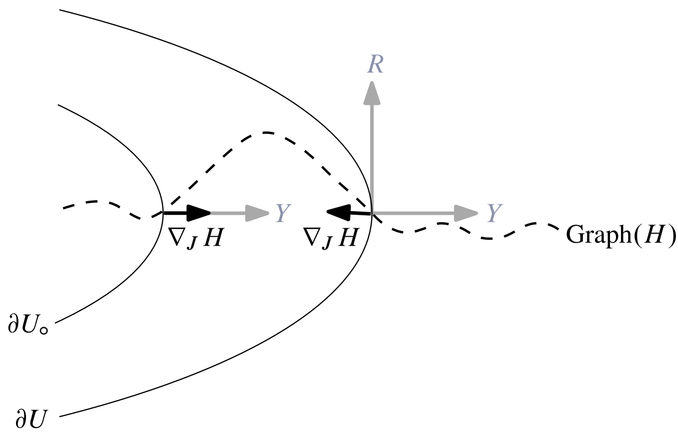

Figure 7: An illustration of a pair with a cylindrical bump.

Definition 3.6 We say that the pair (H, J) admits a *cylindrical bump of width* $r > 0$ *and slope* $\delta > 0$ *around* ∂U (abbreviate to (r, δ)–bump around ∂U) if

(i) $\partial_s H \leq 0$,

(ii) $H = 0$ on $\partial U \times S^1 \times \mathbb{R}$ and on $\partial U_\circ \times S^1 \times \mathbb{R}$, where $U_\circ := \psi^{-r} U$,

(iii) J is cylindrical on a neighborhood of ∂U which contains $\partial U_\circ$, namely, $JY = R$ on an open neighborhood of $\partial U \cup \partial U_\circ$,

(iv) on $\psi^{r'} \partial U \times S^1 \times \mathbb{R}$, we have $\nabla_J H = \delta e^{r'} Y$ if r' is near $-r$, and $\nabla_J H = -\delta e^{r'} Y$ if r' is near 0, and

(v) the only 1–periodic orbits of $H_\pm$ that are not contained in $U_\circ$ are critical points with values in $(-\delta, \delta)$.

In analogy with the discussion in Morse theory, we show that a pair with a cylindrical bump has a barricade.

Proposition 3.7 *Let* (H, J) *be a pair with a cylindrical bump of width* r *and slope* δ*. Then, the pair* (H, J) *has a barricade in* U *around* $U_\circ := \psi^{-r} U$*.*

Proof The proof essentially follows from Lemmas 3.4 and 3.5, together with the fact that a pair (H, J) with a (r, δ)–bump around ∂U is in particular cylindrical near both ∂U and $\partial U_\circ$. As explained in Remark 3.3, in this case Lemmas 3.4 and 3.5 apply for $\partial U_\circ$ as well. Let u be a solution of the s–dependent Floer equation with respect to H and J, which connects $x_\pm \in \mathcal{P}(H_\pm)$. We need to show that u satisfies the constraints from Definition 1.7, and therefore we split into two cases:

(a) Suppose $x_- \subset U_\circ$. If $x_+ \subset U_\circ$ we may apply Lemma 3.5 to H, J and $U_\circ$, and conclude that $\mathrm{im}(u) \subset U_\circ$, as required. Otherwise, $x_+ \subset U_\circ^c$ is a critical point of H_+ and its value lies in the interval $(-\delta, \delta)$. On the other hand, applying Lemma 3.4 to H, J and $U_\circ$ yields that $\mathcal{A}_{H_+}(x_+) < -\delta$, a contradiction.

(b) Suppose $x_+ \subset U$. As before, if $x_- \subset U$ then applying Lemma 3.5 to H, J and U yields $u \subset U$, as required. Otherwise, $x_- \subset U^c$ is a critical point of H_- and its value lies in $(-\delta, \delta)$. On the other hand, applying Lemma 3.4 to H, J and U, and noticing that $\nabla_J H = -\delta Y$ on ∂U, we find that $\mathcal{A}_{H_-}(x_-) > \delta$, a contradiction. ◻

In order to prove Theorem 5, it remains to guarantee the regularity assertion, for which we use the result from Section 9.3.1 below.

Proof of Theorem 5 Let H be a homotopy of Hamiltonians that is supported in $U \times S^1 \times \mathbb{R}$. Then there exists $r > 0$ small enough that H is supported inside $\psi^{-r} U =: U_\circ$. Fix an almost complex structure J that is cylindrical near both ∂U and $\partial U_\circ$ (see Definition 3.6(iii) above), and let h be a $\mathcal{C}^\infty$–small perturbation of H such that the pair (h, J) admits a (r, δ)–bump around ∂U and $h_\pm$ are nondegenerate. Notice that, by definition, the pairs $(h_\pm, J)$ also admit a (r, δ)–bump around ∂U. By Proposition 3.7, the pairs (h, J) and $(h_\pm, J)$ have a barricade in U around $U_\circ$.

The pairs (h, J), $(h_\pm, J)$ constructed above are not necessarily Floer-regular. In order to achieve regularity, we perturb the homotopy h and its ends. Proposition 9.21 below states that for a homotopy h' that satisfies $\mathcal{P}(h'_\pm) = \mathcal{P}(h_\pm)$ and $\mathrm{supp}(\partial_s h') \subset M \times S^1 \times I$ for some fixed finite interval I, if h' is close enough to h, then (h', J) also has a barricade in U around $U_\circ$. Therefore, it remains to describe a perturbation that satisfies the above constraints, and ensures regularity. Starting with the ends and recalling that the $h_\pm$ are nondegenerate, we perturb them without changing their periodic orbits to guarantee that the pairs $(h_\pm, J)$ are Floer-regular — the fact that this is possible is a well-known result from Floer theory, cited in Claim 9.1 below. If the homotopy h is constant, that is, corresponds to a single Hamiltonian, we are done. Otherwise, let us perturb h so that its ends will agree with the regular perturbations of $h_\pm$. Finally, we perturb the resulting homotopy on the set $M \times S^1 \times I$, for some fixed finite interval I, to make the pair (h, J) Floer-regular. This is possible due to Proposition 9.2 below, which is a slight modification of standard claims from Floer theory, and is proved in Section 9.1. ◻

Remark 3.8 Proposition 3.7 suggests that, when given a homotopy (or a Hamiltonian) H that is supported in $U \times S^1 \times \mathbb{R}$, we have some freedom in choosing the pair (h, J)

from Theorem 5. Let us mention some additional properties that can be granted for the perturbation h and the almost complex structure J, and will be useful in applications.

(i) The almost complex structure J can be taken to be time-independent. Moreover, if one of the ends of H, say H_-, is zero, then h can be chosen such that h_- is any time-independent small Morse function that has a cylindrical bump around ∂U. To see this, choose $h \approx H$ and J such that (h, J) has a cylindrical bump around ∂U, and J and h_- are time-independent. Then, the pair (h_-, J) is Floer-regular and, by perturbing h_+ first and then replacing the homotopy by a compactly supported perturbation, we end up with a pair (h, J) that is Floer-regular, as well as its ends, and (h_-, J) is time-independent.

(ii) When the homotopy H is constant on some domain, we can choose the perturbation h such that, on this domain, its ends $h_\pm$ agree on their 1–periodic orbits up to second order. This follows from the use of Claim 9.1 in the proof of Theorem 5.

(iii) Given an interval $[a,b] \subset \mathbb{R}$ such that H is a constant homotopy for $s \notin [a,b]$, we can chose the perturbation h of H to be also constant outside of $[a,b]$, namely $\operatorname{supp}(\partial_s h) \subset M \times S^1 \times [a,b]$. This follows from the use of Proposition 9.2 in the proof of Theorem 5.

(iv) Proposition 3.7 also holds when considering a homotopy of almost complex structures $\{J_s\}_{s\in\mathbb{R}}$, but the demand on (h, J) to have a (r,δ)–bump around ∂U limits the dependence of J_s on s there.

4 Locality of spectral invariants, Schwarz's capacities and superheavy sets

In this section we use barricades to prove Theorem 1 and derive Corollaries 1.5 and 1.6. We will use the definitions and notation from Section 2, in particular Notation 2.1 and formula (11). We will also use the following properties of spectral invariants (see for example [16, Proposition 12.5.3]):

- **Spectrality** $c(F;\alpha) \in \operatorname{spec}(F)$.
- **Stability/continuity** For any Hamiltonians F, G and homology class $\alpha \in H_*(M)$,
$$\int_0^1 \min_{x\in M}(F(x,t)-G(x,t))\,dt \le c(F;\alpha)-c(G;\alpha) \le \int_0^1 \max_{x\in M}(F(x,t)-G(x,t))\,dt.$$
In particular, the functional $c(\cdot\,;\alpha)\colon \mathcal{C}^\infty(M\times S^1)\to\mathbb{R}$ is continuous.

- **Poincaré duality** For any Hamiltonian F, we have $c(F;[M]) = -c(-F;[\mathrm{pt}])$.
- **Energy-capacity inequality** If the support of F is displaceable, then $c(F;[M])$ is bounded by the displacement energy of the support in M, namely $c(F;[M]) \le e(\mathrm{supp}(F);M)$. Recall that a subset X of a symplectic manifold is displaceable if there exists a Hamiltonian G such that $\varphi_G^1(X) \cap X = \varnothing$. In this case, the displacement energy of X is given by

$$e(X;M) := \inf_{G:\varphi_G^1(X)\cap X=\varnothing} \int_0^1 \big(\max_M G(\,\cdot\,,t) - \min_M G(\,\cdot\,,t)\big)\,dt. \tag{20}$$

Let us sketch the idea of the proof of Theorem 1 before giving the details. We will prove the statement for the class of a point, and use Poincaré duality to deduce the same for the fundamental class. We start by showing that the spectral invariant, with respect to [pt], of a Hamiltonian supported in a CIB domain is nonpositive (Lemma 4.1). Then, after properly choosing regular perturbations with barricades (Lemma 4.4), we consider a representative of [pt] of negative action on M. Such a representative must be a combination of orbits in $U_\circ$ and thus can be pushed to a cycle on N. Finally, we use continuation maps, induced by homotopies to small Morse functions, to conclude that the cycle on N represents [pt] there.

As mentioned above, our first step towards proving Theorem 1 is showing that the spectral invariant with respect to [pt] of a Hamiltonian supported in a CIB domain is always nonpositive.

Lemma 4.1 *Let $F\colon M \times S^1 \to \mathbb{R}$ be a Hamiltonian supported in a CIB domain U. Then $c(F;[\mathrm{pt}]) \le 0$.*

Proof Let H be a linear homotopy[10] from $H_- := 0$ to $H_+ := F$. By Theorem 5, there exist a small perturbation h of H and an almost complex structure J such that (h,J) and $(h_\pm,J)$ are Floer-regular and have a barricade in U around $U_\circ$, where $U_\circ$ contains the support of F. By Remark 3.8(i), we can choose J to be time independent and h so that h_- is a time-independent small Morse function. Moreover, we may assume that h_- has a minimum point p that is contained in U^c. Since the Floer complex and differential of (h_-,J) agree with the Morse ones, the point p represents [pt] in $CF_*(h_-) \cong CM_{*+n}(h_-)$. Denoting by $\Phi_{(h,J)}\colon CF_*(h_-) \to CF_*(h_+)$ the continuation map associated to the pair (h,J), the presence of the barricade guarantees

[10] A *linear homotopy* is a homotopy of the form $H(x,t,s) = H_-(x,t) + \beta(s)(H_+(x,t) - H_-(x,t))$, where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a smooth step function.

that $\Phi_{(h,J)}(p) \subset C_{U^c}(h_+)$. Indeed, otherwise, we would have a continuation solution starting at $p \subset U^c$ and ending at some $x_+ \subset U$, a contradiction. The image $\Phi_{(h,J)}(p)$ is a cycle representing [pt] in $CF_*(h_+)$ and its action level is close to zero. Indeed, since h_+ approximates F, which is supported in $U_\circ$, the restriction $h_+|_{U_\circ^c}$ is a small Morse function. Its 1–periodic orbits there are critical points and their actions are the critical values. Therefore, using the stability property of spectral invariants, we conclude that $c(F;[\mathrm{pt}]) \leq c(h_+;[\mathrm{pt}]) + \delta \leq \lambda_{h_+}(\Phi_{(h,J)}(p)) + \delta \leq 2\delta$ for small $\delta > 0$. □

Remark 4.2 • Using Poincaré duality for spectral invariants, the above lemma implies that $c(F;[M]) \geq 0$ for every Hamiltonian F supported in a CIB domain. This is already known for Hamiltonians supported in the interiors of incompressible Liouville domains. Indeed, it follows easily from the max formula, proved in [12], when applied to the functions $F_1 = F$ and $F_2 = 0$:

$$c(F+0;[M]) = \max\{c(F;[M]), c(0;[M])\} \geq 0.$$

• Lemma 4.1 does not hold if M is not symplectically aspherical. For example, the equator in S^2 is known to be superheavy. Therefore, if F is a Hamiltonian on S^2 which is supported on a disk containing the equator, then

$$\zeta(F) = \lim_{k\to\infty} c(kF;[M])/k$$

is not greater than the maximal value that F attains on the equator; see [16, Chapter 6]. Therefore, one can construct a Hamiltonian supported in a disk on S^2 with a negative spectral invariant with respect to the fundamental class.

Our next step towards the proof of Theorem 1 is choosing suitable perturbations for the Hamiltonians F and $\Psi_* F$, as well as homotopies from them to small Morse functions. Before that, we use the embedding Ψ to define a linear map between subspaces of Floer complexes of Hamiltonians on M and on N, which agree on U through Ψ.

Definition 4.3 Consider nondegenerate Hamiltonians f_M on M and f_N on N such that f_M and $f_N \circ \Psi$ have the same 1–periodic orbits in U. For an element $a \in C_U(f_M) \subset CF_*(f_M)$ that is a combination of orbits contained in U, we define its pushforward with respect to the embedding Ψ to be

$$\Psi_* a := \sum_{x\in a} a_x \cdot \Psi(x) \in C_{\Psi(U)}(f_N) \subset CF_*(f_N).$$

Lemma 4.4 (setup) *There exist homotopies and time-independent almost complex structures h_M and J_M on M, and h_N and J_N on N, such that the following hold*:

(i) *The pairs (h_M, J_M), $(h_{M\pm}, J_M)$, (h_N, J_N) and $(h_{N\pm}, J_N)$ are all Floer-regular and have barricades in U around $U_\circ$ and in $\Psi(U)$ around $\Psi(U_\circ)$, respectively, for some $U_\circ \Subset U$ containing the support of F.*

(ii) *h_{M-} and h_{N-} are small perturbations of F and $\Psi_* F$, respectively, and h_{M+} and h_{N+} are small time-independent Morse functions.*

(iii) *On $\Psi(U)$, the Hamiltonians h_{N-} and $h_{M-} \circ \Psi^{-1}$ agree on their periodic orbits up to second order and $J_N = \Psi_* \circ J_M \circ \Psi_*^{-1}$ (abbreviate to $J_N = \Psi_* J_M$).*

(iv) *The differentials and continuation maps commute with the pushforward map Ψ_* when restricted to $U_\circ$:*

$$\Phi_{(h_N, J_N)} \circ \Psi_* \circ \pi_{U_\circ} = \Psi_* \circ \Phi_{(h_M, J_M)} \circ \pi_{U_\circ}, \tag{21}$$

$$\partial_{(h_{N\pm}, J_N)} \circ \Psi_* \circ \pi_{U_\circ} = \Psi_* \circ \partial_{(h_{M\pm}, J_M)} \circ \pi_{U_\circ}. \tag{22}$$

We postpone the proof of Lemma 4.4, and prove Theorem 1 first.

Proof of Theorem 1 We will prove that $c_M(F; [\mathrm{pt}]) = c_N(\Psi_* F; [\mathrm{pt}])$, and the claim for the fundamental class will follow from Poincaré duality for spectral invariants. Suppose that at least one of $c_M(F; [\mathrm{pt}])$ and $c_N(\Psi_* F; [\mathrm{pt}])$ is nonzero, otherwise there is nothing to prove. Without loss of generality, assume that $c_M(F; [\mathrm{pt}]) \neq 0$; then, by Lemma 4.1, $c_M(F; [\mathrm{pt}]) < 0$. We will show that $c_M(F; [\mathrm{pt}]) \geq c_N(\Psi_* F; [\mathrm{pt}])$. This will imply that $c_N(\Psi_* F; [\mathrm{pt}]) < 0$ and equality will follow by symmetry. Let (h_M, J_M) and (h_N, J_N) be pairs of homotopies and almost complex structures on M and N, respectively, that satisfy the assertions of Lemma 4.4, and write $f_M := h_{M-}$ and $f_N := h_{N-}$. By the continuity of spectral invariants, it is enough to prove the claim for f_M and f_N.

Since $c_M(F; [\mathrm{pt}]) < 0$ and $F|_{U_\circ^c} = 0$, by taking f_M to be close enough to F and $F|_{U_\circ^c} = 0$, we may assume that $c_M(f_M; [\mathrm{pt}]) < \min_{U_\circ^c} f_M$. Recalling that f_M is a small Morse function on $U_\circ^c$, its 1–periodic orbits there are its critical points, and their actions are the critical values. As a consequence, a representative $a \in CF_*(f_M)$ of [pt] of action level $\lambda_{f_M}(a) = c_M(f_M; [\mathrm{pt}])$ is a combination of orbits that are contained in $U_\circ$, namely $a \in C_{U_\circ}(f_M)$. Therefore, the pushforward $\Psi_* a \in CF_*(f_N)$ is defined, and by (22), $\Psi_* a$ is closed in $CF_*(f_N)$. To see that $\Psi_* a$ represents the class of a point, we will use (21). Indeed, since a represents [pt] on M, and continuation maps induce isomorphism on homologies, $\Phi_{(h_M,J_M)}(a)$ is a representative of [pt] in $CF_*(h_{M+})$. Since h_{M+} is a small time-independent Morse function (and J_M is time-independent), its Floer complex and differential coincide with the Morse ones,

$$(CF_*(h_{M+}), \partial_{(h_{M+},J_M)}) \cong (CM_{*+n}(h_{M+}), \partial^{\mathrm{Morse}}_{(h_{M+},g_{J_M})}).$$

As a consequence, $\Phi_{(h_M,J_M)}(a)$ is a sum of an odd number of minima.[11] Using (21), we find that $\Phi_{(h_N,J_N)}(\Psi_* a) = \Psi_*(\Phi_{(h_M,J_M)}a)$ is also a sum of an odd number of minima, and as such, represents the point class in $CM_{*+n}(h_{N+}) \cong CF_*(h_{N+})$. Since $\Psi_* a$ is closed, we conclude that it represents [pt] in $CF_*(f_N)$. Together with the fact that, in $\Psi(U)$, $f_M \circ \Psi^{-1}$ and f_N agree on their 1–periodic orbits, this implies that

$$c_N(f_N; [\mathrm{pt}]) \le \lambda_{f_N}(\Psi_* a) = \lambda_{f_M}(a) = c_M(f_M; [\mathrm{pt}]),$$

where the equality $\lambda_{f_N}(\Psi_* a) = \lambda_{f_M}(a)$ follows from the fact that U is incompressible; see Remark 1.3 and Proposition A.35. □

Proof of Lemma 4.4 Let $H_M : M \times S^1 \times \mathbb{R} \to \mathbb{R}$ be a linear homotopy from F to zero, that is constant outside of [0, 1], ie $\partial_s H_M|_{s \notin [0,1]} = 0$. Then H_M is supported in U, and its pushforward $H_N := \Psi_* H_M$ is a linear homotopy from $\Psi_* F$ to zero on N. Let J_M be a time-independent almost complex structure on M and let $h^\flat_M$ be a homotopy with nondegenerate ends, that is constant outside of [0, 1] and approximates H_M, and is such that the pair $(h^\flat_M, J_M)$ has a (r,δ)–bump around ∂U for some r and δ. Set $U_\circ = \psi^{-r} U$. Let J_N be a time-independent almost complex structure obtained as an extension of $\Psi_* J_M$ from $\Psi(U)$ to N.[12] Extending $h^\flat_M \circ \Psi^{-1}$ to N by a homotopy of small Morse functions with critical values in $(-\delta,\delta)$, we obtain a pair $(h^\flat_N, J_N)$ with a (r,δ)–bump around $\Psi(\partial U) = \partial \Psi(U)$. Moreover, $h^\flat_N$ is a homotopy with nondegenerate ends, it approximates H_N, and we can choose it to be constant for $s \notin [0,1]$. Noticing that the ends of these homotopies have (r,δ)–bumps as well, Proposition 3.7 guarantees that the pairs $(h^\flat_M, J_M)$, $(h^\flat_{M\pm}, J_M)$, $(h^\flat_N, J_N)$ and $(h^\flat_{N\pm}, J_N)$ all have barricades in U around $U_\circ$ and in $\Psi(U)$ around $\Psi(U_\circ)$, respectively.

Let us now perturb $h^\flat_M$ to make all of the pairs defined on M regular. As in the proof of Theorem 5, we first perturb the ends $h^\flat_{M\pm}$ into $h_{M\pm}$, without changing their periodic orbits, so that the pairs $(h_{M\pm}, J_M)$ are Floer-regular (as cited in Claim 9.1 below).

[11] See, for example, the proof of Proposition 4.5.1 in [3].

[12] The fact that $\Psi_* J_M$ can be extended to an almost complex structure on N can be deduced from the path-connectivity of the set of almost complex structures on symplectic vector bundles (see eg [14, Proposition 2.63]), together with the fact that ∂U has a tubular neighborhood.

Then, perturb the homotopy $h_M^\flat$ to obtain a homotopy h_M whose ends are the regular perturbations $h_{M\pm}$ and which is constant for $s \notin [0,1]$. Finally, Proposition 9.2 below states that we can perturb the homotopy h_M on the set $M \times S^1 \times [0,1]$ to make the pair (h_M, J_M) Floer-regular. We stress that after the perturbations the regular homotopy h_M is constant for $s \notin [0,1]$ as well. Proposition 9.21 guarantees that every small enough perturbation of $h_M^\flat$ that is constant outside of $[0,1]$ and whose ends have the same periodic orbits as the ends of $h_M^\flat$, also has a barricade in U around $U_\circ$, when paired with J_M. Arguing similarly for the ends $h_{M\pm}$ we conclude that the pairs (h_M, J_M) and $(h_{M\pm}, J_M)$ all have barricades in U around $U_\circ$.

We turn to construct the pairs on N. Let h'_N be an extension to N of the homotopy $h_M \circ \Psi^{-1}$, which is defined on $\Psi(U)$. Notice that by replacing h_M with a smaller perturbation of $h_M^\flat$ if necessary, h'_N can be taken to be arbitrarily close to $h_N^\flat$. This way, we can use Proposition 9.21 again to conclude that (h'_N, J_N) has a barricade in $\Psi(U)$ around $\Psi(U_\circ)$. Finally, we repeat the arguments made above and perturb h'_N to make all of the pairs on N Floer-regular. We obtain a homotopy h_N that is constant for $s \notin [0,1]$, approximates $h_M \circ \Psi^{-1}$ on $\Psi(U)$ and is such that the pairs (h_N, J_N) and $(h_{N\pm}, J_N)$ are all Floer-regular and have barricades in $\Psi(U)$ around $\Psi(U_\circ)$.

It remains to prove that, in $U_\circ$, the pushforward map commutes with the continuation maps and the differentials for the homotopies h_M, h_N and their ends, respectively. We will write the proof for the continuations maps; the proof for the differentials is analogous. We first show that the continuation maps of h_N and h'_N agree on $\Psi(U_\circ)$, and then prove that the commutation relation (21) holds for h_M and h'_N, which agree on U through Ψ. Proposition 9.31 (for the differentials, Proposition 9.25) states that the restriction of the continuation map to $C_{\Psi(U_\circ)}$ does not change under small perturbations, when the pairs have a barricade and satisfy a certain regularity assumption on $\Psi(U)$. This assumption holds for Floer-regular pairs, as well as for pairs that coincide on U with a Floer-regular pair. Therefore, recalling that h_N is a small perturbation of h'_N, and that the pair (h'_N, J_N) agrees on $\Psi(U)$, through a symplectomorphism, with the Floer-regular pair (h_M, J_M), we may apply Proposition 9.31 and conclude that $\Phi_{(h_N, J_N)} \circ \pi_{\Psi(U_\circ)} = \Phi_{(h'_N, J_N)} \circ \pi_{\Psi(U_\circ)}$. In order to prove $\Phi_{(h'_N, J_N)} \circ \Psi_* \circ \pi_{U_\circ} = \Psi_* \circ \Phi_{(h_M, J_M)} \circ \pi_{U_\circ}$, recall the definitions of Ψ_* and the continuation maps (11). We need to show that for every $x_\pm \in \mathcal{P}(h_{M\pm})$ such that $x_- \subset U_\circ$, it holds that $\#_2\mathcal{M}_{(h_M, J_M)}(x_-, x_+) = \#_2\mathcal{M}_{(h'_N, J_N)}(\Psi(x_-), \Psi(x_+))$. This essentially follows from the fact that both pairs (h_M, J_M) and (h'_N, J_N) have barricades, and that $h_M = h'_N \circ \Psi$ and $J_M = J_N \circ \Psi$ on U. Indeed, it follows from $x_- \subset U_\circ$ that $\Psi(x_-) \subset \Psi(U_\circ)$

and thus the barricades guarantee that all of the elements of $\mathcal{M}_{(h_M,J_M)}(x_-,x_+)$ and $\mathcal{M}_{(h'_N,J_N)}(\Psi(x_-),\Psi(x_+))$ are contained in $U_\circ$ and $\Psi(U_\circ)$, respectively. The symplectic embedding Ψ induces a bijection between these two sets, and so it follows that the counts of their elements coincide. □

Having established Theorem 1, we now explain how to derive Corollaries 1.5–1.6. Let us start by recalling the definition of a symplectic capacity:

Definition 4.5 (see eg [4; 11]) Given a class $\mathcal{S}$ of symplectic manifolds, a symplectic capacity on $\mathcal{S}$ is a map $c\colon \mathcal{S}\to[0,\infty]$ that satisfies the following properties:

- **Monotonicity** $c(U,\omega)\le c(V,\Omega)$ if there exists a symplectic embedding $(U,\omega)\hookrightarrow(V,\Omega)$.
- **Conformality** $c(U,\tau\omega)=|\tau|\cdot c(U,\omega)$ for all $\tau\in\mathbb{R}\setminus\{0\}$.
- **Nontriviality** $c(B^{2n}(1),\omega_0)>0$ and $c(Z^{2n}(1),\omega_0)<\infty$, where $B^{2n}(1)$ is the unit ball in $\mathbb{R}^{2n}$ and $Z^{2n}(1)=B^2(1)\times\mathbb{R}^{2n-2}$ is the symplectic cylinder.

Recall the class $\mathcal{S}$ of contractible compact symplectic manifolds with contact-type boundaries that can be embedded into symplectically aspherical manifolds, and consider the class $\widetilde{\mathcal{S}}$ of symplectic manifolds X exhaustible by elements from $\mathcal{S}$, namely such that there exist $A_i\in\mathcal{S}$ such that $A_1\subset A_2\subset\cdots\subset X$ and $X=\bigcup_i A_i$. Let us use Theorem 1 to show that Schwarz's relative capacities, which are defined for subsets of a given closed symplectically aspherical manifold, induce a capacity on the class $\widetilde{\mathcal{S}}$.

Proof of Corollary 1.5 Let $A\in\mathcal{S}$ be a contractible symplectic manifold with a contact-type boundary that can be embedded into a symplectically aspherical manifold (M,ω). Abusing notation, we write $A\subset M$. We start by showing that such an A is an incompressible Liouville domain, and hence a CIB domain, in M. If $\dim A=2$, then A is symplectomorphic to a disc of the same area (see [6] for example) and in particular is an incompressible Liouville domain. Suppose now that $\dim A>2$ and let us show that the Liouville form (and hence the Liouville vector field) extends to A. Denote by λ the Liouville form defined near the boundary of A. Since A is contractible, ω is exact on A. Let θ be a primitive of ω on A. Then $\theta-\lambda$ is a closed form on a collar neighborhood of ∂A on which λ is defined. Since $\dim A>2$, its boundary has a vanishing first homology group.[13] As a consequence, the closed form $\theta-\lambda$ defined

[13] The boundary of a $2n$–dimensional contractible manifold with boundary has the homology groups of the $(2n-1)$–dimensional sphere. This follows from the Lefschetz duality $H^{2n-k}(M)\cong H_k(M,\partial M)$ together with the long exact sequence of a pair.

near ∂A is exact. Let f be a primitive, $df = \theta - \lambda$, and let χ be a cutoff function supported in the collar neighborhood of ∂A that is equal to 1 on a smaller neighborhood of the boundary. Then $\lambda' := \theta - d(\chi f)$ is a Liouville form which coincides with λ near ∂A. We conclude that A is a Liouville domain which is incompressible in M, and hence is a CIB domain.

Recall the definition of Schwarz's relative capacity (2),

$$c_\gamma(A; M) := \sup\{c(F;[M]) - c(F;[\mathrm{pt}]) : \operatorname{supp} X_F \subset A \times S^1\}.$$

Consider a Hamiltonian F on M such that X_F is supported in $A \times S^1$. Since A is contractible, its boundary connected and therefore F is constant on ∂A, as well as on the complement, $M \setminus A$. Denoting $C := F|_{M\setminus A}$, the difference $F - C$ is supported in A. Moreover, it follows from the spectrality and stability of spectral invariants that $c_M(F - C;\alpha) = c_M(F;\alpha) - C$ for every homology class $\alpha \in H_*(M)$. In particular, $c_M(F - C;[M]) - c_M(F - C;[\mathrm{pt}]) = c_M(F;[M]) - c_M(F;[\mathrm{pt}])$ and hence, by replacing F with $F - C$, we may assume that F is supported in A. Suppose that A can be embedded into another symplectically aspherical manifold (N, Ω). By Theorem 1, the spectral invariants of Hamiltonians supported in A on M and N coincide, and therefore the relative capacities of A with respect to M and N agree, and we can define

$$c_\gamma(A) := c_\gamma(A; M) = c_\gamma(A; N).$$

We may extend this definition to elements of the class $\widetilde{\mathcal{S}}$ by taking the supremum over all elements $A \in \mathcal{S}$ in the exhaustion. Before proving that c_γ satisfies the axioms of a symplectic capacity, let us prove the second assertion of the corollary. Given an $A \in \mathcal{S}$ that can be symplectically embedded into $(\mathbb{R}^{2n}, \omega_0)$, we need to show that $c_\gamma(A; M) \le 2e(A;\mathbb{R}^{2n})$. Let $Q = [-R, R]^{2n} \subset \mathbb{R}^{2n}$ be a large cube such that the embedding of A into $\mathbb{R}^{2n}$ satisfies $e(A; Q) = e(A;\mathbb{R}^{2n})$. Then, embedding Q into a large torus $N = \mathbb{R}^{2n}/(3R\mathbb{Z}^{2n}) \cong \mathbb{T}^{2n}$, we conclude that $e(A; N) = e(A;\mathbb{R}^{2n})$. By the energy-capacity inequality, for every Hamiltonian F supported in the embedding of A into N and for every homology class α, one has $c(F;\alpha) \le e(A; N) = e(A;\mathbb{R}^{2n})$. Using Theorem 1 we conclude that for every symplectically aspherical M and an embedding of A into M, $c_\gamma(A; M) = c_\gamma(A) \le 2e(A;\mathbb{R}^{2n})$.

We now briefly explain why c_γ satisfies the axioms of a capacity. Nontriviality follows from the fact that Schwarz's capacities are not smaller than the Hofer–Zehnder capacity, and are not greater than twice the displacement energy; see [18]. Monotonicity follows from the definition of $c_\gamma(\,\cdot\,; M)$, together with the fact that the image of every embedding of a domain in $\mathcal{S}$ into a symplectically aspherical manifold is a CIB domain. To prove the conformality property, suppose that $(A,\Omega)\in\mathcal{S}$ is embedded into (M,ω), and then $(A,\tau\cdot\Omega)$ is embedded into $(M,\tau\cdot\omega)$. In order to prove that

$$c_\gamma((A,\tau\Omega),(M,\tau\omega)) = |\tau|\cdot c_\gamma((A,\Omega),(M,\omega)),$$

we show that for every F such that $\operatorname{supp}(X_F)\subset A\times S^1$ and for every homology class $\alpha\in H_*(M)$, it holds that

$$c_{(M,\tau\omega)}(|\tau|\cdot F;\alpha) = |\tau|\cdot c_{(M,\omega)}(F;\alpha). \tag{23}$$

Starting from the case where $\tau>0$, we notice that the action functional with respect to the form $\tau\omega$ and the Hamiltonian τF is proportional to the action functional with respect to ω and F. The Floer complexes of (ω,J,F) and $(\tau\omega,J,\tau F)$ coincide, while the action filtration is rescaled by τ, and therefore (23) holds. It remains to deal with $\tau=-1$. In this case, the Floer complexes of (ω,J,F) and $(-\omega,-J,F)$ are isomorphic via the map $t\mapsto -t$, and the action filtration is the same. This implies that (23) holds for negative τ as well. □

Proof of Corollary 1.6 Let $A\subset M$ be a contractible domain with a contact-type boundary that can be symplectically embedded in $(\mathbb{R}^{2n},\omega_0)$. As in the proof of Corollary 1.5, let $Q\subset\mathbb{R}^{2n}$ be a large enough cube such that the image of A in $\mathbb{R}^{2n}$ is displaceable in Q. Embedding Q into a large torus $N\cong\mathbb{T}^{2n}$, we denote by $\Psi\colon A\hookrightarrow N$ the composition of the embeddings. As $\Psi(A)$ is displaceable in N, it follows from nonnegativity of $c(\cdot\,;[M])$ (Lemma 4.1), Theorem 1 and the energy capacity inequality that for every Hamiltonian $F\colon M\times S^1\to\mathbb{R}$ supported in A,

$$0\le c_M(F;[M]) = c_N(\Psi_*F;[N]) \le e(A;N) < \infty.$$

As a consequence, the partial symplectic quasistate ζ associated to c vanishes on functions supported in A. The fact that the complement of A is superheavy follows from the following equivalent description of superheavy sets:

Definition [16, Definition 6.1.10] A closed subset $X\subset M$ is superheavy if $\zeta(F)=0$ for every Hamiltonian F that vanishes on X.

The fact that A cannot contain a heavy set can be seen directly from the definition. Alternatively, this fact follows from the intersection property of heavy and superheavy sets, established by Entov and Polterovich in [7]: Every superheavy set intersects every heavy set. □

We conclude this section with two examples, showing that Theorem 1 does not hold in a more general setting.

Example 4.6 The conditions on the manifolds M, N and the domain U in Theorem 1 are necessary:

- The condition on M and N being symplectically aspherical in Theorem 1 is necessary. A simple example is to embed the unit disk $D \subset \mathbb{R}^2$ into a small sphere and into a large sphere. Namely, take M and N to be spheres of areas 1.5π and 2π, respectively. Then there exist Hamiltonians, supported in the embedding of D into M, with arbitrarily large spectral invariants with respect to the fundamental class. This follows from the fact that the embedding of D into M contains the equator, which is a heavy set; see [16, Chapter 6]. A Hamiltonian F that attains large values on the equator in M has a large spectral invariant.

On the other hand, the spectral invariant of any Hamiltonian that is supported in the embedding of D into N is bounded by the displacement energy of this embedded disc in N, which is equal to π.

- The condition that ∂U be incompressible is also necessary. Consider the two embeddings of the annulus $A := \operatorname{int}\bigl(D \setminus \frac{1}{2}D\bigr)$ into a torus of large area, illustrated in Figure 2. The image under the first embedding contains the meridian and therefore is heavy [7] (in this case the boundary is incompressible). The image under the second embedding is displaceable (and the boundary is not incompressible). As mentioned above, in the first case one can construct Hamiltonians with arbitrarily large spectral invariants (with respect to the fundamental class), and in the second case, the spectral invariant is bounded by the (finite) displacement energy. In particular, the assertion of Theorem 1 cannot hold in this case.

5 Relation to certain open symplectic manifolds

In this section we discuss an extension of Theorem 1 to CIB domains in certain open symplectic manifolds. We start by briefly reviewing Floer homology on such manifolds, following [9].[14] Let (W, ω) be a $2n$–dimensional compact symplectic manifold with a contact-type boundary. Using the Liouville vector field Y, we can symplectically

[14]A lot of our sign choices are opposite to those of [9]. Essentially, the complex defined in [9] for a Hamiltonian F coincides with the complex defined here for $-F$.

identify a neighborhood of the boundary in W with $\partial W \times (\varepsilon, 0]$ endowed with the symplectic form $d(e^r\lambda)$, where $\lambda = \iota_Y\omega$ and r is the coordinate on the interval. The *completion* of (W,ω) is defined to be

$$\widehat{W} := W \cup_{\partial W} \partial W \times [0,\infty), \qquad \widehat{\omega} := \begin{cases} \omega & \text{on } W, \\ d(e^r\lambda) & \text{on } \partial W \times (-\varepsilon,\infty). \end{cases}$$

Let J be an $\widehat{\omega}$–compatible almost complex structure on $\widehat{W}$ that, on ∂W, maps Y to the Reeb vector field R and, on $\partial W \times [0,\infty)$, is time-independent and is invariant under r–translations. A time-dependent Hamiltonian F on $\widehat{W}$ is called *admissible* if it coincides on $\partial W \times [0,\infty)$ with $\rho(e^r)$ for a function $\rho\colon [0,\infty) \to \mathbb{R}$ whose derivative on $(0,\infty)$ is positive and smaller than the minimal period of a periodic Reeb orbit—note that in this case, F has no 1–periodic orbits in $W \times (0,\infty)$. For a generic admissible Hamiltonian, the Floer complex of the pair (F,J) on the open manifold $(\widehat{W},\widehat{\omega})$ is generated by the 1–periodic orbits of F in W, and the differential is defined by counting solutions of the Floer equation, as in the closed case; see Section 2. The above assumptions on F and J guarantee that finite-energy solutions are contained in W. This follows from a standard application of the max-principle (see for example [23, Lemma 1.8; 17, Lemma 2.1]), or from Lemma 3.5 above. The homology of this complex is independent of F and J and is isomorphic to the homology of W. Spectral invariants on open manifolds were defined in [9, Section 5] in complete analogy with the closed case.[15] These invariants extend by continuity to any Hamiltonian supported in W.

Remark 5.1 It was suggested to us by Schlenk that Theorem 1 holds for the spectral invariant with respect to the point class on the above open manifolds as well. Namely, given a CIB domain U in W and a symplectic embedding $\Psi\colon (U,\omega) \to (W',\omega')$ whose image is a CIB domain in W', then for every Hamiltonian F supported in U,

$$c_W(F;[\mathrm{pt}]) = c_{W'}(\Psi_*F;[\mathrm{pt}]),$$

where $\Psi_*F\colon W' \times S^1 \to \mathbb{R}$ is the extension by zero of $F \circ \Psi^{-1}$.

5.1 The homology of the subcomplex $C_{U_\circ}(f)$

In what follows, (M,ω) denotes a closed symplectic manifold, as always. Given a Hamiltonian F supported in U, let (f,J) be a Floer-regular pair on M with a barricade

[15] The definition in [9] is given for the point class, but generalizes as is to any $\alpha \in H_*(W)$.

in U around $U_\circ$ for some $U_\circ \Subset U$. The block form (7) of the differential implies that the differential restricts to $C_{U_\circ}(f) \subset CF(f)$. In this section we study the homology of this subcomplex. We show that for a properly chosen pair (f, J), the homology of $(C_{U_\circ}(f), \partial|_{U_\circ})$ coincides with the homology of U, namely,

$$H_*(C_{U_\circ}(f), \partial|_{U_\circ}) \cong H_*(U). \tag{24}$$

To that end, consider a perturbation $f^\flat$ of F such that $(f^\flat, J)$ has a (r, δ)–bump around ∂U (in the sense of Definition 3.6). In particular, we assume that J is cylindrical. Let f be a C^2–small perturbation of $f^\flat$ such that the pair (f, J) is Floer-regular. As argued in the proof of Theorem 5, it follows from Propositions 3.7 and 9.21 that the pair (f, J) has a barricade in U around $U_\circ := \psi^{-r} U$. Taking f to be close enough to $f^\flat$, the restriction $f|_{U_\circ}$ of f to $U_\circ$ can be extended to an admissible Hamiltonian $\hat{f} := \widehat{f|_{U_\circ}}$ on $\hat{U}$ that has no additional 1–periodic orbits. Here $(\hat{U}, \hat{\omega})$ is the open symplectic manifold obtained as the completion of U. As the 1–periodic orbits of $\hat{f}$ in $\hat{U}$ coincide with the 1–periodic orbits of f that are contained in $U_\circ$, the Floer complex of $\hat{f}$ on the open manifold $\hat{U}$ coincides with $C_{U_\circ}(f)$. Since both in M and in $\hat{U}$ all finite-energy solutions of the Floer equation among orbits in $U_\circ$ are contained in $U_\circ$, the differentials coincide. We conclude that the homology of $(C_{U_\circ}(f), \partial|_{U_\circ})$ indeed coincides with $H_*(U)$.

5.2 Locality of spectral invariants with respect to other homology classes

In this section we show how Floer homology on open manifolds is useful in the study of Floer complexes of Hamiltonians supported in CIB domains in closed manifolds.[16] In particular, we explain how to extend Theorem 1 to homology classes in the image of the map induced by the inclusion $\iota\colon U \hookrightarrow M$.

Remark 5.2 The map $\iota_*\colon H_*(U) \to H_*(M)$ induced by the inclusion of U into M coincides (under standard isomorphisms) with the map induced by the inclusion of Floer complexes $C_{U_\circ}(f) \to CF(f)$ when (f, J) has a barricade in U around $U_\circ$. This is clear for the case where f is a small Morse function, since its Floer complex and the $U_\circ$–subcomplex coincide with the Morse ones. To see this for a general Hamiltonian f, consider a homotopy h between f and a small Morse function h_+ such that (h, J) has a barricade in U around $U_\circ$. Denoting by $\Phi\colon CF(f) \to CF(h_+) \cong CM_*(h_+)$ the

[16] The results in this section can be achieved within the scope of Floer homology on closed manifolds, but the proof is slightly more complicated and less natural.

corresponding continuation map, it restricts to the subcomplexes generated by elements in $U_\circ$:

$$\Phi_{U_\circ} := \Phi|_{C_{U_\circ}(f)} \colon C_{U_\circ}(f) \to C_{U_\circ}(h_+).$$

Moreover, the solutions of the Floer equation counted by the map $\Phi_{U_\circ}$ are all contained in $U_\circ$ due to the barricade — these observations are an adaptation of (7) to continuation maps instead of differentials. In fact, the map $\Phi_{U_\circ}$ coincides with the continuation map between the Floer and Morse complexes on the completion of $U_\circ$, with respect to a homotopy obtained as a constant (in the s parameter) extension of $h|_{U_\circ}$ to the completion. Such maps are known to be isomorphisms; see [17, Section 2.9; 24, Theorem 1.4], for example. Since the diagram

$$\begin{tikzcd} C_{U_\circ}(f) \arrow[r] \arrow[d, "\Phi_{U_\circ}"'] & CF(f) \arrow[d, "\Phi"] \\ C_{U_\circ}(h_+) \arrow[r] & CF(h_+) \end{tikzcd}$$

commutes, the maps induced by inclusions of the Floer and Morse complexes coincide under the isomorphisms $\Phi_{U_\circ}$ and Φ.

Claim 5.3 *For every class $\alpha \in \operatorname{im}(\iota_*) \subset H_*(M)$ and a Hamiltonian F supported in U,*

$$c_M(F;\alpha) = \min_{\substack{\beta \in H_*(U)\\ \iota_*(\beta)=\alpha}} c_{\widehat{U}}(\widehat{F};\beta), \tag{25}$$

where c_M and $c_{\widehat{U}}$ are the spectral invariants in the manifolds (M,ω) and $(\widehat{U},\widehat{\omega})$, respectively, and $\widehat{F}$ is the extension by zero of $F|_U$ to $\widehat{U}$.

Proof The proof relies on the observations of Section 5.1: Let f be a perturbation of F and J an almost complex structure such that (f,J) has a barricade in U around $U_\circ$. Assume in addition that the perturbation is chosen to be arbitrarily close to some $f^\flat$ for which the pair $(f^\flat, J)$ has a cylindrical bump around ∂U. As explained previously, the Floer complex of $\widehat{f} := \widehat{f|_{U_\circ}}$ on $(\widehat{U},\widehat{\omega})$ coincides with the subcomplex $C_{U_\circ}(f)$ of $CF(f)$ in M. We will show that formula (25) holds for f and $\widehat{f}$ up to 2δ for some δ which can be made arbitrarily small by shrinking the size of the perturbations.

We start by noticing that given a class $\beta \in \iota_*^{-1}(\alpha)$, every representative $b \in C_{U_\circ}(f)$ of β is a representative of α in $CF_*(f)$. This immediately implies that

$$c_M(f;\alpha) \le \min_{\beta \in \iota_*^{-1}(\alpha)} c_{U_\circ}(\widehat{f};\beta).$$

To prove inequality in the other direction, let $\beta \in \iota_*^{-1}(\alpha)$ be a class on which the minimum in the right-hand side of (25) is attained, and let $a \in CF_*(f)$ and $b \in C_{U_\circ}(f)$ be representatives of α and β of minimal action levels. We need to show that $\lambda_f(b) \le \lambda_f(a) + 2\delta$, where $\lambda_f\colon CF_*(f) \to \mathbb{R}$ is the maximal action of an orbit, as defined in Notation 2.1. Notice that if $a \in C_{U_\circ}(f)$, then it represents in $C_{U_\circ}(f)$ a class in $\iota_*^{-1}(a)$ and, by our choice of b, $\lambda_f(b) \le \lambda_f(a)$, which concludes the proof for this case. Therefore we suppose that a contains critical points in $M \setminus U_\circ$, which implies that $\lambda_f(a) > -\delta$. Assume for the sake of contradiction that $\lambda_f(b) > \lambda_f(a) + 2\delta$; then $\lambda_f(b) > \delta$. Recalling that a and b are homologous in $CF_*(f)$ (they both represent α), there exists $c \in CF_*(f)$ such that $\partial c = a - b$. Consider the decomposition $c = \pi_{U_\circ} c + \pi_{U_\circ^c} c$. Then

$$b' := b + \partial \pi_{U_\circ} c = a - \partial \pi_{U_\circ^c} c \in C_{U_\circ}(f)$$

is homologous to b in $C_{U_\circ}(f)$. This follows from the fact that $\partial \circ \pi_{U_\circ} = \partial|_{U_\circ}$, since (f, J) has a barricade in U around $U_\circ$. Therefore, b' represents in $C_{U_\circ}(f)$ a class in $\iota_*^{-1}(\alpha)$, and by our choice of b, it holds that $\lambda_f(b) \le \lambda_f(b')$. On the other hand,

$$\lambda_f(b') = \lambda_f(a - \partial \pi_{U_\circ^c} c) \le \max\{\lambda_f(a), \lambda_f(\partial \pi_{U_\circ^c} c)\} \le \max\{\lambda_f(a), \delta\} < \lambda_f(b),$$

a contradiction. □

Remark 5.4 When U is a disjoint union of $\{U_i\}$ and $\alpha = [\mathrm{pt}] \in H_*(M)$, equality (25) implies the min formula for the point class, which is equivalent, by Poincaré duality, to Theorem 45 in [12] (the max formula).

6 Spectral invariants of disjointly supported Hamiltonians

In this section we use barricades to prove Theorem 2, which states that a max inequality holds for spectral invariants of Hamiltonians supported in disjoint CIB domains, with respect to a general class $\alpha \in H_*(M)$, and that equality holds when $\alpha = [M]$. Suppose F and G are two Hamiltonians supported in disjoint CIB domains. In order to prove the max inequality (4) for a homology class $\alpha \in H_*(M)$, we construct a representative of α in the Floer complex of (a perturbation of) the sum $F + G$, out of representatives from the Floer complexes of (perturbations of) F and G. The communication between the different Floer complexes is through continuation maps, corresponding to (perturbations of) linear homotopies. The barricades will be used to study the continuation maps, or, more accurately, their restrictions to the CIB domains. In particular, we will use the observation that having a barricade for a disjoint union implies having a barricade for each component:

Remark 6.1 Consider two disjoint domains U and V in M, and a pair (H, J) of a homotopy (or a Hamiltonian) and an almost complex structure, that has a barricade in $U \cup V$ around $U_\circ \cup V_\circ$ for some $U_\circ \Subset U$ and $V_\circ \Subset V$. It follows from Definition 1.7 of the barricade that the pair (H, J) has a barricade in U around $U_\circ$ (and, similarly, in V around $V_\circ$).

We start by arranging the setup required for the proof of Theorem 2.

Lemma 6.2 (setup) *Let F and G be Hamiltonians supported in disjoint CIB domains U and V, respectively. Then there exist an almost complex structure J and homotopies h_F and h_G such that the following hold*:

(1) *The pairs (h_F, J), $(h_{F\pm}, J)$, (h_G, J) and $(h_{G\pm}, J)$ are all Floer-regular and have barricades in $U \cup V$ around $U_\circ \cup V_\circ$ for some $U_\circ \Subset U$ and $V_\circ \Subset V$ containing the supports of F and G, respectively.*

(2) *The left ends h_{F-} and h_{G-} are small perturbations of F and G, respectively. The right ends coincide — $h_{F+} = h_{G+}$ — and are a small perturbation of the sum $F + G$.*

(3) *On $U \times S^1$ (resp. $V \times S^1$) the homotopy h_F (resp. h_G) is a small perturbation of a constant homotopy, and its ends agree on their 1–periodic orbits up to second order. In particular, h_{F-} and h_{F+} (resp. h_{G-} and h_{G+}) have the same 1–periodic orbits in U (resp. V).*

Proof Let H_F and H_G be linear homotopies from F and G, respectively, to the sum $F + G$. As in the proof of Lemma 4.4, we consider perturbations $h^\flat_F$ and $h^\flat_G$ of the linear homotopies that, when paired with J, have a cylindrical bump around $\partial U \cup \partial V$. We demand in addition that all ends are nondegenerate, that the right ends coincide, $h^\flat_{F+} = h^\flat_{G+}$, and that the homotopies are constant on U and V, respectively, $h^\flat_F|_U \equiv h^\flat_{F\_}|_U$ and $h^\flat_G|_V \equiv h^\flat_{G\_}|_V$. By Proposition 3.7, these homotopies and their ends, when paired with J, have barricades in $U \cup V$ around $U_\circ \cup V_\circ$. It remains to perturb again to ensure regularity. As in the proof of Theorem 5, we replace the ends with regular perturbations h_{F-}, h_{G-} and $h_{F+} = h_{G+}$, without changing their periodic orbits (as cited in Claim 9.1, for example), then perturb the homotopies to glue to these regular perturbed Hamiltonians, and finally perturb the homotopies on the set $M \times S^1 \times I$ for some fixed finite interval I, to obtain homotopies that are Floer-regular when paired with J. The last step is possible due to Proposition 9.2 below. Proposition 9.21 states that barricades survive under perturbations that do not change the periodic orbits of the ends and are constant (as homotopies) outside of some fixed finite interval. □

The following lemma is actually a part of the proof of Theorem 2 but, in our opinion, might be interesting on its own.

Lemma 6.3 *Let $\alpha \in H_*(M)$ and let $F, G\colon M \times S^1 \to \mathbb{R}$ be Hamiltonians supported in disjoint CIB domains U and V, respectively. Assume in addition that $c(F;\alpha) < 0$. Then*

$$c(F+G;\alpha) \le \min\{c(F;\alpha), c(G;\alpha)\}. \tag{26}$$

Proof Let us show that $c(F+G;\alpha) \le c(F;\alpha)$. The result will follow by symmetry, since, if $c(G;\alpha) < c(F;\alpha)$, then it is in particular negative.

Let h_F and J be the homotopy and almost complex structure from the setup lemma, Lemma 6.2 (we will not use h_G in this proof), and denote the left end of the homotopy by $f := h_{F-}$. Then f approximates F and, since $c(F;\alpha) < 0$ and $F|_{U_\circ^c} = 0$, we may assume that $c(f;\alpha) < \min_{U_\circ^c} f$. Outside of $U_\circ$, f is a small Morse function and hence its 1–periodic orbits there are critical points, and their actions are the critical values. As a consequence, a representative $a \in CF_*(f)$ of the class α of action level $\lambda_f(a) = c(f;\alpha)$ must be a combination of orbits that are contained in $U_\circ$, namely, $a \in C_{U_\circ}(f)$. As the continuation map $\Phi_{(h_F,J)}\colon CF_*(f) \to CF_*(h_{F+})$ induces isomorphism on homologies, the image $\Phi_{(h_F,J)}(a)$ of a represents the class α in $CF_*(h_{F+})$. Recalling that, on U, the homotopy h_F is a small perturbation of a constant homotopy, it follows from Corollary 9.34 that the restriction of the continuation map $\Phi_{(h_F,J)}$ to orbits contained in $U_\circ$ is the identity map,

$$\Phi_{(h_F,J)} \circ \pi_{U_\circ} = \mathbb{1} \circ \pi_{U_\circ}.$$

Therefore, $\Phi_{(h_F,J)}(a) = a$ is a representative of the class α of action level

$$\lambda_{h_{F+}}(\Phi_{(h_F,J)}(a)) = \lambda_f(a) = c(f;\alpha).$$

We conclude that $c(h_{F+};\alpha) \le c(f;\alpha)$, as required. □

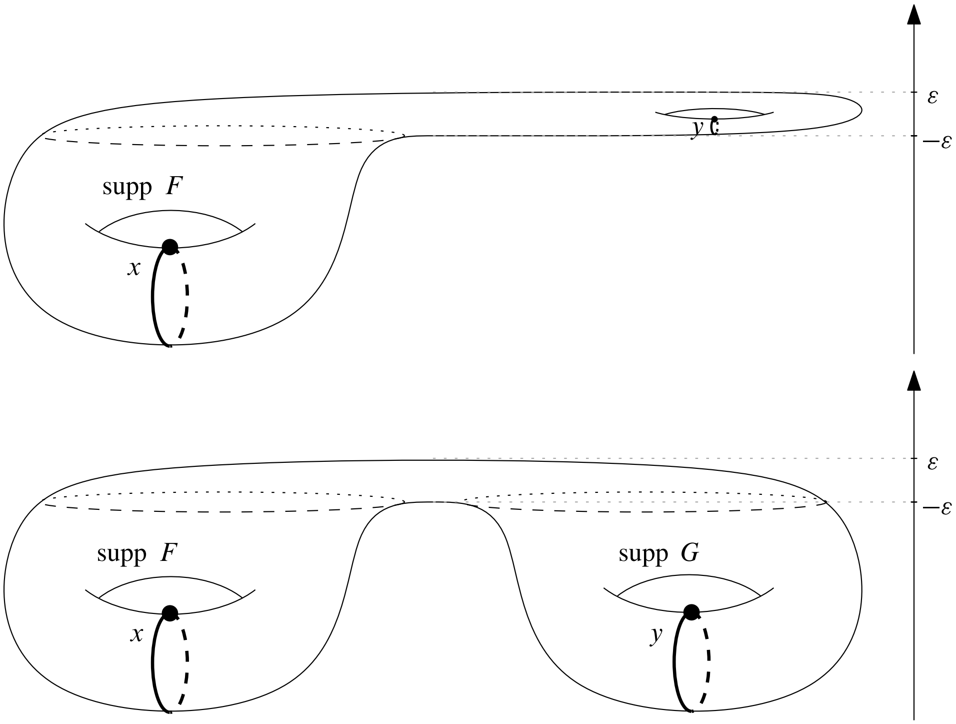

Figure 8: An illustration of nondegenerate perturbations of F, top, and $F+G$, bottom. A representative of the class $\alpha+\beta$ appears at level ≈ 0 for F, and at a negative value for $F+G$.

The following example shows that a strict inequality can be attained in (26).

Example 6.4 Let (M,ω) be a genus-2 surface endowed with an area form, and let $x, y\colon S^1 \to M$ be two disjoint noncontractible loops representing two different homology classes $\alpha,\beta \in H_1(M;\mathbb{Z}_2)$, respectively. Let $F, G\colon M \to \mathbb{R}$ be two small Morse functions with disjoint supports, which are such that F vanishes on y and takes a negative value on x, whereas G vanishes on x and is negative on y. See Figure 8 for an illustration. After perturbing F, G and $F+G$ into Morse functions, representatives of the sum $\alpha+\beta$ first appear for F and G on a sublevel set of values approximately zero. However, this sum of classes appears for $F+G$ in a sublevel set with negative value. We therefore conclude that the spectral invariants of both F and G with respect to the sum $\alpha+\beta$ vanish. On the other hand, the spectral invariant of $F+G$ is negative, and thus

$$c(F+G;\alpha+\beta) < 0 = \min\{c(F;\alpha+\beta), c(G;\alpha+\beta)\}.$$

The following inequality is a simple application of Lemmas 4.1 and 6.3, and will be used to prove that equality holds in (4) for the fundamental class.

Lemma 6.5 *Let $F, G\colon M \times S^1 \to \mathbb{R}$ be Hamiltonians supported in disjoint CIB domains. Then*

$$c(F+G;[M]) \geq \max\{c(F;[M]), c(G;[M])\}.$$

Proof By Lemma 4.1, the spectral invariants of F, G and $F+G$ with respect to [pt] are nonpositive, and thus, using the Poincaré duality property for spectral invariants, we conclude that $c(F+G;[M])$, $c(F;[M])$ and $c(G;[M])$ are all nonnegative. If both $c(F;[M])$ and $c(G;[M])$ are equal to zero, the claim is trivial. Thus we assume, without loss of generality, that $c(F;[M]) > 0$. By Poincaré duality,

$$c(-F;[\mathrm{pt}]) = -c(F;[M]) < 0$$

and we may apply Lemma 6.3 to $-F$, $-G$ and $\alpha = [\mathrm{pt}]$:

$$\begin{aligned} c(F+G;[M]) &= -c(-F-G;[\mathrm{pt}]) \\ &\geq -\min\{c(-F;[\mathrm{pt}]), c(-G;[\mathrm{pt}])\} \\ &= -\min\{-c(F;[M]), -c(G;[M])\} \\ &= \max\{c(F;[M]), c(G;[M])\}. \end{aligned}$$ □

Proof of Theorem 2 In what follows we prove that the spectral invariant of the sum $F+G$ with respect to a homology class α is not greater than the maximum. The equality for the fundamental class will follow from Lemma 6.5. Consider the almost complex structure J, and the homotopies, h_F and h_G, from the setup lemma, Lemma 6.2, and write

$$f := h_{F-} \approx F, \quad g := h_{G-} \approx G, \quad h_+ := h_{F+} = h_{G+} \approx F+G.$$

Set $\lambda := \max\{c(f;\alpha), c(g;\alpha)\}$ and notice that, due to Lemma 6.3 and the continuity of spectral invariants, we may assume that $\lambda \geq -\delta$ if $\delta > 0$ is small enough. Let $\tilde{a} \in CF_*(f)$, $\tilde{b} \in CF_*(g)$ be representatives of α of action levels $\lambda_f(\tilde{a}), \lambda_g(\tilde{b}) \leq \lambda$. Then $a := \Phi_{(h_F,J)}\tilde{a}$ and $b := \Phi_{(h_G,J)}\tilde{b}$ are both representatives of α in $CF_*(h_+)$. Notice that a and b might be of action level higher than λ. We wish to construct out of a and b a representative of α of action level approximately bounded by λ. Let p be a primitive of $a-b$, and set $d := (\partial_{(h_+,J)}\pi_V - \pi_V\partial_{(h_+,J)})p$. We claim that

$$e := \pi_{V^c}a + \pi_V b - d$$

is a representative of α of the required action level. Indeed,

$$\begin{aligned}[\pi_{V^c}a+\pi_V b-d] &= [\pi_{V^c}a+\pi_V b-\partial_{(h_+,J)}(\pi_V p)+\pi_V(\partial_{(h_+,J)}p)]\\ &= [\pi_{V^c}a+\pi_V b-\partial_{(h_+,J)}(\pi_V p)+\pi_V(a-b)]\\ &= [\pi_{V^c}a+\pi_V b-\partial_{(h_+,J)}(\pi_V p)+\pi_V a-\pi_V b]\\ &= [\pi_{V^c}a+\pi_V a-\partial_{(h_+,J)}(\pi_V p)]=[a]=\alpha.\end{aligned}$$

Let us now bound the action level of e. First, notice that outside of $U_\circ\cup V_\circ$, h_+ is a small Morse function (as it approximates a Hamiltonian that is supported in $U_\circ\cup V_\circ$). Therefore, its 1–periodic orbits there are its critical points and their actions are the critical values, which we may assume to be bounded by δ. It follows that the action level of the projection $\pi_{U_\circ^c\cap V_\circ^c}(e)$ is bounded by δ, and so it remains to bound the action levels of $\pi_{U_\circ}e$ and $\pi_{V_\circ}e$. It follows from the fact that (h_+,J) has a barricade in U around $U_\circ$ and in V around $V_\circ$ (more specifically, from (7)), that

(27) $$\pi_{U_\circ}\circ\partial_{(h_+,J)}\circ\pi_{U^c}=0 \quad\text{and}\quad \pi_{V_\circ}\circ\partial_{(h_+,J)}\circ\pi_{V^c}=0.$$

Using this observation, we bound the action levels of the projections of e:

- **Bounding $\boldsymbol{\lambda_{h_+}(\pi_{U_\circ}e)}$** Notice that $\pi_{U_\circ}d=0$. Indeed,

$$\pi_{U_\circ}d=\pi_{U_\circ}\circ(\partial_{(h_+,J)}\circ\pi_V-\pi_V\circ\partial_{(h_+,J)})p=\pi_{U_\circ}\circ\partial_{(h_+,J)}\circ\pi_V p\overset{(27)}{=}0.$$

As a consequence, $\pi_{U_\circ}e=\pi_{U_\circ}a=\pi_{U_\circ}\Phi_{(h_F,J)}\widetilde{a}$. Since, on U, the homotopy h_F is a perturbation of the constant homotopy, we can apply Corollary 9.34 and conclude that $\pi_{U_\circ}\circ\Phi_{(h_F,J)}=\pi_{U_\circ}$. Overall we obtain

$$\lambda_{h_+}(\pi_{U_\circ}e)=\lambda_{h_+}(\pi_{U_\circ}\circ\Phi_{(h_F,J)}\widetilde{a})=\lambda_{h_+}(\pi_{U_\circ}\widetilde{a})=\lambda_f(\pi_{U_\circ}\widetilde{a})\le\lambda_f(\widetilde{a})\le\lambda,$$

where we used the fact that in U, $f=h_{F-}$ and $h_+=h_{F+}$ agree on their 1–periodic orbits, and hence the action of $\pi_{U_\circ}\widetilde{a}$ with respect to h_+ coincides with the action with respect to f.

- **Bounding $\boldsymbol{\lambda_{h_+}(\pi_{V_\circ}e)}$** Here $\pi_{V_\circ}d=0$ as well, but the computation is a little different:

$$\begin{aligned}\pi_{V_\circ}d &= \pi_{V_\circ}\circ(\partial_{(h_+,J)}\circ\pi_V-\pi_V\circ\partial_{(h_+,J)})p\\ &= (\pi_{V_\circ}\circ\partial_{(h_+,J)}\circ\pi_V-\pi_{V_\circ}\circ\partial_{(h_+,J)})p\\ &= (\pi_{V_\circ}\circ\partial_{(h_+,J)}-\pi_{V_\circ}\circ\partial_{(h_+,J)}\circ\pi_{V^c}-\pi_{V_\circ}\circ\partial_{(h_+,J)})p\overset{(27)}{=}0.\end{aligned}$$

Therefore, $\pi_{V_\circ}e=\pi_{V_\circ}b=\pi_{V_\circ}\Phi_{(h_G,J)}\widetilde{b}$, and since on V, the homotopy h_G is a perturbation of the constant homotopy, we apply Corollary 9.34 and conclude that

$\pi_{V_\circ} \circ \Phi_{(h_G,J)} = \pi_{V_\circ}$. Overall,

$$\lambda_{h_+}(\pi_{V_\circ} e) = \lambda_{h_+}(\pi_{V_\circ} \circ \Phi_{(h_G,J)} \widetilde{b}) = \lambda_{h_+}(\pi_{V_\circ} \widetilde{b}) = \lambda_g(\pi_{V_\circ} \widetilde{b}) \le \lambda_g(\widetilde{b}) \le \lambda,$$

where we used the fact that on V, $g = h_{G-}$ and $h_+ = h_{G+}$ agree on their 1–periodic orbits, and hence the action of $\pi_{V_\circ} \widetilde{a}$ with respect to h_+ coincides with the action with respect to g.

We conclude that

$$\begin{aligned} c(h_+;\alpha) \le \lambda_{h_+}(e) &\le \max\{\lambda_{h_+}(\pi_{U_\circ} e), \lambda_{h_+}(\pi_{V_\circ} e), \lambda_{h_+}(\pi_{U_\circ^c \cap V_\circ^c} e)\} \\ &= \max\{\lambda, \delta\} \le \lambda + 2\delta. \end{aligned}$$ □

7 Boundary depth of disjointly supported Hamiltonians

In this section, we use barricades to compare the boundary depths of disjointly supported Hamiltonians and that of their sum. As in the previous section, the communication between Floer complexes of different Hamiltonians is through continuation maps corresponding to homotopies that have barricades. Since we replace the Hamiltonians and their sum by regular perturbations, we will use the continuity property of the boundary-depth:

Theorem [21, Theorem 1.1] *Given two Hamiltonians F and G,*

$$|\beta(F) - \beta(G)| \le \int_0^1 \left(\max_M (F - G) - \min_M (F - G)\right) dt.$$

As before, we use Notation 2.1. Let us start with a lemma that will enable us to push certain boundary terms from one Floer complex to another.

Lemma 7.1 *Let J be an almost complex structure and h a homotopy such that the pairs (h, J) and $(h_\pm, J)$ are Floer-regular and have a barricade in U around $U_\circ$. Assume in addition that on U, h is a small perturbation of a constant homotopy, and that its ends $h_\pm$ agree up to second order on their 1–periodic orbits in U. Then every boundary term $a \in \partial_{(h_+,J)} CF_*(h_+)$ that is a combination of orbits in $U_\circ$, namely $a \in C_{U_\circ}(h_+)$, is also a boundary term in $CF_*(h_-)$.*

Proof We start with the observation that, since h_- and h_+ are close on U and agree on their periodic orbits there, the vector spaces $C_U(h_-)$ and $C_U(h_+)$ coincide. Therefore, a boundary term $a \in CF_*(h_+)$ that is a combination of orbits from $U_\circ$ is also an element of $C_{U_\circ}(h_-)$. Let us show that a is a boundary term in the Floer complex of (h_-, J). As the homotopy h is close to a constant homotopy on U, we may use Corollary 9.34 and conclude that $\Phi_{(h,J)} \circ \pi_{U_\circ} = \pi_{U_\circ}$. Applying this equality to a, we obtain

$$\Phi_{(h,J)} a = \Phi_{(h,J)} \circ \pi_{U_\circ} a = \pi_{U_\circ} a = a;$$

namely, $a \in CF_*(h_+)$ is the image of itself under the continuation map. As $\Phi_{(h,J)}$ induces isomorphism on homologies, it is enough to show that a is closed in $CF_*(h_-)$, and it will then follow that it is a boundary term. To see that a is closed in $CF_*(h_-)$, notice that the presence of a barricade (in particular, (7)) implies $\partial_{(h_-,J)} a \in C_{U_0}(h_-)$; namely, $\partial_{(h_-,J)} a = \pi_{U_\circ} \partial_{(h_-,J)} a$. Therefore, $\partial_{(h_-,J)} a = \pi_{U_\circ} \partial_{(h_-,J)} a = \pi_{U_\circ} \Phi_{(h,J)} \partial_{(h_-,J)} a = \pi_{U_\circ} \partial_{(h_+,J)} \Phi_{(h,J)} a = \pi_{U_\circ} \partial_{(h_+,J)} a = 0$. □

We are now ready to prove Theorem 3.

Proof of Theorem 3 In what follows we show that $\beta(F+G) \geq \beta(F)$. Inequality (5) follows by symmetry. Let H be a linear homotopy from $F+G$ to F. Notice that, since F and $F+G$ agree on U, H is a constant homotopy there. By Theorem 5, there exist a perturbation h of H and an almost complex structure J such that the pairs (h, J) and $(h_\pm, J)$ are Floer-regular and have a barricade in $U \cup V$ around $U_\circ \cup V_\circ$ for $U_\circ \Subset U$ and $V_\circ \Subset V$ containing the supports of F and G, respectively. Since H is a constant homotopy on U, it follows from Remark 3.8(ii) that h can be chosen such that, in U, the $h_\pm$ agree on their 1–periodic orbits up to second order. We stress that h_- approximates $F+G$ and that $f := h_+$ approximates F. Hence, fixing an arbitrarily small $\delta > 0$, we may assume (by taking h to be close enough to H) that $h_-|_{U_\circ^c \cap V_\circ^c}$ and $f|_{U_\circ^c}$ are small Morse functions with values in $(-\delta, \delta)$. Due to the continuity of the boundary depth, it is enough to prove that $\beta(f)$ is approximately bounded by $\beta(h_-)$.

Fix a boundary term $a \in CF_*(f)$, and let us show that there exists a primitive of a whose action level is bounded by $\lambda_f(a) + \beta(h_-) + 4\delta$, for δ that was fixed above. We prove this claim in two steps:

Step 1 Assume that a is a combination of orbits that are contained in $U_\circ$, namely $a \in C_{U_\circ}(f)$. Applying Lemma 7.1 to (h, J), we find that $a \in CF_*(h_-)$ is also a boundary term. Therefore, there exists $b \in CF_*(h_-)$ such that $\partial_{(h_-,J)} b = a$ and $\lambda_{h_-}(b) \leq \lambda_{h_-}(a) + \beta(h_-)$. Let us split into two cases:

- $\boldsymbol{\lambda_{h_-}(b) < -\delta}$ Since h_- is a small Morse function outside of $U_\circ \cup V_\circ$, its 1–periodic orbits there are its critical points, and their actions are the critical values, which are all contained in the interval $(-\delta, \delta)$. As a consequence, b is necessarily a combination of orbits that are contained in $U_\circ \cup V_\circ$, namely, $b \in C_{U_\circ \cup V_\circ}(h_-)$. Writing $b = \pi_{U_\circ} b + \pi_{V_\circ} b$, the presence of the barricade (in particular, (7)) guarantees that

$$\partial_{(h_-,J)} \pi_{U_\circ} b \in C_{U_\circ}(h_-) \quad \text{and} \quad \partial_{(h_-,J)} \pi_{V_\circ} b \in C_{V_\circ}(h_-).$$

Recalling that $\partial_{(h_-,J)} b = a \in C_{U_\circ}(h_-)$, we conclude that $\partial_{(h_-,J)} \pi_{V_\circ} b = 0$:

$$\begin{aligned}\partial_{(h_-,J)} \pi_{V_\circ} b &= \pi_{V_\circ}(\partial_{(h_-,J)} \pi_{V_\circ} b) \\ &= \pi_{V_\circ}(\partial_{(h_-,J)} b - \partial_{(h_-,J)} \pi_{U_\circ} b) \\ &= \pi_{V_\circ}(a - \pi_{U_\circ} \partial_{(h_-,J)} \pi_{U_\circ} b) = 0.\end{aligned}$$

Replacing b by $\pi_{U_\circ} b$, we still have a primitive of a of no greater action level, as $\lambda_{h_-}(b) = \max\{\lambda_{h_-}(\pi_{U_\circ} b), \lambda_{h_-}(\pi_{V_\circ} b)\}$. Therefore, we may assume that $b \in C_{U_\circ}(h_-)$, and so it is also an element of $C_{U_\circ}(f)$. Recalling that h is a perturbation of a constant homotopy on U, Corollary 9.34 states that $\Phi_{(h,J)} \circ \pi_{U_\circ} = \pi_{U_\circ}$, and hence $\Phi_{(h,J)} b = b$ and $\Phi_{(h,J)} a = a$. Thus,

$$\partial_{(f,J)}(b) = \partial_{(f,J)}(\Phi_{(h,J)} b) = \Phi_{(h,J)}(\partial_{(h_-,J)} b) = \Phi_{(h,J)} a = a,$$

ie b is a primitive of a in $CF_*(f)$ with small enough action level: $\lambda_f(b) = \lambda_{h_-}(b) \le \lambda_f(a) + \beta(h_-)$.

- $\boldsymbol{\lambda_{h_-}(b) \geq -\delta}$ Then, writing $b = \pi_U b + \pi_{U^c} b$, the presence of a barricade (in particular, (7)) implies that $\Phi_{(h,J)} \pi_{U^c} b \in C_{U^c}(f)$ and hence $\lambda_f(\Phi_{(h,J)} \pi_{U^c} b) \le \delta$. Turning to bound the action of the projection onto U, recall that h is a perturbation of a constant homotopy on U, and by Corollary 9.34, $\pi_U \circ \Phi_{(h,J)} = \pi_U$. Overall,

$$\begin{aligned}\lambda_f(\Phi_{(h,J)} b) &\le \max\{\lambda_f(\Phi_{(h,J)} \pi_{U^c} b), \lambda_f(\Phi_{(h,J)} \pi_U b)\} \\ &\le \max\{\delta, \lambda_{h_-}(b)\} \\ &\le \lambda_f(a) + \beta(h_-) + 2\delta.\end{aligned}$$

Step 2 Let us prove the claim for general a. Note that if $\lambda_f(a) < -\delta$ then $a \in C_{U_\circ}(f)$ and the claim follows from the previous step. Therefore, we assume that $\lambda_f(a) \ge -\delta$. Let b be any primitive of a in $CF_*(f)$, namely, $\partial_{(f,J)} b = a$, and write $b = \pi_{U_\circ} b + \pi_{U_\circ^c} b$. Both $\pi_{U_\circ^c} b$ and $\partial_{(f,J)} \pi_{U_\circ^c} b$ have action levels bounded by δ. Set $a' := \partial_{(f,J)} \pi_{U_\circ} b$. Then

$$\lambda_f(a') = \lambda_f(a - \partial_{(f,J)} \pi_{U_\circ^c} b) \le \max\{\lambda_f(a), \lambda_f(\partial_{(f,J)} \pi_{U_\circ^c} b)\} \le \lambda_f(a) + 2\delta.$$

Moreover, the presence of the barricade implies that $a' \in C_{U_\circ}(f)$. Therefore, we may apply the previous step to a' and obtain $b' \in CF_*(f)$ such that $\partial_{(f,J)} b' = a'$ and

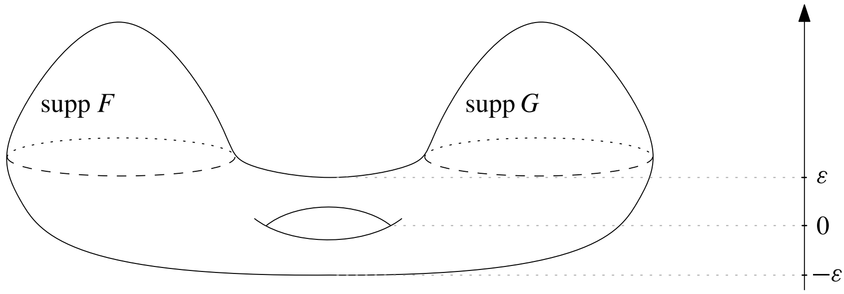

Figure 9: An illustration of a nondegenerate perturbation of the sum $F+G$ from Example 7.2.

$\lambda_f(b') \le \lambda_f(a') + \beta(h_-) + 2\delta \le \lambda_f(a) + \beta(h_-) + 4\delta$. To conclude the proof, notice that $b' + \pi_{U_\circ^c} b$ is a primitive of a and

$$\lambda_f(b' + \pi_{U_\circ^c} b) \le \max\{\lambda_f(b'), \lambda_f(\pi_{U_\circ} b)\} \le \lambda_f(a) + \beta(h_-) + 4\delta. \qquad \square$$

The following example shows that equality does not hold in (5).

Example 7.2 Let $M = \mathbb{T}^2$ be the two-dimensional torus equipped with an area form and take F and G be disjointly supported $\mathcal{C}^2$–small nonnegative bumps; see Figure 9. Approximating F, G and $F+G$ by small Morse functions, their Floer complexes and differentials are equal to the Morse complexes and differentials. Hence, the Floer differentials of both F and G vanish and in particular, $\beta(F) = 0 = \beta(G)$. On the other hand, $\beta(F+G) = \min\{\max F, \max G\}$.

8 Min inequality for the AHS action selector

In this section, we use barricades to prove a "min inequality" for the action selector defined by Abbondandolo, Haug and Schlenk in [1], on symplectically aspherical manifolds. We start by reviewing the construction of this action selector, which we denote by c_{AHS}, and state a few of its properties.

Let $H : M \times S^1 \times \mathbb{R} \to \mathbb{R}$ be a homotopy of Hamiltonians and let $J : S^1 \times \mathbb{R} \to \mathcal{J}_\omega$ be a homotopy of time-dependent almost complex structures (that are compatible with ω). Assume that $\partial_s H$ and $\partial_s J$ have compact support and denote by $H_\pm$ and $J_\pm$ the ends of the homotopies. As before, we denote by $\mathcal{M}_{(H,J)}$ the set of all finite-energy solutions of the Floer equation (FE) with respect to (H, J). On this space, define the functional

$a_{H_-}\colon \mathcal{M}_{(H,J)} \to \mathbb{R}$ by $a_{H_-}(u) := \lim_{s\to-\infty} \mathcal{A}_{H_-}(u(s,\cdot))$. The existence of this limit follows from the fact that the homotopies H and J are constant outside of a compact set, and hence, when s approaches $-\infty$, the function $s \to \mathcal{A}_{H_-}(u(s,\cdot))$ is nonincreasing and bounded; see for example [1, page 8]. Given a Hamiltonian $F\colon M \times S^1 \to \mathbb{R}$, denote by $\mathcal{D}(F) := \{(H,J) \mid H_- = F\}$ the set of all pairs of homotopies that are constant outside of some compact set, and such that F is the left end of H.

Definition 8.1 [1, Definition 3.1] Let $F\colon M \times S^1 \to \mathbb{R}$ be any Hamiltonian and let $(H,J) \in \mathcal{D}(F)$. Set

$$A(H,J) := \min_{u\in\mathcal{M}_{(H,J)}} a_F(u) \quad\text{and}\quad c_{\mathrm{AHS}}(F) := \sup_{(H,J)\in\mathcal{D}(F)} A(H,J). \tag{28}$$

In [1], Abbondandolo, Haug and Schlenk proved that the functional c_{AHS} is continuous and monotone, and that it takes values in the action spectrum, namely $c_{\mathrm{AHS}}(F) \in \operatorname{spec}(F)$. Let us state the result establishing the continuity of c_{AHS}:

Claim 8.2 [1, Proposition 3.4] *For all $F, G \in C^\infty(M\times S^1)$, we have*

$$\int_{S^1} \min_{x\in M}(F(x,t)-G(x,t))\,dt \le c_{\mathrm{AHS}}(F)-c_{\mathrm{AHS}}(G) \le \int_{S^1} \max_{x\in M}(F(x,t)-G(x,t))\,dt.$$

In addition, they proved that the action selector takes nonpositive values on Hamiltonians supported in incompressible Liouville domains.

Claim 8.3 [1, Proposition 5.4] *If F has support in an incompressible Liouville domain, then $c_{\mathrm{AHS}}(F) \le 0$. In particular, $c_{\mathrm{AHS}}(F) = 0$ for all nonnegative Hamiltonians which are supported in an incompressible Liouville domain.*

Using these claims, together with the barricades construction and ideas from the proof of Proposition 3.3 from [1], one can prove that a min inequality holds for c_{AHS}.

Proof of Theorem 4 Let F and G be Hamiltonians supported in the interiors of disjoint incompressible Liouville domains, which we denote by U and V, respectively. Fixing an arbitrarily small $\delta > 0$, we will prove that $c_{\mathrm{AHS}}(F+G) \le c_{\mathrm{AHS}}(F) + 3\delta$. The claim for G will follow by symmetry. We remark that by Claim 8.3, $c_{\mathrm{AHS}}(F+G) \le 0$, and hence the result is immediate if $c_{\mathrm{AHS}}(F) \ge -3\delta$. Therefore, we assume that $c_{\mathrm{AHS}}(F) < -3\delta$. We break the proof into several steps.

Step 1 Our first step is to perturb F and $F+G$ (as well as a homotopy between them) to create barricades. Let H be a linear homotopy from F to $F+G$ that is constant outside of $[0,1]$, that is, $\partial_s H|_{s\notin[0,1]} = 0$. Then H is supported in $U \cup V$, which, as the union of the interiors of incompressible Liouville domains, is also a CIB domain. Applying Theorem 5 to the homotopy H and the domain $U \cup V$, we conclude that there exists a perturbation h of H, an almost complex structure $J^\flat$ and subsets $U_\circ \Subset U$ and $V_\circ \Subset V$ containing the supports of F and G, respectively, such that the pairs $(h, J^\flat)$ and $(h_\pm, J^\flat)$ are Floer-regular and have a barricade in $U \cup V$ around $U_\circ \cup V_\circ$. In particular, the ends of h are nondegenerate, $f := h_-$ approximates F and h_+ approximates $F+G$. By taking h to be close enough to H, we can assume that, outside of $U_\circ$, f is a small Morse function with values in $(-\delta, \delta)$. Moreover, by Remark 3.8(iii), we can choose the perturbation h such that the homotopy h is constant outside of $[0,1]$, namely, $\partial_s h|_{s\notin[0,1]} = 0$. Finally, taking these perturbations to be small enough, it follows from Claim 8.2 that $c_{\mathrm{AHS}}(f) < -2\delta$, and it is sufficient to prove that $c_{\mathrm{AHS}}(h_+) \le c_{\mathrm{AHS}}(f) + \delta$.

Step 2 Recalling the definition of the action selector c_{AHS}, we need to show that for every $(K, J) \in \mathcal{D}(h_+)$, it holds that $A(K, J) \le c_{\mathrm{AHS}}(f) + \delta$. Therefore, our second step is to construct pairs in $\mathcal{D}(f)$ out of a given pair in $\mathcal{D}(h_+)$. Fix $(K, J) \in \mathcal{D}(h_+)$ and assume, without loss of generality, that K and J stabilize for $s \le 0$, namely, $K(x,t,s) = h_+(x,t)$ and $J(s) = J_-$ for $s \le 0$. We construct a sequence of pairs in $\mathcal{D}(f)$ by concatenating the homotopies (K, J) with shifts of the homotopy h and a homotopy $\widetilde{J} = \{\widetilde{J}^s\}_{s\in\mathbb{R}}$ of almost complex structures from $J^\flat$ to J_- which is constant outside of $[0,1]$, namely, $\partial_s \widetilde{J}|_{s\notin[0,1]} = 0$. More precisely, for $s \in \mathbb{R}$, denote by τ_s the shift by s, namely, $\tau_s h(\cdot,\cdot,\cdot) = h(\cdot,\cdot,\cdot+s)$ and $\tau_s \widetilde{J}(\cdot,\cdot) = \widetilde{J}(\cdot,\cdot+s)$, and consider the sequences

$$(29)\quad K_n := \begin{cases} K, & s \ge 0, \\ h_+, & s \in [-2n+1, 0], \\ \tau_{2n} h, & s \in [-2n, -2n+1], \\ f, & s \le -2n, \end{cases} \quad\text{and}\quad J_n := \begin{cases} J, & s \ge 0, \\ J_-, & s \in [-n+1, 0], \\ \tau_n \widetilde{J}, & s \in [-n, -n+1], \\ J^\flat, & s \le -n. \end{cases}$$

See Figure 10 for an illustration. Noticing that $(K_n, J_n) \in \mathcal{D}(f)$ for all n, we wish to show that there exists an $n \in \mathbb{N}$ for which $A(K, J) \le A(K_n, J_n) + \delta$.

Step 3 In this step we choose, for each n, a solution minimizing a_f, and extract a subsequence that *partially converges to a broken trajectory*. Namely, there exists a broken trajectory $\bar{v} = (v_1, \dots, v_N)$ whose pieces v_i are solutions of (FE) with

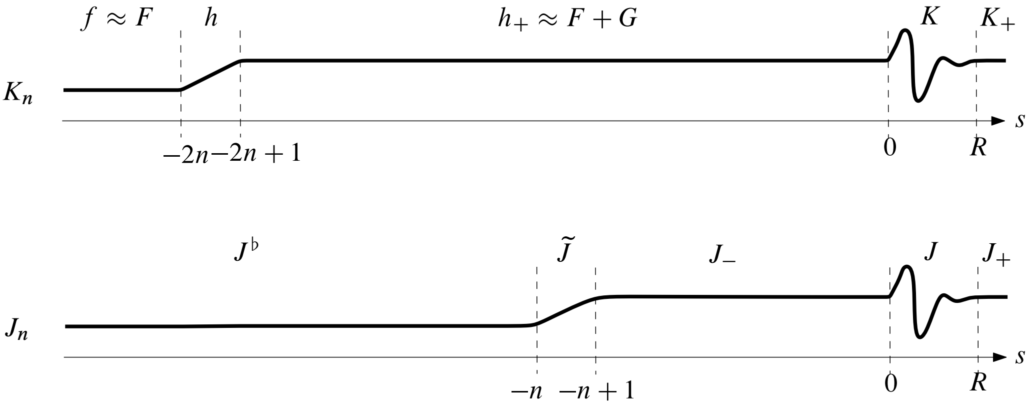

Figure 10: An illustration of the pair of homotopies $(K_n, J_n) \in \mathcal{D}(f)$ constructed out of a given pair $(K, J) \in \mathcal{D}(h_+)$.

respect to the homotopies concatenated in (K_n, J_n), and are obtained as limits of nonpositive shifts of elements from $\{u_n\}$. In particular, for each $i < N$, the solution v_i converges to periodic orbits at the ends that match the limits of the adjacent pieces, ie $\lim_{s\to+\infty} v_i(s,t) = \lim_{s\to-\infty} v_{i+1}(s,t)$. Moreover, the left end of the first piece, $\lim_{s\to-\infty} v_1(s,t)$, coincides with the left end of each element from the subsequence. We stress that unlike the standard convergence to a broken trajectory, in our case, the right end of the last piece in $\overline{v}$ (as well as the right ends of the solutions u_n) does not necessarily converge. The notion of partial convergence to a broken trajectory is defined formally in Proposition 9.17 below.

Let $u_n \in \mathcal{M}_{(K_n,J_n)}$ be a minimizer of the functional a_f, namely

$$a_f(u_n) = \min_{u\in\mathcal{M}_{(K_n,J_n)}} a_f(u) = A(K_n, J_n).$$

Since the supports of the homotopies (H_n, J_n) are not uniformly bounded and the ends are not all nondegenerate, the sequence of solutions $\{u_n\}_n$ does not necessarily converge to a broken trajectory. However, noticing that for $s \le 0$, (H_n, J_n) are concatenations of homotopies with nondegenerate ends, one can prove a (weaker) convergence statement, as we do in Section 9.2.2. In this case, Proposition 9.17 guarantees that there exists a subsequence of $\{u_n\}$, which we still denote by $\{u_n\}$, partially converging to a broken trajectory

$$\overline{v} = (\{v^{(f,J^\flat),\ell}\}_{\ell=1}^{L_1}, w^{(h,J^\flat)}, \{v^{(h_+,J^\flat),\ell}\}_{\ell=1}^{L_2}, w^{(h_+,\widetilde{J})}, \{v^{(h_+,J_-),\ell}\}_{\ell=1}^{L_3},\ w^{(K,J)}),$$

where the $v^{(\cdot,\cdot),\ell} \in \mathcal{M}_{(\cdot,\cdot)}$ denote solutions of s–independent Floer equations, and the $w^{(\cdot,\cdot)} \in \mathcal{M}_{(\cdot,\cdot)}$ denote solutions of s–dependent Floer equations. Moreover, the subsequence $\{u_n\}$ is chosen such that for each n, $\lim_{s\to-\infty} u_n(s,\cdot) = x^{1,0}(\cdot)$, where $x^{1,0} := \lim_{s\to-\infty} v^{(f,J^\flat),1}(s,\cdot) \in \mathcal{P}(f)$.

Step 4 We now use the barricades in order to show that the first few pieces of the broken trajectory $\overline{v}$ are contained in $U_\circ$. It follows from the arguments made above that

$$\mathcal{A}_f(x^{1,0}) = a_f(u_n) = A(K_n, J_n) \le c_{\mathrm{AHS}}(f) < -2\delta,$$

which implies, by our assumptions on f, that $x^{1,0} \subset U_\circ$. We claim that, since $(f, J^\flat)$ and $(h, J^\flat)$ have barricades in U around $U_\circ$, the trajectories $\{v^{(f,J^\flat),\ell}\}_{\ell=1}^{L_1}$ and $w^{(h,J^\flat)}$ are contained in $U_\circ$. Indeed, $\lim_{s\to-\infty} v^{(f,J^\flat),1}(s,\cdot) = x^{1,0} \subset U_\circ$ implies that $v^{(f,J^\flat),1} \subset U_\circ$ and, in particular, the image of $x^{1,1}(\cdot) := \lim_{s\to\infty} v^{(f,J^\flat),1}(s,\cdot)$ is contained in U_0. Since $x^{1,1}$ is the left end of $v^{(f,J^\flat),2}$, we can repeat this argument and conclude that $v^{(f,J^\flat),2}$ is contained in $U_\circ$. Continuing by induction, we find that $\{v^{(f,J^\flat),\ell}\}_\ell$ are all contained in $U_\circ$ and, in particular,

$$x^{1,L_1} := \lim_{s\to\infty} v^{(f,J^\flat),L_1}(s,\cdot) = \lim_{s\to-\infty} w^{(h,J^\flat)}(s,\cdot) \subset U_\circ.$$

Now, since $(h, J^\flat)$ has a barricade in U around $U_\circ$, we conclude that $w^{(h,J^\flat)} \subset U_\circ$ as well.

Step 5 Let us now show that $a_{h_+}(w^{(K,J)}) \le \mathcal{A}_f(x^{1,0}) + \delta = a_f(u_n) + \delta$. To that end, we bound the action growth along the broken trajectory $\overline{v}$:

(i) Along $v^{(\cdot,\cdot),\ell}$: these are solutions of the s–independent Floer equations and, by the energy identity (8), the action is clearly nonincreasing.

(ii) Along $w^{(h,J^\flat)}$: this trajectory is contained in $U_\circ$, where h approximates a constant homotopy, as $F|_U = F+G|_U$. Taking h to be close enough to H, we may assume that the derivative $\partial_s h|_{U_\circ}$ is bounded by δ. Denoting by $x^{1,L_1} \in \mathcal{P}(f)$ and $x^{2,0} \in \mathcal{P}(h_+)$ the orbits to which $w^{(h,J^\flat)}$ converges at the ends, it follows from the energy identity (10) that

$$\begin{aligned}
\mathcal{A}_{h_+}(x^{2,0}) - \mathcal{A}_f(x^{1,L_1}) &\le \left| \int_{\mathbb{R}\times S^1} (\partial_s h) \circ w^{(h,J^\flat)}\, ds\, dt \right| \\
&\le \int_{[0,1]\times S^1} \max_{U_\circ} |\partial_s h|\, ds\, dt \le \delta.
\end{aligned} \tag{30}$$

(iii) Along $w^{(h_+,\tilde{J})}$: it follows from the energy identity (10) that the action is nonincreasing, since $\partial_s h_+ = 0$.

Overall, we conclude that

$$\begin{aligned} a_{h_+}(w^{(K,J)}) &= \mathcal{A}_{h_+}(x^{3,L_3}) \le \cdots \le \mathcal{A}_{h_+}(x^{2,0}) \\ &\overset{(30)}{\le} \mathcal{A}_f(x^{1,L_1}) + \delta \le \cdots \le \mathcal{A}_f(x^{1,0}) + \delta = a_f(u_n) + \delta. \end{aligned}$$

Since u_n were chosen to be minimizers, $a_f(u_n) = A(K_n, J_n) \le c_{\mathrm{AHS}}(f)$. On the other hand, the fact that $w^{(K,J)} \in \mathcal{M}_{(K,J)}$ implies that $a_{h_+}(w^{(K,J)}) \ge \min_{\mathcal{M}(K,J)}(a_{h_+}) = A(K,J)$. We thus have proved that for any $(K,J) \in \mathcal{D}(h_+)$, $A(K,J) \le c_{\mathrm{AHS}}(f) + \delta$, which yields that $c_{\mathrm{AHS}}(h_+) \le c_{\mathrm{AHS}}(f) + \delta$, as required. □

9 The required transversality and compactness results

9.1 Perturbing homotopies and Hamiltonians to achieve regularity

Let (M,ω) be a closed symplectically aspherical manifold. Given a nondegenerate Hamiltonian H and an almost complex structure J, we say that a pair (H,J) is *Floer-regular* if for every pair of 1–periodic orbits $x_\pm$ of $H_\pm$ and for every $u \in \mathcal{M}_{(H,J)}(x_-, x_+)$, the differential $(d\mathcal{F})_u \colon W^{1,p}(u^*TM) \to L^p(u^*TM)$ of the Floer map (see Notation 9.9 below) is surjective. In this case, the space of solutions $\mathcal{M}_{(H,J)}(x_-, x_+)$ is a smooth manifold of dimension $\mu(x_-) - \mu(x_+)$. It is well known that for any nondegenerate Hamiltonian H and an almost complex structure J, one can perturb H, without changing its periodic orbits, in order to make the pair (H,J) Floer-regular. Let us cite a formal statement of this fact.

Claim 9.1 [8, Theorem 5.1] *Let H be a nondegenerate Hamiltonian and let J be an almost complex structure on M, and let $\mathcal{C}^\infty_\varepsilon(H)$ be the space of perturbations which vanish on $\mathcal{P}(H)$ up to second order.*[17] *Then there exist a neighborhood of zero in $\mathcal{C}^\infty_\varepsilon(H)$, and a residual set $\mathcal{H}_{\mathrm{reg}}$ in this neighborhood, such that for every $h \in \mathcal{H}_{\mathrm{reg}}$, the pair $(H+h, J)$ is Floer-regular.*

When H is a homotopy whose ends, $H_\pm$, are Floer-regular with respect to J, one can perturb H on a compact set to guarantee that the pair (H,J) is Floer-regular. For the purposes of this paper, we need to control the size of the support of the perturbation. In this section we prove that one can take the support of the perturbation to be any closed interval with nonempty interior. Before making a formal claim, let us fix

[17] This space is endowed with Floer's ε–norm, which is defined below.

some notation. Throughout this section, we consider homotopies of Hamiltonians, $H\colon M\times S^1\times\mathbb{R}\to\mathbb{R}$, that are constant with respect to the $\mathbb{R}$–coordinate, s, outside of a compact set, namely $\operatorname{supp}(\partial_s H)\subset M\times S^1\times[-R,R]$ for some $R>0$. We assume that the ends $H_\pm(\cdot,\cdot):=\lim_{s\to\pm\infty}H(\cdot,\cdot,s)$ are Floer-regular with respect to a fixed almost complex structure J. For a closed finite interval $I\subset\mathbb{R}$ with nonempty interior, we consider the space $\mathcal{C}^\infty_\varepsilon(I)$ of perturbations with support in $M\times S^1\times I$, whose definition is given in Section 9.1.1 below. Our main goal for this section is to prove the following proposition.

Proposition 9.2 *Let H be a homotopy such that $(H_\pm,J)$ are Floer-regular, where J is an almost complex structure on M, and let $I\subset\mathbb{R}$ be a closed, finite interval with a nonempty interior. Then there exists a residual subset $\mathcal{H}_{\mathrm{reg}}\subset\mathcal{C}^\infty_\varepsilon(I)$ such that for every $h\in\mathcal{H}_{\mathrm{reg}}$, the pair $(H+h,J)$ is Floer-regular.*

The proof of this proposition is postponed to Section 9.1.2. We start by describing the space of perturbations and its relevant properties.

9.1.1 The Banach space $\mathcal{C}^\infty_\varepsilon(I)$ In this section we define the perturbation space $\mathcal{C}^\infty_\varepsilon(I)$ and prove useful properties.

Definition 9.3 Let $\varepsilon=\{\varepsilon_n\}$ be a sequence of positive numbers.

- For $h\in\mathcal{C}^\infty(M\times S^1\times\mathbb{R})$, Floer's ε–norm is defined to be
$$\|h\|_\varepsilon:=\sum_{k\geq 0}\varepsilon_k\sup_{M\times S^1\times\mathbb{R}}|d^kh|.$$
See [3, page 230] for details.
- For a closed and finite interval $I\subset\mathbb{R}$ with a nonempty interior, let $\mathcal{C}^\infty_\varepsilon(I)$ be the space of functions $h\in\mathcal{C}^\infty(M\times S^1\times\mathbb{R})$, supported in $M\times S^1\times I$, whose ε–norm is finite, namely $\|h\|_\varepsilon<\infty$. Then $\mathcal{C}^\infty_\varepsilon(I)$ is a Banach space; see [20, Theorem B.2]. In what follows we identify between the tangent space $T_h\mathcal{C}^\infty_\varepsilon(I)$ at a point h, and the space $\mathcal{C}^\infty_\varepsilon(I)$ itself.

The following claims guarantee that the properties that are required of a space of perturbations hold for $\mathcal{C}^\infty_\varepsilon(I)$.

Claim 9.4 *There exists a sequence ε for which $\mathcal{C}^\infty_\varepsilon(I)$ is dense in $\mathcal{C}^\infty(I)$.*

Claim 9.5 *The Banach space $\mathcal{C}^\infty_\varepsilon(I)$ is separable.*

In order to prove these claims we first state and prove two lemmas. We use notation and ideas from [3, Section 8.3; 20, Appendix B].

Lemma 9.6 *Let E be a finite-dimensional vector bundle over $M \times S^1 \times \mathbb{R}$. Then the space $\mathcal{C}^0_I(E)$ of continuous sections of E that are supported in $M \times S^1 \times I$ is second-countable with respect to the uniform norm.*

Proof Embedding $M \times S^1 \times I$ into $[-N,N]^m$ for some large N and m, the space $\mathcal{C}^0_I(E)$ is isometrically embedded into $\mathcal{C}^0([-N,N]^m;\mathbb{R}^k)$ for some $k \in \mathbb{N}$.[18] By the Weierstrass approximation theorem, the latter space is separable, and hence (being a normed space) is also second-countable. We conclude that the same holds for the closed subspace $\mathcal{C}^0_I(E)$. □

Following [20, Appendix B], set $E^{(0)} := E$ and $E^{(k+1)} := \mathrm{Hom}(T(M \times S^1 \times \mathbb{R}); E^{(k)})$. Then, fixing connections and bundle metrics on both $T(M \times S^1 \times \mathbb{R})$ and E, any section $\eta \in \Gamma(E^{(k)})$ has a covariant derivative $\nabla\eta \in \Gamma(E^{(k+1)})$. Set $F^{(k)} := E^{(0)} \oplus \cdots \oplus E^{(k)}$ and consider the countable product $\prod_{k\in\mathbb{N}} \mathcal{C}^0_I(F^{(k)})$ endowed with the product topology. By Lemma 9.6, each factor is second-countable and therefore so is the product.

Lemma 9.7 *The space $\mathcal{C}^\infty(I)$ of smooth functions $M \times S^1 \times \mathbb{R} \to \mathbb{R}$ supported on $M \times S^1 \times I$ is separable with respect to the $\mathcal{C}^\infty$–topology.*

Proof The space $\mathcal{C}^\infty(I)$ can be embedded into the product $\prod_{k\in\mathbb{N}} \mathcal{C}^0_I(F^{(k)})$ by

$$\eta \mapsto (\eta, (\eta, \nabla\eta), (\eta, \nabla\eta, \nabla^2\eta), \dots).$$

As explained above, the product $\prod_{k\in\mathbb{N}} \mathcal{C}^0_I(F^{(k)})$ is second-countable and hence so is any closed subspace of it. In particular, $\mathcal{C}^\infty(I)$ is separable. □

We can now prove Claim 9.4. The proof is exactly that of [3, Proposition 8.3.1].

Proof of Claim 9.4 Let $f_n \in \mathcal{C}^\infty(I)$ be a dense sequence, whose existence is guaranteed by Lemma 9.7. Let

$$\varepsilon_n := \frac{1}{2^n \cdot \max_{k\le n} \|f_k\|_{\mathcal{C}^n(M\times S^1\times\mathbb{R})}}.$$

For this choice of a sequence ε, it holds that $\|f_n\|_\varepsilon < \infty$ for all n, namely, $f_n \in \mathcal{C}^\infty_\varepsilon(I)$. □

[18] This uses the fact that every vector bundle over a compact base is a subbundle of a trivial vector bundle; see [10, Proposition 1.4].

The proof of Claim 9.5 is essentially that of Lemma B.4 and Theorem B.5 from [20]; we include it for the convenience of the reader.

Proof of Claim 9.5 Consider again the product $\prod_{k\in\mathbb{N}} \mathcal{C}^0_I(F^{(k)})$ and let X_ε be the space of sequences $\xi := (\xi^0, \xi^1, \xi^2, \dots) \in \prod_{k\in\mathbb{N}} \mathcal{C}^0_I(F^{(k)})$ such that

$$\|\xi\|_{X_\varepsilon} := \sum_{k=0}^{\infty} \varepsilon_k \cdot \|\xi^k\|_{\mathcal{C}^0} < \infty.$$

We will first show that X_ε is separable and then embed $\mathcal{C}^\infty_\varepsilon(I)$ into X_ε in order to prove the claim. Indeed, since $\mathcal{C}^0_I(F^{(k)})$ is separable for each k (by Lemma 9.6), we can fix a dense countable subset $P^k \subset \mathcal{C}^0_I(F^{(k)})$. The set

$$P := \{(\xi^0, \dots, \xi^N, 0, 0, \dots) \in X_\varepsilon \mid N \geq 0 \text{ and for all } 0 \leq k \leq N,\ \xi^k \in P^k\}$$

is countable and dense in X_ε. Now consider the injective linear map

$$\mathcal{C}^\infty_\varepsilon(I) \hookrightarrow X_\varepsilon, \quad \eta \mapsto (\eta, (\eta, \nabla\eta), (\eta, \nabla\eta, \nabla^2\eta), \dots).$$

It is an isometric embedding, and hence we may view $\mathcal{C}^\infty_\varepsilon(I)$ as a closed subspace of the separable space X_ε. The latter is also second-countable (being a normed space) and hence so is $\mathcal{C}^\infty_\varepsilon(I)$. □

Remark 9.8 The proof of Claim 9.5 shows that spaces of perturbations with compact support are separable in general. This observation will be used in Section 9.3.2.

9.1.2 Proof of Proposition 9.2 We follow the proofs from Chapters 8 and 11 of [3] and make the necessary changes. Let us start by recalling the relevant notation.

Notation 9.9 Let H be a homotopy, let J be an almost complex structure, and let $x_\pm$ be 1–periodic orbits of $H_\pm$, respectively.

- We denote by $\mathcal{M}_{(H,J)}(x_-, x_+)$ the set of solutions of the (s–dependent) Floer equation with respect to H, J that converge to $x_\pm$ at the ends. We denote by $\mathcal{M}_{(H,J)}$ the set of all finite-energy solutions.
- [3, Definition 8.2.2] Denote by $\mathcal{P}(x_-, x_+)$ the space of maps $\mathbb{R} \times S^1 \to M$ of the form
$$(s,t) \mapsto \exp_{w(s,t)} Y(s,t)$$
for $Y \in W^{1,p}(w^*TM)$ and $w \in \mathcal{C}^\infty_\searrow(x_-, x_+)$. The latter is the space of smooth maps $\mathbb{R} \times S^1 \to M$ converging to $x_\pm$ at the ends with exponentially decaying derivatives. We denote by $\mathcal{L}^p(x_-, x_+)$ the fiber bundle over $\mathcal{P}(x_-, x_+)$ whose fiber at u is $L^p(u^*TM)$.

- The Floer map with respect to H is

(31) $$\mathcal{F}^H: \mathcal{P}(x_-,x_+) \to \mathcal{L}^p(x_-,x_+),$$
$$u \mapsto \frac{\partial u}{\partial s} + J\Big(\frac{\partial u}{\partial t} - X_H \circ u\Big) = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u H,$$

  where $(\operatorname{grad}_u H)(s,t)$ is the gradient of $H(\cdot,t,s)$ with respect to J, restricted to u. In unitary (ie symplectic, orthonormal) coordinates, the differential of the Floer map, $(d\mathcal{F}^H)_u: W^{1,p}(\mathbb{R}\times S^1; u^*TM) \to L^p(\mathbb{R}\times S^1; u^*TM)$, can be written as $(d\mathcal{F})_u(Y) = \bar\partial Y + SY$, where $S: \mathbb{R}\times S^1 \to \operatorname{End}(\mathbb{R}^{2n})$; see [3, Section 8.4 and page 389].

- Set

(32) $$\mathcal{Z}(x_-,x_+) := \{(u,h) \in \mathcal{P}(x_-,x_+) \times \mathcal{C}^\infty_\varepsilon(I) \mid u \in \mathcal{M}_{(H+h,J)}(x_-,x_+)\}.$$

The main ingredients in the proof of Proposition 9.2 are the following two lemmas.

Lemma 9.10 *The set $\mathcal{Z}(x_-,x_+)$ is a Banach manifold.*

Lemma 9.11 *The projection*

$$\pi: \mathcal{Z}(x_-,x_+) \to \mathcal{C}^\infty_\varepsilon(I), \quad (u,h) \mapsto h,$$

is a Fredholm map.

The outline of the proof is as follows: We first prove that the set $\mathcal{Z}(x_-,x_+)$ is a Banach manifold (Lemma 9.10), and then we show that the projection $\pi: \mathcal{Z}(x_-,x_+) \to \mathcal{C}^\infty_\varepsilon(I)$ is a Fredholm map (Lemma 9.11). Taking $\mathcal{H}_{\mathrm{reg}}$ to be the set of regular values of π, the Sard–Smale theorem guarantees that it is a residual set. We will use the following claim from [3].

Claim 9.12 [3, Theorem 11.1.7] *For every homotopy H such that $(H_\pm, J)$ are Floer-regular and every $u \in \mathcal{M}_{(H,J)}(x_-,x_+)$, the differential $(d\mathcal{F}^H)_u$ of the Floer map at u is a Fredholm operator of index $\mu(x_-)-\mu(x_+)$.*

In order to prove Lemma 9.10, we present $\mathcal{Z}(x_-,x_+)$ as an intersection of a certain section with the zero section in a certain vector bundle. The following lemma will be used to guarantee that this intersection is transversal. Its proof, which is a combination of the proofs of [3, Propositions 8.1.4 and 11.1.8], contains the main difference between the proof of Proposition 9.2 and that of [3, Theorem 11.1.6].

Lemma 9.13 *For $(u,h)\in\mathcal{Z}(x_-,x_+)$, the linear operator*

$$\Gamma\colon W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})\times\mathcal{C}^\infty_\varepsilon(I)\to L^p(\mathbb{R}\times S^1;\mathbb{R}^{2n}),\\ (Y,\eta)\mapsto(d\mathcal{F}^{H+h})_u(Y)+\operatorname{grad}_u\eta, \tag{33}$$

is surjective and has a continuous right inverse.

Proof Assume for the sake of contradiction that Γ is not surjective. By Lemma 8.5.1 of [3],[19] there exists a nonzero vector field $Z\in L^q(\mathbb{R}\times S^1;\mathbb{R}^{2n})$[20] of class $\mathcal{C}^\infty$ such that for every $Y\in W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})$ and $\eta\in\mathcal{C}^\infty_\varepsilon(I)$,

$$\langle Z,(d\mathcal{F}^{H+h})_u(Y)\rangle=0, \tag{34}$$

$$\langle Z,\operatorname{grad}_u\eta\rangle=0, \tag{35}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the pairing of L^q and L^p. As mentioned above, the differential of the Floer map can be written in unitary coordinates as $\bar\partial+S(s,t)$. Since Z is of class $\mathcal{C}^\infty$, it follows from (34) that Z is a zero of the dual operator of $(d\mathcal{F})_u$, which is of a "perturbed Cauchy–Riemann" type. The continuation principle [3, Proposition 8.6.6] now implies that if Z has an infinite-order zero, then it is identically zero, $Z\equiv 0$.

Therefore, let us show that (35) guarantees that Z vanishes on $I\times S^1$, and conclude that it vanishes identically, since we assumed that the interior of I is not empty. The proof is roughly the same as that of [3, Lemma 11.1.9], but we include it for the sake of completeness. An equivalent reformulation of (35) is

$$\int_{\mathbb{R}\times S^1} d\eta(Z)\,ds\,dt=0\quad\text{for every }\ \eta\in\mathcal{C}^\infty_\varepsilon(I).$$

Consider the map $\tilde u\colon\mathbb{R}\times S^1\to M\times\mathbb{R}\times S^1$ defined by $(s,t)\mapsto(u(s,t),s,t)$. It is easy to see that $\tilde u$ is an embedding. Viewing Z as a vector field along $\tilde u$ on $M\times\mathbb{R}\times S^1$ that does not have components in the directions $\partial/\partial t\in TS^1$ and $\partial/\partial s\in T\mathbb{R}$, we see that it is not tangent to $\tilde u$ at the points where it is not zero. Assume for the sake of contradiction that there exists a point $(s_0,t_0)\in I\times S^1$ at which Z does not vanish. Since Z is continuous, there exists a small neighborhood C_δ of (s_0,t_0) in which $Z(s,t)$ does not vanish and therefore is transversal to $\tilde u$ for all $(s,t)\in C_\delta$. Notice that if (s_0,t_0) is not in the interior of $I\times S^1$, we may replace it with a point in $C_\delta\cap(\operatorname{int}(I)\times S^1)$, and then replace C_δ by a smaller neighborhood that is contained in $\operatorname{int}(I)\times S^1$. Therefore we assume, without loss of generality, that $C_\delta\subset\operatorname{int}(I)\times S^1$. Let $\beta\colon\mathbb{R}\times S^1\to\mathbb{R}$ be a

[19] This lemma is formulated for a slightly different space, but its proof applies to our case as it is.
[20] Here $1/p+1/q=1$.

smooth function supported in C_δ, whose integral is not zero, $\int_{\mathbb{R}\times S^1} \beta(s,t)\,ds\,dt \neq 0$. Define $\eta\colon M\times S^1\times\mathbb{R}\to\mathbb{R}$ with support in a tubular neighborhood B of $\tilde{u}(C_\delta)$ in such a way that if $\gamma_{(s,t)}(\sigma)$ is a parametrized integral curve of Z passing through $\tilde{u}(s,t)$ at $\sigma=0$, then

$$\eta(\gamma_{(s,t)}(\sigma),t,s) := \beta(s,t)\cdot\sigma \quad \text{for } |\sigma|\le\epsilon.$$

The fact that Z is transversal to $\tilde{u}(C_\delta)$ guarantees that η is well defined. We also assume that $B\cap\operatorname{im}(\tilde{u}) = \tilde{u}(C_\delta)$, which means that $\operatorname{supp}(\eta)\cap\operatorname{im}(\tilde{u})\subset\tilde{u}(C_\delta)$. Let us compute the integral of $d\eta(Z)$:

$$\begin{aligned}\int_{\mathbb{R}\times S^1} d\eta_{s,t}(Z(s,t))\,ds\,dt &= \int_{C_\delta} d\eta_{s,t}(Z(s,t))\,ds\,dt\\ &= \int_{C_\delta} d\eta_{s,t}\left(\frac{\partial\gamma_{s,t}(\sigma)}{\partial\sigma}\bigg|_{\sigma=0}\right)ds\,dt\\ &= \int_{C_\delta}\frac{\partial}{\partial\sigma}(\eta(\gamma_{s,t}(\sigma),t,s))\big|_{\sigma=0}\,ds\,dt\\ &= \int_{C_\delta}\frac{\partial}{\partial\sigma}(\beta(s,t)\cdot\sigma)\big|_{\sigma=0}\,ds\,dt\\ &= \int_{C_\delta}\beta(s,t)\,ds\,dt.\end{aligned}$$

As we chose β to be a function with a nonvanishing integral, we find that (35) does not hold for the function η constructed above. Note that η is a smooth function, supported in $M\times S^1\times I$, but its ε–norm is not necessarily finite. Therefore, to arrive at a contradiction, it remains to approximate η by $\eta'\in\mathcal{C}^\infty_\varepsilon(I)$. This is possible due to Claim 9.4. When η' is close to η, the integral of $d\eta'(Z)$ will be close to that of $d\eta(Z)$ (since their supports are contained in the compact set $M\times S^1\times I$), and hence equality (35) will not hold for $\eta'\in\mathcal{C}^\infty_\varepsilon(I)$, a contradiction.

This shows that Γ is surjective. The fact that it has a continuous right inverse follows from [3, Lemma 8.5.6] and Claim 9.12. ◻

Having Lemma 9.13, the proof of Lemma 9.10, which asserts that $\mathcal{Z}(x_-,x_+)$ is a Banach manifold, is precisely that of [3, Proposition 8.1.3]:

Proof of Lemma 9.10 Let $\mathcal{E} := \{(u,h,Y) \mid Y\in L^p(u^*TM)\}$ be a vector bundle over $\mathcal{P}(x_-,x_+)\times\mathcal{C}^\infty_\varepsilon(I)$, and consider the section induced by $\mathcal{F}^{H+h}$:

$$\sigma\colon\mathcal{P}(x_-,x_+)\times\mathcal{C}^\infty_\varepsilon(I)\to\mathcal{E},\quad (u,h)\mapsto\Big(u,h,\frac{\partial u}{\partial s}+J\frac{\partial u}{\partial t}+\operatorname{grad}_u(H+h)\Big).$$

Notice that the space $\mathcal{Z}(x_-, x_+)$ is the intersection of σ with the zero section in $\mathcal{E}$. Therefore, in order to prove that $\mathcal{Z}(x_-, x_+)$ is a Banach manifold, it is sufficient to show that σ intersects the zero section transversely, or, equivalently, that $d\sigma$ composed with the projection onto the fiber is surjective and has a right inverse, at all points for which $\sigma(u,h) = 0$. But this composition is precisely the operator Γ, whose surjectivity and right invertibility are guaranteed by Lemma 9.13. $\square$

Our next goal is to show that π is a Fredholm map, that is, to prove Lemma 9.11.

Proof of Lemma 9.11 The projection $\pi\colon \mathcal{Z}(x_-, x_+) \to \mathcal{C}^\infty_\varepsilon(I)$ given by $\pi(u,h) = h$ is clearly smooth. Let us show that its differential, $d\pi$, has a finite-dimensional kernel and a closed image of finite codimension.

- $\ker(d\pi)_{(u,h)} = \{(Y, 0) \in T_{(u,h)}\mathcal{Z}(x_-, x_+)\}$. The tangent space of $\mathcal{Z}(x_-, x_+)$ is
$$T_{(u,h)}\mathcal{Z}(x_-, x_+) = \{(Y, \eta) \mid (d\mathcal{F}^{H+h})_u(Y) + \operatorname{grad}_u \eta = 0\},$$
and therefore, the kernel of $(d\pi)_{(u,h)}$ agrees with the kernel of $(d\mathcal{F}^{H+h})_u$, which is finite-dimensional by Claim 9.12.

- $\operatorname{im}(d\pi)_{(u,h)} = \{\eta \mid \exists\, Y \in W^{1,p}(\mathbb{R}\times S^1; u^*TM) \text{ with } \operatorname{grad}_u \eta = -(d\mathcal{F}^{H+h})_u(Y)\}$. Consider the linear map $G\colon \mathcal{C}^\infty_\varepsilon(I) \to L^p(\mathbb{R}\times S^1; u^*TM)$ defined by $G(\eta) = \operatorname{grad}_u \eta$. Then

$$\operatorname{im}(d\pi)_{(u,h)} = \{\eta \mid \operatorname{grad}_u \eta \in \operatorname{im}(d\mathcal{F}^{H+h})_u\} = G^{-1}(\operatorname{im}(d\mathcal{F}^{H+h})_u). \tag{36}$$

By Claim 9.12, the image of $(d\mathcal{F}^{H+h})_u$ is closed and of finite codimension. Let us show that the same properties hold for the image of $(d\pi)_{(u,h)}$. Consider the map induced by G on the quotients,

$$A := \frac{\mathcal{C}^\infty_\varepsilon(I)}{\operatorname{im}(d\pi)_{(u,h)}} \xrightarrow{G'} B := \frac{L^p(\mathbb{R}\times S^1; u^*TM)}{\operatorname{im}(d\mathcal{F}^{H+h})_u},$$

which is well defined due to (36). It is easy to see that G' is injective and, together with the fact that B is finite-dimensional, this yields that $\operatorname{codim}(\operatorname{im}(d\pi)_{(u,h)}) = \dim(A)$ is finite. This now implies that the image of $(d\pi)_{(u,h)}$ is also closed and hence $(d\pi)_{(u,h)}$ is a Fredholm operator. $\square$

Having proved Lemmas 9.10 and 9.11, we are ready to prove the main proposition.

Proof of Proposition 9.2 By Lemma 9.11, the projection $\pi\colon \mathcal{Z}(x_-,x_+)\to\mathcal{C}^\infty_\varepsilon(I)$ is a (smooth) Fredholm map. By Claim 9.5, the space $\mathcal{C}^\infty_\varepsilon(I)$ is separable. To see that $\mathcal{Z}(x_-,x_+)$ is a separable Banach manifold, recall that it is modeled over a subspace of the Banach space $W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})\times\mathcal{C}^\infty_\varepsilon(I)$. The latter is a separable metric space, and therefore second-countable. As any subspace of a second-countable space is also second-countable, and, in particular, separable, we conclude that $\mathcal{Z}(x_-,x_+)$ is separable. It follows that we may apply the Sard–Smale theorem to π and conclude that the set of regular values of π is a countable intersection of open dense sets in $\mathcal{C}^\infty_\varepsilon(I)$. The set $\mathcal{H}_{\mathrm{reg}}\subset\mathcal{C}^\infty_\varepsilon(I)$ is defined to be the intersection of the regular values of the projections for all choices of 1–periodic orbits $x_\pm$.

Let us show that for each $h\in\mathcal{H}_{\mathrm{reg}}$, the pair $(H+h,J)$ is Floer-regular. Fix 1–periodic orbits $x_\pm$. Then h is a regular value of the projection $\pi\colon\mathcal{Z}(x_-,x_+)\to\mathcal{C}^\infty_\varepsilon(I)$. Let us show that for every $u\in\mathcal{M}_{(H+h,J)}(x_-,x_+)$, the differential of the Floer map, $(d\mathcal{F}^{H+h})_u$, is surjective. Indeed, otherwise, arguing as in the proof of Lemma 9.13, there exists $Z\in L^q(\mathbb{R}\times S^1;\mathbb{R}^{2n})$, where $\frac{1}{p}+\frac{1}{q}=1$, such that $\langle Z,(d\mathcal{F}^{H+h})_u(Y)\rangle=0$ for all Y. Since $(d\pi)_{(u,h)}$ is surjective, for every $\eta\in\mathcal{C}^\infty_\varepsilon(I)$ there exists Y such that $\mathrm{grad}_u\,\eta=-(d\mathcal{F}^{H+h})_u(Y)$, and hence $\langle Z,\mathrm{grad}_u\,\eta\rangle=0$ as well. We conclude that Z satisfies both equations (34) and (35), and, proceeding as in the proof of Lemma 9.13, we find $Z=0$. Thus $(d\mathcal{F}^{H+h})_u$ is indeed surjective.

It remains to show that $\mathcal{M}_{(H+h,J)}(x_-,x_+)$ is a smooth manifold of the correct dimension. The inverse image $\pi^{-1}(h)$ is the space of maps $u\in\mathcal{P}(x_-,x_+)$ of class $W^{1,p}$ that are solutions of the Floer equation, $\mathcal{F}^{H+h}(u)=0$. By elliptic regularity, these solutions are all smooth, and hence $\pi^{-1}(h)=\mathcal{M}_{(H+h,J)}(x_-,x_+)$. Since h is a regular value of π, we therefore conclude that $\mathcal{M}_{(H+h,J)}(x_-,x_+)$ is indeed a smooth manifold. Its dimension is

$$\dim\ker(d\pi)_{(u,h)}=\dim\ker(d\mathcal{F}^{H+h})_u=\mathrm{ind}(d\mathcal{F}^{H+h})_u=\mu(x_-)-\mu(x_+),$$

where the last equality follows from Claim 9.12 above. □

9.2 Convergence to broken trajectories

A well-known phenomenon in Floer theory on symplectically aspherical manifolds is the convergence of sequences of solutions to a *broken trajectory*. In this section we formulate and prove results of this sort for the settings that are considered throughout the paper.

9.2.1 Convergence for homotopies with nondegenerate ends In what follows we consider homotopies with nondegenerate ends. We remark that the same arguments apply for nondegenerate Hamiltonians, when one considers them as constant homotopies. Let H be a homotopy that is constant outside of $M \times S^1 \times [-R, R]$ for some fixed $R > 0$, namely, $\partial_s H|_{|s|>R} = 0$. Let H_n be a sequence of homotopies $\mathcal{C}^\infty$–converging to H such that for each n,

$$\operatorname{supp}(\partial_s H_n) \subset M \times S^1 \times [-R, R] \quad \text{and} \quad \mathcal{P}(H_{n\pm}) = \mathcal{P}(H_\pm). \tag{37}$$

Recall that $\mathcal{M}_{(H,J)}$ denotes the set of finite-energy solutions of the Floer equation (FE) with respect to H and J; for $x_\pm \in \mathcal{P}(H_\pm)$, we denote by $\mathcal{M}_{(H,J)}(x_-, x_+) \subset \mathcal{M}_{(H,J)}$ the subset of solutions connecting $x_\pm$. Let

$$\mathcal{M}(x_-, x_+) := \bigcup_n \mathcal{M}_{(H_n,J)}(x_-, x_+) \cup \mathcal{M}_{(H,J)}(x_-, x_+)$$

be the space of all finite-energy solutions connecting $x_\pm$ with respect to (H, J) and (H_n, J) for all n, and set $\mathcal{M} := \bigcup_{x_\pm \in \mathcal{P}(H_\pm)} \mathcal{M}(x_-, x_+)$. The following proposition is an adjustment of [3, Theorems 11.1.10 and 11.3.10] to our case.

Proposition 9.14 *Let H be a homotopy with nondegenerate ends, and let H_n be a sequence converging to H in $\mathcal{C}^\infty(M \times S^1 \times \mathbb{R})$ that satisfies (37) for each n. Given a sequence $u_n \in \mathcal{M}_{(H_n,J)}(x_-, x_+)$ of solutions and a sequence of real numbers $\{\sigma_n\}$, there exist*

- *subsequences of $\{u_n\}$ and $\{\sigma_n\}$, which we still denote by $\{u_n\}$ and $\{\sigma_n\}$,*
- *periodic orbits $x_- = x_0, x_1, \dots, x_k \in \mathcal{P}(H_-)$ and $y_0, y_1, \dots, y_\ell = x_+ \in \mathcal{P}(H_+)$,*
- *sequences of real numbers $\{s_n^i\}_n$ for $1 \le i \le k$ and $\{s_n'^j\}_n$ for $1 \le j \le \ell$,*
- *solutions $v_i \in \mathcal{M}_{(H_-,J)}(x_{i-1}, x_i)$ for $1 \le i \le k$ and $v_j' \in \mathcal{M}_{(H_+,J)}(y_{j-1}, y_j)$ for $1 \le j \le \ell$,*
- *a solution $w \in \mathcal{M}_{(H,J)}(x_k, y_0)$,*

such that in $\mathcal{C}^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1; M)$, for $1 \le i \le k$ and $1 \le j \le \ell$, we have

$$\lim_{n\to\infty} u_n(\cdot + s_n^i, \cdot) = v_i, \quad \lim_{n\to\infty} u_n(\cdot + s_n'^j, \cdot) = v_j', \quad \lim_{n\to\infty} u_n = w,$$

and the sequence $u_n(\cdot + \sigma_n, \cdot)$ converges to one of v_i, w or v_j', perhaps up to a shift in the s–coordinate.

The finite sequence $(v_1, \dots, v_k, w, v'_1, \dots, v'_\ell)$ is called a broken trajectory of (H, J). Before proving the above proposition, we state and prove two lemmas. The first is an analogous statement to [3, Theorem 11.2.7], and gives a uniform bound for the J–gradient of a solution u of the Floer equation with respect to (H, J) or (H_n, J).

Lemma 9.15 *There exists a constant $A > 0$ such that for every $u \in \mathcal{M}$ and every $(s,t) \in \mathbb{R} \times S^1$,*

$$\left\| \frac{\partial u}{\partial s} \right\|_J^2 + \left\| \frac{\partial u}{\partial t} \right\|_J^2 \leq A.$$

Proof For convenience we set $H_0 := H$. Let $x_\pm \in \mathcal{P}(H_\pm)$ be periodic orbits such that $u \in \mathcal{M}(x_-, x_+)$. Then, by the energy identity (10),

$$E(u) \leq \mathcal{A}_{H_-}(x_-) - \mathcal{A}_{H_+}(x_+) + 2R \cdot C', \tag{38}$$

where

$$C' := \sup \left\{ \frac{\partial H_n}{\partial s}(x,t,s) \;\middle|\; (x,t,s) \in M \times S^1 \times \mathbb{R},\ n \geq 0 \right\},$$

and $R > 0$ is the constant from (37). The fact that C' is finite follows from the uniform convergence (with derivatives) of H_n to $H_0 = H$. Setting

$$C := \max_{x_\pm \in \mathcal{P}(H_\pm)} (\mathcal{A}_{H_-}(x_-) - \mathcal{A}_{H_+}(x_+)) + 2R \cdot C',$$

we obtain a uniform bound for the energy, $E(u) \leq C$ for all $u \in \mathcal{M}$. As in [3, Propositions 6.6.2 and 11.1.5], we conclude that there exists $A > 0$ such that

$$\left\| \frac{\partial u}{\partial s} \right\|_J^2 + \left\| \frac{\partial u}{\partial t} \right\|_J^2 \leq A. \qquad \square$$

The next lemma uses the Arzelà–Ascoli theorem and elliptic regularity to show that every sequence of shifted solutions has a converging subsequence. It is an adjustment of Theorem 11.3.7 and Lemma 11.3.9 from [3] to our setting.

Lemma 9.16 *Let $u_n \in \mathcal{M}_{(H_n,J)}(x_-, x_+)$ be a sequence of solutions and let $s_n \in \mathbb{R}$ be a sequence of numbers. Then the sequence of shifted solutions $\tau_{s_n} u_n(\cdot,\cdot) = u_n(\cdot + s_n, \cdot)$ has a subsequence that converges in the $\mathcal{C}^\infty_{\mathrm{loc}}$ topology to a limit v. Moreover:*

(i) *If $s_n \to \sigma \in \mathbb{R}$, then $v \in \mathcal{M}_{(\tau_\sigma H, J)}$, where $\tau_\sigma H(x,t,s) := H(x,t,s+\sigma)$.*

(ii) *If $s_n \to -\infty$, then $v \in \mathcal{M}_{(H_-,J)}$.*

(iii) *If $s_n \to +\infty$, then $v \in \mathcal{M}_{(H_+,J)}$.*

Proof Lemma 9.15 implies that the sequence $v_n := \tau_{s_n} u_n$ is equicontinuous. By the Arzelà–Ascoli theorem and elliptic regularity (see [3, Lemma 12.1.1]), there exists a subsequence, which we still denote by $\{v_n\}$, that converges to a limit v in the $\mathcal{C}^\infty_{\mathrm{loc}}$ topology. The fact that the energy of v is finite follows from the uniform bound (38) on the energies of u_n. It remains to show that the limit v is a solution of the corresponding equation for the above choices of shifts s_n. For each n, v_n is a solution of the equation

$$\begin{aligned} 0 &= \frac{\partial v_n}{\partial s} + J\frac{\partial v_n}{\partial t} + \mathrm{grad}_{v_n}(\tau_{s_n} H_n) \\ &= \left(\frac{\partial v_n}{\partial s} + J\frac{\partial v_n}{\partial t} + \mathrm{grad}_{v_n}(\tau_{s_n} H)\right) + \mathrm{grad}_{v_n}(\tau_{s_n}(H_n - H)). \end{aligned}$$

Since the sequence H_n converges to H uniformly with the derivatives, for every $\epsilon > 0$ there exists N such that for $n \geq N$,

$$\left\|\frac{\partial v_n}{\partial s} + J\frac{\partial v_n}{\partial t} + \mathrm{grad}_{v_n}(\tau_{s_n} H)\right\| < \epsilon. \tag{39}$$

Let us split into cases:

(i) Assume $s_n \to \sigma \in \mathbb{R}$. Fix an arbitrarily large $r > |\sigma|$. Then the derivatives of H are uniformly continuous on the compact set $M \times S^1 \times [-r, r]$. Using (39) together with our assumption that $s_n \to \sigma$, we have

$$\max_{[-r,r]\times S^1}\left|\frac{\partial v_n}{\partial s} + J\frac{\partial v_n}{\partial t} + \mathrm{grad}_{v_n}(\tau_\sigma H)\right| < \epsilon + \max_{[-r,r]\times S^1}|\mathrm{grad}_{v_n}(\tau_\sigma H - \tau_{s_n} H)| < 2\epsilon$$

when n is large enough. It follows that the limit v of the sequence v_n is a solution of the s–dependent Floer equation with respect to $\tau_\sigma H$ and J.

(ii) Assume $s_n \to -\infty$. Recalling that the homotopy H is constant for $|s| \geq R$, we have $H(x,t,s) = H_-(x,t)$ whenever $s \leq -R$. Since $s_n \to -\infty$, for every $r > 0$ there exists N large enough that $s_n < -R - r$ whenever $n \geq N$. For such n, the restriction of (39) to the compact subset $[-r, r] \times S^1$ is

$$\max_{[-r,r]\times S^1}\left|\frac{\partial v_n}{\partial s} + J\frac{\partial v_n}{\partial t} + \mathrm{grad}_{v_n} H_-\right| < \epsilon,$$

since $\tau_{s_n} H(x,t,s) = H(x,t,s+s_n) = H_-$ when $s \in [-r, r]$. Taking the limit when $n \to \infty$ (and $\epsilon \to 0$), we conclude that v is a solution of the Floer equation with respect to (H_-, J).

(iii) When $s_n \to \infty$, the proof is as in the previous case. □

Having Lemma 9.16, the proof of Proposition 9.14 (namely, the convergence to a broken trajectory) is similar to the that of [3, Theorem 11.1.10]. We follow it and make the necessary adjustments.

Proof of Proposition 9.14 Let us prove the claim for the case where $\sigma_n \to -\infty$; the other cases are analogous. We start by fixing $\epsilon > 0$ small enough that the open balls

$$B(x,\epsilon) := \{\gamma \in \mathcal{L}M \mid d_\infty(x,\gamma) < \epsilon\}$$

are disjoint for $x \in \mathcal{P}(H_-)$. Here $\mathcal{L}M$ is the space of contractible loops in M, endowed with the uniform metric d_∞. By shrinking ϵ if necessary, we assume that the balls $\{B(y,\epsilon)\}_{y \in \mathcal{P}(H_+)}$ are also disjoint. Lemma 9.16 guarantees that after passing to a subsequence, the sequence $\tau_{\sigma_n} u_n$ converges in $\mathcal{C}^\infty_{\mathrm{loc}}$ to a finite-energy solution $v \in \mathcal{M}_{(H_-,J)}$. Since H_- is nondegenerate, there exist periodic orbits $x_0, x_1 \in \mathcal{P}(H_-)$ such that $v \in \mathcal{M}_{(H_-,J)}(x_0,x_1)$. Moreover, applying Lemma 9.16 to the sequence u_n with zero shifts, we conclude that after extracting a subsequence, it converges to a finite-energy solution $w \in \mathcal{M}_{(H,J)}(x_k, y_0)$ for some $x_k \in \mathcal{P}(H_-)$ and $y_0 \in \mathcal{P}(H_+)$. Let us find the solutions preceding v, connecting v to w and following w in the broken trajectory:

- **Solutions preceding v** There exists $s_\star \le 0$ such that for any $s \le s_\star$, it holds that $v(s,\cdot) \in B(x_0,\epsilon)$. Since $v = \lim \tau_{\sigma_n} u_n$, when n is large enough, it holds that $u_n(s_\star + \sigma_n, \cdot) \in B(x_0,\epsilon)$ as well. If $x_0 = x_-$, there are no preceding solutions and we are done. Otherwise, $x_0 \ne x_-$, and since u_n converges to x_- when $s \to -\infty$, it must exit the ball $B(x_0,\epsilon)$ for $s \le s_\star$. Let us denote by s_n the first exit point,

$$s_n := \inf\{s \le s_\star \mid u_n(\sigma_n + s', \cdot) \in B(x_0,\epsilon) \text{ for } s' \in [s, s_\star]\}.$$

Let us now show that $s_n \to -\infty$. Indeed, if $\{s_n\}$ were bounded, it would have had a subsequence converging to some $s_\circ \in \mathbb{R}$. Since $\tau_{\sigma_n} u_n$ converges to v in $\mathcal{C}^\infty_{\mathrm{loc}}$ and since $s_\circ \le s_\star$, we would get

$$\lim_{n\to\infty} u_n(s_n + \sigma_n, \cdot) = v(s_\circ, \cdot) \in B(x_0,\epsilon),$$

in contradiction to our choice of s_n, namely, that $u_n(\sigma_n + s_n, \cdot) \in \partial B(x_0,\epsilon)$. We conclude therefore that $s_n \to -\infty$ and, in particular, $s_n + \sigma_n \to -\infty$ as well. Using Lemma 9.16 for $\tau_{s_n+\sigma_n} u_n$, we conclude that, after passing to a subsequence, this shifted sequence converges to some $v_{-1} \in \mathcal{M}_{(H_-,J)}$. We need to prove that v_{-1} converges to x_0 when $s \to \infty$. Fix $s > 0$. Then for n sufficiently large, $s_n < s + s_n < s_\star$ and

$$\tau_{s_n+\sigma_n} u_n(s,\cdot) \in B(x_0,\epsilon).$$

This implies that $v_{-1}(s,\cdot)\in\overline{B(x_0,\epsilon)}$ for all $s>0$, and hence $v_{-1}\in\mathcal{M}_{(H_-,J)}(x_{-1},x_0)$ for some $x_{-1}\in\mathcal{P}(H_-)$.

Continuing in this way we find v_{-2}, v_{-3} and so on, until $x_{-k'}=x_-$. This process is finite, since there are finitely many orbits in $\mathcal{P}(H_-)$ and the action is strictly decreasing in each step; namely, $\mathcal{A}_{H_-}(x_{-i})>\mathcal{A}_{H_-}(x_{-i-1})$ for $0\le i\le k'$.

- **Solutions connecting v to w** Recall that $\tau_{\sigma_n}u_n$ converges to $v\in\mathcal{M}_{(H_-,J)}(x_0,x_1)$ and that u_n converges to $w\in\mathcal{M}_{(H,J)}(x_k,y_0)$. Let us find the solutions that connect v to w (or prove that $x_1=x_k$). In analogy with the previous case, pick $s_\star\ge 0$ such that $v(s,\cdot)\in B(x_1,\epsilon)$ for all $s\ge s_\star$. Then, for n large enough, $u_n(s_\star+\sigma_n,\cdot)\in B(x_1,\epsilon)$ as well. Arguing similarly for $w\in\mathcal{M}_{(H,J)}(x_k,y_0)$, there exists $s_\dagger\le 0$ such that $w(s,\cdot)\in B(x_k,\epsilon)$ for all $s\le s_\dagger$ and, since the u_n converge to w, for n large enough $u_n(s_\dagger,\cdot)\in B(x_k,\epsilon)$ as well. As $\sigma_n\to-\infty$, we have $s_\star+\sigma_n<s_\dagger$ for large n. Consider the first exit of u_n from $B(x_1,\epsilon)$,

$$s_n:=\sup\{s\ge s_\star\mid u_n(\sigma_n+s',\cdot)\in B(x_1,\epsilon)\text{ for }s'\in[s_\star,s]\}.$$

Then, repeating the arguments from the previous step, one sees that $s_n\to\infty$. Moreover, it follows from the definitions of s_n and $s_\dagger$ that $s_n+\sigma_n<s_\dagger$. Therefore, the sequence $\{\sigma_n+s_n\}$ is either bounded or tends to $-\infty$. In the first case, it converges, after passing to a subsequence, to some number $s_\circ\in\mathbb{R}$. Moreover, since u_n converges to w on compacts, we conclude that $\tau_{\sigma_n+s_n}u_n$ converges to $\tau_{s_\circ}w$. In particular, this implies that $x_1=x_k$. Indeed, for every $s<s_\circ$ and n sufficiently large, $s\in[\sigma_n+s_\star,\sigma_n+s_n]$, and thus $u_n(s,\cdot)\in B(x_1,\epsilon)$. As a consequence, $w(s,\cdot)\in B(x_1,\epsilon)$ for all $s<s_\circ$, which means that $x_k=x_1$ and we are done. Let us now deal with the case where $s_n+\sigma_n\to-\infty$. By Lemma 9.16, there exists a subsequence of $\tau_{s_n+\sigma_n}u_n$ that converges to a finite-energy solution $v_1\in\mathcal{M}_{(H_-,J)}$. We need to show that the left end of v_1 converges to x_1, namely, that $v_1\in\mathcal{M}_{(H_-,J)}(x_1,x_2)$ for some $x_2\in\mathcal{P}(H_-)$. Fix $s<0$ and let us show that $v_1(s,\cdot)\in B(x_1,\epsilon)$. Since $s_n\to\infty$, when n is large enough we have that $s+s_n\in[s_\star,s_n]$. As we saw above, this implies that $\tau_{\sigma_n+s_n}u_n(s,\cdot)=u_n(s+s_n+\sigma_n,\cdot)\in B(x_1,\epsilon)$, and thus $v_1(s,\cdot)\in\overline{B(x_1,\epsilon)}$ as required. Repeating this process, we find solutions $v_2,\dots,v_{k-1}$ such that $v_i\in\mathcal{M}_{(H_-,J)}(x_i,x_{i+1})$, and therefore these connect v to w. As in the previous case, this process is finite since every solution v_i is action-decreasing and H_- has finitely many 1–periodic orbits.

- **Solutions following w** The right end of w converges to $y_0\in\mathcal{P}(H_+)$, and hence there exists $s_\star\ge 0$ such that for every $s\ge s_\star$, $w(s,\cdot)\in B(y_0,\epsilon)$. As the u_n converge to w in C^∞_{loc}, for n large enough $u_n(s_\star,\cdot)\in B(y_0,\epsilon)$ as well. Assume that $y_0\ne x_+$,

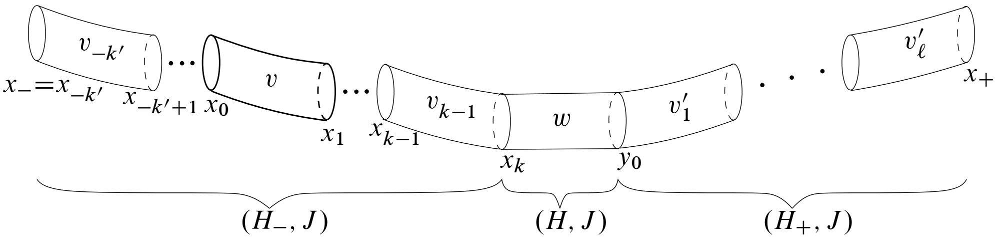

Figure 11: An illustration of the broken trajectory as constructed in the proof of Proposition 9.14.

otherwise there is nothing to prove. Then, since the u_n converge to x_+ for each n, it must leave the ball $B(y_0,\epsilon)$ at some point. Consider the first exit,

$$s_n := \sup\{s \geq s_\star \mid u_n(s',\cdot) \in B(y_0,\epsilon) \text{ for } s' \in [s_\star, s]\}.$$

Then, arguing as above, $s_n \to \infty$. Applying Lemma 9.16 to the sequence u_n shifted by s_n, it converges (up to a subsequence) to a finite-energy solution $v'_1 \in \mathcal{M}_{(H_+,J)}$. We need to show that the left end of v'_1 converges to y_0, namely, that $v'_1 \in \mathcal{M}_{(H_+,J)}(y_0,y_1)$ for some $y_1 \in \mathcal{P}(H_+)$. As before, fix any $s<0$. Then when n is large enough, $s+s_n \in [s_\star, s_n]$ and therefore, $\tau_{s_n}u_n(s,\cdot) = u_n(s+s_n,\cdot) \in B(y_0,\epsilon)$. Again, we conclude that $v'_1(s,\cdot) \in \overline{B(y_0,\epsilon)}$, which guarantees that v'_1 converges to y_0. Continuing by induction and using the fact that each v'_j reduces the action concludes the proof. □

9.2.2 Concatenation of homotopies with possibly degenerate ends In what follows, we study the breaking mechanism for solutions of (FE) with respect to homotopies of Hamiltonians that are obtained as concatenations of finitely many given homotopies, with possibly degenerate ends. In addition, we consider homotopies of almost complex structures, as opposed to the constant structures considered previously. When the ends of the first few concatenated homotopies are nondegenerate, we prove what we call a *partial convergence to a broken trajectory.*

Let $(H^1,J^1),\dots,(H^K,J^K)$ be pairs of homotopies of Hamiltonians and homotopies of almost complex structures, respectively, which are constant outside of $[0,1]$, namely

$$\partial_s H^k = 0 \quad \text{and} \quad \partial_s J^k = 0 \quad \text{for } s \notin [0,1] \text{ and } k = 1,\dots,K.$$

Assume in addition that $H^k_+ = H^{k+1}_-$ and $J^k_+ = J^{k+1}_-$ for $k=1,\dots,K-1$. Let $\{\sigma^1_n\}_n,\dots,\{\sigma^K_n\}_n$ be monotone sequences of real numbers such that for each n we have $\sigma^1_n < \dots < \sigma^K_n$, and for each $k \neq j$, the sequence of differences $\{\sigma^k_n - \sigma^j_n\}_n$ is unbounded. For the rest of this section, we consider the sequences $\{H_n\}$ and $\{J_n\}$ of

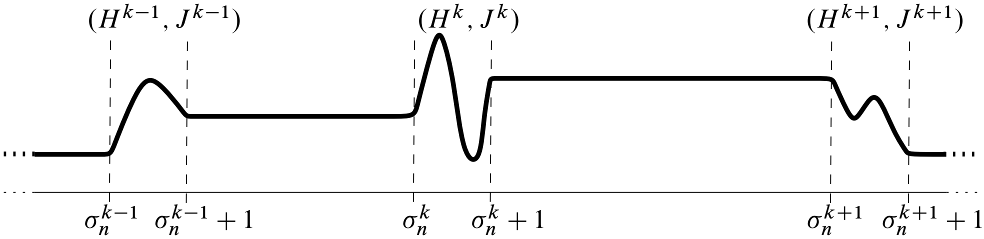

Figure 12: An illustration of the pair (H_n, J_n), which is a concatenation of the homotopies (H^k, J^k) shifted by $\{\sigma_n^k\}$.

homotopies of Hamiltonians and almost complex structures obtained by concatenating the shifts of $\{H^k\}$ and $\{J^k\}$ by the sequences $\{-\sigma_n^k\}$. More formally, H_n and J_n are the sequences which satisfy, for each $k = 1, \dots, K$,

$$\begin{aligned} H_n &= \tau_{-\sigma_n^k} H^k \quad \text{on } M \times S^1 \times [\sigma_n^k, \sigma_n^k + 1], \\ J_n &= \tau_{-\sigma_n^k} J^k \quad \text{on } S^1 \times [\sigma_n^k, \sigma_n^k + 1], \end{aligned}$$

and which are locally constant elsewhere; see Figure 12. Since the ends of the homotopies H^k might be degenerate, a sequence of solutions $u_n \in \mathcal{M}_{(H_n, J_n)}$ does not necessarily admit a subsequence converging to a broken trajectory. However, when some of the homotopies have nondegenerate ends, a slightly weaker statement holds:

Proposition 9.17 *Assume that there exists $1 < K' \le K$ such that for every $k < K'$, the ends of the homotopy H^k are nondegenerate. Then, for every sequence $u_n \in \mathcal{M}_{(H_n, J_n)}$, there exist*

- *a subsequence of $\{u_n\}$, which we still denote by $\{u_n\}$,*
- *periodic orbits $x^{k,\ell} \in \mathcal{P}(H_-^{k+1})$ for $\ell = 0, \dots, L_k$ and $k = 1, \dots, K'$, where $x^{1,0} = \lim_{s \to -\infty} u_n(s, \cdot)$ for all n,*
- *real numbers $s_n^{k,\ell} \in \mathbb{R}$ for $\ell = 1, \dots, L_k$ and $k = 1, \dots, K'$ which are such that $s_n^{k,\ell} < \sigma_n^{k+1} < s_n^{k+1,\ell'}$ for all $\ell = 1, \dots, L_k$ and $\ell' = 1, \dots, L_{k+1}$,*
- *solutions of s–independent Floer equations $v^{k,\ell} \in \mathcal{M}_{(H_-^k, J_-^k)}(x^{k,\ell-1}, x^{k,\ell})$ for $\ell = 1, \dots, L_k$ and $k = 1, \dots, K'$,*
- *solutions of s–dependent Floer equations $w^k \in \mathcal{M}_{(H^k, J^k)}(x^{k,L_k}, x^{k+1,0})$ for $k = 1, \dots, K' - 1$, and $w^{K'} \in \mathcal{M}_{(H^{K'}, J^{K'})}$ such that $\lim_{s \to -\infty} w^{K'}(s, \cdot) = x^{K', L_{K'}}$,*

such that, in $\mathcal{C}^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1; M)$*,*

$$\lim_{n\to\infty} u_n(\cdot + \sigma_n^k, \cdot) = w^k \quad \textit{and} \quad \lim_{n\to\infty} u_n(\cdot + s_n^{k,\ell}, \cdot) = v^{k,\ell}$$

for $1 \le \ell \le L_k$ *and* $1 \le k \le K'$*.*

In this case, we say that $\{u_n\}$ partially converges to the broken trajectory

$$\overline{v} = (\{v^{1,\ell}\}_{\ell=1}^{L_1}, w^1, \{v^{2,\ell}\}_{\ell=1}^{L_2}, w^2, \dots, w^{K'-1}, \{v^{K',\ell}\}_{\ell=1}^{L_{K'}}, w^{K'}).$$

In order to prove Proposition 9.17, we need statements analogous to Lemmas 9.15 and 9.16 adapted for the current setting. Notice that due to our assumption that H^k has nondegenerate ends for $1 \le k < K'$, the left end of the homotopies H_n, which is equal to H_-^1, is nondegenerate. On the other hand, the right end, $H_{n+} = H_+^K$, might be degenerate. A solution u of the Floer equation with respect to a homotopy with degenerate ends does not necessarily converge to periodic orbits at the ends. However, the following lemma asserts that the action of $u(s,\cdot)$ converges, as $s \to \pm\infty$, to a limit that belongs to the action spectrum of the corresponding Hamiltonian. The following statement is proved in the proof of Proposition 2.1(ii) from [1] for the left end of u, namely, $\lim_{s\to-\infty} \mathcal{A}_{H_-}(u(s,\cdot)) \in \operatorname{spec}(H_-)$. The proof for the right end is completely analogous and we therefore omit it.

Lemma 9.18 [1] *Let* (H, J) *be a pair of homotopies of Hamiltonians and almost complex structures. Then, for every finite-energy solution* $u \in \mathcal{M}_{(H,J)}$*,*

$$\lim_{s\to\pm\infty} \mathcal{A}_{H_\pm}(u(s,\cdot)) \in \operatorname{spec}(H_\pm).$$

Denoting by $\mathcal{M} := \bigcup_n \mathcal{M}_{(H_n,J_n)}$ the set of finite-energy solutions, the next lemma provides a uniform bound for the energy of $u \in \mathcal{M}$ and is an adjustment of Lemma 9.15 to the current setting.

Lemma 9.19 *There exists a constant* $A > 0$ *such that for every* $u \in \mathcal{M}$ *and* $(s,t) \in \mathbb{R} \times S^1$*, one has* $\|\operatorname{grad}_{(s,t)} u\| \le A$*.*

Proof For a finite-energy solution u of a homotopy with possibly degenerate ends, the limits $\lim_{s\to\pm\infty} \mathcal{A}_{H_\pm}(u(s,\cdot))$ exist and u satisfies the energy identity

$$E(u) = \lim_{s\to-\infty} \mathcal{A}_{H_{n-}}(u(s,\cdot)) - \lim_{s\to\infty} \mathcal{A}_{H_{n+}}(u(s,\cdot)) + \int_{\mathbb{R}\times S^1} \partial_s H_n(u(s,t),t,s)\,ds\,dt.$$

See, for example, [1, page 8]. When $u \in \mathcal{M}_{(H_n,J_n)}$, it follows from Lemma 9.18, together with the fact that the action spectrum is a compact subset of $\mathbb{R}$, that

$$\begin{aligned}E(u) &\le \max \operatorname{spec}(H_{n-}) - \min \operatorname{spec}(H_{n+}) + \int_{\mathbb{R}\times S^1} \partial_s H_n(u(s,t),t,s)\,ds\,dt \\ &= \max \operatorname{spec}(H_-^1) - \min \operatorname{spec}(H_+^K) + \int_{\mathbb{R}\times S^1} \partial_s H_n(u(s,t),t,s)\,ds\,dt \\ &\le \max \operatorname{spec}(H_-^1) - \min \operatorname{spec}(H_+^K) + K\cdot C,\end{aligned}$$

where, by our construction, K bounds the area of the support of $\max_{x\in M} \partial_s H_n(x,t,s)$ in $S^1\times\mathbb{R}$, and C is defined by

$$\begin{aligned}C &:= \sup\left\{ \frac{\partial H_n}{\partial s}(x,t,s) \;\middle|\; (x,t,s)\in M\times S^1\times\mathbb{R},\, n\ge 0\right\} \\ &= \max_{k\le K}\, \sup\left\{ \frac{\partial H^k}{\partial s}(x,t,s) \;\middle|\; (x,t,s)\in M\times S^1\times[0,1]\right\}.\end{aligned} \tag{40}$$

We therefore have obtained a uniform bound on the energies of solutions in $\mathcal{M}$. Arguing as in [3, Propositions 6.6.2 and 11.1.5], we conclude that there exists $A>0$ such that $\|\operatorname{grad}_{(s,t)} u\| \le A$. □

The last lemma for this section is analogous to Lemma 9.16. It can be viewed as a special case of Proposition 2.1 from [1], but we include the proof for the sake of completeness.

Lemma 9.20 *Let $u_n \in \mathcal{M}_{(H_n,J_n)}$ be a sequence of solutions and let $s_n\in\mathbb{R}$ be a sequence of numbers such that for some $0\le k\le K$ and for every n, $\sigma_n^k \le s_n \le \sigma_n^{k+1}$, where we set $\sigma_n^0=-\infty$ and $\sigma_n^{K+1}=+\infty$ to simplify notation. Then the sequence of shifted solutions $\tau_{s_n}u_n(\cdot,\cdot)=u_n(\cdot+s_n,\cdot)$ has a subsequence that converges in the $\mathcal{C}^\infty_{\mathrm{loc}}$ topology to a limit v. Moreover:*

(i) *If $s_n-\sigma_n^k\to\sigma\in\mathbb{R}$, then $v\in\mathcal{M}_{(\tau_\sigma H^k,\tau_\sigma J^k)}$.*

(ii) *If $s_n-\sigma_n^{k+1}\to\sigma\in\mathbb{R}$, then $v\in\mathcal{M}_{(\tau_\sigma H^{k+1},\tau_\sigma J^{k+1})}$.*

(iii) *If $s_n-\sigma_n^{k+1}\to-\infty$ and $s_n-\sigma_n^k\to\infty$, then $v\in\mathcal{M}_{(H_+^k,J_+^k)}=\mathcal{M}_{(H_-^{k+1},J_-^{k+1})}$.*

Proof The proof is very similar to that of Lemma 9.16 and therefore we only sketch the changes. As before, Lemma 9.19 implies that the sequence $v_n:=\tau_{s_n}u_n$ is equicontinuous, and by the Arzelà–Ascoli theorem and elliptic regularity there exists a subsequence converging to v. The maps v_n solve the Floer equation with respect to the translated pair $(\tau_{s_n} H_n, \tau_{s_n} J_n)$,

$$0 = \frac{\partial v_n}{\partial s} + (\tau_{s_n} J_n)\frac{\partial v_n}{\partial t} + \operatorname{grad}(\tau_{s_n} H_n) \circ v_n.$$

In each case, in order to prove that v is a solution of the corresponding equation, one shows that the translated homotopies $(\tau_{s_n} H_n, \tau_{s_n} J_n)$ converge uniformly on compacts to the required pair. For example, in the first case, where $s_n - \sigma_n^k \to \sigma \in \mathbb{R}$, it follows from the definition of (H_n, J_n) that, given $r > 0$, the sequence $(\tau_{\sigma_n^k} H_n, \tau_{\sigma_n^k} J_n)$ eventually stabilizes to (H^k, J^k) on $\{|s| \le r\}$. As a consequence,

$$(\tau_{s_n} H_n, \tau_{s_n} J_n) \xrightarrow{\mathcal{C}^\infty_{\text{loc}}} (\tau_\sigma H^k, \tau_\sigma J^k). \qquad \square$$

We are now ready to sketch the proof of Proposition 9.17. Note that we will skip some of the details appearing in the proof of Proposition 9.14.

Proof of Proposition 9.17 As mentioned above, for each n, the left end of H_n is a nondegenerate Hamiltonian. As a consequence, the left end of u_n converges to a periodic orbit, namely, there exist $x_{n-} \in \mathcal{P}(H_{n-})$ such that $\lim_{s\to-\infty} u_n(s,\cdot) = x_{n-}(\cdot)$; see, for example, the proof of Theorem 6.5.6 from [3]. Since $\mathcal{P}(H_{n-}) = \mathcal{P}(H^1_-)$ is a finite set, we may assume, by passing to a subsequence, that $x^{1,0} := x_{n-}$ is independent of n.

Next, let us apply Lemma 9.20 to the sequence $\{u_n\}$ with the shifts σ_n^k. Then, after passing to a subsequence, for each $k = 1, \dots, K'$ we obtain $w^k \in \mathcal{M}_{(H^k, J^k)}$ such that

$$\tau_{\sigma_n^k} u_n \xrightarrow{\mathcal{C}^\infty_{\text{loc}}} w^k.$$

Fixing $1 \le k \le K'$, we need to find solutions $\{v^{k,\ell}\}_{\ell=1}^{L_k}$ connecting w^{k-1} to w^k (and $x^{1,0}$ to w^1). The nondegeneracy of H^k_- implies that $\mathcal{P}(H^k_-) = \mathcal{P}(H^{k-1}_+)$ is a finite set (notice that the left end of $H^{K'}$ is nondegenerate, as it coincides with the right end of $H^{K'-1}$). Therefore, we can repeat the arguments from the proof of Proposition 9.14. For $\epsilon > 0$ small enough, the balls $\{B(x_-, \epsilon)\}_{x_- \in \mathcal{P}(H^k_-)}$ are disjoint, and writing $y^k := \lim_{s\to-\infty} w^k(s,\cdot)$ and $x^{k,0} = \lim_{s\to\infty} w^{k-1}(s,\cdot)$, there exists $s_\star \in \mathbb{R}$ such that $w^{k-1}(s,\cdot) \in B(x^{k,0}, \epsilon)$ for $s \ge s_\star$. It follows from the convergence of $\tau_{\sigma_n^{k-1}} u_n$ to w^{k-1} that $u_n(s_\star + \sigma_n^{k-1}, \cdot) \in B(x^{k,0}, \epsilon)$ when n is large. Denoting the first exit by

$$s_n^{k,1} := \sup\{s \ge s_\star + \sigma_n^{k-1} \mid u_n(s', \cdot) \in B(y^k, \epsilon) \text{ for } s' \in [s_\star + \sigma_n^{k-1}, s]\},$$

one can argue as in the proof of Proposition 9.14 to show that

$$s_n^{k,1} - \sigma_n^{k-1} \xrightarrow[n\to\infty]{} \infty.$$

Applying Lemma 9.20 to $\{u_n\}$ shifted by $s_n^{k,1}$, we conclude that $\tau_{s_n^{k,1}} u_n$ either converges to $\tau_\sigma w^k$ for some $\sigma \in \mathbb{R}$, or to $v^{k,1} \in \mathcal{M}_{(H_-^k, J_-^k)} = \mathcal{M}_{(H_+^{k-1}, J_+^{k-1})}$. In the first case, the right end of w^{k-1} and the left end of w^k coincide, namely $x^{k,0} = y^k$, and we are done. Otherwise, we continue by induction and find $v^{k,\ell} \in \mathcal{M}_{(H_-^k, J_-^k)}$ connecting w^{k-1} to w^k. As argued previously, this process is finite since each $v^{k,\ell}$ decreases the action, and $\operatorname{spec}(H_-^k)$ is a finite set. □

9.3 Barricades and perturbations

Throughout this section, we fix an almost complex structure J on M, a CIB domain U, and $U_\circ \Subset U$. We will consider nondegenerate Hamiltonians, or homotopies with nondegenerate ends, which have a barricade in U around $U_\circ$, when paired with J.

9.3.1 Barricades survive under small enough perturbations In this section we show that barricades survive under perturbations of H. Here H denotes a homotopy with nondegenerate ends and we consider Hamiltonians as a special case, by identifying them with constant homotopies.

Proposition 9.21 *Let H be a homotopy with nondegenerate ends, which is such that $\partial_s H|_{|s|>R} = 0$ for some $R > 0$ (in particular, H can be a nondegenerate Hamiltonian), and such that the pairs (H, J), $(H_\pm, J)$ have a barricade in U around $U_\circ$. Then, for every $\mathcal{C}^\infty$–small enough perturbation H' of H that satisfies $\mathcal{P}(H_\pm) = \mathcal{P}(H'_\pm)$ and $\partial_s H'|_{|s|>R} = 0$, the pair (H', J) has a barricade in U around $U_\circ$.*

In order to prove this proposition, we will use the convergence to broken trajectories, which was established in Section 9.2.1. Therefore, we start by showing that barricades also restrict broken trajectories.

Lemma 9.22 *Let H be a homotopy with nondegenerate ends (or, in particular, a nondegenerate Hamiltonian) such that the pairs (H, J) and $(H_\pm, J)$ have a barricade in U around $U_\circ$. Then, for a broken trajectory $\overline{v} = (v_1, \dots, v_k, w, v'_1, \dots, v'_\ell)$ connecting $x_\pm \in \mathcal{P}(H_\pm)$, we have:*

- *If $x_- \subset U_\circ$, then $\overline{v} \subset U_\circ$.*
- *If $x_+ \subset U$, then $\overline{v} \subset U$.*

Proof We prove the first statement; the second statement is completely analogous. Let

$$\overline{v} := (v_1, \dots, v_k, w, v'_1, \dots, v'_\ell)$$

be a broken trajectory of (H, J) such that the periodic orbit $x_0 := \lim_{s\to-\infty} v_1(s,\cdot)$ is contained in $U_\circ$. Then, since (H_-, J) has a barricade in U around $U_\circ$, it holds that $v_1 \subset U_\circ$ and in particular, the periodic orbit $x_1 := \lim_{s\to+\infty} v_1(s,\cdot)$ is contained in $\overline{U}_\circ$. Moreover, by Definition 1.7 of the barricade, the periodic orbits of H_- do not intersect $\partial U_\circ$, and therefore $x_1 \subset U_\circ$. As $\overline{v}$ is a broken trajectory (see Proposition 9.14), x_1 is the negative end of v_2, namely $x_1 = \lim_{s\to-\infty} v_2(s,\cdot)$. Applying the same argument again and again, we conclude that $v_2, \dots, v_k \subset U_\circ$. Now, $x_k := \lim_{s\to+\infty} v_k(s,\cdot) = \lim_{s\to-\infty} w(s,\cdot)$ is also contained in $U_\circ$ and since (H, J) has a barricade in U around $U_\circ$, this means that $w \subset U_\circ$. Arguing the same way and using the fact that (H_+, J) has a barricade in U around $U_\circ$, we conclude that $v'_j \subset U_\circ$ for all $1 \le j \le \ell$, and so the broken trajectory is completely contained in $U_\circ$. □

Given the above lemma, the proof of Proposition 9.21 is a simple application of Proposition 9.14.

Proof of Proposition 9.21 Let $\{H_n\}$ be a sequence of regular homotopies converging to H, such that for each n, $\mathcal{P}(H_{n\pm}) = \mathcal{P}(H_\pm)$ and $\partial_s H_n|_{|s|>R} = 0$. Assume for the sake of contradiction that, for each n, there exists a solution $u_n \in \mathcal{M}_{(H_n,J)}$ such that $x_-^n := \lim_{s\to-\infty} u_n(s,\cdot)$ is contained in $U_\circ$ but u_n is not. For each n, let $\sigma_n \in \mathbb{R}$ be such that $u_n(\sigma_n,\cdot)$ is not contained in $U_\circ$. Since $x_\pm^n \in \mathcal{P}(H_{n\pm}) = \mathcal{P}(H_\pm)$ are elements of a finite sets, by passing to a subsequence, we may assume that $x_\pm^n = x_\pm$ are independent of n, which means that $u_n \in \mathcal{M}(x_-, x_+)$ for all n. Applying Proposition 9.14 to the sequence of solutions $\{u_n\}$ and the sequence of shifts $\{\sigma_n\}$, after passing again to a subsequence, $\{u_n\}$ converges to a broken trajectory $\overline{v}$ of (H, J), and the sequence $u_n(\cdot+\sigma_n,\cdot)$ converges to one of the solutions in $\overline{v}$ (perhaps up to a shift). Lemma 9.22, together with our assumption that $x_- = x_0 \subset U_\circ$, guarantees that the entire broken trajectory $\overline{v}$ is contained in $U_\circ$, and in particular $\lim_{n\to\infty} u_n(\cdot+\sigma_n,\cdot) \subset U_\circ$. Since the latter limit is uniform on compacts, it follows that

$$\lim_{n\to\infty} u_n(\sigma_n,\cdot) = \lim_{n\to\infty} u_n(0+\sigma_n,\cdot)$$

is also contained in $U_\circ$. Recalling that we chose σ_n so that, for each n, the loop $u_n(\sigma_n,\cdot)$ is not contained in the open set $U_\circ$, we arrive at a contradiction.

Similarly, one can prove that when n is large enough, every solution u_n of the Floer equation with respect to (H_n, J) ending in U is contained in U. □

9.3.2 Perturbing Hamiltonians that are regular on a subset In this section, we define the notion of regularity on a subset, $U \subset M$, for a pair (H, J) of a Hamiltonian and an almost complex structure that has a barricade in U around some $U_\circ \Subset U$. We prove that for such a pair, the restriction of the Floer differential to the set is well defined, and is stable under (regular) perturbations. Since Floer-regularity concerns the differential of the Floer map we start with a reminder. Given a Hamiltonian H and an almost complex structure J, the Floer map associated to the pair (H, J) is

$$\mathcal{F} = \mathcal{F}^H : \mathcal{C}^\infty(\mathbb{R} \times S^1; M) \to \mathcal{C}^\infty(\mathbb{R} \times S^1; TM), \quad u \mapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \operatorname{grad}_u(H_t),$$

where $\operatorname{grad}_u H := \nabla_J H \circ u$ is the gradient of H with respect to J, composed with u.

Definition 9.23 Let H be a nondegenerate Hamiltonian such that the pair (H, J) has a barricade in U around $U_\circ$.

(i) We say that the pair (H, J) is *regular on U* if for every solution u of the Floer equation that is contained in U, the linearization $(d\mathcal{F})_u$ of the Floer map $\mathcal{F}$ at u is surjective.
In particular, by [3, Theorem 8.1.2], for every $x_\pm \in \mathcal{P}(H)$ such that $x_+ \subset U$, the space of solutions $\mathcal{M}_{(H,J)}(x_-, x_+)$ is a smooth manifold of dimension $\mu(x_-) - \mu(x_+)$.

(ii) We say that the pair (H, J) is *semiregular on U* if for every $x_\pm \in \mathcal{P}(H)$ with $\mu(x_-) \le \mu(x_+)$ and such that $x_+ \subset U$, we have:
  (a) If $x_- \neq x_+$, then $\mathcal{M}_{(H,J)}(x_-, x_+) = \varnothing$.
  (b) If $x_- = x_+$, then $\mathcal{M}_{(H,J)}(x_-, x_+)$ contains only the constant solution $u(s,t) = x_-(t)$.

Remark 9.24 • If (H, J) is regular on U, then it is also semiregular on U.

• If (H, J) has a barricade in U around $U_\circ$ and agrees, on U, with a Floer-regular pair, then it is regular on U.

• For a pair (H, J) that is regular on U, the differential of the Floer complex might not be defined everywhere. However, using Proposition 9.14 (see also the proof of Lemma 9.26 below), one can show that when $\mu(x_-) - \mu(x_+) = 1$ and $x_+ \subset U$, the quotient manifold $\mathcal{M}_{(H,J)}(x_-, x_+)/\mathbb{R}$ is compact and of dimension 0, and hence finite. Therefore, the composition $\pi_U \circ \partial_{(H,J)}$ can be defined by counting the elements of the latter quotients. This is a slight abuse of notation, as the map $\partial_{(H,J)}$ is not defined on its own. Similarly, one can define the composition $\partial_{(H,J)} \circ \pi_{U_\circ}$ using the fact that $x_- \subset U_\circ$ implies that $x_+ \subset U_\circ \subset U$, due to the barricade.

Our main goal for this section is to prove the following statement.

Proposition 9.25 *Suppose that H is a nondegenerate Hamiltonian such that (H,J) is regular on U. Let H' be a small perturbation of H such that the pair (H',J) is Floer-regular and H' agrees with H on $\mathcal{P}(H)$ up to second order. Then the compositions of the differentials and projections agree*:

(41) $$\pi_U \circ \partial_{(H,J)} = \pi_U \circ \partial_{(H',J)}, \quad \partial_{(H,J)} \circ \pi_{U_\circ} = \partial_{(H',J)} \circ \pi_{U_\circ}.$$

We remark that the second equation in (41) follows immediately from the first. Indeed, due to Proposition 9.21, (H',J) also has a barricade, and $\partial \circ \pi_{U_\circ} = \pi_U \circ \partial \circ \pi_{U_\circ}$ for both (H,J) and (H',J). In order to prove Proposition 9.25, we connect H and H' by a path of Hamiltonians $\{H_\lambda\}_{\lambda\in[0,1]}$ such that, for each $\lambda \in [0,1]$, H_λ agrees with H on the 1–periodic orbits up to second order, and the pair (H_λ, J) is semiregular on U. Note that the first condition implies that, for each λ, $\mathcal{P}(H_\lambda) = \mathcal{P}(H)$. Given $x_\pm \in \mathcal{P}(H)$ such that $x_+ \subset U$, the space

(42) $$\mathcal{M}_\Lambda(x_-,x_+) := \{(\lambda,u) \mid u \in \mathcal{M}_{(H_\lambda,J)}(x_-,x_+)\}$$

is invariant under the $\mathbb{R}$ action $u(\cdot,\cdot) \mapsto u(\sigma+\cdot,\cdot)$. We show that when

$$\mu(x_-) - \mu(x_+) = 1,$$

the quotient $\overline{\mathcal{M}}_\Lambda(x_-,x_+) = \mathcal{M}_\Lambda(x_-,x_+)/\mathbb{R}$ is a smooth, compact 1–dimensional manifold with boundary, which realizes a cobordism between $\mathcal{M}_{(H,J)}(x_-,x_+)/\mathbb{R}$ and $\mathcal{M}_{(H',J)}(x_-,x_+)/\mathbb{R}$. We will then conclude that the number of elements in the quotients $\mathcal{M}_{(H,J)}(x_-,x_+)/\mathbb{R}$ and $\mathcal{M}_{(H',J)}(x_-,x_+)/\mathbb{R}$ coincides modulo 2.

The existence of a semiregular path between H and H' follows from the fact that semiregularity is an open condition.

Lemma 9.26 *Suppose that (H,J) is semiregular on U. Then, for every Hamiltonian H' that is close enough to H and agrees with H on $\mathcal{P}(H)$ up to second order, the pair (H',J) is also semiregular on U.*

Proof Consider a sequence H_n converging to H which is such that for each n, H_n agrees with H on $\mathcal{P}(H)$. Then, in particular, $\mathcal{P}(H_n) = \mathcal{P}(H)$. Suppose that for each n, there exists a solution $u_n \in \mathcal{M}_{(H_n,J)}(x_-^n, x_+^n)$ for some $x_\pm^n \in \mathcal{P}(H_n)$ such that $\mu(x_-^n) \le \mu(x_+^n)$ and $x_+^n \subset U$. Moreover, we assume that if $x_-^n = x_+^n$, then u_n is nonconstant. Since $x_\pm^n \in \mathcal{P}(H)$ are elements of a finite set, we may assume, by passing to a subsequence, that $x_\pm^n = x_\pm$ are independent of n. By Proposition 9.14, there exists a subsequence of the solutions u_n that converges to a broken trajectory $\overline{v}$

of (H, J). Moreover, the ends of the broken trajectory are $x_\pm$. Since x_+ is contained in U and (H, J) has a barricade in U around $U_\circ$, it follows from Lemma 9.22 that the broken trajectory $\overline{v}$ is contained in U. As the pair (H, J) is semiregular on U, for every nonconstant solution in the broken trajectory, the index difference between the left end and the right end is positive. Therefore, in the notation of Proposition 9.14, we have

$$\mu(x_-) = \mu(x_0) > \mu(x_1) > \cdots > \mu(x_+).$$

Together with our assumption that $\mu(x_-) = \mu(x_-^n) \le \mu(x_+^n) = \mu(x_+)$, this implies that the broken trajectory $\overline{v}$ contains only one solution: $v_1(s,t) = x_-(t) = x_+(t)$. In particular, we conclude that $u_n \in \mathcal{M}_{(H_n,J)}(x_-, x_+)$ are Floer solutions with equal ends. By the energy identity (8), the energy of u_n vanishes,

$$E(u_n) = \mathcal{A}_{H_n}(x_-) - \mathcal{A}_{H_n}(x_-) = \mathcal{A}_H(x_-) - \mathcal{A}_H(x_-) = 0,$$

guaranteeing that u_n is a constant solution $u_n(s,t) = x_-(t)$ for all n, a contradiction. □

Our next aim is to show that for a suitable choice of a path of Hamiltonians $\{H_\lambda\}$, the set (42) is a smooth manifold. Let us start with preliminary definitions. Let $\{H_\lambda\}_{\lambda\in[0,1]}$ be a path of Hamiltonians that is stationary for $\lambda \notin [\delta, 1-\delta]$ for some fixed $\delta > 0$, and such that H_λ agrees with H_0 on $\mathcal{P}(H_0)$ up to second order for all $\lambda \in [0,1]$. We will consider the space $\mathcal{C}_\varepsilon^\infty(\{H_\lambda\}_\lambda)$ (of perturbations) consisting of maps

$$h\colon M \times S^1 \times [0,1] \to \mathbb{R}$$

with compact support in $M \times S^1 \times [\delta, 1-\delta]$ that vanish on $\mathcal{P}(H_0) \times [0,1]$ up to second order and are such that $\|h\|_\varepsilon < \infty$. Here $\|\cdot\|_\varepsilon$ is Floer's ε–norm; see Definition 9.3 and [3, page 230]. We identify the map h with the path of time-dependent Hamiltonians $\{h_\lambda(\cdot,\cdot) := h(\cdot,\cdot,\lambda)\}_\lambda$.

The next claim is an adjustment of [3, Theorem 11.3.2] to our setting and is proved similarly. For the sake of completeness we include the proof, but we postpone it until the end of this section.

Claim 9.27 *Let $\{H_\lambda\}_{\lambda\in[0,1]}$ be a path of Hamiltonians as above, and assume that (H_0, J) and (H_1, J) are regular on U. Then there exist a neighborhood of 0 in $\mathcal{C}_\varepsilon^\infty(\{H_\lambda\}_\lambda)$ and a residual set $\mathcal{H}_{\mathrm{reg}}$ in this neighborhood, such that if $h \in \mathcal{H}_{\mathrm{reg}}$, then for $\Lambda = (\{H_\lambda + h_\lambda\}_\lambda, J)$ and every $x_\pm \in \mathcal{P}(H_0)$ with $x_+ \subset U$, the space $\mathcal{M}_\Lambda(x_-, x_+)$ is a manifold-with-boundary of dimension $\mu(x_-) - \mu(x_+) + 1$, and its boundary is*

$$\partial\mathcal{M}_\Lambda(x_-, x_+) = \{0\} \times \mathcal{M}_{(H,J)}(x_-, x_+) \,\cup\, \{1\} \times \mathcal{M}_{(H',J)}(x_-, x_+). \tag{43}$$

Proof of Proposition 9.25 Recall that H is a nondegenerate Hamiltonian such that (H, J) is regular on U. Let H' be a small perturbation of H that agrees with H on $\mathcal{P}(H)$ up to second order, and such that the pair (H', J) is Floer-regular. We wish to show that the compositions of the differentials with respect to (H, J) and (H', J) with the projections onto C_U and $C_{U_\circ}$ agree. Let H_λ be a linear path (or a linear homotopy) between H and H' that is stationary near $\lambda = 0, 1$, and such that for each λ, H_λ agrees with H on $\mathcal{P}(H)$ up to second order (in particular, $\mathcal{P}(H_\lambda) = \mathcal{P}(H)$). Taking H' to be close enough to H, and using Lemma 9.26, one can guarantee that all of the pairs (H_λ, J) are semiregular on U.

By Claim 9.27, there exists a small perturbation of the path $\{H_\lambda\}$ such that for $\Lambda = (\{H_\lambda + h_\lambda\}_\lambda, J)$ and for every $x_\pm \in \mathcal{P}(H_0)$ with $x_+ \subset U$, the space $\mathcal{M}_\Lambda(x_-, x_+)$ is a manifold with boundary of dimension $\mu(x_-) - \mu(x_+) + 1$. Let us show that when $\mu(x_-) - \mu(x_+) = 1$, the quotient $\overline{\mathcal{M}}_\Lambda(x_-, x_+) = \mathcal{M}_\Lambda(x_-, x_+)/\mathbb{R}$ of this manifold by the $\mathbb{R}$ action is compact. Let $(\lambda_n, u_n) \in \mathcal{M}_\Lambda(x_-, x_+)$ be any sequence. Since $\lambda_n \in [0, 1]$, we may assume, by passing to a subsequence, that the sequence λ_n converges to a number $\lambda_\star \in [0, 1]$. By the definition of the space $\mathcal{M}_\Lambda(x_-, x_+)$, $u_n \in \mathcal{M}_{(H_{\lambda_n}, J)}(x_-, x_+)$ are solutions to the Floer equation with respect to Hamiltonians converging to $H_{\lambda_\star}$. By Proposition 9.14, there exists a subsequence of u_n converging to a broken Floer trajectory $\overline{v} = \{v_1, \dots, v_k\}$ of $(H_{\lambda_\star}, J)$. Since the pair $(H_{\lambda_\star}, J)$ is semiregular on U, and $x_+ \subset U$, every solution in $\overline{v}$ that is nonconstant (in the s–coordinate) decreases the index:

$$\mu(x_-) = \mu(x_0) > \mu(x_1) > \cdots > \mu(x_k) = \mu(x_+).$$

Recalling that $\mu(x_-) - \mu(x_+) = 1$, we conclude that $\overline{v}$ contains exactly one nonconstant solution, $\overline{v} = v_1 \in \mathcal{M}_{(H_{\lambda_\star}, J)}(x_-, x_+)$. In other words, given the sequence $(\lambda_n, u_n) \in \mathcal{M}_\Lambda(x_-, x_+)$, there exists a sequence of shifts $s_n \in \mathbb{R}$ such that, after passing to a subsequence,

$$(\lambda_n, \tau_{s_n} u_n) \xrightarrow[n\to\infty]{C^\infty_{\mathrm{loc}}} (\lambda_\star, v_1).$$

In particular, after dividing by the (free, proper and smooth) $\mathbb{R}$–action, the subsequence $(\lambda_n, [u_n]) \in \overline{\mathcal{M}}_\Lambda(x_-, x_+)$ converges to an element of the same space,

$$(\lambda_n, [u_n]) \xrightarrow[n\to\infty]{} (\lambda_\star, [v_1]) \in \overline{\mathcal{M}}_\Lambda(x_-, x_+),$$

and therefore $\overline{\mathcal{M}}_\Lambda(x_-, x_+)$ is compact. Overall, $\overline{\mathcal{M}}_\Lambda(x_-, x_+)$ is a smooth, compact manifold of dimension $\mu(x_-) - \mu(x_+) + 1 - 1 = 1$, and its boundary is the zero-dimensional compact manifold

$$\partial \overline{\mathcal{M}}_\Lambda(x_-, x_+) = \{0\} \times \overline{\mathcal{M}}_{(H,J)}(x_-, x_+) \cup \{1\} \times \overline{\mathcal{M}}_{(H',J)}(x_-, x_+).$$

Hence, the latter are finite sets with an equal number of elements mod 2:

$$\#_2\overline{\mathcal{M}}_{(H,J)}(x_-,x_+)=\#_2\overline{\mathcal{M}}_{(H',J)}(x_-,x_+).$$

The equalities (41) now follow from the definition of the differential map. □

We sketch the proof of Claim 9.27, which follows the arguments in [3, Chapter 11.3.b].

Proof of Claim 9.27 Fix $x_\pm \in \mathcal{P}(H_0)$ such that $x_+ \subset U$. We first show that the space $\mathcal{M}_\Lambda(x_-,x_+)$ has a structure of a manifold-with-boundary near the boundary (43), and afterwards we prove that for perturbed paths the interior is a smooth manifold.

Let $\delta > 0$ be such that the path $\{H_\lambda\}$ is stationary for $\lambda \notin [\delta, 1-\delta]$ and every $h \in \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ satisfies $\operatorname{supp}(h) \subset M \times S^1 \times [\delta, 1-\delta]$. In this case,

$$H_\lambda + h_\lambda = \begin{cases} H_0, & \lambda \le \delta, \\ H_1, & \lambda \ge 1-\delta, \end{cases}$$

for all $h \in \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$. Fixing such an h and setting $\Lambda = (\{H_\lambda + h_\lambda\}_\lambda, J)$, we have that

$$\mathcal{M}_\Lambda(x_-,x_+) \cap \{\lambda < \delta\} = [0,\delta) \times \mathcal{M}_{(H_0,J)}(x_-,x_+),$$
$$\mathcal{M}_\Lambda(x_-,x_+) \cap \{\lambda > 1-\delta\} = (1-\delta, 1] \times \mathcal{M}_{(H_1,J)}(x_-,x_+)$$

are smooth manifolds with boundary, since the pairs (H_0, J), (H_1, J) are regular on U and $x_+ \subset U$. We conclude that near $\{0\} \times \mathcal{M}_{(H,J)}(x_-,x_+) \cup \{1\} \times \mathcal{M}_{(H',J)}(x_-,x_+)$ the space $\mathcal{M}_\Lambda(x_-,x_+)$ has a structure of a manifold with boundary.

Let us now show that the interior of $\mathcal{M}_\Lambda(x_-,x_+)$ is a smooth manifold. Since the spaces $\mathcal{M}_{(H,J)}(x_-,x_+)$ and $\mathcal{M}_{(H',J)}(x_-,x_+)$ composing the boundary are one-dimensional, it will follow that $\dim \mathcal{M}_\Lambda(x_-,x_+) = 2$. The following statement is taken from [3], and states that the linearization $(d\mathcal{F})_u$ of the Floer map $\mathcal{F}$ is a Fredholm operator.

Lemma 9.28 [3, Theorem 8.1.5] *For every nondegenerate Hamiltonian H, every almost complex structure J compatible with ω, and every $u \in \mathcal{M}_{(H,J)}(x_-,x_+)$, the linearization $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x_-) - \mu(x_+)$.*

As in Notation 9.9, we denote by $\mathcal{P}(x_-,x_+)$ the space of maps $(s,t) \mapsto \exp_{w(s,t)} Y(s,t)$, where $Y \in W^{1,p}(w^*TM)$ for $p > 2$, and $w \in \mathcal{C}^\infty(\mathbb{R} \times S^1; M)$ converges to $x_\pm$ with exponential decay. Consider the vector bundle $\mathcal{E} \to \mathcal{P}(x_-,x_+) \times \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ given by

$$\mathcal{E} = \{(u,h,Y) \mid (u,h) \in \mathcal{P}(x_-,x_+) \times \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda),\ Y \in L^p(u^*TM)\}.$$

We define a family of sections $\{\sigma_\lambda\}_{\lambda\in(0,1)}$ by

$$\sigma_\lambda(u,h)=\Big(u,h,\frac{\partial u}{\partial s}+J(u)\frac{\partial u}{\partial t}+\mathrm{grad}_u(H_\lambda+h_\lambda)\Big).$$

For fixed $\lambda_0\in(0,1)$, the map σ_{λ_0} is transversal to the zero section of the vector bundle $\mathcal{E}$ if and only if, when $\sigma_{\lambda_0}(u,h)=(u,h,0)$, the linearized map $(d\sigma_{\lambda_0})_{(u,h)}$ composed with the projection onto the fiber, namely,

$$\widetilde{\Gamma}_{\lambda_0}\colon W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})\times\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)\to L^p(\mathbb{R}\times S^1;\mathbb{R}^{2n}),\qquad (Y,\eta)\mapsto(d\mathcal{F}^{H_{\lambda_0}+h_{\lambda_0}})_u(Y)+\mathrm{grad}_u\,\eta_{\lambda_0},$$

is surjective. Here, and in what follows, we identify the linear space $\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ with its tangent space. If $\lambda_0\notin(\delta,1-\delta)$, then $h_{\lambda_0}=0$ and H_{λ_0} is equal to either H_0 or H_1, which are both regular on U, when paired with J. In this case, the surjectivity of $\widetilde{\Gamma}_{\lambda_0}$ follows from that of $d\mathcal{F}^{H_{\lambda_0}+h_{\lambda_0}}$, which is guaranteed due to the regularity of H_0 and H_1. Let us prove the surjectivity of $\widetilde{\Gamma}_{\lambda_0}$ for $\lambda_0\in(\delta,1-\delta)$. To do this, we embed $\mathcal{C}^\infty_\varepsilon(H_0)=\mathcal{C}^\infty_\varepsilon(H_{\lambda_0})$ into $\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ by mapping $h_{\lambda_0}\in\mathcal{C}^\infty_\varepsilon(H_{\lambda_0})$ to a locally constant path, $h(\cdot,\cdot,\lambda)=h_{\lambda_0}(\cdot,\cdot)$ near $\lambda=\lambda_0$. Here we have used our assumption that $\{H_\lambda\}_\lambda$ all have the same periodic orbits as H_0. It is now clear that the surjectivity of the restricted map,

$$\Gamma\colon W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})\times\mathcal{C}^\infty_\varepsilon(H_{\lambda_0})\to L^p(\mathbb{R}\times S^1;\mathbb{R}^{2n}),\qquad (Y,\zeta)\mapsto(d\mathcal{F}^{H_{\lambda_0}+h_{\lambda_0}})_u(Y)+\mathrm{grad}_u\,\zeta_{\lambda_0},$$

which is guaranteed by [3, Proposition 8.1.4], implies the surjectivity of $\widetilde{\Gamma}_{\lambda_0}$. We conclude that for every $\lambda_0\in(0,1)$, the section σ_{λ_0} intersects the zero section transversely. As a consequence, the section

$$\sigma\colon(\delta,1-\delta)\times\mathcal{P}(x_-,x_+)\times\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)\to(\delta,1-\delta)\times\mathcal{E},\quad(\lambda,u,h)\mapsto(\lambda,\sigma_\lambda(u,h)),$$

also intersects the zero section transversely and we conclude that the intersection

$$\mathcal{Z}(x_-,x_+)=\{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)\mid\lambda\in(\delta,1-\delta),u\in\mathcal{M}_{(H_\lambda+h_\lambda,J)}(x_-,x_+)\}$$

is a Banach manifold; see [3, Propositions 8.1.3 and 11.3.4] for the analogous statements. The tangent space of $\mathcal{Z}(x_-,x_+)$ at a point $(\lambda,u,\{H_\lambda+h_\lambda\})$ consists of all $(a,Y,\eta)\in\mathbb{R}\times W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})\times\mathcal{C}^\infty_\varepsilon(\{H_\lambda\})$ that satisfy the equation

$$a\cdot\mathrm{grad}_u\frac{\partial(H_\lambda+h_\lambda)}{\partial\lambda}+(d\mathcal{F}^{H_\lambda+h_\lambda})_u(Y)+\mathrm{grad}_u(\eta_\lambda)=0. \tag{44}$$

Let $\pi\colon\mathcal{Z}(x_-,x_+)\to\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ be the projection. In order to conclude the proof of the claim, it is sufficient to show that the set of regular values of π is a residual subset of $\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$. This will follow from the Sard–Smale theorem (see [3, Theorem 8.5.7]) as soon as we show that π is a Fredholm map; the separability of the spaces follows from Claim 9.5 and Remark 9.8 above. Let us therefore show that for every $(\lambda, u, \{H_\lambda + h_\lambda\}_\lambda) \in \mathcal{Z}(x_-, x_+)$, the operator

$$(d\pi)_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)}\colon T_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)}\mathcal{Z}(x_-, x_+) \to \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda), \quad (a, Y, \eta) \mapsto \eta,$$

is a Fredholm operator. In analogy with the proof of [3, Proposition 11.3.5], denote by

$$V := \operatorname{grad}_u \frac{\partial(H_\lambda + h_\lambda)}{\partial\lambda} \in L^p(u^*TM)$$

the vector field multiplying a in (44). Then the kernel of $(d\pi)_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)}$ is the space

$$\{(a, Y, 0) \mid (a, Y) \in \mathbb{R} \times W^{1,p}(u^*TM) \text{ and } aV + (d\mathcal{F}^{H_\lambda+h_\lambda})_u(Y) = 0\}.$$

Let us show that this space is finite-dimensional by splitting into two cases:

(i) Suppose $V \notin \operatorname{im}((d\mathcal{F}^{H_\lambda+h_\lambda})_u)$. In this case we find

$$\ker(d\pi)_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)} = \{(0, Y, 0) \mid Y \in \ker((d\mathcal{F}^{H_\lambda+h_\lambda})_u)\},$$

which is finite-dimensional by Lemma 9.28.

(ii) Suppose $V \in \operatorname{im}((d\mathcal{F}^{H_\lambda+h_\lambda})_u)$. Choose a $Y_0 \in W^{1,p}(u^*TM)$ such that

$$(d\mathcal{F}^{H_\lambda+h_\lambda})_u(Y_0) = V.$$

It follows that

$$\ker(d\pi)_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)} = \{(a, Y, 0) \mid a(d\mathcal{F}^{H_\lambda+h_\lambda})_u(Y_0) + (d\mathcal{F}^{H_\lambda+h_\lambda})_u(Y) = 0\}.$$

This space is isomorphic to $\mathbb{R}Y_0 + \ker((d\mathcal{F}^{H_\lambda+h_\lambda})_u)$, which is also finite-dimensional.

Next, let us show that the image of $(d\pi)_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)}$ is closed and has finite codimension. Indeed, it is the inverse image of the subspace

$$\mathbb{R}V + \operatorname{im}((d\mathcal{F}^{H_\lambda+h_\lambda})_u) \subset L^p(u^*TM) \tag{45}$$

under the linear map $\eta \mapsto \operatorname{grad}_u \eta$, viewed as a map $\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda) \to L^p(u^*TM)$. By Lemma 9.28, the subspace (45) is closed and of finite codimension, and hence we conclude the same for the image of $(d\pi)_{(\lambda,u,\{H_\lambda+h_\lambda\}_\lambda)}$. Consequently, π is indeed a Fredholm map, and by the Sard–Smale theorem, the set of its regular values is a residual subset $\mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$.

Denote by $\mathcal{H}_{\mathrm{reg}} \subset \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ the set of regular values of π. Then for any $h \in \mathcal{H}_{\mathrm{reg}}$, setting $\Lambda = (\{h_\lambda + h_\lambda\}_\lambda, J)$, the set

$$\pi^{-1}(h) = \mathcal{M}_\Lambda(x_-, x_+) \cap \{\lambda \in (0,1)\}$$

is a smooth manifold (with respect to the $\mathcal{C}^\infty_{\mathrm{loc}}$ topology). Together with the discussion from the beginning of the proof, this implies that $\mathcal{M}_\Lambda(x_-, x_+)$ is a manifold with boundary. □

9.3.3 Perturbing homotopies that are regular on a subset In this section we state and prove results which are analogous to the ones from Section 9.3.2, but for homotopies instead of Hamiltonians. Fix an almost complex structure J on M, a CIB domain U, and $U_\circ \Subset U$.

Definition 9.29 Let H be a homotopy of Hamiltonians such that the pair (H, J) has a barricade in U around $U_\circ$.

(i) We say that the pair (H, J) is *regular on* U if $(H_\pm, J)$ are regular on U (see Definition 9.23) and for every solution u of the s–dependent Floer equation with respect to (H, J), the linearization $(d\mathcal{F})_u$ of the Floer map $\mathcal{F}$ is surjective. In particular, by [3, Theorem 8.1.2], for every $x_\pm \in \mathcal{P}(H_\pm)$ such that $x_+ \subset U$, the space of solutions $\mathcal{M}_{(H,J)}(x_-, x_+)$ is a smooth manifold of dimension $\mu(x_-) - \mu(x_+)$.

(ii) We say that the pair (H, J) is *semiregular on* U if $(H_\pm, J)$ are semiregular on U (as in Definition 9.23) and for every $x_\pm \in \mathcal{P}(H_\pm)$ with $\mu(x_-) < \mu(x_+)$ and such that $x_+ \subset U$, we have $\mathcal{M}_{(H,J)}(x_-, x_+) = \varnothing$.

As in Section 9.3.2, if a pair is regular on U, then it is also semiregular on U, and every Floer-regular pair with a barricade is regular on U.

Remark 9.30 For a pair (H, J) that is regular on U, the continuation map might not be defined everywhere. However, using Proposition 9.14, one can see that when $\mu(x_-) = \mu(x_+)$ and $x_+ \subset U$, the zero-dimensional manifold $\mathcal{M}_{(H,J)}(x_-, x_+)$ is compact and hence finite. The composition $\pi_U \circ \Phi_{(H,J)}$ can be defined by counting the elements of such manifolds. We remark that this is a slight abuse of notation, as the continuation map $\Phi_{(H,J)}$ is not necessarily defined on its own. Due to the barricade, if $x_- \subset U_\circ$ then $x_+ \subset U_\circ \subset U$. It follows that the composition $\Phi_{(H,J)} \circ \pi_{U_\circ}$ is well defined as well.

Our main goal for this section is to prove the following statement.

Proposition 9.31 *Suppose that H is a homotopy such that (H, J) is regular on U, and let $R > 0$ be such that $\partial_s H|_{|s|>R} = 0$. Let H' be a homotopy such that*

(i) $\partial_s H'|_{|s|>R} = 0$,

(ii) *H' is $\mathcal{C}^\infty$–close to H, and the $H'_\pm$ agree with $H_\pm$ on their 1–periodic orbits up to second order,*

(iii) *(H', J) is regular on U.*

Then the compositions of the continuation maps and projections agree:

$$\pi_U \circ \Phi_{(H,J)} = \pi_U \circ \Phi_{(H',J)}, \quad \Phi_{(H,J)} \circ \pi_{U_\circ} = \Phi_{(H',J)} \circ \pi_{U_\circ}. \tag{46}$$

As before, the second equation in (46) follows from the first, since both (H, J) and (H', J) have a barricade in U around $U_\circ$ and thus $\Phi \circ \pi_{U_\circ} = \pi_U \circ \Phi \circ \pi_{U_\circ}$. In analogy with the previous section, in order to prove Proposition 9.31, we connect H and H' by a linear path (or linear homotopy) of homotopies $\{H_\lambda\}_{\lambda \in [0,1]}$ such that the pairs (H_λ, J) are all semiregular on U. Then, given $x_\pm \in \mathcal{P}(H_\pm)$, with $\mu(x_-) = \mu(x_+)$ and $x_+ \subset U$, we show that the space

$$\mathcal{M}_\Lambda(x_-, x_+) := \{(\lambda, u) \mid u \in \mathcal{M}_{(H_\lambda,J)}(x_-, x_+)\} \tag{47}$$

is a smooth, compact, 1–dimensional manifold-with-boundary that realizes a cobordism between $\mathcal{M}_{(H,J)}(x_-, x_+)$ and $\mathcal{M}_{(H',J)}(x_-, x_+)$. We will then conclude that the number of elements in $\mathcal{M}_{(H,J)}(x_-, x_+)$ and $\mathcal{M}_{(H',J)}(x_-, x_+)$ coincides modulo 2.

As for the case of Hamiltonians, semiregularity of homotopies is also an open condition, as the following lemma guarantees.

Lemma 9.32 *Suppose that (H, J) is semiregular on U, and fix $R > 0$. Then for every homotopy H' that is close enough to H, which is such that $\partial_s H'|_{|s|>R} = 0$ and the $H'_\pm$ agree with $H_\pm$ on their 1–periodic orbits up to second order, the pair (H', J) is also semiregular on U.*

Proof First, notice that by Proposition 9.21, for every homotopy H' that satisfies the conditions of the lemma, the pair (H', J) has a barricade in U around $U_\circ$. Assume for the sake of contradiction that there exists a sequence of homotopies H_n, converging to H, such that for each n, H_n satisfies the conditions of the lemma, and (H_n, J) is not semiregular on U. Then, for each n, there exist $x^n_\pm$ satisfying $\mu(x^n_-) < \mu(x^n_+)$ and $x^n_+ \subset U$, and a solution $u_n \in \mathcal{M}_{(H_n,J)}(x^n_-, x^n_+)$. Since $x^n_\pm \in \mathcal{P}(H_{n\pm}) = \mathcal{P}(H_\pm)$ are elements of finite sets, we may assume, by passing to a subsequence, that $x_\pm^n = x_\pm$ are independent of n. By Proposition 9.14, there exists a subsequence of the solutions u_n that converges to a broken trajectory

$$\overline{v} = (v_1, \dots, v_k, w, v_1', \dots, v_\ell')$$

of (H, J). Here v_i and v_j' are Floer solutions with respect to the Hamiltonians H_- and H_+, respectively, and $w \in \mathcal{M}_{(H,J)}$ is a solution with respect to the homotopy H. Moreover, the ends of the broken trajectory are $x_\pm$. Since x_+ is contained in U and the pairs (H, J) and $(H_\pm, J)$ all have barricades in U around $U_\circ$, it follows from Lemma 9.22 that the broken trajectory $\overline{v}$ is contained in U. As the pair (H, J) is semiregular on U, for every nonconstant v_i or v_j', the index difference between the left end and the right end is positive. Moreover, the index difference between the ends of w is nonnegative. Therefore, under the notation of Proposition 9.14, we have

$$\mu(x_-) = \mu(x_0) > \cdots > \mu(x_k) \geq \mu(y_0) > \cdots > \mu(y_\ell) = \mu(x_+),$$

which contradicts our assumption that $\mu(x_-^n) < \mu(x_+^n)$. □

As in the previous section, we show that for a suitable choice of a path of homotopies $\{H_\lambda\}$, the set (47) is a smooth manifold. Our starting point is a path $\{H_\lambda\}_{\lambda \in [0,1]}$ that is stationary for $\lambda \notin [\delta, 1-\delta]$ and is such that for all $\lambda \in [0, 1]$, H_λ satisfies properties (i)–(ii) from Proposition 9.31. This time the space of perturbations $\mathcal{C}_\varepsilon^\infty(\{H_\lambda\}_\lambda)$ will consist of maps

$$h \colon M \times S^1 \times \mathbb{R} \times [0, 1] \to \mathbb{R}$$

supported in $M \times S^1 \times [-R, R] \times [\delta, 1-\delta]$, which are such that $\|h\|_\varepsilon < \infty$, where again $\|\cdot\|_\varepsilon$ is Floer's norm from Definition 9.3. We identify the map h with the path of homotopies $\{h_\lambda(\cdot,\cdot) := h(\cdot,\cdot,\lambda)\}_\lambda$.

The following claim is an adjustment of [3, Theorem 11.3.2] to the case where the ends of the path, (H_0, J) and (H_1, J), are not necessarily Floer-regular, but are regular on U, and the support of the perturbations is uniformly bounded. The proof is completely analogous to that of Claim 9.27 above, with the single difference that the surjectivity of the operator Γ for homotopies is guaranteed by Lemma 9.13, instead of by [3, Proposition 8.1.4]. We therefore omit the proof.

Claim 9.33 *Let $\{H_\lambda\}_{\lambda \in [0,1]}$ be a path of homotopies as above, and assume that (H_0, J) and (H_1, J) are regular on U. Then there exists a residual subset $\mathcal{H}_{\mathrm{reg}} \subset \mathcal{C}_\varepsilon^\infty(\{H_\lambda\}_\lambda)$ such that if $h \in \mathcal{H}_{\mathrm{reg}}$, then for $\Lambda = (\{H_\lambda + h_\lambda\}_\lambda, J)$ and for every*

$x_\pm \in \mathcal{P}(H_{0\pm})$ with $x_+ \subset U$, the space $\mathcal{M}_\Lambda(x_-, x_+)$ is a manifold-with-boundary of dimension $\mu(x_-) - \mu(x_+) + 1$, and its boundary is

$$\partial \mathcal{M}_\Lambda(x_-, x_+) = \{0\} \times \mathcal{M}_{(H,J)}(x_-, x_+) \cup \{1\} \times \mathcal{M}_{(H',J)}(x_-, x_+). \tag{48}$$

Proof of Proposition 9.31 Recall that H is a homotopy such that (H, J) is regular on U, and H' is a homotopy satisfying properties (i)–(iii) above. Let H_λ be a linear path (or linear homotopy) between the homotopies H and H' which is stationary for $\lambda \notin [\delta, 1-\delta]$. Then, for each λ, the homotopy H_λ is close to H and its ends, $H_{\lambda\pm}$, agree with the ends of H on $\mathcal{P}(H_\pm)$. In particular, $\mathcal{P}(H_{\lambda\pm}) = \mathcal{P}(H_\pm)$ for all $\lambda \in [0, 1]$. Taking H' to be close enough to H, Lemma 9.32 guarantees that all of the homotopies H_λ are semiregular on U when paired with J. In particular, for each λ, the pairs (H_λ, J) and $(H_{\lambda\pm}, J)$ have a barricade in U around $U_\circ$.

By Claim 9.33, there exists a small perturbation $h \in \mathcal{C}^\infty_\varepsilon(\{H_\lambda\}_\lambda)$ such that for $\Lambda = (\{H_\lambda + h_\lambda\}_\lambda, J)$ and for every $x_\pm \in \mathcal{P}(H_{0\pm})$ with $x_+ \subset U$, the space $\mathcal{M}_\Lambda(x_-, x_+)$ is a manifold-with-boundary of dimension $\mu(x_-) - \mu(x_+) + 1$. Let us show that when $\mu(x_-) - \mu(x_+) = 0$, the manifold $\mathcal{M}_\Lambda(x_-, x_+)$ is compact. Let $(\lambda_n, u_n) \in \mathcal{M}_\Lambda(x_-, x_+)$ be any sequence. After passing to a subsequence, we have $\lambda_n \to \lambda_\star \in [0, 1]$, and hence $u_n \in \mathcal{M}_{(H_{\lambda_n}, J)}(x_-, x_+)$ are solutions with respect to homotopies that converge to $H_{\lambda_\star}$. By Proposition 9.14, there exists a subsequence of u_n converging to a broken trajectory $\overline{v} = \{v_1, \dots, v_k, w, v'_1, \dots, v'_\ell\}$ of $(H_{\lambda_\star}, J)$. Since the pairs $(H_{\lambda_\star}, J)$ and $(H_{\lambda_\star\pm}, J)$ have a barricade in U around $U_\circ$, and since $x_+ \subset U$, Lemma 9.22 guarantees that the broken trajectory is completely contained in U. The fact that $(H_{\lambda_\star}, J)$ is semiregular on U now implies that v_i and v'_j are index-decreasing, and w is index-nonincreasing:

$$\mu(x_-) = \mu(x_0) > \cdots > \mu(x_k) \geq \mu(y_0) > \cdots > \mu(y_\ell) = \mu(x_+).$$

Recalling that $\mu(x_-) - \mu(x_+) = 0$, we conclude that $\overline{v}$ does not contain nonconstant solutions of the s–independent Floer equations, and hence $\overline{v} = w \in \mathcal{M}_{(H_{\lambda_\star}, J)}(x_-, x_+)$. This implies that the above subsequence converges to an element of the space,

$$(\lambda_n, u_n) \xrightarrow[n\to\infty]{} (\lambda_\star, w) \in \mathcal{M}_\Lambda(x_-, x_+),$$

and therefore $\mathcal{M}_\Lambda(x_-, x_+)$ is compact.

Overall, $\mathcal{M}_\Lambda(x_-, x_+)$ is a smooth, compact manifold of dimension

$$\mu(x_-) - \mu(x_+) + 1 = 1,$$

and its boundary is

$$\partial\mathcal{M}_\Lambda(x_-,x_+) = \{0\}\times\mathcal{M}_{(H,J)}(x_-,x_+)\cup\{1\}\times\mathcal{M}_{(H',J)}(x_-,x_+).$$

Consequently, the latter are finite sets with an equal number of elements mod 2:

$$\#_2\mathcal{M}_{(H,J)}(x_-,x_+) = \#_2\mathcal{M}_{(H',J)}(x_-,x_+).$$

The equalities (46) follow immediately from the definition of the continuation maps. □

Perturbing homotopies that are constant on a subset A particular application of Proposition 9.31 that will be useful is when H is a homotopy that is constant on the set U, and whose ends, $H_\pm$, are regular on U when paired with J. In this case, it follows from Definition 9.29 that the pair (H,J) is also regular on U. Moreover, for periodic orbits $x_\pm\in\mathcal{P}(H_\pm)$ such that $x_+\subset U$, the space $\mathcal{M}_{(H,J)}(x_-,x_+)$ coincides with $\mathcal{M}_{(H_-,J)}(x_-,x_+)$. As a consequence, when $\mu(x_-)=\mu(x_+)$, the space $\mathcal{M}_{(H,J)}(x_-,x_+)$ is empty if $x_-\neq x_+$, and contains only constant solutions otherwise. We conclude that the continuation map with respect to (H,J) agrees with the identity map after composing with the projections:

$$\pi_U\circ\Phi_{(H,J)} = \pi_U\circ\mathbb{1} \quad\text{and}\quad \Phi_{(H,J)}\circ\pi_{U_\circ} = \mathbb{1}\circ\pi_{U_\circ}.$$

Applying Proposition 9.31 we conclude that the same holds for perturbations of H.

Corollary 9.34 *Suppose that H is a homotopy between two nondegenerate Hamiltonians $H_\pm$ such that (H,J) is constant on U, namely $\partial_s H|_U = 0$, and the $(H_\pm,J)$ are regular on U. Fix $R>0$ and let H' be a $\mathcal{C}^\infty$–small perturbation of H such that*

(i) $\partial_s H'|_{|s|>R} = 0$,

(ii) *the $H'_\pm$ agree with $H_\pm$ on their 1–periodic orbits up to second order*, and

(iii) *(H',J) is regular on U.*

Then

$$\Phi_{(H',J)}\circ\pi_{U_\circ} = \mathbb{1}\circ\pi_{U_\circ} \quad\text{and}\quad \pi_U\circ\Phi_{(H',J)} = \pi_U\circ\mathbb{1}. \tag{49}$$

Appendix Incompressibility of domains with incompressible boundaries

Let M^n be a smooth n–dimensional orientable manifold, and let N^n be a smooth n–dimensional orientable manifold with boundary such that there exists an embedding $\iota\colon N\to M$. Write $U := \operatorname{Im}(\iota(N\setminus\partial N))$, and note that $\partial U = \operatorname{Im}(\iota(\partial N))$.

Proposition A.35 *If ∂U is incompressible in M, then U is incompressible in M.*

Proof In order to show that $\iota_*\colon \pi_1(U) \to \pi_1(M)$ is injective, it is sufficient to prove that if a loop γ in U is contractible in M then it is contractible in U. Let $\gamma\colon S^1 \to U$ be a loop that is contractible in M. Then there exists a map $u\colon D \to M$ such that $u|_{\partial D} \equiv \gamma$, where $D \subset \mathbb{R}^2$ denotes the unit disk.

Without loss of generality we may assume that γ and u are smooth, and that $u \pitchfork \partial U$. Indeed, by Whitney's smooth approximation theorem, u is homotopic to a smooth map $\tilde{u}$. Since $\operatorname{Im}\gamma$ is compact and U is open, we can choose the smooth approximation so that $\tilde{\gamma} := \tilde{u}|_{\partial D}$ is homotopic to γ in U. Applying Thom's transversality theorem, we may assume that $\tilde{u} \pitchfork \partial U$. We replace the maps γ and u by $\tilde{\gamma}$ and $\tilde{u}$, in order to keep the notation.

Under the assumptions above, the preimage $C = u^{-1}(\partial U)$ is a compact one-dimensional submanifold of D, hence a disjoint union of embedded closed curves, $C = \bigsqcup_j C_j$. Some of the curves C_j may encompass others. We call a curve C_j *a maximal curve* if it is not encompassed by any other component of C. More formally, for each component C_j, denote by $D_j \subset D$ the embedded topological disk such that $\partial D_j = C_j$. The curve C_j is maximal if $C_j \not\subseteq D_k$ for all $k \neq j$. We denote the set of maximal curves by $\mathcal{C}_{\max} := \{C_{j_1}, \dots, C_{j_\ell}\}$, and by $\mathcal{D}_{\max} := \{D_{j_1}, \dots, D_{j_\ell}\}$ the set of the corresponding topological disks.

For every $1 \le i \le \ell$, the restriction $u|_{C_{j_i}}$ is a loop in ∂U which is contractible in M by $u|_{D_{j_i}}$. By the incompressibility of ∂U, the loop $u|_{C_{j_i}}$ is contractible in ∂U, namely there exists a map $v_i : D_{j_i} \to \partial U$ such that $v_i|_{C_{j_i}} \equiv u|_{C_{j_i}}$. Using the maps u and v_i we can define a map that contracts γ inside $\overline{U}$:

$$\hat{u} = \begin{cases} v_i(x) & \text{if } x \in D_{j_i}, \\ u(x) & \text{otherwise.} \end{cases}$$

Let us check that $\hat{u}$ is a contraction of γ in $\overline{U}$. Indeed, recalling that $u(\partial D) = \gamma \subset U$ and that $C := u^{-1}(\partial U)$, it follows from the maximality of the curves in $\mathcal{C}_{\max}$ that for all $x \in D \setminus \bigsqcup_{\mathcal{D}_{\max}} D_{j_i}$, one has $u(x) \in U$, and therefore $\hat{u}(x) \in U$. Moreover, for every $x \in D_{j_i}$ we have $\hat{u}(x) \in \partial U$, and we conclude that $\operatorname{Im}(\hat{u}) \subseteq U \cup \partial U$.

Using the fact that ∂U has a collar neighborhood in $\overline{U}$, one can construct a continuous map $w\colon \overline{U} \to U$ which restricts to the identity on $\operatorname{im}(\gamma)$. The composition $w \circ \hat{u}$ is the desired contraction of γ in U. $\square$

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Realization of graded monomial ideal rings modulo torsion

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Let A be the quotient of a graded polynomial ring $\mathbb{Z}[x_1,\dots,x_m]\otimes\Lambda[y_1,\dots,y_n]$ by an ideal generated by monomials with leading coefficients 1. We construct a space X_A such that A is isomorphic to $H^*(X_A)$ modulo torsion elements.

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1 Introduction

A classical problem in algebraic topology asks: which commutative graded R–algebras A are isomorphic to $H^*(X_A;R)$ for some space X_A? The space X_A, if it exists, is called a realization of A. According to Aguadé [1] the problem goes back to at least Hopf, and was later explicitly stated by Steenrod [14]. To solve the problem in general is probably too ambitious, but many special cases have been proven.

One of Quillen's motivations for his seminal work on rational homotopy theory [13] was to solve this problem over $\mathbb{Q}$. He showed that all simply connected graded $\mathbb{Q}$–algebras have a realization. The problem of which polynomial algebras over $\mathbb{Z}$ have realizations has a long history, and a complete solution was given by Anderson and Grodal [2]; see also Notbohm [12]. More recently Trevisan [15] and later Bahri, Bendersky, Cohen and Gitler [4] constructed realizations of $\mathbb{Z}[x_1,\dots,x_m]/I$, where $|x_i|=2$ and I is an ideal generated by monomials with leading coefficient 1.

We want to consider a related problem that lies between the solved realization problem over $\mathbb{Q}$ and the very difficult realization problem over $\mathbb{Z}$. We do this by modding out torsion.

Problem 1.1 *Which commutative graded R–algebras A are isomorphic to*

$$H^*(X_A;R)/torsion$$

for some space X_A?

Such an X_A is called a realization modulo torsion of A. For example, a polynomial ring $\mathbb{Z}[x]$ has a realization modulo torsion given by the Eilenberg–Mac Lane space $K(\mathbb{Z},|x|)$ if $|x|$ is even, while $\mathbb{Z}[x]$ has a realization (before modding out torsion) if and only if $|x|=2$ or 4 [14]. Here we ask: do all finite type connected commutative graded $\mathbb{Z}$–algebras have a realization modulo torsion?

Notice that modding out by torsion is different from taking rational coefficients. For example, both $H^*(\Omega S^{2n+1};\mathbb{Q})$ and $H^*(K(\mathbb{Z},2n);\mathbb{Q})$ are $\mathbb{Q}[x]$ generated by x of degree $2n$. But $H^*(K(\mathbb{Z},2n))/\text{torsion}$ is $\mathbb{Z}[x]$, while $H^*(\Omega S^{2n+1})\cong\Gamma[x]$ is free as a $\mathbb{Z}$–module and is the divided polynomial algebra generated by x.

In this paper, we construct realizations modulo torsion of graded monomial ideal rings A which are tensors of polynomial algebras and exterior algebras modulo monomial ideals. More precisely, let $P=\mathbb{Z}[x_1,\dots,x_m]\otimes\Lambda[y_1,\dots,y_n]$ be a graded polynomial ring where the x_i's have arbitrary positive even degrees and the y_j's have arbitrary positive odd degrees, and let $I=(M_1,\dots,M_r)$ be an ideal generated by r minimal monomials

$$M_j=x_1^{a_{1j}}x_2^{a_{2j}}\cdots x_m^{a_{mj}}\otimes y_1^{b_{1j}}\cdots y_n^{b_{nj}},\quad 1\le j\le r,$$

where the indices a_{ij} are nonnegative integers and b_{ij} are either 0 or 1. Then the quotient algebra $A=P/I$ is called a *graded monomial ideal ring*.

Theorem 1.2 (main theorem) *Let A be a graded monomial ideal ring. Then there exists a space X_A such that $H^*(X_A)/T$ is isomorphic to A, where T is the ideal consisting of torsion elements in $H^*(X_A)$. Moreover, there is a ring morphism $A\to H^*(X_A)$ that is right inverse to the quotient map $H^*(X_A)\to H^*(X_A)/T\cong A$.*

If all of the even degree generators are in degree 2, then we do not need to mod out by torsion and so we get a generalization (Theorem 4.6) of the results of Bahri, Bendersky, Cohen and Gitler [4, Theorem 2.2] and Trevisan [15, Theorem 3.6].

The structure of the paper is as follows. Section 2 contains preliminaries, algebraic tools and lemmas that are used in later sections. In Section 3 we recall the definition of polyhedral products and modify a result of Bahri, Bendersky, Cohen and Gitler [3] to compute $H^*((\underline{X},*)^K)/T$. In Sections 4 and 5 we prove Theorem 1.2 in several steps. First, we prove it in the special case where the ideal I is square-free. Then for the general case, we construct a fibration sequence inspired by algebraic polarization method and show that the fiber X_A is a realization modulo torsion of A. In Section 6 we illustrate how to construct X_A for an easy example of A.

2 Preliminaries

2.1 Quotients of algebras by torsion elements

It is natural to study an algebra A by factoring out the torsion elements since the quotient algebra is torsion-free and has a simpler structure. Driven by this, we start investigating the quotients of cohomology rings of spaces by their torsion elements. Since we cannot find related references in the literature, here we fix the notation and develop lemmas for our purpose.

A graded module $A = \{A_i\}_{i \in S}$ is a family of indexed modules A_i. Since we are interested in cochain complexes and cohomology rings of connected, finite type CW–complexes, we assume A to be a connected, finite type graded module with nonpositive degrees. That is, $S = \mathbb{N}_{\leq 0}$, $A_0 = \mathbb{Z}$ and each component A_i is finitely generated. We follow the convention and denote A_i by A^{-i}.

Remark 2.1 Equivalently we can define a graded module to be a module with a grading structure, that is the direct sum $A = \bigoplus_{i \in S} A_i$ of a family of indexed modules. This definition is slightly different from the definition above. We will use both definitions interchangeably.

An element $x \in A$ is *torsion* if $cx = 0$ for some nonzero integer c, and is *torsion-free* otherwise. The torsion submodule A_t of A is the graded submodule consisting of torsion elements and the torsion-free quotient module $A_f = A/A_t$ is their quotient. If B is another graded module and $g\colon A \to B$ is a morphism, then it induces a morphism $g_f\colon A_f \to B_f$ sending $a + A_t \in A_f$ to $g(a) + B_t \in B_f$. This kind of structure is important in abelian categories and was formalized with Dixon's notion of a torsion theory [6], but in this paper we only use the structure in a naive way.

Lemma 2.2 *If* $0 \to A \xrightarrow{g} B \xrightarrow{h} C \to 0$ *is a short exact sequence of graded modules, then* $C_f \cong (B_f/A_f)_f$*. Furthermore, if the sequence is split exact, then so is*

$$0 \to A_f \xrightarrow{g_f} B_f \xrightarrow{h_f} C_f \to 0.$$

Proof Consider a commutative diagram as in Figure 1, where g_t is the restriction of g to A_t, p and q are the quotient maps, and u and v are the induced maps. By construction all rows and columns are exact sequences except for the right column. A diagram chase implies that u is injective and v is surjective. We claim that the column is exact at C.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_t \arrow[r, "g_t"] \arrow[d] & B_t \arrow[r, "p"] \arrow[d] & B_t/A_t \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & A \arrow[r, "g"] \arrow[d] & B \arrow[r, "h"] \arrow[d] & C \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & A_f \arrow[r, "g_f"] \arrow[d] & B_f \arrow[r, "q"] \arrow[d] & B_f/A_f \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 1

Obviously $v \circ u$ is trivial. Take an element $c \in \ker(v)$ and its preimage $b \in B$. A diagram chase implies $b = g(a) + b'$ for some $a \in A$ and $b' \in B_t$. So $c = h(b') = u \circ p(b')$ is in $\mathrm{Im}(u)$ and the right column $0 \to B_t/A_t \xrightarrow{u} C \xrightarrow{v} B_f/A_f \to 0$ is exact.

For the first part of the lemma, we show that $v_f : C_f \to (B_f/A_f)_f$ is an isomorphism. Since v is surjective, so is v_f. Take $c' \in \ker(v_f)$ and its preimage $\tilde{c}' \in C$. Then $v(\tilde{c}')$ is a torsion element in B_f/A_f and $mv(\tilde{c}') = 0$ for some nonzero integer m. So $m\tilde{c}' \in \ker(v)$. As $\ker(v) = \mathrm{Im}(u)$ consists of torsion elements, $m\tilde{c}'$ is torsion and so is $\tilde{c}'$. Therefore $c' = 0$ in C_f and v_f is injective.

Notice that an exact sequence being split is equivalent to $B \cong A \oplus C$. So $B_f \cong A_f \oplus C_f$ and $0 \to A_f \xrightarrow{g_f} B_f \xrightarrow{h_f} C_f \to 0$ is a split exact sequence. □

A graded algebra (A, m) consists of a graded module A and an associative bilinear multiplication $m = \{m^{i,j} : A^i \otimes A^j \to A^{i+j}\}$ such that $1 \in A^0$ is the multiplicative identity. A pair (M, μ) is a left (resp. right) A–module if M is a graded module and μ is an associative bilinear multiplication $\mu = \{\mu^{i,j} : A^i \otimes M^j \to M^{i+j}\}$ such that $\mu(1 \otimes x) = x$ (resp. $\mu = \{\mu^{i,j} : M^i \otimes A^j \to M^{i+j}\}$ such that $\mu(1, x) = x$) for all $x \in M$. We check that modding out torsion and multiplications are compatible.

Lemma 2.3 *If A and M are graded modules (not necessarily of finite type), then there is a unique isomorphism $\theta : (A \otimes M)_f \to A_f \otimes M_f$ of graded modules making the diagram*

$$\begin{tikzcd}
A \otimes M \arrow[r] \arrow[d] & A_f \otimes M_f \\
(A \otimes M)_f \arrow[ur, "\theta"'] &
\end{tikzcd}$$

commute, where the vertical and the horizontal maps are quotient maps.

Proof It suffices to show that $(A^i \otimes M^j)_f \cong A^i_f \otimes M^j_f$ for any positive integers i and j. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^i_t \otimes M^j \oplus A^i \otimes M^j_t \arrow[r, "\iota_1"] \arrow[d, "a"] & A^i \otimes M^j \arrow[r, "\pi_1"] \arrow[d, equal] & A^i_f \otimes M^j_f \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & (A^i \otimes M^j)_t \arrow[r, "\iota_2"] & A^i \otimes M^j \arrow[r, "\pi_2"] & (A^i \otimes M^j)_f \arrow[r] & 0
\end{tikzcd}$$

where a, ι_1 and ι_2 are inclusions, π_1 and π_2 are quotient maps, and b is the induced map. We want to show that b is an isomorphism, which is equivalent to showing that a is an isomorphism. If A and M are of finite type, then a is an isomorphism since A^i and M^j are finitely generated abelian groups. In the general case, a is an isomorphism by [9, Theorem 61.5]. □

Corollary 2.4 *Let (A, m) be a graded algebra and let m'_f be the composition*

$$m'_f \colon A_f \otimes A_f \cong (A \otimes A)_f \xrightarrow{m_f} A_f.$$

Then (A_f, m'_f) is a graded algebra and there is a commutative diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d] & A \arrow[d] \\
A_f \otimes A_f \arrow[r, "m'_f"] & A_f
\end{tikzcd}$$

where the vertical maps are quotient maps.

Let (M, μ) be a left or right A–module and let μ'_f be the composition

$$\mu'_f \colon A_f \otimes M_f \cong (A \otimes M)_f \xrightarrow{\mu_f} M_f \quad \text{or} \quad \mu'_f \colon M_f \otimes A_f \cong (M \otimes A)_f \xrightarrow{\mu_f} M_f),$$

respectively. Then (M_f, μ'_f) is respectively a left or right A_f–module and there is a commutative diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\mu"] \arrow[d] & M \arrow[d] \\
A_f \otimes M_f \arrow[r, "\mu'_f"] & M_f
\end{tikzcd} \quad \text{or} \quad \begin{tikzcd}
M \otimes A \arrow[r, "\mu"] \arrow[d] & M \arrow[d] \\
M_f \otimes A_f \arrow[r, "\mu'_f"] & M_f
\end{tikzcd}$$

respectively, where the vertical maps are quotient maps.

A *cochain complex* (A,d) consists of a graded module A and a differential

$$d=\{d^i\colon A^i\to A^{i+1}\}$$

such that $d\circ d=0$. Let $d_f=\{d_f^i\colon A_f^i\to A_f^{i+1}\}$ be the induced differential on A_f. Then (A_f,d_f) forms a cochain complex and its cohomology

$$H^*(A_f,d_f)=\{H^i(A_f,d_f)\}_{i\geq 0}$$

is a graded module.

A *differential graded algebra* (A,m,d) is a cochain complex (A,d) such that (A,m) is a graded algebra and d and m satisfy the Leibniz rule. Let d_t be the restriction of d to A_t. Then (A_t,d_t) is a differential ideal and (A_f,d_f) is a differential graded algebra, so $H^*(A_f,d_f)$ is a graded algebra.

A left (resp. right) *dg-algebra module* (M,μ,δ) over (A,m,d) if (M,μ) is a left (resp. right) (A,m)–module, (M,δ) is a cochain complex and δ and μ satisfy the Leibniz rule. Then $H^*(M_f,\delta_f)$ is a left (resp. right) $H^*(A_f)$–module.

Even if (A_f,d_f) is torsion-free, $H^*(A_f,d_f)$ is not necessarily torsion-free. Denote $(H^*(A,d))_f$ by $H_f^*(A,d)$. The following lemma compares $H_f^*(A,d)$ and $H_f^*(A_f,d_f)$.

Lemma 2.5 *Let (A,d) be a cochain complex. Then there is a monomorphism of modules*

$$\psi\colon H_f^*(A,d)\to H_f^*(A_f,d_f).$$

If $H^{i+1}(A_t,d_t)=0$, then $\psi\colon H_f^i(A,d)\to H_f^i(A_f,d_f)$ is an isomorphism. Moreover, suppose (A,m,d) is a differential graded algebra. Then ψ is a morphism of algebras.

Proof Assume (A,d) is a cochain complex. Let $\iota\colon(A_t,d_t)\to(A,d)$ be the inclusion and let $\pi\colon(A,d)\to(A_f,d_f)$ be the quotient map. Then the short exact sequence of cochain complexes $0\to(A_t,d_t)\xrightarrow{\iota}(A,d)\xrightarrow{\pi}(A_f,d_f)\to 0$ induces a long exact sequence

$$\cdots\to H^{i-1}(A_f,d_f)\to H^i(A_t,d_t)\xrightarrow{\iota^*}H^i(A,d)\xrightarrow{\pi^*}H^i(A_f,d_f)\to H^{i+1}(A_t,d_t)\to\cdots.$$

Take $\psi\colon H_f^*(A,d)\to H_f^*(A_f,d_f)$ to be the morphism induced by

$$\pi^*\colon H^*(A,d)\to H^*(A_f,d_f).$$

We show that it has the asserted properties.

To show the injectivity of ψ, take an equivalence class $[a] \in H_f^*(A,d)$ such that $\psi[a] = 0$. Represent it by a cocycle class $a \in H^i(A,d)$. Then $\pi^*(a)$ is torsion and $\pi^*(ca) = 0$ for some nonzero number c. By exactness, $ca \in \mathrm{Im}(\iota^*)$. Since $H^i(A_t, d_t)$ is torsion, so is $\mathrm{Im}(\iota^*)$ and ca is a torsion. Therefore $a \in H^i(A,d)$ is a torsion. By definition, $[a] \in H_f^i(A,d)$ is zero. So ψ is injective.

Suppose A^{i+1} has no torsion elements. Then $A_t^{i+1} = 0$ and $H^{i+1}(A_t, d_t) = 0$. So π^* is surjective. By definition we have commutative diagram

$$\begin{array}{ccc} H^i(A,d) & \xrightarrow{\pi^*} & H^i(A_f, d_f) \\ \big\downarrow & & \big\downarrow \\ H_f^i(A,d) & \xrightarrow{\psi} & H_f^i(A_f, d_f) \end{array}$$

where vertical arrows are quotient maps and are surjective. So

$$\psi\colon H_f^i(A,d) \to H_f^i(A_f, d_f)$$

is surjective and hence isomorphic.

If A is a differential graded algebra, then $\pi^*\colon H^*(A,d) \to H^*(A_f, d_f)$ is a morphism of graded algebras. By Corollary 2.4, the induced morphism ψ is multiplicative. □

Example The surjectivity of $\psi\colon H_f^i(A,d) \to H_f^i(A_f, d_f)$ may fail if A^{i+1} contains torsion elements. Let (A,d) be a cochain complex where

$$A^i = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and d^i are trivial for all i except for $d^0\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ being the quotient map. Then $H^0(A)$ and $H^0(A_f)$ are $\mathbb{Z}$ while $\psi\colon H^0(A) \to H_f^0(A)$ is multiplication $2\colon \mathbb{Z} \to \mathbb{Z}$.

2.2 Eilenberg–Moore spectral sequence

Given a differential graded algebra (A,d) and a right A–module (M, d_M), we first define the bar bicomplex $\mathrm{B}^{*,*}(M,A)$ as follows. For any positive integer i, let $\mathrm{B}^{-i}(M,A) = M \otimes (\bar{A})^{\otimes i}$ where $\bar{A} = \{A^n\}_{n>0}$. Denote an element in $\mathrm{B}^{-i}(M,A)$ by $x[a_1|\cdots|a_i]$ for $x \in M$ and $a_i \in \bar{A}$. Let $\mathrm{B}^{-i,j}(M,A)$ be the submodule of $\mathrm{B}^{-i}(M,A)$

consisting elements $x[a_1|\cdots|a_i]$ such that $|x|+\sum_{k=1}^{i}|a_k|=j$. The internal and external differentials

$$d_I\colon \mathrm{B}^{-i,j}(M,A)\to \mathrm{B}^{-i,j+1}(M,A) \quad\text{and}\quad d_E\colon \mathrm{B}^{-i,j}(M,A)\to \mathrm{B}^{-i+1,j}(M,A)$$

are given by

$$d_I(x[a_1|\cdots|a_i])=(d_Mx)[a_1|\cdots|a_i]+\sum_{j=1}^{i}(-1)^{\epsilon_{j-1}}x[a_1|\cdots|a_{j-1}|d_Aa_j|a_{j+1}|\cdots|a_i],$$

$$d_E(x[a_1|\cdots|a_i])=(-1)^{|x|}(xa_1)[a_2|\cdots|a_i]+\sum_{j=1}^{i-1}(-1)^{\epsilon_j}x[a_1|\cdots|a_{j-1}|a_j\cdot a_{j+1}|\cdots|a_i],$$

where $\epsilon_k=k+|x|+\sum_{j=1}^{k}|a_j|$. Then we define the bar construction $(\mathcal{B}(M,A),d_{\mathcal{B}})$ to be a graded module where

$$\mathcal{B}(M,A)^n=\bigoplus_{-i+j=n}\mathrm{B}^{-i,j}(M,A) \quad\text{and}\quad d_{\mathcal{B}}=\bigoplus_{-i+j=n}(d_I+d_E)$$

for $n\geq 0$.

Take the filtration $\mathscr{F}^{-p}=\bigoplus_{0\leq i\leq p}\mathrm{B}^{-i}(M,A)$. The associated spectral sequence $\{E_r^{*,*}\}_{r=0}^{\infty}$ is the Eilenberg–Moore spectral sequence converging to $H^*(\mathcal{B}(M,A))$; see [7, Remark 2.3] and [11, Corollary 7.9].

Lemma 2.6 *Let A be a simply connected differential graded algebra and M be a right A–module such that A and M are free as $\mathbb{Z}$–modules. Then there is a monomorphism of modules*

$$\psi\colon (E_2^{-p,q})_f\to \bigl(\mathrm{Tor}_{H_f(A)}^{-p,q}(H_f(M),\mathbb{Z})\bigr)_f$$

which is an isomorphism for $p=0$. Moreover, if $H(A)$ and $H(M)$ are free modules, then $E_2^{-p,q}\cong \mathrm{Tor}_{H(A)}^{-p,q}(H(M),\mathbb{Z})$.

Proof The E_0–page is given by

$$E_0^{-p,*}=\mathscr{F}^{-p}/\mathscr{F}^{-p+1}=M\otimes(\bar{A}^{\otimes p})$$

and $d_0=d_I$. By the Künneth theorem, the E_1–page is given by

$$E_1^{-p,*}\cong H(M)\otimes(\widetilde{H}(A)^{\otimes p})\oplus T\cong \mathrm{B}^{-p}(H(M),H(A))\oplus T,$$

where T is a torsion term and d_1 is induced by d_E. Denote $H(M)$ by M' and $H(A)$ by A' for short. By Lemma 2.3, there is an isomorphism of graded modules

$$\theta\colon (E_1^{-p,*})_f\cong(\mathrm{B}^{-p}(M',A'))_f\to \mathrm{B}^{-p}(M'_f,A'_f)$$

such that

$$\begin{tikzcd} \mathrm{B}^{-p}(M', A') \arrow[d] \arrow[dr] & \\ (\mathrm{B}^{-p}(M', A'))_f \arrow[r, "\theta"] & \mathrm{B}^{-p}(M'_f, A'_f) \end{tikzcd}$$

where the downward maps are quotient maps. Let d' be the external differential of $\mathrm{B}^*(M'_f, A'_f)$. Then $\theta\colon ((\mathrm{B}^{-p}(M', A'))_f, (d_1)_f) \to (\mathrm{B}^*(M'_f, A'_f), d')$ is an isomorphism of cochain complexes. By Lemma 2.5, there is a monomorphism of graded modules

$$\psi\colon (E_2^{-p,q})_f = H_f^{-p}(E_1^{*,q}, d_1) \to H_f^{-p}((\mathrm{B}^{*,q}(M', A'))_f, (d_1)_f) \cong H_f^{-p}(\mathrm{B}^{*,q}(M'_f, A'_f), d').$$

Notice that $\mathrm{B}^*(M'_f, A'_f) \cong M'_f \otimes_{A'_f} \mathrm{B}^*(A'_f, A'_f)$ and $d' = \mathbb{1} \otimes_{A'_f} d''$, where d'' is the external differential of $\mathrm{B}^*(A'_f, A'_f)$. Since, by [11, Proposition 7.8],

$$\cdots \to \mathrm{B}^{-1}(A'_f, A'_f) \xrightarrow{d''} \mathrm{B}^0(A'_f, A'_f) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}$ over A'_f–modules where $\epsilon\colon \mathrm{B}^0(A'_f, A'_f) \cong A'_f \to \mathbb{Z}$ is the augmentation, the monomorphism becomes

$$\psi\colon (E_2^{-p,q})_f \to (\mathrm{Tor}_{A'_f}^{-p,q}(M'_f, \mathbb{Z}))_f.$$

Since $\mathrm{B}^1(M', A') = 0$, ψ is isomorphic for $p = 0$ by Lemma 2.5.

Suppose $H(A)$ and $H(M)$ are free $\mathbb{Z}$–modules. By the Künneth theorem,

$$E_1^{*,*} \cong \mathrm{B}^{*,*}(H(M), H(A))$$

and d_1 is the external differential. So $E_2^{-p,q} \cong \mathrm{Tor}_{H(A)}^{-p,q}(H(M), \mathbb{Z})$. □

Let $F \to E \xrightarrow{\pi} X$ be a fibration sequence where all spaces are connected, finite type CW–complexes, and X is simply connected. In [7, Theorem III] there is a quasi-isomorphism

$$\Theta\colon \Omega(C_*^{\pi}(E), C_*(X)) \to CN_*(F)$$

of dg-algebra modules, which is natural in π. Here $\Omega(-,-)$ is the cobar construction, $C_*^{\pi}(E)$ is a nonnegative chain complex, $C_*(X)$ is a simply connected chain complex, $CN_*(F)$ is a chain complex, and $C_*^{\pi}(E)$, $C_*(X)$ and $CN_*(F)$ are quasi-isomorphic to the singular chain complexes of E, X and F, respectively.

Denote the dual of a (co)chain complex C by $C^{\vee} = \mathrm{Hom}(C, \mathbb{Z})$. Since X is simply connected, $H^1(X) = 0$ and $H^2(X)$ is free. By [7, Propositions 4.2 and 4.6] there are finite type graded free modules $V=\{V^i\}_{i\geq 2}$ and $W=\{W^j\}_{j\geq 0}$, a quasi-isomorphism of dg-algebras

$$\phi\colon T(V)\to (C_*(X))^\vee$$

and a quasi-isomorphism of dg-algebra modules

$$\varphi\colon T(V)\otimes W\to (C^\pi_*(E))^\vee,$$

where $T(V)$ is the tensor algebra on V. Write $\widetilde{X}=T(V)$ and $\widetilde{E}=T(V)\otimes W$ for short. Then the compositions

$$C_*(X)\xrightarrow{\text{incl}}(C^*(X))^\vee\xrightarrow{\phi^\vee}\widetilde{X}^\vee \quad\text{and}\quad C^\pi_*(E)\xrightarrow{\text{incl}}(C^*(E))^\vee\xrightarrow{\varphi^\vee}\widetilde{E}^\vee$$

are quasi-isomorphisms of dg-coalgebras and of dg-coalgebra modules. Since $C_*(X)$ and $\widetilde{X}^\vee$ are simply connected free chain complexes, and $C^\pi_*(E)$ and $\widetilde{E}^\vee$ are nonnegative chain complexes, we have a zig-zag of quasi-isomorphisms

$$\Omega(\widetilde{E}^\vee,\widetilde{X}^\vee)\xleftarrow{\simeq}\Omega(C^\pi_*(E),C_*(X))\xrightarrow{\Theta}CN_*(F).$$

Since $\widetilde{E}$ and $\widetilde{X}$ are of finite type, dualize the zig-zag and take cohomology to get an isomorphism

$$H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))\xrightarrow{\cong}H^*(F).$$

The Eilenberg–Moore spectral sequence $\{E_r^{*,*}\}_{r=0}^\infty$ on $F\to E\xrightarrow{\pi}X$ is the Eilenberg–Moore spectral sequence given by $A=\widetilde{X}$ and $M=\widetilde{E}$. Note that this definition depends on the choice of the pair $(\widetilde{X},\widetilde{E},\phi,\varphi)$. Any two choices may give spectral sequences with different E_0–pages, but their E_r–pages are isomorphic for $r\geq 1$.

Lemma 2.7 *Let $F\to E\xrightarrow{\pi}X$ be a fibration sequence such that all spaces are finite type spaces and X is simply connected, and let $\{E_2^{-p,q}\}$ be the E_2–page of Eilenberg–Moore spectral sequence on this fibration. Then there is a monomorphism*

$$\psi\colon (E_2^{-p,q})_f\to\big(\mathrm{Tor}^{-p,q}_{H^*_f(X)}(H^*_f(E),\mathbb{Z})\big)_f$$

as modules such that ψ is an isomorphism for $p=0$.

Proof Since $H(\widetilde{E})\cong H^*(E)$ and $H(\widetilde{X})\cong H^*(X)$, Lemma 2.6 implies that there is a monomorphism $\psi\colon (E_2^{-p,q})_f\to\big(\mathrm{Tor}^{-p,q}_{H^*_f(X)}(H^*_f(E),\mathbb{Z})\big)_f$ such that ψ is an isomorphism at $p=0$. □

Recall that the E_0–page is given by $E_0^{p,*} = \mathscr{F}^{-p}/\mathscr{F}^{-p+1} \cong \widetilde{E} \otimes (\overline{\widetilde{X}})^{\otimes p}$. In particular, if $p = 0$, then $E_0^{0,*} \cong \widetilde{E}$. On the other hand, $\{E_r^{*,*}\}_{r=0}^{\infty}$ is a second quadrant spectral sequence. So $E_r^{0,*}$ is the kernel of the differential map and $E_{r+1}^{0,*}$ is a quotient group of $E_r^{0,*}$. For $r \in \mathbb{N} \cup \{\infty\}$, define the edge homomorphism e_r to be the composition

$$e_r \colon H(E) \cong H(\widetilde{E}) \cong E_1^{0,*} \to E_r^{0,*}$$

where the unnamed arrow is the quotient map. The following lemma tells how the edge homomorphisms relate the E_r–page to $H^*(E)$ and $H^*(F)$.

Lemma 2.8 *Under the hypotheses of Lemma 2.7, the edge homomorphisms make the diagram*

$$\begin{tikzcd} H^*(E) \arrow[r, equal] \arrow[d, "e_1"] & H^*(E) \arrow[r, equal] \arrow[d, "e_2"] & \cdots \arrow[r] & H^*(E) \arrow[r, equal] \arrow[d, "e_\infty"] & H^*(E) \arrow[d, "\iota^*"] \\ E_1^{0,*} \arrow[r, "\jmath_1"] & E_2^{0,*} \arrow[r, "\jmath_2"] & \cdots \arrow[r] & E_\infty^{0,*} \arrow[r, "\jmath"] & H^*(F) \end{tikzcd}$$

commute, where ι^* is induced by $\iota\colon F \to E$, $\jmath$ is the inclusion and the $\jmath_r$'s are the quotient maps.

Proof We use the notation above. Consider the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\iota"] \arrow[d, "\iota"] & E \arrow[r, "\pi"] \arrow[d, equal] & X \arrow[d] \\ E \arrow[r, equal] & E \arrow[r, "c"] & \mathrm{pt} \end{tikzcd}$$

where c is the constant map. We have

$$\begin{tikzcd} \Omega(C_*^{\pi}(E), C_*(X)) \arrow[r, "\Theta"] \arrow[d] & CN_*(F) \arrow[d, "\iota_*"] \\ \Omega(C_*^{c}(E), C_*(\mathrm{pt})) \arrow[r, "\Theta"] & CN_*(E) \end{tikzcd}$$

since the quasi-isomorphism Θ is natural. The supplement $\mathbb{Z} \to (C_*(\mathrm{pt}))^\vee$ is a quasi-isomorphism of dg-algebras and $\varphi\colon \widetilde{E} \to (C_*^{\pi}(E))^\vee$ is a quasi-isomorphism of dg-algebra modules. Using this replacement and taking dual and cohomology of the diagram, we obtain

$$\begin{tikzcd} H^*(\widetilde{E}) \arrow[r, "\cong"] \arrow[d, "e^*"] & H^*(E) \arrow[d, "\iota^*"] \\ H^*(\mathcal{B}(\widetilde{E}, \widetilde{X})) \arrow[r, "\cong"] & H^*(F) \end{tikzcd} \tag{1}$$

where e^* is the composition

$$e^*\colon H^*(\widetilde{E}) \cong H^*(\mathcal{B}(\widetilde{E},\mathbb{Z})) \xrightarrow{e'} H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))$$

and e' is induced by the inclusion $e\colon \mathrm{B}^{*,*}(\widetilde{E},\mathbb{Z}) \to \mathrm{B}^{*,*}(\widetilde{E},\widetilde{X})$. Let $\{\widehat{E}_r^{*,*}\}_{r=0}^{\infty}$ be the Eilenberg–Moore spectral sequence on $E \xrightarrow{=} E \xrightarrow{c} \mathrm{pt}$. Then $\widehat{E}_0^{*,*} \cong \mathrm{B}^{*,*}(\widetilde{E},\mathbb{Z})$ and the $\widehat{E}_1$–page collapses to $H^*(\widetilde{E})$. The inclusion $e\colon \mathrm{B}^{*,*}(\widetilde{E},\mathbb{Z}) \to \mathrm{B}^{*,*}(\widetilde{E},\widetilde{X})$ gives the commutative diagram

$$\begin{tikzcd}
H^*(\widetilde{E}) \arrow[r, equal] \arrow[d, "\tilde{e}_1"] & H^*(\widetilde{E}) \arrow[r, equal] \arrow[d, "\tilde{e}_2"] & \cdots \arrow[r] & H^*(\widetilde{E}) \arrow[r, equal] \arrow[d, "\tilde{e}_\infty"] & H^*(\widetilde{E}) \arrow[d, "e^*"] \\
E_1^{0,*} \arrow[r, "J1"] & E_2^{0,*} \arrow[r, "J2"] & \cdots \arrow[r] & E_\infty^{0,*} \arrow[r, "\tilde{J}"] & H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))
\end{tikzcd}$$

where $\tilde{e}_r\colon H^*(\widetilde{E}) \cong H^*(E) \xrightarrow{e_r} E_r^{0,*}$ and $\tilde{J}\colon E_\infty^{0,*} \xrightarrow{J} H^*(F) \cong H^*(\mathcal{B}(\widetilde{E},\widetilde{X}))$. Combine this with (1) and obtain the asserted commutative diagram. □

2.3 Regular sequences and freeness

Here we use the alternative description of graded objects. A commutative graded algebra $A = \bigoplus_{i\geq 0} A_i$ is an algebra with a grading such that $ab = (-1)^{ij} ba$ for $a \in A^i$ and $b \in A^j$, and a graded A–module $M = \bigoplus_{j\geq 0} M_j$ is the direct sum of a family of A–modules. A set $\{r_1,\dots,r_n\}$ of elements in M is called an *M–regular sequence* if the ideal $(r_1,\dots,r_n)M$ is not equal to M and the multiplication

$$r_i\colon M/(r_1,\dots,r_{i-1})M \to M/(r_1,\dots,r_{i-1})M$$

is injective for $1 \leq i \leq n$. In the special case where M is a $\mathbb{K}[x_1,\dots,x_n]$–module for some field $\mathbb{K}$ and the grading of M has a lower bound, M is a free $\mathbb{K}[x_1,\dots,x_n]$–module if $\{x_i\}_{i=1}^n$ is a regular sequence in M. We want to extend this fact to the case where M is a $\mathbb{Z}[x_1,\dots,x_n]$–module. Recall a corollary of the graded Nakayama lemma.

Lemma 2.9 *Let A be a graded ring and let M be an A–module. Suppose A and M are nonnegatively graded, and $I = (r_1,\dots,r_n) \subset A$ is an ideal generated by homogeneous elements r_i of positive degrees. If $\{m_\alpha\}_{\alpha\in S}$ is a set of homogeneous elements in M whose images generate M/IM, then $\{m_\alpha\}_{\alpha\in S}$ generates M.*

Lemma 2.10 *Let M be a $\mathbb{Z}[x_1,\dots,x_n]$–module with nonnegative degrees. If*

$$M/(x_1,\dots,x_n)M$$

is a free $\mathbb{Z}$–module and $\{x_1,\dots,x_n\}$ is an M–regular sequence, then M is a free $\mathbb{Z}[x_1,\dots,x_n]$–module.

Proof Let $I=(x_1,\dots,x_n)$. By assumption there is a set $\{m_\alpha\}_{\alpha\in S}$ of homogeneous elements in M such that their quotient images form a basis in M/IM. By Lemma 2.9, $\{m_\alpha\}_{\alpha\in S}$ generates M. We need to show that $\{m_\alpha\}_{\alpha\in S}$ is linear independent over $\mathbb{Z}[x_1,\dots,x_n]$.

For $0\le i\le n$, let $M_i=M/(x_1,\dots,x_{n-i})M$, $A_i=\mathbb{Z}[x_{n+1-i},\dots,x_n]$ and $m_{\alpha,i}$ be the quotient image of m_α in M_i. We prove that M_i is a free A_i–module with a basis $\{m_{\alpha,i}\}_{\alpha\in S}$ by induction on i. For $i=0$, $M_0=M/IM$ and $A_0=\mathbb{Z}$. The statement is true since $\{m_{\alpha,0}\}_{\alpha\in S}$ is a basis by construction. Assume the statement holds for $i\le k$. For $i=k+1$, if there is a collection $\{f_\alpha\}_{\alpha\in S}$ of polynomials satisfying

$$\sum_{\alpha\in S} f_\alpha\cdot m_{\alpha,k+1}=0, \tag{2}$$

we show that all f_α's are zero.

If not, then there are finitely many nonzero polynomials $f_{j_1},\dots,f_{j_r}$. Quotient M_{k+1} and A_{k+1} by the ideal (x_{n-k}) and let $\bar f_{j_i}$ be the image of f_{j_i} in A_k. Then (2) becomes

$$\sum_{i=1}^{r}\bar f_{j_i}\cdot m_{j_i,k}=0.$$

By our inductive assumption, $\{m_{\alpha,k}\}$ is a basis in M_k. So $\bar f_{j_i}=0$ and $f_{j_i}=x_{n-k}g_{j_i}$ for some polynomial $g_{j_i}\in A_{k+1}$. Since x_{n-k} is not a zero-divisor, putting $f_{j_i}=x_{n-k}g_{j_i}$ in (2) gives

$$\sum_{i=1}^{r} g_{j_i}\cdot m_{j_i,k+1}=0.$$

So $g_{j_1},\dots,g_{j_r}$ are nonzero polynomials satisfying (2) and $|g_{j_i}|<|f_{j_i}|$ for $1\le i\le r$. Iterating this argument implies that the $|f_{j_i}|$'s are arbitrarily large, but this is impossible. So the f_{j_i}'s must be zero and $\{m_{\alpha,k+1}\}$ is linearly independent. It follows that M_{k+1} is a free A_{k+1}–module. □

3 Cohomology rings of polyhedral products

Let $[m]=\{1,\dots,m\}$, K be a simplicial complex on $[m]$ and $(\underline{X},\underline{A})=\{(X_i,A_i)\}_{i=1}^m$ be a sequence of pairs of relative CW–complexes. For any simplex $\sigma\in K$, define

$$(\underline{X},\underline{A})^\sigma=\left\{(x_1,\dots,x_m)\in\prod_{i=1}^m X_i \;\middle|\; x_i\in A_i \text{ for } i\notin\sigma\right\}$$

as a subspace of $\prod_{i=1}^m X_i$, and define the *polyhedral product*

$$(\underline{X},\underline{A})^K=\bigcup_{\sigma\in K}(\underline{X},A)^\sigma$$

to be the union of $(\underline{X},\underline{A})^\sigma$ over $\sigma\in K$.

If $X_i=\mathbb{CP}^\infty$ and $A_i=*$ for all i, then $(\mathbb{CP}^\infty,*)^K$ is homotopy equivalent to Davis–Januszkiewicz space [5, Theorem 4.3.2]. For any principal ideal domain R, $H^*((\mathbb{CP}^\infty,*)^K;R)$ is isomorphic to the Stanley–Reisner ring $R[x_1,\dots,x_m]/I_K$. Here I_K is the ideal generated by $x_{j_1}\cdots x_{j_k}$ for $x_{j_i}\in\widetilde{H}^*(X_{j_i};R)$ and $\{j_1,\dots,j_k\}\notin K$, and is called the *Stanley–Reisner ideal of* K. In general, a similar formula holds for $H^*((\underline{X},*)^K)$ whenever the X_i's are any spaces with free cohomology.

Theorem 3.1 [3] *Let R be a principal ideal domain, K be a simplicial complex on $[m]$ and $\underline{X}=\{X_i\}_{i=1}^m$ be a sequence of CW–complexes. If $H^*(X_i;R)$ is a free R–module for all i, then*

$$H^*((\underline{X},*)^K;R)\cong\bigotimes_{i=1}^m H^*(X_i;R)/I_K,$$

where I_K is generated by $x_{j_1}\otimes\cdots\otimes x_{j_k}$ for $x_{j_i}\in\widetilde{H}^*(X_{j_i};R)$ and $\{j_1,\dots,j_k\}\notin K$ and is called the generalized Stanley–Reisner ideal of K.

The proof of Theorem 3.1 uses the strong form of the Künneth theorem, which says that

$$\mu\colon\bigotimes_{i=1}^m H^*(X_i;R)\to H^*\left(\prod_{i=1}^m X_i;R\right),\qquad x_1\otimes\cdots\otimes x_m\mapsto\pi_1^*(x_1)\cup\cdots\cup\pi_m^*(x_m),$$

where π_j^* is induced by the projection $\pi_j:\prod_{i=1}^m X_i\to X_j$, is an isomorphism if all $H^*(X_i;R)$'s are free. In the reduced version of the Künneth theorem,

$$\bar{\mu}\colon\bigotimes_{i=1}^m\widetilde{H}^*(X_i)\to\widetilde{H}^*\left(\bigwedge_{i=1}^m X_i\right)$$

is also an isomorphism if all $\widetilde{H}^*(X_i;R)$'s are free. The goal of this section is to modify Theorem 3.1 by removing the freeness assumption on $H^*(X_i)$. As a trade-off, we need to mod out the torsion elements of $H^*(X_i)$. First let us refine the Künneth theorem.

Lemma 3.2 *Let $\underline{X}=\{X_i\}_{i=1}^m$ be a sequence of spaces X_i. Then the induced morphisms*

$$\mu_f\colon \bigotimes_{i=1}^m H_f^*(X_i)\to H_f^*\Big(\prod_{i=1}^m X_i\Big) \quad\text{and}\quad \bar{\mu}_f\colon \bigotimes_{i=1}^m \widetilde{H}_f^*(X_i)\to \widetilde{H}_f^*\Big(\bigwedge_{i=1}^m X_i\Big).$$

are isomorphisms as algebras, and there is a commutative diagram

$$\begin{tikzcd} \bigotimes_{i=1}^m \widetilde{H}_f^*(X_i) \arrow[r, "\bar{\mu}_f"] \arrow[d] & \widetilde{H}_f^*(\bigwedge_{i=1}^m X_i) \arrow[d, "q_f^*"] \\ \bigotimes_{i=1}^m H_f^*(X_i) \arrow[r, "\mu_f"] & H_f^*(\prod_{i=1}^m X_i) \end{tikzcd}$$

where q_f^* is induced by the quotient map $q\colon \prod_{i=1}^m X_i\to\bigwedge_{i=1}^m X_i$.

Proof It suffices to show the $m=2$ case. Let (X,A) and (Y,B) be pairs of relative CW–complexes and let $\pi_X\colon(X\times Y, A\times Y)\to(X,A)$ and $\pi_Y\colon(X\times Y, X\times B)\to(Y,B)$ be projections. By the generalized version of Künneth theorem [10, Chapter XIII, Theorem 11.2], the sequence

$$0\to\bigoplus_{i+j=n} H^i(X,A)\otimes H^j(Y,B)\xrightarrow{\mu'} H^n(X\times Y, X\times B\cup A\times Y)\to T\to 0,$$

where T is a torsion term and μ' sends $u\otimes v\in H^i(X,A)\otimes H^j(Y,B)$ to $\pi_X^*(u)\cup\pi_Y^*(v)$, is split exact. By Lemma 2.3 $(H^*(X,A)\otimes H^*(Y,B))_f\cong H_f^*(X,A)\otimes H_f^*(Y,B)$ and by Lemma 2.2

$$\mu'_f\colon H_f^*(X,A)\otimes H_f^*(Y,B)\to H_f^*(X\times Y, X\times B\cup A\times Y)$$

is an isomorphism. Since μ' is multiplicative, so is μ'_f. Letting A and B be the basepoints of X and Y, or be the empty set, gives the isomorphisms

$$\mu_f\colon H_f^*(X)\otimes H_f^*(Y)\cong H_f^*(X\times Y) \quad\text{and}\quad \bar{\mu}_f\colon \widetilde{H}_f^*(X)\otimes\widetilde{H}_f^*(Y)\cong\widetilde{H}_f^*(X\wedge Y).$$

The commutative diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^m \widetilde{H}^*(X_i) \arrow[r, "\bar{\mu}"] \arrow[d] & \widetilde{H}^*\bigl(\bigwedge_{i=1}^m X_i\bigr) \arrow[d, "q^*"] \\
\bigotimes_{i=1}^m H^*(X_i) \arrow[r, "\mu"] & H^*\bigl(\prod_{i=1}^m X_i\bigr)
\end{tikzcd}$$

leads to the asserted commutative diagram. □

Proposition 3.3 *Let $\underline{X} = \{X_i\}_{i=1}^m$ be a sequence of spaces X_i, and let K be a simplicial complex on $[m]$. Then the inclusion $\iota : (\underline{X}, *)^K \to \prod_{i=1}^m X_i$ induces a ring isomorphism*

$$H_f^*((\underline{X}, *)^K) \cong \Bigl(\bigotimes_{i=1}^m H_f^*(X_i)\Bigr)/I_K$$

where I_K is generated by $x_{j_1} \otimes \cdots \otimes x_{j_k}$ for $x_{j_i} \in \widetilde{H}_f^*(X_{j_i})$ and $\{j_1, \dots, j_k\} \notin K$.

Proof This proof modifies the proofs in [3; 5]. Consider the homotopy cofibration sequence

$$(\underline{X}, *)^K \xrightarrow{\iota} \prod_{i=1}^m X_i \xrightarrow{\jmath} C,$$

where C is the mapping cone of ι and $\jmath$ is the inclusion. Suspend it and obtain a diagram of homotopy cofibration sequences

$$\begin{tikzcd}
\Sigma(\underline{X}, *)^K \arrow[r, "\Sigma\iota"] \arrow[d, "a"] & \Sigma\bigl(\prod_{i=1}^m X_i\bigr) \arrow[r, "\Sigma\jmath"] \arrow[d, "b"] & \Sigma C \arrow[d, "c"] \\
\bigvee_{J \in K} \Sigma \underline{X}^{\wedge J} \arrow[r, "\bar{\iota}"] & \bigvee_{J \in [m]} \Sigma \underline{X}^{\wedge J} \arrow[r, "\bar{\jmath}"] & \bigvee_{J \notin K} \Sigma \underline{X}^{\wedge J}
\end{tikzcd} \tag{3}$$

where $\underline{X}^{\wedge J} = X_{j_1} \wedge \cdots \wedge X_{j_k}$ for $J = \{j_1, \dots, j_k\}$, $\bar{\iota}$ is the inclusion, $\bar{\jmath}$ is the pinch map, a is a homotopy equivalence by [3, Theorem 2.21], b is a homotopy equivalence, and c is an induced homotopy equivalence. Take cohomology and get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{J \notin K} \widetilde{H}^*(\underline{X}^{\wedge J}) \arrow[r, "\bar{\jmath}^*"] \arrow[d, "c^*"] & \bigoplus_{J \in [m]} \widetilde{H}^*(\underline{X}^{\wedge J}) \arrow[r, "\bar{\iota}^*"] \arrow[d, "b^*"] & \bigoplus_{J \in K} \widetilde{H}^*(\underline{X}^{\wedge J}) \arrow[r] \arrow[d, "a^*"] & 0 \\
0 \arrow[r] & \widetilde{H}^*(C) \arrow[r, "\jmath^*"] & \widetilde{H}^*\bigl(\prod_{i=1}^m X_i\bigr) \arrow[r, "\iota^*"] & \widetilde{H}^*((\underline{X}, *)^K) \arrow[r] & 0
\end{tikzcd}$$

where the rows are split exact sequences, all vertical maps are isomorphisms, and all maps are additive while $\imath^*$ is multiplicative. Apply Lemma 2.2 to the diagram and get:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{J\notin K}\widetilde{H}^*_f(\underline{X}^{\wedge J}) \arrow[r, "\bar{\jmath}^*_f"] \arrow[d, "c^*_f"] & \bigoplus_{J\in[m]}\widetilde{H}^*_f(\underline{X}^{\wedge J}) \arrow[r, "\bar{\imath}^*_f"] \arrow[d, "b^*_f"] & \bigoplus_{J\in K}\widetilde{H}^*_f(\underline{X}^{\wedge J}) \arrow[r] \arrow[d, "a^*_f"] & 0 \\
0 \arrow[r] & \widetilde{H}^*_f(C) \arrow[r, "\jmath^*_f"] & \widetilde{H}^*_f\big(\prod_{i=1}^m X_i\big) \arrow[r, "\imath^*_f"] & \widetilde{H}^*_f((\underline{X},*)^K) \arrow[r] & 0
\end{tikzcd} \tag{4}$$

By Lemma 3.2, $H^*_f\big(\prod_{i=1}^m X_i\big)\cong\bigotimes_{i=1}^m H^*_f(X_i)$ so there is a ring isomorphism

$$H^*_f((\underline{X},*)^K)\cong\left(\bigotimes_{i=1}^m H^*_f(X_i)\right)/\ker(\imath^*_f).$$

Since the rows are split exact and the vertical maps are isomorphic in (4), $\ker(\imath^*_f)$ is generated by $x_{j_1}\otimes\cdots\otimes x_{j_k}$ for $x_{j_i}\in\widetilde{H}^*_f(X_{j_i})$ and $\{j_1,\dots,j_k\}\notin K$. Therefore $\ker(\imath^*_f)=I_K$ and $H^*_f((\underline{X},*)^K)\cong\big(\bigotimes_{i=1}^m H^*_f(X_i)\big)/I_K$. $\square$

Proposition 3.3 can be refined as follows. If the quotient map $H^*(X_i)\to H^*_f(X_i)$ has right inverse for all i, then so does $H^*((\underline{X},*)^K)\to H^*_f((\underline{X},*)^K)$. To formulate this, we introduce new definition.

Definition 3.3.1 A graded algebra $\mathcal{A}$ is *free split* if the quotient map $\pi\colon\mathcal{A}\to\mathcal{A}_f$ has a section as algebras. In other words, there is a ring morphism $s\colon\mathcal{A}_f\to\mathcal{A}$ making the diagrams

$$\begin{tikzcd}
\mathcal{A}_f \arrow[r, "s"] \arrow[dr, equal] & \mathcal{A} \arrow[d, "\pi"] \\
& \mathcal{A}_f
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathcal{A}_f\otimes\mathcal{A}_f \arrow[r, "m_f"] \arrow[d, "s\otimes s"] & \mathcal{A}_f \arrow[d, "s"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}$$

commute, where m and m_f are multiplications in $\mathcal{A}$ and $\mathcal{A}_f$. We call s a *free splitting* of $\mathcal{A}$.

In general, a free splitting of $\mathcal{A}$ is not unique. Any two free splittings s_1 and s_2 differ by a torsion element.

Remark 3.4 Not all cohomology rings of spaces are free split. Let C be the mapping cone of the composite

$$P^3(2)\xrightarrow{\rho}S^3\xrightarrow{[\imath_1,\imath_2]}S^2\vee S^2,$$

where $P^3(2)$ is the mapping cone of degree map $2\colon S^2 \to S^2$, ρ is the quotient map and $[\iota_1, \iota_2]$ is the Whitehead product. Then $H^*(C) \cong \mathbb{Z}[a,b]/(a^2 = b^2 = 2ab = 0)$ where $|a| = |b| = 2$, and it is not free split.

Lemma 3.5 *Under the conditions of Proposition 3.3, if $H^*(X_i)$ is free split for all i, then $H^*((\underline{X},*)^K)$ is free split.*

Proof Use the notations in the proof of Proposition 3.3. Let $s_i : H_f^*(X_i) \to H^*(X_i)$, for $1 \le i \le m$, be a free splitting and let s be the composite

$$s\colon \bigotimes_{i=1}^m H_f^*(X_i) \xrightarrow{\bigotimes_{i=1}^m s_i} \bigotimes_{i=1}^m H^*(X_i) \xrightarrow{\mu} H^*\Bigl(\prod_{i=1}^m X_i\Bigr).$$

Then s is a free splitting of $H^*\bigl(\prod_{i=1}^m X_i\bigr)$. As $\iota_f^*\colon H_f^*\bigl(\prod_{i=1}^m X_i\bigr) \to H_f^*((\underline{X},*)^K)$ is surjective, define $s'\colon H_f^*((\underline{X},*)^K) \to H^*((\underline{X},*)^K)$ by the diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^m H_f^*(X_i) \arrow[r, "s"] \arrow[d, "\iota_f^*"] & H^*\bigl(\prod_{i=1}^m X_i\bigr) \arrow[d, "\iota^*"] \\
H_f^*((\underline{X},*)^K) \arrow[r, "s'"] & H^*((\underline{X},*)^K)
\end{tikzcd}$$

We need to show that s' is well defined. For $x \in H_f^*(\underline{X},*)^K$, let $y, y' \in \bigotimes_{i=1}^m H_f^*(X_i)$ be two preimages of x. Then $y - y' \in \ker(\iota_f^*) = I_K$. For $J \notin K$, s sends $\widetilde{H}_f^*(\underline{X})^{\otimes J}$ to $\mu(\widetilde{H}^*(\underline{X})^{\otimes J})$ which is contained in $\ker(\iota^*)$. So $\iota^* \circ s(y - y') = 0$ and s' is well defined. Since s, ι^* and ι_f^* are multiplicative, so is s'. Furthermore, s' is right inverse to the quotient map $H^*((\underline{X},*)^K) \to H_f^*((\underline{X},*)^K)$. So s' is a free splitting. □

4 Realization of graded monomial ideal rings

We follow the idea of [3] and prove Theorem 1.2 in several steps. In Section 4.1 we use Proposition 3.3 to prove the special case where the ideal I of A is square-free. In Sections 4.2 and 4.3 we construct a fibration sequence inspired by algebraic polarization method and show that the fiber X_A is a realization modulo torsion of A. More precisely, we apply the Eilenberg–Moore spectral sequence defined in Section 2.2 to calculate $H_f^*(X_A)$ and give the E_∞–page by the end of this section. The extension problem is long and complicated and will be discussed in Section 5.

4.1 Quotient rings of square-free ideals

Let $P = \mathbb{Z}[x_1, \dots, x_m] \otimes \Lambda[y_1, \dots, y_n]$ be a graded polynomial ring where the x_i's have arbitrary positive even degrees and the y_j's have arbitrary positive odd degrees, and let $I = (M_1, \dots, M_r)$ be an ideal generated by monomials

$$M_j = x_1^{a_{1j}} \cdots x_m^{a_{mj}} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}},$$

where the a_{ij}'s are nonnegative integers and the b_{ij}'s are either 0 or 1. Then $A = P/I$ is a *graded monomial ideal ring*. We say that I is *square-free* if the M_j's are square-free monomials, that is all a_{ij}'s are either 0 or 1.

In the following let

- $\{i_1, \dots, i_k\} + \{j_1, \dots, j_l\} = \{i_1, \dots, i_k, j_1+m, \dots, j_l+m\}$ for $\{i_1, \dots, i_k\} \subset [m]$ and $\{j_1, \dots, j_l\} \subset [n]$, and
- $\underline{X} + \underline{Y} = \{X_1, \dots, X_m, Y_1 \dots, Y_n\}$ for sequences of spaces $\underline{X} = \{X_i\}_{i=1}^m$ and $\underline{Y} = \{Y_j\}_{j=1}^n$.

Given a square-free ideal I of A, take K to be a simplicial complex on $[m+n]$ by removing faces $\{i_1, \dots, i_k\} + \{j_1, \dots, j_l\}$ whenever $x_{i_1} \cdots x_{i_k} \otimes y_{j_1} \cdots y_{j_l} \in I$. Then I is the generalized Stanley–Reisner ideal of K.

Lemma 4.1 *Let $\underline{X} = \{K(\mathbb{Z}, |x_i|)\}_{i=1}^m$ and $\underline{Y} = \{S^{|y_j|}\}_{j=1}^n$ and let K be the simplicial complex defined as above. Then there is a ring isomorphism $H_f^*((\underline{X} + \underline{Y}, *)^K) \cong A$. Furthermore, $H^*((\underline{X} + \underline{Y}, *)^K)$ is free split.*

Proof Since $H_f^*(X_i) \cong \mathbb{Z}[x_i]$ and $H^*(Y_j) \cong \Lambda[y_j]$, the first part follows from Proposition 3.3.

For the second part, it suffices to show that $H^*(X_i)$ and $H^*(Y_j)$ are free split by Lemma 3.5. For $1 \le j \le n$, $H^*(Y_j)$ is free and hence free split. For $1 \le i \le m$, let x_i' be a generator of $H^{|x_i|}(X_i) \cong \mathbb{Z}$. Then inclusion $\iota\colon \mathbb{Z}\langle x_i' \rangle \to H^*(X_i)$ extends to a ring morphism

$$s\colon \mathbb{Z}[x_i'] \cong H_f^*(X_i) \to H^*(X_i).$$

Let $\pi\colon H^*(X_i) \to H_f^*(X_i)$ be the quotient map. Since $\pi \circ \iota$ sends x_i' to itself, by the universal property $\pi \circ s$ is the identity map. So s is a free splitting of $H^*(X_i)$. □

4.2 Polarization of graded monomial ideal rings

Now drop the square-free assumption on $I = (x_1^{a_{1j}} \cdots x_m^{a_{mj}} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}} \mid 1 \le j \le r)$ and suppose some a_{ij}'s are greater than 1. Following ideas from [3] and [15], we use polarization to reduce the realization problem of A to the special case when I is square-free.

For $1 \le i \le m$, let $a_i = \max\{a_{i1}, \dots, a_{ir}\}$ be the largest index of x_i among the M_j's, and let

$$\Omega = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i \le m, 1 \le j \le a_i\}$$

where $(i, j) \in \Omega$ are ordered in left lexicographical order, that is $(i, j) < (i', j')$ if $i < i'$, or if $i = i'$ and $j < j'$. Let

$$\begin{aligned} P' &= \mathbb{Z}[x_{ij} \mid (i, j) \in \Omega] \otimes \Lambda[y_1, \dots, y_n] \\ &= \mathbb{Z}[x_{11}, \dots, x_{1a_1}, x_{21}, \dots, x_{2a_2}, \dots, x_{m1}, \dots, x_{ma_m}] \otimes \Lambda[y_1, \dots, y_n], \end{aligned}$$

be a graded polynomial ring where $|x_{ij}| = |x_i|$, let

$$M'_j = (x_{11}x_{12} \cdots x_{1a_{1j}})(x_{21}x_{22} \cdots x_{2a_{2j}}) \cdots (x_{m1}x_{m2} \cdots x_{ma_{mj}}) \otimes (y_1^{b_{1j}} \cdots y_n^{b_{nj}})$$

and let $I' = (M'_1, \dots, M'_r)$. Then I' is square-free and $A' = P'/I'$ is called the *polarization* of A.

Conversely, we can reverse the polarization process and obtain A back from A'. Let

$$\overline{\Omega} = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i \le m, 2 \le j \le a_i\}$$

where $(i, j) \in \overline{\Omega}$ are ordered in left lexicographical order, and let W be a graded polynomial ring

$$W = \mathbb{Z}[w_{ij} | (i, j) \in \overline{\Omega}] = \mathbb{Z}[w_{12}, \dots, w_{1a_1}, w_{22}, \dots, w_{2a_2}, \dots, w_{m2}, \dots, w_{ma_m}],$$

where $|w_{ij}| = |x_i|$. Define a ring morphism $\delta\colon W \to P'$ by $\delta(w_{ij}) = x_{ij} - x_{i1}$ and make P' a W–module via δ. Then A' is a W–module and $A \cong A'/\overline{W} A'$, where $\overline{W} = \{W^i\}_{i>0}$.

Lemma 4.2 *Let A' be a square-free graded monomial ideal ring and let W and δ be defined as above. Then A' is a free W–module.*

Proof Since $A'/\overline{W} A'$ is a free $\mathbb{Z}$–module, by Lemma 2.10 it suffices to show that $\{w_{ij}\}_{(i,j)\in\overline{\Omega}}$ is a A'–regular sequence. Set $N = |\overline{\Omega}| = \sum_{i=1}^m a_i - m$. For $1 \le k \le N$, let $(i_k, j_k) \in \overline{\Omega}$ be the k^{th} pair under lexicographical order and let

$$I_k = (w_{12}, w_{13}, \dots, w_{i_k j_k}).$$

We need to show that multiplication $w_{i_{k+1}j_{k+1}}\colon A'/I_kA' \to A'/I_kA'$ is injective. Observe that $A'/I_kA' = \widetilde{P}/\tilde{I}$, where

$$\widetilde{P} = \mathbb{Z}[x_{11}, x_{21}, \dots, x_{m1}, x_{i_{k+1}j_{k+1}}, x_{i_{k+2}j_{k+2}}, \dots x_{i_Nj_N}] \otimes \Lambda[y_1, \dots, y_n]$$

and $\tilde{I} = (\widetilde{M}_1, \dots, \widetilde{M}_r)$ is generated by monomials $\widetilde{M}_j$ obtained by identifying x_{ij} with x_{i1} in M'_j for $(i,j) \le (i_k, j_k)$. Suppose there is a polynomial $p \in \widetilde{P}$ such that

$$(x_{i_{k+1}j_{k+1}} - x_{i_{k+1}1}) \cdot p \in \tilde{I}.$$

We can use the combinatorial argument of [8, page 31] to show $p \in \tilde{I}$. Here is an outline of the argument. Write $p = \sum_\alpha p_\alpha$ as a sum of monomials p_α. For each monomial p_α, it can be shown that $x_{i_{k+1}j_{k+1}} p_\alpha$ and $x_{i_{k+1}1} p_\alpha$ are in $\tilde{I}$. Counting the indices of variables implies $p_\alpha \in \tilde{I}$. So p is in $\tilde{I}$ and multiplication $w_{i_{k+1}j_{k+1}}\colon A'/I_kA' \to A'/I_kA'$ is injective. Therefore $\{w_{ij}\}_{(i,j)\in\overline{\Omega}}$ is a regular sequence and A' is a free W–module. □

4.3 Constructing a realization modulo torsion X_A

Let $A' = P'/I'$ be the polarization of A and let K be a simplicial complex on $\left(\sum_{i=1}^m a_i + n\right)$ vertices such that I' is the generalized Stanley–Reisner ideal of K. Construct a polyhedral product to realize A'. Take

$$\begin{aligned}\underline{X} &= \{X_{ij} = K(\mathbb{Z}, |x_i|) \mid (i,j) \in \Omega\}\\ &= \{\underbrace{K(\mathbb{Z}, |x_1|), \dots, K(\mathbb{Z}, |x_1|)}_{a_1}, \underbrace{K(\mathbb{Z}, |x_2|), \dots, K(\mathbb{Z}, |x_2|)}_{a_2},\\ &\qquad\qquad \dots, \underbrace{K(\mathbb{Z}, |x_m|), \dots, K(\mathbb{Z}, |x_m|)}_{a_m}\}\end{aligned}$$

and

$$\underline{Y} = \{Y_k = S^{|y_k|} \mid 1 \le k \le n\} = \{S^{|y_1|}, S^{|y_2|}, \dots, S^{|y_n|}\}.$$

By Lemma 4.1, $H^*_f((\underline{X} + \underline{Y}, *)^K)$ is isomorphic to A'.

For $1 \le i \le m$, define $\delta_i\colon \prod_{j=1}^{a_i} X_{ij} \to \prod_{j=2}^{a_i} X_{ij}$ by

$$\delta_i(u_1, \dots, u_{a_i}) = (u_2 \cdot u_1^{-1}, \dots, u_{a_i} \cdot u_1^{-1}),$$

and define $\delta\colon (\underline{X} + \underline{Y}, *)^K \to \prod_{(i,j)\in\overline{\Omega}} X_{ij}$ to be the composite

$$\delta\colon (\underline{X} + \underline{Y}, *)^K \hookrightarrow \prod_{(i,j)\in\Omega} X_{ij} \times \prod_{k=1}^{n} Y_k \xrightarrow{\text{proj}} \prod_{(i,j)\in\Omega} X_{ij} \xrightarrow{\prod_{i=1}^m \delta_i} \prod_{(i,j)\in\overline{\Omega}} X_{ij}.$$

As δ is a fibration, take X_A to be its fiber. We claim that $H^*_f(X_A) \cong A$.

Notation 4.3 Let $\{E_r^{*,*}\}_{r=0}^{\infty}$ be the Eilenberg–Moore spectral sequence defined in Section 2.2 on the fibration sequence

$$X_A \to (\underline{X}+\underline{Y},*)^K \xrightarrow{\delta} \prod_{(ij)\in\overline{\Omega}} X_{ij}, \tag{5}$$

where $H^*((\underline{X}+\underline{Y},*)^K)$ is an $H^*\bigl(\prod_{(ij)\in\overline{\Omega}} X_{ij}\bigr)$–module via δ^*.

Lemma 4.4 *For the E_∞–page, $(E_\infty^{0,q})_f \cong A^q$ as modules and $(E_\infty^{-p,q})_f = 0$ for $p\neq 0$.*

Proof The E_2–page is given by $E_2^{-p,*} = \mathrm{Tor}^{-p,*}_{H^*(\prod_{(ij)\in\overline{\Omega}} X_{ij})}(H^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z})$. By Lemma 2.6, there is a monomorphism

$$\pi'\colon (E_2^{-p,*})_f \to \bigl(\mathrm{Tor}^{-p,*}_{H_f^*(\prod_{(ij)\in\overline{\Omega}} X_{ij})}(H_f^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z})\bigr)_f,$$

which is an isomorphism for $p=0$. By Lemmas 3.2 and 3.3, $H_f^*((\underline{X}+\underline{Y},*)^K)\cong A'$ and

$$H_f^*\biggl(\prod_{(ij)\in\overline{\Omega}} X_{ij}\biggr) \cong \mathbb{Z}[w_{12},\dots,w_{1a_1},w_{22},\dots,w_{2a_2},\dots,w_{m2},\dots,w_{ma_m}].$$

Denote $H_f^*\bigl(\prod_{(ij)\in\overline{\Omega}} X_{ij}\bigr)$ by W. So A' is a W–module via δ^*. By Lemma 4.2, A' is a free W–module, so

$$\mathrm{Tor}_W^{-p,q}(A',\mathbb{Z}) \cong \begin{cases} A^q & \text{if } p=0,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $(E_2^{-p,q})_f$ is A^q for $p=0$ and is zero otherwise.

Suppose $(E_r^{-p,q})_f$ is A^q for $p=0$ and is zero otherwise. Since $(E_r^{-p,*})_f$ is concentrated in the column $p=0$, any differentials d_r in and out of torsion-free elements are trivial. So we have $\ker(d_r)_f = (E_r^{-p,q})_f$ and $\mathrm{Im}(d_r)_f = 0$. By Lemma 2.2, $(E_{r+1}^{-p,q})_f \cong (E_r^{-p,q})_f$. Therefore $(E_\infty^{-p,q})_f$ is isomorphic to A^q for $p=0$ and is zero otherwise. □

Lemma 4.5 *There is an additive isomorphism $H_f^q(X_A)\cong A^q$.*

Proof Since the Eilenberg–Moore spectral sequence strongly converges to $H^*(X_A)$, for any fixed q there is a decreasing filtration $\{\mathscr{F}^{-p}\}$ of $H^q(X_A)$ such that

$$\mathscr{F}^{-\infty} = H^q(X_A),\quad \mathscr{F}^1 = 0,\quad E_\infty^{-p,p+q} \cong \mathscr{F}^{-p}/\mathscr{F}^{-p+1}.$$

By Lemma 2.2, $(E_\infty^{-p,p+q})_f \cong \bigl((\mathscr{F}^{-p})_f/(\mathscr{F}^{-p+1})_f\bigr)_f$. By Lemma 4.4, $(E_\infty^{-p,p+q})_f$ is zero unless $p=0$, so $H_f^q(X_A)\cong (E_\infty^{0,q})_f \cong A^q$ as modules. □

Before going to the extension problem of the E_∞–page, we consider the special case where all of the even degree generators of A are in degree 2. The following theorem refines Lemma 4.5 and shows that $H^*(X_A) \cong A$ as algebras without modding out the cohomology by torsion. This generalizes the results of Bahri, Bendersky, Cohen and Gitler [4, Theorem 2.2] and Trevisan [15, Theorem 3.6].

Theorem 4.6 *Let A be a graded monomial ideal ring where its generators have either degree 2 or arbitrary positive odd degrees. Then $H^*(X_A) \cong A$ as algebras.*

Proof The E_2–page is given by

$$E_2^{-p,*} = \operatorname{Tor}^{-p,*}_{H^*(\prod_{(ij)\in\overline{\Omega}} X_{ij})}(H^*((\underline{X}+\underline{Y},*)^K),\mathbb{Z}).$$

By hypothesis, $X_{ij} = \mathbb{CP}^\infty$ for $(i,j)\in\Omega$, and $H^*\big(\prod_{(ij)\in\overline{\Omega}} X_{ij}\big)$ and $H^*((\underline{X}+\underline{Y},*)^K$ are free. Following the argument in the proof of Lemma 4.4, $E_2^{-p,q}$ is A^q for $p=0$ and is zero otherwise. Since the E_2–page is concentrated in the column $p=0$, the spectral sequence collapses and $H^*(X_A)\cong A$ as modules.

Let $\phi\colon X_A \to (\underline{X}+\underline{Y},*)^K$ be the fiber inclusion. Lemma 2.8 implies the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "\cong"] \arrow[d, "e"] & H^*((\underline{X}+\underline{Y},*)^K) \arrow[d, "\phi^*"] \\ E_2^{*,*} \arrow[r, "\cong"] & H^*(X_A) \end{tikzcd}$$

where e is surjective. Since ϕ^* is surjective and multiplicative and its kernel is W, $H^*(X_A)\cong A'/W \cong A$ as algebras. □

5 The extension problem

In this section we continue using Notation 4.3. Lemma 4.5 shows that $H^*_f(X_A)$ and A are free $\mathbb{Z}$–modules of same rank at each degree. We claim that they are isomorphic as algebras. The idea is to construct a space Z_A related to X_A such that $H^*(Z_A)$ is free and computable. Then we define a map $g_A\colon Z_A \to X_A$ and compare $H^*(X_A)$ with $H^*(Z_A)$ via g_A^*.

Construction of Z_A For $1 \le i \le m$ let $|x_i| = 2c_i$, and let

$$\begin{aligned}\underline{Z} &= \{Z_{ij} = (\mathbb{CP}^\infty)^{c_i} \mid (i,j) \in \Omega\} \\ &= \{\underbrace{(\mathbb{CP}^\infty)^{c_1}, \dots, (\mathbb{CP}^\infty)^{c_1}}_{a_1}, \underbrace{(\mathbb{CP}^\infty)^{c_2}, \dots, (\mathbb{CP}^\infty)^{c_2}}_{a_2}, \\ &\qquad \dots, \underbrace{(\mathbb{CP}^\infty)^{c_m}, \dots, (\mathbb{CP}^\infty)^{c_m}}_{a_m}\}\end{aligned}$$

and construct the polyhedral product $(\underline{Z} + \underline{Y}, *)^K$. Fix a generator z of $H^2(\mathbb{CP}^\infty)$. For $(i,j) \in \Omega$ and $1 \le k \le c_i$, let $\pi_{ijk}\colon Z_{ij} \to \mathbb{CP}^\infty$ be the projection onto the k^{th} copy of $\mathbb{CP}^\infty$ and let $z_{ijk} = \pi_{ijk}^*(z)$. By Theorem 3.1,

$$H^*((\underline{Z} + \underline{Y}, *)^K) \cong Q'/L',$$

where $Q' = \mathbb{Z}[z_{ijk} \mid (i,j) \in \Omega, 1 \le k \le c_i] \otimes \Lambda[y_1, \dots, y_n]$ and L' is the ideal generated by monomials

$$z_{i_1 j_1 k_1} \cdots z_{i_t j_t k_t} \otimes y_{l_1} \cdots y_{l_\tau}$$

for $\{j_1 + \sum_{s=1}^{i_1 - 1} a_s, \dots, j_t + \sum_{s=1}^{i_t - 1} a_s\} + \{l_1, \dots, l_\tau\} \notin K$. For $1 \le i \le m$, define

$$\tilde{\delta}_i : \prod_{j=1}^{a_i} Z_{ij} \to \prod_{j=2}^{a_i} Z_{ij}, \quad \tilde{\delta}_i(u_1, \dots, u_{a_i}) = (u_2 \cdot u_1^{-1}, \dots, u_{a_i} \cdot u_1^{-1}),$$

and define $\tilde{\delta}\colon (\underline{Z} + \underline{Y}, *)^K \to \prod_{(i,j) \in \overline{\Omega}} Z_{ij}$ to be the composite

$$\tilde{\delta}\colon (\underline{Z} + \underline{Y}, *)^K \hookrightarrow \prod_{(i,j) \in \Omega} Z_{ij} \times \prod_{k=1}^{n} Y_k \xrightarrow{\text{proj}} \prod_{(i,j) \in \Omega} Z_{ij} \xrightarrow{\prod_{i=1}^m \tilde{\delta}_i} \prod_{(i,j) \in \overline{\Omega}} Z_{ij}.$$

Lemma 5.1 *Let Z_A be the fiber of δ'. Then $H^*(Z_A) \cong Q/L$, where*

$$Q = \mathbb{Z}[z_{ik} \mid 1 \le i \le m, 1 \le k \le c_i] \otimes \Lambda[y_1, \dots, y_n]$$

with $|z_{ik}| = 2$ and L is generated by monomials $z_{i_1 k_1} \cdots z_{i_N k_N} \otimes y_1^{b_{1j}} \cdots y_n^{b_{nj}}$ satisfying

$$1 \le j \le r, \quad 1 \le k_l \le c_{i_l} \quad \text{and} \quad (i_1, \dots, i_N) = (\underbrace{1, \dots, 1}_{a_{1j}}, \underbrace{2, \dots, 2}_{a_{2j}}, \dots, \underbrace{m, \dots, m}_{a_{mj}}).$$

Proof Apply the Eilenberg–Moore spectral sequence to the fibration sequence

$$Z_A \to (\underline{Z} + \underline{Y}, *)^K \xrightarrow{\tilde{\delta}} \prod_{(i,j) \in \overline{\Omega}} Z_{ij}.$$

The E_2–page is given by $\widetilde{E}_2^{-p,*} = \mathrm{Tor}^{-p,*}_{H^*(\prod_{(i,j)\in\overline{\Omega}} Z_{ij})}(\mathbb{Z}, H^*((\underline{Z}+\underline{Y},*)^K))$. By the Künneth theorem,

$$H^*\Bigl(\prod_{(i,j)\in\overline{\Omega}} Z_{ij}\Bigr) \cong \mathbb{Z}[v_{ijk} \mid (i,j)\in\overline{\Omega}, 1\le k\le c_i],$$

where $|v_{ijk}|=2$. Denote $H^*(\prod_{(i,j)\in\overline{\Omega}} Z_{ij})$ by V. By definition $\tilde{\delta}^*(v_{ijk}) = z_{ijk}-z_{i1k}$. This gives an action of V on Q'. By Lemma 4.2, Q'/L' is a free V–module, so

$$\mathrm{Tor}_V^{-p,*}(Q'/L',\mathbb{Z}) = \begin{cases} (Q'/L')/(z_{ijk}-z_{ilk}) & \text{if } p=0,\\ 0 & \text{otherwise.}\end{cases}$$

Modding out $(z_{ijk}-z_{ilk})$ identifies z_{ijk} with z_{ilk} in Q'/L', so

$$(Q'/L')/(z_{ijk}-z_{ilk}) \cong Q/L.$$

Since the E_2–page is concentrated in the column $p=0$, $H^*(Z_A)\cong Q/L$.

Lemma 2.8 implies a commutative diagram

$$\begin{tikzcd} Q' \arrow[r, "\cong"] \arrow[d, "e"] & H^*((\underline{Z}+\underline{Y})^K) \arrow[d, "\phi^*"] \\ E_2^{*,*} \arrow[r, "\cong"] & H^*(Z_A) \end{tikzcd}$$

where e is surjective and ϕ^* is induced by the fiber inclusion $\phi\colon Z_A\to(\underline{Z}+\underline{Y})^K$. This implies ϕ^* is surjective. Since ϕ^* is multiplicative, $H^*(Z_A)\cong Q/L$ as algebras. □

Construction of g_A Fix a generator $z\in H^2(\mathbb{CP}^\infty)$. Let $\pi_j\colon(\mathbb{CP}^\infty)^{c_i}\to\mathbb{CP}^\infty$, for $1\le j\le c_i$, be the projection onto the j^{th} copy of $\mathbb{CP}^\infty$ and let $z_j=\pi_j^*(z)$. For $1\le i\le m$, take a map $g_i\colon(\mathbb{CP}^\infty)^{c_i}\to K(\mathbb{Z},2c_i)$ that represents the cocycle class $z_1\cdots z_{c_i}\in H^{2c_i}((\mathbb{CP}^\infty)^{c_i})$. For $(i,j)\in\Omega$, let $g_{ij}\colon Z_{ij}\to X_{ij}$ be g_i, and for $1\le k\le n$, let $h_k\colon Y_k\to Y_k$ be the identity map. Then $\{g_{ij},h_k\mid(i,j)\in\Omega, 1\le k\le n\}$ induces a map $g_K\colon(\underline{Z}+\underline{Y},*)^K\to(\underline{X}+\underline{Y},*)^K$ by the functoriality of polyhedral products.

Lemma 5.2 *Let $\{x_{ij},y_k\mid(i,j)\in\Omega, 1\le k\le n\}$ be generators of*

$$H_f^*((\underline{X}+\underline{Y},*)^K)\cong P'/I'$$

and $\{z_{ijl},y'_k\mid(i,j)\in\Omega, 1\le l\le c_i, 1\le k\le n\}$ be generators of

$$H^*((\underline{Z}+\underline{Y},*)^K)\cong Q'/L'.$$

Then $(g_K^*)_f(x_{ij})=\prod_{l=1}^{c_i} z_{ijl}$ and $(g_K^*)_f(y_k)=y'_k$.

Proof There is a commutative diagram

$$\begin{array}{ccc} (\underline{Z}+\underline{Y},*)^K & \xrightarrow{\jmath} & \prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k \\ \Big\downarrow{\scriptstyle g_K} & & \Big\downarrow{\scriptstyle g} \\ (\underline{X}+\underline{Y},*)^K & \xrightarrow{\imath} & \prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k \end{array}$$

where $\imath$ and $\jmath$ are inclusions, and $g=\prod_{(i,j)\in\Omega} g_{ij}\times\prod_{k=1}^n h_k$. Taking cohomology and modding out torsion elements, we obtain the commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{\imath_f^*} & P'/I' \\ {\scriptstyle g_f^*}\Big\downarrow & & \Big\downarrow{\scriptstyle (g_K^*)_f} \\ Q' & \xrightarrow{\jmath^*} & Q'/L' \end{array}$$

where $\imath_f^*$ and $\jmath^*$ are the quotient maps. Let $\tilde{x}_{ij},\tilde{y}_k\in P'$ and $\tilde{z}_{ijl},\tilde{y}'_k\in Q'$ be generators such that $\imath_f^*(\tilde{x}_{ij})=x_{ij}$, $\imath_f^*(\tilde{y}_k)=y_k$, $\jmath_f^*(\tilde{y}'_k)=y'_k$ and $\jmath_f^*(\tilde{z}_{ijl})=z_{ijl}$. By construction $g_f^*(\tilde{x}_{ij})=\prod_{l=1}^{c_i}\tilde{z}_{ijl}$ and $g_f^*(\tilde{y}_k)=\tilde{y}'_k$, so we have $(g_K^*)_f(x_{ij})=\prod_{l=1}^{c_i} z_{ijl}$ and $(g_K^*)_f(y_k)=y'_k$. □

Lemma 5.3 *There is a map $g_A\colon Z_A\to X_A$ making the diagram*

$$\begin{array}{ccc} Z_A & \longrightarrow & (\underline{Z}+\underline{Y},*)^K \\ \Big\downarrow{\scriptstyle g_A} & & \Big\downarrow{\scriptstyle g_K} \\ X_A & \longrightarrow & (\underline{X}+\underline{Y},*)^K \end{array}$$

commute, where the horizontal maps are the inclusion maps.

Proof One may want to construct g_A by showing the diagram

$$\begin{array}{ccc} (\underline{Z}+\underline{Y},*)^K & \xrightarrow{\bar{\delta}} & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\ \Big\downarrow{\scriptstyle g_K} & & \Big\downarrow{\scriptstyle \prod_{(i,j)\in\overline{\Omega}} g_{ij}} \\ (\underline{X}+\underline{Y},*)^K & \xrightarrow{\delta} & \prod_{(i,j)\in\overline{\Omega}} X_{ij} \end{array}$$

commutes. However, as $\left(\prod_{(i,j)\in\overline{\Omega}} g_{ij}\right)\circ\bar{\delta}$ and $\delta\circ g_K$ induce different morphisms on cohomology, the diagram cannot commute. Instead, we show that the composite

$$Z_A\to(\underline{Z}+\underline{Y},*)^K\xrightarrow{g_K}(\underline{X}+\underline{Y},*)^K\xrightarrow{\delta}\prod_{(i,j)\in\overline{\Omega}} X_{ij}$$

is trivial. If so, there will exist a map $g_A\colon Z_A\to X_A$ as asserted since X_A is the fiber of δ.

By definition of $\bar{\delta}$ there is a commutative diagram

$$\begin{tikzcd}
(\underline{Z}+\underline{Y},*)^K \arrow[r, "\tilde{\delta}"] \arrow[d, "\jmath"] & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \\
\prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k \arrow[r, "\mathrm{proj}"] & \prod_{(i,j)\in\Omega} Z_{ij} \arrow[u, "\prod_{i=1}^m \tilde{\delta}_i"']
\end{tikzcd}$$

where $\jmath$ is the inclusion. Denote $\left(\prod_{i=1}^m \tilde{\delta}_i\right)\circ\mathrm{proj}$ by $\tilde{\delta}'$ and extend the diagram to

$$\begin{tikzcd}
Z_A \arrow[r] \arrow[d, "e"] & (\underline{Z}+\underline{Y},*)^K \arrow[r, "\tilde{\delta}"] \arrow[d, "\jmath"] & \prod_{(i,j)\in\overline{\Omega}} Z_{ij} \arrow[d, equal] \\
\prod_{i=1}^m(\mathbb{CP}^\infty)^{c_i}\times\prod_{k=1}^n Y_k \arrow[r, "\triangle'\times h"] & \prod_{(i,j)\in\Omega} Z_{ij}\times\prod_{k=1}^n Y_k \arrow[r, "\tilde{\delta}'"] & \prod_{(i,j)\in\overline{\Omega}} Z_{ij}
\end{tikzcd}$$

where $\triangle'\colon\prod_{i=1}^m(\mathbb{CP}^\infty)^{c_i}\to\prod_{j=1}^{a_i} Z_{ij}$ is the diagonal map, $h\colon\prod_{k=1}^n Y_k\to\prod_{k=1}^n Y_k$ is the identity map, and e is an induced map. The top and the bottom row are fibration sequences. The left square fits into the commutative diagram

$$\begin{tikzcd}
Z_A \arrow[r] \arrow[d, "e"] & (\underline{Z}+\underline{Y},*)^K \arrow[r, "g_K"] \arrow[d, "\jmath"] & (\underline{X}+\underline{Y},*)^K \arrow[r, "\delta"] \arrow[d, "\imath"] & \prod_{\overline{\Omega}} X_{ij} \arrow[d, equal] \\
\prod_{i=1}^m(\mathbb{CP}^\infty)^{c_i}\times\prod_{j=1}^n Y_j \arrow[r, "\triangle'\times h"] \arrow[dr, "\prod g_i\times h"'] & \prod_{\Omega} Z_{ij}\times\prod_{j=1}^n Y_j \arrow[r, "\prod g_{ij}\times h"] & \prod_{\Omega} X_{ij}\times\prod_{k=1}^n Y_k \arrow[r, "\delta'"] & \prod_{\overline{\Omega}} X_{ij} \\
& \prod_{i=1}^m K(\mathbb{Z},|x_i|)\times\prod_{j=1}^n Y_j \arrow[ur, "\triangle\times h"'] & &
\end{tikzcd}$$

where $\imath$ is the inclusion, $\triangle\colon\prod_{i=1}^m K(\mathbb{Z},|x_i|)\to\prod_{j=1}^{a_i} X_{ij}$ is the diagonal map, and δ' is the composite

$$\delta'\colon\prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k\xrightarrow{\mathrm{proj}}\prod_{(i,j)\in\Omega} X_{ij}\xrightarrow{\prod_{i=1}^m\delta_i}\prod_{(i,j)\in\overline{\Omega}} X_{ij}.$$

The middle square is due to the functoriality of polyhedral products, the right square is due to the definition of δ and the bottom triangle is due to the naturality of diagonal maps.

The composite of maps from Z_A to $\prod_{(i,j)\in\overline{\Omega}} X_{ij}$ round the bottom triangle is trivial since

$$\prod_{i=1}^m K(\mathbb{Z},|x_i|)\times\prod_{k=1}^n Y_k\xrightarrow{\triangle\times h}\prod_{(i,j)\in\Omega} X_{ij}\times\prod_{k=1}^n Y_k\xrightarrow{\delta'}\prod_{(i,j)\in\overline{\Omega}} X_{ij}$$

is a fibration sequence. So the composite in the top row is trivial and this induces a map $g_A\colon Z_A \to X_A$ as asserted. $\square$

Since $g_K^*\colon H^*((\underline{X}+\underline{Y})^K) \to H^*((\underline{Z}+\underline{Y})^K)$ is multiplicative and $H^*(Z_A)$ is a quotient algebra of $H^*((\underline{Z}+\underline{Y})^K)$, we use g_A to compare $H^*(X_A)$ and $H^*(Z_A)$ and show that $H_f^*(X_A)$ is a quotient algebra of $H_f^*((\underline{X}+\underline{Y})^K)$.

Lemma 5.4 *Let $\phi\colon X_A \to ((\underline{X}+\underline{Y},*)^K)$ be the inclusion. Then the induced morphism*

$$\phi_f^*\colon H_f^*((\underline{X}+\underline{Y},*)^K) \to H_f^*(X_A)$$

is surjective and $\ker(\phi_f^*)$ is generated by $x_{ij}-x_{i1}$ for $(i,j)\in\overline{\Omega}$.

Proof Fix a positive integer q and let $\psi\colon Z_A \to (\underline{Z}+\underline{Y},*)^K$ be the inclusion. Consider the commutative diagram

$$\begin{tikzcd} & H^q((\underline{X}+\underline{Y},*)^K) \arrow[r, "g_K^*"] \arrow[dl, "e"'] \arrow[d, "\phi^*"] & H^q((\underline{Z}+\underline{Y},*)^K) \arrow[d, "\psi^*"] \\ E_\infty^{0,q} \arrow[r, "h"] & H^q(X_A) \arrow[r, "g_A^*"] & H^q(Z_A) \end{tikzcd}$$

where e is surjective and h is injective. The left triangle commutes due to Lemma 2.8 and the right square commutes due to Lemma 5.3. Mod out torsion elements and take a generator

$$x_{i_1j_1}\cdots x_{i_sj_s}\otimes y_{l_1}\cdots y_{l_t}\in H_f^q((\underline{X}+\underline{Y},*)^K).$$

By Lemma 5.2 and the above diagram,

$$(g_A^*\circ h\circ e)_f(x_{i_1j_1}\cdots x_{i_sj_s}\otimes y_{l_1}\cdots y_{l_t}) = (\psi^*\circ g_K^*)_f(x_{i_1j_1}\cdots x_{i_sj_s}\otimes y_{l_1}\cdots y_{l_t}),$$

$$(g_A^*\circ h)_f(x_{i_1}\cdots x_{i_s}\otimes y_{l_1}\cdots y_{l_t}) = \left(\prod_{u=1}^{s}\prod_{k=1}^{c_{i_u}} z_{i_uj_uk}\right)\otimes y_{l_1}\cdots y_{l_t}.$$

Since $x_{i_1}\cdots x_{i_s}\otimes y_{l_1}\cdots y_{l_t}$ and $\left(\prod_{u=1}^{s}\prod_{k=1}^{c_{i_u}} z_{i_uj_uk}\right)\otimes y_{l_1}\cdots y_{l_t}$ are generators, $(g_A\circ h)_f^*$ is the inclusion of a direct summand into $H_f^q(Z_A)$. By Lemma 4.4, $(E_\infty^{0,q})_f$ and $H_f^q(X_A)$ are free modules of same rank, so h_f is an isomorphism. Since e_f is a surjection, so is ϕ_f^*.

For the second part of the lemma, suppose there is a polynomial $p\in\ker(\phi_f^*)$ not contained in $(x_{ij}-x_{i1})_{(i,j)\in\overline{\Omega}}$. Since ϕ_f^* is a degree 0 morphism, we assume $p=\sum_\alpha p_\alpha$ is a sum of monomials p_α of some fixed degree q. Then the p_α's are linearly dependent. So the rank of $H_f^q(X_A)$ is less than the rank of A^q, contradicting to Lemma 4.4. Thus $\ker(\phi_f^*)=(x_{ij}-x_{i1})_{(i,j)\in\overline{\Omega}}$. $\square$

Next we restate our main theorem (Theorem 1.2) and prove it.

Theorem 5.5 *Let A be a graded monomial ideal ring. Then there exists a space X_A such that $H_f^*(X_A)$ is ring isomorphic to A. Moreover, $H^*(X_A)$ is free split.*

Proof For the first part of the statement, the ring isomorphism $H_f^*(X_A) \cong A$ follows from Lemma 5.4.

In Lemma 4.1 we construct a free splitting $s_K\colon H_f^*(\underline{X}+\underline{Y},*)^K \to H^*(\underline{X}+\underline{Y},*)^K$ out of free splittings $s_{ij}\colon H_f^*(X_{ij}) \to H^*(X_{ij})$ and the identity maps on $H^*(Y_k)$. Define a map $s\colon H_f^*(X_A) \to H^*(X_A)$ by

$$\begin{tikzcd}
H_f^*((\underline{X}+\underline{Y},*)^K) \arrow[r, "s_K"] \arrow[d, "\phi_f^*"] & H^*((\underline{X}+\underline{Y},*)^K) \arrow[d, "\phi^*"] \\
H_f^*(X_A) \arrow[r, "s"] & H^*(X_A)
\end{tikzcd}$$

We need to show that s is well defined. By Lemma 5.4, ϕ_f^* is a surjection and $\ker(\phi_f^*)$ is generated by polynomials $x_{ij}-x_{i1}$ for $(i,j)\in\overline{\Omega}$. It suffices to show that $\phi^*\circ s_K(x_{ij}-x_{i1})=0$. Let $\tilde{x}_{ij}\in H^{2c_i}(X_{ij})$ and $\tilde{x}'_{ij}\in H_f^{2c_i}(X_{ij})$ be generators such that $s_{ij}(\tilde{x}'_{ij})=\tilde{x}_{ij}$. There is a string of equations

$$\begin{aligned}
\phi^*\circ s_K(x_{ij}-x_{i1}) &= \phi^*\circ\mu(s_{ij}(\tilde{x}'_{ij})-s_{i1}(\tilde{x}'_{i1}))\\
&= \phi^*\circ\mu(\tilde{x}_{ij}-\tilde{x}_{i1})\\
&= \phi^*\circ\delta^*\circ\mu(1\otimes\cdots\otimes\tilde{x}_{ij}\otimes\cdots\otimes 1)\\
&= 0,
\end{aligned}$$

where the first line is due to the definition of s_K, the third line is due to the naturality of μ, and the last line is due to the fact that δ and ϕ are two consecutive maps in the fibration sequence $X_A \xrightarrow{\phi} (\underline{X}+\underline{Y},*)^K \xrightarrow{\delta} \prod_{(i,j)\in\overline{\Omega}} X_{ij}$. So s is well defined.

Obviously s is right inverse to the quotient map $H^*(X_A)\to H_f^*(X_A)$. Since ϕ_f^*, ϕ^* and s_K are multiplicative, so is s. Therefore s is a free splitting. □

6 An example

Now we illustrate how to construct X_A for $A=\mathbb{Z}[x]\otimes\Lambda[y]/(x^2y)$, where $|x|=4$ and $|y|=1$. First, polarize A by introducing two new variables x_1 and x_2 of degree 4 and let

$$A' = \mathbb{Z}[x_1, x_2] \otimes \Lambda[y]/(x_1 x_2 y).$$

Let K be the boundary of a 2–simplex. Then $(x_1 x_2 y)$ is the Stanley–Reisner ideal of K. Take

$$\underline{X} = \{K(\mathbb{Z},4), K(\mathbb{Z},4)\}, \quad \underline{Y} = \{S^1\}$$

and construct polyhedral product $(\underline{X} + \underline{Y}, *)^K$. By Proposition 3.3,

$$H_f^*((\underline{X} + \underline{Y}, *)^K) \cong \mathbb{Z}[x_1, x_2] \otimes \Lambda[y]/(x_1 x_2 y).$$

Define $\delta\colon (\underline{X} + \underline{Y}, *)^K \to K(\mathbb{Z},4)$ by $\delta_1(u_1, u_2, t) = u_2 \cdot u_1^{-1}$, and define X_A to be the fiber of δ. By Theorem 5.5, $H_f^*(X_A) \cong A$.

Next, we construct Z_A and g_A to illustrate the proof of the extension problem. In this case, take $\underline{Z} = \{(\mathbb{CP}^\infty)^2, (\mathbb{CP}^\infty)^2\}$. Denote the first $(\mathbb{CP}^\infty)^2$ by Z_1 and the second $(\mathbb{CP}^\infty)^2$ by Z_2. Then $H^*(Z_1) = \mathbb{Z}[z_{11}, z_{12}]$ and $H^*(Z_2) = \mathbb{Z}[z_{21}, z_{22}]$, where $|z_{ij}| = 2$ for $i, j \in \{1, 2\}$, and

$$H^*((\underline{Z} + \underline{Y}, *)^K) \cong \mathbb{Z}[z_{11}, z_{12}, z_{21}, z_{22}] \otimes \Lambda[y]/L',$$

where $L' = (z_{11} z_{21} y, z_{11} z_{22} y, z_{12} z_{21} y, z_{12} z_{22} y)$. Define

$$\tilde{\delta}\colon (\underline{Z} + \underline{Y}, *)^K \to (\mathbb{CP}^\infty)^2, \quad \tilde{\delta}(v_1, v_2, t) = v_2 \cdot v_1^{-1},$$

and define Z_A to be the fiber of $\tilde{\delta}$. Then $H_f^*(Z_A) \cong \mathbb{Z}[z_1, z_2] \otimes \Lambda[y]/L$, where $|z_1| = |z_2| = 2$ and $L = (z_1^2 y, z_2^2 y, z_1 z_2 y)$.

For $i = \{1, 2\}$, let $g_i : Z_i \to K(\mathbb{Z},4)$ be a map representing $z_{i1} z_{i2} \in H^4(Z_i)$, and let $h\colon S^1 \to S^1$ be the identity map. Then g_1, g_2 and h induce

$$g_K\colon (\underline{Z} + \underline{Y}, *)^K \to (\underline{X} + \underline{Y}, *)^K$$

such that $g_K^*(x_i) = z_{i1} z_{i2}$ and $g_K^*(y) = y$. Lemma 5.3 gives a map $g_A\colon Z_A \to X_A$ making the diagram

$$\begin{tikzcd}
Z_A \arrow[r] \arrow[d, "g_A"] & (\underline{Z} + \underline{Y}, *)^K \arrow[d, "g_K"] \\
X_A \arrow[r] & (\underline{X} + \underline{Y}, *)^K
\end{tikzcd}$$

commute.

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Nonslice linear combinations of iterated torus knots

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In 1976, Rudolph asked whether algebraic knots are linearly independent in the knot concordance group. We use twisted Blanchfield pairings to answer this question in the affirmative for new large families of algebraic knots.

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1 Introduction

A knot is *algebraic* if it arises as a link of an isolated singularity of a complex curve. Algebraic knots are special cases of iterated torus knots. In 1976, Rudolph [28] asked whether the set of algebraic knots is linearly independent in the knot concordance group. For ease of reference and because of the later literature on the subject, we refer to this question as a conjecture.

Conjecture 1 (Rudolph's conjecture [28]) *The set of algebraic knots is linearly independent in the smooth knot concordance group $\mathcal{C}$.*

This question has been of particular interest due to its relevance to the slice-ribbon conjecture: a result of Miyazaki shows that nontrivial linear combinations of iterated torus knots are not ribbon [26, Corollary 8.4]. In particular, if the slice-ribbon conjecture holds, then Rudolph's conjecture holds. Baker [2] and Abe and Tagami [1] recently noticed that the slice-ribbon conjecture implies a statement stronger than Rudolph's conjecture:

Conjecture 2 (Abe and Tagami [1] and Baker [2]) *The set of prime fibred strongly quasipositive knots is linearly independent in the smooth knot concordance group $\mathcal{C}$.*

This paper exhibits new large families of knots for which Conjectures 1 and 2 hold.

1.1 Statement of the results

Evidence of Rudolph's conjecture was first provided in 1979 by Litherland, who proved that positive torus knots are linearly independent in $\mathcal{C}$ [21]. In 2010, Hedden, Kirk and Livingston showed that, for an appropriate choice of positive integers $\{q_n\}_{n=1}^{\infty}$, the set $\{T(2,q_n), T(2,3;2,q_n)\}_{n=1}^{\infty}$ is linearly independent in $\mathcal{C}$, where $T(p,q)$ and $T(p,q;r,s)$ denote the (p,q)–torus knot and the (r,s)–cable of $T(p,q)$, respectively, and p is coprime to qrs. It is known that an *iterated torus knot* $T(p_1,q_1;\dots;p_k,q_k)$ is algebraic if and only if $p_i,q_i > 0$ and $q_{i+1} > q_i p_{i+1} p_i$ for each i. Our main result, which relies on metabelian twisted Blanchfield pairings — see Miller and Powell [25] and Borodzik, Conway and Politarczyk [3; 4; 5] — reads as follows:

Theorem 1.1 *Fix a prime power p. Let $\mathcal{S}_p$ be the set of iterated torus knots*

$$T(p,q_1;p,q_2;\dots;p,q_\ell),$$

where the sequences $(q_1,q_2,\dots,q_\ell)$ of positive integers satisfy

(i) *for $i=1,\dots,\ell$, the integer q_i is coprime to p;*

(ii) *q_ℓ is a prime;*

(iii) *for $i=1,\dots,\ell-1$, the integer q_i is coprime to q_ℓ when $\ell>1$.*

The set $\mathcal{S}_p$ is linearly independent in the topological knot concordance group $\mathcal{C}^{\mathrm{top}}$.

As an immediate corollary of Theorem 1.1, we obtain the following:

Corollary 1.2 *For every prime power p, the subset $\mathcal{S}_p^{\mathrm{alg}} \subset \mathcal{S}_p$ of algebraic knots in $\mathcal{S}_p$ is linearly independent in $\mathcal{C}^{\mathrm{top}}$ and therefore satisfies Conjecture 1.*

Since positively iterated torus knots are strongly quasipositive — see Hedden [11, Theorem 1.2; 13, Proposition 2.1] — Theorem 1.1 also gives infinite families of knots satisfying Conjecture 2.

Corollary 1.3 *For every prime power p, the set $\mathcal{S}_p$ satisfies Conjecture 2, and $\mathcal{S}_p \smallsetminus \mathcal{S}_p^{\mathrm{alg}}$ is an infinite family of nonalgebraic knots satisfying Conjecture 2.*

Abe and Tagami also conjecture that the set of L–space knots is linearly independent in $\mathcal{C}$ [1, Conjecture 3.4]. For a knot K with Seifert genus g, the (p,q)–cable $K_{p,q}$ is an L–space knot if and only if K is an L–space knot and $(2g-1)p \le q$; see Hedden [12] and Hom [16]. Since torus knots are L–space knots, we also obtain the following result:

Corollary 1.4 *For every prime power p, the subset $\mathcal{S}_p^L \subset \mathcal{S}_p$ of L–space knots in $\mathcal{S}_p$ is linearly independent in $\mathcal{C}^{\mathrm{top}}$, and this statement also holds for the infinite family $\mathcal{S}_p^L \smallsetminus \mathcal{S}_p^{\mathrm{alg}}$ of nonalgebraic L–space knots.*

Note however that not all our examples are L–spaces knots: since the cable of an iterated torus knot need not be an L–space knot, Corollary 1.3 shows that the infinite set $\mathcal{S}_p \smallsetminus \mathcal{S}_p^L$ contains no L–spaces knots but is nevertheless linearly independent in $\mathcal{C}^{\mathrm{top}}$.

1.2 Context and comparison with smooth techniques

Litherland used the Levine–Tristram signature to show that torus knots are linearly independent in $\mathcal{C}$ [21]. This approach is insufficient to answer Rudolph's conjecture, since Livingston and Melvin showed in [23] that the following linear combinations of iterated torus knots are algebraically slice:

(1) $$J(p,q,q_1,q_2) := T(p,q;p,q_1)\,\#-T(p,q_1)\,\#-T(p,q;p,q_2)\,\#\,T(p,q_2).$$

Classical knot invariants can thus not obstruct $J(p,q,q_1,q_2)$ from being slice.

Hedden, Kirk and Livingston managed to leverage the Casson–Gordon invariants to provide further evidence of Rudolph's conjecture [14]. Indeed, they showed that, for an appropriate choice of $\{q_n\}_{n=1}^{\infty}$, the knots $\{J(2,3,q_{2n-1},q_{2n})\}_{n=1}^{\infty}$ generate an infinite-rank subgroup in $\mathcal{C}$. This result is particularly notable since they observe that the s–invariant from Khovanov homology and the τ–invariant from Heegaard Floer homology both vanish on $J(2,3,q_{2n-1},q_{2n})$ [14, Proposition 8.2]. In fact, their argument (combined with Proposition 5.3) generalises to show that, if K is a linear combination of algebraically slice knots belonging to $\mathcal{S}_p$, then $\tau(K)=0$ and $s(K)=0$.

Next, we observe that the Upsilon invariant $\Upsilon_K\colon [0,2]\to\mathbb{R}$ from Ozsváth, Stipsicz and Szabó's knot Floer homology [27] is also insufficient to prove Theorem 1.1. First note that, if $q_1,q_2 > p(p-1)(q-1)$, then $T(p,q;p,q_i)$ is an L–space knot [12], and thus a result of Tange shows that $\Upsilon_{T(p,q;p,q_i)}(t) = \Upsilon_{T(p,q)}(pt) + \Upsilon_{T(p,q_i)}(t)$ for all $t\in[0,2]$ [29, Theorem 3]. The additivity of Υ then establishes that $\Upsilon_{J(p,q,q_1,q_2)}(t)=0$ for all $t\in[0,2]$ whenever $q_1,q_2 > p(p-1)(q-1)$.

1.3 Strategy and ingredients of the proof

The proof of Theorem 1.1 relies on Casson–Gordon theory [6; 7] — see also Kirk and Livingston [18] — and more specifically on the metabelian Blanchfield pairings introduced by Miller and Powell [25] and further developed by the first and third authors with Maciej Borodzik [3; 4; 5]. Since these invariants are somewhat technical, the next paragraphs describe some background and ideas that go into the proof of Theorem 1.1. For notational simplicity, however, we restrict ourselves to a very particular case: we apply our strategy to the knot $J(p,q,q_1,q_2)$ described in (1).

The sliceness obstruction Let p be a prime power, let $\Sigma_p(J)$ be the p–fold branched cover of the knot $J := J(p,q,q_1,q_2)$, let χ be a character on $H_1(\Sigma_p(J))$, and let M_J be the 0–framed surgery of J. Associated to these data, there is a nonsingular sesquilinear and Hermitian *metabelian Blanchfield pairing*

$$\mathrm{Bl}_{\alpha(p,\chi)}(J)\colon H_1(M_J;\mathbb{C}[t^{\pm1}]^p)\times H_1(M_J;\mathbb{C}[t^{\pm1}]^p)\to\mathbb{C}(t)/\mathbb{C}[t^{\pm1}].$$

Here $H_1(M_J;\mathbb{C}[t^{\pm1}]^p)$ denotes the homology of M_J twisted by a metabelian representation $\alpha(p,\chi)\colon \pi_1(M_J)\to \mathrm{GL}_p(\mathbb{C}[t^{\pm1}])$, whose definition will be recalled in Section 3. The precise definition of $\mathrm{Bl}_{\alpha(p,\chi)}(J)$ is irrelevant in this paper: only its properties are required. Informally, however, the pairing $\mathrm{Bl}_{\alpha(p,\chi)}(J)$ contains the information from both twisted polynomial invariants and twisted signature invariants. We now describe how $\mathrm{Bl}_{\alpha(p,\chi)}(J)$ provides a sliceness obstruction.

Let $\lambda_p(J)$ denote the $\mathbb{Q}/\mathbb{Z}$–valued linking form on $H_1(\Sigma_p(J))$. Miller and Powell show that if, for every $\mathbb{Z}_p$–invariant metaboliser G of $\lambda_p(J)$, there exists a prime power–order character χ that vanishes on G and is such that $\mathrm{Bl}_{\alpha(p,\chi)}(J)$ is not metabolic, then J is not slice [25, Theorem 6.10]. In order to make this obstruction more concrete, we now recall some terminology on linking forms and their metabolisers.

The Witt group of linking forms We focus on linking forms over $\mathbb{C}[t^{\pm1}]$, referring to Section 4 for a discussion over more general rings. A *linking form* over $\mathbb{C}[t^{\pm1}]$ is a sesquilinear Hermitian pairing $V\times V\to\mathbb{C}(t)/\mathbb{C}[t^{\pm1}]$, where V is a torsion $\mathbb{C}[t^{\pm1}]$–module. A linking form (V,λ) is *metabolic* if there is a submodule $L\subset V$ such that $L=L^\perp$; such an L is called a *metaboliser*. The *Witt group of linking forms*, denoted by $W(\mathbb{C}(t),\mathbb{C}[t^{\pm1}])$, consists of the monoid of nonsingular linking forms modulo the submonoid of metabolic linking forms. We write $\lambda_1\sim\lambda_2$ if two linking forms agree in $W(\mathbb{C}(t),\mathbb{C}[t^{\pm1}])$. The Miller–Powell obstruction to sliceness, therefore, consists of deciding whether a certain twisted Blanchfield pairing $\mathrm{Bl}_{\alpha(p,\chi)}(J)$ is zero in the group $W(\mathbb{C}(t),\mathbb{C}[t^{\pm1}])$. As we will now describe, one of our main ideas is to transfer a problem of linear independence in $\mathcal{C}^{\mathrm{top}}$ (namely Rudolph's conjecture) into a problem of linear independence in $W(\mathbb{C}(t),\mathbb{C}[t^{\pm1}])$.

From linear independence in $\mathcal{C}^{\mathrm{top}}$ to linear independence in $W(\mathbb{C}(t),\mathbb{C}[t^{\pm1}])$ Since the knot $J=T(p,q;p,q_1)\,\#\,{-T(p,q_1)}\,\#\,{-T(p,q;p,q_2)}\,\#\,T(p,q_2)$ is a connected sum of four knots, both $H_1(\Sigma_p(J))$ and $\lambda_p(J)$ can be decomposed into four direct summands:

$$\lambda_p(J) = \lambda_p(T(p,q_1)) \oplus -\lambda_p(T(p,q_1)) \oplus \lambda_p(T(p,q_2)) \oplus -\lambda_p(T(p,q_2)).$$

In particular, any character on $H_1(\Sigma_p(J))$ can be written as $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$. For each given $\mathbb{Z}_p$–invariant metaboliser M of $\lambda_p(J)$, the "sliceness-obstructing character" that we will produce will be of the form $\chi = \chi_1 \oplus \chi_2 \oplus \theta \oplus \theta$, where θ denotes the trivial character. Using the definition of J, together with the direct sum decomposition of [4, Corollary 4.21], the Witt class of the metabelian Blanchfield pairing of J is given by

$$\begin{aligned}(2)\quad \mathrm{Bl}_{\alpha(p,\chi)}(J) \sim \mathrm{Bl}_{\alpha(p,\chi_1)}(T(p,q;p,q_1)) \oplus -\mathrm{Bl}_{\alpha(p,\chi_2)}(T(p,q_1))\\ \oplus -\mathrm{Bl}_{\alpha(p,\theta)}(T(p,q;p,q_2)) \oplus \mathrm{Bl}_{\alpha(p,\theta)}(T(p,q_2)).\end{aligned}$$

This expression can be further decomposed by applying the satellite formula for the metabelian Blanchfield forms given in [4, Theorem 4.19]. Regardless of the final expression, the problem has been converted into a question of linear independence in $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$. In Proposition 4.3, we describe a criterion for linear independence in terms of roots of the orders of the underlying modules (recall that the order of a module over $\mathbb{C}[t^{\pm 1}]$ is a Laurent polynomial in $\mathbb{C}[t^{\pm 1}]$; it is defined up to multiplication by units of $\mathbb{C}[t^{\pm 1}]$). Here is a simplified version of this statement:

Proposition 1.5 *If (V_1, λ_1) and (V_2, λ_2) are two nonmetabolic linking forms over $\mathbb{C}[t^{\pm 1}]$ such that $\mathrm{Ord}(V_1)$ and $\mathrm{Ord}(V_2)$ have distinct roots, then the Witt classes $[V_1, \lambda_1]$ and $[V_2, \lambda_2]$ are linearly independent in $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$.*

1.3.1 Computation of twisted Alexander polynomials In order to apply Proposition 1.5, we must therefore understand the roots of the metabelian twisted Alexander polynomials of $T(p,q)$ associated to characters on $H_1\big(\Sigma_p(T(p,q))\big)$ (indeed, these twisted polynomial arise as orders of modules of the form $H_1(M_{T(p,q)}; \mathbb{C}[t^{\pm 1}]^p)$, the same modules on which twisted Blanchfield pairings are defined). This is carried out in Section 3 and relies on our explicit understanding of the p–fold cover $E_p(T(p,q)) \to E(T(p,q))$ from Section 2; here $E(K)$ denotes the exterior of a knot K. Since this computation of twisted polynomials might be of independent interest, we summarise it as follows.

Proposition 1.6 (Lemma 3.1 and Corollary 3.4) *Let $p, q > 0$ be two coprime integers, and set $\xi_p = e^{2\pi i/p}$. The abelian group of characters on $H_1\big(\Sigma_p(T(p,q))\big) \cong \mathbb{Z}_q^{p-1}$ is isomorphic to*

$$\{\boldsymbol{a} := (a_1, \dots, a_p) \in \mathbb{Z}_q^p \mid a_1 + \cdots + a_p = 0\}.$$

We write $\chi_{\boldsymbol{a}}$ for the character associated to $\boldsymbol{a}$. The metabelian twisted Alexander polynomial of the 0–framed surgery $M_{T(p,q)}$ associated to the character

$$\chi_{\boldsymbol{a}} \colon H_1\big(\Sigma_p(T(p,q))\big) \to \mathbb{Z}_q$$

is given by

$$\Delta_1^{\alpha(p,\chi_{\boldsymbol{a}})}(M_{T(p,q)}) = \frac{(-1)^{p-1}(1-t^q)^{p-1}}{(t\xi_q^{a_1}-1)(t\xi_q^{a_2}-1)\cdots(t\xi_q^{a_p}-1)(t-1)}.$$

Main steps of the proof We now return to the knot $J = J(p,q,q_1,q_2)$ from (1). Obstructing J from being slice has three main steps. In fact, the proof of Theorem 1.1 in its full generality, as described in Section 5, follows more complicated versions of these same steps:

(i) Firstly, we use the previously described ingredients to study the implications of $\mathrm{Bl}_{\alpha(p,\chi)}(J)$ being metabolic on the characters χ_1 and χ_2; here $\chi = \chi_1 \oplus \chi_2 \oplus \theta \oplus \theta$ with θ the trivial character. This is the content of Section 5.2.2.

(ii) Secondly, we show that, for every metaboliser L of $\lambda_p(T_{p,q_1}) \oplus -\lambda_p(T_{p,q_1})$, it is possible to build characters χ_1 and χ_2 that violate these conditions and are such that $\chi_1 \oplus \chi_2$ vanishes on L. This is the content of Section 5.2.3.

(iii) Finally, we combine these two steps to obstruct the sliceness of J: for every metaboliser G of $\lambda_p(J)$, we are able to build a character $\chi = \chi_1 \oplus \chi_2 \oplus \theta \oplus \theta$ that vanishes on G and such that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is not metabolic. This is the content of Section 5.2.4.

Remark 1.7 When $p = 2$, Hedden, Kirk and Livingston also use an obstruction based on the Casson–Gordon setup to show that, for an appropriate choice of positive integers $\{q_n\}_{n=1}^{\infty}$, the set $\{T(2,q_n), T(2,3;2,q_n)\}_{n=1}^{\infty}$ is linearly independent in $\mathcal{C}^{\mathrm{top}}$ [14]. Our work differs from theirs in two main points:

- While [14] uses a blend of discriminants and signatures to prove its linear independence result, we use metabelian Blanchfield pairings. In a nutshell, the Blanchfield pairing encapsulates both the discriminant and (most of) the signature invariants allowing us to both streamline and generalise several of the arguments from [14].
- The result of [14] is proved without having to study invariant metabolisers; see also [5, Section 6]. This is a feature of iterated torus knots $T(p,Q)$ with $p = 2$ and fails when $p > 2$.

Passing from our outline to obstruct the sliceness of $J(p,q,q_1,q_2)$ to the proof of Theorem 1.1 requires additional steps. As often in Casson–Gordon theory, the main technical difficulty to overcome concerns the metabolisers of the linking form of the knot in question. Regarding these metabolisers, our strategy can be summarised as follows:

(i) Given a metaboliser, we isolate certain technical conditions which guarantee that a character violates the sliceness obstruction. This is the content of Lemma 5.8.

(ii) We distinguish a certain family of metabolisers, called *graph metabolisers*, see Section 4.2.

(iii) The construction of the required character, for any fixed nongraph metaboliser is not overly challenging; see Cases 1 and 2 in the proof of Lemma 5.8.

(iv) Dealing with graph metabolisers requires more work. In Case 3, we show that either there exists a character satisfying the conditions from Lemma 5.8, or the knot in question contains a slice summand $K \# -K$, for some knot K. Consequently, once we cancel all the summands of the form $K \# -K$, we are able to construct the desired obstructing character for any graph metaboliser, and finish the proof.

1.4 Assumptions and outlook

We conclude this introduction by commenting on the various technical assumptions that appear in Theorem 1.1.

(i) The assumption that the integers q_i are coprime to q_ℓ is used in Proposition 5.7 to ensure that certain Witt classes are linearly independent in $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$. This hypothesis has its roots in the notion of *p–independence* introduced in [14, Definition 6.2].

(ii) We assume that p is a prime power in order to use Casson–Gordon theory [6; 7].

(iii) We require that the q_i be positive mostly because of our interest in Rudolph's conjecture: algebraic knots are iterated torus knots with *positive* cabling parameters.

(iv) We use that q_ℓ is prime in order to obtain the decomposition in (21) and to ensure that $\mathbb{F}_{q_\ell}$ is a field.

Summarising, our assumptions are made for technical reasons: we have so far not encountered linear combinations of (algebraically slice) iterated torus knots whose sliceness is not obstructed by some Casson–Gordon invariants. Furthermore, this paper does not fully use the techniques developed in [3; 4; 5] to compute the Casson–Gordon Witt class. Therefore, it would be interesting to study how far these methods can be pushed to investigate Rudolph's conjecture.

Organisation

This paper is organised as follows. In Section 2, we collect several results on the algebraic topology of the exterior of the torus knot $T(p,q)$. In Section 3, we use these results to compute Alexander polynomials of $T(p,q)$ twisted by metabelian representations. In Section 4, we review some facts about linking forms. Finally in Section 5, we prove Theorem 1.1.

Conventions

Manifolds are assumed to be compact and oriented. Throughout the paper, the p–fold branched cover of a knot is denoted by $\Sigma_p(K)$, and $\lambda_p(K)$ denotes the linking form on $H_1(\Sigma_p(K))$.

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2 Branched covers of torus knots

The aim of this section is to describe the $\mathbb{Z}[\mathbb{Z}_p]$–module structure of $H_1\big(\Sigma_p(T(p,q))\big)$ induced from the $\mathbb{Z}_p$–covering action on $\Sigma_p(T(p,q))$ when q is prime. Let $E(T(p,q))$ be the complement of the torus knot $T(p,q)$, and let $E_p(T(p,q))$ be its p–fold cyclic cover. In Section 2.1, to set up some notation, we recall the decomposition of $E(T(p,q))$ coming from the standard genus 1 Heegaard splitting of S^3, as described in [10, Example 1.24]. In Section 2.2, this decomposition of $E(T(p,q))$ is used to decompose $E_p(T(p,q))$; after that, $H_1\big(\Sigma_p(T(p,q))\big)$ can be computed via a Mayer–Vietoris sequence argument since $\Sigma_p(T(p,q))$ is a union of $E_p(T(p,q))$ with a solid torus glued along the torus boundary.

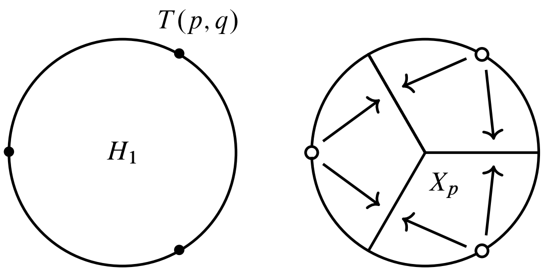

Figure 1: Left: the intersection $T(p,q) \cap (\{x\} \times D^2)$. Right: the complement $H_1 \smallsetminus T(p,q)$ deformation retracts onto a 2–complex X_p.

2.1 The homotopy type of $E(T(p,q))$

The goal of this subsection is to describe the homotopy type of $E(T(p,q))$, as well as describe explicit generators for $\pi_1\big(E(T(p,q))\big)$. To achieve this, we follow closely [10, Example 1.24].

Consider the standard decomposition $S^3 = S^1 \times D^2 \cup D^2 \times S^1$ and denote $S^1 \times D^2$ and $D^2 \times S^1$ by H_1 and H_2, respectively, $H_1 \subset \mathbb{R}^3$ being the solid torus. We parametrise the (p,q)–torus knot $T(p,q)$ on the torus $H_1 \cap H_2$ as follows:

$$T(p,q) = \{(e^{2\pi ipt}, e^{2\pi iqt}) \mid t \in [0,1]\} \subset S^1 \times S^1 = H_1 \cap H_2. \tag{3}$$

Using this description of $T(p,q)$, for each $x \in S^1$, we see that $T(p,q)$ intersects $\{x\} \times D^2 \subset H_1$ in p equidistributed points of $\{x\} \times \partial D^2$; see Figure 1 for $p = 3$.

As depicted in Figure 1, right, the complement $H_1 \smallsetminus T(p,q)$ deformation retracts onto a 2–complex $X_p \subset H_1$ which is the mapping cylinder of the degree p map $f_p \colon S^1 \to c_1$, where c_1 is the core circle of H_1. The same argument shows that $H_2 \smallsetminus T(p,q)$ deformation retracts onto the mapping cylinder X_q of the degree q map $f_q \colon S^1 \to c_2$, where c_2 is the core circle of H_2. By perturbing X_p near $H_1 \cap H_2$, we can arrange that X_p and X_q match up on $H_1 \cap H_2$. Next, let $X_{p,q}$ be the union of X_p and X_q. Note that $X_{p,q}$ is homeomorphic to the double mapping cylinder of the maps f_p and f_q, defined by

$$X_{p,q} := S^1 \times [0,1] \cup c_1 \cup c_2/{\sim},$$

where $(z,0) \sim f_p(z)$ and $(z,1) \sim f_q(z)$ for all $z \in S^1$ (see Figure 2). By van Kampen's theorem,

$$\pi_1(X_{p,q}) \cong \langle c_1, c_2 \mid c_1^p = c_2^q \rangle.$$

Summarising, we have the following proposition, which is implicit in Hatcher:

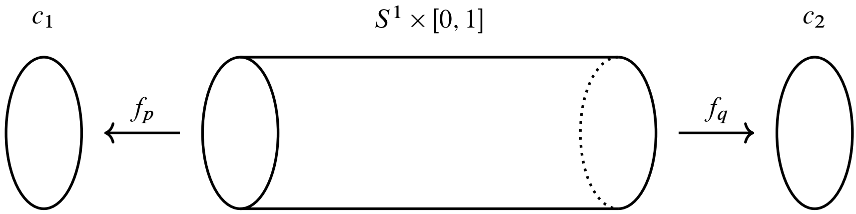

Figure 2: The double mapping cylinder $X_{p,q}$ obtained by gluing $S^1 \times [0,1]$ with the circles c_1 and c_2 by the degree p and q maps f_p and f_q.

Proposition 2.1 [10, Example 1.24] *There is a deformation retraction $E(T(p,q)) \to X_{p,q}$ sending $H_1 \setminus T(p,q)$ and $H_2 \setminus T(p,q)$ to X_p and X_q, respectively. In particular, $\pi_1\big(E(T(p,q))\big) \cong \langle c_1, c_2 \mid c_1^p = c_2^q \rangle$, where c_i is the core circle of H_i for $i = 1, 2$.*

2.2 The computation of $H_1\big(\Sigma_p(T(p,q))\big)$ as a $\mathbb{Z}[\mathbb{Z}_p]$–module

In this subsection, we describe the $\mathbb{Z}[\mathbb{Z}_p]$–module structure of $H_1\big(\Sigma_p(T(p,q))\big)$. To do so, we first study the p–fold cyclic covering map $\pi\colon E_p(T(p,q)) \to E(T(p,q))$, then we compute $\pi_1\big(E_p(T(p,q))\big)$, and finally we describe $H_1\big(\Sigma_p(T(p,q))\big)$.

We first use Section 2.1 to describe a deformation retract of $E_p(T(p,q))$. Using (3), we see that the torus knot $T(p,q)$ links respectively q and p times the core circles c_1 and c_2. Consequently, c_1 and c_2 are homologous to $q\mu$ and $p\mu$ in $H_1\big(E(T(p,q))\big)$, where $\mu = c_1^k c_2^l$ is a meridian of $T(p,q)$ and $pk + ql = 1$. Use $(X_{p,q})_p$ to denote the preimage $\pi^{-1}(X_{p,q})$, and observe that, by Proposition 2.1, $E_p(T(p,q))$ deformation retracts onto $(X_{p,q})_p$.

To describe $\pi_1\big(E_p(T(p,q))\big)$ we study the homotopy type of $(X_{p,q})_p$. The (restricted) covering map $\pi\colon (X_{p,q})_p \to X_{p,q}$ corresponds to the homomorphism $\pi_1(X_{p,q}) \to \mathbb{Z}_p$ sending c_1 to $q \in \mathbb{Z}_p$ and c_2 to $0 \in \mathbb{Z}_p$. We use $\pi_*\colon \pi_1((X_{p,q})_p) \to \pi_1(X_{p,q})$ to denote the induced map. Let a be the preimage $\pi^{-1}(c_1)$ and let $b_0, \dots, b_{p-1}$ be the components of the preimage $\pi^{-1}(c_2)$; we choose the indices of the b_i so that

$$\pi_*(b_i) = \mu^i c_2 \mu^{-i} \quad \text{for } i = 0, \dots, p-1. \tag{4}$$

Since π is a covering map, the induced map $\pi_*\colon \pi_1((X_{p,q})_p) \to \pi_1(X_{p,q})$ is injective. For this reason, we shall often identify b_i with $\mu^i c_2 \mu^{-i}$. Since $X_{p,q}$ is a double mapping cylinder, so is $(X_{p,q})_p$. More precisely, as illustrated in Figure 3,

$$(X_{p,q})_p = \bigcup_{i=0}^{p-1} S_i^1 \times [0,1] \cup a \cup b_0 \cup \dots \cup b_{p-1}/\sim,$$

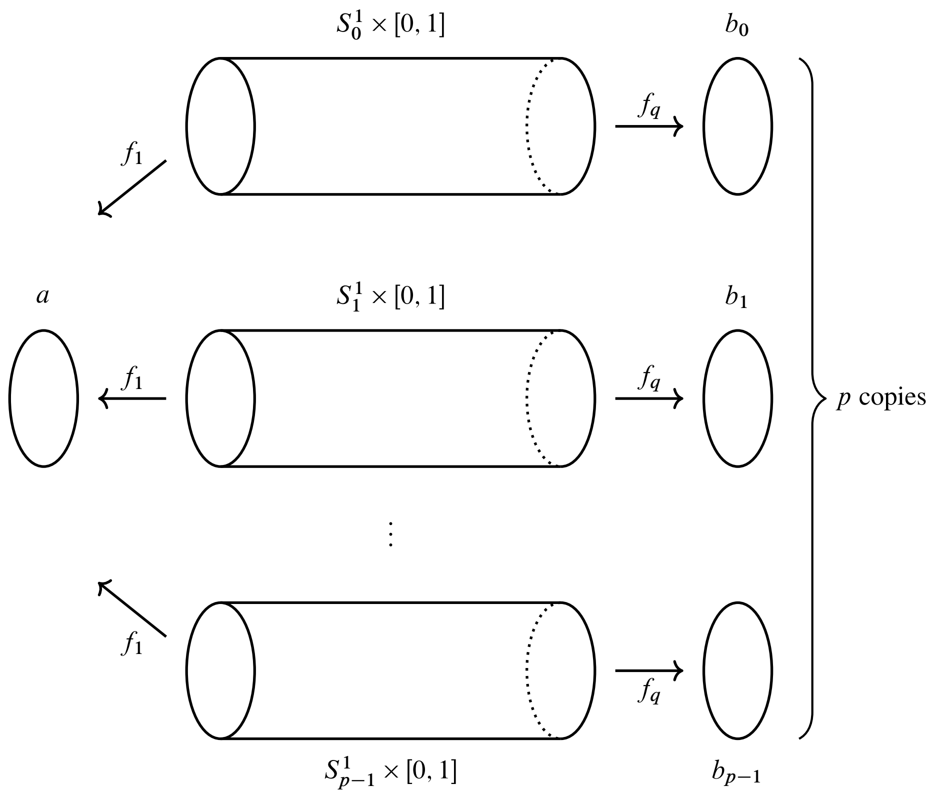

Figure 3: The p–fold cyclic cover $(X_{p,q})_p$ of $X_{p,q}$ is also a double mapping cylinder, where f_1 and f_q denote the degree 1 and the degree q maps, respectively.

where each $S^1_i \times \{0\}$ is identified with the circle a by the identity map, and $S^1_i \times \{1\}$ is identified with the circle b_i by the degree q map. By van Kampen's theorem, we deduce that

$$\pi_1((X_{p,q})_p) \cong \langle b_0, b_1, \dots, b_{p-1} \mid b_i^q = b_j^q \text{ for } 0 \le i \ne j \le p-1 \rangle.$$

Since $E_p(T(p,q))$ deformation retracts onto $(X_{p,q})_p$, we obtain the following proposition:

Proposition 2.2 *Let $\pi \colon E_p(T(p,q)) \to E(T(p,q))$ be the p–fold cyclic covering and let $b_0, b_1, \dots, b_{p-1}$ be the homotopy classes of the components of $\pi^{-1}(c_2)$ such that $\pi_*(b_i) = \mu^i c_2 \mu^{-i}$. Then*

$$\pi_1\bigl(E_p(T(p,q))\bigr) = \langle b_0, b_1, \dots, b_{p-1} \mid b_i^q = b_j^q \text{ for } 0 \le i \ne j \le p-1 \rangle.$$

Next, we use this description of $\pi_1\big(E_p(T(p,q))\big)$ to obtain generators of the finite abelian group $H_1\big(\Sigma_p(T(p,q))\big) = TH_1\big(E_p(T(p,q))\big)$. First, note that Proposition 2.2 shows that $H_1\big(E_p(T(p,q))\big) \cong \mathbb{Z} \oplus \mathbb{Z}_q^{p-1}$ has generators $b_0, b_1, \dots, b_{p-1}$ and relations $qb_i = qb_j$ for each i and j. In the remainder of this section, we describe a set of generators that will be more convenient for the twisted Alexander polynomial computations of Section 3.

Remark 2.3 While the meridian μ of $T(p,q)$ does not lift to $E_p(T(p,q))$, a loop representing μ^p does. Since the projection-induced map $\pi_*\colon \pi_1\big(E_p(T(p,q))\big) \to \pi_1\big(E(T(p,q))\big)$ is injective, we slightly abuse notation and also write μ^p for the homotopy class of this lift in $\pi_1\big(E_p(T(p,q))\big)$.

We will make no notational distinction between elements in $\pi_1\big(E_p(T(p,q))\big)$ and elements in $H_1\big(E_p(T(p,q))\big)$, despite switching from multiplicative to additive notation. In some rare instances, we will also use the multiplicative notation in homology. Keeping this in mind, for $i = 0, \dots, p-1$, we consider $\mu^{-p} b_i$ in $\pi_1\big(E_p(T(p,q))\big)$ and $x_i := b_i - \mu^p$ in $H_1\big(E_p(T(p,q))\big)$. The next proposition describes the homology group $H_1\big(\Sigma_p(T(p,q))\big)$ as a $\mathbb{Z}[\mathbb{Z}_p]$–module.

Proposition 2.4 *The abelian group $H_1\big(\Sigma_p(T(p,q))\big) \cong \mathbb{Z}_q^{p-1}$ is generated by the $x_i = b_i - \mu^p$, and these elements satisfy the following relations*:

(i) $x_0 + x_1 + \dots + x_{p-1} = 0$.

(ii) $x_i = t^i x_0$ *for* $i = 0, \dots, p-1$, *where* t *denotes the covering transformation of* $\Sigma_p(T(p,q))$.

In particular, there exists an isomorphism of $\mathbb{Z}[\mathbb{Z}_p]$–modules

$$H_1\big(\Sigma_p(T(p,q))\big) \cong \mathbb{Z}_q[t]/(1 + t + t^2 + \dots + t^{p-1}).$$

Proof The proof has four steps. Firstly, we establish a criterion for an element in $H_1\big(\Sigma_p(T(p,q))\big)$ to be torsion; secondly, we prove that the x_i are torsion; thirdly, we show that that x_i generate $TH_1\Sigma_p(T(p,q))$ as an abelian group; fourthly and finally we prove that the x_i satisfy the two identities stated in the lemma.

We assert that an element $x = \sum_{i=0}^{p-1} a_i b_i$ in $H_1\big(E_p(T(p,q))\big)$ is torsion if and only if $\sum_{i=0}^{p-1} a_i = 0$. The map $\pi_*\colon H_1\big(E_p(T(p,q))\big) \to H_1\big(E(T(p,q))\big)$ maps $TH_1\big(E_p(T(p,q))\big)$ to zero and maps the infinite cyclic summand isomorphically onto $p\mathbb{Z} \cong \mathbb{Z}\langle c_2\rangle$.[1] In particular, a class $x \in H_1\big(E_p(T(p,q))\big)$ is torsion if and only if $\pi_*(x) = 0$. On the other hand, using Proposition 2.2, we deduce that π induces the following map on homology, concluding the proof of the assertion:

$$\pi_*\colon H_1\big(E_p(T(p,q))\big) \to p\mathbb{Z} \subset \mathbb{Z} = H_1\big(E(T(p,q))\big), \quad \sum_{i=0}^{p-q} a_i b_i \mapsto \sum_{i=0}^{p-1} a_i.$$

We move on to the second step: we prove that the homology classes $x_0, \dots, x_{p-1}$ are torsion. Using the criterion, we must show that $\pi_*(x_i) = 0$ for each i. Since $\pi_*(b_i) = 1$, this reduces to showing that $\pi_*(\mu^p) = 1$. We start by computing the abelianisation of μ^p. Since $\mu = c_1^k c_2^l$, we notice that, in $\pi_1\big(E_p(T(p,q))\big)$,

$$\mu^p = (c_1^k c_2^l c_1^{-k}) \cdot (c_1^{2k} c_2^l c_1^{-2k}) \cdots (c_1^{(p-1)k} c_2^l c_1^{-(p-1)k}) c_1^{pk} c_2^l. \tag{5}$$

In order to compute the abelianisation of this expression, we claim that, for any $0 \le s \le p-1$ and any k, the equation $\mu^s c_2 \mu^{-s} = c_1^{ks} c_2 c_1^{-ks}$ holds in $H_1\big(E_p(T(p,q))\big) = \pi_1\big(E_p(T(p,q))\big)^{\mathrm{ab}}$. This claim is a consequence of the following direct computation in $\pi_1\big(E_p(T(p,q))\big)$:

$$\mu^s c_2 \mu^{-s} = \left(\prod_{i=1}^{s-1} c_1^{ki} c_2^l c_1^{-ki}\right) \cdot (c_1^{ks} c_2 c_1^{-ks}) \cdot \left(\prod_{i=1}^{s-1} c_1^{ki} c_2^{-l} c_1^{-ki}\right).$$

Using consecutively (5), the equation $\mu^s c_2 \mu^{-s} = c_1^{ks} c_2 c_1^{-ks}$ that we just established, and the identification $b_i = \mu^i c_2 \mu^{-i}$ from (4) (as well as the presentation in Proposition 2.1 and $qk + pl = 1$), we obtain the sequence of equalities, in $H_1\big(E_p(T(p,q))\big)$,

$$\begin{aligned} \mu^p &= (c_1^k c_2^l c_1^{-k}) \cdot (c_1^{2k} c_2^l c_1^{-2k}) \cdots (c_1^{(p-1)k} c_2^l c_1^{-(p-1)k}) c_1^{pk} c_2^l \\ &= (\mu c_2^l \mu^{-1})(\mu^2 c_2^l \mu^{-2}) \cdots (\mu^{(p-1)} c_2^l \mu^{-(p-1)}) c_1^{pk} c_2^l \\ &= l(b_0 + b_1 + \cdots + b_{p-1}) + qk b_0. \end{aligned} \tag{6}$$

As $\pi_*(b_i) = 1$ for each i, this implies that $\pi_*(\mu^p) = 1$. It follows that $\pi_*(x_i) = \pi_*(b_i) - \pi_*(\mu^p) = 0$, and therefore each of the x_i is torsion. This concludes the second step of the proof.

Thirdly, we show that every element of $TH_1\big(E_p(T(p,q))\big)$ can be written as a linear combination of the x_i for $i = 0, 1, \dots, p-1$: given $x = \sum_{i=0}^{p-1} a_i b_i$, adding and

[1] For any knot K and prime power n, one has the decomposition $H_1(E_n(K)) = TH_1(E_n(K)) \oplus \mathbb{Z}$, where the $\mathbb{Z}$ summand is generated by a lift of the n–fold power of the meridian.

subtracting μ^p, using $\sum_{i=0}^{p-1} a_i = 0$ (which holds thanks to the first step) and the definition of x_i, we obtain

$$x = \sum_{i=0}^{p-1} a_i b_i = \sum_{i=0}^{p-1} a_i \mu^p + \sum_{i=0}^{p-1} a_i (b_i - \mu^p) = \sum_{i=0}^{p-1} a_i x_i.$$

Fourthly and finally, we establish the relations $x_0 + x_1 + \cdots + x_{p-1} = 0$ and $x_i = t^i x_0$. The latter relation is clear (since $b_i = t^i b_0$ and $t\mu^p = \mu^p$) and so we focus on the former. Using consecutively (6), the relation $qb_i = qb_j$, and the fact that $pl + qk = 1$, we notice that, in $H_1\big(E_p(T(p,q))\big)$,

$$\begin{aligned} p\mu^p &= pl(b_0 + b_1 + \cdots + b_{p-1}) + pqkb_0 \\ &= pl(b_0 + b_1 + \cdots + b_{p-1}) + qk(b_0 + b_1 + \cdots + b_{p-1}) \\ &= (b_0 + b_1 + \cdots + b_{p-1}). \end{aligned}$$

The conclusion now promptly follows from the definition of the x_i, establishing the proposition. □

Assume that q is a prime. In this case $H_1\big(\Sigma_p(T(p,q))\big)$ becomes an $\mathbb{F}_q$–vector space. The covering action t is then an $\mathbb{F}_q$–linear endomorphism of $V_{p,q}$.

3 Twisted polynomials of torus knots

In this section, we compute the Alexander polynomial of the 0–framed surgery $M_{T(p,q)}$ twisted by a metabelian representation $\alpha_{T(p,q)}(p,\chi)\colon \pi_1(M_{T(p,q)}) \to \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$ that frequently appears in Casson–Gordon theory [15]. In Section 3.1, we recall the definition of $\alpha_K(p,\chi)$ for a general knot K. In Section 3.2, we study this representation in the case of torus knots. Finally, in Section 3.3, we compute the relevant twisted Alexander polynomials.

3.1 The metabelian representation $\alpha_K(p,\chi)$

In this subsection, given a knot K and a positive integer p, we recall the definition of the representation $\alpha_K(p,\chi)\colon \pi_1(M_K) \to \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$ from [15], where $\chi\colon H_1(\Sigma_p(K)) \to \mathbb{Z}_m$ is a character. In what follows, E_K denotes the exterior of K and M_K denotes its 0–framed surgery. Finally, we use $\xi_m := e^{2\pi i/m}$ to denote the m^{th} primitive root of unity.

We use

$$H_1(E(K);\mathbb{Z}[t_K^{\pm 1}]) \cong \pi_1(E(K))^{(1)}/\pi_1(E(K))^{(2)}$$

to denote the Alexander module of K. In what follows, we shall frequently identify $H_1(\Sigma_p(K))$ with $H_1(E(K);\mathbb{Z}[t_K^{\pm 1}])/(t_K^p-1)$, as for instance in [8, Corollary 2.4]. Consider the composition of canonical projections

$$q_K\colon \pi_1(M_K)^{(1)} \to H_1(E(K);\mathbb{Z}[t_K^{\pm 1}]) \to H_1(\Sigma_p(K)). \tag{7}$$

Use $\phi_K\colon \pi_1(E(K)) \to H_1(E(K);\mathbb{Z}) \cong \mathbb{Z} = \langle t_K\rangle$ to denote the abelianisation homomorphism, and fix an element μ_K in $\pi_1(E(K))$ such that $\phi_K(\mu_K)=t_K$. Note that, for every $g\in\pi_1(E(K))$, we have $\phi_K(\mu_K^{-\phi_K(g)}g)=1$. Since ϕ_K is the abelianisation map, we deduce that $\mu_K^{-\phi_K(g)}g$ belongs to $\pi_1(E(K))^{(1)}$. Combining this notation, we consider the representation

$$\alpha_K(p,\chi)\colon \pi_1(E(K)) \to \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$$

given by

$$\begin{aligned}\alpha_K(p,\chi)(g) &= \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ t & 0 & \cdots & 0 \end{pmatrix}^{\phi_K(g)} \\ &\quad \cdot \begin{pmatrix} \xi_m^{\chi(q_K(\mu_K^{-\phi_K(g)}g))} & 0 & \cdots & 0 \\ 0 & \xi_m^{\chi(t_K\cdot q_K(\mu_K^{-\phi_K(g)}g))} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_m^{\chi(t_K^{p-1}\cdot q_K(\mu_K^{-\phi_K(g)}g))} \end{pmatrix} \\ &=: A_p(t)^{\phi_K(g)} D_g. \end{aligned} \tag{8}$$

Note that $\alpha(p,\chi)$ can equally well be defined on $\pi_1(M_K)$ instead of $\pi_1(E(K))$: the definition can be adapted verbatim, and we use the same notation

$$\alpha_K(p,\chi)\colon \pi_1(M_K) \to \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}]).$$

A closely related observation is that $\alpha(p,\chi)$ is a metabelian representation and therefore vanishes on the longitude of K; this also explains why $\alpha_K(p,\chi)$ descends to $\pi_1(M_K)$.

3.2 An explicit description of $\alpha_{T(p,q)}(p,\chi)$

We use the presentation of $\pi_1\big(E(T(p,q))\big)$ from Proposition 2.1 to describe the representation $\alpha_{T(p,q)}(p,\chi)$. Throughout this subsection, we set $K := T(p,q)$ in order to avoid cumbersome notation such as $q_{T(p,q)}$.

We recall the definition of the generators $x_0,\dots,x_{p-1}$ of $H_1(\Sigma_p(K)) \cong \mathbb{Z}_q^{p-1}$ described in Proposition 2.4, referring to Section 2 for further details. Using the notation of that section, we set $x_i = b_i - \mu^p$, where μ is a meridian of K. For $i=0,\dots,p-1$, thinking of x_i as the abelianisation of $\mu^{-p} b_i$, and using Proposition 2.2 to identify b_i with $\mu^i c_2 \mu^{-i}$,

$$t_K^i q_K(\mu^{-p} c_2) = q_K(\mu^{-p} \mu^i c_2 \mu^{-i}) = q_K(\mu^{-p} b_i) = x_i. \tag{9}$$

Recall furthermore that Proposition 2.4 also established the relations $x_0 + \cdots + x_{p-1} = 0$ as well as $t_K x_i = x_{i+1}$ for $i = 0, \ldots, p-1$. The next result follows immediately from these considerations.

Lemma 3.1 *Fix two coprime integers $p, q > 0$ and set $K := T(p,q)$. The abelian group of characters on $H_1(\Sigma_p(K))$ is isomorphic to*

$$\{\boldsymbol{a} := (a_1, \ldots, a_p) \in \mathbb{Z}_q^p \mid a_1 + \cdots + a_p = 0\}.$$

The isomorphism maps a character χ to $(\chi(x_0), \ldots, \chi(x_{p-1}))$, and we write $\chi_{\boldsymbol{a}}$ for the character associated to $\boldsymbol{a}$.

Recall that Proposition 2.1 described a two-generator, one-relation presentation for the knot group $\pi_1(E(K))$; the generators were denoted by c_1 and c_2 and the unique relator was $c_1^p c_2^{-q}$. The next proposition describes the image of these generators under $\alpha(p,\chi) := \alpha_K(p,\chi)$. This will be useful in Proposition 3.3 when we compute the twisted Alexander polynomial of $E(K)$.

Proposition 3.2 *Fix two coprime integers $p, q > 0$ and set $K := T(p,q)$. For a character $\chi = \chi_{\boldsymbol{a}} \colon H_1(\Sigma_p(K)) \to \mathbb{Z}_q$, the representation $\alpha(p,\chi)$ is conjugated to a representation $\alpha'(p,\chi)$ such that*

$$\alpha'(p,\chi)(c_2) = t \cdot \operatorname{diag}(\xi_q^{a_1}, \ldots, \xi_q^{a_p}), \quad \alpha'(p,\chi)(c_1) = A_p(t)^q.$$

Proof We first compute $\alpha(p,\chi)(c_2)$. We know that $\phi_K(c_2) = p$ and $A_p(t)^p = t \cdot \mathrm{id}$. In order to compute the diagonal matrix which appears in the definition of $\alpha(p,\chi)(c_2)$ (recall (8)), we use (9) and Lemma 3.1 to obtain $\chi(t_K^{i-1} q_K(\mu^{-p} c_2)) = \chi(x_{i-1}) = a_i$. The first assertion follows:

$$\alpha(p,\chi)(c_2) = t \cdot \operatorname{diag}(\xi_q^{\chi(q_K(\mu^{-p}c_2))}, \ldots, \xi_q^{\chi(t_K^{p-1} q_K(\mu^{-p}c_2))}) = t \cdot \operatorname{diag}(\xi_q^{a_1}, \ldots, \xi_q^{a_p}).$$

Next, we study the conjugacy class of $\alpha(p,\chi)(c_1)$: we must find an invertible matrix X such that

$$X\alpha(p,\chi)(c_1)X^{-1} = A_p(t)^q, \tag{10}$$

$$X\alpha(p,\chi)(c_2)X^{-1} = t \cdot \operatorname{diag}(\xi_q^{a_1}, \ldots, \xi_q^{a_p}). \tag{11}$$

For $v \in H_1(\Sigma_p(K))$, we define $\widetilde{\alpha}(v) := \operatorname{diag}(\xi_q^{\chi(v)}, \xi_q^{\chi(t_K v)}, \ldots, \xi_q^{\chi(t_K^{p-1} v)})$. Observe that, if we set $X := \widetilde{\alpha}(z)$, then (11) is satisfied for any $z \in H_1(\Sigma_p(K))$; indeed,

$\alpha(p,\chi)(c_2)$ commutes with X since both are diagonal. Therefore, we just have to establish the existence of a $z \in H_1(\Sigma_p(K))$ such that (10) is satisfied for $X = \widetilde{\alpha}(z)$.

First, for any $x \in H_1(\Sigma_p(K))$, a computation shows that

$$\widetilde{\alpha}(x) A_p(t)^q \widetilde{\alpha}(x)^{-1} = A_p(t)^q \widetilde{\alpha}((t_K^{-q}-1)x).$$

Define $y := q_K(\mu^{-q}c_1)$, so that $\alpha(p,\chi)(c_1) = A_p(t)^q\widetilde{\alpha}(y)$. Consequently, if we set $X := \widetilde{\alpha}(z)$ (for any $z \in H_1(\Sigma_p(K))$), use the definition of y, the fact that $\widetilde{\alpha}(y)$ and X commute (both are diagonal), and the aforementioned identity, then we obtain

$$\begin{aligned} X\alpha(p,\chi)(c_1)X^{-1} = XA_p(t)^q\widetilde{\alpha}(y)X^{-1} &= XA_p(t)^qX^{-1}\widetilde{\alpha}(y) \\ &= A_p(t)^q\widetilde{\alpha}((t_K^{-q}-1)z+y). \end{aligned}$$

Therefore, if we choose $z := -(t_K^{-q}-1)^{-1}y$, then (10) holds. For this to make sense, however, we must argue that $t_K^{-q}-1$ is an automorphism of $H_1(\Sigma_p(K))$. This is indeed the case: as $t_K - 1$ is an automorphism of $H_1(\Sigma_p(K))$, the inverse is given by $(t_K^{-1}-1)^{-1}(1+t_K^{-q}+t_K^{-2q}+\cdots+t_K^{-(k-1)q})$, where $qk \equiv 1 \bmod p$. Such a k exists because p and q are coprime. We have therefore found X such that (10) and (11) hold, and this concludes the proof of the proposition. □

3.3 The computation of the twisted polynomial

In this subsection, we compute the twisted Alexander polynomial of the 0–framed surgery $M_{T(p,q)}$ with respect to $\alpha(p,\chi)$.

Recall that, given a space X and a representation $\beta\colon \pi_1(X) \to \mathrm{GL}_p(\mathbb{C}[t^{\pm 1}])$, the *twisted Alexander polynomial* $\Delta_1^\beta(X)$ is defined as the order of the twisted Alexander module $H_1(X;\mathbb{C}[t^{\pm1}]^p_\beta)$. More generally, we write $\Delta_i^\beta(X)$ for the order of the $\mathbb{C}[t^{\pm1}]$–module $H_i(X;\mathbb{C}[t^{\pm1}]^p_\beta)$. Recall that the $\Delta_i^\beta(X)$ are defined up to multiplication by units of $\mathbb{C}[t^{\pm1}]$.

The next proposition describes $\Delta_1^{\alpha(p,\chi)}\big(E(T(p,q))\big)$, where $E(T(p,q))$ denotes the exterior of $T(p,q)$.

Proposition 3.3 *Let $p,q>0$ be coprime integers. For $\chi = \chi_{\boldsymbol{a}}\colon H_1\big(\Sigma_p(T(p,q))\big) \to \mathbb{Z}_q$, the metabelian twisted Alexander polynomial of $E(T(p,q))$ is given by*

$$\Delta_1^{\alpha(p,\chi)}\big(E(T(p,q))\big) = \frac{(1-t^q)^{p-1}}{(t\xi_q^{a_1}-1)(t\xi_q^{a_2}-1)\cdots(t\xi_q^{a_p}-1)}.$$

Proof We use $\tau^{\alpha(p,\chi)}(E(K))$ to denote the Reidemeister torsion of a knot exterior $E(K)$ twisted by $\alpha(p,\chi) := \alpha_K(p,\chi)$. We refer to [9] for more on the subject, but simply note that $\tau^{\alpha(p,\chi)}(E(K))$ is defined since the chain complex $C_*(E(K);\mathbb{C}(t)^p)$ of left $\mathbb{C}(t)$–modules is acyclic [6, Corollary after Lemma 4]. Since $E(K)$ has torus boundary, by [9, Proposition 2(5)], the twisted Reidemeister torsion and twisted Alexander polynomial are related by

$$\tau^{\alpha(p,\chi)}(E(K)) = \frac{\Delta_1^{\alpha(p,\chi)}(E(K))}{\Delta_0^{\alpha(p,\chi)}(E(K))}.$$

Since $\Delta_0^{\alpha(p,\chi)}(E(K)) = 1$ for every knot K [4, Lemma 4.1], we are reduced to computing $\tau^{\alpha(p,\chi)}\big(E(T(p,q))\big)$. By [19, Theorem A], this torsion invariant can be expressed via Fox calculus. In our case, using the presentation of $\pi_1\big(E(T(p,q))\big)$ resulting from Proposition 2.1, we obtain

$$\Delta_1^{\alpha(p,\chi)}\big(E(T(p,q))\big) = \tau^{\alpha(p,\chi)}\big(E(T(p,q))\big) = \frac{\det\big(\alpha(p,\chi)(\partial(c_1^p c_2^{-q})/\partial c_1)\big)}{\det(\alpha(p,\chi)(c_2)-\mathrm{id})}. \tag{12}$$

Since this expression does not depend on the conjugacy class of $\alpha(p,\chi)$, we can work with the representation $\alpha'(p,\chi)$ described in Proposition 3.2. Using the first item of Proposition 3.2, the denominator of (12) is given by the formula

$$\begin{aligned} \det(\alpha(p,\chi)(c_2)-\mathrm{id}) &= \det(\mathrm{diag}(t\xi_q^{a_1}-1, t\xi_q^{a_2}-1,\dots,t\xi_q^{a_p}-1)) \\ &= \prod_{i=1}^{p}(t\xi_q^{a_i}-1). \end{aligned} \tag{13}$$

We will now compute the numerator of (12) and show that it equals $(1-t^q)^{p-1}$. Recall from (8) that, for $g\in\pi_1(E(K))$, the metabelian representation $\alpha_K(p,\chi)$ is given by $\alpha_K(p,\chi)(g) = A_p(t)^{\phi_K(g)}D_g$. An inductive argument involving the properties of the Fox derivative shows that

$$\frac{\partial(c_1^p c_2^{-q})}{\partial c_1} = \frac{\partial c_1^p}{\partial c_1} = 1 + c_1 + c_1^2 + \cdots + c_1^{p-1} =: g.$$

We will now apply $\alpha(p,\chi)$ to g. We recall from Proposition 3.2 that $\alpha'(p,\chi)(c_1) = A_p(t)^q$, and we now work over $\mathbb{C}[t^{\pm 1/p}]$. Indeed, as observed in [15, page 935], in this ring, the matrix $A_p(t)$ is conjugated to the diagonal matrix

$$B_p(t) := \mathrm{diag}(t^{1/p}, \xi_p t^{1/p}, \xi_p^2 t^{1/p},\dots,\xi_p^{p-1}t^{1/p}).$$

Since (12) only depends on the conjugacy class of the representation $\alpha(p,\chi)$, we can work with $B_p(t)$ instead of $A_p(t)$. We use $\sim$ to denote the conjugacy relation. Since $B_p(t)$ is diagonal, its powers are easy to compute and, as a consequence, we obtain

$$\begin{aligned}\alpha'(p,\chi)(g) &\sim \mathrm{id} + B_p(t)^q + B_p(t)^{2q} + \cdots + B_p(t)^{(p-1)q}\\ &= \mathrm{diag}\left(\frac{1-t^q}{1-t^{q/p}}, \frac{1-t^q}{1-\xi_p^q t^{q/p}}, \frac{1-t^q}{1-\xi_p^{2q} t^{q/p}}, \dots, \frac{1-t^q}{1-\xi_p^{q(p-1)} t^{q/p}}\right).\end{aligned}$$

Taking the determinant of this expression, we deduce that

$$\det\left(\alpha(p,\chi)\left(\frac{\partial(c_1^p c_2^{-q})}{\partial c_1}\right)\right) = \prod_{j=0}^{p-1} \frac{1-t^q}{1-\xi_p^{jq} t^{q/p}} = \frac{(1-t^q)^p}{1-t^q} = (1-t^q)^{p-1}. \tag{14}$$

Plugging (13) and (14) into (12) concludes the proof of the proposition. □

Using Proposition 3.3, we can compute the twisted polynomial of the 0–framed surgery $M_{T(p,q)}$.

Corollary 3.4 *Let $p, q > 0$ be coprime integers. For $\chi = \chi_{\boldsymbol{a}} : H_1(\Sigma_p(T(p,q))) \to \mathbb{Z}_q$, the metabelian twisted Alexander polynomial of $M_{T(p,q)}$ is given by*

$$\Delta_1^{\alpha(p,\chi)}(M_{T(p,q)}) = \frac{(-1)^{p-1}(1-t^q)^{p-1}}{(t\xi_q^{a_1}-1)(t\xi_q^{a_2}-1)\cdots(t\xi_q^{a_p}-1)(t-1)}.$$

Proof By Proposition 3.3, we need only show that $(-1)^{p-1}(t-1)\Delta_1^{\alpha(p,\chi)}(M_K) = \Delta_1^{\alpha(p,\chi)}(E(K))$ for every knot K, where $\alpha(p,\chi) := \alpha_K(p,\chi)$. Using the equality $\Delta_1^{\alpha(p,\chi)}(E(K)) = \tau^{\alpha(p,\chi)}(E(K))$ that was obtained in the proof of Proposition 3.3, Lemma 3 of [9], as well as [9, Propositions 2(8) and 5] and the fact that $\Delta_0^{\alpha(p,\chi)}(M_K) = 1$ (by [4, Lemma 4.1]), we obtain the sequence of equalities

$$\begin{aligned}\Delta_1^{\alpha(p,\chi)}(E(K)) = \tau^{\alpha(p,\chi)}(E(K)) &= \det(\alpha(p,\chi)(\mu_K) - \mathrm{id})\tau^{\alpha(p,\chi)}(M_K)\\ &= \det(\alpha(p,\chi)(\mu_K) - \mathrm{id})\frac{\Delta_1^{\alpha(p,\chi)}(M_K)}{\Delta_0^{\alpha(p,\chi)}(M_K)\Delta_2^{\alpha(p,\chi)}(M_K)}\\ &= \det(\alpha(p,\chi)(\mu_K) - \mathrm{id})\frac{\Delta_1^{\alpha(p,\chi)}(M_K)}{\Delta_0^{\alpha(p,\chi)}(M_K)\overline{\Delta_0^{\alpha(p,\chi)}(M_K)}}\\ &= \det(\alpha(p,\chi)(\mu_K) - \mathrm{id})\Delta_1^{\alpha(p,\chi)}(M_K).\end{aligned}$$

It thus remains to show that $\det(\alpha(p,\chi)(\mu_K) - \mathrm{id}) = (-1)^{p-1}(t-1)$; this follows from the definition of $\alpha(p,\chi)$ (recall (8)) since $\alpha(p,\chi)(\mu_K) = A_p(t)$. □

4 Linking forms and their metabolisers

This section collects some facts about linking forms and their metabolisers. This will be useful in Section 5 since both the metabelian Blanchfield pairing and $\lambda_p(T(p,q))$

are linking forms. In Section 4.1, we recall some basics on linking forms and their Witt groups. In Section 4.2, we prove a result on metabolisers of linking forms of the type $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$.

4.1 The Witt group of linking forms

Let R be a PID with involution and let Q denote its field of fractions. This subsection is concerned with linking forms. Firstly, we recall the definition of the Witt group $W(Q,R)$ of linking forms. Secondly, we collect some facts about $W(\mathbb{C}(t),\mathbb{C}[t^{\pm 1}])$ that are used in Section 5 below.

A *linking form* over R is a pair (V,λ), where V is a torsion R–module and $\lambda\colon V \times V \to Q/R$ is a sesquilinear and Hermitian pairing. The set of linking forms over R forms a monoid under the direct sum. A linking form (V,λ) is *nonsingular* if its *adjoint* $\lambda^\bullet\colon V \to V^*$, $x \mapsto \lambda(x,-)$ is an isomorphism. In the sequel, our linking forms will be either over $\mathbb{Z}$ or $\mathbb{C}[t^{\pm 1}]$. From now on, we also assume that all linking forms are nonsingular. Given a linking form (V,λ) over R, a submodule $L \subset V$ is *isotropic* if $L \subset L^\perp$ and is a *metaboliser* if $L = L^\perp$. A linking form is *metabolic* if it admits a metaboliser. The set of metabolic linking forms over R forms a submonoid of the monoid of linking forms over R.

Definition 4.1 The *Witt group of linking forms*, denoted by $W(Q,R)$, consists of the monoid of linking forms modulo the submonoid of metabolic linking forms. Two linking forms (V,λ) and (V',λ') are called *Witt equivalent* if they represent the same element in $W(Q,R)$.

The Witt group of linking forms is known to be an abelian group under direct sum, where the inverse of the class $[(V,\lambda)]$ is represented by $(V,-\lambda)$. Next, we collect some facts on $W(\mathbb{C}(t),\mathbb{C}[t^{\pm 1}])$ that will be used in Section 5 below.

Remark 4.2 The Witt group $W(\mathbb{C}(t),\mathbb{C}[t^{\pm 1}])$ is known to be free abelian and is detected by the signature jumps $\delta\sigma_{(V,\lambda)}$ [3, Sections 4 and 5]. In particular, a linking form (V,λ) over $\mathbb{C}[t^{\pm 1}]$ is metabolic if and only if all its signature jumps vanish [3, Theorem 5.3]. Reformulating, $[V,\lambda] = 0$ in $W(\mathbb{C}(t),\mathbb{C}[t^{\pm 1}])$ if and only if $\delta\sigma_{(V,\lambda)}(\omega) = 0$ for all $\omega \in S^1$. We refer to [3, Sections 4 and 5] for further details regarding signatures of linking forms but note that a linking form (V,λ) will have a trivial jump at $\omega \in S^1$ if the order $\operatorname{Ord}(T)$ of the $\mathbb{C}[t^{\pm 1}]$–module T does not have a root at ω.

In particular, Remark 4.2 implies the following result about linear independence in $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$:

Proposition 4.3 *If (V_1, λ_1) and (V_2, λ_2) are two linking forms over $\mathbb{C}[t^{\pm 1}]$ such that* $\operatorname{Ord}(V_1)$ *and* $\operatorname{Ord}(V_2)$ *have distinct roots, then the following assertions hold*:

- (i) *If (V_1, λ_1) and (V_2, λ_2) are not metabolic, then the Witt classes $[V_1, \lambda_1]$ and $[V_2, \lambda_2]$ are linearly independent in $W(\mathbb{C}(t), \mathbb{C}[t^{\pm 1}])$.*
- (ii) *If $(V_1, \lambda_1) \oplus (V_2, \lambda_2)$ is metabolic, then (V_1, λ_1) and (V_2, λ_2) are both metabolic.*

Proof We only prove the first assertion as the second assertion follows immediately. Assume that $n_1[V_1, \lambda_1] + n_2[V_2, \lambda_2] = 0$ for some integers n_1 and n_2. Remark 4.2 implies that all the signature jumps of $n_1\lambda_1 \oplus n_2\lambda_2$ must vanish. Since λ_1 is not metabolic, Remark 4.2 also implies that λ_1 admits a nontrivial signature jump at some $\omega_1 \in S^1$. As a consequence of these two assertions, we infer that $n_1\lambda_1$ and $n_2\lambda_2$ must have a nontrivial signature jump at ω_1. Since $\operatorname{Ord}(V_1)$ and $\operatorname{Ord}(V_2)$ have distinct roots, we deduce that $n_1 = 0$. The same reasoning shows that $n_2 = 0$, thus establishing the linear independence of $[V_1, \lambda_1]$ and $[V_2, \lambda_2]$ and establishing the proposition. □

4.2 Graph metabolisers

Given linking forms $(V_1, \lambda_1), (V_2, \lambda)$, we prove a result on metabolisers of linking forms of the type $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$. More precisely, Proposition 4.4 provides a criterion for when such a metaboliser must be a graph. This result will be used in Section 5 when we study metabolisers of $\lambda_p(T(p,q))^N \oplus -\lambda_p(T(p,q))^N$.

Given linking forms (V_1, λ_1) and (V_2, λ_2), a *morphism* of linking forms is an R–linear homomorphism $f : V_1 \to V_2$ such that $\lambda_2(f(x), f(y)) = \lambda_1(x, y)$ for all $x, y \in V_1$. If the forms are nonsingular, then a morphism is necessarily injective. An *isometry* of linking forms is a bijective morphism of linking forms. The graph

$$\Gamma_f = \{(v, f(v)) \in V_1 \oplus V_2 \mid v \in V_1\}$$

of a morphism $f : (V_1, \lambda_1) \to (V_2, \lambda_2)$ is an isotropic submodule of $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$. If f is an isometry, then Γ_f is in fact a metaboliser of $(V_1 \oplus V_2, \lambda_1 \oplus -\lambda_2)$. The next proposition provides an assumption under which the converse also holds.

Proposition 4.4 *Let (V_1, λ_1) and (V_2, λ_2) be linking forms over R and let $L \subset V_1 \oplus V_2$ be a metaboliser of $\lambda_1 \oplus -\lambda_2$. The following assertions hold*:

(i) *If $L\cap(V_1\oplus 0)=0=L\cap(0\oplus V_2)$, then L is the graph of an isometry $f\colon V_1\to V_2$,*

$$L=\{(v,f(v))\in V_1\oplus V_2 \mid v\in V_1\}.$$

(ii) *If we additionally work over $R=\mathbb{Z}$, suppose that V_1 and V_2 are equipped with an isometric $\mathbb{Z}_p$–action, and L is a $\mathbb{Z}_p$–invariant metaboliser, then the isometry f is $\mathbb{Z}_p$–equivariant.*

Proof We prove the first assertion. The isometry f will be defined by using the canonical projections $\mathrm{pr}_i\colon V_1\oplus V_2\to V_i$ for $i=1,2$. Since $L\cap(V_1\oplus 0)=0=L\cap(0\oplus V_2)$, it follows that $\mathrm{pr}_i|_L$ is injective, for $i=1,2$. Set $W_i:=\mathrm{pr}_i(L)$, for $i=1,2$, and define f as the composition

$$f\colon W_1\xrightarrow[\cong]{\mathrm{pr}_1^{-1}} L\xrightarrow[\cong]{\mathrm{pr}_2} W_2.$$

Since f is an isomorphism of R–modules, it remains to check that it is a morphism of linking forms. First, however, we use the definition of f to observe that

$$L=\{(v,f(v))\in V_1\oplus V_2 \mid v\in W_1\}\subset V_1\oplus V_2. \tag{15}$$

The fact that f is a morphism now follows from the fact that L is isotropic: for any $v,w\in W_1$, the pairs $(v,f(v)),(w,f(w))$ belong to L, and therefore

$$0=(\lambda_1\oplus-\lambda_2)\big((v,f(v)),(w,f(w))\big)=\lambda_1(v,w)-\lambda_2(f(v),f(w)).$$

Looking at (15), it only remains to show that $V_1=W_1$ and $V_2=W_2$. Since f is an isomorphism, we have $\mathrm{ord}(W_1)=\mathrm{ord}(W_2)$ and therefore (15) implies that $\mathrm{ord}(L)^2=\mathrm{ord}(W_1)\,\mathrm{ord}(W_2)$. Since L is a metaboliser, we deduce that

$$\mathrm{ord}(V_1)\,\mathrm{ord}(V_2)=\mathrm{ord}(L)^2=\mathrm{ord}(W_1)\,\mathrm{ord}(W_2). \tag{16}$$

By way of contradiction, assume that $\mathrm{ord}(W_1)$ divides $\mathrm{ord}(V_1)$ with $\mathrm{ord}(W_1)\neq\mathrm{ord}(V_1)$; we write $\mathrm{ord}(W_1)\nmid\mathrm{ord}(V_1)$. A glance at (16) shows that $\mathrm{ord}(V_2)\nmid\mathrm{ord}(W_2)$, contradicting the inclusion $W_2\subset V_2$. We conclude that $\mathrm{ord}(W_i)=\mathrm{ord}(V_i)$ and consequently $W_i=V_i$ for $i=1,2$. This concludes the proof of the first assertion.

We prove the second assertion. Use t to denote a generator of $\mathbb{Z}_p$. As the metaboliser L is $\mathbb{Z}_p$–invariant, if $(v,f(v))\in L$, then $(tv,tf(v))\in L$ for any $v\in V_1$. Moreover, as $(tv,f(tv))\in L$ and $L\cap(0\oplus V_2)=0$, it follows that $(tv,f(tv))=(tv,tf(v))$. We have therefore established that $f(tv)=tf(v)$ for any $v\in V_1$, and thus f is $\mathbb{Z}_p$–equivariant, as desired. □

5 Nonslice linear combinations of iterated torus knots

This section aims to prove Theorem 1.1, whose statement we now recall. For an integer $p \geq 2$ and a sequence $Q = (q_1, q_2, \dots, q_\ell)$ of integers that are relatively prime to p, we write iterated torus knots as $T(p, Q) := T(p, q_1; p, q_2; \dots; p, q_\ell)$. Our main result reads as follows:

Theorem 1.1 *Fix a prime power p. Let $\mathcal{S}_p$ be the set of iterated torus knots*

$$T(p, q_1; p, q_2; \dots; p, q_\ell),$$

where the sequences $(q_1, q_2, \dots, q_\ell)$ of positive integers satisfy

(i) *for $i = 1, \dots, \ell$, the integer q_i is coprime to p;*

(ii) *q_ℓ is a prime;*

(iii) *for $i = 1, \dots, \ell - 1$, the integer q_i is coprime to q_ℓ when $\ell > 1$.*

The set $\mathcal{S}_p$ is linearly independent in the topological knot concordance group $\mathcal{C}^{\mathrm{top}}$.

To prove Theorem 1.1, we must obstruct the sliceness of linear combinations of knots belonging to $\mathcal{S}_p$. The first step, which is carried out in Section 5.1, is to determine which of these linear combinations are algebraically slice. In Section 5.2, we use metabelian twisted Blanchfield pairings to obstruct the sliceness of such algebraically slice linear combinations.

5.1 Algebraically slice linear combinations of algebraic knots

Fix an integer $p \geq 2$. For $i = 1, \dots, k$, fix sequences $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i})$ of ℓ_i positive integers each of which is coprime to p, and let $n_1, \dots, n_k \in \mathbb{Z}$. The goal of this subsection is to determine when the following knot is algebraically slice:

$$K = n_1 T(p, Q_1) \# n_2 T(p, Q_2) \# \cdots \# n_k T(p, Q_k). \tag{17}$$

In order to provide a convenient criterion, we define the *s–level* of K to be the knot

$$\mathcal{K}_s(K) := n_1 T(p, q_{1,\ell_1 - s}) \# n_2 T(p, q_{2,\ell_2 - s}) \# \cdots \# n_k T(p, q_{k,\ell_k - s}).$$

Here, it is understood that $T(p, q_{i,\ell_i - s})$ is the unknot U if $\ell_i - s < 1$. As an example of this notation, we see that, if $Q = (q_1, \dots, q_\ell)$, then $\mathcal{K}_s(T(p, Q)) = T(p, q_{\ell - s})$ for $0 \leq s \leq \ell - 1$ and $\mathcal{K}_s(T(p, Q)) = U$ for $s \geq \ell$. In particular, the cabling formula for the classical Blanchfield form implies that

$$\mathrm{Bl}(T(p, Q)) \cong \bigoplus_{s \geq 0} \mathrm{Bl}\big(\mathcal{K}_s(T(p, Q))\big)(t^{p^s}). \tag{18}$$

Indeed, for a knot L, the cabling formula reads as [24]

$$\mathrm{Bl}(L_{p,q})(t) = \mathrm{Bl}(T(p,q))(t) \oplus \mathrm{Bl}(L)(t^p).$$

Next, we move on to a slightly more involved example.

Example 5.1 The s–levels of

$$J := T(p,q_1;p,q_2)\,\#\,T(p,q_3)\,\#-T(p,q_1;p,q_3)\,\#-T(p,q_2)$$

are given by

$$\mathcal{K}_s(J) = \begin{cases} T(p,q_2)\,\#\,T(p,q_3)\,\#-T(p,q_3)\,\#-T(p,q_2) & \text{if } s=0, \\ T(p,q_1)\,\#-T(p,q_1) & \text{if } s=1, \\ U & \text{if } s \geq 2. \end{cases}$$

For $s=1$, we used that $\mathcal{K}_1(J) = T(p,q_1)\#U\#-T(p,q_1)\#-U$ is $T(p,q_1)\#-T(p,q_1)$. In particular, the formula displayed in (18) also holds for J. As we shall use in Proposition 5.3 below, it holds for the linear combination of (17).

For later use, we note that the 0–level of K is the most important to us: the first homology of its p–fold branched cover equals that of K.

Remark 5.2 Since $H_1(\Sigma_p(J_{p,q})) = H_1\big(\Sigma_p(T(p,q))\big)$ for any knot J, we deduce

$$H_1(\Sigma_p(K)) = H_1\big(\Sigma_p(\mathcal{K}_0(K))\big) = \bigoplus_{i=1}^{k} H_1\big(\Sigma_p(T(p,q_{i,\ell_i}))\big).$$

The analogous decomposition holds for the linking form $\lambda_p(K)$ [22, Lemma 4].

The next proposition uses s–levels to exhibit a criterion for the algebraic sliceness of K.

Proposition 5.3 *Fix an integer $p \geq 2$ and choose sequences of positive integers $Q_i = (q_{i,1},\ldots,q_{i,\ell_i})$ that are relatively prime to p for $i = 1,2,\ldots,k$. The following statements are equivalent*:

(i) *The knot $K = n_1T(p,Q_1)\#\cdots\#n_kT(p,Q_k)$ is algebraically slice.*

(ii) *Each $\mathcal{K}_s(K)$ is slice.*

Proof We first assert that the polynomials $\Delta_{\mathcal{K}_s(K)}(t^{p^s})$ and $\Delta_{\mathcal{K}_u(K)}(t^{p^u})$ have distinct roots if $s \neq u$. For a positive integer m, we set $\xi_m := e^{2\pi i/m}$. The roots of $\Delta_{T(p,q)}(t)$ occur at those ξ_{pq}^a where the integer $1 \leq a \leq pq$ is such that neither p nor q divides a, ie $(\xi_{pq}^a)^p \neq 1$ and $(\xi_{pq}^a)^q \neq 1$. Consequently, the roots of $\Delta_{T(p,q)}(t^{p^s})$ occur at $\xi_{p^{s+1}q}^a$ such that $1 \leq a \leq p^{s+1}q$ and neither p nor q divides a.

We argue that, if $s \neq u$, then $\Delta_{T(p,q_1)}(t^{p^s})$ and $\Delta_{T(p,q_2)}(t^{p^u})$ have distinct roots. Assume to the contrary that they have a common root. This root must be of the form $\xi_{p^{s+1}q_1}^a = \xi_{p^{u+1}q_2}^b$, where q_1 and p (resp. q_2 and p) do not divide a (resp. b). Without loss of generality, assume that $s < u$, so that $1 = (\xi_{p^{s+1}q_1}^a)^{p^{s+1}q_1} = (\xi_{p^{u+1}q_2}^b)^{p^{s+1}q_1} = \xi_{p^{u-s}q_2}^{bq_1}$. This implies that $p^{u-s}q_2$ divides bq_1. However, by assumption, p divides neither q_1 nor b, yielding the desired contradiction.

Next, recall from the definition of the s–level that

$$\mathcal{K}_s(K) := n_1 T(p, q_{1,\ell_1-s}) \# n_2 T(p, q_{2,\ell_2-s}) \# \cdots \# n_k T(p, q_{k,\ell_k-s}).$$

Thus, if $s \neq u$, then $\Delta_{\mathcal{K}_s(K)}(t^{p^s})$ and $\Delta_{\mathcal{K}_u(K)}(t^{p^u})$ have distinct roots. This proves the assertion.

Assume that K is algebraically slice. By the cabling formula for the Blanchfield pairing (see Example 5.1),

$$\mathrm{Bl}(K)(t) \cong \bigoplus_{s \geq 0} \mathrm{Bl}(\mathcal{K}_s(K))(t^{p^s}) \tag{19}$$

is metabolic. By the assertion and Proposition 4.3, we deduce that each $\mathrm{Bl}(\mathcal{K}_s(K))(t^{p^s})$ is metabolic. It follows that the jump function of each $\mathrm{Bl}(\mathcal{K}_s(K))(t^{p^s})$ is trivial, which is simply a reparametrisation of the jump function of $\mathrm{Bl}(\mathcal{K}_s(K))(t)$, where the parameter $t \in S^1$ is changed to t^{p^r}. Hence, $\mathcal{K}_s(K)$ is a connected sum of torus knots such that the jump function of $\sigma_\omega(\mathcal{K}_s)$ is trivial. Since Litherland showed in [21, Lemma 1] that the jump functions of $\sigma_\omega(T(p,q))$ are linearly independent, $\mathcal{K}_s(K)$ is slice, as desired.

Assume that each $\mathcal{K}_s(K)$ is slice. As a linking form over $\mathbb{Z}[t^{\pm 1}]$, $\mathrm{Bl}(\mathcal{K}_s(K))$ is metabolic. Combining this with the decomposition displayed in (19), we deduce that $\mathrm{Bl}(K)$ is metabolic, as a linking form over $\mathbb{Z}[t^{\pm 1}]$. This is equivalent to K being algebraically slice [17], completing the proof of Proposition 5.3. □

When K is algebraically slice, we obtain a convenient description of the 0–level of K.

Corollary 5.4 *Suppose that K, p, ℓ_i and Q_i for $i = 1, \ldots, k$ are as in Proposition 5.3. If K is algebraically slice, then k is even and, after renumbering if necessary, the 0–level of K is*

$$\mathcal{K}_0(K) = \mathop{\#}_{j=1}^{k/2} m_j (T(p, q_{j,\ell_j}) \# -T(p, q_{j,\ell_j})).$$

Proof By Proposition 5.3, $\mathcal{K}_0(K)$ is a slice linear combination of torus knots. Since torus knots are linearly independent in the knot concordance group, the conclusion follows. □

5.2 Linearly independent families of iterated torus knots

Fix a prime power p. The goal of this section is to prove Theorem 1.1.

For $i = 1, \dots, k$, we choose sequences $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i})$ of positive integers, where q_{i,ℓ_i} is prime for all i, and the integer $q_{i,j}$ is coprime to p and to q_{i,ℓ_i} for all j. We also let $n_1, \dots, n_k \in \mathbb{Z}$ be integers. We will use metabelian Blanchfield pairings [25; 3; 4; 5] to obstruct the sliceness of the knot

$$K = n_1 T(p, Q_1) \# n_2 T(p, Q_2) \# \cdots \# n_k T(p, Q_k).$$

The sliceness obstruction that we will use, due to Miller and Powell [25, Theorem 6.10], reads as follows. If, for every $\mathbb{Z}_p$–invariant metaboliser G of $\lambda_p(K)$, there exists a prime power–order character χ that vanishes on G and is such that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is not metabolic, then K is not slice. Here, we use $\alpha(p,\chi) := \alpha_K(p,\chi)$ to denote the metabelian representation that was described in Section 3.1.

Remark 5.5 The *metabelian Blanchfield pairing* is a linking form

$$\mathrm{Bl}_{\alpha(p,\chi)}(K)\colon H_1(M_K; \mathbb{C}[t^{\pm 1}]^p_{\alpha(p,\chi)}) \times H_1(M_K; \mathbb{C}[t^{\pm 1}]^p_{\alpha(p,\chi)}) \to \mathbb{C}(t)/\mathbb{C}[t^{\pm 1}],$$

where $H_1(M_K; \mathbb{C}[t^{\pm 1}]^p_{\alpha(p,\chi)})$ denotes the homology of the 0–framed surgery of K twisted by $\alpha(p,\chi)$. The precise definition of $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is not needed in this paper (the interested reader can find it in [25; 3; 4; 5]). All we need is the behaviour of $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ under satellite operations, and this will be recalled as the argument proceeds.

The strategy behind the proof of Theorem 1.1 is as follows:

(i) Firstly, we study the characters on $H_1(\Sigma_p(K))$.

(ii) Secondly, we study the consequences of $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ being metabolic. This will impose substantial restrictions on χ.

(iii) Thirdly, we build characters that violate these restrictions.

(iv) Finally, we combine these first three steps to conclude the proof.

The reader that wishes to see how these steps combine might glance at the end of the argument, after the conclusion of the proof of Lemma 5.8; see Section 5.2.4.

5.2.1 Characters on $H_1(\Sigma_p(K))$ Assume that K is slice. The first step is to study the possible characters on the p–fold branched cover of K. Since K is algebraically slice, Corollary 5.4 implies that k is even and, after renumbering if necessary, for some prime r (which is one of the q_{j,ℓ_j}) and some integers $m_1,\dots,m_{k/2}$, we can write

$$\mathcal{K}_0(K)=m_1(T(p,r)\#-T(p,r))\#\mathop{\#}_{j=2}^{k/2}m_j(T(p,q_{j,\ell_j})\#-T(p,q_{j,\ell_j})),$$

where $q_{i,\ell_i}=r$ if and only if $1\le i\le 2m_1$. It follows that, if we set $M_j=m_1+\dots+m_{j-1}$ for $j=2,\dots,\frac{1}{2}k$, then, after further possible renumbering, the knot K can be rewritten as

$$K=\mathop{\#}_{i=1}^{m_1}(T(p,Q_{2i-1})\#-T(p,Q_{2i})) \\ \#\mathop{\#}_{j=2}^{k/2}\mathop{\#}_{i=1}^{m_j}(T(p,Q_{2M_j+2i-1})\#-T(p,Q_{2M_j+2i})). \tag{20}$$

As Remark 5.2 implies that $H_1(\Sigma_p(K))\cong H_1\big(\Sigma_p(\mathcal{K}_0(K))\big)$, the description of $\mathcal{K}_0(K)$, the primary decomposition and the fact that the q_{i,ℓ_i} are prime show that

$$H_1(\Sigma_p(K))=H_1\big(\Sigma_p(T(p,r))\big)^{m_1}\oplus H_1\big(\Sigma_p(-T(p,r))\big)^{m_1} \\ \oplus\bigoplus_{j=2}^{k/2}\big(H_1\big(\Sigma_p(T(p,q_{j,\ell_j}))\big)^{m_j}\oplus H_1\big(\Sigma_p(-T(p,q_{j,\ell_j}))\big)^{m_j}\big). \tag{21}$$

The linking form $\lambda_p(K)$ on $\Sigma_p(K)$ decomposes analogously.

From now on, θ denotes the trivial character. Also, since $H_1\big(\Sigma_p(T(p,r))\big)\cong\mathbb{Z}_r^{p-1}$, we write characters $H_1\big(\Sigma_p(T(p,r))\big)\to\mathbb{Z}_r$ as $\chi_{\boldsymbol{a}}$, where $\boldsymbol{a}\in\mathbb{Z}_r^p$. Since r is distinct from q_{i,ℓ_i} for $i>2m_1$, the decomposition of (21) implies that any character $\chi\colon H_1(\Sigma_p(K))\to\mathbb{Z}_r$ must be of the form

$$\chi=\bigoplus_{i=1}^{m_1}(\chi_{\boldsymbol{a}^i}\oplus\chi_{\boldsymbol{b}^i})\oplus\bigoplus_{j=2}^{k/2}\bigoplus_{i=1}^{m_j}\theta\oplus\theta, \tag{22}$$

where $\{\boldsymbol{a}^j\}_{j=1}^{m_1}$ and $\{\boldsymbol{b}^j\}_{j=1}^{m_1}$ are sequences of p elements in $\mathbb{Z}_r$.

Remark 5.6 Recall that the Miller–Powell obstruction requires that, for every $\mathbb{Z}_p$–invariant metaboliser G of $\lambda_p(K)$, we construct a prime power–order character χ that vanishes on G and is such that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is not metabolic. The primary decomposition implies that every such metaboliser decomposes as a direct sum of metabolisers of the summands in (21).

Consequently, thanks to the form of the character in (22), it suffices to prove the following result: for every $\mathbb{Z}_p$–invariant metaboliser L of $\lambda_p(T(p,r))^{m_1} \oplus -\lambda_p(T(p,r))^{m_1}$, there is a prime power–order character $\bigoplus_{i=1}^{m_1}(\chi_{\boldsymbol{a}^i} \oplus \chi_{\boldsymbol{b}^i})$ that vanishes on L and is such that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is not metabolic, with χ as in (22).

5.2.2 The metabelian Blanchfield pairing of K We now study the metabelian Blanchfield pairing of K. We first use satellite formulas to decompose it, and we then study the implications of it being metabolic. We use $\alpha(p,\chi) := \alpha_K(p,\chi)$ to denote the metabelian representation that was described in Section 3.1. The behaviour of metabelian Blanchfield pairings under connected sums [4, Corollary 4.21] implies that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is Witt equivalent to the linking form

$$\begin{aligned}(23)\quad &\mathrm{Bl}_{\alpha(p,\chi)}(K)\\ &\sim \bigoplus_{i=1}^{m_1}\big(\mathrm{Bl}_{\alpha(p,\chi_{\boldsymbol{a}^i})}(T(p,Q_{2i-1})) \oplus -\mathrm{Bl}_{\alpha(p,\chi_{\boldsymbol{b}^i})}(T(p,Q_{2i}))\big)\\ &\qquad \oplus \bigoplus_{j=2}^{k/2}\bigoplus_{i=1}^{m_j}\big(\mathrm{Bl}_{\alpha(p,\theta)}(T(p,Q_{2M_j+2i-1})) \oplus -\mathrm{Bl}_{\alpha(p,\theta)}(T(p,Q_{2M_j+2i}))\big).\end{aligned}$$

For a sequence $S=(q_1,\dots,q_k)$, we use $T(p,\widehat{S})$ to denote the iterated torus knot $T(p,q_1;\dots;p,q_{k-1})$. Next, we apply the satellite formula for the metabelian Blanchfield pairing [4, Theorem 4.19] to both expressions in (23). As we are working with p–fold covers and the sequences Q_{2i-1} and Q_{2i} (resp. Q_{2M_j+2i-1} and Q_{2M_j+2i}) both have r (resp. q_{j,ℓ_j}) as the prime in last position, we claim

$$\begin{aligned}(24)\quad &\mathrm{Bl}_{\alpha(p,\chi)}(K)\\ &\sim \bigoplus_{i=1}^{m_1}\big(\mathrm{Bl}_{\alpha(p,\chi_{\boldsymbol{a}^i})}(T(p,r)) \oplus -\mathrm{Bl}_{\alpha(p,\chi_{\boldsymbol{b}^i})}(T(p,r))\big)\\ &\qquad \oplus \bigoplus_{i=1}^{m_1}\bigoplus_{u=1}^{p}\big(\mathrm{Bl}(T(p,\widehat{Q}_{2i-1}))(\xi_r^{\boldsymbol{a}^i_u}t) \oplus -\mathrm{Bl}(T(p,\widehat{Q}_{2i}))(\xi_r^{\boldsymbol{b}^i_u}t)\big)\\ &\qquad \oplus \bigoplus_{j=2}^{k/2}\bigoplus_{i=1}^{m_j}\big(\mathrm{Bl}_{\alpha(p,\theta)}(T(p,q_{j,\ell_j})) \oplus -\mathrm{Bl}_{\alpha(p,\theta)}(T(p,q_{j,\ell_j}))\big)\\ &\qquad \oplus \bigoplus_{j=2}^{k/2}\bigoplus_{i=1}^{m_j}\bigoplus_{u=1}^{p}\big(\mathrm{Bl}(T(p,\widehat{Q}_{2M_j+2i-1}))(t) \oplus -\mathrm{Bl}(T(p,\widehat{Q}_{2M_j+2i}))(t)\big).\end{aligned}$$

The satellite formula of [4, Theorem 4.19] involves $\mathrm{Bl}(K)\big(\xi_{q_1}^{\chi(t_Q^{i-1} q_Q(\mu_Q^{-w}\eta))} t\big)$, where μ_Q denotes the meridian of the satellite knot $Q = P_\eta(K)$ with pattern P, companion K and infection curve η; furthermore, $q_Q\colon \pi_1(M_Q) \to H_1(\Sigma_p(Q))$ denotes the map described in (7). Recalling the notation of Section 2, we see that, in our case, η coincides with the curve c_2, and $\mu_Q = \mu_{T(p,q)}$. Thus, as explained in (9) for $\chi = \chi_{\boldsymbol{a}}$, we deduce that $\chi(t_Q^{u-1} q_Q(\mu_Q^{-w}\eta)) = \boldsymbol{a}_u$, and this explains the second summand of (24). The decomposition in (24) is now justified, concluding the claim.

Next, we wish to apply the cabling formula $\mathrm{Bl}(J_{p,q})(t) = \mathrm{Bl}(T(p,q))(t) \oplus \mathrm{Bl}(J)(t^p)$ for the classical Blanchfield pairing. To make notation more manageable, however, for $s \geq 1$, coprime integers p and q, and $\boldsymbol{a} \in \mathbb{Z}_r^p$, we consider the linking form

$$\Lambda(p,q,\chi_{\boldsymbol{a}},s) := \bigoplus_{u=0}^{p-1} \mathrm{Bl}(T(p,q))(\xi_r^{p^{s-1}\boldsymbol{a}_u} t^{p^{s-1}}).$$

If the character $\chi_{\boldsymbol{a}}$ is trivial, then we write $\Lambda(p,q,s)$ instead of $\Lambda(p,q,\theta,s)$. These pairings appear as summands of the Blanchfield pairing of a cable. Indeed, using this notation and the aforementioned untwisted cabling formula, we deduce from (24) that

$$\begin{aligned}
&\mathrm{Bl}_{\alpha(p,\chi)}(K)\\
&\sim \bigoplus_{i=1}^{m_1} \big(\mathrm{Bl}_{\alpha(p,\chi_{\boldsymbol{a}^i})}(T(p,r)) \oplus -\mathrm{Bl}_{\alpha(p,\chi_{\boldsymbol{b}^i})}(T(p,r))\big) \dots\\
&\quad \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} \big(\mathrm{Bl}_{\alpha(p,\theta)}(T(p,q_{j,\ell_j})) \oplus -\mathrm{Bl}_{\alpha(p,\theta)}(T(p,q_{j,\ell_j}))\big)\\
&\quad \oplus \bigoplus_{i=1}^{m_1} \bigoplus_{s\geq 1} (\Lambda(p,q_{2i-1,\ell_{2i-1}-s},\chi_{\boldsymbol{a}^i},s) \oplus -\Lambda(p,q_{2i,\ell_{2i}-s},\chi_{\boldsymbol{b}^i},s))\\
&\quad \oplus \bigoplus_{j=1}^{k/2} \bigoplus_{i=2}^{m_j} \bigoplus_{s\geq 1} (\Lambda(p,q_{2M_j+2i-1,\ell_{2M_j+2i-1}-s},s) \oplus -\Lambda(p,q_{2M_j+2i,\ell_{2M_j+2i}-s},s))\\
&=: B_1^\chi \oplus B_2 \oplus B_3^\chi \oplus B_4.
\end{aligned}$$

Now that we have decomposed $\mathrm{Bl}_{\alpha(p,\chi)}(K)$, we study the consequences of it being metabolic.

Claim 1 *If* $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ *is metabolic, then* B_1^χ *and* $B_3^\chi \oplus B_4$ *are metabolic.*

Proof As $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ and B_2 are metabolic, $B_1^\chi \oplus (B_3^\chi \oplus B_4)$ is metabolic. By Proposition 4.3, it suffices to prove that the orders of B_1^χ and $B_3^\chi \oplus B_4$ have distinct roots:

the roots of the twisted polynomial occur at prime powers of unity (by Proposition 3.3), while this is never the case for the classical Alexander polynomial [8, proof of Proposition 3.3(3)].[2] This proves Claim 1. □

In order to study the consequences of $B_3^\chi \oplus B_4$ being metabolic, for $s \geq 1$, we set

$$B_3^\chi(s) := \bigoplus_{i=1}^{m_1} (\Lambda(p, q_{2i-1,\ell_{2i-1}-s}, \chi_{\boldsymbol{a}^i}, s) \oplus -\Lambda(p, q_{2i,\ell_{2i}-s}, \chi_{\boldsymbol{b}^i}, s)),$$
$$B_4(s) := \bigoplus_{j=1}^{k/2} \bigoplus_{i=2}^{m_j} (\Lambda(p, q_{2M_j+2i-1,\ell_{2M_j+2i-1}-s}, s) \oplus -\Lambda(p, q_{2M_j+2i,\ell_{2M_j+2i}-s}, s)).$$

Using these forms, we derive a further consequence of $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ being metabolic.

Claim 2 *If $B_3^\chi \oplus B_4$ is metabolic, then $B_3^\chi(s) \oplus B_4(s)$ is metabolic for each s.*

Proof By definition, we have the decompositions $B_3^\chi = \bigoplus_{s \geq 1} B_3^\chi(s)$ and $B_4 = \bigoplus_{s \geq 1} B_4(s)$. For $u \neq v$, the order of $B_3^\chi(u) \oplus B_4(u)$ and the order of $B_3^\chi(v) \oplus B_4(v)$ have distinct roots. By Proposition 4.3, Claim 2 follows. □

Consequently, it is sufficient to study the linking forms $B_3^\chi(s) \oplus B_4(s)$ for a fixed $s \geq 1$. To further decompose $B_3^\chi(s) \oplus B_4(s)$, we want to group these linking forms according to the torus knots that appear. We also need to be attentive to the fact that the torus knot $T(p, q_{i,\ell_i-s})$ is trivial when $i \leq \ell_i$. As a consequence, for $s \geq 1$, we consider the sets

$$\mathcal{I}_1(q,s) := \{1 \leq i \leq m_1 \mid \ell_{2i-1} > s,\ q_{2i-1,\ell_{2i-1}-s} = q\},$$
$$\mathcal{I}_2(q,s) := \{1 \leq i \leq m_1 \mid \ell_{2i} > s,\ q_{2i,\ell_{2i}-s} = q\},$$
$$\mathcal{I}_3(q,s) := \bigcup_{j=2}^{k/2} \{1 \leq i \leq m_j \mid \ell_{2M_j+2i-1} > s,\ q_{2M_j+2i-1,\ell_{2M_j+2i-1}-s} = q\}, \tag{25}$$
$$\mathcal{I}_4(q,s) := \bigcup_{j=2}^{k/2} \{1 \leq i \leq m_j \mid \ell_{2M_j+2i} > s,\ q_{2M_j+2i,\ell_{2M_j+2i}-s} = q\}.$$

[2] Here is a topological proof of this fact: for a knot K and an integer q, the order of $H_1(\Sigma_q(K))$ is $\prod_{a=1}^{q-1} \Delta_K(\xi_q^a)$ [20, Corollary 9.8]; since q is a prime power, $H_1(\Sigma_q(K))$ is a finite group, and thus none of the $\Delta_K(\xi_q^a)$ can vanish.

Note that, for some q, the set $\mathcal{I}_i(q,s)$ may well be empty. However, from now on, we will implicitly assume that we only consider q for which this is not the case. In order to study the consequences of $B_3^\chi(s) \oplus B_4(s)$ being metabolic, we set

$$B_3^\chi(q,s) := \bigoplus_{k \in \mathcal{I}_1(q,s)} \Lambda(p,q,\chi_{\boldsymbol{a}^k},s) \oplus - \bigoplus_{k \in \mathcal{I}_2(q,s)} \Lambda(p,q,\chi_{\boldsymbol{b}^k},s),$$
$$B_4(q,s) := \bigoplus_{k \in \mathcal{I}_3(q,s)} \Lambda(p,q,s) \oplus - \bigoplus_{k \in \mathcal{I}_4(q,s)} \Lambda(p,q,s).$$

Note that $B_4(q,s)$ is not automatically metabolic as the cardinality of $\mathcal{I}_3(q,s)$ need not agree with that of $\mathcal{I}_4(q,s)$. Observe however that, if K is algebraically slice, Proposition 5.3 implies that

$$\#\mathcal{I}_1(q,s) - \#\mathcal{I}_2(q,s) + \#\mathcal{I}_3(q,s) - \#\mathcal{I}_4(q,s) = 0. \tag{26}$$

Indeed, note that the sets $\mathcal{I}_i(q,s)$ record where $T(p,q)$ appears in the s–level of K. Using the $B_i(q,s)$, we now derive a further consequence of $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ being metabolic.

Claim 3 *If $B_3^\chi(s) \oplus B_4(s)$ is metabolic, then $B_3^\chi(q,s) \oplus B_4(q,s)$ is metabolic for each q.*

Proof We have decompositions $B_3^\chi(s) = \bigoplus_{q \geq 1} B_3^\chi(q,s)$ and $B_4(s) = \bigoplus_{q \geq 1} B_4(q,s)$. Since all the q_i are positive, for $u \neq v$, the order of $B_3^\chi(u,s) \oplus B_4(u,s)$ and the order of $B_3^\chi(v,s) \oplus B_4(v,s)$ have distinct roots. By Proposition 4.3, Claim 3 follows. □

Summarising these claims, we have shown that, if the metabelian Blanchfield pairing $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is metabolic, then the linking forms $B_3^\chi(q,s) \oplus B_4(q,s)$ are metabolic for all q and s. This concludes the second part of the proof.

5.2.3 Building the characters that vanish on metabolisers The third part consists in showing that, for every $\mathbb{Z}_p$–invariant metaboliser L of $\lambda_p(T(p,r))^{m_1} \oplus -\lambda_p(T(p,r))^{m_1}$, there are characters $\chi_{\boldsymbol{a}} = \bigoplus_{i=1}^{m_1} \chi_{\boldsymbol{a}^i}$ and $\chi_{\boldsymbol{b}} = \bigoplus_{i=1}^{m_1} \chi_{\boldsymbol{b}^i}$ such that $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishes on L, but for which the linking forms $B_3^\chi(q,s) \oplus B_4(q,s)$ are not all metabolic, where $\chi = \chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}} \oplus \theta$ is as in (22).

The next proposition describes characters for which $B_3^\chi(q,s) \oplus B_4(q,s)$ is not metabolic.

Proposition 5.7 *Let $q,s>0$ be positive integers with q coprime to p. If a character $\bigoplus_{i=1}^{m_1}\chi_{\boldsymbol{a}^i}\oplus\chi_{\boldsymbol{b}^i}$ satisfies either*

(i) *$\chi_{\boldsymbol{b}^k}=\theta$ for every $k\in I_2(q,s)$ and $\chi_{\boldsymbol{a}^{k_0}}\neq\theta$ for some $k_0\in I_1(q,s)$, or*

(ii) *$\chi_{\boldsymbol{a}^k}=\theta$ for every $k\in I_1(q,s)$ and $\chi_{\boldsymbol{b}^{k_0}}\neq\theta$ for some $k_0\in I_2(q,s)$,*

then the linking form $B_3^{\chi}(q,s)\oplus B_4(q,s)$ is not metabolic.

Proof We will only consider case (i). In order to give the proof in case (ii), just exchange the roles of $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$. Assume that $\chi_{\boldsymbol{b}^k}=\theta$ for every $k\in\mathcal{I}_2(q,s)$ and $\chi_{\boldsymbol{a}^{k_0}}\neq\theta$ for some $k_0\in\mathcal{I}_1(q,s)$. Since K is algebraically slice, recall from (26) that

$$\#\mathcal{I}_1(q,s)-\#\mathcal{I}_2(q,s)+\#\mathcal{I}_3(q,s)-\#\mathcal{I}_4(q,s)=0.$$

We thus define $N:=\#\mathcal{I}_1(q,s)=\#\mathcal{I}_2(q,s)-\#\mathcal{I}_3(q,s)+\#\mathcal{I}_4(q,s)$, leading to the Witt equivalence

$$B_3^{\chi}(q,s)\oplus B_4(q,s)\sim\bigoplus_{k\in I_1(q,s)}\Lambda(p,q,\chi_{\boldsymbol{a}^i},s)\oplus-\bigoplus_{i=1}^{p\cdot N}\mathrm{Bl}(T(p,q))(t^{p^{s-1}}). \tag{27}$$

We assert that the orders of the modules underlying the summands of the right-hand side of (27) have distinct roots. First, note that r is coprime to q: as $k\in\mathcal{I}_1(q,s)$, we know that $q\in Q_i$ for some $i<2m_1$ and, since $Q_i=(q_{i,1},q_{i,2},\dots,q_{i,\ell_i-1},r)$ for $i<2m_1$, this follows from the assumption of Theorem 1.1. It is known that $\Delta_{T(p,q)}(\xi_r^{a_1}t)$ and $\Delta_{T(p,q)}(\xi_r^{a_2}t)$ have distinct roots whenever $a_1\neq a_2$ and r and q are coprime [14, Theorem 7.1]. This establishes the assertion.

Thanks to the assertion, we may apply Proposition 4.3. Indeed, the fact that $\chi_{\boldsymbol{a}^{k_0}}\neq\theta$ and Proposition 4.3 now guarantees that the linking form on the right-hand side of (27) is not metabolic. □

Before constructing the required characters, we introduce some terminology. We say that the knot K is *simplified* if there are no indices $k_1\in\mathcal{I}_1(q,s)$ and $k_2\in\mathcal{I}_2(q,s)$ such that $Q_{2k_1-1}=Q_{2k_2}$. If K is not simplified, then it contains a slice connected summand $T(p,Q_{2k_1-1})\#-T(p,Q_{2k_1-1})$.

Lemma 5.8 *Let p be a prime power. If the knot K is simplified, then, for any $\mathbb{Z}_p$–invariant metaboliser $L\subset H_1\big(\Sigma_p(T(p,r))\big)^{m_1}\oplus H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$, there exist q, s and a character $\chi_{\boldsymbol{a}}\oplus\chi_{\boldsymbol{b}}=\bigoplus_{i=1}^{m_1}\chi_{\boldsymbol{a}^i}\oplus\chi_{\boldsymbol{b}^i}$ vanishing on L such that either*

(i) *$\chi_{\boldsymbol{b}^k}=\theta$ for every $k\in\mathcal{I}_2(q,s)$ and $\chi_{\boldsymbol{a}^{k_0}}\neq\theta$ for some $k_0\in\mathcal{I}_1(q,s)$, or*

(ii) *$\chi_{\boldsymbol{a}^k}=\theta$ for every $k\in\mathcal{I}_1(q,s)$ and $\chi_{\boldsymbol{b}^{k_0}}\neq\theta$ for some $k_0\in\mathcal{I}_2(q,s)$.*

Proof Fix a metaboliser $L \subset H_1\big(\Sigma_p(T(p,r))\big)^{m_1} \oplus H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$ of

$$\lambda_p(T(p,r))^{m_1} \oplus -\lambda_p(T(p,r))^{m_1}.$$

For $i=1,2$, consider the projection $\mathrm{pr}_i : H_1\big(\Sigma_p(T(p,r))\big)^{m_1} \oplus H_1\big(\Sigma_p(T(p,r))\big)^{m_1} \to H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$ onto the i^{th} factor. The proof is divided into three separate cases.

Case 1 $\big(\mathrm{pr}_1(L)$ is a proper subspace of $H_1\big(\Sigma_p(T(p,r))\big)^{m_1}\big)$ In this case, we can define the characters $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ as $\chi_{\boldsymbol{b}} = \theta$ and

$$\chi_{\boldsymbol{a}} \colon H_1\big(\Sigma_p(T(p,r))\big)^{m_1} \to H_1\big(\Sigma_p(T(p,r))\big)^{m_1}/\mathrm{pr}_1(L) \xrightarrow{\text{nontrivial character}} \mathbb{Z}_r.$$

It is not difficult to see that $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ satisfy (i) and are such that $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishes on L.

Case 2 $\big(\mathrm{pr}_2(L)$ is a proper subspace of $H_1\big(\Sigma_p(T(p,r))\big)^{m_1}\big)$ In this case, we exchange the roles of $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ and repeat the argument from the first case. This way, we obtain characters $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ that satisfy (ii) and are such that $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishes on L.

Case 3 $\big(\mathrm{pr}_1(L) = H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$ and $\mathrm{pr}_2(L) = H_1\big(\Sigma_p(T(p,r))\big)^{m_1}\big)$ We wish to apply Proposition 4.4 in order to prove that L is a graph. We verify the hypothesis of this proposition. Using the assumption of Case 3 and the definition of the projections, we have

$$\begin{aligned} 0 &= \ker(\mathrm{pr}_1|_L) = L \cap \big(0 \oplus H_1\big(\Sigma_p(T(p,r))\big)^{m_1}\big), \\ 0 &= \ker(\mathrm{pr}_2|_L) = L \cap \big(H_1\big(\Sigma_p(T(p,r))\big)^{m_1} \oplus 0\big). \end{aligned}$$

Consequently, by Proposition 4.4, L is the graph of an isometry

$$g \colon \big(H_1\big(\Sigma_p(T(p,r))\big)^{m_1}, \lambda_p(T(p,r))^{m_1}\big) \to \big(H_1\big(\Sigma_p(T(p,r))\big)^{m_1}, \lambda_p(T(p,r))^{m_1}\big).$$

For each q, s and $j=1,2$, consider the subsets of $H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$

$$\begin{aligned} S_{\mathcal{I}_j(q,s)} &= \{(v_1, v_2, \dots, v_{m_1}) \in H_1\big(\Sigma_p(T(p,r))\big)^{m_1} : v_i = 0 \text{ for } i \notin \mathcal{I}_j(q,s)\} \\ &= \bigoplus_{k \in \mathcal{I}_j(q,s)} H_1\big(\Sigma_p(T(p,Q_k))\big), \end{aligned}$$

where $\mathcal{I}_j(q,s)$ is as defined in (25).

Next, we use these sets and the isometry g to describe a sufficient criterion to obtain the characters $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ required by the statement of Lemma 5.8.

Claim 4 *If there exist q and s such that $g(S_{\mathcal{I}_1(q,s)}) \neq S_{\mathcal{I}_2(q,s)}$, then there are characters $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ satisfying either* (i) *or* (ii) *and such that $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishes on L.*

Proof If $g(S_{\mathcal{I}_1(q,s)}) \setminus S_{\mathcal{I}_2(q,s)} \neq \varnothing$, then choose $v \in S_{\mathcal{I}_1(q,s)}$ such that $g(v) \notin S_{\mathcal{I}_2(q,s)}$. Since r is a prime, $H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$ is an $\mathbb{F}_r$–vector space and so we obtain a direct-sum decomposition $H_1\big(\Sigma_p(T(p,r))\big)^{m_1} = \langle v \rangle \oplus W$ for some $\mathbb{F}_r$–vector space W. We can then define the characters as

$$\chi_{\boldsymbol{a}}(v) = 1, \quad \chi_{\boldsymbol{a}}|_W = \theta, \quad \chi_{\boldsymbol{b}}(x) = -\chi_{\boldsymbol{a}}(g^{-1}(x)).$$

Such choices of $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ satisfy condition (i). We verify that $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishes on L, where we recall that L is the graph of g. For an element $(h, g(h)) \in L$ of this graph, one has $(\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}})(h, g(h)) = \chi_{\boldsymbol{a}}(h) - \chi_{\boldsymbol{a}}\big(g^{-1}(g(h))\big) = 0$. This concludes the proof in this case.

If, on the other hand, we assume that $S_{\mathcal{I}_2(q,s)} \setminus g(S_{\mathcal{I}_1(q,s)}) \neq \varnothing$, the argument is nearly identical. Choose $v \in S_{\mathcal{I}_2(q,s)} \setminus g(S_{\mathcal{I}_1(q,s)})$ and write once more $H_1\big(\Sigma_p(T(p,r))\big)^{m_1} = \langle v \rangle \oplus W$ and define the required characters as

$$\chi_{\boldsymbol{b}}(v) = 1, \quad \chi_{\boldsymbol{b}}|_W = \theta, \quad \chi_{\boldsymbol{a}}(x) = -\chi_{\boldsymbol{b}}(g(x)).$$

These choices of $\chi_{\boldsymbol{a}}$ and $\chi_{\boldsymbol{b}}$ satisfy condition (ii) and $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishes on L. □

By Claim 4, to prove Lemma 5.8, it is enough to show that there always exist q and s such that $g(S_{\mathcal{I}_1(q,s)}) \neq S_{\mathcal{I}_2(q,s)}$. Assume by way of contradiction that $g(S_{\mathcal{I}_1(q,s)}) = S_{\mathcal{I}_2(q,s)}$ for all q and s. We will show in Claim 5 below that this assumption implies that K is not simplified. This is a contradiction since we assumed that K is simplified. This proves Lemma 5.8 modulo Claim 5. □

Claim 5 *If $g(S_{\mathcal{I}_1(q,s)}) = S_{\mathcal{I}_2(q,s)}$ for all q and s, then K is not simplified.*

Proof We will observe that, under the assumption of the claim, K contains a summand of the form $T(p, Q_{2k_0-1}) \# -T(p, Q_{2k_0-1})$ for some integer k_0. To be precise, choose $1 \le k_0 \le m_1$ such that the length ℓ_{2k_0-1} of the sequence of Q_{2k_0-1} is maximal among all the ℓ_{2k-1} for $k = 1, \dots, m_1$, and define[3]

$$X(k_0) = \{1 \le k \le m_1 \mid Q_{2k_0-1} = Q_{2k-1}\} = \bigcap_{s=1}^{\ell_{2k_0-1}} \mathcal{I}_1(q_{2k_0-1,\ell_{2k_0-1}-s}, s),$$

$$Y(k_0) = \{1 \le k \le m_1 \mid Q_{2k_0-1} = Q_{2k}\} = \bigcap_{s=1}^{\ell_{2k_0-1}} \mathcal{I}_2(q_{2k_0-1,\ell_{2k_0-1}-s}, s).$$

We will need the following properties of these sets:

(a) Since $k_0 \in X(k_0)$, $X(k_0)$ is nonempty.

(b) If $j \in X(k_0)$, then $T(p, Q_{2j-1}) = T(p, Q_{2k_0-1})$.

(c) If $j \in Y(k_0)$, then $T(p, Q_{2j}) = T(p, Q_{2k_0-1})$.

It is enough to show that $Y(k_0) \neq \varnothing$. By (a)–(c), this would imply that K is not simplified since K contains a summand of the form $T(p, Q_{2k_0-1}) \# -T(p, Q_{2k_0-1})$.

To show that $Y(k_0) \neq \varnothing$, consider the subspaces of $H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$

$$\begin{aligned} S_{X(k_0)} &:= \{(v_1, v_2, \dots, v_{m_1}) \in H_1\big(\Sigma_p(T(p,r))\big)^{m_1} : v_i = 0 \text{ for } i \notin X(k_0)\}, \\ &= \bigoplus_{k \in X(k_0)} H_1\big(\Sigma_p(T(p, Q_{2k-1}))\big), \\ S_{Y(k_0)} &:= \{(v_1, v_2, \dots, v_{m_1}) \in H_1\big(\Sigma_p(T(p,r))\big)^{m_1} : v_i = 0 \text{ for } i \notin Y(k_0)\} \\ &= \bigoplus_{k \in Y(k_0)} H_1\big(\Sigma_p(T(p, Q_{2k}))\big). \end{aligned}$$

The advantage of writing $X(k_0)$ and $Y(k_0)$ as intersections of the $\mathcal{I}_j(q_{k_0,\ell_{k_0-s}}, s)$ is that the action of g on $S_{X(k_0)}$ can be described as

$$g(S_{X(k_0)}) = \bigcap_{s \ge 1} S_{g(\mathcal{I}_1(q_{2k_0-1,\ell_{2k_0-1}-s}, s))} = \bigcap_{s \ge 1} S_{\mathcal{I}_2(q_{2k_0-1,\ell_{2k_0-1}-s}, s)} = S_{Y(k_0)},$$

where the second equality follows from the assumption. As g is an $\mathbb{F}_r$–linear automorphism, $\dim S_{X(k_0)} = \dim S_{Y(k_0)}$. Since the $\mathbb{F}_r$–dimension of $H_1\big(\Sigma_p(T(p,r))\big)$ is $p-1$, we deduce that

$$(p-1)\,\#X(k_0) = \dim S_{X(k_0)} = \dim S_{Y(k_0)} = (p-1)\,\#Y(k_0).$$

It follows that $\#X(k_0) = \#Y(k_0)$. Since $X(k_0) \neq \varnothing$ by (a), it follows that $Y(k_0) \neq \varnothing$. As we mentioned, this implies that K is not simplified by (a)–(c) and Claim 5 is proved. □

[3]Without the maximality assumption on ℓ_{2k_0-1}, we would have had to replace the condition $Q_{k_0} = Q_k$ by $Q_{k_0} \subset Q_k$.

This concludes the third part of the proof.

5.2.4 Conclusion of the proof of Theorem 1.1 Let K be a (nontrivial) linear combination of iterated torus knots of the form $T(p, Q_i)$ for $i = 1, \dots, k$. Here, the $Q_i = (q_{i,1}, q_{i,2}, \dots, q_{i,\ell_i})$ are sequences of ℓ_i positive integers where q_{i,ℓ_i} is prime for all i and the integer $q_{i,j}$ is coprime to p and to q_{i,ℓ_i} for all j. Assume that K is slice to obtain a contradiction. In particular K is algebraically slice and, as we saw in (20), we can therefore assume without loss generality that it is of the form

$$K = \mathop{\#}_{i=1}^{m_1} (T(p, Q_{2i-1}) \# -T(p, Q_{2i})) \\ \# \mathop{\#}_{j=2}^{k/2} \mathop{\#}_{i=1}^{m_j} (T(p, Q_{2M_j+2i-1}) \# -T(p, Q_{2M_j+2i})). \tag{28}$$

Here we arranged that $q_{i,\ell_i} = r$ if and only if $1 \le i \le 2m_1$. Furthermore, we can assume that K is simplified by cancelling terms of the form $J \# -J$ if any such term appears in (28). We can also assume that there is an index i such that $\ell_i > 1$: otherwise, K would be a linear combination of torus knots, which is impossible since the latter are linearly independent in $\mathcal{C}^{\mathrm{top}}$ [21]. To prove that K is not slice, we saw that it is enough to show that, for every $\mathbb{Z}_p$–invariant metaboliser L of $\lambda_p(T(p,r))^{m_1} \oplus -\lambda_p(T(p,r))^{m_1}$, there is a character $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}} = \bigoplus_{k=1}^{m_1} (\chi_{\boldsymbol{a}^k} \oplus \chi_{\boldsymbol{b}^k})$ that vanishes on L such that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is not metabolic, where $\chi = \chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}} \oplus \bigoplus_{j=2}^{k/2} \bigoplus_{i=1}^{m_j} \theta \oplus \theta$; recall Remark 5.6. We then applied satellite formulas to show that $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ decomposes (up to Witt equivalence) as

$$\mathrm{Bl}_{\alpha(p,\chi)}(K) \sim B_1^\chi \oplus B_2 \oplus B_3^\chi \oplus B_4 = B_1^\chi \oplus B_2 \oplus \bigoplus_{q,s} B_3^\chi(q,s) \oplus \bigoplus_{q,s} B_4(q,s).$$

Claim 1 shows that, if $\mathrm{Bl}_{\alpha(p,\chi)}(K)$ is metabolic, then B_1^χ and $B_3^\chi \oplus B_4$ are metabolic. By Claims 2 and 3, it follows that $B_3^\chi(q,s) \oplus B_3^\chi(q,s)$ must be metabolic *for all* q and s and all characters $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$. On the other hand, as the knot K is simplified, Lemma 5.8 implies that, for any $\mathbb{Z}_p$–invariant metaboliser $L \subset H_1\big(\Sigma_p(T(p,r))\big)^{m_1} \oplus H_1\big(\Sigma_p(T(p,r))\big)^{m_1}$, there exist q, s and a character $\chi_{\boldsymbol{a}} \oplus \chi_{\boldsymbol{b}}$ vanishing on L such that either

(i) $\chi_{\boldsymbol{b}^k} = \theta$ for every $k \in \mathcal{I}_2(q,s)$ and $\chi_{\boldsymbol{a}^{k_0}} \neq \theta$ for some $k_0 \in \mathcal{I}_1(q,s)$, or

(ii) $\chi_{\boldsymbol{a}^k} = \theta$ for every $k \in \mathcal{I}_1(q,s)$ and $\chi_{\boldsymbol{b}^{k_0}} \neq \theta$ for some $k_0 \in \mathcal{I}_2(q,s)$.

Applying Proposition 5.7, we deduce that for such characters and such integers q and s, the linking form $B_3^\chi(q,s) \oplus B_4(q,s)$ is not metabolic. This is the desired contradiction, and Theorem 1.1 is proved. □

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Rectification of interleavings and a persistent Whitehead theorem

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The homotopy interleaving distance, a distance between persistent spaces, was introduced by Blumberg and Lesnick and shown to be universal, in the sense that it is the largest homotopy-invariant distance for which sublevel-set filtrations of close-by real-valued functions are close-by. There are other ways of constructing homotopy-invariant distances, but not much is known about the relationships between these choices. We show that other natural distances differ from the homotopy interleaving distance in at most a multiplicative constant, and prove versions of the persistent Whitehead theorem, a conjecture of Blumberg and Lesnick that relates morphisms that induce interleavings in persistent homotopy groups to stronger homotopy-invariant notions of interleaving.

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1 Introduction

Context Many of the main theoretical tools of topological data analysis (TDA) come in the form of stability theorems. One of the best known stability theorems, due to Cohen-Steiner, Edelsbrunner and Harer [7], implies that if $f, g : X \to \mathbb{R}$ are sufficiently tame functions, such as piecewise linear functions on the geometric realization of a finite simplicial complex, then

$$d_B(D_n(f), D_n(g)) \leq \|f - g\|_\infty.$$

Here, $D_n(f)$ denotes the n–dimensional *persistence diagram* of f. This consists of a multiset of points of the extended plane $\overline{\mathbb{R}}^2$ that captures the isomorphism type of the n^{th} *persistent homology* of the sublevel sets of f, that is, of the functor $\mathbb{R} \to \mathrm{Vec}$ obtained by composing the sublevel-set filtration $r \mapsto f^{-1}(-\infty, r] : \mathbb{R} \to \mathrm{Top}$ with the

n^{th} homology functor $H_n\colon \mathrm{Top} \to \mathrm{Vec}$, where $\mathbb{R}$ denotes the poset of real numbers and Vec denotes the category of vector spaces over some fixed field. The distance d_B is the *bottleneck distance*, a combinatorial way of comparing persistence diagrams.

This result was later refined by Chazal, Cohen-Steiner, Glisse, Guibas and Oudot in [4] to the *algebraic stability theorem*, which says that for $F, G\colon \mathbb{R} \to \mathrm{Vec}$ sufficiently tame functors, one has

$$d_B(D(F), D(G)) \le d_I(F, G),$$

where, as before, $D(F)$ denotes the persistence diagram of F, which describes the isomorphism type of F, and d_I denotes the *interleaving distance*, a distance between functors $\mathbb{R} \to C$ for any fixed category C, which we recall below.

Stability theorems imply that pipelines like the following, popular in TDA, are robust to perturbations of the input data and can be used for inference purposes:

$$\boxed{\text{data}} \to \boxed{\text{persistent spaces}} \xrightarrow{H_n} \boxed{\text{persistent vector spaces}} \xrightarrow{D} \boxed{\text{persistence diagrams}}$$

For example, the algebraic stability theorem tells us that the last step is stable, if we endow persistent vector spaces ($\mathrm{Vec}^{\mathbb{R}}$) with the interleaving distance and persistence diagrams with the bottleneck distance, while functoriality implies that the second step is stable, if we also endow persistent spaces ($\mathrm{Top}^{\mathbb{R}}$) with the interleaving distance; see Bubenik and Scott [3].

Problem statement Although useful in some applications, the interleaving distance on $\mathrm{Top}^{\mathbb{R}}$ is often too fine; for instance, it is easy to see that Vietoris–Rips and other functors $S\colon \mathrm{Met} \to \mathrm{Top}^{\mathbb{R}}$ are not stable with respect to the Gromov–Hausdorff distance on metric spaces and the interleaving distance on $\mathrm{Top}^{\mathbb{R}}$. However, when one composes these functors with a homotopy-invariant functor, such as homology $H_n\colon \mathrm{Top}^{\mathbb{R}} \to \mathrm{Vec}^{\mathbb{R}}$, the composite $H_n \circ S\colon \mathrm{Met} \to \mathrm{Vec}^{\mathbb{R}}$ turns out to be stable; see Chazal, Cohen-Steiner, Guibas, Mémoli and Oudot [5]. So, in these cases, one way to make the first step in the pipeline above stable is to force the interleaving distance on $\mathrm{Top}^{\mathbb{R}}$ to be homotopy-invariant [2, Section 1.2]. For this reason, many homotopy-invariant adaptations of the interleaving distance on $\mathrm{Top}^{\mathbb{R}}$ and related categories have been proposed; see eg Blumberg and Lesnick [2], Frosini, Landi and Mémoli [8] and Lesnick [15]. In order to describe some of these adaptations, we recall the definition of the interleaving distance d_I.

Let C be a category. Given $\delta \ge 0 \in \mathbb{R}$ and $F\colon \mathbb{R} \to C$, let $F^\delta\colon \mathbb{R} \to C$ be given by $F^\delta(r) := F(r+\delta)$, with the obvious structure morphisms. One says that $F, G \in C^{\mathbb{R}}$ are

δ*–interleaved* if there exist natural transformations $f: F \to G^\delta$ and $g: G \to F^\delta$ such that $g^\delta \circ f: F \to F^{2\delta}$ equals the natural transformation $F \to F^{2\delta}$ given by the structure morphisms of F, and such that $f^\delta \circ g: G \to G^{2\delta}$ equals the natural transformation $G \to G^{2\delta}$ given by the structure morphisms of G. Then

$$d_I(F,G) := \inf(\{\delta \geq 0 : F \text{ and } G \text{ are } \delta\text{–interleaved}\} \cup \{\infty\}).$$

Blumberg and Lesnick [2] define $X, Y \in \mathrm{Top}^{\mathbb{R}}$ to be δ*–homotopy interleaved* if there exist weakly equivalent persistent spaces $X' \simeq X$ and $Y' \simeq Y$ such that X' and Y' are δ–interleaved, and use homotopy interleavings to define the *homotopy interleaving distance*, denoted d_{HI}. The homotopy interleaving distance is the (metric) quotient of the interleaving distance by the equivalence relation given by weak equivalence, in the sense that d_{HI} is the largest homotopy-invariant distance that is bounded above by the interleaving distance.

Instead of taking a metric quotient, one can take the categorical quotient of $\mathrm{Top}^{\mathbb{R}}$ by weak equivalences, and define interleavings directly in the homotopy category, similar to what is done in eg Frosini, Landi and Mémoli [8] Kashiwara and Schapira [14] and Lesnick [15]. In order to do this, one notes that the shift functors $(-)^\delta: \mathrm{Top}^{\mathbb{R}} \to \mathrm{Top}^{\mathbb{R}}$ preserve weak equivalences and thus induce functors $(-)^\delta: \mathsf{Ho}(\mathrm{Top}^{\mathbb{R}}) \to \mathsf{Ho}(\mathrm{Top}^{\mathbb{R}})$. This lets one copy the definition of interleaving, but in the homotopy category, which gives the notions of *interleaving in the homotopy category* and of *interleaving distance in the homotopy category*, denoted by d_{IHC}.

A third option, also introduced in [2], is to compare objects of $\mathrm{Top}^{\mathbb{R}}$ using interleavings in $\mathsf{Ho}(\mathrm{Top})^{\mathbb{R}}$, called *homotopy commutative interleavings*, which give rise to the *homotopy commutative interleaving distance*, denoted by d_{HC}.

We have described three homotopy-invariant notions of interleaving in decreasing order of coherence. On one end, homotopy interleavings can be equivalently described as homotopy coherent diagrams of spaces [2, Section 7]. On the other end, homotopy commutative interleavings correspond to diagrams in the homotopy category of spaces. It is clear that $d_{\mathrm{HI}} \geq d_{\mathrm{IHC}} \geq d_{\mathrm{HC}}$, and that any of the homotopy-invariant interleavings induce interleavings in homotopy groups.

Two questions arise: Are the three distances in some sense equivalent or are they fundamentally different? If a map induces interleavings in homotopy groups, does it follow that the map is part of one of the homotopy-invariant notions of interleaving? A conjectural answer to the second question is given in [2, Conjecture 8.6], where it is conjectured that when X and Y are a kind of persistent CW–complex of finite dimension $d \in \mathbb{N}$, if there exists a morphism between them inducing a δ–interleaving in homotopy groups, then X and Y are $c\delta$–homotopy interleaved for a constant c that only depends on d.

Contributions Homotopy interleavings compose in any functor category of the form $\mathcal{M}^{\mathbb{R}^m}$ for $\mathcal{M}$ a cofibrantly generated model category (Proposition 2.3). This allows us to state some of our results for any cofibrantly generated model category $\mathcal{M}$, or for a category of spaces $\mathbb{S}$, which can be instantiated to be any of the Quillen equivalent model categories of topological spaces or simplicial sets (Remark 2.1). Our first theorem is the following rectification result.

Theorem A *Let $\mathcal{M}$ be a cofibrantly generated model category, let $X, Y \in \mathcal{M}^{\mathbb{R}}$, and let $\delta > 0 \in \mathbb{R}$. If X and Y are δ–homotopy commutative interleaved, then they are $c\delta$–homotopy interleaved for every $c > 2$.*

It follows that we have $2d_{\mathrm{HC}} \geq d_{\mathrm{HI}} \geq d_{\mathrm{IHC}} \geq d_{\mathrm{HC}}$. The above rectification result is different from many such results in homotopy theory, where a diagram of a certain shape, in the homotopy category, is lifted to a strict diagram of the same shape. The difference lies in the fact that the shape of the strict diagram we construct is different from the shape of the diagram in the homotopy category. In fact, building on the suggestion in [2] of using Toda brackets to give a lower bound for the above rectification, we show (Proposition 3.12) that for $\mathcal{M} = \mathrm{Top}$, if $c d_{\mathrm{HC}} \geq d_{\mathrm{HI}}$ then $c \geq \frac{3}{2}$, so that, in particular, $d_{\mathrm{HC}} \neq d_{\mathrm{HI}}$. This means that rectification in the usual sense is not possible in general, and thus standard results are not directly applicable. We also show that Theorem A has no analogue for multipersistent spaces (Section 3.3).

Our second theorem relates morphisms inducing interleavings in homotopy groups to interleavings in the homotopy category. See Definition 5.7 for the notion of persistent CW–complex and Definition 5.2 for the notion of interleaving induced in persistent homotopy groups.

Theorem B *Fix $m \geq 1 \in \mathbb{N}$ and $d \in \mathbb{N}$. Let $X, Y \in \mathbb{S}^{\mathbb{R}^m}$ be (multi)persistent spaces that are assumed to be projective cofibrant and d–skeletal if $\mathbb{S} = \mathrm{sSet}$, or persistent CW–complexes of dimension d if $\mathbb{S} = \mathrm{Top}$. Let $\delta \geq 0 \in \mathbb{R}^m$. If there exists a morphism in the homotopy category $X \to Y^{\delta} \in \mathsf{Ho}(\mathbb{S}^{\mathbb{R}^m})$ that induces δ–interleavings in all homotopy groups, then X and Y are $(4(d+1)\delta)$–interleaved in the homotopy category.*

Together, Theorems A and B give a positive answer to a version of the persistent Whitehead conjecture [2, Conjecture 8.6] (see Remark 5.14 for a discussion and Conjecture 5.15 for a statement of the conjecture).

Structure of the paper In Section 2, we recall and give references for the necessary background. In Section 3, we prove Theorem A, we provide a lower bound for the rectification of homotopy commutative interleavings between persistent spaces, and we show that Theorem A has no analogue for multipersistent spaces. In Section 4, we characterize projective cofibrant (multi)persistent simplicial sets as filtered simplicial sets. In Section 5, we prove Theorem B.

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2 Background and conventions

The main purpose of this section is to fix notation and to provide the reader with references. This section can be referred to as needed, but we do recommend going over Section 2.2 as it contains the notions of interleaving relevant to us.

We assume that the reader is comfortable with the language of category theory. Throughout the paper, we will use the term *distance* to refer to any *extended pseudometric* on a (possibly large) set X, that is, to any function $d_X\colon X \times X \to [0, \infty]$ that is symmetric, satisfies the triangle inequality, and is 0 on the diagonal.

2.1 Spaces and model categories

2.1.1 Spaces We work model-independently whenever possible. This means that whenever we say *space* we will mean either topological space or simplicial set. Results stated for spaces will hold for both possible models. The category of spaces will be denoted by $\mathbb{S}$.

For a general introduction to simplicial sets, see eg [9] or [12, Chapter 3]. We denote the geometric realization functor for simplicial sets by $|-|\colon \mathrm{sSet} \to \mathrm{Top}$.

2.1.2 Model categories The theory of model categories was introduced in [18]; for a modern and thorough development of this theory we recommend [11] and [12].

We recall that two objects $x, y \in \mathcal{M}$ of a model category $\mathcal{M}$ are said to be *weakly equivalent* if they are isomorphic in $\mathsf{Ho}(\mathcal{M})$, which happens if and only if they are connected by a zigzag of weak equivalences in $\mathcal{M}$. This is an equivalence relation, which we denote by $x \simeq y$. When there is risk of confusion, morphisms in $\mathsf{Ho}(\mathcal{M})$ will be surrounded by square brackets $[f]$, to distinguish them from morphisms in $\mathcal{M}$.

Two of the main model structures of interest to us are the *Quillen model structure* on Top, the category of topological spaces [12, Chapter 1, Section 2.4], and the *Kan–Quillen model structure* on sSet, the category of simplicial sets [12, Chapter 3]. We recall that the geometric realization functor $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ is left adjoint to the singular functor $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{sSet}$, and that, together, they form a *Quillen equivalence* [12, Chapter 1, Section 1.3, Theorem 3.6.7]. For completeness, we mention that there is a subcategory $\mathrm{Top}_{\mathsf{CGWH}} \subseteq \mathrm{Top}$, the category of compactly generated weakly Hausdorff topological spaces (called *compactly generated spaces* in [12, Definition 2.4.21]), that is often used instead of Top. The Quillen model structure on Top restricts to a model structure on $\mathrm{Top}_{\mathsf{CGWH}}$, and the inclusion $\mathrm{Top}_{\mathsf{CGWH}} \to \mathrm{Top}$ is part of a Quillen equivalence [12, Theorem 2.4.25]. This model structure is, in some respects, better behaved than the Quillen model structure on topological spaces, and is in fact the model of space used in [2]. We will not concern ourselves with these subtleties since, by the observations in Remark 2.1, there is no essential difference between using Top or $\mathrm{Top}_{\mathsf{CGWH}}$ when studying homotopy-invariant notions of interleaving.

We will make use of the notion of *cofibrantly generated model category* [12, Chapter 2, Section 2.1]. Recall that the Kan–Quillen model structure on simplicial sets is cofibrantly generated, where a set of generating cofibrations consists of the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$ [12, Theorem 3.6.5]. The Quillen model structure on topological spaces is also cofibrantly generated, with a set of generating cofibrations given by $\{S^{n-1} \hookrightarrow D^n\}_{n\geq 0}$ [12, Theorem 2.4.19].

We conclude by recalling the basic properties of projective model structures. Given a model category $\mathcal{M}$ and a small category $\mathbb{I}$, the *projective model structure* on the functor category $\mathcal{M}^{\mathbb{I}}$ is, when it exists, the model structure whose fibrations (resp. weak equivalences) are those which are pointwise fibrations (resp. weak equivalences) of $\mathcal{M}$.

The projective model structure on $\mathcal{M}^{\mathbb{I}}$ exists, and is cofibrantly generated, whenever $\mathcal{M}$ is cofibrantly generated. Moreover, if $\mathcal{I}$ and $\mathcal{J}$ are, respectively, generating cofibrations and generating trivial cofibration for the model structure of $\mathcal{M}$, then

$$\{\mathbb{I}(i,-) \odot f : i \in \mathbb{I}, f \in \mathcal{I}\} \quad \text{and} \quad \{\mathbb{I}(i,-) \odot g : i \in \mathbb{I}, g \in \mathcal{J}\}$$

are, respectively, generating cofibrations and generating trivial cofibrations for the projective model structure, where, given a functor $F\colon \mathbb{I} \to \mathrm{Set}$ and an object $X \in \mathcal{M}$, the functor $F \odot X\colon \mathbb{I} \to \mathcal{M}$ is defined by $i \mapsto \coprod_{a \in F(i)} X$ [11, Section 11.6]. For simplicity, we denote $\mathbb{I}(i,-) \odot X$ by $i \odot X$.

We are especially interested in the projective model structure when the indexing category is a poset $(P, \leq)$. In this case, if $r \in P$ and $X \in \mathcal{M}$, then $r \odot X$ is the functor that takes the value X on every $s \geq r$, and has as value the initial object of $\mathcal{M}$ when $s \not\geq r$. The nontrivial structure morphisms of this functor are the identity of X.

Note that we have a functor $\mathsf{h}\colon \mathsf{Ho}(\mathcal{M}^{\mathbb{I}}) \to \mathsf{Ho}(\mathcal{M})^{\mathbb{I}}$ by the universal property of $\mathsf{Ho}(\mathcal{M}^{\mathbb{I}})$.

2.2 Interleavings and interleavings up to homotopy

2.2.1 Strict interleavings We denote the poset of real numbers with their standard order by $\mathbb{R}$, and for $m \in \mathbb{N}$, we let $\mathbb{R}^m$ be the set of m–tuples of real numbers with the product order. We set $\overline{m} = \{i : 1 \leq i \leq m\}$, so that $(\varepsilon_i)_{i \in \overline{m}} \leq (\delta_i)_{i \in \overline{m}} \in \mathbb{R}^m$ if and only if $\varepsilon_i \leq \delta_i$ for all $1 \leq i \leq m$. We denote the element $(0, \dots, 0) \in \mathbb{R}^m$ by 0.

Fix a category C and a natural number $m \geq 1$. An *m–persistent object* of C is any functor of the form $\mathbb{R}^m \to C$. We often refer to m–persistent objects simply as *persistent objects* or as *multipersistent objects* when we want to stress the fact that m is not necessarily 1. Fix persistent objects $X, Y, Z \in C^{\mathbb{R}^m}$, $r, s \in \mathbb{R}^m$, and $\varepsilon, \delta \geq 0 \in \mathbb{R}^m$. We use the following conventions.

- For $f\colon X \to Y$ a natural transformation, denote the r–component of f by $f_r\colon X(r) \to Y(r)$.
- Assume $r \leq s$. The structure morphism $X(r) \to X(s)$ will be denoted by $\varphi^X_{r,s}$.
- The *δ–shift to the left* of X is the functor $X^\delta\colon \mathbb{R}^m \to C$ defined by $X^\delta(r) = X(r+\delta)$, with structure morphisms $\varphi^{X^\delta}_{r,s} := \varphi^X_{r+\delta,s+\delta}$. Shifting to the left gives a functor $(-)^\delta\colon C^{\mathbb{R}^m} \to C^{\mathbb{R}^m}$. Dually, there is a *$\delta$–shift to the right* functor $\delta \cdot (-)\colon C^{\mathbb{R}^m} \to C^{\mathbb{R}^m}$ defined by mapping X to the persistent object $\delta \cdot X$, with values given by $(\delta \cdot X)(r) = X(r - \delta)$.

- Natural transformations $f\colon X \to Y^\delta$ will be referred to as *δ–morphisms*, and will often be denoted by $f\colon X \to_\delta Y$. Since we have natural bijections
$$\mathrm{Hom}(\varepsilon \cdot X, Y^\delta) \cong \mathrm{Hom}(X, Y^{\varepsilon+\delta}) \cong \mathrm{Hom}((\varepsilon+\delta)\cdot X, Y),$$
we can treat a δ–morphism $f\colon X \to_\delta Y$ as $f\colon X \to Y^\delta$ or as $f\colon \delta \cdot X \to Y$.
- Assume $\varepsilon \le \delta$ and let $f\colon X \to_\varepsilon Y$. We can compose the r–component of f with $\varphi^Y_{r+\varepsilon,r+\delta}\colon Y(r+\varepsilon) \to Y(r+\delta)$, giving $\varphi^Y_{r+\varepsilon,r+\delta} \circ f_r\colon X(r) \to Y(r+\delta)$. Together, these components define the *shift* from ε to δ of f, which is a δ–morphism denoted $\mathsf{S}_{\varepsilon,\delta}(f)\colon X \to_\delta Y$.
- Note that an ε–morphism $f\colon X \to_\varepsilon Y$ can be composed with a δ–morphism $g\colon Y \to_\delta Z$, yielding an $(\varepsilon+\delta)$–morphism $g^\varepsilon \circ f\colon X \to_{\varepsilon+\delta} Y$. This composition is associative and unital, and is natural with respect to shifts of morphisms.
- An (ε,δ)*–interleaving* between X and Y consists of an ε–morphism $f\colon X \to_\varepsilon Y$ together with a δ–morphism $g\colon Y \to_\delta X$ such that $g^\varepsilon \circ f = \mathsf{S}_{0,\varepsilon+\delta}(\mathsf{id}_X)$ and $f^\delta \circ g = \mathsf{S}_{0,\varepsilon+\delta}(\mathsf{id}_Y)$. By *$\delta$–interleaving* we mean a (δ,δ)–interleaving.
- If $f\colon X \to_\varepsilon Y$ and $g\colon Y \to_\delta X$ form an (ε,δ)–interleaving, then we write $f\colon X\; {}_\delta\!\longleftrightarrow_\varepsilon Y : g$.

Let $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 \ge 0 \in \mathbb{R}^m$. Note that an $(\varepsilon_1,\varepsilon_2)$–interleaving between X and Y can be composed with any (δ_1,δ_2)–interleaving between Y and Z, yielding an $(\varepsilon_1+\delta_1, \varepsilon_2+\delta_2)$–interleaving. The fact that interleavings compose implies that, when $m=1$, the formula
$$d_I(X,Y) = \inf(\{\delta \ge 0 \in \mathbb{R} : X \text{ and } Y \text{ are } \delta\text{–interleaved}\} \cup \{\infty\})$$
defines an extended pseudometric $d_I\colon \mathsf{Obj}(C^{\mathbb{R}}) \times \mathsf{Obj}(C^{\mathbb{R}}) \to [0,\infty]$. This is the *interleaving distance* on the class of objects of the category $C^{\mathbb{R}}$. This notion of distance can be extended to objects of the functor category $C^{\mathbb{R}^m}$ [15], but we will not make use of this extension.

2.2.2 Interleavings up to homotopy If one is comparing objects of a category of functors of the form $\mathbb{R}^m \to \mathcal{M}$, for $\mathcal{M}$ a model category, it makes sense to want to find a homotopy-invariant notion of interleaving. In this paper, we consider the following three homotopy-invariant relaxations of the notion of interleaving. Let $\mathcal{M}$ be a cofibrantly generated model category and endow $\mathcal{M}^{\mathbb{R}^m}$ with the projective model structure. Let $X, Y \in \mathcal{M}^{\mathbb{R}^m}$ and let $\varepsilon, \delta \ge 0 \in \mathbb{R}^m$.

(1) Following [2], we say that X and Y are (ε,δ)*–homotopy interleaved* if there exist $X \simeq X'$ and $Y \simeq Y'$ such that X' and Y' are (ε,δ)–interleaved.

(2) Note that the shift functor $(-)^\delta\colon \mathcal{M}^{\mathbb{R}^m} \to \mathcal{M}^{\mathbb{R}^m}$ maps weak equivalences to weak equivalences. This implies that all the notions in Section 2.2.1 have analogues in the category $\mathsf{Ho}(\mathcal{M}^{\mathbb{R}^m})$. We say that X and Y are (ε,δ)–*interleaved in the homotopy category* if they are (ε,δ)–interleaved as objects of $\mathsf{Ho}(\mathcal{M}^{\mathbb{R}^m})$.

(3) Finally, as also done in [2], we say that X and Y are (ε,δ)–*homotopy commutative interleaved* if their images $\mathsf{h}X, \mathsf{h}Y\colon \mathbb{R}^m \to \mathsf{Ho}(\mathcal{M})$ are (ε,δ)–interleaved.

An (ε,δ)–homotopy interleaving gives rise to an (ε,δ)–interleaving in the homotopy category, which, in turn, gives rise to an (ε,δ)–homotopy commutative interleaving.

For each of the three homotopy-invariant notions of interleaving introduced above, we have a corresponding extended pseudometric on the collection of objects of the category $\mathcal{M}^{\mathbb{R}}$. Let $X, Y \in \mathcal{M}^{\mathbb{R}}$. Following [2], we define the *homotopy interleaving distance* as

$$d_{\mathrm{HI}}(X,Y) = \inf(\{\delta \geq 0 \in \mathbb{R} : X \text{ and } Y \text{ are } \delta\text{–homotopy interleaved}\} \cup \{\infty\}).$$

The fact that the homotopy interleaving distance satisfies the triangle inequality follows from Proposition 2.3. The *interleaving distance in the homotopy category* is

$$d_{\mathrm{IHC}}(X,Y) = \inf(\{\delta \geq 0 \in \mathbb{R} : X, Y \text{ are } \delta\text{–interleaved in the homotopy category}\} \cup \{\infty\}).$$

Again following [2], the *homotopy commutative interleaving distance* is defined as

$$d_{\mathrm{HC}}(X,Y) = \inf(\{\delta \geq 0 \in \mathbb{R} : X, Y \text{ are } \delta\text{–homotopy commutative interleaved}\} \cup \{\infty\}).$$

Remark 2.1 If $\mathcal{M} \rightleftarrows \mathcal{N}$ is a Quillen equivalence between cofibrantly generated model categories, then the induced Quillen equivalence [11, Theorem 11.6.5] $\mathcal{M}^{\mathbb{R}^m} \rightleftarrows \mathcal{N}^{\mathbb{R}^m}$ between the projective model structures respects interleavings, in the sense that shifts commute with both the left and right adjoints. This implies that, for any of the three homotopy-invariant notions of interleaving described above, we have that two functors on one side of the adjunction are (ε,δ)–interleaved if and only if their images (along the derived adjunction) on the other side are (ε,δ)–interleaved. In particular, if $m = 1$, the two adjoints give an isometry between $\mathcal{M}^{\mathbb{R}}$ and $\mathcal{N}^{\mathbb{R}}$ independently of whether we use d_{HI}, d_{IHC} or d_{HC}.

2.2.3 Composability of homotopy interleavings In this short section, we give a simplified proof of a generalization of the fact that homotopy interleavings can be composed, originally proved in [2, Section 4]. This is generalized further in [20, Theorem 4.1.4].

Lemma 2.2 *Let C admit pullbacks. Fix $m \geq 1 \in \mathbb{N}$, objects $X, Y, B\colon \mathbb{R}^m \to C$, elements $\varepsilon, \delta \geq 0 \in \mathbb{R}^m$, an (ε,δ)–interleaving $f\colon X \;{}_{\delta}\!\!\longleftrightarrow_{\varepsilon} Y : g$, and a map $h\colon B \to Y$. The pullback of $f\colon X \to Y^{\varepsilon}$ along $h^{\varepsilon}\colon B^{\varepsilon} \to Y^{\varepsilon}$, denoted by $k\colon A \to B^{\varepsilon}$, is part of an (ε,δ)–interleaving $k\colon A \;{}_{\delta}\!\!\longleftrightarrow_{\varepsilon} B : l$.*

Proof We start by depicting the pullback square in the statement:

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d] & B^{\varepsilon} \arrow[d, "h^{\varepsilon}"] \\ X \arrow[r, "f"] & Y^{\varepsilon} \end{tikzcd}$$

Consider the morphisms $i = \mathsf{S}_{0,\varepsilon+\delta}(\mathsf{id}_B)\colon \delta\cdot B \to B^{\varepsilon}$ and $g \circ (\delta \cdot h)\colon \delta \cdot B \to X$. Since $f \circ g \circ (\delta \cdot h) = h^{\varepsilon} \circ i$, the universal property of A gives us a map $l\colon \delta\cdot B \to A$, or equivalently, a map $l\colon B \to A^{\delta}$. By construction, $k^{\delta} \circ l = \mathsf{S}_{0,\varepsilon+\delta}(\mathsf{id}_B)\colon B \to B^{\varepsilon+\delta}$. To prove that $l^{\varepsilon} \circ k = \mathsf{S}_{0,\varepsilon+\delta}(\mathsf{id}_A)\colon A \to A^{\varepsilon+\delta}$, or equivalently that

$$l^{\varepsilon} \circ k = \mathsf{S}_{0,\varepsilon+\delta}(\mathsf{id}_A)\colon \varepsilon\cdot A \to A^{\delta},$$

apply the functor $(-)^{\delta}\colon C^{\mathbb{R}^m} \to C^{\mathbb{R}^m}$ to the pullback square above, and use the uniqueness part of its universal property. □

Proposition 2.3 (cf [2, Section 4]) *Let $\mathcal{M}$ be cofibrantly generated, fix $m \geq 1$, let $X, Y, Z\colon \mathbb{R}^m \to \mathcal{M}$, and let $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2 \geq 0 \in \mathbb{R}^m$. If X and Y are $(\varepsilon_1,\varepsilon_2)$–homotopy interleaved and Y and Z are (δ_1,δ_2)–homotopy interleaved, then X and Z are $(\varepsilon_1{+}\delta_1, \varepsilon_2{+}\delta_2)$–homotopy interleaved.*

Proof Given interleavings $X' \;{}_{\varepsilon_2}\!\!\longleftrightarrow_{\varepsilon_1} Y'$ and $Y'' \;{}_{\delta_2}\!\!\longleftrightarrow_{\delta_1} Z'$ with

$$X \simeq X', \quad Y' \simeq Y \simeq Y', \quad Z' \simeq Z,$$

we must construct an interleaving $X'' \;{}_{\varepsilon_2+\delta_2}\!\!\longleftrightarrow_{\varepsilon_1+\delta_1} Z''$ with $X'' \simeq X$ and $Z'' \simeq Z$.

Since $\mathcal{M}$ is cofibrantly generated, the projective model structure on $\mathcal{M}^{\mathbb{R}^m}$ exists, and, by applying a functorial fibrant replacement $\mathcal{M} \to \mathcal{M}$ pointwise, we get a functorial fibrant replacement $\mathcal{M}^{\mathbb{R}^m} \to \mathcal{M}^{\mathbb{R}^m}$. By construction, the fibrant replacement $\mathcal{M}^{\mathbb{R}^m} \to \mathcal{M}^{\mathbb{R}^m}$ commutes with $(-)^{\delta}\colon \mathcal{M}^{\mathbb{R}^m} \to \mathcal{M}^{\mathbb{R}^m}$ so, in particular, it preserves interleavings. With this in mind, we can assume that Y' and Y'' are fibrant, which implies — and this is a general fact — that we have $C \in \mathcal{M}^{\mathbb{R}^m}$ and trivial fibrations $C \to Y'$ and $C \to Y''$.

Using Lemma 2.2, we can pull back the interleavings we were given along the trivial fibrations, as follows:

$$
\begin{tikzcd}
 & X'' \arrow[dl, dotted] \arrow[rr, dotted, leftrightarrow, "\varepsilon_2" near start, "\varepsilon_1" near end] & & C \arrow[dl] \arrow[dr] \arrow[rr, dotted, leftrightarrow, "\delta_2" near start, "\delta_1" near end] & & Z'' \arrow[dr, dotted] & \\
X' \arrow[rr, leftrightarrow, "\varepsilon_2"' near start, "\varepsilon_1"' near end] & & Y' & & Y'' \arrow[rr, leftrightarrow, "\delta_2"' near start, "\delta_1"' near end] & & Z'
\end{tikzcd}
$$

Since trivial fibrations are stable under pullback, we have that $X'' \simeq X$ and $Z'' \simeq Z$, and since interleavings compose, we have that X'' and Z'' are $(\varepsilon_1+\delta_1, \varepsilon_2+\delta_2)$–interleaved, as required. □

We remark that the idea of using pullbacks to prove a triangle inequality appears in [17].

3 Interleavings in $\mathcal{M}^{\mathbb{R}}$ and in $\mathsf{Ho}(\mathcal{M})^{\mathbb{R}}$

This section is concerned with the rectification of homotopy commutative interleavings into homotopy interleavings. In Section 3.1, we prove Theorem A, which allows one to construct, for any $c > 2$, a $c\delta$–homotopy interleaving out of a δ–homotopy commutative interleaving, when working with 1–persistent objects of any cofibrantly generated model category $\mathcal{M}$. We think of this result as giving a multiplicative upper bound of 2 for this rectification. In Section 3.2, we give a multiplicative lower bound of $\frac{3}{2}$ for the rectification, when $\mathcal{M}$ is the category of spaces. In Section 3.3, we show that Theorem A has no analogue for multipersistent spaces.

3.1 Upper bound

Let $\mathbb{Z} \subseteq \mathbb{R}$ denote the posets of integers and real numbers respectively. The inclusion $i\colon \mathbb{Z} \to \mathbb{R}$ induces a restriction functor $i^*\colon C^{\mathbb{R}} \to C^{\mathbb{Z}}$ for any category C. Given $A\colon \mathbb{Z} \to C$, let $i_*(A)\colon \mathbb{R} \to C$ be given by A precomposed with the functor $\lfloor - \rfloor\colon \mathbb{R} \to \mathbb{Z}$, where $\lfloor r \rfloor$ is the largest integer bounded above by r. Note that, given $m \geq 0 \in \mathbb{Z}$, one has a notion of m–interleaving between functors $A, B\colon \mathbb{Z} \to C$, and that $i_*\colon C^{\mathbb{Z}} \to C^{\mathbb{R}}$ preserves these interleavings.

We start with a few simplifications. For $\delta > 0$, let $M_\delta\colon \mathbb{R} \to \mathbb{R}$ be given by $M_\delta(r) = \delta \times r$. The following lemma allows us to work with integer-valued interleavings instead of δ–interleavings, and its proof is immediate.

Lemma 3.1 *Let $\delta > 0 \in \mathbb{R}$ and $m \geq 1 \in \mathbb{Z}$. Then $X, Y \in C^{\mathbb{R}}$ are δ–interleaved if and only if $(M_{\delta/m})^*(X)$ and $(M_{\delta/m})^*(Y)$ are m–interleaved.* □

The following lemma allows us to work with $\mathbb{Z}$–indexed persistent objects instead of $\mathbb{R}$–indexed ones. Here, by homotopy interleaving between $\mathbb{Z}$–indexed functors we mean the obvious adaptation of the notion of homotopy interleaving to $\mathbb{Z}$–indexed functors with values in a model category.

Lemma 3.2 *Let $\mathcal{M}$ be cofibrantly generated. Let $X, Y \in \mathcal{M}^{\mathbb{R}}$ and let $m \geq 1 \in \mathbb{Z}$. If $i^*(X), i^*(Y) \in \mathcal{M}^{\mathbb{Z}}$ are m–homotopy interleaved, then X and Y are $(m+2)$–homotopy interleaved.*

Proof Note that X is 1–interleaved with $i_*(i^*(X))$, as, for all $r \in \mathbb{R}$, we have $r - 1 \leq \lfloor r \rfloor \leq r \leq \lfloor r \rfloor + 1$. Since i_* preserves interleavings and weak equivalences, it is enough to show that homotopy interleavings between $\mathbb{Z}$–indexed functors with values in a cofibrantly generated model category compose, which is a straightforward adaptation of Proposition 2.3 to $\mathbb{Z}$–indexed functors. □

The next straightforward lemma gives us a special replacement of an object of the category $\mathcal{M}^{\mathbb{Z}}$, with $\mathcal{M}$ a model category, that will be useful when lifting structure from $\mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$ to $\mathcal{M}^{\mathbb{Z}}$.

Lemma 3.3 *Given a model category $\mathcal{M}$ and $X \in \mathcal{M}^{\mathbb{Z}}$, there exists $\overline{X} \in \mathcal{M}^{\mathbb{Z}}$ and a weak equivalence $\overline{X} \to X$ such that*

- *$\overline{X}(i)$ is cofibrant in $\mathcal{M}$ for every $i \in \mathbb{N}$;*
- *for every $i \geq 0$, the structure morphism $f_i \colon \overline{X}(i) \to \overline{X}(i+1)$ is a cofibration in $\mathcal{M}$.*

Dually, we can replace $Y \in \mathcal{M}^{\mathbb{Z}}$ by a pointwise fibrant $\overline{Y}$ whose "negative" maps are fibrations.

The following lemma will allow us to lift interleavings in $\mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$ to homotopy interleavings in $\mathcal{M}^{\mathbb{Z}}$.

Lemma 3.4 *Let $\mathcal{M}$ be a model category. The functor $\mathsf{h} \colon \mathsf{Ho}(\mathcal{M}^{\mathbb{Z}}) \to \mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$ is essentially surjective, conservative and full. In particular, if $A, B \in \mathcal{M}^{\mathbb{Z}}$ become isomorphic in $\mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$, then they are weakly equivalent.*

Proof It is clear that the functor is essentially surjective and full, so we only prove the last property. Assume we are given $X, Y \in \mathsf{Ho}(\mathcal{M}^{\mathbb{Z}})$ together with a map $f : \mathsf{h}X \to \mathsf{h}Y$. Thanks to Lemma 3.3, we can assume that X (resp. Y) is pointwise cofibrant (resp. fibrant) in $\mathcal{M}$, and that all the nonnegative (resp. negative) structural maps in X (resp. Y) are cofibrations (resp. fibrations). The map f can therefore be represented as a family $\{[f_i]\}_{i\in\mathbb{Z}}$ of homotopy classes of maps of $\mathcal{M}$. We construct a preimage of f under h inductively, starting with a choice of representatives f_i' for the homotopy classes $[f_i]$. The squares

$$\begin{array}{ccccc} X(-1) & \xrightarrow{x_{-1}} & X(0) & \xrightarrow{x_0} & X(1) \\ \downarrow{\scriptstyle f'_{-1}} & & \downarrow{\scriptstyle f'_0} & & \downarrow{\scriptstyle f'_1} \\ Y(-1) & \xrightarrow{y_{-1}} & Y(0) & \xrightarrow{y_0} & Y(1) \end{array}$$

commute up to homotopy, and since x_0 and y_{-1} are, respectively, a cofibration and a fibration, we can deform f_1' and f_{-1}' into homotopic maps $f_1 : X_1 \to Y_1$ and $f_{-1} : X_{-1} \to Y_{-1}$, which render the above squares commutative. Inductively, we can iterate this procedure to find the desired preimage of f under h. □

The next result is the main rectification step involved in lifting interleavings in $\mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$ to homotopy interleavings in $\mathcal{M}^{\mathbb{Z}}$.

Proposition 3.5 *Let $\mathcal{M}$ be a model category and let $A, B \in \mathcal{M}^{\mathbb{Z}}$. Let $m \geq 1 \in \mathbb{Z}$. If $\mathsf{h}A$ and $\mathsf{h}B$ are m–interleaved in $\mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$, then A and B are $2m$–homotopy interleaved in $\mathcal{M}^{\mathbb{Z}}$.*

Proof We start by giving the proof for the case $m = 1$, as in this case the main idea is more clear. We will use the following constructions. Let $\mathsf{e} : \mathbb{Z} \to \mathbb{Z}$ be the functor that maps even numbers to themselves and an odd number n to $n-1$. Similarly, let $\mathsf{o} : \mathbb{Z} \to \mathbb{Z}$ be the functor that maps odd numbers to themselves and an even number n to $n-1$.

Note that, for every $C \in \mathcal{M}^{\mathbb{Z}}$, we have that C is $(1,0)$–interleaved with $\mathsf{e}^*(C)$ and with $\mathsf{o}^*(C)$, and that $\mathsf{e}^*(C)$ and $\mathsf{o}^*(C)$ are 1–interleaved.

Now assume given a 1–interleaving between $\mathsf{h}A$ and $\mathsf{h}B$ in $\mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$, that is, assume that there are morphisms $f_i : \mathsf{h}A(i) \to \mathsf{h}B(i+1)$ and $g_i : \mathsf{h}B(i) \to \mathsf{h}A(i+1)$ in $\mathsf{Ho}(\mathcal{M})$

rendering the following diagram commutative:

$$\begin{tikzcd}
\cdots \arrow[r, "{[\alpha_{-2}]}"'] \arrow[dr, "f_{-2}"'] & \mathsf{h}A(-1) \arrow[r, "{[\alpha_{-1}]}"'] \arrow[dr, "f_{-1}"'] & \mathsf{h}A(0) \arrow[r, "{[\alpha_0]}"'] \arrow[dr, "f_0"'] & \mathsf{h}A(1) \arrow[r, "{[\alpha_1]}"'] \arrow[dr, "f_1"'] & \cdots \\
\cdots \arrow[r, "{[\beta_{-2}]}"] \arrow[ur, "g_{-2}"] & \mathsf{h}B(-1) \arrow[r, "{[\beta_{-1}]}"] \arrow[ur, "g_{-1}"] & \mathsf{h}B(0) \arrow[r, "{[\beta_0]}"] \arrow[ur, "g_0"] & \mathsf{h}B(1) \arrow[r, "{[\beta_1]}"] \arrow[ur, "g_1"] & \cdots
\end{tikzcd}$$

Consider the object $C' \in \mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$ given by one of the two diagonal zigzags of the diagram above; namely, let

$$C' = \quad \cdots \xrightarrow{f_{-2}} \mathsf{h}B(-1) \xrightarrow{g_{-1}} \mathsf{h}A(0) \xrightarrow{f_0} \mathsf{h}B(1) \xrightarrow{g_1} \mathsf{h}A(2) \xrightarrow{f_2} \cdots .$$

Using Lemma 3.4, construct $C \in \mathcal{M}^{\mathbb{Z}}$ such that $\mathsf{h}C \cong C'$.

Now, by construction, we have that $\mathsf{h}(\mathsf{e}^*(A)) = \mathsf{e}^*(\mathsf{h}A) = \mathsf{e}^*(C') \cong \mathsf{e}^*(\mathsf{h}C) = \mathsf{h}(\mathsf{e}^*(C))$, so from Lemma 3.4 it follows that $\mathsf{e}^*(A) \simeq \mathsf{e}^*(C)$. Similarly, we have $\mathsf{o}^*(B) \simeq \mathsf{o}^*(C)$. Since A is $(1,0)$–interleaved with $\mathsf{e}^*(A)$, $\mathsf{e}^*(C)$ is 1–interleaved with $\mathsf{o}^*(C)$, and $\mathsf{o}^*(B)$ is $(0,1)$–interleaved with B, Proposition 2.3 implies that A and B are 2–homotopy interleaved, concluding the proof for the case $m = 1$.

The proof for general $m \geq 1 \in \mathbb{Z}$ is analogous, replacing the functor $\mathsf{e}\colon \mathbb{Z} \to \mathbb{Z}$ with $\mathsf{e}_m\colon \mathbb{Z} \to \mathbb{Z}$ given by $\mathsf{e}_m(n) = \mathsf{e}(n /\!\!/ m) \times m$, the functor $\mathsf{o}\colon \mathbb{Z} \to \mathbb{Z}$ with $\mathsf{o}_m\colon \mathbb{Z} \to \mathbb{Z}$ given by $\mathsf{o}_m(n) = \mathsf{o}(n /\!\!/ m) \times m$, and $C' \in \mathsf{Ho}(\mathcal{M})^{\mathbb{Z}}$ with

$$C'(n) = \begin{cases} \mathsf{h}(\mathsf{e}_m^*(A))(n) & \text{if } n /\!\!/ m \text{ is even,} \\ \mathsf{h}(\mathsf{o}_m^*(B))(n) & \text{if } n /\!\!/ m \text{ is odd,} \end{cases}$$

where $n /\!\!/ m$ denotes the largest integer l such that $l \times m \leq n$. □

We are now ready to prove the main result of this section.

Theorem A *Let $\mathcal{M}$ be a cofibrantly generated model category, let $X, Y \in \mathcal{M}^{\mathbb{R}}$, and let $\delta > 0 \in \mathbb{R}$. If X and Y are δ–homotopy commutative interleaved, then they are $c\delta$–homotopy interleaved for every $c > 2$.*

Proof Let $c > 2$ and let $m \geq 1 \in \mathbb{Z}$ be large enough so that $(2m+2)/m \leq c$. By Lemma 3.1, we may assume that $X, Y \in \mathcal{M}^{\mathbb{R}}$ are m–homotopy commutative interleaved and we must show that they are cm–homotopy interleaved. Since $2m + 2 \leq mc$, Lemma 3.2 reduces the problem to showing that $i^*(X)$ and $i^*(Y)$ are $2m$–homotopy interleaved in $\mathcal{M}^{\mathbb{Z}}$, knowing that they are m–homotopy commutative interleaved. Proposition 3.5 now finishes the proof. □

3.2 Lower bound

Theorem A implies that we have $d_{\mathrm{HI}} \le c\, d_{\mathrm{HC}}$ as distances on $\mathcal{M}^{\mathbb{R}}$, for $c = 2$ and for every cofibrantly generated model category $\mathcal{M}$. One could wonder if the constant $c = 2$ can be improved. In this section we show that, when $\mathcal{M} = \mathbb{S}$, we have $c \ge \frac{3}{2}$. We do this by characterizing three-object persistent spaces which are 1–homotopy interleaved with a trivial persistent space in terms of the vanishing of a Toda bracket. The idea of using Toda brackets to prove that $d_{\mathrm{HI}} \ne d_{\mathrm{HC}}$ is suggested in [2, Example 7.3].

The Toda bracket is an operation on composable triples of homotopy classes of pointed maps, and was originally defined to compute homotopy groups of spheres [21]. We are interested in the use of Toda brackets as an algebraic obstruction to the rectification of diagrams. We now describe the fundamental procedure involved in the definition of Toda brackets, and the few properties that we are interested in; see eg [1].

Let $\mathbb{S}_\bullet$ denote the category of pointed spaces. For concreteness, in the arguments of this section we use $\mathbb{S} = \mathrm{Top}$. Let $[3]$ denote the category freely generated by the graph

$$\bullet \to \bullet \to \bullet \to \bullet.$$

A diagram $X \in \mathsf{Ho}(\mathbb{S}_\bullet)^{[3]}$, which is given by $X(0), X(1), X(2), X(3) \in \mathsf{Ho}(\mathbb{S}_\bullet)$ and homotopy classes of pointed maps $[f_0]\colon X(0) \to X(1)$, $[f_1]\colon X(1) \to X(2)$, and $[f_2]\colon X(2) \to X(3)$, is a *bracket sequence* if $[f_1] \circ [f_0]$ and $[f_2] \circ [f_1]$ are equal to the null map, that is, to the homotopy class of the constant pointed map.

Let $X' \in \mathsf{Ho}(\mathbb{S}_\bullet)^{[3]}$ be a bracket sequence and let $X \in \mathbb{S}_\bullet^{[3]}$ be such that $\mathsf{h}X \cong X'$, which exists by Lemma 3.4. We can, and do, assume that X takes values in CW–complexes. Consider the diagram of pointed spaces and pointed maps

$$\begin{tikzcd} X(0) \arrow[r, "f_0"] \arrow[d] & X(1) \arrow[r] \arrow[d, "f_1"] & * \arrow[d] \\ * \arrow[r] & X(2) \arrow[r, "f_2"] & X(3) \end{tikzcd}$$

Since X' is a bracket sequence, we know that there exist (pointed) homotopies filling the squares in the diagram above. For Y a pointed space, let CY denote its reduced cone. Each pair of such homotopies gives us pointed maps $\alpha\colon CX(0) \to X(2)$ and $\beta\colon CX(1) \to X(3)$ such that $\alpha \circ i = f_1 \circ f_0\colon X(0) \to X(2)$ and $\beta \circ i = f_2 \circ f_1$, where

i is the inclusion into the cone. In particular, we have a commutative square

$$\begin{tikzcd} X(0) \arrow[r, "i \circ f_0"] \arrow[d, "i"'] & CX(1) \arrow[d, "\beta"] \\ CX(0) \arrow[r, "f_2 \circ \alpha"] & X(3) \end{tikzcd} \tag{3-1}$$

which, by noticing that the pushout of the top and left morphisms is a model for the reduced suspension of $X(0)$, gives us an element of $[\Sigma X'(0), X'(3)]$, where $[-,-]$ denotes homotopy classes of pointed maps.

Definition 3.6 Let $X \in \mathsf{Ho}(\mathbb{S}_\bullet)^{[3]}$ be a bracket sequence. Consider the subset of $[\Sigma X(0), X(3)]$ consisting of all elements that can be obtained using the procedure above. This is the *Toda bracket* of X. We say that the Toda bracket *vanishes* if it contains the null map.

It is well known — see eg [1, Section 1] — that the nonvanishing of a Toda bracket is an obstruction to the rectification of the bracket sequence, in the following sense.

Proposition 3.7 *The Toda bracket of a bracket sequence $X' \in \mathsf{Ho}(\mathbb{S}_\bullet)^{[3]}$ vanishes if and only if there exists $X \in \mathbb{S}_\bullet^{[3]}$ with $\mathsf{h}X \cong X'$ and with $f_1 \circ f_0$ and $f_2 \circ f_1$ equal to the null map.* □

Although Toda brackets are defined for diagrams of pointed spaces, one can extend them to unpointed spaces, provided the spaces are simply connected. This is what we do now. A *simply connected space* is a nonempty, connected space whose fundamental groupoid is trivial. Let $\mathbb{S}_{\mathsf{sc}}$ and $\mathbb{S}_{\mathsf{sc},\bullet}$ denote the categories of simply connected spaces and of pointed, simply connected spaces, respectively. We have the following well-known fact and corollary.

Lemma 3.8 *The forgetful functor $U\colon \mathsf{Ho}(\mathbb{S}_{\mathsf{sc},\bullet}) \to \mathsf{Ho}(\mathbb{S}_{\mathsf{sc}})$ is an equivalence of categories.* □

Corollary 3.9 *If $X \in \mathsf{Ho}(\mathbb{S}_{\mathsf{sc},\bullet})^{[3]}$ is such that the composite of consecutive maps of $U_*(X)$ are null-homotopic, then X is a bracket sequence.*

Let $X, X' \in \mathsf{Ho}(\mathbb{S}_{\mathsf{sc},\bullet})^{[3]}$ be such that $U_*(X) \cong U_*(X')$. Then X is a bracket sequence if and only if X' is; in that case, the Toda bracket of X vanishes if and only if the Toda bracket of X' does. □

Corollary 3.9 implies that, for $X \in \mathsf{Ho}(\mathbb{S}_{\mathsf{sc}})^{[3]}$, there is a well-defined notion of X being a bracket sequence, namely that any lift $X' \in \mathsf{Ho}(\mathbb{S}_{\mathsf{sc},\bullet})^{[3]}$ is a bracket sequence; in that case, we say that the Toda bracket of X vanishes if the Toda bracket of X' does.

Let $\mathsf{j}\colon \mathbb{S}^{[3]} \to \mathbb{S}^{\mathbb{Z}}$ be given by extending $X \in \mathbb{S}^{[3]}$ to the right with the singleton space and to the left with the empty space. Let $* \in \mathbb{S}^{[3]}$ be the constant singleton space.

Proposition 3.10 *Let $X \in \mathbb{S}_{\mathsf{sc}}^{[3]}$. Then*

(1) $\mathsf{h}(\mathsf{j}(X)) \in \mathsf{Ho}(\mathbb{S})^{\mathbb{Z}}$ *is 1–interleaved with* $\mathsf{h}(\mathsf{j}(*))$ *if and only if* $\mathsf{h}X$ *is a bracket sequence*;

(2) *if* $\mathsf{h}(\mathsf{j}(X)) \in \mathsf{Ho}(\mathbb{S})^{\mathbb{Z}}$ *is 1–interleaved with* $\mathsf{h}(\mathsf{j}(*))$*, then* $\mathsf{j}(X)$ *is 1–homotopy interleaved with* $\mathsf{j}(*)$ *if and only if the Toda bracket of* $\mathsf{h}X$ *vanishes.*

Proof Statement (1) follows directly from Corollary 3.9. For (2), note that if the Toda bracket of $\mathsf{h}X$ vanishes, then, by Proposition 3.7, there exists $X' \in \mathbb{S}_{\mathsf{sc},\bullet}^{[3]}$ such that $\mathsf{h}X' \cong \mathsf{h}X$ and such that the composite of consecutive maps of X' are null maps. In particular, $\mathsf{j}(X')$ is 1–interleaved with $\mathsf{j}(*)$, and, since $\mathsf{h}(\mathsf{j}(X')) \cong \mathsf{h}(\mathsf{j}(X))$, we have that $\mathsf{j}(X') \simeq \mathsf{j}(X)$ by Lemma 3.4, so $\mathsf{j}(X)$ and $\mathsf{j}(*)$ are 1–homotopy interleaved.

For the converse of (2), assume that $\mathsf{j}(X)$ and $\mathsf{j}(*)$ are 1–homotopy interleaved. It follows that there exists a commutative diagram of pointed spaces and pointed maps

$$\begin{tikzcd}
X'(0) \arrow[r, "f_0'"] \arrow[d] & X'(1) \arrow[r] \arrow[d, "f_1'"] & B \arrow[d] \\
A \arrow[r] \arrow[urr] & X'(2) \arrow[r, "f_2'"] & X'(3)
\end{tikzcd}$$

with A and B contractible and $X' \in \mathbb{S}_{\bullet}^{[3]}$ such that $X' \simeq X$, as diagrams of unpointed spaces. It suffices to show that the Toda bracket of X' vanishes. For this, note that, using the diagonal morphism $A \to B$, we can find maps $\alpha\colon CX'(0) \to X'(2)$ and $\beta\colon CX'(1) \to X'(3)$ such that $\beta \circ Cf_1' = f_2' \circ \alpha$. In particular, in this case, there is a diagonal filler for the square (3-1) and thus the induced map $\Sigma X'(0) \to X'(3)$ is nullhomotopic, as required. $\square$

The following lemma is clear.

Lemma 3.11 *Let $X, Y \in C^{\mathbb{Z}}$. If $i_*(X), i_*(Y) \in C^{\mathbb{R}}$ are r–interleaved for some $0 \le r < \frac{3}{2}$, then $X, Y \in C^{\mathbb{Z}}$ are 1–interleaved.* $\square$

We are now ready to prove the lower bound.

Proposition 3.12 *Let $\mathcal{M} = \mathbb{S}$. If $d_{\mathrm{HI}} \leq c d_{\mathrm{HC}}$ then $c \geq \frac{3}{2}$.*

Proof By Lemma 3.11 and Proposition 3.10, it suffices to find a bracket sequence $X \in \mathsf{Ho}(\mathbb{S}_\bullet)^{[3]}$ valued in simply connected spaces such that its Toda bracket does not vanish. Examples of this are given in [21]. A classical example, referenced in [2], is $S^4 \to S^4 \to S^3 \to S^3$ with the first and last maps degree 2 maps, and the middle map the suspension of the Hopf map. □

Remark 3.13 Proposition 3.12 implies in particular that $d_{\mathrm{HI}} \neq d_{\mathrm{HC}}$. As mentioned in the introduction, we know that we have $d_{\mathrm{HI}} \geq d_{\mathrm{IHC}} \geq d_{\mathrm{HC}}$, so it is natural to wonder whether we have $d_{\mathrm{HI}} \neq d_{\mathrm{IHC}}$ or $d_{\mathrm{IHC}} \neq d_{\mathrm{HC}}$, or both. We leave these as open questions.

3.3 Impossibility of rectification in higher dimensions

In this section, we show that Theorem A has no analogue for multipersistent spaces; we thank Alex Rolle for pointing this out to us. We prove this for $m = 2$ and remark that a similar argument works for $m > 2$.

Proposition 3.14 *If $m = 2$, there is no constant $c > 0 \in \mathbb{R}$ such that for all $\delta > 0 \in \mathbb{R}^m$, if $X, Y \in \mathbb{S}^{\mathbb{R}^m}$ are δ–homotopy commutative interleaved, then they are $c\delta$–homotopy interleaved.*

Let sq denote the subposet of $\mathbb{R}^2$ spanned by $\{(0,0), (0,1), (1,0), (1,1)\}$, so that a functor $\mathrm{sq} \to C$ from sq to a category C corresponds to a commutative square in C. We will use the following well-known fact, which says that a homotopy commutative diagram can have different, nonequivalent lifts. For a specific instance see eg [10].

Lemma 3.15 *There exist $A, B\colon \mathrm{sq} \to \mathbb{S}$ such that $\mathsf{h}A \cong \mathsf{h}B \in \mathsf{Ho}(\mathbb{S})^{\mathrm{sq}}$ and such that $A \not\simeq B$.* □

Proof of Proposition 3.14 Given a diagram $A\colon \mathrm{sq} \to \mathbb{S}$, consider the bipersistent space $A'\colon \mathbb{R}^2 \to \mathbb{S}$ such that $A'(r,s) = \varnothing$ whenever r or s are negative, $A'(r,s) = A(\lfloor r \rfloor, \lfloor s \rfloor)$ whenever $0 \leq r, s < 2$, and $A'(r,s)$ is the singleton space whenever $0 \leq r, s$ and $2 \leq \max(r,s)$. Let $A, B\colon \mathrm{sq} \to \mathbb{S}$. Note that if $(0,0) \leq \delta < \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$ and $A', B' \in \mathbb{S}^{\mathbb{R}^2}$ are δ–homotopy interleaved, then we have $A \simeq B$.

To prove the result, it is enough to show that there exist bipersistent spaces $X, Y \in \mathbb{S}^{\mathbb{R}^2}$ that are 0–homotopy commutative interleaved, ie such that $\mathsf{h}X \cong \mathsf{h}Y$, which are not δ–homotopy interleaved for any $0 \leq \delta < \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$. In order to do this, we can let A and B be as in Lemma 3.15 and take $X = A'$ and $Y = B'$. □

4 Projective cofibrant persistent simplicial sets

The purpose of this section is to characterize projective cofibrant persistent simplicial sets as filtered simplicial sets (Proposition 4.5). We work with simplicial sets indexed by an arbitrary poset $(P, \leq)$.

Definition 4.1 A *P–filtered simplicial set* (*filtered simplicial set* when there is no risk of confusion) is a simplicial set X equipped with functions $\beta_n \colon X_n \to P$, satisfying

- $\beta_{n-1}(d_i(\sigma)) \leq \beta_n(\sigma)$ for every $n \geq 1$, $\sigma \in X_n$, and boundary map $d_i : X_n \to X_{n-1}$;
- $\beta_{n+1}(s_i(\sigma)) \leq \beta_n(\sigma)$ for every $n \geq 0$, $\sigma \in X_n$, and degeneracy map $s_i : X_n \to X_{n+1}$.

When there is no risk of ambiguity, we denote the filtered simplicial set (X, β) simply by X.

Definition 4.2 Given a filtered simplicial set (X, β), define a persistent simplicial set $\widehat{(X, \beta)} \in \mathrm{sSet}^{\boldsymbol{P}}$ such that for $r \in P$ we have $\widehat{(X, \beta)}(r)_n = \{\sigma \in X_n : \beta_n(\sigma) \leq r\}$, with faces and degeneracies given by restricting the ones of X.

By a standard abuse of language, We say that a persistent simplicial set *is a filtered simplicial set* if it is isomorphic to $\widehat{Y}$ for Y a filtered simplicial set.

The following result is a characterization of filtered simplicial sets among persistent simplicial sets by means of easily verified point-set conditions.

Lemma 4.3 *A persistent simplicial set $X \in \mathrm{sSet}^{\boldsymbol{P}}$ is a filtered simplicial set if and only if the following conditions are satisfied*:

(1) *The structure morphism $X(r) \to X(r')$ is a monomorphism for every $r \leq r'$ in P. In particular, up to isomorphism, we may, and do, assume that $X(r)$ is a subsimplicial set of $X(r')$.*

(2) *For every simplex $\sigma \in \bigcup_{r \in P} X(r)$, the set $\{t \in P : \sigma \in X(t)\}$ has a minimum.*

Proof The only nontrivial part is that if X satisfies the two conditions in the statement then it is filtered. Set $Y = \bigcup_{r \in P} X(r)$, which makes sense thanks to condition (1). Given $\sigma \in Y_n$, define $\beta_n(\sigma) := \min\{t \in P : \sigma \in X(t)\}$, which is well defined thanks to condition (2). We then have $X \cong \widehat{Y}$. The rest of the proof is clear. □

The proof of the following lemma a straightforward application of Lemma 4.3. We use the term *cell attachment* to indicate any pushout of a generating cofibration $r \odot \partial\Delta^n \to r \odot \Delta^n$.

Lemma 4.4 (1) *A retract of a filtered simplicial set is filtered.*

(2) *If the domain of a cell attachment is a filtered simplicial set, then the codomain is too.*

(3) *Let ζ be a limit ordinal and let $X_\bullet : \zeta \to \mathrm{sSet}^P$ be a diagram of persistent simplicial sets, where for each $\gamma < \zeta$ we have that the map $X_\gamma \to X_{\gamma+1}$ is a cell attachment. If X_γ is a filtered simplicial set for every $\gamma < \zeta$, then $X_\zeta = \mathrm{colim}_{\gamma<\zeta} X_\gamma$ is a filtered simplicial set.* □

The recognition principle for projective cofibrant persistent simplicial sets is now a consequence of Lemma 4.4 and the fact that the cofibrant objects in a cofibrantly generated model category are precisely the retracts of transfinite compositions of cell attachments [12, Proposition 2.1.18(b)].

Proposition 4.5 *A persistent simplicial set is filtered if and only if it is projective cofibrant.* □

In practice, many of the persistent spaces relevant to topological data analysis are filtered simplicial sets.

Example 4.6 The Vietoris–Rips complex associated to a metric space (X, d_X), usually defined to be a persistent simplicial complex, can be turned into a persistent simplicial set by choosing a total order on X. It follows directly from its definition that this persistent simplicial set is filtered. Other examples of this form include the Čech complex and the filtrations of [6].

An example of a filtered multipersistent simplicial set is the following. Given a metric space (X, d_X) together with a real-valued function $f : X \to \mathbb{R}$, one can construct a bifiltered simplicial set as follows. For each $s \in \mathbb{R}$, consider $X_s = f^{-1}(-\infty, s]$ and let $F_{s,r}$ be the Vietoris–Rips complex of X_s at scale r.

We remark that persistent simplicial sets whose structure maps are monomorphisms are not necessarily filtered. This happens in practice when the same simplex "appears at different times", that is, when condition (2) in Lemma 4.3 is not satisfied. Examples of this include the *degree-Rips* bifiltration [16], and Vietoris–Rips applied to the *kernel density filtration* of [19].

5 Interleaving in $\mathsf{Ho}(\mathbb{S}^{\mathbb{R}^m})$ and in homotopy groups

In this section, we prove Theorem B. We start by defining the notions of persistent homotopy groups of a persistent space, and of morphism inducing an interleaving in persistent homotopy groups. The notion of persistent homotopy group we use is essentially the same as that of Jardine [13].

We model the n^{th} homotopy group $\pi_n(W,w)$ of a pointed space (W,w) by the set of pointed homotopy classes of pointed maps from the n–dimensional sphere S^n into W.

Definition 5.1 Let $X \in \mathbb{S}^{\mathbb{R}^m}$. The persistent set $\pi_0(X)\colon \mathbb{R}^m \to \mathrm{Set}$ is defined by $\pi_0 \circ X$. Let $n \geq 1$, $r \in \mathbb{R}^m$ and $x \in X(r)$. The n^{th} *persistent homotopy group* of X based at x is the persistent group $\pi_n(X,x)\colon \mathbb{R}^m \to \mathrm{Grp}$ that is trivial at $s \not\geq r$, and that is $\pi_n(X(s), \varphi^X_{r,s}(x)) \in \mathrm{Grp}$ at $s \geq r$.

Note that π_n is functorial for every $n \in \mathbb{N}$.

Definition 5.2 Let $\varepsilon, \delta \geq 0 \in \mathbb{R}^m$. Assume given a homotopy class of morphisms $[f]\colon X' \to Y'^{\varepsilon} \in \mathsf{Ho}(\mathbb{S}^{\mathbb{R}^m})$. Let $X' \simeq X$ be a cofibrant replacement, let $Y' \simeq Y$ be a fibrant replacement, and let $f : X \to_{\varepsilon} Y$ be a representative of f. We say that $[f]$ *induces an* (ε,δ)*–interleaving in homotopy groups* if the induced map $\pi_0(f)\colon \pi_0(X) \to_{\varepsilon} \pi_0(Y)$ is part of an (ε,δ)–interleaving of persistent sets, and if for every $r \in \mathbb{R}^m$, every $x \in X(r)$, and every $n \geq 1 \in \mathbb{N}$, the induced map $\pi_n(f)\colon \pi_n(X,x) \to_{\varepsilon} \pi_n(Y, f(x))$ is part of an (ε,δ)–interleaving of persistent groups.

It is clear that the definition above is independent of the choices of representatives.

A standard result in classical homotopy theory is that a fibration of Kan complexes inducing an isomorphism in all homotopy groups has the right lifting property with respect to cofibrations [9, Theorem I.7.10]. An analogous, persistent, result (Corollary 5.13), says that, for a fibration of fibrant objects inducing a δ–interleaving in homotopy groups, the lift exists up to a shift, which depends on both δ and on a certain "length" $n \in \mathbb{N}$ associated to the cofibration. To make this precise, we introduce the notion of n–dimensional extension.

Definition 5.3 Let $A, B \in \mathbb{S}^{\mathbb{R}^m}$ and let $n \in \mathbb{N}$. A map $j : A \to B$ is an *n–dimensional extension* (of A) if there exists a set I, a family of tuples of real numbers $\{r_i \in \mathbb{R}^m\}_{i \in I}$, and commutative squares of the form depicted on the left below, that together give rise to the pushout square on the right:

$$\begin{tikzcd} \partial D^n \arrow[r, "f_i"] \arrow[d, hook] & A(r_i) \arrow[d, "j_{r_i}"] \\ D^n \arrow[r, "g_i"] & B(r_i) \end{tikzcd} \qquad \begin{tikzcd} \coprod_{i \in I} r_i \odot (\partial D^n) \arrow[r, "f"] \arrow[d] & A \arrow[d, "j"] \\ \coprod_i r_i \odot (D^n) \arrow[r, "g"] & B \end{tikzcd}$$

Here, $\partial D^n \hookrightarrow D^n$ stands for $S^{n-1} \hookrightarrow D^n$ if $\mathbb{S} = \mathrm{Top}$, and for $\partial \Delta^n \hookrightarrow \Delta^n$ if $\mathbb{S} = \mathrm{sSet}$.

A *single-dimensional extension* is an n–dimensional extension for some $n \in \mathbb{N}$.

Definition 5.4 Let $\iota\colon A \to B$ be a projective cofibration of $\mathbb{S}^{\mathbb{R}^m}$ and let $n \in \mathbb{N}$. We say that ι is an *n–cofibration* if it factors as the composite of $n+1$ maps $f_0, \dots, f_n$, with f_i an n_i–dimensional extension for some $n_i \in \mathbb{N}$. We say that $A \in \mathbb{S}^{\mathbb{R}^m}$ is n–cofibrant if the map $\varnothing \to A$ is an n–cofibration.

The next lemma, which follows directly from Proposition 4.5, gives a rich family of examples of n–cofibrant persistent simplicial sets. Recall that a simplicial set is n–skeletal if all its simplices in dimensions above n are degenerate.

Lemma 5.5 *Let $A \in \mathrm{sSet}^{\mathbb{R}^m}$ and let $n \in \mathbb{N}$. If A is projective cofibrant and pointwise n–skeletal, then it is n–cofibrant.* □

Example 5.6 The Vietoris–Rips complex $\mathsf{VR}(X)$ of a metric space X, as defined in Example 4.6, is n–cofibrant if the underlying set of X has finite cardinality $|X| = n+1$.

If one is interested in persistent (co)homology of some bounded degree n, then one can restrict computations to the $(n+1)$–skeleton of a Vietoris–Rips complex, which is $(n+1)$–cofibrant.

A result analogous to Lemma 5.5, but for persistent topological spaces, does not hold, as cells are not necessarily attached in order of dimension. This motivates the following definition.

Definition 5.7 Let $n \in \mathbb{N}$. A persistent topological space $A \in \mathrm{Top}^{\mathbb{R}^m}$ is an *n–dimensional persistent CW–complex* if the map $\varnothing \to X$ can be factored as a composite of maps $f_0, \dots, f_n$, with f_i an i–dimensional extension.

Example 5.8 The geometric realization of any n–cofibrant persistent simplicial set is an n–dimensional persistent CW–complex.

Lemma 5.9 *Every n–dimensional persistent CW–complex is n–cofibrant.* □

We now make precise the notion of lifting property up to a shift.

Definition 5.10 Let $i: A \to B$ and $p: Y \to X$ be morphisms in $\mathbb{S}^{\mathbb{R}^m}$ and let $\delta \geq 0$. We say that p has the *right δ–lifting property* with respect to i if for all morphisms $A \to Y$ and $B \to X$ making the square on the left below commute, there exists a diagonal δ–morphism $f: B \to_\delta Y$ rendering the diagram commutative. The diagram on the left is shorthand for the one on the right:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "\delta"] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Y \arrow[r, "\mathsf{S}_{0,\delta}(\mathrm{id}_Y)"] \arrow[d, "p"] & Y^\delta \arrow[d, "p^\delta"] \\ B \arrow[r] \arrow[urr, dotted] & X \arrow[r, "\mathsf{S}_{0,\delta}(\mathrm{id}_X)"'] & X^\delta \end{tikzcd}$$

We now prove Lemma 5.12, an adaptation of a result of Jardine, which says that fibrations inducing interleavings in homotopy groups have a shifted right lifting property, as defined above. The main difference is that we work in the multipersistent setting. We use simplicial notation and observe that the corresponding statement for persistent topological spaces follows from the simplicial one by using the singular complex-realization adjunction. We recall a standard, technical lemma whose proof is given within that of eg [9, Theorem I.7.10].

Lemma 5.11 *Consider a commutative square of simplicial sets*

(5-1)
$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"] & Y \end{tikzcd}$$

where p is a Kan fibration between Kan complexes. If there is commutative diagram like the one on the left below, for which the lifting problem on the right admits a solution,

$$\begin{tikzcd}
\partial\Delta^n \arrow[dd, hook] \arrow[dr, "(\mathrm{id}_{\partial\Delta^n}\times\{1\})"] \arrow[drr, bend left, "\alpha"] & & \\
& \partial\Delta^n\times\Delta^1 \arrow[r, "h"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[ur, "(\mathrm{id}_{\Delta^n}\times\{1\})"'] \arrow[rr, bend right, "\beta"'] & \Delta^n\times\Delta^1 \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial\Delta^n \arrow[r, "h\circ(\mathrm{id}_{\partial\Delta^n}\times\{0\})"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "g\circ(\mathrm{id}_{\Delta^n}\times\{0\})"] & Y
\end{tikzcd}$$

then the initial square (5-1) admits a solution.

Lemma 5.12 (cf [13, Lemma 14]) *Let $\delta \geq 0$, and let $f\colon X \to Y \in \mathbb{S}^{\mathbb{R}^m}$ induce a $(0,\delta)$–interleaving in homotopy groups. If X and Y are projective fibrant and f is a projective fibration, then f has the right 2δ–lifting property with respect to boundary inclusions $r \odot \partial D^n \to r \odot D^n$ for every $r \in \mathbb{R}^m$ and every $n \in \mathbb{N}$.*

Proof Consider a commutative diagram as on the left below, which corresponds to the one on the right:

(5-2)
$$\begin{tikzcd}
r\odot\partial\Delta^n \arrow[r, "a"] \arrow[d] & X \arrow[d, "p"] \\
r\odot\Delta^n \arrow[r, "b"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d] & X(r) \arrow[d, "p_r"] \\
\Delta^n \arrow[r, "\beta"] & Y(r)
\end{tikzcd}$$

We must find a 2δ–lift for the diagram on the right. The proof strategy is to appeal to Lemma 5.11 to simplify α, then prove that at the cost of a δ–shift we can further reduce α to a constant map, and then show that the simplified lifting problem can be solved at the cost of another δ–shift. So we end up with a 2δ–lift, as in the statement. We proceed by proving the claims in opposite order.

We start by showing that (5-2) can be solved up to a δ–shift whenever α is constant. Let us assume that α is of the form $\alpha = *$ for some $* \in X(r)_0$. Since, then, β represents an element $[\beta] \in \pi_n(Y(r), *)$, there exists a map $\alpha'\colon \Delta^n \to X(r+\delta)$ whose restriction to $\partial\Delta^n$ is constant on $* \in X(r)_0$, and such that there is a homotopy $h\colon \beta \simeq p\alpha'$ relative to $\partial\Delta^n$. We can thus consider

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\mathrm{id}_{\partial\Delta^n}\times\{1\}"] \arrow[d, "i"] & (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r, "(*,\alpha')"] \arrow[d] & X(r+\delta) \arrow[d, "p_{r+\delta}"] \\
\Delta^n \arrow[r, "\mathrm{id}_{\Delta^n}\times\{1\}"] & \Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[ur, dotted, "H"] & Y(r+\delta)
\end{tikzcd}$$

where H is a diagonal filler for the right-hand side square, which exists since the middle vertical map is a trivial cofibration of simplicial sets and $p_{r+\delta}$ is a Kan fibration by assumption. The composite map $H \circ \mathrm{id}_{\Delta^n} \times \{1\}$ is a lift for (5-2).

We now assume that α is of a specific, simplified form, and prove that, up to a δ–shift, we can reduce the lifting problem (5-2) to the case in which α is constant. Let us assume that $d_i(\alpha) = * \in X(r)_0$ for every $0 < i \leq n$, and set $\alpha_0 = d_0(\alpha)$. Then α_0 represents an element $[\alpha_0] \in \pi_{n-1}(X(r), *)$, with the property that $p[\alpha_0] = 0 \in \pi_{n-1}(Y(r), *)$. Since p induces a $(0,\delta)$–interleaving in homotopy groups,

$$\varphi^X_{r,r+\delta}([\alpha_0]) = 0 \in \pi_{n-1}(X(r+\delta), *),$$

witnessed by a homotopy $h_0\colon \Delta^{n-1} \times \Delta^1 \to X(r+\delta)$, constant on $\partial\Delta^{n-1}$. If we set $h'_i = *\colon \Delta^{n-1} \to X(r+\delta)$ for every $0 < i \leq n-1$ and $h'_0 = h_0$, we get a map $h'\colon \partial\Delta^n \times \Delta^1 \to X(r+\delta)$. We can now extend

$$(\varphi^Y_{r,r+\delta} \circ \beta, ph')\colon (\Delta^n \times \{1\}) \cup (\partial\Delta^n \times \Delta^1) \to Y(r+\delta)$$

to a homotopy $H'\colon \Delta^n \times \Delta^1 \to Y(r+\delta)$. Now observe that the lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "h'_1\circ(\mathrm{id}_{\partial\Delta^n}\times\{0\})"] \arrow[d] & X(r+\delta) \arrow[d, "p_{r+\delta}"] \\
\Delta^n \arrow[r, "H'\circ(\mathrm{id}_{\Delta^n}\times\{0\})"] & Y(r+\delta)
\end{tikzcd}$$

is such that $h' \circ (\mathrm{id}_{\partial\Delta^n} \times \{0\}) = *$, so, thanks to Lemma 5.11, we have reduced this case to the case in which α is constant.

To conclude, we must show that we can reduce the original lifting problem (5-2) to one in which all but the 0^{th} faces of α are constant on a point $* \in X(r)_0$. Let $K\colon \Lambda^n_0 \times \Delta^1 \to \Lambda^n_0$ be the homotopy that contracts the simplicial horn onto its vertex 0, which determines a diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\mathrm{id}_{\Lambda^n_0}\times\{1\}"] \arrow[d, hook] & \Lambda^n_0 \times \Delta^1 \arrow[d, "k_1"] & \Lambda^n_0 \arrow[l, "\mathrm{id}_{\Lambda^n_0}\times\{0\}"'] \arrow[d] \\
\partial\Delta^n \arrow[r, "\alpha"] & X & \Delta^0 \arrow[l, "\alpha(0)"']
\end{tikzcd}$$

with $k_1 = \alpha \circ j \circ K$, where $j : \Lambda^n_0 \to \partial\Delta^n$ is the inclusion of the horn into the boundary. We can now extend the map $(\alpha, k_1) \colon (\partial\Delta^n \times \{1\}) \cup (\Lambda^n_0 \times \Delta^1) \to X(r)$ to a homotopy $k \colon \partial\Delta^n \times \Delta^1 \to X(r)$. Similarly, we extend the map

$$(\beta, p \circ k) \colon (\Delta^n \times \{1\}) \cup (\partial\Delta^n \times \Delta^1) \to Y(r)$$

to a homotopy $g \colon \Delta^n \times \Delta^1 \to Y(r)$. It now suffices to consider the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[dr, "(\mathrm{id}_{\partial\Delta^n}\times\{1\})"] \arrow[ddd, hook] \arrow[rrd, bend left, "\alpha"] & & \\
& \partial\Delta^n \times \Delta^1 \arrow[r, "k"] \arrow[d, hook] & X \arrow[d, "p"] \\
& \Delta^n \times \Delta^1 \arrow[r, "g"] & Y \\
\Delta^n \arrow[ur, "(\mathrm{id}_{\Delta^n}\times\{1\})"'] \arrow[rru, bend right, "\beta"'] & &
\end{tikzcd}$$

observing that $\alpha' := k_{|\partial\Delta^n \times \{0\}}$ satisfies $d_i(\alpha') = *$ for $0 < i \le n$, and appeal to Lemma 5.11. □

Corollary 5.13 *Let $\delta \ge 0$ and let $f : X \to_\delta Y$ induce a δ–interleaving in homotopy groups. If X and Y are projective fibrant and f is a projective fibration, then f has the right $(4(n+1)\delta)$–lifting property with respect to n–cofibrations for all $n \in \mathbb{N}$.*

Proof By assumption, $f : X \to Y^\delta$ induces a $(0, 2\delta)$–interleaving in all homotopy groups. Now, an n–cofibration can be written as a composite of $n+1$ single-dimensional extensions, and any shift of a single-dimensional extension is again a single-dimensional extension, so it is enough to show that f has the right 4δ–lifting property with respect to single-dimensional extensions.

A single-dimensional extension is the pushout of a coproduct

$$\coprod_{i\in I} r_i \odot (\partial D^n) \to \coprod_{i\in I} r_i \odot D^n,$$

so it suffices that f has the right 4δ–lifting property with respect to coproducts of that form, which follows from Lemma 5.12 and the universal property of coproducts. $\square$

We are ready to prove Theorem B.

Theorem B *Let $X, Y \in \mathbb{S}^{\mathbb{R}^m}$ be persistent spaces that are assumed to be projective cofibrant and d–skeletal if $\mathbb{S} = \mathrm{sSet}$, or persistent CW–complexes of dimension at most d if $\mathbb{S} = \mathrm{Top}$. Let $\delta \geq 0 \in \mathbb{R}^m$. If there exists a morphism in the homotopy category $X \to Y^\delta \in \mathsf{Ho}(\mathbb{S}^{\mathbb{R}^m})$ that induces δ–interleavings in all homotopy groups, then X and Y are $(4(d+1)\delta)$–interleaved in the homotopy category.*

Proof By Lemmas 5.5 and 5.9, X and Y are d–cofibrant. Let $[f]\colon X \to Y^\delta$ be as in the statement. Since $[f]$ is a morphism in the homotopy category, we begin by choosing a convenient representative of it. We let $p\colon X' \to Y'$ be a projective fibration between projective fibrant objects such that there exist trivial cofibrations $i : X \to X'$ and $j : Y \to Y'$ with $[p] \circ [i] = [j] \circ [f]$, in $\mathsf{Ho}(\mathbb{S}^{\mathbb{R}^m})$.

Note that $[p]$ induces a $(0, 2\delta)$–interleaving in homotopy groups, between X' and Y'^δ. Since Y^δ is d–cofibrant, Corollary 5.13 guarantees that we can find a $(4(d+1)\delta)$–lift g' of p against $\varnothing \to Y$. We can then construct the lift

$$\begin{tikzcd} Y \arrow[r, "g'"] \arrow[d, "j"'] & X'^{(4d+3)\delta} \\ Y' \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

using the fact that $j : Y \to Y'$ is a trivial cofibration and X' is fibrant. We will show that $\mathsf{S}_{\delta,4(d+1)\delta}([p])\colon X' \to_{4(d+1)\delta} Y'$ and $\mathsf{S}_{(4d+3)\delta,4(d+1)\delta}([g])\colon Y' \to_{4(d+1)\delta} X'$ form a $(4(d+1)\delta)$–interleaving in the homotopy category between X' and Y'.

On the one hand, note that, by construction, $p^{(4d+3)\delta} \circ g \circ j = p^{(4d+3)\delta} \circ g' = j$, so, since $[j]$ is an isomorphism, it follows that $[p]^{(4d+3)\delta} \circ [g] = \mathsf{S}_{4(d+1)\delta}([\mathrm{id}_{Y'}])$, and thus that

$$\mathsf{S}_{\delta,4(d+1)\delta}([p])^{4(d+1)\delta} \circ \mathsf{S}_{(4d+3)\delta,4(d+1)\delta}([g]) = \mathsf{S}_{8(d+1)\delta}([\mathrm{id}_{Y'}]).$$

On the other hand, since X is cofibrant and Y' is fibrant, it follows from the previous paragraph that

$$p^{4(d+1)\delta} \circ g^{\delta} \circ p \circ i : X \to Y'^{(4d+5)\delta}$$

is homotopic to $p^{4(d+1)\delta} \circ \mathsf{S}_{0,4(d+1)\delta}(i)$. Let $H\colon I \times X \to Y'^{(4d+5)\delta}$ be a homotopy between these maps, which gives the commutative diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "(\mathsf{S}_{0,4(d+1)\delta}(i)\text{,}g^{\delta}\circ p\circ i)"] \arrow[d, "(i_0\text{,}i_1)"'] & X'^{4(d+1)\delta} \arrow[d, "p^{4(d+1)\delta}"] \\ I \times X \arrow[r, "H"] & Y'^{(4d+5)\delta} \end{tikzcd}$$

where the left vertical map is the inclusion of into the cylinder. We claim that, since X is d–cofibrant, the inclusion into the cylinder is a d–cofibration. Indeed, a cell decomposition of this map is obtained by attaching an $(n+1)$–cell for each n–cell in the decomposition of X. Now, by Corollary 5.13, we can find a $(4(d+1)\delta)$–lift of the diagram, which shows that

$$\mathsf{S}_{4(d+1)\delta,8(n+1)\delta}([g]^{\delta} \circ [p] \circ [i]) = \mathsf{S}_{0,8(d+1)\delta}([i])\colon X \to X'^{8(d+1)\delta}.$$

Since the left-hand side equals $\mathsf{S}_{(4d+3)\delta,4(d+1)\delta}([g])^{4(d+1)\delta} \circ \mathsf{S}_{\delta,4(d+1)\delta}([p] \circ [i])$, and $[i]$ is an isomorphism, it follows that $[g]^{4(d+1)\delta} \circ \mathsf{S}_{\delta,4(d+1)\delta}([p]) = \mathsf{S}_{8(d+1)\delta}([\mathrm{id}_{X'}])$. $\square$

Remark 5.14 Together, Theorems A and B imply a version of the persistent Whitehead conjecture, which we recall as Conjecture 5.15. Our result is, in a sense, stronger than the one conjectured, since Theorem B, which addresses part (i) of the conjecture, applies to arbitrary multipersistent spaces. In another respect, our result is slightly weaker, as the conjecture is stated for cofibrant, pointwise CW–complexes, which does not necessarily imply being a persistent CW–complex in our sense. We believe that this is not an issue, as many of the cofibrant, pointwise CW–complexes persistent topological spaces that appear in applications are in fact persistent CW–complexes, as they are usually the geometric realization of a filtered simplicial complex.

Conjecture 5.15 [2, Conjecture 8.6] *Suppose we are given connected, cofibrant $X, Y : \mathbb{R} \to \mathrm{Top}$, with each $X(r)$ and $Y(r)$ CW–complexes of dimension at most d, and $f : X \to Y^{\delta}$ inducing a δ–interleaving in all homotopy groups. Then there is a constant c, depending only on d, such that*

(i) *f induces a $c\delta$–interleaving in the homotopy category $\mathsf{Ho}(\mathrm{Top}^{\mathbb{R}})$;*

(ii) *X and Y are $c\delta$–homotopy interleaved.*

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Operadic actions on long knots and 2–string links

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We realize the space of 2–string links $\mathcal{L}$ as a free algebra over a colored operad denoted by $\mathcal{SCL}$ (for "Swiss-cheese for links"). This result extends works of Burke and Koytcheff about the quotient of $\mathcal{L}$ by its center, and is compatible with Budney's freeness theorem for long knots. From an algebraic point of view, our main result refines Blaire, Burke and Koytcheff's theorem on the monoid of isotopy classes of string links. Topologically, it expresses the homotopy type of the isotopy class of a 2–string link in terms of the homotopy types of the classes of its prime factors.

57R40; 55P48, 55U40, 57M99

Introduction

Motivation and context

The study of knots and links is a vast subject that emerged in the late nineteenth century and saw several renewals in the past thirty years. It is subject to many different approaches, being at the crossroads of topology, geometry, algebra, combinatorics and physics. The central theme in classical knot theory is the study of the isotopy classes of knots, ie the isotopy classes of embeddings in $\mathrm{Emb}(S^1, S^3)$. They are the set of components $\pi_0 \mathrm{Emb}(S^1, S^3)$. A common method is to try to chop the knots into simpler pieces. Two ways of performing such a decomposition have proved themselves particularly fruitful: the prime decomposition and the satellite decomposition. The former splits a knot as the connected sum of other knots called its prime factors. The connected sum is a binary operation on $\pi_0 \mathrm{Emb}(S^1, S^3)$ denoted by #. It endows the isotopy classes with a unital commutative monoidal structure. Intuitively, the product $k_1 \# k_2$ is the knot obtained by cutting open k_1 and k_2 and closing them up into a single knot. This decomposition is fairly well understood thanks to a theorem of Schubert [28] stating that $\pi_0 \mathrm{Emb}(S^1, S^3)$ is freely generated as a monoid by the isotopy classes of the prime knots. There are infinitely many prime knots and they can be very different in nature. This is why it is often useful to further decompose the prime knots as satellites of simpler knots. The satellite construction also originates in Schubert's work. It consists of a wide family of operations one can carry out on several knots at a time. Its rigorous definition is a bit more involved but clearly dispensed for example in Cromwell's book [10]. Jaco, Shalen, Johannson and Thurston played a major role in the study of satellite knots, with the results in [19; 20; 31]. Although they are quite complicated, the satellite operations have the advantage of generating the whole space of knots from a fairly small and classifiable class of knots. Namely, Budney shows in [4] — refining a result of [31] — that every knot can be obtained via successive satellite operations on hyperbolic and torus knots.

This depicts a duality between the two methods of decomposition: one is simple but leads to potentially complicated primes, while the other is more elaborate but has easier irreducible pieces. A similar story can be told for links, ie for the isotopy classes of embeddings in $\mathrm{Emb}(S^1 \amalg \cdots \amalg S^1, S^3)$. Adapting the connected sum # to this setting takes some work: problems arise once the components of a link are cut open, as there is no canonical way to close them up. However, a step-by-step adaptation of the satellite construction works for links, and a decomposition theorem exists as well. Its building blocks are the hyperbolic and Seifert-fibered links.

Nowadays, it is more common to study not only the set of components

$$\pi_0 \mathrm{Emb}(S^1 \amalg \cdots \amalg S^1, S^3),$$

but the full homotopy type of the spaces of knots and links. To adapt the decomposition approach described above, one needs to define an analogue of the connected sum and satellite operations on the space level, ie directly on the embeddings and not between isotopy classes. To rigorously carry out this task, one uses the space of long knots $\mathcal{K}$ and the space of string links $\mathcal{L}$. Coupled with the language of operads, these spaces enable one to extend the existing operations on π_0 to the space level. More precisely, the different types of operations one can carry out on knots and links can be encoded in the action of an operad on a space of embeddings. This new framework is due to Budney in [5; 6] for the case of long knots. Namely, Budney constructs in [5] an action of the little 2–cubes operad $\mathcal{C}_2$ on a space $\widehat{\mathcal{K}}$ homotopy equivalent to $\mathcal{K}$, in such a way that all the induced operations descend to the connected sum on π_0. He builds in [6] another action on $\widehat{\mathcal{K}}$ of a more intricate operad which he calls the splicing operad. These operations correspond to the satellite constructions in many ways. In the case of string links, Burke and Koytcheff build in [8] a complex operad called the infection operad. It is an adaptation of Budney's splicing operad and deals with the satellite operations between string links. The authors mentioned above not only prove the existence of such actions but also their freeness, refining the unique decomposition results on π_0. It remains to find an operadic encoding of the connected sum of links, if possible, leading to a free algebra.

Present work

The present paper aims to fill in this gap in the case of 2–stranded string links. Unlike $\pi_0\mathcal{K}$, the monoid of isotopy classes of 2–string links is neither free nor commutative. Indeed, as explained by Blair, Burke and Koytcheff in [2], $\pi_0\mathcal{L}$ contains invertible elements in the form of the pure braid group KB_2. Together with three copies of $\pi_0\mathcal{K}$, these invertible elements generate the center of the monoid. Burke and Koytcheff state a partial result in [8] by building a free action of the little 1–cubes operad $\mathcal{C}_1$ on a subspace of 2–string links that ignores the homotopy center. They mention as an open problem the extension of such a structure to the whole space of 2–string links. Our main result provides an answer to this particular question. For this purpose, we introduce a four-colored operad $\mathcal{SCL}$ (standing for "Swiss-cheese for links") with set of colors $S = \{o, \uparrow, \downarrow, \updownarrow\}$. An $\mathcal{SCL}$–algebra is in particular a family of spaces $(X, A_\uparrow, A_\downarrow, A_\updownarrow)$, where X is a $\mathcal{C}_1$–algebra and each A_s, $s \in \{\uparrow, \downarrow, \updownarrow\}$, is a $\mathcal{C}_2$–algebra acting on X. One can think of the A_s as independent parts of the homotopy center of X. As in the case of Budney's $\mathcal{C}_2$–action on long knots, we consider a space $\widehat{\mathcal{L}}$ homotopy equivalent to $\mathcal{L}$ to prove a first result which can be summarized as follows:

Theorem 3.9 *The family $(\widehat{\mathcal{L}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is an $\mathcal{SCL}$–algebra. In particular, the family $(\mathcal{L}, \mathcal{K}, \mathcal{K}, \mathcal{K})$ is homotopy equivalent to an explicit $\mathcal{SCL}$–algebra.*

The structure so obtained is compatible with Budney's action on long knots and Burke and Koytcheff's action on their subspace of noncentral string links. It provides a complete understanding of the connected sum of 2–string links. As expected, we also prove a freeness result, refining the structure theorem for the monoid of isotopy classes proved in [2]. In order to do this, we discard the invertible elements by splitting $\pi_0\widehat{\mathcal{L}}$ as a product $\pi_0\widehat{\mathcal{L}}^{\mathbf{0}} \times KB_2$ and prove:

Theorem 4.11 *The quadruplet of spaces $(\widehat{\mathcal{L}}^{\mathbf{0}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is homotopy equivalent as an $\mathcal{SCL}$–algebra to the free $\mathcal{SCL}$–algebra generated by prime knots and links.*

In addition to these algebraic considerations, this result has a homotopical significance as it expresses the homotopy type of the isotopy class of a 2–string link as a function of the homotopy types of the classes of its prime factors. This reduces the computation of the homotopy type of the whole $\mathcal{L}$ to figuring out the homotopy types of the components of the primes. As mentioned above, the latter can be further decomposed thanks to Burke and Koytcheff's infection operad defined in [8].

Organization of the paper

We define in a first section the different spaces of embeddings at stake here: long knots, string links and their fattened versions $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{L}}$. The second section sets up the operadic framework we use. We recall in particular the notion of colored operad and discuss the resulting algebras. We introduce along the way the operad that will appear in our main result, the Swiss-cheese operad for links $\mathcal{SCL}$. The third section aims to define various operadic actions on the spaces of knots and links. We recall the constructions of Budney, Burke and Koytcheff's actions and unify them in a single action of $\mathcal{SCL}$ on 2–string links. Finally, the fourth section proves the freeness result sketched above using low-dimensional and homotopical tools.

Upcoming projects

Our main statements concern the space of string links on two strands. One might naturally wonder what happens in the k–stranded case for some $k > 2$. The conjecture that Theorems 3.9 and 4.11 have adaptations to arbitrary string links seems reasonable, since most of the techniques used in Section 4.1 naturally generalize to the k–stranded case. However, lots of difficulties arise, even at the level of isotopy classes. Theorem 4.11 generalizes Blair, Burke and Koytcheff's explicit model for the monoid $\pi_0\mathcal{L}$, but it does not provide an alternative proof for it. Actually, Blair, Burke and Koytcheff's result is used in the very first line of the proof of our Theorem 4.11. Thus, if one wants to adapt Theorem 4.11 to the k–stranded case, some preliminary work on the monoid of isotopy classes of string links on k strands is necessary. A key point is the understanding of its center. The commutation between string links on k strands has already been characterized in [2], but there are other relations among prime links that remain to be understood. For example, the invertible k–stranded string links are the pure braids on k strands, which already admit a wide family of fairly complex relations amongst themselves. Moreover, these units are not central anymore, which makes it harder to study prime decompositions. A solution to these difficulties could be to accept a less explicit construction for the replacement of $\mathcal{SCL}$, maybe a definition by induction on k. The operad for k–stranded string links would rely on a large set of colors, and would restrict to the operad for $(k-1)$–stranded string links on some subcollections of colors. The existence and freeness of an action could then be easier to prove, but the difficulty is only shifted towards understanding these potentially massive operads.

Another interesting question concerns the Goodwillie–Weiss calculus, introduced to study embedding spaces in [12; 34]. In the context of knots and links, this theory gives rise to two towers of fibrations $\{T_k\widehat{\mathcal{K}}\}$ and $\{T_k\widehat{\mathcal{L}}\}$, converging to the so-called polynomial approximations $T_\infty\widehat{\mathcal{K}}$ and $T_\infty\widehat{\mathcal{L}}$, respectively. Unfortunately, the natural applications $\iota_{\widehat{\mathcal{K}}}\colon \widehat{\mathcal{K}} \to T_\infty\widehat{\mathcal{K}}$ and $\iota_{\widehat{\mathcal{L}}}\colon \widehat{\mathcal{L}} \to T_\infty\widehat{\mathcal{L}}$ are not weak equivalences, but they preserve a lot of homotopical information. In particular, we know from Budney, Conant, Koytcheff and Sinha [7] that the map $\widehat{\mathcal{K}} \to T_k\widehat{\mathcal{K}}$ is a finite type-$(k-1)$ knot invariant and it has been conjectured that $T_k\widehat{\mathcal{K}}$ is actually the universal finite type-$(k-1)$ invariant. This conjecture is already proved rationally in Volić's thesis [32]. Moreover, the polynomial approximations can be simplified and identified to homotopy totalizations using the multiplicative Kontsevich operad K_3 obtained as a compactification of configurations of points in $\mathbb{R}^3$. Briefly speaking, one has the identifications

$$\widehat{\mathcal{K}} \xrightarrow{\iota_{\widehat{\mathcal{K}}}} T_\infty\widehat{\mathcal{K}} \xleftarrow{\mu_{\widehat{\mathcal{K}}}} \mathrm{hoTot}(K_3 \circ \mathrm{SO}(3)), \quad \widehat{\mathcal{L}} \xrightarrow{\iota_{\widehat{\mathcal{L}}}} T_\infty\widehat{\mathcal{L}} \xleftarrow{\mu_{\widehat{\mathcal{L}}}} \mathrm{hoTot}(K_3^2 \circ \mathrm{SO}(3)),$$

where $K_3^2(k) = K_3(2k)$ is a shifted version of the Kontsevich operad. The applications $\mu_{\widehat{\mathcal{K}}}$ and $\mu_{\widehat{\mathcal{L}}}$ have been proved to be weak equivalences by Sinha in [29] and Munson and Volić in [25], respectively. We know that the spaces $\widehat{\mathcal{K}}$, $T_\infty\widehat{\mathcal{K}}$ and $\mathrm{hoTot}(K_3 \circ \mathrm{SO}(3))$ are $\mathcal{C}_2$–algebras by [5], [3] and [11], respectively. However, it is still unknown if $\iota_{\widehat{\mathcal{K}}}$ and $\mu_{\widehat{\mathcal{K}}}$ are $\mathcal{C}_2$–algebra maps. All these questions can be extended to the colored case. From the present work, the family $(\widehat{\mathcal{L}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is equipped with an explicit $\mathcal{SCL}$–algebra structure. We believe that similar structures exist for the families

$$\big(T_\infty\widehat{\mathcal{L}}, T_\infty\widehat{\mathcal{K}}, T_\infty\widehat{\mathcal{K}}, T_\infty\widehat{\mathcal{K}}\big),$$

$$\big(\mathrm{hoTot}(K_3^2\circ \mathrm{SO}(3)), \mathrm{hoTot}(K_3\circ \mathrm{SO}(3)), \mathrm{hoTot}(K_3\circ \mathrm{SO}(3)), \mathrm{hoTot}(K_3\circ \mathrm{SO}(3))\big),$$

and that the zigzag of morphisms induced by $\iota_{\widehat{\mathcal{K}}}$, $\mu_{\widehat{\mathcal{K}}}$, $\iota_{\widehat{\mathcal{L}}}$ and $\mu_{\widehat{\mathcal{L}}}$ between the corresponding families are morphisms of $\mathcal{SCL}$–algebras.

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General framework and notation

We set up here the global framework we work in as well as some notation that might not be completely standard.

Topological spaces

By spaces, we understand compactly generated Hausdorff spaces. They form a full subcategory of topological spaces that we denote by **Top** by slight abuse of notation. Many useful properties of **Top** have been introduced by Steenrod in [30]. The standard Quillen model structure has then been adapted for it by Hovey in [18]. It is a convenient category in the sense that the natural curryfication map

$$\mathbf{Top}(X\times Y, Z)\cong \mathbf{Top}(X, \mathbf{Top}(Y, Z))$$

is a homeomorphism for any three spaces X, Y and Z in **Top**. The need to restrict ourselves to such a subcategory arises from the following fact: when defining an action of an algebraic structure that it also a topological space A on a space X, one can ask for the continuity of either $A\times X\to X$ or $A\to \mathbf{Top}(X, X)$. The homeomorphism above gives the equivalence between these two approaches and enables one to go back and forth between both frameworks. This will be useful when dealing with operadic actions.

Operations on maps

Let $f: A \to X$ and $g: B \to Y$ be maps between spaces. We use the following notation:

- $f \amalg g$ is the map between coproducts $A \amalg B \to X \amalg Y$.
- $f \oplus g$ is the map $A \amalg B \to X$ when $X = Y$.
- $f \times g$ is the map between products $A \times B \to X \times Y$.
- (f, g) is the map $A \to X \times Y$ when $A = B$.
- $A^{\times n}$ is the product of n copies of A and $f^{\times n}$ is the map $A^{\times n} \to B^{\times n}$.
- $B^{\amalg n}$ is the coproduct of n copies of B and $f^{\amalg n}$ is the map $A^{\amalg n} \to B^{\amalg n}$.

Smooth manifolds

When discussing manifolds, we think of usual (possibly bordered) C^∞ manifolds. We write $I = [0, 1]$ for the unit interval, $J = [-1, 1]$ for the 1–dimensional unit disk and $J^k = J^{\times k}$ for the k–dimensional unit cube. The set of C^∞ maps between two manifolds M and N is denoted by $C^\infty(M, N)$ and topologized with the usual C^∞–topology described in [17]. The space of embeddings, immersions, submersions or diffeomorphisms between manifolds are topologized as subspaces of the latter. This turns diffeomorphism groups into topological groups and makes every composition map continuous.

1 Embedding spaces

This section aims to review the construction of various spaces of embeddings, namely spaces of knots and 2–stranded links. We start by recalling the definition of the usual space of knots and introduce three variations: the long knots $\mathcal{K}$, the framed long knots $\mathrm{EC}(1, D^2)$ and the fat long knots $\widehat{\mathcal{K}}$. These spaces are meant to ease algebraic and homotopical manipulations. We also discuss the classical monoid structure on the space of knots, its interactions with these spaces and finally adapt these constructions to 2–stranded links.

1.1 Knot spaces

The first instance of a space of knots arises as the space of embeddings $\mathrm{Emb}(S^1, S^3)$. Its components $\pi_0 \mathrm{Emb}(S^1, S^3)$ are the isotopy classes of knots in the 3–sphere and are the central object of study in knot theory. The class of the standardly embedded

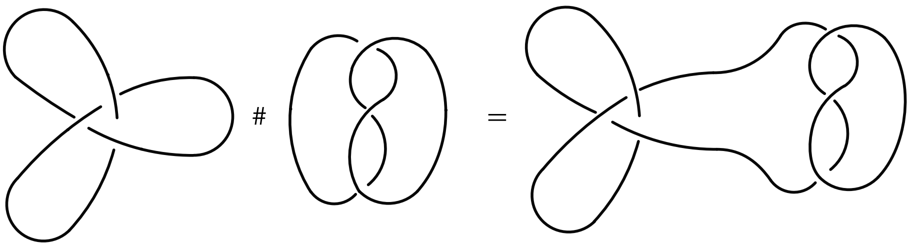

Figure 1: Illustration of the connected sum of two knots.

circle $S^1 \hookrightarrow \mathbb{R}^3 \subset S^3$ is called the trivial knot or unknot. Given two (isotopy classes of) knots k_1 and k_2, one can define the connected sum $k_1 \# k_2$ in various ways, as done for instance in [10]. Intuitively, $k_1 \# k_2$ is obtained by cutting open k_1 and k_2 and closing them back into a single knot. An example is provided in Figure 1. This operation turns out to be associative, commutative and unital with the unknot as unit. This turns $\pi_0 \operatorname{Emb}(S^1, S^3)$ into a commutative monoid. The nontrivial elements k which admit no nontrivial factorization $k = k_1 \# k_2$ are called prime. They are in a sense the most elementary knots. However, there are infinitely many of them and a further decomposition developed in [4] suggests that they form a fairly wide class of knots. The monoid structure on $\pi_0 \operatorname{Emb}(S^1, S^3)$ is completely understood thanks to a theorem of Schubert:

Theorem 1.1 (Schubert [28]) *The monoid $\pi_0 \operatorname{Emb}(S^1, S^3)$ is the free commutative monoid generated by prime knots.*

We now introduce long knots. They are a mild variation of usual knots for which the connected sum is easier to deal with. Let $\iota \colon \mathbb{R} \to \mathbb{R}^3$, with $\iota(t) = (t, 0, 0)$ be the standard embedding of the real line in $\mathbb{R}^3$.

Definition 1.2 A *long knot* is an embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$ that agrees with the standard embedding ι outside of $J = [-1, 1]$ and maps the interior of J in the interior of $J \times D^2 \subset \mathbb{R}^3$. The space of long knots is denoted by $\mathcal{K}$. One can alternatively think of a long knot as a proper embedding $J \hookrightarrow J \times D^2$ whose values and derivatives at ∂J match those of ι.

With these conditions on the embeddings, it is natural to define a binary stacking operation between long knots as follows. Let $L, R \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the maps sending (x, y, z) to $\left(\frac{1}{2}(x-1), y, z\right)$ and $\left(\frac{1}{2}(x+1), y, z\right)$, respectively. Given k_1 and k_2 in $\mathcal{K}$, we

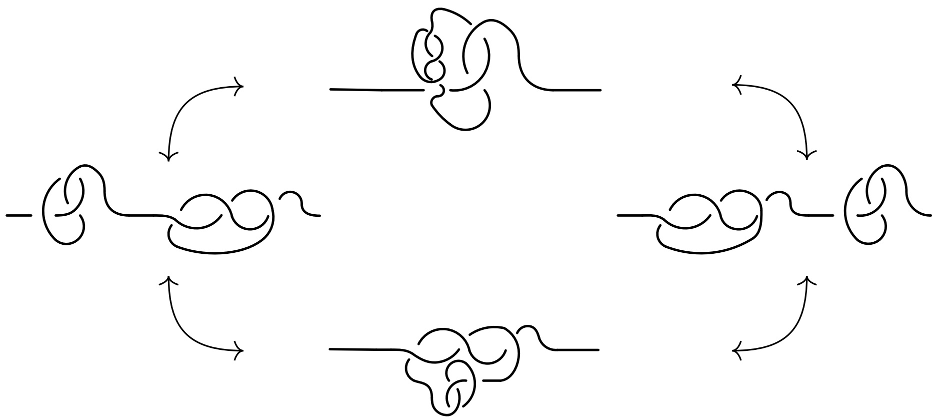

Figure 2: Illustration of the commutativity in the monoid $\pi_0\mathcal{K}$.

define the concatenation of k_1 and k_2 as the long knot that restricts to $t \mapsto L \circ k_1(2t+1)$ on $[-1,0]$ and to $t \mapsto R \circ k_2(2t-1)$ on $[0,1]$. This operation and its commutativity up to homotopy are illustrated in Figure 2. We still denote this operation by # as it is the analogue of the connected sum in the following sense. Each long knot is linear outside of J and can therefore be extended to an embedding $S^1 \hookrightarrow S^3$ by compactifying the domain and codomain. This specifies an inclusion $\mathcal{K} \hookrightarrow \mathrm{Emb}(S^1, S^3)$ which turns out to be a bijection on π_0. It is easy to verify that the concatenation of two knots is sent to their connected sum. The isotopy classes $\pi_0\mathcal{K}$ therefore inherit a monoid structure. When $\mathcal{P}$ denotes the collection of long knots which are prime, Schubert's theorem applies and gives:

Theorem 1.3 *The monoid $\pi_0\mathcal{K}$ is the free commutative monoid on the basis $\pi_0\mathcal{P}$.*

We now define framed long knots. The latter are meant to approximate the long knots defined above while being composable. The idea is to thicken the strand $\mathbb{R}$ into a long tube $\mathbb{R} \times D^2$. This definition is due to Budney and originates in [5]. We define:

Definition 1.4 (Budney [5]) A framed long knot is an embedding $\mathbb{R} \times D^2 \to \mathbb{R} \times D^2$ that restricts to the identity outside of $J \times D^2$. The space of framed long knots is denoted by $\mathrm{EC}(1, D^2)$. When $\mathrm{supp}(f)$ denotes the support of an embedding $f : \mathbb{R} \times D^2 \hookrightarrow \mathbb{R} \times D^2$, ie the closure of $\{(t,x) \in \mathbb{R} \times D^2 \mid f(t,x) \neq (t,x)\}$, the condition for f to lie in $\mathrm{EC}(1, D^2)$ can be reformulated as $\mathrm{supp}(f) \subset J \times D^2$.

Note that this space is still equipped with a stacking operation # defined just as in the case of long knots. It also still descends to an associative, commutative unital pairing on

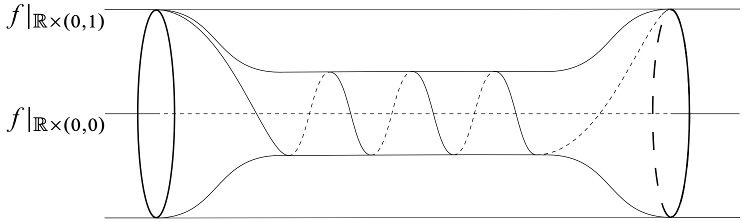

Figure 3: A framed long knot f with framing number 4.

isotopy classes. There is a restriction map $\mathrm{EC}(1, D^2) \to \mathcal{K}$, defined by $f \mapsto f|_{\mathbb{R}\times(0,0)}$, that preserves the concatenation. But, it does not induce a bijection on π_0. To see this, consider the diffeomorphism r of $J \times D^2$ that progressively performs a full turn rotation on the disk factor. One can parametrize $r\colon (t, x) \mapsto (t, e^{i\pi(t+1)}x)$. This diffeomorphism can be isotoped about $\partial J \times D^2$ to be extended into an element of $\mathrm{EC}(1, D^2)$. Now, for any long knot k and f an extension in $\mathrm{EC}(1, D^2)$, each composite $f \circ r^{\circ n}$ also extends k but no two are isotopic. This produces infinitely many components in the fiber over k, which shows that the restriction map does not induce a bijection on π_0. As of here, this twisting phenomenon is an unwanted byproduct of the thickening process. We will get rid of it when defining $\widehat{\mathcal{K}}$ as an unframed subspace of $\mathrm{EC}(1, D^2)$. To do so, we need to quantify the framing of a knot, which we do via the framing number.

Definition 1.5 We define the *framing number* $\omega(f)$ of a framed long knot f of $\mathrm{EC}(1, D^2)$ as the linking number $\mathrm{lk}(f|_{\mathbb{R}\times(0,0)}, f|_{\mathbb{R}\times(0,1)})$. Strictly speaking, the linking number is only defined between disjoint closed curves. We deal with this problem by identifying the curves above with their extension by compactification $S^1 \to S^3$ and isotoping $f|_{\mathbb{R}\times(0,1)}$ about the point at infinity to keep the disjointness.

Intuitively, the linking number counts the number of times a closed curve winds around another. Here, we think of $\omega(f)$ as the number of times a curve on the surface of the knot wraps around the core $f|_{\mathbb{R}\times(0,0)}$. This provides a way to quantify the framing of the elements of $\mathrm{EC}(1, D^2)$ and we have $\omega(r^{\circ n}) = n$. Another example is provided in Figure 3. Since the linking number can be computed by counting crossings in diagrams, it is easy to see that ω is additive with respect to the stacking product. It is also isotopy invariant and therefore descends to a morphism of monoids $\pi_0\mathrm{EC}(1, D^2) \to \mathbb{Z}$.

We are finally able to define $\widehat{\mathcal{K}}$, our preferred model to approximate $\mathcal{K}$. We simply set:

Definition 1.6 (Budney [5]) The space of *fat long knots* is the subspace $\widehat{\mathcal{K}} = \omega^{-1}(0)$.

Fat long knots are stable under # thanks to the additivity of ω. The primes in $\widehat{\mathcal{K}}$ are denoted by $\widehat{\mathcal{P}}$. There still is a restriction map $\widehat{\mathcal{K}} \to \mathcal{K}$ which preserves this structure. $\widehat{\mathcal{K}}$ is a good approximation for $\mathcal{K}$ in the sense that:

Proposition 1.7 (Budney [5]) *The restriction map $\widehat{\mathcal{K}} \to \mathcal{K}$ is a homotopy equivalence.*

In particular, we get an isomorphism on π_0 which enables us to transfer Schubert's theorem to fat long knots:

Corollary 1.8 *The monoid $\pi_0\widehat{\mathcal{K}}$ is the free commutative monoid on the basis $\pi_0\widehat{\mathcal{P}}$.*

1.2 Link spaces

We now adapt these constructions for 2–links. Most of this generalization work has already been carried out by Burke and Koytcheff in [8]. The space of usual 2–links arises as $\mathrm{Emb}(S^1 \amalg S^1, S^3)$. There is no canonical version of a connected sum operation here, as there is no preferred strand in each link for one to merge. As in the case of long knots, the space of string links $\mathcal{L}$ deals with this problem by setting a framework where a stacking operation is naturally defined. Let $\iota_2 : \mathbb{R} \amalg \mathbb{R} \hookrightarrow \mathbb{R}^3$ be the embedding of two copies of the real line in $\mathbb{R}^3$ mapping the first copy as $t \mapsto \left(t, 0, \frac{1}{2}\right)$ and the other one as $t \mapsto \left(t, 0, \frac{-1}{2}\right)$. We refer to ι_2 as the standard embedding for links with two strands. We then define:

Definition 1.9 A 2–*string link* is an embedding $\mathbb{R} \amalg \mathbb{R} \hookrightarrow \mathbb{R}^3$ that agrees with ι_2 outside of $J \amalg J$ and maps the interior of $J \amalg J$ in the interior of $J \times D^2 \subset \mathbb{R}^3$. The space of 2–string links is denoted by $\mathcal{L}$. One can alternatively think of a 2–string link as a proper embedding $J \amalg J \hookrightarrow J \times D^2$ whose values and derivatives at $\partial J \amalg \partial J$ match those of ι_2.

A binary stacking operation # can now be defined on $\mathcal{L}$ as in the case of long knots. It turns $\pi_0\mathcal{L}$ into a monoid with unit ι_2. A 2–string link is said to be prime if it is not invertible but cannot be factored without an invertible element. There is an injection $\mathcal{L} \hookrightarrow \mathrm{Emb}(S^1 \amalg S^1, S^3)$ obtained by closing a truncated link $f|_{J \amalg J}$ with two fixed smooth curves from $\left(-1, \frac{\pm 1}{2}\right)$ to $\left(1, \frac{\pm 1}{2}\right)$ as illustrated in Figure 4. But, this inclusion does not induce a bijection on isotopy classes. Indeed, there are pairs of nonisotopic 2–string links that yield isotopic links once closed as shown in Figure 4. Therefore, studying string links slightly differs from usual link theory.

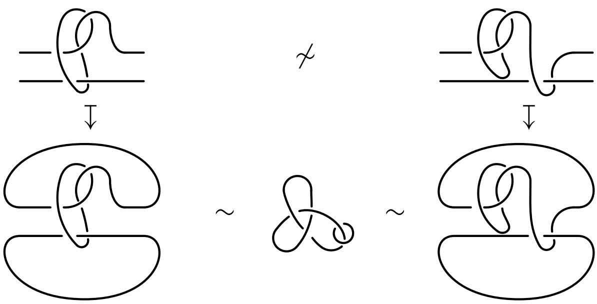

Figure 4: The two top string links are not isotopic (closing the first one up with two vertical lines results in a trivial knot, doing so with the second one yields a trefoil knot). The corresponding links are however isotopic.

Let us spend some time to investigate $\pi_0\mathcal{L}$. We first identify the invertible elements. There is a canonical map from the pure braid group on two strands to $\pi_0\mathcal{L}$ sending a pure braid to its isotopy class as a string link. It is a morphism of monoids that only maps to units in $\pi_0\mathcal{L}$ since braids form a group. This association is easily shown to be injective: the linking number map $\mathrm{lk}\colon \mathcal{L} \to \mathrm{Emb}(S^1 \amalg S^1, S^3) \to \mathbb{Z}$ descends to a left inverse when one identifies the pure braids on two strands with the integers in the natural way. This provides a whole collection of invertible elements and it turns out that every unit in $\pi_0\mathcal{L}$ is of this form. Observe as well that these invertible links commute with every other link: an isotopy exhibiting this relation is suggested by Figure 5. Let now $\mathcal{L}^0$ be the preimage of 0 through the linking number map lk. The injection of the braid group provides a section in the short exact sequence

$$\pi_0\mathcal{L}^0 \to \pi_0\mathcal{L} \to \mathbb{Z}.$$

Thus $\pi_0\mathcal{L}$ splits as $\pi_0\mathcal{L}^0 \times \mathbb{Z}$ and we can focus on the first factor.

The monoid $\pi_0\mathcal{L}^0$ is not commutative but it contains several copies of $\pi_0\mathcal{K}$ in its center. Indeed, consider the injective morphism $\varphi^\uparrow\colon \pi_0\mathcal{K} \to \pi_0\mathcal{L}^0$ mapping a long knot k to the string link whose upper strand is knotted according to k and does not interact

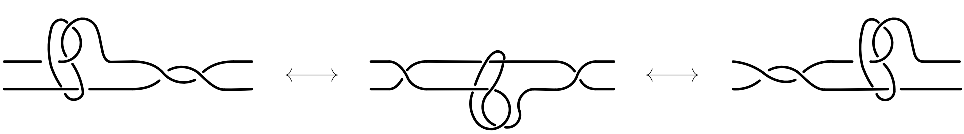

Figure 5: Illustration of the commutation between a braid and an arbitrary string link.

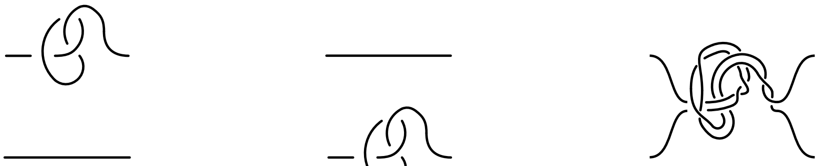
$\varphi^{\uparrow}(f)$ $\varphi^{\downarrow}(f)$ $\varphi^{\updownarrow}(f)$

Figure 6: Illustration of the morphisms φ^s, $s \in \{\uparrow, \downarrow, \updownarrow\}$.

with the unknotted lower strand. An illustration of $\varphi^{\uparrow}$ is given in Figure 6. The image of $\varphi^{\uparrow}$ is a copy of $\pi_0\mathcal{K}$ lying in $\pi_0\mathcal{L}^0$, and one can build a similar morphism $\varphi^{\downarrow}$ by switching the roles of the strands. A third copy of the knot monoid can be found as follows. Consider the morphism $\varphi^{\updownarrow}$ that sends a knot k to the link whose strands are parallel and unlinked but knotted according to k. This $\varphi^{\updownarrow}$ maps to a third copy of $\pi_0\mathcal{K}$ in $\pi_0\mathcal{L}^0$.

The images of any two of these morphisms intersect only in the component of ι_2 so that $\pi_0\mathcal{K}^{\times 3}$ actually lives in $\pi_0\mathcal{L}^0$. Every string link in the image of a φ^s, for $s \in \{\uparrow, \downarrow, \updownarrow\}$, commutes with any other link. The structure theorem for 2–string links proved by Blair, Burke and Koytcheff in [2] actually shows that the center of $\pi_0\mathcal{L}^0$ is generated by the images of the maps φ^s, alongside the fact that the remaining links are freely generated by some prime elements. More precisely, when $\mathcal{Q}$ denotes the prime 2–string links in $\mathcal{L}$ that do not belong to the image of a φ^s and when $\mathcal{Q}^0 = \mathcal{Q} \cap \mathcal{L}^0$, one has the following result:

Theorem 1.10 (Blair, Burke and Koytcheff [2]) *The monoid $\pi_0\mathcal{L}^0$ is isomorphic to the product of $\pi_0\mathcal{K}^{\times 3}$ and the free* (*noncommutative*) *monoid on the basis $\pi_0\mathcal{Q}^0$. Moreover, an isomorphism is induced by the inclusion $\pi_0\mathcal{Q}^0 \hookrightarrow \pi_0\mathcal{L}^0$ and the maps φ^s.*

The string links generated by the images of $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ are called split. Such a string link can also be characterized by the existence of a properly embedded disk separating the two strands in the complement. The string links generated by the image of $\varphi^{\updownarrow}$ and invertible elements are called double cables. They can alternatively be defined as the links whose strands are parallel. Note that Theorem 1.10 above tells us that the center of $\pi_0\mathcal{L}$ precisely consists of the split links, double cables and their products, and that any other string link only commutes with central elements.

We now introduce framed 2–string links. They are the 2–stranded analogue of framed long knots as they arise by thickening the two strands. Let $\iota\colon D^2 \amalg D^2 \hookrightarrow D^2$ be

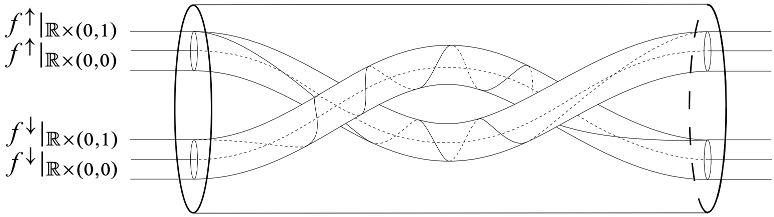

Figure 7: A framed 2–string link f with framing number $(3,-5)$.

the embedding that rescales the disks to make their radii $\frac{1}{8}$ and translates them so that they are centered at $\bigl(0, \frac{\pm 1}{2}\bigr)$. We refer to $\mathrm{id}_{\mathbb{R}} \times \iota$ as the standard embedding $(\mathbb{R} \times D^2) \amalg (\mathbb{R} \times D^2) \hookrightarrow \mathbb{R} \times D^2$.

Definition 1.11 A *framed 2–string link* is an embedding

$$(\mathbb{R} \times D^2) \amalg (\mathbb{R} \times D^2) \hookrightarrow \mathbb{R} \times D^2$$

that restricts to the standard embedding outside of $(J \times D^2) \amalg (J \times D^2)$ and maps the interior of $(J \times D^2) \amalg (J \times D^2)$ in the interior of $J \times D^2$. The space of framed 2–string links is denoted by $\mathrm{EC}_\iota(1, D^2)$. When $\mathrm{supp}_\iota(f)$ denotes the closure of $\{(t,x) \in (\mathbb{R}\times D^2) \amalg (\mathbb{R}\times D^2) \mid f(t,x) \neq (t, \iota(x))\}$ for any embedding f, the condition for f to lie in $\mathrm{EC}_\iota(1, D^2)$ can be reformulated as $\mathrm{supp}_\iota(f) \subset (J \times D^2) \amalg (J \times D^2)$ and $f\bigl(\mathrm{int}((J \times D^2) \amalg (J \times D^2))\bigr) \subset J \times D^2$.

Framed 2–string links again dispose of a binary concatenation operation # and a restriction map $\mathrm{EC}_\iota(1, D^2) \to \mathcal{L}$ preserving it. This endows the isotopy classes $\pi_0\mathrm{EC}_\iota(1, D^2)$ with a monoid structure with the standard embedding as unit. An obstruction for the restriction map to be a homotopy equivalence is again the framing of each strand. We define the framing number ω of an element of $\mathrm{EC}_\iota(1, D^2)$ as in Definition 1.5, except that it now consists of a pair of integers, one for each strand.

Definition 1.12 Let f be a framed 2–string link with strands $f^{\uparrow}$ and $f^{\downarrow}$. We define the upper framing number $\omega_{\uparrow}(f)$ as the linking number $\mathrm{lk}(f^{\uparrow}|_{\mathbb{R}\times(0,0)}, f^{\uparrow}|_{\mathbb{R}\times(0,1)})$. The lower framing number $\omega_{\downarrow}(f)$ is defined the same way and the whole framing number $\omega(f)$ is the pair of integers $(\omega_{\uparrow}(f), \omega_{\downarrow}(f))$.

The framing number is isotopy invariant and additive with respect to the concatenation for the same reasons as before. This makes it descend to a monoid morphism $\pi_0\mathrm{EC}_\iota(1, D^2) \to \mathbb{Z}^{\times 2}$. We are now able to get rid of this twisting phenomenon by defining:

Definition 1.13 The space of *fat 2–string links* $\widehat{\mathcal{L}}$ is the subspace

$$\omega^{-1}(0,0) \subset \mathrm{EC}_\iota(1, D^2).$$

Fat 2–string links are stable under concatenation. We denote by $\widehat{\mathcal{L}}^0$, $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{Q}}^0$ the elements of $\widehat{\mathcal{L}}$ whose restrictions to $(\mathbb{R} \times (0,0)) \amalg (\mathbb{R} \times (0,0))$ lie in $\mathcal{L}^0$, $\mathcal{Q}$ and $\mathcal{Q}^0$, respectively. $\widehat{\mathcal{L}}$ is a good approximation for $\mathcal{L}$ in the sense that:

Proposition 1.14 (Burke and Koytcheff [8]) *The restriction map $\widehat{\mathcal{L}} \to \mathcal{L}$ is a homotopy equivalence.*

In particular, the monoid $\pi_0\widehat{\mathcal{L}}$ is completely understood and has the same structure as $\pi_0\mathcal{L}$, made explicit in Theorem 1.10.

The remainder of this paper is dedicated to the elaboration of an algebraic structure on the space level of long knots and string links. The stacking products are examples of binary operations on the space level that relate to the monoids $\pi_0\mathcal{K}$ and $\pi_0\mathcal{L}$. We aim to find a refinement of these operations into a more subtle structure, in order to generalize Theorems 1.8 and 1.10 to the space level. These structures will be formalized as operadic actions and the isomorphisms described in the theorems above will generalize as equivariant homotopy equivalences. Budney's Theorem 11 in [5] precisely answers this problem in the case of knots, and Burke and Koytcheff's Theorem 6.8 in [8] partially deals with the case of 2–string links. In the following sections, we recall the work presented in these two papers and treat the case of 2–string links in a wider manner.

2 Operads

The purpose of this section is to recall the definition of (colored) operads, set up some notation and introduce the two operads of prime interest in this paper: the little cubes operad $\mathcal{C}_n$ and a 4–colored version of the Swiss-cheese operad that we call $\mathcal{SCL}$ for "Swiss-cheese for links". We also discuss the different types of algebras these objects encode and review free models for these structures.

2.1 Colored operads and their algebras

We start by reviewing the notion of colored operad. Let S be a set of colors. We denote by $S^\star$ the collection of tuples of elements of S. In other words, $S^\star$ is the union

$\coprod_{n\geq 0} S^{\times n}$. Each $S^{\times n}$ is naturally a right Σ_n–space. The length of a tuple $\underline{t}\in S^\star$ is denoted by $|\underline{t}|$ and, for every $s\in S$, $|\underline{t}|_s$ is the number of times s appears in $\underline{t}$. Given two tuples of colors $\underline{t}$, $\underline{u}$ and an integer $i\leq|\underline{t}|$, we denote by $\underline{t}\circ_i\underline{u}$ the $(|\underline{t}|+|\underline{u}|-1)$–tuple

$$\underline{t}\circ_i\underline{u}=(t_1,\dots,t_{i-1},u_1,\dots,u_{|\underline{u}|},t_{i+1},\dots,t_{|\underline{t}|}).$$

Note that $|\underline{t}\circ_i\underline{u}|_s=|\underline{t}|_s+|\underline{u}|_s$ whenever $s\neq t_i$ and $|\underline{t}|_s+|\underline{u}|_s-1$ for $s=t_i$. We write s^n for the tuple $(s,\dots,s)\in S^{\times n}$ for every $s\in S$ and $n\geq 0$. We are now in the right framework to define:

Definition 2.1 A *colored operad* $\mathcal{O}$ over the colors S consists of the following combined data:

(i) for every $\underline{t}\in S^\star$ and $s\in S$, a space $\mathcal{O}(\underline{t};s)$;

(ii) for every $\underline{t}$, $\underline{u}\in S^\star$ and $i\leq|\underline{t}|$, operadic compositions

$$\circ_i\colon\mathcal{O}(\underline{t};s)\times\mathcal{O}(\underline{u};t_i)\to\mathcal{O}(\underline{t}\circ_i\underline{u};s)$$

satisfying the usual associativity, symmetric and unital conditions — the latter are thoroughly detailed in [35];

(iii) for every permutation $\sigma\in\Sigma_n$, $\underline{t}$ an n–tuple of colors and s a color, a map

$$\sigma^*\colon\mathcal{O}(\underline{t};s)\to\mathcal{O}(\underline{t}\sigma;s)$$

such that $\tau^*\circ\sigma^*=(\sigma\circ\tau)^*$ and $\mathrm{id}^*=\mathrm{id}$;

(iv) for every color s, a unit $1_s\in\mathcal{O}(s;s)$.

The elements of $\mathcal{O}(\underline{t};s)$ are called operations with inputs $\underline{t}$ and output s. The units 1_s are sometimes referred to as identities. We also often write $a\sigma$ for $\sigma^*(a)$ and $a\circ_i b$ for $\circ_i(a,b)$. A morphism of colored operads $f\colon\mathcal{O}\to\mathcal{P}$ is a collection of maps $f_{(\underline{t};s)}\colon\mathcal{O}(\underline{t};s)\to\mathcal{P}(\underline{t};s)$ preserving operadic compositions, symmetric actions and units.

Example 2.2 The prototypical examples of colored operads are the endomorphism operads. Let $X=(X_s)_{s\in S}$ be an S–tuple of spaces. Given a vector of colors $\underline{t}$, we write $X^{\times\underline{t}}$ for the product $\prod_i X_{t_i}$. Now, for any output color s, we set $\mathcal{E}_X(\underline{t};s)$ to be the mapping space $\mathbf{Top}(X^{\times\underline{t}},X_s)$. An element $\sigma\in\Sigma_{|\underline{t}|}$ can act on an $f\in\mathcal{E}_X(\underline{t};s)$ by permuting its entries, resulting in an element of $\mathcal{E}_X(\underline{t}\sigma;s)$. Also, when $\underline{u}$ is another vector of colors and g lies in $\mathcal{E}_X(\underline{u};t_i)$, one can inject g into the i^{th} entry of f to get the composite $f\circ_i g\in\mathcal{E}_X(\underline{t}\circ_i\underline{u};s)$. This specifies the data of a colored operad $\mathcal{E}_X$ on the colors S.

Definition 2.3 A colored operad $\mathcal{O}$ over a single color s is called an *operad*. In this case, we write $\mathcal{O}(n)$ for $\mathcal{O}(s^n; s)$ and call it the space in arity n. The unit 1_s is simply denoted by 1. The operadic compositions are now maps $\mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ and the symmetric structure turns each $\mathcal{O}(n)$ into a right Σ_n–space.

Operads are useful to specify categories of algebraic objects. This is formalized via operadic actions which we define now.

Definition 2.4 Let $\mathcal{O}$ be a colored operad over the set of colors S and $X = (X_s)_{s \in S}$ an S–tuple of spaces. We say that X is an *$\mathcal{O}$–algebra* if it comes with a morphism of operads $\kappa \colon \mathcal{O} \to \mathcal{E}_X$. In other words, an *$\mathcal{O}$–algebra structure* on X is a collection of maps

$$\kappa_{(\underline{t};s)} \colon \mathcal{O}(\underline{t}; s) \to \mathbf{Top}(X^{\times \underline{t}}, X_s)$$

preserving the operadic compositions, identities and symmetric actions described in [35]. They may also be thought of as maps $\mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}} \to X_s$, and we shall use each framework when it is more convenient. A *morphism of $\mathcal{O}$–algebras* $f \colon X \to Y$ is a collection of maps $f_s \colon X_s \to Y_s$ preserving the operadic actions.

Example 2.5 Consider the operad obtained with the one point space in every arity and trivial symmetric actions and operadic compositions. An action of this operad on a space X is the data of a single map $X^{\times n} \to X$ for every nonnegative n. One readily checks that the required relations listed in [35] correspond to the associativity and commutativity of $X^{\times 2} \to X$, as well as the fact that the element specified by the zeroth map $X^{\times 0} \to X$ acts as a unit. In other words, an action of this operad on X is a commutative topological monoid structure on X. This justifies the terminology $\mathcal{C}om$ for this operad.

Example 2.6 Consider the operad whose space in arity n is the discrete symmetric group Σ_n as an evident right Σ_n–space with the following operadic composition. For every $\sigma \in \Sigma_n$, $\tau \in \Sigma_m$ and $i \leq n$, $\sigma \circ_i \tau$ permutes $\{1, \dots, n+m-1\}$ according to σ while treating $\{i, \dots, i+m-1\}$ as a single block, then shuffles the latter internally according to τ. An action of this operad on a space X is a data of a map $X^{\times n} \to X$ for every ordering of $\{1, \dots, n\}$. One readily checks that the required relations correspond to the associativity of $X^{\times 2} \to X$ and the fact that the element $X^{\times 0} \to X$ acts as a unit. In other words, the algebras over this operad are the not necessarily commutative topological monoids. This operad is called the associative operad and is denoted by $\mathcal{A}s$.

2.2 Free algebras

Before we introduce the two operads that will act on the spaces of knots and links, we take some time to discuss free algebras. When $\mathcal{O}$ is a colored operad over the set of colors S, the $\mathcal{O}$–algebras and their morphisms form a category denoted by $\mathcal{O}$–**Alg**. There is a forgetful functor $\mathcal{U}\colon \mathcal{O}\text{–}\mathbf{Alg} \to \mathbf{Top}^{\times S}$ mapping an $\mathcal{O}$–algebra to its underlying S–tuple of spaces. By free $\mathcal{O}$–algebra, we understand the left adjoint $\mathcal{O}[\_]$ to the forgetful functor $\mathcal{U}$. In other words, the free $\mathcal{O}$–algebra generated by $X = (X_s)_{s\in S}$ should lead to a bijection

$$\mathcal{O}\text{–}\mathbf{Alg}(\mathcal{O}[X], Y) \cong \mathbf{Top}^{\times S}(X, \mathcal{U}(Y))$$

for every $\mathcal{O}$–algebra Y. A well-known model for $\mathcal{O}[X]$ is obtained as follows. For every vector $\boldsymbol{x} = (x_1, \dots, x_n) \in X^{\times \underline{t}}$ and permutation σ, we will write $\sigma\boldsymbol{x}$ for $(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) \in X^{\times \underline{t}\sigma^{-1}}$. We set

$$\mathcal{O}[X]_s = \coprod_{\underline{t}} \mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}} /{\sim},$$

where $\sim$ identifies each $(a, \boldsymbol{x})$ with $(a\sigma, \sigma^{-1}\boldsymbol{x})$ for every permutation σ. The action of $\mathcal{O}$ is obtained by composing in the $\mathcal{O}(\underline{t}; s)$ factor. The desired bijection above is then a formal verification. When $\mathcal{O}$ is an uncolored operad, $\sim$ corresponds to the Σ_n–orbits and we can simplify

$$\mathcal{O}[X] = \coprod_n \mathcal{O}(n) \times_{\Sigma_n} X^{\times n}.$$

We conclude this subsection with a quick observation about free algebras. When $\mathcal{O}$ is a colored operad with set of colors S, the components $\pi_0\mathcal{O}$ naturally inherit an operad structure. Similarly, if $X = (X_s)_{s\in S}$ is an $\mathcal{O}$–algebra, then the components $\pi_0 X = (\pi_0 X_s)_{s\in S}$ inherit a $\pi_0\mathcal{O}$–algebra structure. Finally, the following result will come in handy when proving that some actions yield free algebras in Section 4.

Proposition 2.7 *Let X be an S–tuple of spaces. Then $\pi_0(\mathcal{O}[X])$ is a model for the free algebra $\pi_0\mathcal{O}[\pi_0 X]$.*

Proof We have the two models

$$\pi_0(\mathcal{O}[X]_s) = \pi_0\biggl(\coprod_{\underline{t}} \mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}} /{\sim}\biggr), \quad \pi_0\mathcal{O}[\pi_0 X]_s = \coprod_{\underline{t}} \pi_0\mathcal{O}(\underline{t}; s) \times \pi_0 X^{\times \underline{t}} /{\sim}.$$

Since π_0 commutes with products and coproducts, the right-hand side is equal to the quotient of $\pi_0\bigl(\coprod_{\underline{t}} \mathcal{O}(\underline{t}; s) \times X^{\times \underline{t}}\bigr)$ by the relations $[a, \boldsymbol{x}] \sim [a\sigma, \sigma^{-1}\boldsymbol{x}]$. The left-hand side also matches this description so both spaces are the same. It is a tautological verification to see that the action of $\pi_0\mathcal{O}$ is the same under these identifications. $\square$

2.3 The little cubes operad

We introduce the little cubes operad $\mathcal{C}_n$ and quickly discuss the algebras it encodes. It is an uncolored operad originated in [24] in order to understand iterated loop spaces. Our treatment is very similar to the one of Budney in [5].

Definition 2.8 The real functions of the form $x \mapsto ax+b$ for some positive a are said to be affine increasing. A *little n–cube* is an application $L\colon J^n \to J^n$ of the form $L = l^1 \times \cdots \times l^n$ for some affine increasing functions l^i. The space of k *overlapping little n–cubes* $\mathcal{C}_n^\infty(k)$ is the set of configurations of k little n–cubes $J^n \amalg \cdots \amalg J^n \to J^n$. We set $\mathcal{C}_n^\infty(0)$ to be the one point space. Given an element $L \in \mathcal{C}_n^\infty(k)$, we write its decomposition in little n–cubes $L = \bigoplus_i L^i$. Each L^i decomposes uniquely in affine increasing functions $l^{i,1} \times \cdots \times l^{i,n}$, so writing $l^{i,j} : x \mapsto a_{i,j}x + b_{i,j}$ gives rise to an injection $\mathcal{C}_n^\infty(k) \hookrightarrow \mathbb{R}^{2nk} : L \mapsto (a_{1,1}, b_{1,1}, \cdots, a_{k,n}, b_{k,n})$. This is used to transfer a topology on $\mathcal{C}_n^\infty(k)$. Considering the C^∞–topology actually has the same outcome.

An element of $\mathcal{C}_n^\infty(k)$ is represented by a drawing of the images of its little n–cubes. We now equip the family of spaces $\mathcal{C}_n^\infty(k)$ with an operadic structure. Thereafter, we define the little n–cubes operad as a suboperad by adding a disjointness conditions on the cubes.

Definition 2.9 The *overlapping little n–cubes operad* $\mathcal{C}_n^\infty$ is the operad specified as follows.

(i) The space in arity k is the set of configurations of little n–cubes $\mathcal{C}_n^\infty(k)$.

(ii) For every positive integer k, l and $i \le k$, the operadic composition is given by

$$\circ_i : \mathcal{C}_n^\infty(k) \times \mathcal{C}_n^\infty(l) \to \mathcal{C}_n^\infty(l+k-1),$$
$$(L, P) \mapsto L^1 \oplus \cdots \oplus L^{i-1} \oplus L^i \circ P^1 \oplus \cdots \oplus L^i \circ P^l \oplus L^{i+1} \oplus \cdots \oplus L^k.$$

Composing $L \in \mathcal{C}_n^\infty(k)$ with the one point in $\mathcal{C}_n^\infty(0)$ discards the i^{th} cube of L.

(iii) The action of $\sigma \in \Sigma_l$ on $L \in \mathcal{C}_n^\infty(l)$ permutes the little n–cubes of L, ie

$$L\sigma = \bigoplus_i L^{\sigma(i)}.$$

(iv) The unit is the identity little n–cube $\mathrm{id}_{J^n} \in \mathcal{C}_n^\infty(1)$.

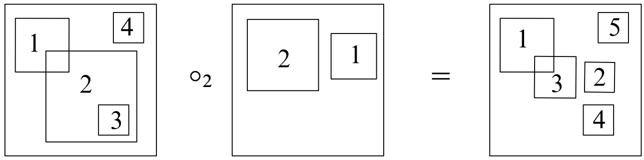

Figure 8: Illustration of the operadic composition $\circ_2 : \mathcal{C}_2^\infty(4) \times \mathcal{C}_2^\infty(2) \to \mathcal{C}_2^\infty(5)$.

Definition 2.10 Two little n–cubes are said to be *almost disjoint* if the interiors of their images are disjoint. The space of k little n–cubes $\mathcal{C}_n(k)$ is the subspace of $\mathcal{C}_n^\infty(k)$ consisting of pairwise almost disjoint little n–cubes. We still set $\mathcal{C}_n(0)$ to be the one point space. Operadic compositions in the overlapping little cubes operad preserve the property of being almost disjoint so the subspaces $\mathcal{C}_n(k)$ forms a suboperad $\mathcal{C}_n$ called the little n–cubes operad.

Remark 2.11 Budney defines in [6] an operad $\mathcal{C}'_n$ which he calls "the operad of overlapping little n–cubes". The resemblance with our terminology for $\mathcal{C}_n^\infty$ is only coincidental. The two objects are not equivalent (Budney's $\mathcal{C}'_n$ is equivalent to $\mathcal{C}_{n+1}$, while $\mathcal{C}_n^\infty$ has contractible underlying spaces). The operad $\mathcal{C}'_n$ does not appear in this article, so there should be no confusion.

We conclude this subsection by taking a look at the operad $\pi_0\mathcal{C}_n$. Given k little n–cubes $L \in \mathcal{C}_n(k)$, restricting L to the center of each cube leads to an injective map $\{1, \dots, k\} \hookrightarrow J^n$, ie an element of the configuration space $\mathrm{conf}_k(J^n)$. Conversely, given k points $\underline{x}$ in $\mathrm{conf}_k(J^n)$, one gets an element of $\mathcal{C}_n(k)$ by considering the identical cubes centered at $\underline{x}$ whose size is the maximal size that keeps them almost disjoint. Intuitive straight line homotopies show that these two maps are homotopy inverses, so that the homotopy type of $\mathcal{C}_n(k)$ is the one of $\mathrm{conf}_k(J^n)$. In particular, when $n > 1$, each $\mathcal{C}_n(k)$ is path connected, so $\pi_0\mathcal{C}_n = \mathcal{C}om$ from Example 2.5. The free algebras over this operad are the free commutative monoids. In dimension 1, the isotopy classes of $\mathcal{C}_1(k)$ are indexed by the orderings of $\{1, \dots, n\}$, so $\pi_0\mathcal{C}_1 = \mathcal{A}s$ from Example 2.6. The free $\mathcal{A}s$–algebras are the free monoids.

2.4 The operad $\mathcal{SCL}$

We go through the construction of the Swiss-cheese operad for links $\mathcal{SCL}$. It is a 4–colored operad that is also defined in terms of little cubes. This terminology is motivated by the fact that $\mathcal{SCL}$ restricts to the 2–colored Swiss-cheese operad on several pairs of colors. We then conclude by investigating the operad of components $\pi_0\mathcal{SCL}$.

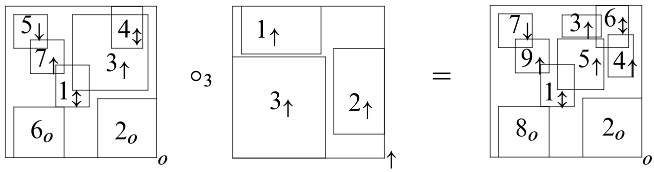

Figure 9: Illustration of the operadic composition in $\mathcal{SCL}$, more precisely of $\circ_3 : \mathcal{SCL}(\updownarrow, o, \uparrow, \updownarrow, \downarrow, o, \uparrow; o) \times \mathcal{SCL}(\uparrow, \uparrow, \uparrow; \uparrow) \to \mathcal{SCL}(\updownarrow, o, \uparrow, \uparrow, \uparrow, \updownarrow, \downarrow, o, \uparrow; o)$.

Definition 2.12 A little n–cube $L = l^1 \times \cdots \times l^n$ is said to *meet the lower face of the unit cube* if $l^n(-1) = -1$. Visually, this happens when the image of L intersects $J^{n-1} \times -1 \subset J^n$. The configurations of k almost disjoint little n–cubes meeting the lower face of J^n is denoted by $\mathcal{C}_n^\circ(k)$ and these spaces form an operad $\mathcal{C}_n^\circ$ just as in the case of $\mathcal{C}_n$.

Consider the set of four colors $S = \{o, \uparrow, \downarrow, \updownarrow\}$. The notation o is meant to remind one of the "open" color in the Swiss-cheese operad as it will play a similar role. The other symbols call up to the upper and lower strands of a string link.

Definition 2.13 The Swiss-cheese operad for links $\mathcal{SCL}$ is specified as follows.

(i) For $s \in \{\uparrow, \downarrow, \updownarrow\}$, the only inputs $\underline{t}$ that do not lead to an empty $\mathcal{SCL}(\underline{t}; s)$ are the monochromatic ones such that $\underline{t} = s^n$. In that case, we set $\mathcal{SCL}(s^n; s) = \mathcal{C}_2(n)$. When $s = o$, we set $\mathcal{SCL}(\underline{t}; o)$ to be those $L \in \mathcal{C}_2^\infty(|\underline{t}|)$ such that each L^i with $t_i = o$ meets the lower face of J^2 while being almost disjoint from any other cube, the L^i with $t_i = \uparrow$ are almost disjoint from each other and the same holds for $t_i = \downarrow$ and $\updownarrow$.

(ii) The operadic compositions, symmetric actions and units are inherited from $\mathcal{C}_2^\infty$.

An element of $\mathcal{SCL}(\underline{t}; s)$ is represented by a drawing of the images of its little n–cubes. We decorate the numbering of each little cube with its associated color to distinguish the cubes that simply happen to meet the lower face of J^n from those that have to. The output color also appears as an index of the whole drawing. Figure 9 gives an example. It is immediate from its definition that $\mathcal{SCL}$ restricts to the (cubic) Swiss-cheese operad on the pairs of colors $\{o, \uparrow\}$, $\{o, \downarrow\}$ and $\{o, \updownarrow\}$. It also clearly restricts to the little cubes operad $\mathcal{C}_2$ on $\uparrow$, $\downarrow$ and $\updownarrow$. Originally, the 2–dimensional Swiss-cheese operad is a 2–colored operad on the set of colors $\{o, c\}$. In some sense, the color o encodes a homotopy associative algebra and the color c describes part of its center. In the case of 2–string links, the center of $\pi_0\mathcal{L}^0$ decomposes as $\pi_0\mathcal{K}^{\times 3}$. To encode this extra information, we split the color c into three independent colors $\{\uparrow, \downarrow, \updownarrow\}$.

We now investigate $\pi_0\mathcal{SCL}$ and its algebras. Observe that for every k and $n > 1$, the projection map $\mathcal{C}_n^{\circ}(k) \to \mathcal{C}_{n-1}(k)$ is a homotopy equivalence. A homotopy inverse is obtained by inflating $(n-1)$–cubes into n–cubes of some fixed height. This reasoning can readily be adapted to show that $\mathcal{SCL}(\underline{t}; o)$ is homotopy equivalent to the product

$$\mathcal{C}_1(|\underline{t}|_o) \times \mathcal{C}_2(|\underline{t}|_\uparrow) \times \mathcal{C}_2(|\underline{t}|_\downarrow) \times \mathcal{C}_2(|\underline{t}|_\updownarrow).$$

In particular, $\pi_0\mathcal{SCL}(\underline{t}; s)$ is either

- empty if $s \neq o$ and $\underline{t} \neq s^n$,
- a single point if $s \neq o$ and $\underline{t} = s^n$,
- the discrete space $\Sigma_{|\underline{t}|_o}$ when $s = o$.

Therefore, an action of $\pi_0\mathcal{SCL}$ on $X = (X_o, X_\uparrow, X_\downarrow, X_\updownarrow)$ is the data of a monoid structure on each space, such that the X_s are commutative for $s \neq o$ and act on X_o with compatible actions. With this description, it is easy to see with the universal property of free objects that the free such quadruplet on the basis (A, B, C, D) is given by

$$\pi_0\mathcal{SCL}[A, B, C, D] = \big(\mathcal{A}s[A] \times \mathcal{C}om[B] \times \mathcal{C}om[C] \times \mathcal{C}om[D], \mathcal{C}om[B], \mathcal{C}om[C], \mathcal{C}om[D]\big),$$

where the last three monoids act on the first one on their respective factor.

3 Operadic actions

We gather here the objects introduced in the previous sections and endow the spaces $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{L}}$ with operadic actions. In the first subsection, the fat long knots $\widehat{\mathcal{K}}$ are equipped with a $\mathcal{C}_2$–algebra structure originally exhibited by Budney in [5]. In the case of fat 2–string links, there is a $\mathcal{C}_1$–algebra structure that follows from Burke and Koytcheff's work in [8]. We recall its construction in the second subsection and extend it to an action of $\mathcal{SCL}$ in a third one.

3.1 Budney's action on fat long knots

We start with the little 2–cubes action on $\widehat{\mathcal{K}}$ originated in [5]. Following Budney's work, we first define an action of the affine increasing automorphisms of $\mathbb{R}$ on the self-embeddings of $\mathbb{R} \times D^2$, then proceed to extend it to the 2–dimensional little cubes operad $\mathcal{C}_2$. We denote by CAut_1 the group of real affine increasing functions. A

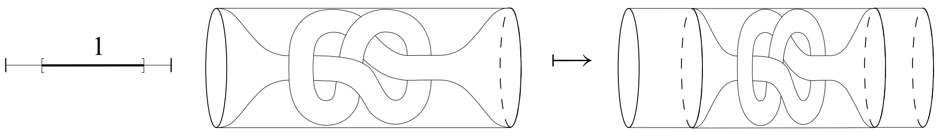

Figure 10: Illustration of the action of $\mathcal{C}_1(1)$ on $\mathrm{EC}(1, D^2)$.

little 1–cube is identified with its natural extension to the real line so that $\mathcal{C}_1(1)$ lives in CAut_1. We topologize CAut_1 as we topologized $\mathcal{C}_1(1)$, which coincides with the C^∞–topology and turns it into a topological group.

Proposition 3.1 (Budney [5]) *The topological group* CAut_1 *acts on the space of embeddings* $\mathrm{Emb}(\mathbb{R} \times D^2, \mathbb{R} \times D^2)$ *via*

$$\begin{aligned}\mathrm{CAut}_1 \times \mathrm{Emb}(\mathbb{R} \times D^2, \mathbb{R} \times D^2) &\to \mathrm{Emb}(\mathbb{R} \times D^2, \mathbb{R} \times D^2),\\ (L, f) &\mapsto (L \times \mathrm{id}_{D^2}) \circ f \circ (L^{-1} \times \mathrm{id}_{D^2}).\end{aligned}$$

Moreover, this restricts to an action of $\mathcal{C}_1(1)$ *on* $\mathrm{EC}(1, D^2)$*, which we write as* $(L, f) \mapsto Lf$*.*

Proof That this map defines a valid action of a topological group is immediate. To prove the statement about the restriction, we just need to check that Lf restricts to the identity outside of $J \times D^2$, provided $L \in \mathcal{C}_1(1)$ and $f \in \mathrm{EC}(1, D^2)$. For any $t \notin J$, $L^{-1}(t)$ does not lie in J because $L(J) \subset J$. So, for every $x \in D^2$, $(L^{-1} \times \mathrm{id}_{D^2})(t, x)$ is outside of $J \times D^2$, where f restricts to the identity. Thus $Lf(t, x) = (t, x)$, which proves the second part of the proposition. □

Definition 3.2 (Budney [5]) We define two operations and a partial order on little 2–cubes.

(i) Given a little 2–cube $L = l^1 \times l^2$, we write L_π for the little 1–cube l^1. When $L \in \mathcal{C}_2(k)$, L_π denotes $\bigoplus_i L_\pi^i$. These little 1–cubes may overlap so L_π lies in $\mathcal{C}_1^\infty(k)$ but not necessarily in $\mathcal{C}_1(k)$.

(ii) For every $L = l^1 \times l^2 \in \mathcal{C}_2(1)$, let L_t denote the number $l^2(-1)$. Again, if $L \in \mathcal{C}_2(k)$, then L_t is the k–tuple of reals $(L_t^1, \dots, L_t^k) \in J^k$.

(iii) We define a partial order on the little cubes L^i of an element $L \in \mathcal{C}_2(k)$. This binary relation is the order generated by setting $L^i < L^j$ if and only if $L_t^i < L_t^j$ and the interiors of L_π^i and L_π^j intersect. Then, a permutation $\sigma \in \Sigma_k$ is said to *order* L if the mapping $i \mapsto L^{\sigma(i)}$ is nondecreasing.

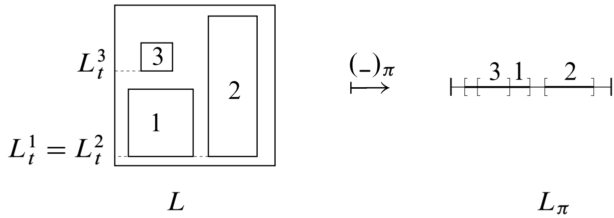

Figure 11: Illustration of the operations $(\_)_\pi$ and $(\_)_t$. The permutations id, (12) and (23) order L.

We are now ready to define an action of $\mathcal{C}_2$ on $\mathrm{EC}(1, D^2)$, ie a map of operads $\kappa\colon \mathcal{C}_2 \to \mathcal{E}_{\mathrm{EC}(1,D^2)}$. For every element $L \in \mathcal{C}_2(k)$ and permutation $\sigma \in \Sigma_k$ that orders L, we set

$$\begin{aligned}\kappa_k(L)\colon \mathrm{EC}(1, D^2)^{\times k} &\to \mathrm{EC}(1, D^2),\\ \boldsymbol{f} = (f_i)_i &\mapsto (L_\pi^{\sigma(1)} f_{\sigma(1)}) \circ \cdots \circ (L_\pi^{\sigma(k)} f_{\sigma(k)}).\end{aligned}$$

One can be assured that $\kappa_k(L)$ does not depend on σ as follows. Two choices for σ differ by a sequence of transpositions (ab) such that L^a and L^b are incomparable, ie such that L_π^a and L_π^b are almost disjoint. Then, $\mathrm{supp}(L_\pi^a f_a)$ and $\mathrm{supp}(L_\pi^b f_b)$ are almost disjoint as well so both orders of composition yield the same outcome. For the continuity, consider for every $\tau \in \Sigma_k$ the map

$$\begin{aligned}\kappa_k^\tau\colon \mathcal{C}_2(j) \times \mathrm{EC}(1, D^2)^{\times k} &\to \mathrm{EC}(1, D^2),\\ (L, \boldsymbol{f}) &\mapsto (L_\pi^{\tau(1)} f_{\tau(1)}) \circ \cdots \circ (L_\pi^{\tau(k)} f_{\tau(k)}).\end{aligned}$$

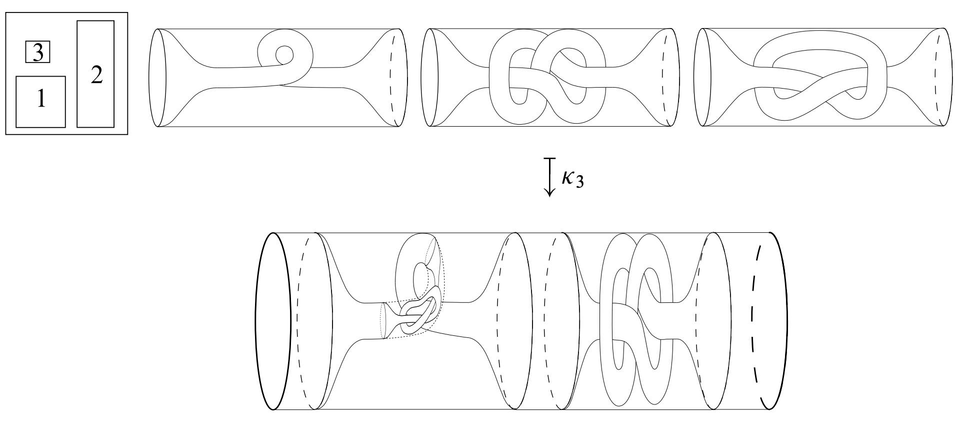

Figure 12: Illustration of Budney's action on fat long knots.

Each κ_k^τ is continuous and coincides with κ_k on

$$F_\tau = \{L \in \mathcal{C}_2(k) \mid \tau \text{ orders } L\} \times \mathrm{EC}(1, D^2)^{\times k}.$$

The sets F_τ are closed and cover $\mathcal{C}_2(k) \times \mathrm{EC}(1, D^2)^{\times k}$ so κ_k is continuous. In arity 0, we set κ_0 to be the map sending the single point in $\mathcal{C}_2(0) \times \mathrm{EC}(1, D^2)^{\times 0}$ to $\mathrm{id}_{\mathbb{R}\times D^2}$.

Theorem 3.3 (Budney [5]) *The operations κ turn $\mathrm{EC}(1, D^2)$ into a $\mathcal{C}_2$–algebra.*

Even though a proof of this result is readily available in [5], we provide one here as the methods and ideas at stake will be reused before the end of this section.

Proof The operation κ_1 clearly maps the basepoint $\mathrm{id}_{J^2} \in \mathcal{C}_2(1)$ to the identity. We need to check the compatibility of κ with the symmetric group actions and the operadic compositions. We start with the symmetric structure. Recall that a permutation τ acting on the right of $L \in \mathcal{C}_2(k)$ yields $\bigoplus_i L^{\tau(i)}$. It also acts on the left of

$$\boldsymbol{f} = (f_i)_i \in \mathrm{EC}(1, D^2)^{\times k}$$

to give $\tau\boldsymbol{f} = (f_{\tau^{-1}(i)})_i$. Thus, if σ is a permutation that orders $L\tau$, then $\tau\circ\sigma$ orders L. This proves the needed equality

$$\kappa_k(L\tau, \boldsymbol{f}) = (L_\pi^{\tau\circ\sigma(1)} f_{\sigma(1)}) \circ \cdots \circ (L_\pi^{\tau\circ\sigma(k)} f_{\sigma(k)}) = \kappa_k(L, \tau\boldsymbol{f}).$$

We are left to prove that κ preserves operadic compositions. Given little 2–cubes $L \in \mathcal{C}_2(k)$, $P \in \mathcal{C}_2(l)$ and an integer $i \le k$, we need to show that

$$\kappa_{k+l-1}(L \circ_i P) = \kappa_k(L) \circ_i \kappa_l(P).$$

Let σ and τ be permutations that respectively order L and P. Unravelling the definition of κ shows that the desired equality boils down to checking that $\sigma \circ_i \tau$ orders $L \circ_i P$. Recall the definitions of $\sigma \circ_i \tau$ and $L \circ_i P$:

- $\sigma \circ_i \tau$ shuffles the interval $\{i, \dots, i+l-1\}$ according to τ, then permutes $\{1, \dots, k+l-1\}$ according to σ while treating the shuffled interval as a single block.
- $L \circ_i P$ is obtained from L by replacing L^i with $\bigoplus_r L^i \circ P^r$.

If L^i and L^j are incomparable, $L^i \circ P^r$ and L^j also are, so the result follows. $\square$

This recently developed structure on the space of framed long knots generalizes the stacking operation in the following sense: acting with two side-by-side rectangles of width 1 on two knots results in their concatenation. In particular, the $\mathcal{C}om$–algebra structure on $\pi_0\mathrm{EC}(1, D^2)$ induced from $\mathcal{C}_2 \to \mathcal{E}_{\mathrm{EC}(1,D^2)}$ is the monoid structure described in Section 1.1.

Theorem 3.4 (Budney [5]) *The fat long knots $\widehat{\mathcal{K}}$ form a sub-$\mathcal{C}_2$–algebra of* $\mathrm{EC}(1, D^2)$.

Proof As mentioned in Section 1.1, the framing number ω descends to a morphism of monoids $\pi_0\mathrm{EC}(1, D^2) \to \mathbb{Z}$. Recall also that $\pi_0\mathcal{C}_2$ is the commutative operad $\mathcal{C}om$. The integers $\mathbb{Z}$ form an abelian group and thus a commutative monoid. They can therefore be seen as a $\mathcal{C}_2$–algebra via the structure map $\mathcal{C}_2 \to \mathcal{C}om \to \mathcal{E}_{\mathbb{Z}}$. In this framework, the framing number ω is a $\mathcal{C}_2$–algebra morphism, hence the result. □

We conclude this subsection with a quick discussion about κ. Let L be an element of $\mathcal{C}_2(k)$. The heights of the little cubes of L only appear in the formula of $\kappa_k(L)$ to dictate a composition order. This is done via an ordering permutation, which we defined as an element $\sigma \in \Sigma_k$ such that the mapping $i \mapsto L^{\sigma(i)}$ is nondecreasing. Here, one can replace the word "decreasing" with "increasing" and define another action with the same formula. We refer to it as *Budney's reverse action*. There is no substantial difference between these two versions of κ, nor is there a reason to prefer one or the other. We still introduce the two of them now, as they will both play a role in the next subsections. Informally, the need for a reversed action arises because knots yielding split links must be tied at the beginning of a composition, while knots yielding cables must be tied at the end.

3.2 Burke and Koytcheff's actions on fat 2–string links

This subsection is a first step towards an adaptation of Budney's work to 2–string links. Namely, we build an action of $\mathcal{C}_1$ on $\mathrm{EC}_\iota(1, D^2)$ and $\widehat{\mathcal{L}}$. This structure has already been exhibited by Burke and Koytcheff [8, Theorem 6.8], with $\mathcal{C}_1$ appearing as a suboperad of a way bigger object called the infection operad. As before, we start with an action of CAut_1 on the embeddings $(\mathbb{R} \times D^2) \amalg (\mathbb{R} \times D^2) \hookrightarrow \mathbb{R} \times D^2$, then proceed to extend it to $\mathcal{C}_1$.

Proposition 3.5 *The topological group* CAut_1 *acts on* $\mathrm{Emb}((\mathbb{R}\times D^2)^{\amalg 2}, \mathbb{R}\times D^2)$ *via*

$$\mathrm{CAut}_1 \times \mathrm{Emb}((\mathbb{R}\times D^2)^{\amalg 2}, \mathbb{R}\times D^2) \to \mathrm{Emb}((\mathbb{R}\times D^2)^{\amalg 2}, \mathbb{R}\times D^2),$$
$$(L, l) \mapsto (L \times \mathrm{id}_{D^2}) \circ l \circ (L^{-1} \times \mathrm{id}_{D^2})^{\amalg 2}.$$

Moreover, this restricts to an action of $\mathcal{C}_1(1)$ on $\mathrm{EC}_\iota(1, D^2)$, which we write as $(L, l) \mapsto Ll$.

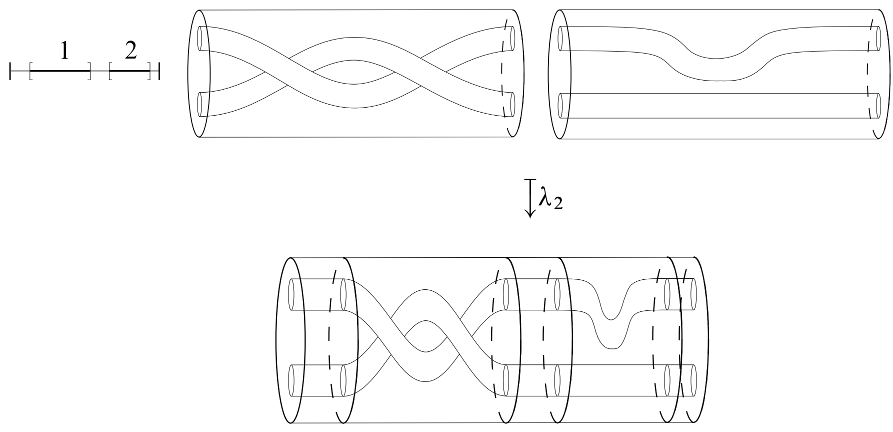

Figure 13: Illustration of Burke and Koytcheff's action on fat 2–string links.

Proof The proof of this is very similar to the proof of Proposition 3.1: the fact that the formula above specifies a valid action of a topological group is still clear and the restriction statement is proved just as in the case of framed long knots. □

To distinguish Budney's action from the one we build now, we denote the structure map by λ. The space $\mathcal{C}_1(0)$ in arity 0 still consists of a single point that λ_0 maps to $\mathrm{id}_{\mathbb{R}} \times \iota$ from Definition 1.11. For any positive integer k and $L \in \mathcal{C}_1(k)$, we set $\lambda_k(L)$ to the map that concatenates k framed string links according to the configuration of intervals L. That is, for every $\boldsymbol{f} = (f_i)_i \in \mathrm{EC}_\iota(1, D^2)^{\times k}$,

$$\lambda_k(L)(\boldsymbol{f})\colon (\mathbb{R} \times D^2) \amalg (\mathbb{R} \times D^2) \to \mathbb{R} \times D^2,$$
$$(t, x) \mapsto \begin{cases} L^i f_i(t, x) & \text{when } t \in L^i(J), \\ (t, \iota(x)) & \text{elsewhere.} \end{cases}$$

The embeddings patch in a differentiable way because the little cubes are almost disjoint. The outcome lies in $\mathrm{EC}_\iota(1, D^2)$ because $\mathrm{supp}_\iota(L^i f_i) = (L^i \times \mathrm{id}_{D^2})^{\amalg 2}(\mathrm{supp}_\iota(f_i))$.

Theorem 3.6 (Burke and Koytcheff [8]) *The operations λ turn $\mathrm{EC}_\iota(1, D^2)$ into a $\mathcal{C}_1$–algebra.*

Proof It is clear that $\lambda_1(\mathrm{id}_J)$ is the identity on $\mathrm{EC}_\iota(1, D^2)$. We check the compatibility with the symmetric actions. Let τ be a permutation, $L \in \mathcal{C}_1(k)$ and $\boldsymbol{f} \in \mathrm{EC}_\iota(1, D^2)^{\times k}$. To prove the desired $\lambda_k(L\tau, \boldsymbol{f}) = \lambda_k(L, \tau\boldsymbol{f})$, we show that these maps agree on the images of every little cube of L. This is enough as they clearly restrict to $\mathrm{id}_{\mathbb{R}} \times \iota$

outside of these intervals. For every $i \leq k$, the left-hand side of the equation restricts to $(L\tau)^i f_i$ on $(L\tau)^i(J^k)$. The right-hand side restricts to $L^{\tau(i)} f_{\tau^{-1}\circ\tau(i)} = (L\tau)^i f_i$ so we are done. The associative compatibility is verified the same way. □

As in the case of knots, the stacking operation arises as a special case of this recently developed action. More precisely, acting with two side-by-side intervals of width 1 on two string links results in their concatenation. Therefore, the $\mathcal{A}s$–algebra structure on $\pi_0\mathrm{EC}_\iota(1, D^2)$ induced from $\lambda\colon \mathcal{C}_1 \to \mathcal{E}_{\mathrm{EC}_\iota(1,D^2)}$ is the monoid structure on $\pi_0\mathrm{EC}_\iota(1, D^2)$ discussed in Section 1.2. Moreover, as in the case of knots, we can restrict ourselves to unframed embeddings:

Theorem 3.7 *The fat 2–string links $\hat{\mathcal{L}}$ form a sub-$\mathcal{C}_1$–algebra of* $\mathrm{EC}_\iota(1, D^2)$.

Proof Just as in Theorem 3.4, $\mathbb{Z}^{\times 2}$ is a group that we can think of as an $\mathcal{A}s$–algebra and therefore a $\mathcal{C}_1$–algebra. This turns the framing number ω into a morphism of $\mathcal{C}_1$–algebras, hence the result. □

3.3 The action of $\mathcal{SCL}$ on fat 2–string links

This section aims to merge the two actions defined above into a single $\mathcal{SCL}$–algebra structure on the spaces of fat long knots and fat 2–string links. More precisely, we build an action of $\mathcal{SCL}$ on the quadruplet of spaces $X = (X_o, X_\uparrow, X_\downarrow, X_\updownarrow)$, where $X_o = \mathrm{EC}_\iota(1, D^2)$ and $X_s = \mathrm{EC}(1, D^2)$ for every $s \in \{\uparrow, \downarrow, \updownarrow\}$. We start with a lemma to ease the construction.

Lemma 3.8 *There is a map*

$$\mathrm{EC}_\iota(1, D^2) \times \mathrm{EC}(1, D^2)^{\times 3} \to \mathrm{EC}_\iota(1, D^2), \quad (l, k_\uparrow, k_\downarrow, k_\updownarrow) \mapsto k_\updownarrow \circ l \circ [k_\uparrow \amalg k_\downarrow].$$

Proof The continuity of this application immediately follows from the continuity of the composition in the C^∞–topology. The purpose of this lemma is actually to check that $k_\updownarrow \circ l \circ [k_\uparrow \amalg k_\downarrow]$ indeed lives in $\mathrm{EC}_\iota(1, D^2)$. This follows from the inclusions $\mathrm{supp}(k_s) \subset J \times D^2$ for $s \in \{\uparrow, \downarrow, \updownarrow\}$, and $l\big(\mathrm{int}((J \times D^2) \amalg (J \times D^2))\big) \subset J \times D^2$. □

This map is in some way a combination of the morphisms φ^s from Section 1.2. Indeed, if one restricts this application to the subspace $\{\mathrm{id}_\mathbb{R} \times \iota\} \times \mathrm{EC}(1, D^2) \times \{\mathrm{id}_{\mathbb{R}\times D^2}\}^{\times 2}$, the formula becomes $k_\uparrow \mapsto (\mathrm{id}_\mathbb{R} \times \iota) \circ [k_\uparrow \amalg \mathrm{id}_{\mathbb{R}\times D^2}]$, which is a fattened version of $\varphi^\uparrow$. We denote it by $\hat{\varphi}^\uparrow$. The same goes for $\downarrow$. In the case of $\updownarrow$, one is left with $k_\updownarrow \mapsto k_\updownarrow \circ (\mathrm{id}_\mathbb{R} \times \iota)$. This map sends a long knot $k_\updownarrow$ to the string link whose strands are parallel and knotted according to $k_{\updownarrow}$. In other words, it is again a fattened version of $\varphi^{\updownarrow}$, which we denote by $\hat{\varphi}^{\updownarrow}$.

We are now ready to define the morphism $\mu\colon \mathcal{SCL} \to \mathcal{E}_X$ for the new action. As the values of $\mu_{(\underline{t};s)}$ heavily depend on $(\underline{t};s)$, defining μ takes several steps.

We start by specifying the values of μ in monochromatic cases. The operad $\mathcal{SCL}$ restricts to the little cubes operad $\mathcal{C}_2$ on the colors $s \in \{\uparrow, \downarrow\}$. We set μ to Budney's action on these colors. In other words, $\mu_{(s^k;s)} = \kappa_k$ from Theorem 3.3. When $s = \updownarrow$, we similarly set $\mu_{(\updownarrow^k;\updownarrow)}$ to Budney's reverse action, which we will still denote by κ by slight abuse of notation. On the color o, $\mathcal{SCL}$ restricts to the operad $\mathcal{C}_2^\circ$. There is a morphism $(\_)_\pi\colon \mathcal{C}_2^\circ \to \mathcal{C}_1$ and we set μ to the composite $\lambda \circ (\_)_\pi$ on this suboperad.

For every $s \in \{\uparrow, \downarrow, \updownarrow\}$, the only input colors $\underline{t}$ that do not lead to an empty $\mathcal{SCL}(\underline{t};s)$ are the monochromatic ones such that $\underline{t} = s^k$. Thus, we are left to specify $\mu_{(\underline{t};s)}$ when $s = o$ and the inputs are mixed. To this end, we introduce color sorting functions. Let $\underline{t} \in S^\star$. Consider four injective (not necessarily increasing) maps

$$\alpha_s\colon [|\underline{t}|_s] = \{1, \dots, |\underline{t}|_s\} \to [|\underline{t}|] = \{1, \dots, |\underline{t}|\}, \quad s \in S,$$

whose disjoint images cover $[|\underline{t}|]$ and such that $t_{\alpha_s(i)} = s$ for every $i \in [|\underline{t}|_s]$. These maps regroup inputs of the same color and are said to sort the colors of $\underline{t}$. Observe that each α_s lifts to a map $\mathcal{SCL}(\underline{t};o) \to \mathcal{C}_2(|\underline{t}|_s)$ that discards the little cubes whose colors are different from s,

$$\alpha_s\colon \mathcal{SCL}(\underline{t};s) \to \mathcal{C}_2(|\underline{t}|_s), \quad L \mapsto \bigoplus_i L^{\alpha_s(i)}.$$

Discarding embeddings also yields a map

$$\alpha_s\colon X^{\times \underline{t}} \to X_s^{\times |\underline{t}|_s}, \quad \boldsymbol{f} \mapsto \alpha_s \boldsymbol{f} = (f_{\alpha_s(1)}, \dots, f_{\alpha_s(|\underline{t}|_s)}).$$

The behavior of these lifts with respect to the symmetric structures on $\mathcal{SCL}$ and $X^{\times \underline{t}}$ is captured by the following relations: for every $\sigma \in \Sigma_{|\underline{t}|}$, $\tau \in \Sigma_{|\underline{t}|_s}$, $L \in \mathcal{SCL}(\underline{t};o)$ and $\boldsymbol{f} \in X^{\times \underline{t}}$,

$$(\sigma \circ \alpha_s)L = \bigoplus_i L^{\sigma\circ\alpha_s(i)} = \bigoplus_i (L\sigma)^{\alpha_s(i)} = \alpha_s(L\sigma), \quad (\sigma \circ \alpha_s)\boldsymbol{f} = \alpha_s(\sigma^{-1}\boldsymbol{f}),$$

$$(\alpha_s \circ \tau)L = \bigoplus_i L^{\alpha_s\circ\tau(i)} = \bigoplus_i (\alpha_s L)^{\tau(i)} = (\alpha_s L)\tau, \quad (\alpha_s \circ \tau)\boldsymbol{f} = \tau^{-1}(\alpha_s \boldsymbol{f}).$$

We can finally combine the previous actions and define $\mu_{(\underline{t};o)}(L)$ as the map

$$\mu_{(\underline{t};o)}(L)\colon X^{\times \underline{t}} \to \mathrm{EC}_\iota(1, D^2),$$
$$\boldsymbol{f} \mapsto \kappa(\alpha_\updownarrow L, \alpha_\updownarrow \boldsymbol{f}) \circ \lambda(\alpha_o L_\pi, \alpha_o \boldsymbol{f}) \circ \bigl[\kappa(\alpha_\uparrow L, \alpha_\uparrow \boldsymbol{f}) \amalg \kappa(\alpha_\downarrow L, \alpha_\downarrow \boldsymbol{f})\bigr],$$

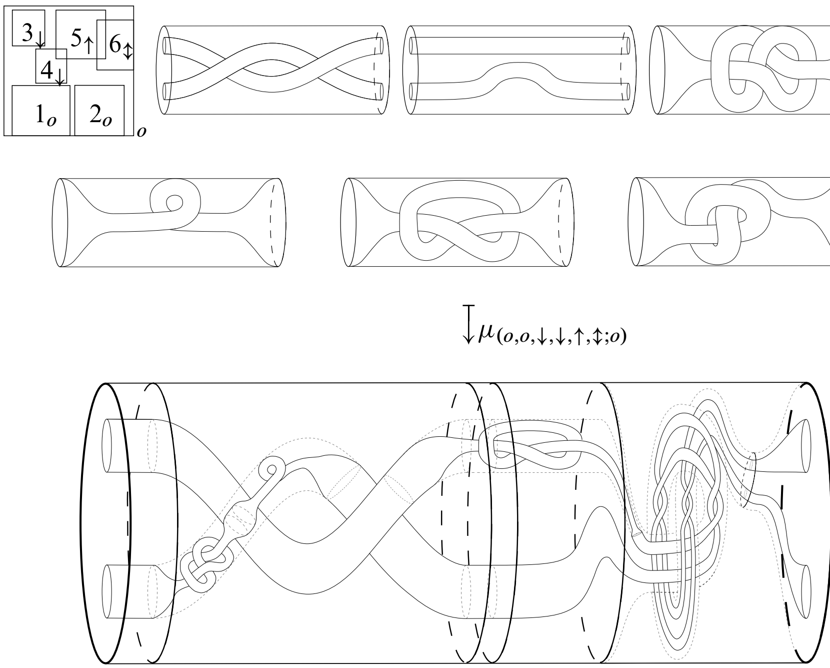

Figure 14: Illustration of the action of $\mathcal{SCL}$ on fat 2–string links and fat long knots.

where the κ on the left-hand side refers to Budney's reverse action and the other two to Budney's regular action.

The continuity of μ is immediate and its values do not depend on the color sorting functions: if one chooses to replace α_o with $\alpha_o{}'$, there is a permutation τ such that $\alpha_o{}' = \alpha_o \circ \tau$. Then, the relations above give

$$\begin{aligned}\lambda(\alpha_o{}' L_\pi, \alpha_o{}' \boldsymbol{f}) &= \lambda((\alpha_o \circ \tau) L_\pi, (\alpha_o \circ \tau) \boldsymbol{f}) \\ &= \lambda((\alpha_o L_\pi)\tau, \tau^{-1}(\alpha_o \boldsymbol{f})) \\ &= \lambda(\alpha_o L_\pi, \alpha_o \boldsymbol{f}).\end{aligned}$$

The same argument with κ shows that the remaining α_s can be replaced as well.

Theorem 3.9 *The operations μ turn the quadruplet X into an $\mathcal{SCL}$–algebra.*

Proof First of all, it is clear that every $\mu_{(s;s)}$ sends id_{J^2} to the identity. The symmetric compatibility is also quickly verified: when functions $(\alpha_s)_s$ sort the colors of $\underline{t}$, the composites $(\sigma^{-1} \circ \alpha_s)_s$ sort the colors of $\underline{t}\sigma$ and the needed equality follows.

We are left to check the compatibility with the operadic composition. Let $L \in \mathcal{SCL}(\underline{t}; s)$ and $P \in \mathcal{SCL}(\underline{u}; t_i)$ for some i. We need to show that $\mu(L \circ_i P) = \mu(L) \circ_i \mu(P)$. The validity of Budney, Burke and Koytcheff's actions (Theorems 3.3 and 3.6) implies the result when $\underline{t}$ and $\underline{u}$ are monochromatic of the same color. Since there is no operation with output color $s \in \{\uparrow, \downarrow, \updownarrow\}$ and input colors $\underline{t} \neq s^k$, we may assume that $s = o$ and $\underline{t} \neq o^k$. When evaluated in embeddings $\boldsymbol{f} \in X^{\times \underline{t} \circ_i \underline{u}}$, the desired equality reads

$$\begin{aligned}\mu_{(\underline{t} \circ_i \underline{u}; o)}(L \circ_i P, \boldsymbol{f}) \\ = \mu_{(\underline{t}; o)}\big(L, f_1, \dots, f_{i-1}, \mu_{(\underline{u}; t_i)}(P, f_i, \dots, f_{i+|\underline{u}|-1}), f_{i+|\underline{u}|}, \dots, f_{|\underline{t} \circ_i \underline{u}|}\big).\end{aligned}$$

We split cases and unravel the definition of μ on both sides of this equation.

Assume first that $t_i = \uparrow$. This forces $\underline{u} = \uparrow^{|\underline{u}|}$. Let $(\alpha_s)_s$ sort the colors of $\underline{t}$. We may ask for $\alpha_\uparrow(|\underline{t}|_\uparrow) = i$. We sort the colors $s \neq \uparrow$ in $\underline{t} \circ_i \underline{u}$ with functions $(\gamma_s)_s$ mapping j to $\alpha_s(j)$ if $\alpha_s(j) < i$ or to $\alpha_s(j) + |\underline{u}|$ if $\alpha_s(j) > i$. The reason for this choice is $\gamma_s(L \circ_i P) = \alpha_s L$. For the remaining $\gamma_\uparrow$, we use the same construction on $[|\underline{t}|_\uparrow - 1]$ and extend it to $[|\underline{t} \circ_i \underline{u}|_\uparrow]$ via the increasing map onto the interval $\{i + j,\ j < |\underline{u}|\}$. The equality reduces to

$$K_\updownarrow \circ \Lambda \circ [K_\uparrow^L \amalg K_\downarrow] = K_\updownarrow \circ \Lambda \circ [K_\uparrow^R \amalg K_\downarrow],$$

where

$$\begin{aligned}\Lambda &= \lambda(\alpha_o L_\pi, \gamma_o \boldsymbol{f}),\\ K_\uparrow^L &= \kappa(\gamma_\uparrow(L \circ_i P), \gamma_\uparrow \boldsymbol{f}),\\ K_\updownarrow &= \kappa(\alpha_\updownarrow L, \gamma_\updownarrow \boldsymbol{f}),\\ K_\uparrow^R &= \kappa(\alpha_\uparrow L, f_{\alpha_\uparrow(1)}, \dots, f_{\alpha_\uparrow(|\underline{t}|_\uparrow - 1)}, \kappa(P, f_i, \dots, f_{|\underline{u}|+i-1})),\\ K_\downarrow &= \kappa(\alpha_\downarrow L, \gamma_\downarrow \boldsymbol{f}).\end{aligned}$$

Furthermore, $\gamma_\uparrow(L \circ_i P) = \alpha_\uparrow L \circ_{|\underline{t}|_\uparrow} P$, so the validity of Budney's action (Theorem 3.3) completes the proof in this case. The same manipulations treat the cases $t_i = \downarrow$ and $\updownarrow$.

We are left to treat the case $s = o$ and $t_i = o$. Let $(\alpha_s)_s$ and $(\beta_s)_s$ be color sorting functions for $\underline{t}$ and $\underline{u}$, respectively. We may again ask for $\alpha_o(|\underline{t}|_o) = i$. Let $(\gamma_s)_s$ be the color sorting functions for $\underline{t} \circ_i \underline{u}$ one naturally builds from $(\alpha_s)_s$ and $(\beta_s)_s$. More precisely, γ_s agrees with α_s on ${\alpha_s}^{-1}(\{l \mid l < i\})$, with $\alpha_s + |\underline{u}|$ on ${\alpha_s}^{-1}(\{l \mid l > i\})$ and maps the remaining interval to the inputs s in $\{i + l \mid l < |\underline{u}|\}$ according to β_s. These choices are the ones giving $\gamma_s(L \circ_i P) = \alpha_s L \oplus (L^i \circ \beta_s P)$ for every s in $\{\uparrow, \downarrow, \updownarrow\}$ and $\gamma_o(L \circ_i P) = \alpha_o L \circ_{|\underline{t}|_o} \beta_o P$. The left-hand side of the equality reads

$K^L_{\updownarrow} \circ \Lambda^L \circ [K^L_{\uparrow} \amalg K^L_{\downarrow}]$, where

$$\Lambda^L = \lambda(\alpha_o L \circ_{|\underline{t}|_o} \beta_o P, \gamma_o \boldsymbol{f}),$$
$$K^L_s = \kappa(\alpha_s L \oplus (L^i \circ \beta_s P), \gamma_s \boldsymbol{f}) \quad \text{for every } s \in \{\uparrow, \downarrow, \updownarrow\}.$$

On the other hand, the right-hand side of the equality is $K^R_{\updownarrow} \circ \Lambda^R \circ [K^R_{\uparrow} \amalg K^R_{\downarrow}]$, where

$$\Lambda^R = \lambda(\alpha_o L, f_{\gamma_o(1)}, \dots, f_{\gamma_o(|\underline{t}|_o - 1)}, \mu_{(\underline{u};o)}(P, f_i, \dots, f_{i+|\underline{u}|-1})),$$
$$K^R_s = \kappa(\alpha_s L, f_{\gamma_s(1)}, \dots, f_{\gamma_s(|\underline{t}|_s)}) \quad \text{for every } s \in \{\uparrow, \downarrow, \updownarrow\}.$$

But $\mu_{(\underline{u};o)}(P, f_i, \dots, f_{i+|\underline{u}|-1})$ is itself of the form $K_{\updownarrow}{}' \circ \Lambda' \circ [K_{\uparrow}{}' \amalg K_{\downarrow}{}']$, where

$$\Lambda' = \lambda(\beta_o P, f_{\gamma_o(|\underline{t}|_o)}, \dots, f_{\gamma_o(|\underline{t} \circ_i \underline{u}|_o)}),$$
$$K_s{}' = \kappa(\beta_o P, f_{\gamma_s(|\underline{t}|_s + 1)}, \dots, f_{\gamma_s(|\underline{t} \circ_i \underline{u}|_s)}) \quad \text{for every } s \in \{\uparrow, \downarrow, \updownarrow\}.$$

It is easy to check from the definition of λ and Theorem 3.6 that

$$\Lambda^R = L^i K_{\updownarrow}{}' \circ \Lambda^L \circ [L^i K_{\uparrow}{}' \amalg L^i K_{\downarrow}{}'].$$

We get the following new expression for the whole right-hand side of the equation:

$$(K^R_{\updownarrow} \circ L^i K_{\updownarrow}{}') \circ \Lambda^L \circ [L^i K_{\uparrow}{}' \circ K^R_{\uparrow} \amalg L^i K_{\downarrow}{}' \circ K^R_{\downarrow}].$$

Thus we are left to identify K factors. We previously computed

$$K^L_{\uparrow} = \kappa(\alpha_{\uparrow} L \oplus (L^i \circ \beta_{\uparrow} P), \gamma_{\uparrow} \boldsymbol{f}).$$

Recall that when evaluating κ, one chooses a permutation that orders $\alpha_{\uparrow} L \oplus (L^i \circ \beta_{\uparrow} P)$ and composes the embeddings accordingly. Here, L^i is a little 2–cube that meets the lower face of the unit cube. In other words, $L^i_t = -1$ and cannot get any lower. Thus, the factors $(L^i \circ \beta_{\uparrow} P)^j f_{\gamma_{\uparrow}(|\underline{t}|_{\uparrow} + j)}$ can be placed in first position when computing $K^L_{\uparrow}$. This ultimately shows that

$$K^L_{\uparrow} = L^i K_{\uparrow}{}' \circ K^R_{\uparrow}.$$

One deals with $\downarrow$ the exact same way. For $\updownarrow$, the same phenomenon with Budney's reverse action shows that the factors $(L^i \circ \beta_{\updownarrow} P)^j f_{\gamma_{\updownarrow}(|\underline{t}|_{\updownarrow} + j)}$ can be placed in last position when computing $K^L_{\updownarrow}$, which again shows the desired

$$K^L_{\updownarrow} = K^R_{\updownarrow} \circ L^i K_{\updownarrow}{}'. \qquad \square$$

Once again, the concatenation comes as a special case with side-by-side cubes of equal width. Budney's action on knots can be recovered and one can also turn a knot into a double cable or a split link using identity cubes in $\mathcal{SCL}(s;o)$ for $s \in \{\uparrow, \downarrow, \updownarrow\}$. More precisely, $\mu_{(s;o)}(\mathrm{id}_{J^2}) = \hat{\varphi}^s$. This shows that the $\pi_0\mathcal{SCL}$–algebra structure on the quadruplet $\pi_0 X$ is the data of the usual monoids $\pi_0\mathrm{EC}(1, D^2)$ and $\pi_0\mathrm{EC}_\iota(1, D^2)$, together with the three distinct independent actions of $\pi_0\mathrm{EC}(1, D^2)$ on $\pi_0\mathrm{EC}_\iota(1, D^2)$ given by the $\hat{\varphi}^s$, $s \in \{\uparrow, \downarrow, \updownarrow\}$. Finally, the spaces of unframed knots and links are still stable:

Theorem 3.10 *The quadruplet $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ forms a sub-$\mathcal{SCL}$–algebra of X.*

Proof Consider the two monoids $\mathbb{Z}$ and $\mathbb{Z}^{\times 2}$. The first one acts on the second one in three different ways: on the first factor of $\mathbb{Z}^{\times 2}$, on the second factor or diagonally. The data of these three actions is precisely that of a $\pi_0\mathcal{SCL}$–algebra structure on the quadruplet $(\mathbb{Z}^{\times 2}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$. One can think of this structure as an $\mathcal{SCL}$–algebra structure. Thanks to the additive properties of the linking number with respect to the concatenation of curves, one easily checks that the framing number ω turns into a morphism of $\mathcal{SCL}$–algebras. The result follows. $\square$

This action of $\mathcal{SCL}$ on $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ combines all the structure we have met on long knots and 2–string links so far. Moreover, the isotopies exhibiting the commutativity relations discussed in Section 1.2 can all be obtained with paths in $\mathcal{SCL}$ from a configuration of cubes to another. The next section aims to show that this correspondence actually follows from a deeper result: a homotopy equivalence between $(\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ and a free algebra over $\mathcal{SCL}$.

Remark 3.11 It is possible to extend Theorem 3.9 to manifolds other than the 2–dimensional disk D^2. More precisely, when M is a manifold of dimension n, we can consider the space $\mathrm{EC}(k, M)$ consisting of the embeddings from $\mathbb{R}^k \times M$ to itself that restrict to the identity outside of $J^k \times M$. The notation "EC" comes from the terminology "embedding" and "cubical". This space has been intensively studied by Budney and proved to be an algebra over the $(k+1)$–dimensional little cubes operad $\mathcal{C}_{k+1}$ in [5].

Similarly, for any fixed embedding $\iota\colon M \amalg M \hookrightarrow M$, one can define $\mathrm{EC}_\iota(k, M)$, the space consisting of embeddings from $\mathbb{R}^k \times M^{\amalg 2}$ to $\mathbb{R}^k \times M$ that restrict to $\mathrm{id}_{\mathbb{R}^k} \times \iota$ outside of $J^k \times M^{\amalg 2}$ and map the interior of $J^k \times M^{\amalg 2}$ to the interior of $J^k \times M$. Burke and Koytcheff mentioned these spaces in [2], alongside their work on the special case corresponding to framed string links.

In order to understand the structure on the quadruplet of spaces

$$X_M = (\mathrm{EC}_\iota(k, M), \mathrm{EC}(k, M), \mathrm{EC}(k, M), \mathrm{EC}(k, M)),$$

we need a higher-dimensional version of the Swiss-cheese operad for links. Roughly speaking, one can define $\mathcal{SCL}_k$ the same way we did $\mathcal{SCL}$, except that the operad $\mathcal{C}_k$ is used in the construction, instead of the 2–dimensional little cubes operad $\mathcal{C}_2$. We can then extend Budney's action on $\mathrm{EC}(k, M)$ in order to get an $\mathcal{SCL}_{k+1}$–algebraic structure on the quadruplet X_M. The precise formula for this action is the same as the one introduced before Theorem 3.9, and checking that is specifies a valid $\mathcal{SCL}_{k+1}$–algebra structure boils down to the verifications already carried out in the proof above.

4 Freeness results

We prove here that the operadic actions constructed in Section 3 lead to free algebras over different operads. More precisely, we first introduce the main result of Budney in [5], which states that $\widehat{\mathcal{K}}$ is homotopy equivalent as a $\mathcal{C}_2$–algebra to $\mathcal{C}_2[\widehat{\mathcal{P}}]$. A second theorem proved by Burke and Koytcheff in [8] provides an analogous statement about the action of $\mathcal{C}_1$ on a subspace of $\widehat{\mathcal{L}}$. We then combine these results to prove the main theorem of this paper, Theorem 4.11, stating that $(\widehat{\mathcal{L}}^{\mathbf{0}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$ is homotopy equivalent to a free $\mathcal{SCL}$–algebra. These three theorems are proved with very similar methods, most of them coming from 3–manifold topology and homotopy theory. The first subsection recalls the concepts we need from these fields, and the following three are each dedicated to a freeness theorem. The proofs of the results of Budney, Burke and Koytcheff are only quickly outlined, since thorough treatments are available in [5] and [8]. We still dispense sketches of proofs as the arguments they involve will be useful for Theorem 4.11.

4.1 Notions of 3–dimensional topology

We introduce some basic concepts of 3–manifold theory. The instances of 3–manifolds we will encounter mostly lie in $\mathbb{R}^3$, so they inherit very nice features. Furthermore, they are compact, orientable, connected and irreducible. It is very common when studying 3–manifolds to deal with embedded surfaces: we denote by S^2 the 2–sphere, D^2 the disk, A the annulus, T^2 the torus and $(T^2)^{\#2}$ the genus 2 oriented surface. We denote by P_n the n–punctured disk, whose boundary splits as an external component $\partial_{\mathrm{ext}} P_n$ and n internal components $\partial_{\mathrm{int}} P_n$. As for common 3–manifolds, we note $B = J \times D^2$ the cylinder, homeomorphic to a 3–ball D^3, $H_n = P_n \times I$ the n–handlebody and $C_f \subset B$ the complement of a fat long knot or a fat 2–string link f. The boundary of C_f is a torus when f is a fat long knot, and a 2–torus when f is a fat 2–string link. A recurring procedure in the upcoming proofs is the cutting of C_f along essential surfaces. We define the latter now.

Definition 4.1 Let S be a (not necessarily connected) orientable surface embedded in an orientable 3–manifold M properly (ie $S \cap \partial M = \partial S$). A disk $D \subset M$ with $D \cap S = \partial D$ is said to be a *compressing disk for* S if its boundary does not bound a disk in S. A surface that admits a compressing disk is said to be *compressible*, and a surface different from S^2 or D^2 admitting no compressing disk is said to be *incompressible*.

Definition 4.2 Let S be a bordered surface properly embedded in a 3–manifold M. A *∂–compressing disk for* S is a disk $D \subset M$ whose boundary consists of two arcs α and β with $\alpha \subset S$ and $\beta \subset \partial M$, whose interior is disjoint from S and ∂M, such that there is no arc γ in ∂S such that $\gamma \cup \alpha$ bounds a disk in S. A surface that admits a ∂–compressing disk is said to be *∂–compressible*. Otherwise, it is *∂–incompressible*.

Definition 4.3 A properly embedded surface S in a 3–manifold M is said to be *∂–parallel* if it can be isotoped to a piece of ∂M.

Definition 4.4 A properly embedded orientable surface S in a 3–manifold M is *essential* if one of the following holds:

(i) S is a sphere and does not bound a ball.

(ii) S is a disk whose boundary does not bound a disk in ∂M.

(iii) S is not a sphere nor a disk, it is incompressible, ∂–incompressible and not ∂–parallel.

Spaces of embeddings of incompressible surfaces have been extensively studied by Hatcher in [14]. He describes in his paper how the homotopy type of such a space depends on S. This result will be used repeatedly so we formulate a precise version here.

Theorem 4.5 (Hatcher [14]) *Let M be an orientable compact connected irreducible 3–manifold and $S \hookrightarrow M$ an essential orientable compact surface in M. Let $\mathrm{Emb}(S, M, \partial S)$ be the space of embeddings of S in M whose values at ∂S are fixed. Then the component $\mathrm{Emb}(S, M, \partial S)_S$ of $S \hookrightarrow M$ in $\mathrm{Emb}(S, M, \partial S)$ is weakly contractible unless S is closed and the fiber of a bundle structure on M, or if S is a torus. In these exceptional cases $\pi_i \,\mathrm{Emb}(S, M) = 0$ for all $i > 1$. In the bundle case, the inclusion of the subspace consisting of embeddings with image a fiber induces an isomorphism on π_1. When S is a torus but not the fiber of a bundle structure, the inclusion $\mathrm{Diff}(S) \hookrightarrow \mathrm{Emb}(S, M)$ obtained by precomposing $S \hookrightarrow M$ induces an isomorphism on π_1.*

Another tool of 3–manifold theory that will come in handy is the JSJ–decomposition. It provides a way to cut an irreducible manifold into simpler ones. The cuts are performed along essential tori, but if one keeps on cutting a manifold until no such torus is available, the obtained decomposition might not be unique. A manifold that admits no essential torus is said to be *atoroidal*. In order to get a unique decomposition, one must agree not to cut the pieces that are Seifert-fibered. The latter are manifolds consisting of disjoint parallel circles forming a particular fibering. The precise definition of this fibering is looser than the notion of fiber bundle with fiber S^1. It is specified for example in [15]. The decomposition theorem we use is the following:

Theorem 4.6 (Jaco, Shalen and Johannson [19; 20]) *Every orientable, compact, irreducible 3–manifold M contains a collection of embedded, incompressible tori T so that if one removes an open tubular neighborhood of T from M, the outcome is a disjoint union of Seifert-fibered and atoroidal manifolds. Moreover, a minimal collection of such tori is unique up to isotopy.*

The minimal collection of tori T from Theorem 4.6 (or sometimes its isotopy class) is called the JSJ–decomposition of M. In the case where ∂M consists of a single component, the piece of the cut M containing ∂M is called the root of the decomposition. The tori of T bounding the root are referred to as the base-level tori of T.

Our main concern while studying an orientable compact 3–manifold M will actually be the homotopy type of the group of its boundary-fixing diffeomorphisms $\mathrm{Diff}(M, \partial M)$. More precisely, we are interested of the subgroup $\mathrm{Diff}_{\mathrm{d}}(M, \partial M)$ consisting of the diffeomorphisms whose derivatives at ∂M agree with those of the identity. This extra condition is relevant for our work because it enables one to postcompose a fat long knot by an element of $\mathrm{Diff}_{\mathrm{d}}(B, \partial B)$ and still end up with a fat long knot. The main ingredient we use to prove the three upcoming freeness theorems is the following proposition:

Proposition 4.7 *Let M be an orientable compact connected irreducible 3–manifold and S an essential surface which cuts M into pieces M_i, such that the component $\mathrm{Emb}(S, M, \partial S)_S$ is weakly contractible and stable under the postcomposition action of $\mathrm{Diff}(M, \partial M)$. Then the inclusion*

$$\prod_i \mathrm{Diff}_{\mathrm{d}}(M_i, \partial M_i) \hookrightarrow \mathrm{Diff}_{\mathrm{d}}(M, \partial M)$$

is a weak homotopy equivalence.

Proof Thanks to the stability condition, we have a well-defined postcomposition map $\mathrm{Diff}(M, \partial M) \to \mathrm{Emb}(S, M, \partial S)_S$. Restriction maps such as this one have been shown to be locally trivial, and in particular fibrations. This problem, as well as the local triviality of the restriction map $\mathrm{Emb}(M, N) \to \mathrm{Emb}(M', N)$ for a submanifold $M' \subset M$, has been treated in several articles: in [26] for the case of closed manifolds and in [9] for the case of bordered manifolds. A more recent exposition is provided in the third section of [21]: the precise result we use here is formulated as Corollary 3.7 in [21]. The fiber over S of this map is the subgroup of diffeomorphisms fixing S, ie

$$\mathrm{Diff}(M, \partial M \cup S) \hookrightarrow \mathrm{Diff}(M, \partial M) \longrightarrow \mathrm{Emb}(S, M, \partial S)_S.$$

The base space is weakly contractible so the inclusion of the fiber is a weak homotopy equivalence. We are left to add the derivative condition on the diffeomorphisms.

It is proved in Kupers' book on diffeomorphism groups [22] that the inclusion of the subgroup $\mathrm{Diff_d}(N, \partial N) \hookrightarrow \mathrm{Diff}(N, \partial N)$ is a weak homotopy equivalence for every compact manifold N. This justifies the bottom left equivalence in the diagram of inclusions

$$\begin{tikzcd}
\mathrm{Diff_d}(M, \partial M \cup S) \arrow[r, hook] \arrow[d, equal] & \mathrm{Diff}(M, \partial M \cup S) \arrow[r, hook, "\simeq"] & \mathrm{Homeo}(M, \partial M \cup S) \arrow[d, equal] \\
\prod_i \mathrm{Diff_d}(M_i, \partial M_i) \arrow[r, hook, "\simeq"] & \prod_i \mathrm{Diff}(M_i, \partial M_i) \arrow[r, hook, "\simeq"] & \prod_i \mathrm{Homeo}(M_i, \partial M_i)
\end{tikzcd}$$

while the right horizontal equivalences come from works of Cerf in [9]. The two-out-of-three rule assures us that the top left inclusion is a weak equivalence as well. This same rule in the diagram

$$\begin{tikzcd}
\mathrm{Diff_d}(M, \partial M \cup S) \arrow[r, hook] \arrow[d, hook, "\simeq"] & \mathrm{Diff_d}(M, \partial M) \arrow[d, hook, "\simeq"] \\
\mathrm{Diff}(M, \partial M \cup S) \arrow[r, hook, "\simeq"] & \mathrm{Diff}(M, \partial M)
\end{tikzcd}$$

concludes the proof. □

The need to study these diffeomorphism groups arises from the following classical result in modern knot theory:

Proposition 4.8 *Let f be a fat long knot or a fat 2–string link. Then the component $\widehat{\mathcal{K}}_f$ or $\widehat{\mathcal{L}}_f$ of f in $\widehat{\mathcal{K}}$ or $\widehat{\mathcal{L}}$ is a model for the classifying space of* $\mathrm{Diff_d}(C_f, \partial C_f)$. *Moreover, it is a $K(G, 1)$.*

Proof We treat the case where f is a fat long knot, the other one is treated identically. When B is the solid cylinder and C_f the complement of f in B, we have the inclusion and restriction maps

$$\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f) \hookrightarrow \mathrm{Diff}_{\mathrm{d}}(B, \partial B) \longrightarrow \widehat{\mathcal{K}}_f.$$

The application on the right-hand side precomposes a diffeomorphism by f and is a fibration thanks to [21, Corollary 3.7]. Now, $\mathrm{Diff}_{\mathrm{d}}(B, \partial B) \simeq \mathrm{Diff}(B, \partial B)$ is weakly contractible, as proved in [13]. Thus, $\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ acts properly and freely on a contractible space, and the quotient of this action can be identified to $\widehat{\mathcal{K}}_f$, so $\widehat{\mathcal{K}}_f$ is a model for $B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$. Moreover, spaces of diffeomorphisms of orientable Haken bordered 3–manifolds which preserve the boundary pointwise always have vanishing higher homotopy groups by [14, Theorem 2]. Thus, the long exact sequence in homotopy coming from the fibration above assures us that $\widehat{\mathcal{K}}_f$ is a $K(\pi_0 \mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f), 1)$. □

4.2 Budney's freeness theorem

We now recall Budney's main theorem in [5], which is a freeness statement about the space of fat long knots as a $\mathcal{C}_2$–algebra. Recall from Section 1.1 that the prime fat long knots are denoted by $\widehat{\mathcal{P}} \subset \widehat{\mathcal{K}}$, and that they are the embeddings whose isotopy classes are prime elements in the monoid $\pi_0 \widehat{\mathcal{K}}$. The result we present appears as Theorem 11 in Budney's paper, and we only dispense a sketch of proof here, as the complete proof is fairly long.

Theorem 4.9 (Budney [5]) *The restriction of the structure map*

$$\kappa : \mathcal{C}_2[\widehat{\mathcal{P}}] \to \widehat{\mathcal{K}}$$

from Theorem 3.3 is a homotopy equivalence.

Sketch of Proof Thanks to Corollary 1.8, Proposition 2.7 and the fact that applying π_0 to the structure map $\kappa : \mathcal{C}_2 \to \mathcal{E}_{\widehat{\mathcal{K}}}$ endows $\pi_0 \widehat{\mathcal{K}}$ with its usual monoid structure, we are assured that κ induces a bijection on components. Thus, we are left to prove that it is a homotopy equivalence on each of these components.

On the component of the unknot, κ restricts to the map $\mathcal{C}_2(0) \times \widehat{\mathcal{P}}^{\times 0} \to \widehat{\mathcal{K}}_{\mathrm{id}_{\mathbb{R} \times D^2}}$. The complement of an unknot is a 1–handlebody H_1, and the diffeomorphism group $\mathrm{Diff}_{\mathrm{d}}(H_1, \partial H_1) \simeq \mathrm{Diff}(H_1, \partial H_1)$ is contractible. Proposition 4.8 therefore gives the contractibility of $\widehat{\mathcal{K}}_{\mathrm{id}_{\mathbb{R} \times D^2}} = B\mathrm{Diff}_{\mathrm{d}}(H_1, \partial H_1)$, so we have an equivalence in this case.

On the component of a prime knot $f \in \widehat{\mathcal{P}}$, κ restricts to the map $\mathcal{C}_2(1)\times\widehat{\mathcal{P}}_f \to \widehat{\mathcal{P}}_f = \widehat{\mathcal{K}}_f$. The homotopy retracting $\mathcal{C}_2(1)$ onto the identity little cube shows that this map is an equivalence as well.

Suppose now that f is a composite knot $f = f_1 \# \cdots \# f_n$. Budney proved in [4] that the base-level tori T of the JSJ–decomposition of the complement of f split C_f into $n+1$ pieces: the complements of the prime factors C_{f_i} and a root homeomorphic to $S^1 \times P_n$. One would like to apply Proposition 4.7 in order to split $\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ as a product of diffeomorphism groups involving each $\mathrm{Diff}_{\mathrm{d}}(C_{f_i}, \partial C_{f_i})$. But, cutting along T is not possible here as the component $\mathrm{Emb}(T^2 \amalg \cdots \amalg T^2, C_f)_T$ is not contractible by Theorem 4.5. It is also not stable under the postcomposition action of $\mathrm{Diff}(C_f, \partial C_f)$. Indeed, when T_i is the torus bounding C_{f_i}, two tori T_i and T_j can be permuted by some diffeomorphism of C_f if and only if f_i and f_j are isotopic. Let Σ_f be the subgroup of Σ_n preserving the partition of $\{1, \dots, n\}$ given by $i \sim j$ if and only if f_i and f_j are isotopic. Then, by considering the components of $\mathrm{Emb}(T^2 \amalg \cdots \amalg T^2, C_f)$ where the image of the i^{th} torus is isotopic to a T_j with $i \sim j$, and by quotienting out these components by the parametrization of each torus, one gets a space homotopy equivalent to Σ_f. It is stable under the postcomposition action of $\mathrm{Diff}(C_f, \partial C_f)$ so one can use an argument similar to the proof of Proposition 4.7 to split $\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ into a product involving each C_{f_i}. Namely, after some manipulations on the diffeomorphisms, Budney manages to fit $\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ up to homotopy in a fibration

$$\mathrm{KDiff}(P_n, \partial P_n) \times \prod_i \mathrm{Diff}_{\mathrm{d}}(C_{f_i}, \partial C_{f_i}) \to \mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f) \to \Sigma_f,$$

for some subgroup $\mathrm{KDiff}(P_n, \partial P_n)$ homotopy equivalent to the pure braid group on n strands KB_n. Since $BKB_n = \mathrm{conf}_n(J^2) \simeq \mathcal{C}_2(n)$, applying the classifying space functor B leads to

$$\begin{aligned} \widehat{\mathcal{K}}_f &\simeq B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f) \\ &\simeq B\,\mathrm{KDiff}(P_n, \partial P_n) \times_{\Sigma_f} \prod_i B\mathrm{Diff}_{\mathrm{d}}(C_{f_i}, \partial C_{f_i}) \\ &\simeq \mathcal{C}_2(n) \times_{\Sigma_f} \prod_i \widehat{\mathcal{K}}_{f_i} = \mathcal{C}_2[\widehat{\mathcal{P}}]_f. \end{aligned}$$

At this stage, this equivalence is merely an abstract map. But the vast majority of the group morphisms we met are inclusion-based, so Budney manages to show that this equivalence coincides with κ via explicit models. Weak equivalences are finally promoted to strong ones via an application of Whitehead's theorem. □

This result can be thought of as a generalization of Schubert's theorem: the connected sum operation # is extended to an algebraic structure on the space level of $\widehat{\mathcal{K}}$, and the isomorphism $\pi_0\widehat{\mathcal{K}} = \mathcal{C}om[\pi_0\widehat{\mathcal{P}}]$ is extended to the $\mathcal{C}_2$–equivariant homotopy equivalence $\widehat{\mathcal{K}} \simeq \mathcal{C}_2[\widehat{\mathcal{P}}]$. Note however that Budney uses Corollary 1.8 in the very first sentence of the proof, so that this generalization is in no way an alternative argument for Schubert's result.

4.3 Burke and Koytcheff's freeness theorem

We now get to Burke and Koytcheff's result about fat 2–string links. Recall from Section 1.2 that $\widehat{\mathcal{Q}}^0 \subset \widehat{\mathcal{L}}^0$ denotes the fat 2–string links which are prime but not in the image of one of the maps $\hat{\varphi}^s$, $s \in \{\uparrow, \downarrow, \updownarrow\}$. The theorem we present here is [8, Theorem 6.8]. We again provide a quick sketch of proof, as the ideas involved will reappear in the proof of our main result, Theorem 4.11.

Theorem 4.10 (Burke and Koytcheff [8]) *Let $\widehat{\mathcal{S}}^0$ be the subspace of $\widehat{\mathcal{L}}^0$ consisting of the fat 2–string links whose prime factors lie in $\widehat{\mathcal{Q}}^0$. Then, the restriction of the structure map*

$$\lambda\colon \mathcal{C}_1[\widehat{\mathcal{Q}}^0] \to \widehat{\mathcal{S}}^0$$

from Theorem 3.6 is a homotopy equivalence.

Sketch of proof Thanks to Theorem 1.10, we are assured that λ induces a bijection on components. We are therefore again left to show that it is an equivalence on each of these components.

On the component of the trivial link, λ restricts to the map $\mathcal{C}_1(0) \times (\widehat{\mathcal{Q}}^0)^{\times 0} \to \widehat{\mathcal{L}}^0_{\mathrm{id}_{\mathbb{R}} \times \iota}$. The complement of $\mathrm{id}_{\mathbb{R}} \times \iota$ is a 2–handlebody H_2, and the diffeomorphism group $\mathrm{Diff}_{\mathrm{d}}(H_2, \partial H_2) \simeq \mathrm{Diff}(H_2, \partial H_2)$ is contractible so we can conclude as in the case of knots.

Let now f be a nontrivial element of $\widehat{\mathcal{S}}^0$ and $f = f_1 \# \cdots \# f_n$ its decomposition in noncentral prime fat 2–string links. According to [2, Theorem 4.1], there are $n-1$ twice-punctured disks separating C_f into the complements of the f_i. Moreover, as shown in the fourth step of the proof of this same theorem, these disks are unique up to isotopy. This enables us to split $\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ as the product $\prod_i \mathrm{Diff}_{\mathrm{d}}(C_{f_i}, \partial C_{f_i})$ thanks to Proposition 4.7. Applying the classifying space functor yields the equivalences

$$\widehat{\mathcal{S}}^0_f = B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f) \simeq \prod_i B\mathrm{Diff}_{\mathrm{d}}(C_{f_i}, \partial C_{f_i}) \simeq \prod_i \widehat{\mathcal{L}}_{f_i}.$$

When we denote by $\mathcal{C}_1(n)_{(1,\dots,n)}$ the (contractible) component of $\mathcal{C}_1(n)$ where the intervals appear from left to right in the order $(1,\dots,n)$, we can go further and write

$$\widehat{\mathcal{S}}^0_f \simeq \prod_i \widehat{\mathcal{L}}_{f_i} \simeq \mathcal{C}_1(1)_{(1,\dots,n)} \times \prod_i \widehat{\mathcal{L}}_{f_i} = \mathcal{C}_1[\widehat{\mathcal{Q}}^0]_f .$$

At this stage, the equivalence is still given by an abstract map but it is easy to keep track of the identifications and see that it coincides with λ. Weak equivalences are finally promoted to strong ones via an application of Whitehead's theorem. □

Again, this result is a generalization of Theorem 1.10 as its provides a free algebraic structure on the space level of $\widehat{\mathcal{S}}^0$, which descends to the usual free monoid on the basis $\widehat{\mathcal{Q}}^0$ on isotopy classes.

4.4 Fat long knots and 2–string links form a free $\mathcal{SCL}$–algebra

We combine in this last subsection the theorems of Budney, Burke and Koytcheff presented above to prove a freeness result for the whole space of fat 2–string links. Namely, we prove:

Theorem 4.11 *The restriction of the structure map*

$$\mu\colon \mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}] \to (\widehat{\mathcal{L}}^0, \widehat{\mathcal{K}}, \widehat{\mathcal{K}}, \widehat{\mathcal{K}})$$

from Theorem 3.9 is a homotopy equivalence.

Just as in the preceding subsections, the proof mainly consists in reducing ourselves to each connected component and splitting the diffeomorphism group of the complement of a link along suitable surfaces. We state two technical lemmas to prepare this cutting process, then proceed to the proof of the theorem.

Lemma 4.12 *Let f be a fat 2–string link decomposing as*

$$f^o \# \hat{\varphi}^{\uparrow}(f^{\uparrow}) \# \hat{\varphi}^{\downarrow}(f^{\downarrow}) \# \hat{\varphi}^{\updownarrow}(f^{\updownarrow})$$

for some element f^o in $\widehat{\mathcal{S}}^0$ and fat long knots f^s, $s \in \{\uparrow, \downarrow, \updownarrow\}$. Then, the three vertical twice-punctured disks D cutting C_f into C_{f^o}, $C_{\hat{\varphi}^{\uparrow}(f^{\uparrow})}$, $C_{\hat{\varphi}^{\downarrow}(f^{\downarrow})}$ and $C_{\hat{\varphi}^{\updownarrow}(f^{\updownarrow})}$ are unique up to isotopy fixing the boundary. Moreover, the component $\mathrm{Emb}(D, C_f, \partial D)_D$ is weakly contractible and stable under the postcomposition action of $\mathrm{Diff}(C_f, \partial C_f)$.

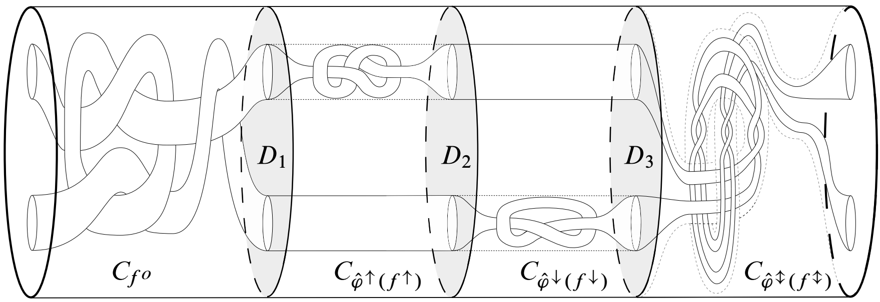

Figure 15: Illustration of the three disks $D = D_1 \amalg D_2 \amalg D_3$ in C_f.

Proof The uniqueness statement is proved in the steps 1 and 3 of Blair, Burke and Koytcheff's proof of their Theorem 4.1 in [2]. Applying a diffeomorphism of $\mathrm{Diff}(C, \partial C_f)$ to the punctured disks D does not change the fact that they split C_f into pieces homeomorphic to the complement of f^o and $\hat{\varphi}^s(f^s)$ for $s \in \{\uparrow, \downarrow, \updownarrow\}$. Thus, the component $\mathrm{Emb}(D, C_f, \partial D)_D$ is stable under the postcomposition action of this diffeomorphism group. Its contractibility immediately follows from Hatcher's work on incompressible surfaces (Theorem 4.5). □

Lemma 4.13 *Let f be a fat long knot and $\hat{\varphi}^s(f)$ the central link obtained from f for $s \in \{\uparrow, \downarrow, \updownarrow\}$. Consider the annulus A_s in the complement $C_{\hat{\varphi}^s(f)}$ specified by*

(i) $A_\uparrow = (\mathrm{id}_{\mathbb{R}} \times \iota)((J \times \partial D^2) \amalg \varnothing)$,

(ii) $A_\downarrow = (\mathrm{id}_{\mathbb{R}} \times \iota)(\varnothing \amalg (J \times \partial D^2))$,

(iii) $A_\updownarrow = f(J \times \partial D^2)$.

Then the isotopy class of A_s is stable under the postcomposition action of

$$\mathrm{Diff}(C_{\hat{\varphi}^s(f)}, \partial C_{\hat{\varphi}^s(f)})$$

and the component $\mathrm{Emb}(A, C_{\hat{\varphi}^s(f)}, \partial A)_{A_s}$ *is weakly contractible.*

Proof We first treat the case where $s = \uparrow$. Consider a horizontal disk $E \subset C_{\hat{\varphi}^{\uparrow}(f)}$ separating the two strands of $\hat{\varphi}^{\uparrow}(f)$. Cutting along E yields two manifolds: an upper piece containing $A_\uparrow$, homeomorphic to C_f, and a lower one that is a 1–handlebody H_1. Any two disks in $C_{\hat{\varphi}^{\uparrow}(f)}$ sharing their boundary are isotopic because $C_{\hat{\varphi}^{\uparrow}(f)}$ is irreducible. Therefore, given a diffeomorphism g in $\mathrm{Diff}(C_{\hat{\varphi}^{\uparrow}(f)}, \partial C_{\hat{\varphi}^{\uparrow}(f)})$, there is an isotopy from $g(E)$ to E. It can be extended to a boundary preserving ambient isotopy and postcomposing g by the latter shows that we may assume that $g(E) = E$. In other

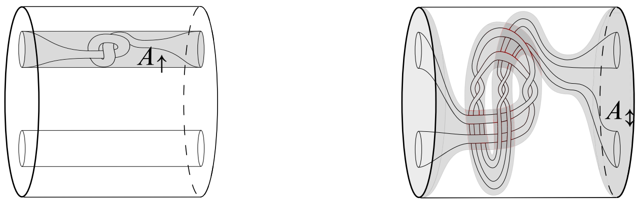

Figure 16: Illustration of the annuli $A_\uparrow$ and $A_\updownarrow$.

words, g preserves the cut and in particular $g(A_\uparrow)$ lies in the upper piece. But $A_\uparrow$ is boundary parallel after the cut, so it is unique up to boundary-fixing isotopy in its piece, which concludes the proof in this case. The weak contractibility statement immediately follows from Theorem 4.5. The case where $s = \downarrow$ is treated the exact same way.

We now deal with the case where $s = \updownarrow$. The proof of the stability statement uses the JSJ–decomposition recalled in Theorem 4.6, especially the uniqueness part. The idea is to show that the JSJ–decomposition of $C_{\hat{\varphi}^\updownarrow(f)}$ admits a single base-level torus $T_\updownarrow$, which must be unique up to isotopy, and that $A_\updownarrow$ is a suitable piece of it.

Let $A_\updownarrow{}'$ be the annulus $J \times \partial D^2 \subset \partial C_{\hat{\varphi}^\updownarrow(f)}$. The two annuli $A_\updownarrow$ and $A_\updownarrow{}'$ share their boundary so that their union is a torus in $C_{\hat{\varphi}^\updownarrow(f)}$. Let $T_\updownarrow$ be the torus obtained by pushing $A_\updownarrow \cup A_\updownarrow{}'$ in the interior of $C_{\hat{\varphi}^\updownarrow(f)}$. It is essential and cuts $C_{\hat{\varphi}^\updownarrow(f)}$ into two manifolds, one containing $\partial C_{\hat{\varphi}^\updownarrow(f)}$, $A_\updownarrow$ and $A_\updownarrow{}'$ that we denote by V, the other one homeomorphic to C_f. We now proceed to show that $T_\updownarrow$ is the base-level torus of the JSJ–decomposition of $C_{\hat{\varphi}^\updownarrow(f)}$.

Claim 1 *Let P_2 be a twice-punctured disk and $\gamma_\updownarrow$ a curve in the interior of P_2 parallel to the external boundary circle. Then V is homeomorphic to a 2–handlebody $J \times P_2$ deprived of a solid torus that is a tubular neighborhood of $0 \times \gamma_\updownarrow$.*

Proof This unknotting process is similar to Budney's "untwisted reembedding" described in the beginning of his paper on knot complements [4]. Cutting V along $A_\updownarrow$ results in two pieces: an external one containing $T_\updownarrow$ and an internal one that is a (knotted) 2–handlebody H_2. The torus $T_\updownarrow$ is boundary parallel in the external part, so this piece is a fattened torus $T^2 \times I$. This shows that V is the manifold obtained by gluing a 2–handlebody H_2 and a fattened torus $T^2 \times I$ along specific annuli in their boundaries. This description also matches our new model for V so we are done. □

This new model makes a lot of considerations easier since it forgets all the complexity of the knotting of f. We call the two unknotted annuli $J \times \partial_{\text{int}} P_2$ the *strands* of ∂V.

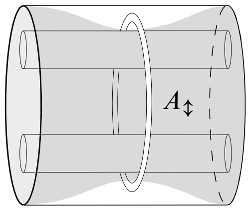

Figure 17: Illustration of the new model for V and $A_\updownarrow$.

Through this identification, $A_\updownarrow$ turns into the annulus bounding a neighborhood of the two strands not containing $T_\updownarrow$, as illustrated in Figure 17. The second annulus $A_\updownarrow{}'$ remains in the boundary as $J \times \partial D^2$.

Claim 2 *V is atoroidal.*

Proof In many cases including that of V, the definition of an atoroidal 3–manifold can be reformulated algebraically using the fundamental group. More precisely, one defines a peripheral subgroup of a 3–manifold M to be a subgroup of $\pi_1 M$ that lies in the image of the inclusion of a boundary component, then declares M to be atoroidal if every subgroup of $\pi_1 M$ isomorphic to $\mathbb{Z}^{\times 2}$ is conjugate to a peripheral subgroup. The equivalence between the two definitions is alluded for example in the beginning of [1], but the implication we are about to use follows from Corollary 5.5 in Waldhausen's article [33].

We start by computing $\pi_1 V$. The application of van Kampen's theorem summarized in Figure 18 gives

$$\pi_1 V = \langle \alpha_1, \alpha_2, \beta \mid \alpha_1 \beta \alpha_1^{-1} = \alpha_2 \beta \alpha_2^{-1} \rangle.$$

We can further write

$$\pi_1 V \cong \langle a, b, c \mid ab = ba \rangle = \mathbb{Z}^{\times 2} * \mathbb{Z}$$

via $\alpha_1 \mapsto c$, $\alpha_2 \mapsto ca^{-1}$ and $\beta \mapsto b$.

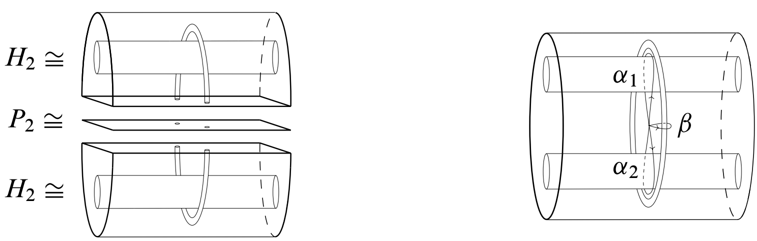

Figure 18: Applying van Kampen's theorem to compute $\pi_1 V$.

The inclusion of the toric boundary component $T_\updownarrow \subset V$ has image in π_1 the subgroup generated by a and b. It corresponds to the $\mathbb{Z}^{\times 2}$ factor under the isomorphism $\pi_1 V \cong \mathbb{Z}^{\times 2} * \mathbb{Z}$. We prove that any subgroup of $\mathbb{Z}^{\times 2} * \mathbb{Z}$ isomorphic to $\mathbb{Z}^{\times 2}$ is conjugate to a subgroup of the latter. Consider an arbitrary injection $\mathbb{Z}^{\times 2} \hookrightarrow \mathbb{Z}^{\times 2} * \mathbb{Z}$ and denote by x and y the images of $(0,1)$ and $(1,0)$. These x and y commute, they are nontrivial and they are not allowed to be powers of some third element. Now, applying Theorem 4.5 on page 209 of the book [23] on groups and presentations yields the following three possibilities for x and y:

(i) x or y may be trivial;

(ii) if neither x nor y is trivial, but x is in the conjugate of a factor, then y is in that same conjugate of a factor;

(iii) if neither x nor y is in a conjugate of a factor, then they are powers of a third element of $\mathbb{Z}^{\times 2} * \mathbb{Z}$.

Thanks to our observations, (i) and (iii) are ruled out and x and y must lie in the same conjugate of a factor of $\mathbb{Z}^{\times 2} * \mathbb{Z}$. The $\mathbb{Z}$ factor does not admit any subgroup isomorphic to $\mathbb{Z}^{\times 2}$, so we are done. $\square$

At this point, we showed that $T_\updownarrow$ and the tori of the JSJ–decomposition of C_f cut $C_{\hat{\varphi}^\updownarrow(f)}$ into atoroidal and Seifert-fibered pieces. Finally, $T_\updownarrow$ cannot be removed to obtain a smaller decomposition because V can never be part of a Seifert-fibered manifold, having a boundary component homeomorphic to a 2–torus. This shows that $T_\updownarrow$ and the tori of the JSJ–decomposition of C_f form a minimal decomposition of $C_{\hat{\varphi}^\updownarrow(f)}$ into atoroidal and Seifert-fibered manifolds, which proves that this collection is the JSJ–decomposition of $C_{\hat{\varphi}^\updownarrow(f)}$. The root V is bounded by $\partial C_{\hat{\varphi}^\updownarrow(f)}$ and $T_\updownarrow$, so $T_\updownarrow$ is the only base-level torus. Its image is therefore unique up to isotopy.

Consider now $g \in \mathrm{Diff}(C_{\hat{\varphi}^\updownarrow(f)}, \partial C_{\hat{\varphi}^\updownarrow(f)})$. Thanks to the work above, there is an isotopy from $g(T_\updownarrow)$ to $T_\updownarrow$. It can be extended to a boundary-fixing ambient isotopy, and postcomposing g by the latter shows that we may assume that $g(T_\updownarrow) = T_\updownarrow$. In other words, g preserves the cut along $T_\updownarrow$ and $g(A_\updownarrow)$ lies in V. Let us now take a look at the (boundary-fixing) isotopy classes of annuli in V with boundary $\partial A_\updownarrow$. We prove that there are only two of them: the one of $A_\updownarrow$ and the one of $A_\updownarrow{}'$. Let $A \subset V$ be an arbitrary annulus with adequate boundary. We may assume that the interiors of A and $A_\updownarrow{}'$ are disjoint because $A_\updownarrow{}'$ is ∂–parallel. Now, $A \cup A_\updownarrow{}'$ is an embedded torus in V. Its image in π_1 is generated by $\alpha_2^{-1}\alpha_1 = a$ and some other element y that commutes with a. Using [23, Theorem 4.5] again, we see that y is either trivial or in the $\mathbb{Z}^{\times 2}$ factor of $\mathbb{Z}^{\times 2} * \mathbb{Z} \cong \pi_1 V$. If y is trivial or a power of a, then $A \cup A_{\updownarrow}'$ bounds a solid torus and the two annuli are isotopic. If y is not just a power of a, then $A \cup A_{\updownarrow}'$ is incompressible and must parallel to $T_{\updownarrow}$ by Claim 2. In this situation, A is isotopic to $A_{\updownarrow}$. The two annuli cannot be permuted by a diffeomorphism of V because $A_{\updownarrow}'$ is ∂–parallel and $A_{\updownarrow}$ is not. This shows that $g|_V(A_{\updownarrow})$ is isotopic to $A_{\updownarrow}$ via an isotopy that fixes the boundary in V. This concludes this second case. The weak contractibility of the component in the embedding space again follows from Theorem 4.5. □

We now implement all the tools at our disposal to complete the proof of Theorem 4.11.

Proof of Theorem 4.11 Thanks to Theorems 1.8 and 1.10 and Proposition 2.7, we are assured that μ induces a bijection on components. Thus, we are left to prove that it is a homotopy equivalence on each of these components.

On the component of the unlink, μ restricts to the map

$$\mathcal{SCL}(\varnothing; o) \times (\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}})^{\times \varnothing} \to \widehat{\mathcal{L}}_{\mathrm{id}_{\mathbb{R}} \times \iota}.$$

The space $\mathcal{SCL}(\varnothing; o)$ consists of a single point. The complement of the unlink is a 2–handlebody H_2, and the diffeomorphism group $\mathrm{Diff}_{\mathrm{d}}(H_2, \partial H_2) \simeq \mathrm{Diff}(H_2, \partial H_2)$ is contractible. Proposition 4.8 then gives the contractibility of $\widehat{\mathcal{L}}_{\mathrm{id}_{\mathbb{R}} \times \iota} = B\mathrm{Diff}_{\mathrm{d}}(H_2, \partial H_2)$, so we have an equivalence in this case.

On the component of a fat long knot or an element of $\widehat{\mathcal{S}}^0$, Theorems 4.9 and 4.10 imply the result because the action of $\mathcal{SCL}$ restricts to κ and λ on these components.

Now let f be a nontrivial fat 2–string link. The action map μ restricts on the component of f to the map $\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f \to \widehat{\mathcal{L}}_f$. Suppose f decomposes as the concatenation $f^o \# \hat{\varphi}^{\uparrow}(f^{\uparrow}) \# \hat{\varphi}^{\downarrow}(f^{\downarrow}) \# \hat{\varphi}^{\updownarrow}(f^{\updownarrow})$ for some element f^o of $\widehat{\mathcal{S}}^0$ and some fat long knots f^s for $s \in \{\uparrow, \downarrow, \updownarrow\}$. We denote by $f^o = \#_{i \leq |f|_o} f_i^o$ and $f^s = \#_{i \leq |f|_s} f_i^s$ the prime decompositions of f^o and f^s. Then, one readily checks with the usual model presented in Section 2.2 that the component $\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f$ in the free algebra is given by

$$\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)} \times_{(\Sigma_{f^\uparrow} \times \Sigma_{f^\downarrow} \times \Sigma_{f^\updownarrow})} \prod_{i \leq |f|_o} \widehat{\mathcal{L}}_{f_i^o} \times \prod_s \prod_{i \leq |f|_s} \widehat{\mathcal{K}}_{f_i^s},$$

where $\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)}$ is the component of

$$\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)$$

where the cubes indexed by o appear from left to right in the order $(1, \dots, |f|_o)$ and where Σ_{f^s} is the subgroup of $\Sigma_{|f|_s}$ preserving the partition specified by $i \sim j$

if and only if f_i^s is isotopic to f_j^s. This group acts on the cubes indexed by s in $\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)}$ and permutes the entries in $\prod_{i\le|f|_s}\widehat{\mathcal{K}}_{f_i^s}$. The inclusion

$$\mathcal{SCL}(o^{|f|_o}, \uparrow^{|f|_\uparrow}, \downarrow^{|f|_\downarrow}, \updownarrow^{|f|_\updownarrow}; o)_{(1,\dots,|f|_o)} \hookrightarrow \mathcal{C}_2^\circ(|f|_o)_{(1,\dots,|f|_o)} \times \prod_s \mathcal{C}_2(|f|_s)$$

is a homotopy equivalence, so that we have, by rearranging terms in the product, the natural equivalences

$$\begin{aligned}\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f &\simeq \Big[\mathcal{C}_2^\circ(|f|_o)_{(1,\dots,|f|_o)} \times \prod_i \widehat{\mathcal{L}}_{f_i^o}\Big] \times \prod_s \Big[\mathcal{C}_2(|f|_s) \times_{\Sigma_{f^s}} \prod_i \widehat{\mathcal{K}}_{f_i^s}\Big] \\ &\simeq \mathcal{C}_1[\widehat{\mathcal{Q}}^0]_{f^o} \times \prod_s \mathcal{C}_2[\widehat{\mathcal{P}}]_{f^s}.\end{aligned}$$

On the other hand, thanks to Proposition 4.8, we know that $\widehat{\mathcal{L}}_f$ is a model for the classifying space of $\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$. The three vertical twice-punctured disks D from Lemma 4.12 splitting C_f as C_{f^o}, $C_{\hat{\varphi}^\uparrow(f^\uparrow)}$, $C_{\hat{\varphi}^\downarrow(f^\downarrow)}$ and $C_{\hat{\varphi}^\updownarrow(f^\updownarrow)}$ are stable under the action of $\mathrm{Diff}(C_f, \partial C_f)$ and have their component in $\mathrm{Emb}(D, C_f, \partial D)$ weakly contractible. Therefore, Proposition 4.7 gives the inclusion-based weak equivalence

$$\mathrm{Diff}_{\mathrm{d}}(C_{f^o}, \partial C_{f^o}) \times \prod_s \mathrm{Diff}_{\mathrm{d}}(C_{\hat{\varphi}^s(f^s)}, \partial C_{\hat{\varphi}^s(f^s)}) \hookrightarrow \mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f).$$

Now, Lemma 4.13 states that the three annuli A_s also satisfy the conditions of Proposition 4.7. They each split $C_{\hat{\varphi}^s(f^s)}$ into a 2–handlebody H_2 and a manifold diffeomorphic to C_{f^s}. Actually, this second piece is precisely the image of C_{f^s} under $\mathrm{id}_{\mathbb{R}} \times \iota$ when $s \in \{\uparrow, \downarrow\}$ and $C_{f^\updownarrow}$ itself when $s = \updownarrow$. The diffeomorphism group $\mathrm{Diff}_{\mathrm{d}}(H_2, \partial H_2)$ is contractible so we get the further natural equivalences

$$\mathrm{Diff}_{\mathrm{d}}(C_{f^s}, \partial C_{f^s}) \hookrightarrow \mathrm{Diff}_{\mathrm{d}}(C_{f^s}, \partial C_{f^s}) \times \mathrm{Diff}_{\mathrm{d}}(H_2, \partial H_2) \hookrightarrow \mathrm{Diff}_{\mathrm{d}}(C_{\hat{\varphi}^s(f^s)}, \partial C_{\hat{\varphi}^s(f^s)}).$$

Composing these results with Proposition 4.8 yields the natural equivalences

$$\begin{aligned}\widehat{\mathcal{L}}_{f^o} \times \prod_s \widehat{\mathcal{K}}_{f^s} &\simeq B\mathrm{Diff}_{\mathrm{d}}(C_{f^o}, \partial C_{f^o}) \times \prod_s B\mathrm{Diff}_{\mathrm{d}}(C_{f^s}, \partial C_{f^s}) \\ &\simeq B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f) \\ &\simeq \widehat{\mathcal{L}}_f.\end{aligned}$$

We are now able to use Budney, Burke and Koytcheff's freeness results (Theorems 4.9 and 4.10) and our previous discussion to get the equivalence

$$\mathcal{SCL}[\widehat{\mathcal{Q}}^0, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}, \widehat{\mathcal{P}}]_f \simeq \mathcal{C}_1[\widehat{\mathcal{Q}}^0]_{f^o} \times \prod_s \mathcal{C}_2[\widehat{\mathcal{P}}]_{f^s} \simeq \widehat{\mathcal{L}}_{f^o} \times \prod_s \widehat{\mathcal{K}}_{f^s} \simeq \widehat{\mathcal{L}}_f.$$

We merely have an abstract equivalence at this stage. To show that it coincides with μ, we need to check the commutativity up to homotopy of the diagram

$$\begin{tikzcd} B\mathrm{Diff}_{\mathrm{d}}(C_{f^o}, \partial C_{f^o}) \times \prod_s B\mathrm{Diff}_{\mathrm{d}}(C_{f^s}, \partial C_{f^s}) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f) \arrow[d, "\simeq"] \\ \widehat{\mathcal{L}}_{f^o} \times \prod_s \widehat{\mathcal{K}}_{f^s} \arrow[r, "\mu"] & \widehat{\mathcal{L}}_f \end{tikzcd}$$

but the spaces at stake are $K(G,1)$'s by Proposition 4.8, so it is enough to check the commutativity in π_1. This verification is very similar to the end of Budney's proof of Theorem 11 in [5]. An element of $\pi_1\widehat{\mathcal{L}}_f$ is (a homotopy class of) a based path in $\widehat{\mathcal{L}}_f$, ie an isotopy from f to f. The elements of $\pi_1 B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ can canonically be identified with $\pi_0\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ in the long exact sequence of the fibration realizing $\widehat{\mathcal{L}}_f$ as $B\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ in Proposition 4.8. In this framework, picking a class $\phi \in \pi_0\mathrm{Diff}_{\mathrm{d}}(C_{f^o}, \partial C_{f^o})$ and chasing the diagram along the clockwise route turns it into an element of $\pi_0\mathrm{Diff}_{\mathrm{d}}(C_f, \partial C_f)$ with its support lying between $-1\times D^2$ and $D_1 \subset C_f$, then converts it into an isotopy of f according to the construction in Proposition 4.8. Chasing ϕ along the counterclockwise route converts it into an isotopy of f^o in $\pi_1\widehat{\mathcal{L}}_{f^o}$, then applies μ to it. The outcome is the same as each $\hat{\varphi}^s(f^s)$ is fixed all along this last isotopy. The same argument shows that picking a class in $\pi_0\mathrm{Diff}_{\mathrm{d}}(C_{f^s}, \partial C_{f^s})$ and chasing the diagram in either direction has the same effect. When evaluated in π_1, the upper-left product is a direct product, on which a factor-by-factor verification is thus sufficient to get the commutativity.

Finally, Hatcher and McCullough proved in [16] that the classifying spaces of the diffeomorphism groups at stake here have the homotopy types of (aspherical finite) CW–complexes (what we use here could also be deduced from Palais' earlier article [27]). Thus, Whitehead's theorem promotes the weak homotopy equivalence μ to a strong one. □

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A short proof that the L^p–diameter of $\mathrm{Diff}_0(S, \text{area})$ is infinite

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We give a short proof that the L^p–diameter of the group of area preserving diffeomorphisms isotopic to the identity of a compact surface is infinite.

37E30, 57K10, 58D05

1 Introduction

Let (M, g) be a Riemannian manifold and let μ be the measure induced by the metric g. We denote the group of all diffeomorphisms of M that preserve μ and are isotopic to the identity by $\mathrm{Diff}_0(M, \mu)$.

In [12] Shnirelman showed that the L^2–diameter of $\mathrm{Diff}_0(M, \mu)$ is finite if M is the n–dimensional ball for $n > 2$ see also Shnirelman [13]. Conjecturally, the same is true for any compact simply connected Riemannian manifold of dimension greater than 2 (it is stated in Eliashberg and Ratiu [8] without proof).

The situation is different for 2–dimensional manifolds. In this case it is customary to denote the measure induced by g by area. For simplicity, let us restrict the discussion to orientable compact connected Riemannian surfaces (S, g). Eliashberg and Ratiu [8] proved that the L^p–diameter $(p \geq 1)$ of $\mathrm{Diff}_0(S, \text{area})$ is infinite if S is a surface with boundary. They show that the Calabi homomorphism is Lipschitz with respect to the L^p–norm. Later Gambaudo and Lagrange [9] obtained a similar result for a huge class of quasimorphisms on $\mathrm{Diff}_0(S, \text{area})$ if S is the closed disc (see as well Brandenbursky [3], Brandenbursky and Shelukhin [6] and Shelukhin [11] for more results concerning quasimorphisms and the L^p geometry). Their proof makes use of the braid group of the disc and inequalities relating the geometric intersection number of a braid and its word-length.

If S has negative Euler characteristic it is relatively easy to show that the L^p–diameter for $p \geq 1$ of $\mathrm{Diff}_0(S, \mathrm{area})$ is infinite; see Proposition 3.2 or Brandenbursky and Kędra [4, Theorem 1.2]. In the case of the torus one needs to know in addition that the group of Hamiltonian diffeomorphisms of the torus is simply connected, which is a nontrivial result from symplectic topology; see Brandenbursky and Shelukhin [7, Appendix A].

The last unsolved case was the sphere. Recently Brandenbursky and Shelukhin [7] showed that in this case the diameter is also infinite. Moreover, for each $p \geq 1$, $\mathrm{Diff}_0(S^2, \mathrm{area})$ contains quasi-isometrically embedded right-angled Artin groups (see Kim and Koberda [10]) and $\mathbb{R}^m$ for each natural m. Their arguments use some new tools along with the ideas from [9]. However, using intersection numbers in the case of the sphere requires considerably more work.

Our aim is to give a short and elementary proof of the following theorem:

Theorem 1 *Let (S, g) be a compact surface (with or without boundary). Then for every $p \geq 1$ the L^p–diameter of $\mathrm{Diff}_0(S, \mathrm{area})$ is infinite.*

Our method gives a unified proof for every compact surface S. It is partially inspired by [9]; in particular Lemma 5.2 can be seen as a generalization of an inequality obtained in [9] for the disk. The main simplification comes from the fact that instead of using the braid group and intersection numbers, we directly look at the geometry of the configuration space $C_n(S)$ with a certain complete metric described in Section 4. In Section 5 we relate the L^1–norm of $f \in \mathrm{Diff}_0(S, \mathrm{area})$ to an L^1–norm, defined by this complete metric, of the diffeomorphism on $C_n(S)$ induced by f. This allows us to apply the simple technique, described in Section 3, of showing the unboundedness of the L^p–norm in the case where the fundamental group of the manifold is complicated enough.

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2 The L^p–norm

Let (M, g) be a Riemannian manifold and let μ be a finite measure on M. Usually one assumes that μ is induced by g, even though the definition of an L^p–norm works as well if μ is any finite measure (then the L^p–norm could be a pseudonorm). We introduce here a more general definition as it is useful for stating results in Section 5.

Suppose $f \in \mathrm{Diff}_0(M,\mu)$ and let $X\colon M \to TM$ be a map to a tangent space of M such that $X(x) \in T_{f(x)}M$. One can think of X as a tangent vector to $\mathrm{Diff}_0(M,\mu)$ at the point f. The L^p–norm of X is defined by the formula

$$\|X\|_p = \left(\int_M |X(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Let $f_t \in \mathrm{Diff}_0(M,\mu)$ for $t \in [0,1]$ be a smooth isotopy, ie it defines a smooth map $M \times [0,1] \to M$. We always assume that isotopies are smooth. The L^p–length of $\{f_t\}$ is defined by

$$l_p(\{f_t\}) = \int_0^1 \|\dot{f}_t\|_p \, dt,$$

where $\dot{f}_t(x) = (d/ds) f_s(x)|_{s=t} \in T_{f_t(x)}M$. Note that if $p = 1$, then $\int_0^1 |\dot{f}_t(x)|\, dt$ is the length of the path $f_t(x)$, thus $l_1(\{f_t\})$ can be interpreted as the μ–average of the lengths of all paths $f_t(x)$.

Letting $f \in \mathrm{Diff}_0(M,\mu)$, we define the L^p–norm of f by

$$l_p(f) = \inf l_p(\{f_t\}),$$

where the infimum is taken over all smooth isotopies $f_t \in \mathrm{Diff}_0(M,\mu)$ connecting the identity on M with f. The assumption that f is μ–preserving was not used in the definition, but it is needed to show that l_p satisfies the triangle inequality.

The L^p–diameter of $\mathrm{Diff}_0(M,\mu)$ equals

$$\sup\{l_p(f) : f \in \mathrm{Diff}_0(M,\mu)\}.$$

It is worth noting that geodesics in $\mathrm{Diff}_0(M,\mu)$ with the L^2–metric are solutions of the Euler equations of an incompressible fluid. For more on the connection between the L^2–metric and hydrodynamics see [1].

3 The base case

In this section we present the basic method which can be used to show that, for $p \geq 1$, the L^p–diameter of $\mathrm{Diff}_0(M,\mu)$ is infinite if $\pi_1(M)$ is complicated enough.

Lemma 3.1 *Let X be a topological space and let $f_t \in \mathrm{Homeo}(X)$ for $t \in [0,1]$ be a loop in $\mathrm{Homeo}(X)$ based at Id_X, ie $f_0 = f_1 = \mathrm{Id}_X$. Then for every $x \in X$, the loop $f_t(x)$ for $t \in [0,1]$ is in the center of $\pi_1(X,x)$.*

Proof Let $x \in X$ and let γ_s for $s \in [0,1]$ be a loop in X based at x. Consider the map $\phi\colon S^1 \times S^1 \to X$ given by $(t,s) \mapsto f_t(\gamma_s)$, where $S^1 = [0,1]/0{\sim}1$. We have that $\phi(t,0) = f_t(x)$ and $\phi(0,s) = \gamma_s$. Thus loops $f_t(x)$ and γ_s are in the image of the torus $S^1 \times S^1$, therefore they commute. □

Let (M,g) be a Riemannian manifold. Suppose $h \in \pi_1(M)$. Let $l(h)$ denote the infimum over lengths of based loops in M that represent h. We denote by $Z(\pi_1(M))$ the center of $\pi_1(M)$.

Proposition 3.2 *Let (M,g) be a Riemannian manifold and μ the measure induced by g. Assume that for every r the set $\{h \in \pi_1(M) : l(h) < r\}$ is finite (it holds eg if M is compact) and $\pi_1(M)/Z(\pi_1(M))$ is infinite. Then for every $p \geq 1$ the L^p–diameter of $\mathrm{Diff}_0(M,\mu)$ is infinite.*

Proof By the Hölder inequality we can assume $p = 1$. Let $z \in M$ be a basepoint and let $h \in \pi_1(M,z)$. We represent h as a loop γ based at z.

Let U be a contractible neighborhood of z and let $f_t \in \mathrm{Diff}_0(M,\mu)$ for $t \in [0,1]$ be a finger-pushing isotopy that moves U all the way along γ. For a detailed construction see [5, proof of Lemma 3.1].

For every $x \in U$ we choose a path ϕ_x contained in U connecting z with x. We can assume that $l(\phi_x) < \mathrm{diam}(U)$, where $l(\phi_x)$ is the length of ϕ_x. We denote by ϕ_x^* the reverse of ϕ_x.

The isotopy f_t is defined so that it satisfies:

(1) For every $x \in U$, $f_1(x) = x$.

(2) For every $x \in U$, the concatenation of ϕ_x, $f_t(x)$ and ϕ_x^* is a loop based at z and its homotopy class equals h.

Let $f_h = f_1$ and define $L_h = \min\{l(hc) : c \in Z(\pi_1(M,z))\}$. We shall show that

$$\mu(U)(L_h - 2\,\mathrm{diam}(U)) \leq l_1(f_h).$$

Let g_t for $t \in [0,1]$ be any isotopy connecting the identity on M with f_h. Due to Lemma 3.1, for every $x \in U$ the paths $g_t(x)$ and $f_t(x)$ represent elements of $\pi_1(M,x)$ that differ by an element of the center. Thus the concatenation of ϕ_x, $g_t(x)$ and ϕ_x^* represents an element of the form $hc \in \pi_1(M,z)$ where $c \in Z(\pi_1(M,z))$. Since $l(\phi_x) < \mathrm{diam}(U)$, we have that $l(g_t(x)) \geq L_h - 2\,\mathrm{diam}(U)$. Indeed, otherwise the concatenation of ϕ_x, $g_t(x)$ and ϕ_x^* would be a loop of length less then $L_h \le l(hc)$, which is impossible.

Since $l(g_t(x)) = \int_0^1 |\dot{g}_t(x)|\, dt$, we have

$$\mu(U)(L_h - 2\,\mathrm{diam}(U)) \le \int_U \int_0^1 |\dot{g}_t(x)|\, dt\, dx \le \int_M \int_0^1 |\dot{g}_t(x)|\, dt\, dx = l_1(\{g_t\}).$$

The isotopy g_t was arbitrary, therefore $\mu(U)(L_h - 2\,\mathrm{diam}(U)) \le l_1(f_h)$.

By assumption, for every r the set $S_h = \{h \in \pi_1(M) : l(h) < r\}$ is finite. Therefore, since $\pi_1(M)/Z(\pi_1(M))$ is infinite, there exists h such that the coset $hZ(\pi_1(M))$ does not intersect S_h. For such h we have $L_h \ge r$. Since the set U does not depend on the choice of h, and L_h can be arbitrary large, we conclude that the L^1–diameter of $\mathrm{Diff}_0(M, \mu)$ is infinite. □

In particular, Proposition 3.2 can be applied when (S, g) is a compact surface of negative Euler characteristic (then $\pi_1(S)$ is infinite and has trivial center). Unfortunately, it says nothing about the L^p–diameter of $\mathrm{Diff}_0(S, \text{area})$ for the remaining surfaces. Our main goal is to find an argument which is still based on the proof of Proposition 3.2, but works for any compact surface S.

To this end, one could pass to the configuration space of n ordered points in S, denoted by $C_n(S) \subset S^n$, with the product Riemannian metric g^n. Its fundamental group is the pure braid group $P_n(S)$, and $P_n(S)/Z(P_n(S))$ is infinite for every S if $n > 3$. However, the problem with this space is that every braid $P_n(S)$ can be represented as a based loop in $(C_n(S), g^n)$ of length at most $2n\,\mathrm{diam}(S) + 1$, thus one cannot apply Proposition 3.2.

We solve this problem by changing the metric on $C_n(S)$. We describe it, in a slightly more general setting, in the next section.

4 A complete metric on a manifold with removed submanifolds

Let (M, g) be a compact Riemannian manifold and let $D = \bigcup_{i=1}^k D_i$, where the D_i are submanifolds of M. The aim of this paragraph is to construct a metric on $M - D$ satisfying the following property: for every L the number of elements in $\pi_1(M - D)$

that can be represented by a based loop of length less then L is finite. For $x \in M$ denote by $d(x)$ the distance of x to D, that is

$$d(x) = d_g(x, D) = \min\{d_g(x, D_i) : i = 1, \ldots, k\},$$

where d_g is the metric on M induced by g.

Rescaling g by $1/d$ we define a new quadratic form g_b on the tangent space of $M-D$ by

$$|v|_{g_b} = \frac{|v|_g}{d(x)},$$

where $v \in T_x(M-D)$ is a vector tangent to a point $x \in M-D$.

Note that $d(x)$, and consequently g_b, are not differentiable. They are only continuous. In this case g_b is called a C^0–Riemannian metric and a smooth manifold with such a quadratic form is called a C^0–Riemannian manifold. A C^0–Riemannian structure allows us to define lengths of paths and a metric d on the underlying manifold. The topology induced by d is equal to the manifold topology.

Lemma 4.1 *$M-D$ with the metric g_b is a complete C^0–Riemannian manifold.*

Proof Let $N = (M-D, g_b)$ and let $B_N(x,r)$ denote the closed ball in N of radius r and center $x \in N$. To show completeness we must show that for every $x \in N$ the ball $B_N\left(x, \frac{1}{2}\right)$ is compact.

Let $x \in N$. We shall show that the distance from $B_N\left(x, \frac{1}{2}\right)$ to D is at least $\frac{1}{2}d(x)$:

$$B_N\left(x, \tfrac{1}{2}\right) \subset L := \left\{y \in N : d(y) \geq \tfrac{1}{2}d(x)\right\}.$$

Since L is compact, it follows that $B_N\left(x, \frac{1}{2}\right)$ is compact.

Suppose $y \in B_N\left(x, \frac{1}{2}\right)$ and $d(y) < d(x)$ (otherwise obviously $y \in L$). Let $\epsilon > 0$ and let $\gamma\colon [0,l] \to N$ be a path connecting x with y such that $|\dot{\gamma}(t)|_{g_b} = 1$ for $t \in [0,l]$ and $l < \frac{1}{2} + \epsilon$.

Let

$$t_0 = \sup\{t \in [0,l] : d(\gamma(t)) \geq d(x)\},$$

ie t_0 is the last time when $d(\gamma(t_0)) = d(x)$. For $t \geq t_0$, we have

$$|\dot{\gamma}(t)|_g = |\dot{\gamma}(t)|_{g_b} d(\gamma(t)) = d(\gamma(t)) \leq d(x).$$

Let γ' be the restriction of γ to the interval $[t_0, l]$. Let $l_g(\gamma')$ be the length of γ' in the metric g. Since $|\dot\gamma(t)|_g \le d(x)$, we have

$$l_g(\gamma') \le (l - t_0)d(x) \le \left(\tfrac{1}{2} + \epsilon\right)d(x).$$

Therefore the distance of y to D in g is at least

$$d(y) \ge d(\gamma(t_0)) - l_g(\gamma') \ge d(x) - \left(\tfrac{1}{2} + \epsilon\right)d(x) = \tfrac{1}{2}d(x) - \epsilon d(x).$$

Since ϵ is arbitrarily small, $y \in L$ and therefore $B_N\left(x, \frac{1}{2}\right) \subset L$. $\square$

Before we proceed we need the following simple lemma. Note that this lemma would be standard if $(M - D, g_b)$ were a complete Riemannian manifold.

Lemma 4.2 *Let $N = (M - D, g_b)$ and let $\widetilde{N}$ be the universal cover of N with the pulled-back C^0–Riemannian metric. Then every closed ball in $\widetilde{N}$ is compact.*

Proof By the Weierstrass approximation theorem there exists $C \in \mathbb{R}$ and a smooth function $f\colon N \to \mathbb{R}$ such that $C^{-1}f(x) < 1/d(x) < Cf(x)$ for every $x \in N$. Let g_s be a Riemannian metric defined by $|v|_{g_s} = f(x)|v|_g$, where $v \in T_xN$. Then $C^{-1}|v|_{g_s} < |v|_{g_b} < C|v|_{g_s}$, thus the metrics induced by g_b and g_s are equivalent. By Lemma 4.1, (N, g_s) is a complete Riemannian manifold and it is a standard fact that closed balls in the universal cover of (N, g_s) are compact. Clearly it holds as well for (N, g_b), since the metrics defined by pullbacks of g_s and g_b to the universal cover are equivalent. $\square$

Let $h \in \pi_1(M - D)$. Denote by $l(h)$ the infimum of lengths (with respect to g_b) of based loops representing $h \in \pi_1(M - D)$.

Lemma 4.3 *For every r, the set $\{h \in \pi_1(M - D) : l(h) < r\}$ is finite.*

Proof Let $N = (M - D, g_b)$, let $x \in N$ be a basepoint and let $p\colon \widetilde{N} \to N$ be the universal cover of N. Choose $y \in p^{-1}(x)$. The preimage $p^{-1}(x)$ is discrete and $B_{\widetilde{N}}(y, r) \subset \widetilde{N}$ is compact by Lemma 4.2. Thus $p^{-1}(x) \cap B_{\widetilde{N}}(y, r)$ is finite for every r and therefore $\{h \in \pi_1(N) : l(h) < r\}$ is finite. $\square$

5 A Lipschitz embedding

In this section we focus on the particular case where $M - D$ is a configuration space. Let (S, g) be a compact Riemannian surface and g^n be the product metric on S^n.

Let $D_{ij} = \{(x_1, \dots, x_n) \in S^n : x_i = x_j\}$. Denote by $C_n(S) = S^n - \bigcup_{i,j} D_{ij}$ the configuration space of n ordered points in S. On S^n and $C_n(S)$ we consider the measure induced by the product metric g^n.

We shall now find a formula for $d_{g^n}(x, D_{ij})$ in terms of the metric on S. Let $x = (x_1, \dots, x_n) \in S^n$ and let m be the midpoint of a geodesic connecting x_i with x_j. If we start moving points x_i and x_j towards m with constant speed, we get a geodesic in S^n connecting x with the closest point in D_{ij}. Since $d_g(m, x_i) = d_g(m, x_j) = \frac{1}{2}\, d_g(x_i, x_j)$ and we are in the product metric,

$$d_{g^n}(x, D_{ij}) = \sqrt{d_g(m, x_i)^2 + d_g(m, x_j)^2} = \frac{1}{\sqrt{2}}\, d_g(x_i, x_j).$$

The distance function d has the form

$$d(x) = \frac{1}{\sqrt{2}} \min\{d_g(x_i, x_j) : 1 \le i < j \le n\}.$$

Let $g_b = (g^n)_b$ be the metric on $C_n(S)$ defined in the previous section, namely $|v|_{g_b} = |v|_{g^n}/d(x)$, where $v \in T_x(C_n(S))$.

Let us fix a point $p \in S$ and let $x = (x_1, \dots, x_{n-1}) \in S^{n-1}$. Then $(p, x) \in S^n$ and $d((p, x))$ is the minimum over $(1/\sqrt{2}) d_g(p, x_i)$ for $1 \le i \le n-1$ and $(1/\sqrt{2}) d_g(x_i, x_j)$ for $1 \le i < j \le n-1$.

We need the following technical lemma.

Lemma 5.1 *There exists $C \in \mathbb{R}$ such that for every $p \in S$ we have*

$$\int_{S^{n-1}} \frac{1}{d((p, x))}\, dx \le C.$$

Proof It can be easily seen using polar coordinates that there exists C' such that for every $p \in D^2$, where D^2 is the euclidean disc,

$$\int_D \frac{1}{|p - x|}\, dx < C'.$$

Since such C' exists for a disc, we have a similar bound for every compact surface S: for every $p \in S$

$$\int_S \frac{1}{d_g(p, x)}\, dx < C'.$$

After integrating over all possible $p \in S$ (we assume area$(S) = 1$),

$$\int_{S^2} \frac{1}{d_g(p, x)}\, dp\, dx < C'.$$

Let $x = (x_1, \dots, x_{n-1})$. Since $d((p,x))$ is the minimum over $(1/\sqrt{2})d_g(p, x_i)$ for $i = 1, \dots, n-1$ and $(1/\sqrt{2})d_g(x_i, x_j)$ for $1 \le i < j \le n-1$,

$$\frac{1}{d((p,x))} \le \sum_i \frac{\sqrt{2}}{d_g(p, x_i)} + \sum_{i \ne j} \frac{\sqrt{2}}{d_g(x_i, x_j)}.$$

Thus

$$\begin{aligned}\int_{S^{n-1}} \frac{1}{d((p,x))}\,dx &\le \sum_i \int_{S^{n-1}} \frac{\sqrt{2}}{d_g(p,x_i)}\,dx + \sum_{i\ne j}\int_{S^{n-1}} \frac{\sqrt{2}}{d_g(x_i,x_j)}\,dx \\ &= (n-1)\int_S \frac{\sqrt{2}}{d_g(p,x)}\,dx + \tfrac{1}{2}n(n-1)\int_{S^2} \frac{\sqrt{2}}{d_g(x_1,x_2)}\,dx_1\,dx_2 \\ &\le \sqrt{2}(n-1)C' + \frac{n(n-1)}{\sqrt{2}}C' =: C. \qquad \square\end{aligned}$$

Let μ be the measure on $C_n(S)$ induced by the product metric g^n. A diffeomorphism $f \in \mathrm{Diff}_0(S, \text{area})$ defines a product diffeomorphism $f_* \in \mathrm{Diff}_0(C_n(S), \mu)$. Namely, for $x = (x_1, \dots, x_n) \in S^n$ we have $f_*(x) = (f(x_1), \dots, f(x_n))$. Thus we have a product embedding $\mathrm{Diff}_0(S, \text{area}) \hookrightarrow \mathrm{Diff}_0(C_n(S), \mu)$.

On $\mathrm{Diff}_0(C_n(S), \mu)$ we consider the L^1–norm defined by the metric g_b and the measure μ. Note that here we are in the case where g_b and μ are not compatible, that is, the measure induced by g_b and the measure μ are different.

The following lemma provides a link between the L^1–norm on $\mathrm{Diff}_0(S, \text{area})$ and the L^1–norm on $\mathrm{Diff}_0(C_n(S), \mu)$ defined above. Note that in the proof it is essential that f preserves the area on S.

Lemma 5.2 *The product embedding* $\mathrm{Diff}_0(S, \text{area}) \hookrightarrow \mathrm{Diff}_0(C_n(S), \mu)$ *is Lipschitz, ie there exists C such that $l_1(f_*) \le C l_1(f)$.*

Proof Let $f \in \mathrm{Diff}_0(S, \text{area})$ and let $X\colon S \to TS$ such that $X(x) \in T_{f(x)}S$. For $x = (x_1, \dots, x_n) \in C_n(S)$ we define $X_*(x) = (X(x_1), \dots, X(x_n)) \in T_{f_*(x)}C_n(S)$.

The set $\bigcup_{i,j} D_{ij} \subset S^n$ is of measure zero. This means that we can regard $|X_*(x)|_{g_b}$ as a measurable function defined on S^n. Thus in what follows, we integrate $|X_*(x)|_{g_b}$ over S^n with the product measure rather then over $C_n(S)$.

To prove the lemma it is enough to show that there exists C such that for every $f \in \mathrm{Diff}_0(S, \text{area})$ and every map $X\colon S \to TS$ such that $X(x) \in T_{f(x)}S$ the following inequality holds:

$$\|X_*\|_1 \le C\|X\|_1.$$

Recall that by definition $\|X_*\|_1 = \int_{S^n} |X_*(x)|_{g_b}\, dx$. We have

$$\int_{S^n} |X_*(x)|_{g_b}\, dx = \int_{S^n} \frac{|X_*(x)|_{g^n}}{d(f_*(x))}\, dx = \int_{S^n} \frac{\sqrt{|X(x_1)|_g^2 + \cdots + |X(x_n)|_g^2}}{d(f_*(x))}\, dx$$
$$\leq \int_{S^n} \frac{|X(x_1)|_g + \cdots + |X(x_n)|_g}{d(f_*(x))}\, dx = n \int_{S^n} \frac{|X(x_1)|_g}{d(f_*(x))}\, dx.$$

Since f_* preserves the measure on S^n,

$$\begin{aligned}\int_{S^n} \frac{|X(x_1)|_g}{d(f_*(x))}\, dx &= \int_{S^n} \frac{|X \circ f^{-1}(x_1)|_g}{d(x)}\, dx \\ &= \int_S |X \circ f^{-1}(x_1)|_g \left(\int_{S^{n-1}} \frac{1}{d(x_1, x)}\, dx \right) dx_1 \\ &\leq C \int_S |X \circ f^{-1}(x_1)|_g\, dx_1 && \text{(by Lemma 5.1)} \\ &= C \int_S |X(x_1)|_g dx_1 = C\|X\|_1. \end{aligned}$$

□

6 Proof of the theorem

Theorem 1 *Let (S, g) be a compact surface (with or without boundary). Then for every $p \geq 1$ the L^p–diameter of $\mathrm{Diff}_0(S, \text{area})$ is infinite.*

Proof By the Hölder inequality we can assume $p = 1$. Fix $n > 3$.

Let $z = (z_1, \dots, z_n) \in C_n(S)$ and let $P_n(S) = \pi_1(C_n(S), z)$ denote the pure braid group on n strings. Suppose $U_i \subset S$ are disjoint discs such that $z_i \in U_i$, then let $U = U_1 \times U_2 \times \cdots \times U_n \subset C_n(S)$.

Choose $h \in P_n(S)$ and γ a loop in $C_n(S)$ representing h. Let $f_t \in \mathrm{Diff}_0(S, \text{area})$ for $t \in [0, 1]$ be an isotopy such that $(f_t)_* \in \mathrm{Diff}_0(C_n(S), \mu)$ moves U all the way along γ and has properties (1) and (2) from the proof of Proposition 3.2. Let $f_h = f_1$.

It is convenient to imagine that f_t moves U_i along the trajectory of z_i given by γ. In fact, to construct f_t satisfying the above properties for a general $h \in P_n(S)$, it is enough to do it for a given finite set of generators of $P_n(S)$ (or generators of the full braid group $B_n(S)$). In [2] one can find a set of generators of $B_n(S)$ for which the construction of f_t is straightforward.

Recall that on $C_n(S)$ we consider the complete metric g_b. By Lemma 4.3, the set $\{h \in \pi_1(C_n(S)) : l(h) < r\}$ is finite for every r and $P_n(S)/Z(P_n(S))$ is infinite. It follows from the proof of Proposition 3.2 that $l_1((f_h)_*)$ can be arbitrarily large.

Therefore, due to Lemma 5.2, $l_1(f_h)$ can be arbitrarily large. Thus the L^1–diameter of $\mathrm{Diff}_0(S, \mathrm{area})$ is infinite. □

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Extension DGAs and topological Hochschild homology

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We study differential graded algebras (DGAs) that arise from ring spectra through the extension of scalars functor. Namely, we study DGAs whose corresponding Eilenberg–Mac Lane ring spectrum is equivalent to $H\mathbb{Z} \wedge E$ for some ring spectrum E. We call these DGAs extension DGAs. We also define and study this notion for E_∞ DGAs.

The topological Hochschild homology (THH) spectrum of an extension DGA splits in a convenient way. We show that formal DGAs with nice homology rings are extension, and therefore their THH groups can be obtained from their Hochschild homology groups in many cases of interest. We also provide interesting examples of DGAs that are not extension.

In the second part, we study properties of extension DGAs. We show that, in various cases, topological equivalences and quasi-isomorphisms agree for extension DGAs. From this, we obtain that dg Morita equivalences and Morita equivalences also agree in these cases.

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1 Introduction

In [27], Stanley shows that the homotopy category of differential graded algebras is equivalent to the homotopy category of $H\mathbb{Z}$–algebras. Later, Shipley [26] improves this equivalence to a zigzag of Quillen equivalences between the model categories of DGAs and $H\mathbb{Z}$–algebras. This opens up a new opportunity to study DGAs, ie to study DGAs using ring spectra.

Dugger and Shipley [9] use this zigzag of Quillen equivalences to define new equivalences between DGAs called topological equivalences; see Definition 1.10 below. They show nontrivial examples of topologically equivalent DGAs and they use topological equivalences to develop a Morita theory for DGAs. In [2], the author uses topological equivalences to obtain classification results for DGAs. Moreover, topological equivalences for E_∞ DGAs are studied by the author in [1].

In this work, we follow this philosophy in a different way. We study what we call extension DGAs which are the DGAs that are obtained from ring spectra through the extension of scalars functor from $\mathbb{S}$–algebras to $H\mathbb{Z}$–algebras, ie the functor $H\mathbb{Z} \wedge -$. More generally, we work in R–DGAs for a discrete commutative ring R. There is a zigzag of Quillen equivalences between R–DGAs and HR–algebras [26]. Composing the corresponding derived functors, one obtains a functor from the category of R–DGAs to the category of HR–algebras. For a given R–DGA X, we let HX denote the HR–algebra corresponding to X under this composite functor. We often abuse notation and denote a cofibrant and fibrant replacement of HX also by HX.

Definition 1.1 An R–DGA X is *R–extension* if the HR–algebra corresponding to X is weakly equivalent to $HR \wedge E$ for some cofibrant $\mathbb{S}$–algebra E. For $R = \mathbb{Z}$, we omit $\mathbb{Z}$ and write extension instead of $\mathbb{Z}$–extension.

To define R–extension E_∞ R–DGAs, we use the zigzag of Quillen equivalences between E_∞ R–DGAs and commutative HR–algebras constructed by Richter and Shipley in [19]. As before, composing the corresponding derived functors, one obtains a functor from the category of E_∞ R–DGAs to the category of commutative HR–algebras. For a given E_∞ R–DGA X, the corresponding commutative HR–algebra, which we denote by $H_{E_\infty}X$, is obtained by applying this composite functor to X. Again, we often abuse notation and denote a cofibrant and fibrant replacement of $H_{E_\infty}X$ also by $H_{E_\infty}X$.

Definition 1.2 An E_∞ R–DGA X is *R–extension* if the commutative HR–algebra corresponding to X is weakly equivalent to $HR \wedge E$ for some cofibrant commutative $\mathbb{S}$–algebra E. For $R = \mathbb{Z}$, we omit $\mathbb{Z}$ and write extension instead of $\mathbb{Z}$–extension.

See Appendix A for a discussion on the compatibility of the two definitions above.

Although we only study the extension problems coming from the definitions above, it is also interesting to consider the following general extension problem. Let $\varphi\colon A \to B$ be a map of commutative $\mathbb{S}$–algebras and let X be a B–algebra. We say X is φ–extension if it is weakly equivalent to $B \wedge_A E$ for some cofibrant A–algebra E. For the map $\mathbb{S} \to HR$, this corresponds to the extension problem coming from Definition 1.1.

Let k be a perfect field of characteristic p and let $W(k)$ denote the Witt ring of k. The extension problem corresponding to the canonical map $\phi\colon HW(k) \to Hk$ is analogous to a classical lifting problem for schemes; see for instance Grothendieck [11, Section 6] and Serre [24]. One of the motivations for the classical Witt-lifting problem is to understand the crystalline cohomology of smooth algebraic varieties over $\mathbb{F}_p$ through the de Rham cohomology of their lifts to $W(\mathbb{F}_p)$ whenever such a lift exists; see Berthelot [5, V.2.3.2]. Following this philosophy, Petrov and Vologodsky [18] recently showed that if an Hk–algebra (ie a k–DGA) X is ϕ–extension, ie $X \simeq Hk \wedge_{HW(k)} E$ for some cofibrant $HW(k)$–algebra (ie a $W(k)$–DGA) E, then the p–completed periodic topological cyclic homology of X agrees with the p–completed periodic cyclic homology of E when $p > 2$. This boils down the computation of a topological homology theory to the computation of an algebraic homology theory.

Similarly, the extension property we study in this work boils down the computation of the topological Hochschild homology of an extension DGA to a Hochschild homology computation. Namely, for an R–extension DGA X (as in Definition 1.1), we have the following splitting at the level of topological Hochschild homology. This splitting is possibly well known to the experts in the field; see Schwänzl, Vogt and Waldhausen [20, Theorem 1] for an instance of this splitting when X is the Eilenberg–Mac Lane spectrum of a discrete ring. In the proposition below, $\mathrm{HH}^R(-)$ denotes $\mathrm{THH}^{HR}(-)$.

Proposition 1.3 *If X is an R–extension R–DGA, then there is an equivalence of spectra*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

If X is an R–extension E_∞ R–DGA, then the equivalence above is an equivalence of commutative $\mathbb{S}$–algebras.

For a map $\varphi\colon A \to B$ of commutative $\mathbb{S}$–algebras, there is a similar splitting of $\mathrm{THH}^A(X)$ whenever X is a φ–extension B–algebra; see Proposition B.1.

The splitting in Proposition 1.3 simplifies THH calculations significantly in many situations. Indeed, it is an important stepping stone in many THH calculations in the literature, particularly for the case where X is a discrete ring, ie a DGA whose homology is concentrated in degree 0. For example, Larsen and Lindenstrauss [16] show that this splitting exists at the level of homotopy groups for various discrete rings of characteristic p. Furthermore, Hesselholt and Madsen [12, Theorem 7.1] prove such a splitting for discrete rings that have a nice basis with respect to the ground ring R. In the following theorem, we generalize this result to connective formal DGAs. Note that a connective DGA is a DGA whose negative homology is trivial.

Theorem 1.4 *Let X be a connective formal R–DGA whose homology has a homogeneous basis as an R–module containing the multiplicative unit such that the multiplication of two basis elements is either zero or a basis element. In this situation, X is R–extension. As a result, we have the equivalence of spectra*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

Section 5 is devoted to the proof of this theorem. Furthermore, for a given R–DGA that satisfies the hypothesis of the theorem above, we provide an explicit description of the corresponding HR–algebra; see Proposition 5.8. The author and Moulinos [3, 4.8 and 6.1] show that for such HR–algebras, one often obtains nontrivial splittings at the level of topological negative cyclic homology and topological periodic homology. Using these splittings, the author and Moulinos compute the algebraic K–theory of $\mathrm{THH}(H\mathbb{F}_p)$, ie the algebraic K–theory of the formal DGA with homology $\mathbb{F}_p[x_2]$. In a future work, the author plans to compute the algebraic K–theory groups of various formal DGAs by using Proposition 5.8 and the splittings provided in [3].

Remark 1.5 Another way to state the hypothesis of Theorem 1.4 is the following. Let M be a monoid in the category of graded pointed sets. From M, one obtains a graded R–algebra $R\langle M\rangle$ whose underlying R–module is the free R–module over the graded set M_- obtained by removing the based point from M. The multiplication on $R\langle M\rangle$ is given by the multiplication on M where the based point of M is considered as the zero element in $R\langle M\rangle$. A graded R–algebra of the form $R\langle M\rangle$ is called a graded monoid R–algebra. With this definition, a connective formal R–DGA satisfies the hypothesis of Theorem 1.4 if and only if its homology is a graded monoid R–algebra.

Remark 1.6 We mention a few examples of graded rings that satisfy the hypothesis of the theorem above as homology of X. The polynomial algebra over R with a nonnegatively graded set S of generators $R[S]$ satisfies the hypothesis if all the elements of S are in even degrees. The basis of $R[S]$ is given by the monomials in S and the unit $1 \in R$. Similarly, many examples of quotients of polynomial rings with even degree generators also satisfy this hypothesis; for example $R[x]/(x^2)$, $R[x,y]/(y^2)$ and $R[x,y]/(x^2y, y^3)$ with even $|x|$ and $|y|$. However, there are rings that do not satisfy this hypothesis. For example, for $R = \mathbb{Z}$, the exterior algebra on two generators $\Lambda[x,y] \cong \Lambda[x] \otimes \Lambda[y]$ with odd $|x|$ and $|y|$ has a basis given by $\{x, y, xy\}$, but $yx = -xy$ and therefore yx is not one of the basis elements. Indeed, $\Lambda[x,y]$ has no basis that satisfies this hypothesis.

We prove the following nonextension results.

Theorem 1.7 *Let Y be an E_∞ DGA. For all primes p, if Y is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA then Y is not an extension E_∞ DGA.*

Theorem 1.8 *Let X be a DGA. If X is quasi-isomorphic to an $\mathbb{F}_2$–DGA then X is not an extension DGA.*

Remark 1.9 These theorems should be compared with the two commutative $H\mathbb{Z}$–algebras X and Y obtained from $H\mathbb{Z} \wedge H\mathbb{F}_p$ through the structure maps

$$H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} \to H\mathbb{Z} \wedge H\mathbb{F}_p \quad \text{and} \quad H\mathbb{Z} \cong \mathbb{S} \wedge H\mathbb{Z} \to \mathbb{S} \wedge H\mathbb{F}_p \to H\mathbb{Z} \wedge H\mathbb{F}_p,$$

respectively. The E_∞ DGA corresponding to X is an extension E_∞ DGA and the E_∞ DGA corresponding to Y is an E_∞ $\mathbb{F}_p$–DGA. Although these two E_∞ DGAs are E_∞ topologically equivalent, they are not quasi-isomorphic due to [1, Theorem 5.3]. For the associative case with $p = 2$, the distinction between the two DGAs corresponding to X and Y is due to [9, Example 5.6].

In the results above, we work with (E_∞) DGAs in mixed characteristic, ie we work in (E_∞) $\mathbb{Z}$–DGAs. A natural question to ask is if there are examples of E_∞ k–DGAs that are not k–extension for a field k. In Example 1.12 below, we show that there are E_∞ $\mathbb{F}_p$–DGAs that are not $\mathbb{F}_p$–extension.

Now we discuss topological equivalences of DGAs and the properties of extension DGAs regarding topological equivalences.

Definition 1.10 Two DGAs X and Y are *topologically equivalent* if the corresponding $H\mathbb{Z}$–algebras HX and HY are weakly equivalent as $\mathbb{S}$–algebras.

The definition of E_∞ topological equivalences is as follows.

Definition 1.11 Two E_∞ DGAs X and Y are E_∞ *topologically equivalent* if the corresponding *commutative* $H\mathbb{Z}$–algebras $H_{E_\infty}X$ and $H_{E_\infty}Y$ are weakly equivalent as *commutative* $\mathbb{S}$–algebras.

It follows from these definitions that quasi-isomorphic (E_∞) DGAs are (E_∞) topologically equivalent. However, there are examples of nontrivially topologically equivalent DGAs, ie DGAs that are topologically equivalent but not quasi-isomorphic [9]. Furthermore, examples of nontrivially E_∞ topologically equivalent E_∞ DGAs are constructed by the author in [1].

Example 1.12 This is an example of E_∞ $\mathbb{F}_p$–DGAs that are not $\mathbb{F}_p$–extension. In [1, Example 5.1], the author constructs nontrivially E_∞ topologically equivalent E_∞ $\mathbb{F}_p$–DGAs that we call X and Y, ie X and Y are E_∞ topologically equivalent but they are not quasi-isomorphic. Although these E_∞ $\mathbb{F}_p$–DGAs are E_∞ topologically equivalent, their Dyer–Lashof operations are different.

For $p=2$, the homology rings of these E_∞ $\mathbb{F}_p$–DGAs are given by

$$\mathbb{F}_2[x]/(x^4)$$

for both X and Y where $|x|=1$. On the homology of X, the first Dyer–Lashof operation is trivial, ie $\mathrm{Q}^1 x = 0$. On the other hand, we have $\mathrm{Q}^1 x = x^3$ on the homology of Y. Using these properties we show (for all primes) that these E_∞ $\mathbb{F}_p$–DGAs are not $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGAs. See Section 3B for a proof of this fact.

By [1, Theorem 1.6], E_∞ topological equivalences between E_∞ $\mathbb{F}_p$–DGAs with trivial first homology preserve Dyer–Lashof operations. We prove a stronger result for $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGAs.

Theorem 1.13 *Let X be an $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGA with $H_1 X = 0$ and let Y be an E_∞ $\mathbb{F}_p$–DGA. Then X and Y are quasi-isomorphic if and only if they are E_∞ topologically equivalent.*

In the following results, we show various situations where topological equivalences and quasi-isomorphisms agree.

Theorem 1.14 *Let Y be an $\mathbb{F}_p$–DGA and let X be an $\mathbb{F}_p$–extension $\mathbb{F}_p$–DGA. For odd p, assume that the homology of X is trivial in degrees $2p^r-2$ for $r \geq 1$ and $2p^s-1$ for $s \geq 0$. For $p=2$, assume that the homology of X is trivial in degree 2^r-1 for $r \geq 1$. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

For the corollary below, note that a coconnective DGA is a DGA with trivial homology in positive degrees.

Corollary 1.15 *Let X be a coconnective extension $\mathbb{F}_p$–DGA and let Y be an $\mathbb{F}_p$–DGA. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Theorem 1.16 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$ and let X be an R–DGA whose corresponding HR–algebra is equivalent to $HR \wedge Z$ for some cofibrant $\mathbb{S}$–algebra Z whose underlying spectrum is equivalent to a coproduct of (de)suspensions of the sphere spectrum. Also, let Y be an R–DGA. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Our main interest for this theorem is due to its corollary stated below. This follows by Proposition 5.8 which implies that an R–DGA that satisfies the hypothesis of Theorem 1.4 also satisfies the hypothesis of the theorem above.

Corollary 1.17 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$, let Y be an R–DGA and let X be as in Theorem 1.4. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Two DGAs X and Y are said to be *Morita equivalent* if the model categories of X–modules and Y–modules are Quillen equivalent. There is a stronger notion of Morita equivalence for DGAs called *dg Morita equivalences* defined by Keller [14, Section 3.8]. Due to [9, 7.7], two DGAs X and Y are dg Morita equivalent if and only if the model categories of X–modules and Y–modules are *additively* Quillen equivalent; see Dugger and Shipley [8] for the definition of additive Quillen equivalences. This is a strictly stronger notion of Morita equivalence since there are examples of DGAs that are Morita equivalent but not dg Morita equivalent [9, Section 8]. However, in the situations where topological equivalences and quasi-isomorphisms agree, these two notions of Morita equivalences also agree [9, Proposition 7.7 and Theorem 7.2]. We obtain the following corollary to Theorems 1.14 and 1.16.

Corollary 1.18 *Assume that X and Y are as in Theorem 1.14 or Theorem 1.16. Then X and Y are Morita equivalent if and only if they are dg Morita equivalent.*

Organization In Section 2, we describe the dual Steenrod algebra and the Dyer–Lashof operations on it. In Section 3, we prove Theorems 1.13, 1.14 and 1.16. Section 4 is devoted to the proof of Theorems 1.7 and 1.8. In Section 5, we prove Theorem 1.4. This section is independent from Sections 2, 3 and 4, and it contains explicit descriptions of the $H\mathbb{Z}$–algebras corresponding to the formal DGAs as in Theorem 1.4, which is of independent interest. We leave the proof of Theorem 1.4 to the end because it uses different tools than the rest of the proofs in this work. Appendix A is devoted to a discussion on the compatibility of Definitions 1.1 and 1.2.

Terminology We work in the setting of symmetric spectra in simplicial sets; see Hovey, Shipley and Smith [13]. For commutative ring spectra, we use the positive $\mathbb{S}$–model structure developed by Shipley in [25]. When we work in the setting of associative ring spectra, we use the stable model structure of [13]. Throughout this work, R denotes a general discrete commutative ring except in Section 3C where R denotes a quotient of $\mathbb{Z}$. When we say (E_∞) DGA, we mean (E_∞) $\mathbb{Z}$–DGA.

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2 The dual Steenrod algebra

Here, we recall the ring structure and the Dyer–Lashof operations on the dual Steenrod algebra. Using the standard notation, we denote the dual Steenrod algebra by $\mathcal{A}_*$. We have $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) \cong \mathcal{A}_*$. Milnor shows that the dual Steenrod algebra is a free graded commutative $\mathbb{F}_p$–algebra [17].

For $p = 2$, $\mathcal{A}_*$ is given by

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r \mid r \geq 1] = \mathbb{F}_2[\zeta_r \mid r \geq 1],$$

where $|\xi_r| = |\zeta_r| = 2^r - 1$. Let χ denote the action of the transpose map of the smash product on $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$. We have $\chi(\xi_r) = \zeta_r$.

For an odd prime p, the dual Steenrod algebra is described by

$$\mathcal{A}_* = \mathbb{F}_p[\xi_r \mid r \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\tau_s \mid s \geq 0) = \mathbb{F}_p[\zeta_r \mid r \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\bar{\tau}_s \mid s \geq 0),$$

where $|\xi_r| = |\zeta_r| = 2(p^r - 1)$ and $|\tau_s| = |\bar{\tau}_s| = 2p^s - 1$. In this case, we have $\chi(\xi_r) = \zeta_r$ and $\chi(\tau_r) = \bar{\tau}_r$.

Dyer–Lashof operations are power operations that act on the homotopy rings of H_∞ $H\mathbb{F}_p$–algebras [7]. By forgetting structure, commutative $H\mathbb{F}_p$–algebras are examples of H_∞ $H\mathbb{F}_p$–algebras and therefore Dyer–Lashof operations are also defined on the homotopy ring of commutative $H\mathbb{F}_p$–algebras, and maps of commutative $H\mathbb{F}_p$–algebras preserve these operations. For $p = 2$, there is a Dyer–Lashof operation denoted by Q^s for ever integer s where Q^s increases the degree by s. For odd p, there are

Dyer–Lashof operations denoted by βQ^s and Q^s for every integer s that increase the degree by $2s(p-1)-1$ and $2s(p-1)$, respectively. See [7, III.1.1] for further properties of these operations.

With the unit map

$$H\mathbb{F}_p \cong H\mathbb{F}_p \wedge \mathbb{S} \to H\mathbb{F}_p \wedge H\mathbb{F}_p,$$

$H\mathbb{F}_p \wedge H\mathbb{F}_p$ is a commutative $H\mathbb{F}_p$–algebra and therefore Dyer–Lashof operations are defined on the dual Steenrod algebra. These operations are first studied in [7, III.2]. Steinberger shows that the degree one element τ_0 for odd p and ξ_1 for $p=2$ generates the dual Steenrod algebra as an algebra with Dyer–Lashof operations, ie as an algebra over the Dyer–Lashof algebra. In particular for $p=2$, we have

$$Q^{2^s-2}\xi_1 = \zeta_s \quad \text{for } s > 1.$$

For odd p, we have

$$Q^{(p^s-1)/(p-1)}\tau_0 = (-1)^s \bar{\tau}_s, \quad \beta Q^{(p^s-1)/(p-1)}\tau_0 = (-1)^s \zeta_s$$

for $s \geq 1$.

3 Proof of the results on topological equivalences and the nonextension example

In this section, we prove Theorems 1.13, 1.14 and 1.16 which provide comparison results on (E_∞) topological equivalences and quasi-isomorphisms of (E_∞) DGAs for various cases. At the end, we prove Proposition 3.2 which justifies the last claim in Example 1.12. This provides examples of E_∞ $\mathbb{F}_p$–DGAs that are not $\mathbb{F}_p$–extension.

These results are obtained using similar arguments. Therefore, we suggest the reader to go through their proof in the order presented in this section.

3A Proof of Theorems 1.13 and 1.14

In the proof of Theorems 1.13 and 1.14 and also in the proof of Theorem 1.16 and Proposition 3.2, we show that for various R–extension (E_∞) R–DGAs, (E_∞) topological equivalences and quasi-isomorphisms agree.

For this, we use the same technique to produce a quasi-isomorphism, ie an HR–algebra equivalence, out of a given topological equivalence, ie an $\mathbb{S}$–algebra equivalence. We start by describing this technique.

Let us focus on the E_∞ case. Assume that we are given commutative HR–algebras Y and $HR \wedge Z$, where Z denotes a cofibrant commutative $\mathbb{S}$–algebra and assume that we are given a weak equivalence

$$\varphi\colon HR \wedge Z \xrightarrow{\sim} Y$$

of commutative $\mathbb{S}$–algebras. Using φ, we produce a map of commutative HR–algebras through the composite

$$\psi\colon HR \wedge Z \cong HR \wedge \mathbb{S} \wedge Z \xrightarrow{i} HR \wedge HR \wedge Z \xrightarrow[\simeq]{HR\wedge\varphi} HR \wedge Y \xrightarrow{m} Y.$$

Here, i is the canonical map induced by the unit map $\mathbb{S} \to HR$ of HR and m is the commutative HR–algebra structure map of Y. Except Y, we provide the objects in the composite above with the commutative HR–algebra structure coming from the first HR factor. The maps i and $HR \wedge \varphi$ are maps of commutative HR–algebras as they are obtained using the functor $HR \wedge -$ from the category of commutative $\mathbb{S}$–algebras to the category of commutative HR–algebras. Furthermore, we assume that HR is cofibrant as a commutative $\mathbb{S}$–algebra in the positive $\mathbb{S}$–model structure of [25]. This implies that HR is cofibrant as an $\mathbb{S}$–module [25, 4.1] in the model structure of [25], ie HR is $\mathbb{S}$–cofibrant in the terminology of [13, 5.3.6]. Therefore, $HR \wedge \varphi$ is a weak equivalence [13, 5.3.10]. Note that m is the left adjoint of the identity map of Y under the adjunction between the categories of commutative $\mathbb{S}$–algebras and commutative HR–algebras whose left adjoint is given by the extension of scalars functor $HR \wedge -$ and whose right adjoint is given by the restriction of scalars functor. In particular, this shows that m is also a map of commutative HR–algebras. We deduce that ψ is a map of commutative HR–algebras as it is given by a composite of such maps. Compared to the commutative case, the definition of the map ψ is slightly more complicated in the associative case as we consider various cofibrant replacements. The results we prove in this section are obtained by showing that ψ is an equivalence under the given hypothesis.

We start with the proof of Theorem 1.13. We provide a restatement of this theorem below.

Theorem 1.13 *Let X be an $\mathbb{F}_p$–extension E_∞ $\mathbb{F}_p$–DGA with $H_1 X = 0$ and let Y be an E_∞ $\mathbb{F}_p$–DGA. Then X and Y are quasi-isomorphic if and only if they are E_∞ topologically equivalent.*

In what follows, we denote the category of commutative E–algebras by $E\text{–}c\mathcal{A}lg$ and the category of associative E–algebras by $E\text{–}\mathcal{A}lg$ for a given commutative ring spectrum E.

Proof Since quasi-isomorphic E_∞ DGAs are always E_∞ topologically equivalent, we only need to show that if X and Y are E_∞ topologically equivalent then they are quasi-isomorphic as E_∞ $\mathbb{F}_p$–DGAs.

Let $H\mathbb{F}_p$ denote a cofibrant model of $H\mathbb{F}_p$ in $\mathbb{S}$–$c\mathcal{A}lg$. The category of commutative $H\mathbb{F}_p$–algebra spectra is the same as the category of commutative $\mathbb{S}$–algebra spectra under $H\mathbb{F}_p$. Therefore we have a model structure on $H\mathbb{F}_p$–$c\mathcal{A}lg$ where the cofibrations, fibrations and weak equivalences are precisely the maps that forget to cofibrations, fibrations and weak equivalences in $\mathbb{S}$–$c\mathcal{A}lg$. We let Y also denote the commutative $H\mathbb{F}_p$–algebra corresponding to the E_∞ DGA Y. Therefore $\pi_1(Y)=0$. Taking a fibrant replacement, we assume Y is fibrant both in $H\mathbb{F}_p$–$c\mathcal{A}lg$ and in $\mathbb{S}$–$c\mathcal{A}lg$. Furthermore, we let $H\mathbb{F}_p\wedge Z$ denote the commutative $H\mathbb{F}_p$–algebra corresponding to the extension E_∞ $\mathbb{F}_p$–DGA X, where Z is a cofibrant object in $\mathbb{S}$–$c\mathcal{A}lg$. This ensures that $H\mathbb{F}_p\wedge Z$ is cofibrant in $H\mathbb{F}_p$–$c\mathcal{A}lg$. Therefore the composite $\mathbb{S}\rightarrowtail H\mathbb{F}_p\rightarrowtail H\mathbb{F}_p\wedge Z$ is also a cofibration in $\mathbb{S}$–$c\mathcal{A}lg$; this shows that $H\mathbb{F}_p\wedge Z$ is also cofibrant in $\mathbb{S}$–$c\mathcal{A}lg$. To prove Theorem 1.13, we need to show that $H\mathbb{F}_p\wedge Z$ and Y are weakly equivalent in $H\mathbb{F}_p$–$c\mathcal{A}lg$.

Because $H\mathbb{F}_p\wedge Z$ and Y are obtained from E_∞ topologically equivalent E_∞ DGAs, they are equivalent as commutative $\mathbb{S}$–algebras. Furthermore $H\mathbb{F}_p\wedge Z$ is cofibrant and Y is fibrant; therefore there is a weak equivalence $\varphi\colon H\mathbb{F}_p\wedge Z\xrightarrow{\sim} Y$ of commutative $\mathbb{S}$–algebras. We consider the composite map

$$(1)\quad \psi\colon H\mathbb{F}_p\wedge Z\cong H\mathbb{F}_p\wedge\mathbb{S}\wedge Z\xrightarrow{i} H\mathbb{F}_p\wedge H\mathbb{F}_p\wedge Z\xrightarrow[\simeq]{H\mathbb{F}_p\wedge\varphi} H\mathbb{F}_p\wedge Y\xrightarrow{m} Y,$$

where the first map is induced by the unit map $u_{H\mathbb{F}_p}\colon\mathbb{S}\to H\mathbb{F}_p$ of $H\mathbb{F}_p$ and the last map is the $H\mathbb{F}_p$ structure map of Y. If we consider all the objects in this composite except Y to have the $H\mathbb{F}_p$ structure coming from the first smash factor, then all objects involved are commutative $H\mathbb{F}_p$–algebras and the maps involved are maps of commutative $H\mathbb{F}_p$–algebras. Note that i and $H\mathbb{F}_p\wedge\varphi$ are maps of commutative $H\mathbb{F}_p$–algebras as they are obtained via the functor $H\mathbb{F}_p\wedge-\colon\mathbb{S}\text{–}c\mathcal{A}lg\to H\mathbb{F}_p\text{–}c\mathcal{A}lg$. The last map m is a map of commutative $H\mathbb{F}_p$–algebras because it is the left adjoint of the identity map of Y under the usual adjunction between $\mathbb{S}$–$c\mathcal{A}lg$ and $H\mathbb{F}_p$–$c\mathcal{A}lg$. Since all the maps in the composite above are maps of commutative $H\mathbb{F}_p$–algebras, we deduce that ψ is a map of commutative $H\mathbb{F}_p$–algebras.

What remains is to show that ψ is a weak equivalence. For this, we take the homotopy groups of the composite defining ψ and show that it is an isomorphism. Firstly, we have a splitting

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z)$$

in $H\mathbb{F}_p$–$c\mathcal{A}lg$ where we consider the object on the right-hand side of the equality with the $H\mathbb{F}_p$ structure given by the first smash factor instead of the canonical one given by the smash product $\wedge_{H\mathbb{F}_p}$. Because the homotopy of $H\mathbb{F}_p$ is a field, we have $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z) \cong \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z)$; see [10, IV.4.1]. With this identification, we obtain that the composite map induced in homotopy by the composite defining ψ is given by

$$\psi_*\colon \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow[\cong]{\pi_*(H\mathbb{F}_p \wedge \varphi)} H\mathbb{F}_{p*}Y \xrightarrow{m_*} Y_*. \tag{2}$$

Note that although we identify the domain of $\pi_*(H\mathbb{F}_p \wedge \varphi)$ as a tensor product, we do not claim that $\pi_*(H\mathbb{F}_p \wedge \varphi)$ splits as a tensor product of two maps.

Below, we state three claims. Afterwards, we assume these claims and prove that ψ_* is an isomorphism by showing $\psi_* = \varphi_*$, ie we prove the theorem assuming the claims below. After that, we provide a proof of the three claims listed below.

Claim 1 *The composite $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)$ maps every element of the form $a \otimes_{\mathbb{F}_p} x$ with $|a| > 0$ to zero in Y_*.*

Claim 2 *We have $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x) = \varphi_*(x)$ for every $x \in \pi_*(H\mathbb{F}_p \wedge Z)$.*

Claim 3 *We have $i_*(x) = 1 \otimes_{\mathbb{F}_p} x + \Sigma_i a_i \otimes_{\mathbb{F}_p} x_i$ for some $a_i \in \mathcal{A}_*$ with $|a_i| > 0$ and $x_i \in \pi_*(H\mathbb{F}_p \wedge Z)$.*

Now we show that ψ is a weak equivalence by assuming the claims above. We have

$$\begin{aligned}\psi_*(x) &= m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi) \circ i_*(x)\\ &= m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x + \Sigma_i a_i \otimes_{\mathbb{F}_p} x_i)\\ &= \varphi_*(x)\end{aligned}$$

for some $a_i \in \mathcal{A}_*$ with $|a_i| > 0$. Here, the first equality follows by the definition of ψ_*, the second equality follows by Claim 3 and the third follows by Claims 1 and 2. This proves that ψ_* is an isomorphism since φ_* is an isomorphism. Therefore, we deduce that ψ is a weak equivalence as desired. What is left to prove is the three claims stated above.

Proof of Claim 1 The map $\mathbb{S} \to Z$ induces a map

$$(H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \to (HF_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z).$$

This map is in $H\mathbb{F}_p$–$c\mathcal{A}lg$, therefore the induced map in homotopy preserves the Dyer–Lashof operations. The induced map in homotopy is given by the inclusion $\mathcal{A}_* \otimes_{\mathbb{F}_p} \mathbb{F}_p \to \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z)$ and this shows that Dyer–Lashof operations on this subset of $\mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z)$ are given by the action of the Dyer–Lashof operations on the dual Steenrod algebra, ie $\mathrm{Q}^s(a \otimes_{\mathbb{F}_p} 1) = (\mathrm{Q}^s a) \otimes_{\mathbb{F}_p} 1$. Let p be an odd prime. Since $\pi_1(Y)$ is trivial, $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(\tau_0 \otimes_{\mathbb{F}_p} 1) = 0$. Because the dual Steenrod algebra is generated with the Dyer–Lashof operations by τ_0, this shows that $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(a \otimes_{\mathbb{F}_p} 1) = 0$ for all $a \in \mathcal{A}_*$ with $|a| > 0$. Since all maps involved are ring maps and $a \otimes_{\mathbb{F}_p} x = (a \otimes_{\mathbb{F}_p} 1)(1 \otimes_{\mathbb{F}_p} x)$, this finishes the proof of our claim $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(a \otimes_{\mathbb{F}_p} x) = 0$ whenever $|a| > 0$. Note that for $p = 2$, one uses ξ_1 instead of τ_0. □

Proof of Claim 2 We consider the commutative diagram

$$\begin{tikzcd}
(\mathbb{S} \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z) \arrow[r, "\cong"] \arrow[d, "h"] & \mathbb{S} \wedge H\mathbb{F}_p \wedge Z \arrow[r, "\mathbb{S}\wedge\varphi"] \arrow[d] & \mathbb{S} \wedge Y \arrow[d, "h_Y"] \arrow[drr, "\mathrm{id}"] & & \\
(H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z) \arrow[r, "\cong"] & H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \arrow[r, "H\mathbb{F}_p\wedge\varphi"] & H\mathbb{F}_p \wedge Y \arrow[rr, "m"] & & Y
\end{tikzcd} \tag{3}$$

Because Y is in $H\mathbb{F}_p$–$c\mathcal{A}lg$, we have $m \circ h_Y = \mathrm{id}$. We also have

$$m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x) = m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi) \circ h_*(x).$$

Carrying x through the top row and then composing with $m \circ h_Y$, we obtain the equality $m_* \circ \pi_*(H\mathbb{F}_p \wedge \varphi)(1 \otimes_{\mathbb{F}_p} x) = \varphi_*(x)$ in our claim. □

Proof of Claim 3 The composite of the maps below is the identity

$$H\mathbb{F}_p \wedge Z \cong H\mathbb{F}_p \wedge \mathbb{S} \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow{m_{H\mathbb{F}_p} \wedge \mathrm{id}} H\mathbb{F}_p \wedge Z,$$

where $m_{H\mathbb{F}_p}$ is the multiplication map of $H\mathbb{F}_p$. With the identification

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge Z),$$

we obtain the composite in homotopy

$$\pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id})} \pi_*(H\mathbb{F}_p \wedge Z), \tag{4}$$

where $\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id})$ is given by the augmentation $\mathcal{A}_* \to \mathbb{F}_p$. This description of $\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id})$ and the fact that $\pi_*(m_{H\mathbb{F}_p} \wedge \mathrm{id}) \circ i_* = \mathrm{id}$ proves our claim. □

This completes the proof of Theorem 1.13. □

Remark 3.1 The proof of Theorem 1.13 is showing slightly more. For a given cofibrant Z in $\mathbb{S}$–$c\mathcal{A}lg$ and a fibrant Y in $H\mathbb{F}_p$–$c\mathcal{A}lg$ with $\pi_1 Y = 0$ and an equivalence $H\mathbb{F}_p \wedge Z \xrightarrow{\sim} Y$ of $\mathbb{S}$–algebras, the map $H\mathbb{F}_p \wedge Z \to Y$ in $H\mathbb{F}_p$–$c\mathcal{A}lg$ given by the structure map of Y on $H\mathbb{F}_p$ and the map $\mathbb{S} \wedge Z \to H\mathbb{F}_p \wedge Z \xrightarrow{\sim} Y$ on Z is also a weak equivalence. Note that to construct this map, we use the fact that $H\mathbb{F}_p \wedge Z$ is a coproduct of $H\mathbb{F}_p$ and Z in $\mathbb{S}$–$c\mathcal{A}lg$.

The proof of Theorem 1.14 (restated below) is similar to the proof of Theorem 1.13. Therefore, in the proof of Theorem 1.14, we assume familiarity with the proof of Theorem 1.13.

Theorem 1.14 *Let Y be an $\mathbb{F}_p$–DGA and let X be an $\mathbb{F}_p$–extension $\mathbb{F}_p$–DGA. For odd p, assume that the homology of X is trivial in degrees $2p^r - 2$ for $r \geq 1$ and $2p^s - 1$ for $s \geq 0$. For $p = 2$, assume that the homology of X is trivial in degree $2^r - 1$ for $r \geq 1$. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Proof Here, we work in the setting of associative algebras. In this case, we need to be more careful with cofibrant replacements since the forgetful functor from $H\mathbb{F}_p$–$\mathcal{A}lg$ to $\mathbb{S}$–$\mathcal{A}lg$ does not necessarily preserve cofibrant objects. Let $H\mathbb{F}_p$ be cofibrant in $\mathbb{S}$–$c\mathcal{A}lg$ (with the model structure of [25]) as before and let Z be cofibrant in $\mathbb{S}$–$\mathcal{A}lg$ such that $H\mathbb{F}_p \wedge Z$ is an $H\mathbb{F}_p$–algebra that corresponds to X. By abuse of notation, let Y be a fibrant $H\mathbb{F}_p$–algebra corresponding to Y. Let $T \xrightarrow{\sim} H\mathbb{F}_p \wedge Z$ be a cofibrant replacement of $H\mathbb{F}_p \wedge Z$ in $\mathbb{S}$–$\mathcal{A}lg$. We have the lift

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d, tail] & T \arrow[d, two heads, "\sim"] \\ Z \arrow[r] \arrow[ur, dashed, "f"] & H\mathbb{F}_p \wedge Z \end{tikzcd} \tag{5}$$

in $\mathbb{S}$–$\mathcal{A}lg$ where the bottom map is given by the map $Z \cong \mathbb{S} \wedge Z \to H\mathbb{F}_p \wedge Z$. Since T and Y are obtained from topologically equivalent DGAs, they are equivalent in $\mathbb{S}$–$\mathcal{A}lg$. Also because T is cofibrant and Y is fibrant, we have a weak equivalence $\varphi\colon T \xrightarrow{\sim} Y$ of $\mathbb{S}$–algebras. We obtain the composite map of $H\mathbb{F}_p$–algebras

$$\psi\colon H\mathbb{F}_p \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge T \xrightarrow[\simeq]{H\mathbb{F}_p \wedge \varphi} H\mathbb{F}_p \wedge Y \xrightarrow{m} Y,$$

where $i = H\mathbb{F}_p \wedge f$ and m is the $H\mathbb{F}_p$ structure map of Y. The map m is a map of $H\mathbb{F}_p$–algebras because it is the left adjoint of the identity map of Y under the usual adjunction between $H\mathbb{F}_p$–$\mathcal{A}lg$ and $\mathbb{S}$–$\mathcal{A}lg$. Note that we denote $H\mathbb{F}_p \wedge f$ by i because the map i in the composite above should be compared to the map i in (1).

Again, what remains is to show that ψ_* is an isomorphism. Note that the functor $H\mathbb{F}_p \wedge -$ preserves weak equivalences [13, 5.3.10]. Identifying homotopy groups of T with homotopy groups of $H\mathbb{F}_p \wedge Z$ through the trivial fibration above, and similarly identifying the homotopy groups of $H\mathbb{F}_p \wedge T$ with those of $H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z$, we obtain a description of ψ_* similar to the one in (2),

$$\psi_* \colon \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow[\cong]{\pi_*(H\mathbb{F}_p \wedge \varphi)} H\mathbb{F}_{p*}Y \xrightarrow{m_*} Y_*.$$

It is sufficient to show that the claims in the proof of Theorem 1.13 also hold in this case. Claim 1 follows by the hypothesis that π_*Y is trivial at the degrees where the algebra generators of the dual Steenrod algebra are. Claim 2 follows similarly. For Claim 3, consider the sequence of maps

$$H\mathbb{F}_p \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge T \xrightarrow{\sim} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow{m_{H\mathbb{F}_p} \wedge \mathrm{id}} H\mathbb{F}_p \wedge Z,$$

where $m_{H\mathbb{F}_p}$ is the multiplication map of $H\mathbb{F}_p$. Due to diagram (5), the composite above is the identity map. Taking homotopy groups of the composite above and omitting the equivalence in the middle, one obtains (4). The rest of the proof of Claim 3 follows as before. □

3B Example 1.12

Here, we show that the E_∞ $\mathbb{F}_p$–DGAs provided in Example 1.12 are not $\mathbb{F}_p$–extension.

Proposition 3.2 *Let X and Y be as in Example 1.12. As E_∞ $\mathbb{F}_p$–DGAs, X and Y are not $\mathbb{F}_p$–extension.*

Proof Recall that in Example 1.12, we provide examples of E_∞ $\mathbb{F}_p$–DGAs that are E_∞ topologically equivalent but not quasi-isomorphic. We prove that X is not an extension E_∞ $\mathbb{F}_p$–DGA. In order to show Y is not extension, it suffices to exchange the roles of X and Y in the proof below.

We assume that X is an extension E_∞ $\mathbb{F}_p$–DGA and obtain a contradiction by showing that X and Y are quasi-isomorphic under this assumption. This is similar to the proof of Theorem 1.13, which we assume familiarity with. Following the constructions there, we obtain a map of commutative $H\mathbb{F}_p$–algebras

$$\psi \colon H\mathbb{F}_p \wedge Z \cong H\mathbb{F}_p \wedge \mathbb{S} \wedge Z \xrightarrow{i} H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge Z \xrightarrow[\simeq]{H\mathbb{F}_p \wedge \varphi} H\mathbb{F}_p \wedge Y \xrightarrow{m} Y$$

as in (1), where $H\mathbb{F}_p \wedge Z$ denotes a commutative $H\mathbb{F}_p$–algebra corresponding to X and Y denotes a commutative $H\mathbb{F}_p$–algebra corresponding to the E_∞ $\mathbb{F}_p$–DGA Y by abusing notation. This is a map of commutative $H\mathbb{F}_p$–algebras as before. Therefore, it is sufficient to show that ψ_* is an isomorphism.

As in (2), ψ_* is given by

$$\psi_*\colon \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow{i_*} \mathcal{A}_* \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge Z) \xrightarrow[\cong]{\pi_*(H\mathbb{F}_p \wedge \varphi)} H\mathbb{F}_{p_*}Y \xrightarrow{m_*} Y_*.$$

By Claim 3 in the proof of Theorem 1.14, for every $x \in \pi_*(H\mathbb{F}_p \wedge Z)$ we have

$$i_*(x) = 1 \otimes_{\mathbb{F}_p} x + \Sigma_i a_i \otimes_{\mathbb{F}_p} x_i \tag{6}$$

for some $a_i \in \mathcal{A}_*$ with $|a_i| > 0$ and $x_i \in \pi_*(H\mathbb{F}_p \wedge Z)$.

For $p = 2$, $\pi_*(H\mathbb{F}_p \wedge Z) \cong \mathbb{F}_2[x]/(x^4)$ with $|x| = 1$. By degree reasons, we either have $i_*(x) = 1 \otimes_{\mathbb{F}_p} x$ or $i_*(x) = 1 \otimes_{\mathbb{F}_p} x + \xi_1 \otimes_{\mathbb{F}_p} 1$. Since $(1 \otimes_{\mathbb{F}_p} x + \xi_1 \otimes_{\mathbb{F}_p} 1)^4 \neq 0$ but $x^4 = 0$, the second option is not possible. Therefore we have $i_*(x) = 1 \otimes_{\mathbb{F}_p} x$. Since i is a map of ring spectra, i_* is multiplicative so $i_*(x^l) = 1 \otimes_{\mathbb{F}_p} x^l$ for every l. By Claim 2 in the proof of Theorem 1.13, this shows that ψ_* is an isomorphism. This provides a contradiction as X and Y are not quasi-isomorphic as E_∞ $\mathbb{F}_2$–DGAs.

For odd p, we have

$$\pi_*Y \cong \pi_*(H\mathbb{F}_p \wedge Z) \cong \Lambda_{\mathbb{F}_p}[x, y]$$

with $|x| = 1$ and $|y| = 2p-2$. By (6) above, either

$$i_*(y) = 1 \otimes_{\mathbb{F}_p} y \quad \text{or} \quad i_*(y) = c\xi_1 \otimes_{\mathbb{F}_p} 1 + 1 \otimes_{\mathbb{F}_p} y$$

for some unit $c \in \mathbb{F}_p$. However, $y^2 = 0$ but $(c\xi_1 \otimes_{\mathbb{F}_p} 1 + 1 \otimes_{\mathbb{F}_p} y)^2 \neq 0$ so only the first option is possible. This shows that $\psi_*(y) = y$ due to Claim 2 in the proof of Theorem 1.13. The $2p-2$ Postnikov sections of Y and $H\mathbb{F}_p \wedge Z$ agrees with that of $H\mathbb{F}_p \wedge H\mathbb{F}_p$ in commutative $H\mathbb{F}_p$–algebras; see [1, Example 5.1]. Using this together with the fact that $\beta \mathrm{Q}^1 \tau_0 = -\zeta_1$ in the dual Steenrod algebra, $\beta \mathrm{Q}^1 x = y$ up to a unit both in $\pi_*(H\mathbb{F}_p \wedge Z)$ and in π_*Y. Because ψ is a map of commutative $H\mathbb{F}_p$–algebras, ψ_* preserves Dyer–Lashof operations. Since $\psi_*(y) = y$, we obtain that $\psi_*(x) = x$ up to a unit of $\mathbb{F}_p$. Because ψ_* is a ring map, we deduce that ψ_* is indeed an isomorphism. Therefore ψ is a weak equivalence of commutative $H\mathbb{F}_p$–algebras between the commutative $H\mathbb{F}_p$–algebras corresponding to the E_∞ $\mathbb{F}_p$–DGAs X and Y. This contradicts the fact that X and Y are not quasi-isomorphic as E_∞ $\mathbb{F}_p$–DGAs and finishes our proof. □

3C Proof of Theorem 1.16

Theorem 1.16 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$ and let X be an R–DGA whose corresponding HR–algebra is equivalent to $HR \wedge Z$ for some cofibrant $\mathbb{S}$–algebra Z whose underlying spectrum is equivalent to a coproduct of (de)suspensions of the sphere spectrum. Also, let Y be an R–DGA. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Proof Let HR be cofibrant as a commutative $\mathbb{S}$–algebra in Shipley's convenient model structure. This guarantees that $HR \wedge -$ preserves weak equivalences [13, 5.3.10]. Since $HR \wedge -$ preserves weak equivalences, we can further assume Z to be fibrant.

Let Y be an R–DGA. Since quasi-isomorphic R–DGAs are always topologically equivalent, we only need to show that X and Y are quasi-isomorphic if they are topologically equivalent. Abusing notation, we also let Y denote a fibrant HR–algebra corresponding to the R–DGA Y. We assume that X and Y are topologically equivalent, ie $HR \wedge Z$ and Y are equivalent as $\mathbb{S}$–algebras. Using this, we are going to show that there is a weak equivalence

$$\psi \colon HR \wedge Z \xrightarrow{\sim} Y$$

of HR–algebras.

Let $g \colon T \xrightarrow{\sim} HR \wedge Z$ be a cofibrant replacement of $HR \wedge Z$ in $\mathbb{S}$–algebras. As in diagram (5), there exists a map $f : Z \to T$ such that

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d, tail] & T \arrow[d, two heads, "g", "\simeq"'] \\ Z \arrow[r, "h_Z"'] \arrow[ur, dashed, "f"] & HR \wedge Z \end{tikzcd} \tag{7}$$

commutes. Here, h_Z denotes the canonical map

$$h_Z \colon Z \cong \mathbb{S} \wedge Z \to HR \wedge Z.$$

Since X and Y are topologically equivalent, T and Y are equivalent as $\mathbb{S}$–algebras. Furthermore, T is cofibrant and Y is fibrant; therefore we have a weak equivalence

$$\varphi \colon T \xrightarrow{\sim} Y$$

of $\mathbb{S}$–algebras.

We obtain the composite map

$$\psi \colon HR \wedge Z \xrightarrow{HR \wedge f} HR \wedge T \xrightarrow{HR \wedge \varphi} HR \wedge Y \xrightarrow{m} Y$$

of HR–algebras where m denotes the HR–module structure map of Y. Note that the last map above is a map of HR–algebras as it is the left adjoint of the identity map of Y under the usual adjunction between the categories of HR–algebras and $\mathbb{S}$–algebras. Since ψ is a map of HR–algebras, it is sufficient to show that ψ induces an isomorphism in homotopy.

We have the commuting diagram

$$\begin{tikzcd}
\mathbb{S} \wedge T \arrow[r, "\mathbb{S}\wedge\varphi"] \arrow[d, "h_T"] & \mathbb{S} \wedge Y \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
HR \wedge T \arrow[r, "HR\wedge\varphi"] & HR \wedge Y \arrow[r, "m"] & Y
\end{tikzcd}$$

where the vertical maps are the canonical maps induced by the unit map $u_R \colon \mathbb{S} \to HR$. This shows that the composite map starting from $T \cong \mathbb{S} \wedge T$ and ending in Y is given by φ and therefore is a weak equivalence. In particular, $\pi_*(m \circ (HR \wedge \varphi))$ is an isomorphism when it is restricted to the image of the Hurewicz map of T

$$\pi_* h_T \colon \pi_*(\mathbb{S} \wedge T) \to \pi_*(HR \wedge T).$$

Therefore, in order to prove that ψ_* is an isomorphism, it is sufficient to show that the map

$$\pi_*(HR \wedge f) \colon \pi_*(HR \wedge Z) \to \pi_*(HR \wedge T)$$

is injective and its image agrees with the image of $\pi_* h_T$. For this, it is sufficient to prove that the corresponding statements are true after composing with the isomorphism

$$\pi_*(HR \wedge g) \colon \pi_*(HR \wedge T) \xrightarrow{\cong} \pi_*(HR \wedge HR \wedge Z).$$

In other words, it is sufficient to show that

$$\pi_*(HR \wedge g) \circ \pi_*(HR \wedge f)$$

is injective and the image of this map agrees with the image of $\pi_*(HR \wedge g) \circ \pi_* h_T$. Due to (7), $g \circ f = h_Z$. Therefore, it is sufficient to show that $\pi_*(HR \wedge h_Z)$ is injective in homotopy and its image agrees with the image of $\pi_*(HR \wedge g) \circ \pi_* h_T$.

The composite

$$HR \wedge Z \xrightarrow{HR\wedge h_Z} HR \wedge HR \wedge Z \xrightarrow{m\wedge \mathrm{id}} HR \wedge Z$$

is the identity map, where m denotes the multiplication map of HR and id denotes the identity map of Z. From this, we deduce that $\pi_*(HR \wedge h_Z)$ is injective in homotopy, as desired. What remains to prove is that the image of $\pi_*(HR \wedge h_Z)$ agrees with the image of $\pi_*(HR \wedge g) \circ \pi_* h_T$.

Due to the commuting diagram

$$\begin{tikzcd}
\mathbb{S} \wedge T \arrow[r, "g", "\simeq"'] \arrow[d, "h_T"] & \mathbb{S} \wedge HR \wedge Z \arrow[d, "h_{HR \wedge Z}"] \\
HR \wedge T \arrow[r, "HR \wedge g", "\simeq"'] & HR \wedge HR \wedge Z
\end{tikzcd}$$

the image of the map $\pi_*(HR \wedge g) \circ \pi_* h_T$ is given by the image of the Hurewicz map

$$\pi_*(h_{HR \wedge Z}) \colon \pi_*(\mathbb{S} \wedge HR \wedge Z) \to \pi_*(HR \wedge HR \wedge Z)$$

of $HR \wedge Z$. Note that $h_{HR \wedge Z}$ is induced by the unit map of HR as usual. Therefore, it is sufficient to show that the image of $\pi_*(HR \wedge h_Z)$ agrees with the image of $\pi_*(h_{HR \wedge Z})$.

The map $HR \wedge h_Z$ is the canonical map

$$HR \wedge Z \cong HR \wedge \mathbb{S} \wedge Z \to HR \wedge HR \wedge Z.$$

This is the same as the composite

$$(8) \qquad HR \wedge Z \cong \mathbb{S} \wedge HR \wedge Z \xrightarrow{h_{HR \wedge Z}} HR \wedge HR \wedge Z \xrightarrow{\tau \wedge \mathrm{id}} HR \wedge HR \wedge Z,$$

where τ is the transposition map of the monoidal structure. Since the map $h_{HR \wedge Z}$ in the middle of the composite in (8) induces $\pi_*(h_{HR \wedge Z})$, it is sufficient to show that $\pi_*(\tau \wedge \mathrm{id})$ is the identity map on the image of $\pi_*(h_{HR \wedge Z})$.

By hypothesis, the underlying spectrum of Z is a wedge of suspensions of the sphere spectrum. Let

$$E = \bigvee_{a \in A} \Sigma^{|a|} \mathbb{S}$$

be weakly equivalent to Z as a spectrum where A is a graded set. Since E is cofibrant and Z is fibrant, there is a weak equivalence of spectra $E \xrightarrow{\sim} Z$.

This equivalence induces the vertical maps in the commuting diagram of $\mathbb{S}$–modules,

$$\begin{tikzcd}
HR \wedge E \arrow[r, "h_{HR \wedge E}"] \arrow[d, "\simeq"] & HR \wedge HR \wedge E \arrow[r, "\tau \wedge \mathrm{id}"] \arrow[d, "\simeq"] & HR \wedge HR \wedge E \arrow[d, "\simeq"] \\
HR \wedge Z \arrow[r, "h_{HR \wedge Z}"] & HR \wedge HR \wedge Z \arrow[r, "\tau \wedge \mathrm{id}"] & HR \wedge HR \wedge Z
\end{tikzcd}$$

where $h_{HR\wedge E}$ denotes the canonical map that induces the Hurewicz map of $HR \wedge E$ in homotopy. In order to show that $\pi_*(\tau \wedge \mathrm{id})$ (of the bottom row) is the identity map on the image of $\pi_*(h_{HR\wedge Z})$, it is sufficient to show that $\pi_*(\tau \wedge \mathrm{id})$ (of the top row) is given by the identity map on the image of $\pi_*(h_{HR\wedge E})$. For this, it is sufficient to show that the composite of the maps on the top row is given by $\pi_*(h_{HR\wedge E})$ in homotopy.

Note that the canonical R–module basis elements of

$$\pi_*(HR \wedge E) = \pi_*\Bigl(HR \wedge \Bigl(\bigvee_{a\in A} \Sigma^{|a|}\mathbb{S}\Bigr)\Bigr) \cong \bigoplus_{a\in A} \Sigma^{|a|} R$$

are also abelian group generators because $R = \mathbb{Z}/(m)$ for some integer m. Therefore, it is sufficient to show that

$$\pi_*(\tau \wedge \mathrm{id}) \circ \pi_*(h_{HR\wedge E})(x) = \pi_*(h_{HR\wedge E})(x)$$

for every canonical basis element x. Such an x is represented by a map

$$u_{HR} \wedge i_a \colon \mathbb{S} \wedge \Sigma^{|a|}\mathbb{S} \to HR \wedge \Bigl(\bigvee_{a\in A} \Sigma^{|a|}\mathbb{S}\Bigr) = HR \wedge E,$$

where i_a is the inclusion of the cofactor corresponding to an $a \in A$.

In other words, it is sufficient to show that the composite

$$\mathbb{S} \wedge \Sigma^{|a|}\mathbb{S} \xrightarrow{u_{HR}\wedge i_a} HR \wedge E \xrightarrow{h_{HR\wedge E}} HR \wedge HR \wedge E \xrightarrow{\tau\wedge \mathrm{id}} HR \wedge HR \wedge E$$

agrees with the composite

$$\mathbb{S} \wedge \Sigma^{|a|}\mathbb{S} \xrightarrow{u_{HR}\wedge i_a} HR \wedge E \xrightarrow{h_{HR\wedge E}} HR \wedge HR \wedge E.$$

To see this, note that the composite maps above are of the form $\upsilon \wedge i_a$ and $\nu \wedge i_a$, respectively, where υ and ν are $\mathbb{S}$–algebra maps from $\mathbb{S}$ to $HR \wedge HR$. Since $\mathbb{S}$ is the initial object in the category of $\mathbb{S}$–algebras, we deduce that $\upsilon = \nu$. Therefore, the two composites above agree, as claimed. □

4 E–infinity $\mathbb{F}_p$–DGAs are not extension

This section is devoted to the proof of Theorems 1.7 and 1.8. We restate these theorems below. Recall that when we say extension (E_∞) DGA, we mean $\mathbb{Z}$–extension (E_∞) $\mathbb{Z}$–DGA.

Theorem 1.7 *Let Y be an E_∞ DGA. For all primes p, if Y is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA then Y is not an extension E_∞ DGA.*

Theorem 1.8 *Let X be a DGA. If X is quasi-isomorphic to an $\mathbb{F}_2$–DGA then X is not an extension DGA.*

In the proof of these theorems, we use the ring structure and the Dyer–Lashof operations on $\pi_*(H\mathbb{F}_p \wedge H\mathbb{Z}) = H\mathbb{F}_{p*}H\mathbb{Z}$. For odd p, the ring structure is given by

$$H\mathbb{F}_{p*}H\mathbb{Z} \cong \mathbb{F}_p[\zeta_r \mid r \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\bar{\tau}_s \mid s \geq 1),$$

where the degrees of the generators are the same as those of the dual Steenrod algebra. Note that $H\mathbb{F}_{p*}H\mathbb{Z}$ has the same generators as the dual Steenrod algebra except that $H\mathbb{F}_{p*}H\mathbb{Z}$ does not contain the degree 1 generator τ_0. Indeed, the map

$$H\mathbb{F}_{p*}H\mathbb{Z} \to H\mathbb{F}_{p*}H\mathbb{F}_p = \mathcal{A}_*$$

induced by $H\mathbb{Z} \to H\mathbb{F}_p$ is the canonical inclusion [21, II.10.26]. This inclusion is induced by a map of commutative $H\mathbb{F}_p$–algebras and therefore it preserves the Dyer–Lashof operations. Therefore through this map, the Dyer–Lashof operations on the dual Steenrod algebra determine the Dyer–Lashof operations on $H\mathbb{F}_{p*}H\mathbb{Z}$; see [7, III.2].

For $p = 2$, we have

$$H\mathbb{F}_{2*}H\mathbb{Z} = \mathbb{F}_2[\zeta_1^2] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\zeta_r \mid r \geq 2],$$

where $|\zeta_i| = 2^i - 1$ for $i \geq 2$ and $|\zeta_1^2| = 2$. Again, the canonical map

$$H\mathbb{F}_{2*}H\mathbb{Z} \to H\mathbb{F}_{2*}H\mathbb{F}_2 = \mathcal{A}_*$$

is the canonical inclusion and this determines the Dyer–Lashof operations on $H\mathbb{F}_{2*}H\mathbb{Z}$.

For the rest of this section, we assume that $H\mathbb{Z}$ is cofibrant as a commutative $\mathbb{S}$–algebra and $H\mathbb{F}_p$ is cofibrant as a commutative $H\mathbb{Z}$–algebra in the model structure developed in [25]. Since the category of commutative $H\mathbb{Z}$–algebras is the same as the category of commutative $\mathbb{S}$–algebras under $H\mathbb{Z}$, cofibrations of commutative $H\mathbb{Z}$–algebras forget to cofibrations of commutative $\mathbb{S}$–algebras. Therefore, $H\mathbb{Z} \to H\mathbb{F}_p$ is also a cofibration of commutative $\mathbb{S}$–algebras. This ensures that $H\mathbb{F}_p$ is also cofibrant as a commutative $\mathbb{S}$–algebra and therefore the functor $H\mathbb{F}_p \wedge -$ preserves all weak equivalences [13, 5.3.10].

We start by proving the following lemma. This lemma is obvious if one assumes that for a map of discrete commutative rings $R \to R'$, the Quillen equivalences of [19; 26] are compatible with the restriction of scalars functors from (E_∞) R'–DGAs to (E_∞) R–DGAs and from (commutative) HR'–algebras to (commutative) HR–algebras. However, there is no such compatibility result available in the literature and proving it is beyond the scope of this work.

Lemma 4.1 *Let X be an E_∞ DGA that is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA. Then there is a map of commutative $H\mathbb{Z}$–algebras*

$$H\mathbb{F}_p \to H_{E_\infty}X,$$

where $H_{E_\infty}X$ denotes a fibrant commutative $H\mathbb{Z}$–algebra corresponding to the E_∞ DGA X.

If X is a DGA that is quasi-isomorphic to an $\mathbb{F}_p$–DGA, then there is a map of $H\mathbb{Z}$–algebras

$$c(H\mathbb{F}_p) \to HX,$$

where HX denotes a fibrant $H\mathbb{Z}$–algebra corresponding to the DGA X. Here, $c(H\mathbb{F}_p)$ denotes a cofibrant replacement of $H\mathbb{F}_p$ in $H\mathbb{Z}$–algebras.

Proof We only prove the E_∞ case; the associative case follows in a similar manner. Assume that we are using a unital E_∞ operad, ie an operad given by the monoidal unit $\mathbb{F}_p$ in operadic degree zero. The Barratt–Eccles operad is an example of a unital E_∞–operad [4]. In this situation, $\mathbb{F}_p$ is the free E_∞ $\mathbb{F}_p$–DGA generated by the trivial $\mathbb{F}_p$–chain complex 0. Therefore, $\mathbb{F}_p$ is the initial object in E_∞ $\mathbb{F}_p$–DGAs. This, together with the fact that X is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA implies that there is a map $\mathbb{F}_p \to X$ in the homotopy category of E_∞ DGAs.

The equivalence of categories between the homotopy categories of commutative $H\mathbb{Z}$–algebras and E_∞ DGAs implies that there is also a map $H\mathbb{F}_p \to H_{E_\infty}X$ in the homotopy category of commutative $H\mathbb{Z}$–algebras. Since $H_{E_\infty}X$ is fibrant in commutative $H\mathbb{Z}$–algebras and $H\mathbb{F}_p$ is cofibrant in commutative $H\mathbb{Z}$–algebras due to our standing assumptions, there is a map $H\mathbb{F}_p \to H_{E_\infty}X$ of commutative $H\mathbb{Z}$–algebras as desired. □

The following starts with the proof of Theorem 1.7, and at the end we mention how this also shows Theorem 1.8.

Proof of Theorems 1.7 and 1.8 Assume to the contrary that there is an extension E_∞ DGA X that is quasi-isomorphic to an E_∞ $\mathbb{F}_p$–DGA. It follows by Lemma 4.1 that there is a map $H\mathbb{F}_p \to H_{E_\infty}X$ of commutative $H\mathbb{Z}$–algebras where $H_{E_\infty}X$ denotes a fibrant commutative $H\mathbb{Z}$–algebra corresponding to the E_∞ DGA X. In particular, the $H\mathbb{Z}$–structure map $H\mathbb{Z} \to H_{E_\infty}X$ of $H_{E_\infty}X$ factors as

$$H\mathbb{Z} \xrightarrow{\varphi_{H\mathbb{F}_p}} H\mathbb{F}_p \to H_{E_\infty}X,$$

where $\varphi_{H\mathbb{F}_p}$ denotes the canonical map.

Since X is a $\mathbb{Z}$–extension E_∞ DGA, there is a cofibrant commutative $\mathbb{S}$–algebra Y such that $H\mathbb{Z} \wedge Y$ is weakly equivalent to $H_{E_\infty} X$ in commutative $H\mathbb{Z}$–algebras.

Note that $H\mathbb{Z} \wedge Y$ is cofibrant as a commutative $H\mathbb{Z}$–algebra; this is the case because $H\mathbb{Z} \wedge -$ is a left Quillen functor from commutative $\mathbb{S}$–algebras to commutative $H\mathbb{Z}$–algebras and therefore it preserves cofibrant objects.

Since $H_{E_\infty} X$ is fibrant and $H\mathbb{Z} \wedge Y$ is cofibrant, there is a weak equivalence of commutative $H\mathbb{Z}$–algebras $\psi\colon H\mathbb{Z} \wedge Y \xrightarrow{\sim} H_{E_\infty} X$. Because ψ is a map of commutative $H\mathbb{Z}$–algebras, we obtain a commutative diagram

$$\begin{tikzcd}
 & H\mathbb{Z} \arrow[dl, "\varphi_{H\mathbb{Z}\wedge Y}"'] \arrow[r, "\varphi_{H\mathbb{F}_p}"] & H\mathbb{F}_p \arrow[dr] & \\
H\mathbb{Z} \wedge Y \arrow[rrr, "\psi", "\simeq"'] & & & H_{E_\infty} X
\end{tikzcd}$$

where the composite on the right from $H\mathbb{Z}$ to $H_{E_\infty} X$ is the composite given above. The map $\varphi_{H\mathbb{Z}\wedge Y}$ is the $H\mathbb{Z}$–structure map of $H\mathbb{Z} \wedge Y$ which is given by

$$H\mathbb{Z} \cong H\mathbb{Z} \wedge \mathbb{S} \to H\mathbb{Z} \wedge Y.$$

Applying the homology functor $H\mathbb{F}_{p*}$ to this diagram and inverting $H\mathbb{F}_{p*}\psi$, we obtain

$$\begin{tikzcd}
 & H\mathbb{F}_{p*}H\mathbb{Z} \arrow[dl, "H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}"'] \arrow[r, "H\mathbb{F}_{p*}\varphi_{H\mathbb{F}_p}"] & H\mathbb{F}_{p*}H\mathbb{F}_p \arrow[dr] & \\
H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y & & & H\mathbb{F}_{p*}H_{E_\infty} X \arrow[lll, "\cong"', "(H\mathbb{F}_{p*}\psi)^{-1}"]
\end{tikzcd}$$

By the Künneth spectral sequence in [10, IV.4.1],

$$H\mathbb{F}_{p*}(H\mathbb{Z} \wedge Y) \cong H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y$$

and the morphism on the left is given by

$$H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}(a) = a \otimes_{\mathbb{F}_p} 1. \tag{9}$$

Since the diagram above commutes, $H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}$ factors as

$$H\mathbb{F}_{p*}\varphi_{H\mathbb{Z}\wedge Y}\colon H\mathbb{F}_{p*}H\mathbb{Z} \xrightarrow{H\mathbb{F}_{p*}\varphi_{H\mathbb{F}_p}} H\mathbb{F}_{p*}H\mathbb{F}_p \xrightarrow{f} H\mathbb{F}_{p*}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{p*}Y, \tag{10}$$

where the second map f is the composite in the triangle above starting from $H\mathbb{F}_{p*}H\mathbb{F}_p$ and ending in the bottom left corner. Both maps in the composite above are ring maps that preserve the Dyer–Lashof operations.

Let p denote an odd prime; we discuss the case $p=2$ at the end of this proof. We have $\beta\mathrm{Q}^1\tau_0=\zeta_1$ (up to a unit we are going to omit) in $H\mathbb{F}_{p*}H\mathbb{F}_p$. Note that $f(\zeta_1)=\zeta_1\otimes_{\mathbb{F}_p}1$. This follows by considering the composite in (10), equality (9) and by noting that $H\mathbb{F}_{p*}\varphi_{H\mathbb{F}_p}$ is the canonical inclusion. Since f preserves Dyer–Lashof operations,

$$\beta\mathrm{Q}^1 f(\tau_0)=f(\beta\mathrm{Q}^1\tau_0)=f(\zeta_1)=\zeta_1\otimes_{\mathbb{F}_p}1.$$

We conclude that $\beta\mathrm{Q}^1 f(\tau_0)=\zeta_1\otimes_{\mathbb{F}_p}1$ in $H\mathbb{F}_{p*}H\mathbb{Z}\otimes_{\mathbb{F}_p}H\mathbb{F}_{p*}Y$.

We obtain a contradiction by showing that there is no z in $H\mathbb{F}_{p*}H\mathbb{Z}\otimes_{\mathbb{F}_p}H\mathbb{F}_{p*}Y$ that satisfies $\beta\mathrm{Q}^1 z=\zeta_1\otimes_{\mathbb{F}_p}1$, ie there is no candidate for $f(\tau_0)$. For an element of the form $1\otimes_{\mathbb{F}_p}y\in H\mathbb{F}_{p*}H\mathbb{Z}\otimes_{\mathbb{F}_p}H\mathbb{F}_{p*}Y$, we have that $\beta\mathrm{Q}^1(1\otimes_{\mathbb{F}_p}y)=1\otimes_{\mathbb{F}_p}\beta\mathrm{Q}^1 y$ does not contain $\zeta_1\otimes_{\mathbb{F}_p}1$ as a summand. Now consider an element of the form $a\otimes_{\mathbb{F}_p}y\in H\mathbb{F}_{p*}H\mathbb{Z}\otimes_{\mathbb{F}_p}H\mathbb{F}_{p*}Y$ with $|a|>0$. By the Cartan formula and the fact that the Bockstein operation is a derivation, $\beta\mathrm{Q}^1(a\otimes_{\mathbb{F}_p}y)$ is a sum of elements of the form $a'\otimes_{\mathbb{F}_p}y'$ where a' is obtained by applying a Dyer–Lashof operation to a. In particular, $|a'|>|a|\geq|\zeta_1|$; therefore $\beta\mathrm{Q}^1(a\otimes_{\mathbb{F}_p}y)$ does not contain $\zeta_1\otimes_{\mathbb{F}_p}1$ as a summand either. We deduce that $\beta\mathrm{Q}^1 z$ does not contain $\zeta_1\otimes_{\mathbb{F}_p}1$ as a summand for all $z\in H\mathbb{F}_{p*}H\mathbb{Z}\otimes_{\mathbb{F}_p}H\mathbb{F}_{p*}Y$.

For $p=2$, we do not need to use the Dyer–Lashof operations. In this case, we have $f(\zeta_1^2)=\zeta_1^2\otimes_{\mathbb{F}_2}1$ due to the composite in (10). We obtain that $f(\zeta_1)^2=\zeta_1^2\otimes_{\mathbb{F}_2}1$. However, there is no element in $H\mathbb{F}_{2*}H\mathbb{Z}\otimes_{\mathbb{F}_2}H\mathbb{F}_{2*}Y$ that squares to $\zeta_1^2\otimes_{\mathbb{F}_2}1$. Since this does not use Dyer–Lashof operations, this argument at $p=2$ also works for DGAs and $H\mathbb{Z}$–algebras and provides a proof of Theorem 1.8. □

5 Formal DGAs to $H\mathbb{Z}$–algebras

This section is devoted to the proof of Proposition 5.8 which provides an explicit description of the HR–algebra corresponding to a formal R–DGA whose homology satisfies the hypothesis of Theorem 1.4. This description provides Theorem 1.4. Recall that we also use Proposition 5.8 to obtain Corollary 1.17.

We work in several different monoidal categories in this section. When we work in the category of chain complexes or in the category of differential graded algebras, we denote the monoidal product by $\otimes$. For the categories of HR–modules and HR–algebras, we denote the smash product by $\wedge_{HR}$ as before. In all the other cases, we let $\wedge$ denote the monoidal product. In this section, HR denotes the Eilenberg–Mac Lane spectrum of a general discrete commutative ring as in [13, 1.2.5].

Let X be an R–DGA satisfying the hypothesis of Theorem 1.4. Recall from Remark 1.5 that there is a monoid M in graded pointed sets for which $H_*(X) \cong R\langle M\rangle$ as R–algebras where the underlying R–module of $R\langle M\rangle$ is the free graded R–module over the graded set M_- obtained by removing the base point of M. Furthermore, the multiplication on $R\langle M\rangle$ is the canonical one induced by that of M. For the rest of this section, let M denote a monoid in nonnegatively graded pointed sets.

5A A monoid object corresponding to M

Here, we construct a monoid in a general monoidal category by using M. Furthermore, we show that this construction is preserved by strong monoidal Quillen pairs.

We start by explaining a notation we use for the symmetric monoidal pointed model categories we consider in this section. For a cofibrant C, ΣC denotes the pushout of the diagram $\bar{*} \leftarrowtail C \rightarrowtail \bar{*}$, where $\bar{*}$ is obtained by a factorization $C \rightarrowtail \bar{*} \xrightarrow{\sim} \!\!\!\!\to * $ of the map $C \to *$ by a cofibration followed by a trivial fibration, and $*$ denotes the final object. For the unit $\mathbb{I}$ of the monoidal structure, $\Sigma^k \mathbb{I}$ denotes $(\Sigma\mathbb{I})^{\wedge k}$ for $k > 0$ and denotes $\mathbb{I}$ for $k = 0$.

Construction 5.1 Let $(\mathcal{C}, \wedge, \mathbb{I})$ denote a pointed cofibrantly generated closed symmetric monoidal model category whose unit $\mathbb{I}$ is cofibrant. Furthermore, assume that $\mathcal{C}$ satisfies the monoid axiom and the smallness axioms of [22]. This implies that the category of modules over a monoid in $\mathcal{C}$ carries an induced model structure where the weak equivalences and the fibrations are those created by the forgetful functor to $\mathcal{C}$ [22, 4.1]. For a given M as above, we construct a monoid structure on

$$\bigvee_{m \in M_-} \Sigma^{|m|}\mathbb{I},$$

where $\vee$ denotes the coproduct in $\mathcal{C}$. The multiplication map

$$\left(\bigvee_{m \in M_-} \Sigma^{|m|}\mathbb{I}\right) \wedge \left(\bigvee_{n \in M_-} \Sigma^{|n|}\mathbb{I}\right) \cong \bigvee_{(m,n) \in M_- \times M_-} \Sigma^{|m|+|n|}\mathbb{I} \to \bigvee_{m \in M_-} \Sigma^{|m|}\mathbb{I} \tag{11}$$

is given (on the cofactor corresponding to $(m,n) \in M \times M$) by the inclusion of the cofactor corresponding to $mn \in M$ if $mn \neq 0$ and given by the zero map if $mn = 0$.

Note that in a pointed model category, there is a unique zero map between every pair of objects which is defined to be the map that factors through the point object. One easily checks that the multiplication above is associative and unital.

If E is a commutative monoid in $\mathcal{C}$, then the category of E–modules is also a symmetric monoidal model category [22, 4.1]. We let $\bigvee_{m \in M_-} \Sigma^{|m|} E$ denote the monoid we obtain by applying the construction above in the category of E–modules. In particular, $\bigvee_{m \in M_-} \Sigma^{|m|} E$ is an E–algebra.

Using the construction above, we obtain an HR–algebra $\bigvee_{m \in M_-} \Sigma^{|m|} HR$. In order to prove Theorem 1.4, we go through the zigzag of Quillen equivalences between the model categories of R–DGAs and HR–algebras to show that the HR–algebra corresponding to the formal R–DGA with homology $R\langle M \rangle$ is given by $\bigvee_{m \in M_-} \Sigma^{|m|} HR$ [26]. We deduce that the formal R–DGA with homology $R\langle M \rangle$ is R–extension by showing that $\bigvee_{m \in M_-} \Sigma^{|m|} HR$ is weakly equivalent to $HR \wedge c\big(\bigvee_{m \in M_-} \Sigma^{|m|} \mathbb{S}\big)$ in HR–algebras where c denotes the cofibrant replacement functor in $\mathbb{S}$–algebras. For this, we start with the following lemmas.

Lemma 5.2 *Assume that $(\mathcal{C}, \wedge, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$ are pointed and closed symmetric monoidal model categories with cofibrant units. Furthermore, let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

be a Quillen pair, where F denotes the left adjoint. If there is a weak equivalence $\upsilon \colon F(\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \mathbb{I}_{\mathcal{D}}$, then there exists a weak equivalence

$$\varphi \colon F(\Sigma \mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \Sigma \mathbb{I}_{\mathcal{D}}.$$

Proof By factoring the map $\mathbb{I}_{\mathcal{C}} \to *$ by a cofibration followed by a trivial fibration, we obtain a factorization $F(\mathbb{I}_{\mathcal{C}}) \rightarrowtail F(\bar{*}) \xrightarrow{\sim} F(*) \cong *$. Note that the isomorphism follows by the fact that F is a left adjoint functor between pointed categories. To see that the second map is a weak equivalence, note that $*$ is cofibrant in the pointed model category $\mathcal{C}$ and that F preserves all weak equivalences between cofibrant objects. Similarly, we have a factorization $\mathbb{I}_{\mathcal{D}} \rightarrowtail \bar{*} \overset{\sim}{\twoheadrightarrow} *$ consisting of a cofibration followed by a trivial fibration. We use the equivalence $\upsilon \colon F(\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \mathbb{I}_{\mathcal{D}}$ and the lift in the square

$$\begin{tikzcd} F(\mathbb{I}_{\mathcal{C}}) \arrow[r, "\upsilon", "\sim"'] \arrow[d, tail] & \mathbb{I}_{\mathcal{D}} \arrow[r, tail] & \bar{*} \arrow[d, two heads, "\sim"] \\ F(\bar{*}) \arrow[urr, dashed, "\sim"] \arrow[rr, "\sim"] & & F(*) \cong * \end{tikzcd}$$

to obtain a weak equivalence of diagrams

$$(F(\bar{*}) \leftarrowtail F(\mathbb{I}_{\mathcal{C}}) \rightarrowtail F(\bar{*})) \xrightarrow{\sim} (\bar{*} \leftarrowtail \mathbb{I}_{\mathcal{D}} \rightarrowtail \bar{*}).$$

This in turn gives a map φ of the corresponding pushouts of these diagrams. This is a weak equivalence because these are diagrams consisting only of cofibrations between cofibrant objects; therefore their pushout is the homotopy pushout. Since the pushout of the diagram on the left-hand side is $F(\Sigma\mathbb{I}_{\mathcal{C}})$ and the pushout of the diagram on the right-hand side is $\Sigma\mathbb{I}_{\mathcal{D}}$, we obtain the weak equivalence

$$\varphi\colon F(\Sigma\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \Sigma\mathbb{I}_{\mathcal{D}}$$

we wanted to construct. □

Lemma 5.3 *Assume that $(\mathcal{C}, \wedge, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \wedge, \mathbb{I}_{\mathcal{D}})$ are pointed and closed symmetric monoidal model categories with cofibrant units as in Construction 5.1. Furthermore, let*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a Quillen pair where the left adjoint F is a strong monoidal functor. In this situation, $Fc\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{D}}$ are weakly equivalent as monoids in $\mathcal{D}$, where c denotes the cofibrant replacement functor in the model category of monoids in $\mathcal{C}$ [22, 4.1].

Proof Since F is a strong monoidal functor, we have a natural isomorphism

$$F(X) \wedge F(Y) \cong F(X \wedge Y)$$

and an isomorphism $F(\mathbb{I}_{\mathcal{C}}) \cong \mathbb{I}_{\mathcal{D}}$. This isomorphism provides the weak equivalence υ in the hypothesis of Lemma 5.2. Thus, there is a weak equivalence $\varphi\colon F(\Sigma\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \Sigma\mathbb{I}_{\mathcal{D}}$.

Using φ, we produce a weak equivalence of monoids,

$$\Phi\colon F\Big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\Big) \cong \bigvee_{m\in M_-} F(\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}) \xrightarrow{\sim} \bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_{\mathcal{D}}.$$

Here, Φ is the coproduct of maps given by the isomorphism $F(\mathbb{I}_{\mathcal{C}}) \cong \mathbb{I}_{\mathcal{D}}$ for $|m| = 0$ and the map

$$F(\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}) = F((\Sigma\mathbb{I}_{\mathcal{C}})^{\wedge|m|}) \cong F(\Sigma\mathbb{I}_{\mathcal{C}})^{\wedge|m|} \xrightarrow[\sim]{\varphi^{\wedge|m|}} (\Sigma\mathbb{I}_{\mathcal{D}})^{\wedge|m|} = \Sigma^{|m|}\mathbb{I}_{\mathcal{D}}$$

for $|m| > 0$, where the first and the last equalities follow by our definition of $\Sigma^k -$ for $k > 0$ and the second isomorphism comes from the strong monoidal structure of F.

Also, note that $\varphi^{\wedge|m|}$ is a weak equivalence because it is a smash product of weak equivalences between cofibrant objects. Since Φ is a coproduct of weak equivalences between cofibrant objects, it is a weak equivalence by [28, Lemma 4.7]. It is clear that Φ is a map of monoids by the definition of the monoidal structure on both sides and from the fact that left adjoint functors between pointed categories preserve the zero maps. This shows that Φ is a weak equivalence of monoids between $F\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{D}}$.

Therefore, in order to finish the proof of the lemma, it is sufficient to show that the monoids $Fc\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $F\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ are weakly equivalent. Since c is the cofibrant replacement functor in the category of monoids, there is a weak equivalence of monoids

$$f: c\Big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\Big) \xrightarrow{\sim} \bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}.$$

By [22, Theorem 4.1], the source of f is cofibrant in $\mathcal{C}$. This means that f is a weak equivalence between cofibrant objects and therefore $F(f)$ is a weak equivalence. Furthermore, $F(f)$ is a weak equivalence of monoids because a strong monoidal functor preserves maps of monoids. Therefore, the monoids $Fc\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ and $F\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_{\mathcal{C}}\big)$ are weakly equivalent as desired. $\square$

5B From DGAs to $H\mathbb{Z}$–algebras

Here, we carry out our discussion for the case $R=\mathbb{Z}$. The case of general discrete commutative ring R follows similarly.

The DGA corresponding to an $H\mathbb{Z}$–algebra is obtained using the zigzag of monoidal Quillen equivalences of [26]

$$H\mathbb{Z}\text{-}\mathcal{M}od \underset{U}{\overset{Z}{\rightleftarrows}} \mathrm{Sp}^{\Sigma}(s\mathcal{A}B) \underset{\phi^*N}{\overset{L}{\leftrightarrows}} \mathrm{Sp}^{\Sigma}(\mathcal{C}h^+) \underset{R}{\overset{D}{\rightleftarrows}} \mathcal{C}h,$$

where the left adjoints are the top arrows and the pairs (Z,U) and (D,R) are both strong monoidal Quillen equivalences. The pair (L,ϕ^*N) is a weak monoidal Quillen equivalence. See [23, 3.6] for the definitions of strong monoidal Quillen equivalences and weak monoidal Quillen equivalences. We often use the fact that the model categories in the zigzag above are pointed.

Since each Quillen equivalence in the zigzag is a monoidal Quillen equivalence, there is an induced zigzag of Quillen equivalences of the corresponding model categories of monoids. This gives the induced derived functors $\mathbb{H}\colon \mathcal{DGA}\to H\mathbb{Z}\text{–}\mathcal{A}lg$ and

$\Theta\colon H\mathbb{Z}\text{–}\mathcal{A}lg \to \mathcal{D}GA$ in [26, Theorem 1.1]. We have

$$\Theta = Dc\phi^* NZc, \quad \mathbb{H} = UL^{\mathrm{mon}}cR,$$

where L^{mon} is the induced left adjoint at the level of monoids and c denotes the cofibrant replacement functors in the corresponding model category of monoids. See [23, Section 3.3] for a definition of the induced left adjoint at the level of monoids. Recall that for a given DGA X, we often write HX to denote $\mathbb{H}X$ or a cofibrant and/or fibrant replacement of $\mathbb{H}X$ as an $H\mathbb{Z}$–algebra.

In the lemmas below, $\mathbb{I}_1$ and $\mathbb{I}_2$ denote the monoidal units of $\mathrm{Sp}^\Sigma(s\mathcal{A}B)$ and $\mathrm{Sp}^\Sigma(\mathcal{C}h^+)$ respectively. Note that the units of the monoidal model categories in the zigzag above are all cofibrant [26, Definition 2.1 and Corollary 3.4]. By Construction 5.1, we have the monoids $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1$ and $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2$ in $\mathrm{Sp}^\Sigma(s\mathcal{A}B)$ and $\mathrm{Sp}^\Sigma(\mathcal{C}h^+)$, respectively.

Lemma 5.4 *In $\mathrm{Sp}^\Sigma(s\mathcal{A}B)$, $Zc\bigl(\bigvee_{m\in M_-}\Sigma^{|m|}H\mathbb{Z}\bigr)$ and $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1$ are weakly equivalent as monoids. In $\mathcal{C}h$, $Dc(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2)$ and the formal DGA with homology $\mathbb{Z}\langle M\rangle$ are quasi-isomorphic as DGAs.*

Proof The first statement is a direct consequence of Lemma 5.3. We prove the second statement of the lemma. It again follows by Lemma 5.3 that $Dc\bigl(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2\bigr)$ and $\bigoplus_{m\in M_-}\Sigma^{|m|}\mathbb{Z}$ are quasi-isomorphic as DGAs (ie weakly equivalent as monoids in $\mathcal{C}h$).

Therefore, it is sufficient to show that $\bigoplus_{m\in M_-}\Sigma^{|m|}\mathbb{Z}$ is quasi-isomorphic to the formal DGA with homology $\mathbb{Z}\langle M\rangle$. Let $\bar{0}$ denote the chain complex consisting of $\mathbb{Z}$ in degrees 0 and 1 and the trivial module in the rest of the degrees; its differentials are trivial except degree 1 where the differential is the identity. There is a factorization $\mathbb{Z} \rightarrowtail \bar{0} \xrightarrow{\sim}\!\!\!\!\twoheadrightarrow 0$ of the trivial map $\mathbb{Z} \to 0$ as a cofibration followed by a trivial fibration.

Let $\sigma\mathbb{Z}$ denote the chain complex consisting of $\mathbb{Z}$ in degree 1 and the trivial module in the rest of the degrees. This is the pushout of the diagram $\bar{0} \leftarrowtail \mathbb{Z} \to 0$.

Note that due to our conventions, $\Sigma\mathbb{Z}$ is the pushout of the diagram $\bar{0} \leftarrowtail \mathbb{Z} \to \bar{0}$. Since the category of chain complexes is left proper, there is a weak equivalence $\varphi\colon \Sigma\mathbb{Z} \xrightarrow{\sim} \sigma\mathbb{Z}$. Let $\sigma^n\mathbb{Z}$ denote $(\sigma\mathbb{Z})^{\otimes n}$. Following Construction 5.1, we obtain a formal DGA $\bigoplus_{m\in M_-}\sigma^{|m|}\mathbb{Z}$. Similar to the map Φ in the proof of Lemma 5.3, we obtain a quasi-isomorphism of DGAs

$$\Phi\colon \bigoplus_{m\in M_-}\Sigma^{|m|}\mathbb{Z} \xrightarrow{\sim} \bigoplus_{m\in M_-}\sigma^{|m|}\mathbb{Z}$$

given by the identity map for $|m|=0$ and given by $\varphi^{|m|}$ for $|m|>0$. This shows that $\bigoplus_{m\in M_-}\Sigma^{|m|}\mathbb{Z}$ and $\bigoplus_{m\in M_-}\sigma^{|m|}\mathbb{Z}$ are quasi-isomorphic as DGAs where the latter is the formal DGA with homology $\mathbb{Z}\langle M\rangle$. □

We state and prove the following two lemmas, which we use in the proof of Lemma 5.7.

Lemma 5.5 *The functor ϕ^*N preserves colimits.*

Proof The category of symmetric spectra in a closed symmetric monoidal model category $\mathcal{C}$ is the category of modules over a monoid in symmetric sequences in $\mathcal{C}$; see [26, Definition 2.7]. Since symmetric sequences in $\mathcal{C}$ is a diagram category in $\mathcal{C}$, the colimits in symmetric sequences are levelwise. Furthermore, the forgetful functor from modules over a monoid to the underlying closed monoidal category preserves colimits. Therefore colimits of symmetric spectra in $\mathcal{C}$ are also levelwise.

Here, N is the normalization functor $s\mathcal{AB}\to\mathcal{C}h^+$ of the Dold–Kan correspondence, an equivalence of categories, applied levelwise. Therefore it preserves colimits. Furthermore, ϕ^* is the restriction of scalars functor between the categories of modules over two monoids induced by a map of these monoids in symmetric sequences in $\mathcal{C}h^+$; see [26, page 358]. Therefore ϕ^* is the identity functor on the underlying symmetric sequences and therefore it also preserves colimits. □

Lemma 5.6 *For every cofibrant A in $\mathrm{Sp}^\Sigma(\mathcal{C}h^+)$ and every B in $\mathrm{Sp}^\Sigma(s\mathcal{AB})$, a map $L(A)\to B$ is a weak equivalence if and only if its adjoint $A\to\phi^*N(B)$ is a weak equivalence.*

Proof This follows from the fact that ϕ^*N preserves weak equivalences. Let $B\xrightarrow{\sim} fB$ be a fibrant replacement of B. The adjoint of the composite $L(A)\to B\xrightarrow{\sim} fB$ is given by the composite $A\to\phi^*N(B)\xrightarrow{\sim}\phi^*N(fB)$ whose first map is the adjoint of the map $L(A)\to B$. Because (L,ϕ^*N) is a Quillen equivalence, the first composite is a weak equivalence if and only if the second composite is a weak equivalence. The result follows by the two-out-of-three property of weak equivalences. □

The following lemma takes care of the middle step in the zigzag of Quillen equivalences between the model categories of $H\mathbb{Z}$–algebras and DGAs. Note that since (L,ϕ^*N) is a weak monoidal Quillen pair, ϕ^*N is a lax monoidal functor; see [23, Definition 3.3]. Therefore, ϕ^*N carries monoids to monoids. In particular, $\phi^*N\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1\big)$ is a monoid.

Lemma 5.7 *In* $\mathrm{Sp}^\Sigma(\mathcal{C}h^+)$, $\phi^* N\big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1\big)$ *and* $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2$ *are weakly equivalent as monoids.*

Proof By Lemma 5.5, ϕ^*N preserves coproducts. Therefore, there is an isomorphism

$$\phi^* N\Big(\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_1\Big)\cong\bigvee_{m\in M_-}\phi^* N(\Sigma^{|m|}\mathbb{I}_1). \tag{12}$$

Similar to Construction 5.1, the object on the right-hand side above carries a canonical monoid structure given by the multiplication on M and the lax monoidal structure of ϕ^*N. Namely, the multiplication map

$$\bigvee_{m\in M_-}\phi^* N(\Sigma^{|m|}\mathbb{I}_1)\wedge\bigvee_{n\in M_-}\phi^* N(\Sigma^{|n|}\mathbb{I}_1)\to\bigvee_{m\in M_-}\phi^* N(\Sigma^{|m|}\mathbb{I}_1)$$

is given (on the cofactor corresponding to $(m,n)\in M_-\times M_-$) by the composite

$$\phi^* N(\Sigma^{|m|}\mathbb{I}_1)\wedge\phi^* N(\Sigma^{|n|}\mathbb{I}_1)\to\phi^* N(\Sigma^{|m|}\mathbb{I}_1\wedge\Sigma^{|n|}\mathbb{I}_1)=\phi^* N(\Sigma^{|mn|}\mathbb{I}_1)$$

followed by the inclusion of the cofactor corresponding to $mn\in M$ if $mn\neq 0$ and given by the zero map if $mn=0$. Note that the map above is the lax monoidal structure map of ϕ^*N and the equality above follows by our definition of Σ^k-. Furthermore, one checks using this definition that the isomorphism in (12) is an isomorphism of monoids. Therefore, in order to prove the lemma, it is sufficient to show that there is an isomorphism of monoids between $\bigvee_{m\in M_-}\phi^* N(\Sigma^{|m|}\mathbb{I}_1)$ and $\bigvee_{m\in M_-}\Sigma^{|m|}\mathbb{I}_2$.

There is a weak equivalence $L(\mathbb{I}_2)\xrightarrow{\sim}\mathbb{I}_1$ since (L,ϕ^*N) is a weak monoidal Quillen pair; see [23, 3.6]. Therefore, there is also a weak equivalence $\varphi\colon L(\Sigma\mathbb{I}_2)\xrightarrow{\sim}\Sigma\mathbb{I}_1$ by Lemma 5.2. Let

$$\psi\colon\Sigma\mathbb{I}_2\to\phi^* N(\Sigma\mathbb{I}_1)$$

be the adjoint of φ.

Let ψ^0 denote the unit $\mathbb{I}_2\to\phi^*N(\mathbb{I}_1)$ of the lax monoidal structure of ϕ^*N and let ψ^1 denote ψ. For $\ell>1$, let ψ^ℓ denote the composite

$$\psi^\ell\colon\Sigma^\ell\mathbb{I}_2=(\Sigma\mathbb{I}_2)^{\wedge\ell}\xrightarrow{\psi^{\wedge\ell}}(\phi^*N(\Sigma\mathbb{I}_1))^{\wedge\ell}\to\phi^*N((\Sigma\mathbb{I}_1)^{\wedge\ell})=\phi^*N(\Sigma^\ell\mathbb{I}_1),$$

where the equalities follow by our definition of Σ^ℓ and the second map is obtained by successive applications of the transformation $\phi^*N(-)\wedge\phi^*N(-)\to\phi^*N(-\wedge-)$ that is a part of the lax monoidal structure of ϕ^*N; see [23, 3.3].

Now we define a map of monoids

$$\Psi\colon \bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{I}_2 \to \bigvee_{m\in M_-} \phi^* N(\Sigma^{|m|}\mathbb{I}_1)$$

as the coproduct of $\psi^{|m|}$ over $m \in M_-$. By the associativity and the unitality of the lax monoidal structure on $\phi^* N$ and by the fact that right adjoint functors preserve the zero maps between pointed categories, Ψ is a map of monoids; see [6, 6.4.1].

Finally, we need to show that Ψ is a weak equivalence. By Lemmas 5.5 and 5.6, it is sufficient to show that the adjoint of Ψ is a weak equivalence. Since both $\phi^* N$ and L preserve coproducts and since Ψ is a coproduct of maps $\psi^{|m|}$, the adjoint of Ψ is a coproduct of the adjoints of the maps $\psi^{|m|}$. Note that a coproduct of weak equivalences of cofibrant objects is again a weak equivalence by [28, 4.7]. Since the adjoint of ψ^ℓ is a map between cofibrant objects, it is sufficient to show that the adjoint of ψ^ℓ is a weak equivalence for each $\ell \geq 0$.

For the case $\ell = 0$, we have that the adjoint of ψ^0 is the weak equivalence $L(\mathbb{I}_2) \xrightarrow{\sim} \mathbb{I}_1$ mentioned above. For $\ell = 1$, the adjoint of ψ^1 is the map φ above which is also a weak equivalence.

We show the $\ell = 2$ case and the rest follow similarly. Let $m_{\phi^* N}$ denote the natural transformation

$$m_{\phi^* N}\colon \phi^* N(-\wedge -) \to \phi^* N(-) \wedge \phi^* N(-)$$

that is part of the lax monoidal structure of $\phi^* N$. We show that the adjoint to the composite defining ψ^2

$$\psi^2\colon \Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\psi\wedge\psi} \phi^* N(\Sigma\mathbb{I}_1) \wedge \phi^* N(\Sigma\mathbb{I}_1) \xrightarrow{m_{\phi^* N}} \phi^* N(\Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1)$$

is the composite map

$$L(\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2) \xrightarrow[\simeq]{c_L} L(\Sigma\mathbb{I}_2) \wedge L(\Sigma\mathbb{I}_2) \xrightarrow[\simeq]{\varphi\wedge\varphi} \Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1. \tag{13}$$

The first map in this composite is the comonoidal map induced by the lax monoidal structure of $\phi^* N$ and this is a weak equivalence since $(L, \phi^* N)$ is a weak monoidal Quillen pair [26, 4.4]. Furthermore, the second map in the composite is a smash product of weak equivalences between cofibrant objects; therefore, it is also a weak equivalence. This shows that the composite is a weak equivalence.

To show that ψ^2 is the adjoint to this composite, first note that by the discussion on equation (3.4) in [23], the comonoidal map c_L is the adjoint of the composite map

$$\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\eta\wedge\eta} \phi^* NL(\Sigma\mathbb{I}_2) \wedge \phi^* NL(\Sigma\mathbb{I}_2) \xrightarrow{m_{\phi^* N}} \phi^* N(L(\Sigma\mathbb{I}_2) \wedge L(\Sigma\mathbb{I}_2)),$$

where η denotes the unit of the adjunction $(L, \phi^* N)$. Considering the adjoint of the composite (13) as the adjoint of the first map c_L in the composite followed by $\phi^* N(\varphi \wedge \varphi)$, we obtain that the adjoint of (13) is given by the composite

$$\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\eta\wedge\eta} \phi^* NL(\Sigma\mathbb{I}_2) \wedge \phi^* NL(\Sigma\mathbb{I}_2) \xrightarrow{m_{\phi^* N}} \phi^* N(L(\Sigma\mathbb{I}_2) \wedge L(\Sigma\mathbb{I}_2)) \xrightarrow{\phi^* N(\varphi\wedge\varphi)} \phi^* N(\Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1).$$

By the naturality of $m_{\phi^* N}$, this composite is equal to the canonical composite

$$\Sigma\mathbb{I}_2 \wedge \Sigma\mathbb{I}_2 \xrightarrow{\eta\wedge\eta} \phi^* NL(\Sigma\mathbb{I}_2) \wedge \phi^* NL(\Sigma\mathbb{I}_2) \xrightarrow{\phi^* N(\varphi)\wedge\phi^* N(\varphi)} \phi^* N(\Sigma\mathbb{I}_1) \wedge \phi^* N(\Sigma\mathbb{I}_1) \xrightarrow{m_{\phi^* N}} \phi^* N(\Sigma\mathbb{I}_1 \wedge \Sigma\mathbb{I}_1).$$

Note that the composition of the first two maps is the smash product of adjoints of φ which is $\psi \wedge \psi$. Therefore, this composite is precisely the composite that defines ψ^2 above. This shows that the adjoint of ψ^2 is the composite weak equivalence in (13). $\square$

5C Proof of Theorem 1.4

We prove the following proposition which provides an explicit description of the HR–algebra corresponding to the formal R–DGA with homology $R\langle M\rangle$. After that, we use this description to prove Theorem 1.4.

Proposition 5.8 *The R–DGA corresponding to the HR–algebra $\bigvee_{m\in M_-} \Sigma^{|m|} HR$ is the formal R–DGA with homology $R\langle M\rangle$. Furthermore, there is an equivalence of HR–algebras*

$$\bigvee_{m\in M_-} \Sigma^{|m|} HR \simeq HR \wedge c\Bigl(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\Bigr),$$

where c denotes the cofibrant replacement functor in $\mathbb{S}$–algebras.

Proof For the first statement, we discuss the case $R = \mathbb{Z}$, the proof for general R follows similarly. The first statement is a consequence of Lemmas 5.4 and 5.7.

Now we prove the second statement. Recall that $HR \wedge -$ is a symmetric monoidal functor between $\mathbb{S}$–modules and HR–modules. Therefore, the second statement is consequence of Lemma 5.3. $\square$

Theorem 1.4 *Let X be a connective formal R–DGA whose homology has a homogeneous basis as an R–module containing the multiplicative unit such that the multiplication of two basis elements is either zero or a basis element. In this situation, X is R–extension. As a result, we have the equivalence of spectra,*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

Proof Recall from Remark 1.5 that the homology of X is $R\langle M\rangle$ for some monoid M in nonnegatively graded pointed sets. In other words, X is the formal R–DGA with homology $R\langle M\rangle$. Using Proposition 5.8, we deduce that the HR–algebra corresponding to X is $\bigvee_{m\in M_-} \Sigma^{|m|} HR$. By the equivalence given in Proposition 5.8, X is an R–extension R–DGA.

Since X is an R–extension R–DGA, the splitting for $\mathrm{THH}(X)$ is a consequence of Proposition 1.3. □

We are ready to prove the following corollaries of our results.

Corollary 1.17 *Let $R = \mathbb{Z}/(m)$ for some integer $m \neq \pm 1$, let Y be an R–DGA and let X be as in Theorem 1.4. Then X and Y are quasi-isomorphic if and only if they are topologically equivalent.*

Proof By Remark 1.5, X is the formal R–DGA with homology $R\langle M\rangle$ for some monoid M in nonnegatively graded pointed sets. Using Proposition 5.8, we deduce that the HR–algebra corresponding to X is given by $HR \wedge c\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\big)$, where c denotes the cofibrant replacement functor in HR–algebras. In particular, $Z = c\big(\bigvee_{m\in M_-} \Sigma^{|m|}\mathbb{S}\big)$ is weakly equivalent as a spectrum to a wedge of suspensions of the sphere spectrum. We deduce that X satisfies the hypothesis of Theorem 1.16. This implies that X and Y are quasi-isomorphic if and only if they are topologically equivalent. □

Corollary 1.18 *Assume that X and Y are as in Theorem 1.14 or Theorem 1.16. Then X and Y are Morita equivalent if and only if they are dg Morita equivalent.*

Proof We need to show that the model categories of X–modules and Y–modules are additively Quillen equivalent if and only if they are Quillen equivalent [9, 7.7]. By definition, additively Quillen equivalent additive model categories are Quillen equivalent [8]. Therefore, we only need to prove one direction.

If the model categories of X–modules and Y–modules are Quillen equivalent then there exists a fibrant and cofibrant representative P of a compact generator of the homotopy category of Y–modules such that the endomorphism DGA $\mathrm{End}_{Y\text{–mod}}(P)$ of P is topologically equivalent to X [9, 7.2]. Since Y is an R–DGA, $\mathrm{End}_{Y\text{–mod}}(P)$ is also an R–DGA. It follows by Theorems 1.14 and 1.16 that $\mathrm{End}_{Y\text{–mod}}(P)$ quasi-isomorphic to X. By [9, 7.2], this implies that the model categories of X–modules and Y–modules are additively Quillen equivalent, as desired. □

Appendix A

Here, we provide a short discussion on the compatibility of Definitions 1.1 and 1.2.

If we choose our E_∞ operad to be the Barratt–Eccles operad, then every E_∞ R–DGA is at the same time an R–DGA; see [4, Section 1.1.1]. Let X be an R–extension E_∞ R–DGA and let $U(X)$ denote its underlying R–DGA. The canonical compatibility question asks if $U(X)$ is R–extension as an R–DGA. In other words, we want to know if every R–extension E_∞ R–DGA forgets to an R–extension R–DGA.

Let $H_{E_\infty}X$ denote the commutative HR–algebra corresponding to X and let $HU(X)$ denote the HR–algebra corresponding to $U(X)$. For the moment, assume that $H_{E_\infty}X$ is weakly equivalent to $HU(X)$ as an HR–algebra. Under this assumption, we conclude that $U(X)$ is R–extension. To see this, let $H_{E_\infty}X \simeq HR \wedge E$ for some cofibrant commutative $\mathbb{S}$–algebra E and let c denote the cofibrant replacement functor in $\mathbb{S}$–algebras. Since cofibrant (commutative) $\mathbb{S}$–algebras forget to cofibrant $\mathbb{S}$–modules [22; 25] and since the left Quillen functor $HR\wedge -$ preserves weak equivalences between cofibrant objects, we deduce that $HR \wedge E$ is equivalent to $HR \wedge cE$ in HR–algebras. Hence, $HU(X)$ is weakly equivalent to $HR \wedge cE$ and therefore $U(X)$ is R–extension, as desired.

However, it is not known whether $H_{E_\infty}X$ and $HU(X)$ are weakly equivalent in HR–algebras. In other words, it is not known if the zigzag of Quillen equivalences between HR–algebras and R–DGAs in [26] is compatible with the zigzag of Quillen equivalences between commutative HR–algebras and E_∞ R–DGAs in [19]. In conclusion, if we assume that these Quillen equivalences are compatible, then Definitions 1.1 and 1.2 are also compatible in the sense described above.

Appendix B

Here, we provide a proof of Proposition 1.3. Indeed, we prove the following more general statement.

Proposition B.1 *Let $\varphi\colon A \to B$ be a map of commutative $\mathbb{S}$–algebras and let X be a B–algebra. If X is φ–extension, ie if $X \simeq B \wedge_A E$ for some cofibrant A–algebra E, then there is the equivalence of spectra*

$$\mathrm{THH}^A(X) \simeq \mathrm{THH}^A(B) \wedge_B \mathrm{THH}^B(X).$$

Furthermore, if X is a commutative B–algebra that is weakly equivalent to $B \wedge_A E$ for some cofibrant commutative A–algebra E, then the equivalence above is an equivalence of commutative ring spectra.

Proof Let $X \simeq B \wedge_A E$ for some cofibrant A–algebra E. The equivalence in the proposition is given by the composite of the chain of equivalences

$$\begin{aligned} \mathrm{THH}^A(B \wedge_A E) &\simeq \mathrm{THH}^A(B) \wedge_A \mathrm{THH}^A(E) \\ &\simeq \mathrm{THH}^A(B) \wedge_B (B \wedge_A \mathrm{THH}^A(E)) \\ &\simeq \mathrm{THH}^A(B) \wedge_B \mathrm{THH}^B(B \wedge_A E). \end{aligned} \tag{14}$$

The first equivalence follows by the fact that $\mathrm{THH}^A(-)$ is a monoidal functor and the last equivalence follows by the base change formula for topological Hochschild homology; see [15, Conventions]. The base change formula and the monoidality of $\mathrm{THH}^A(-)$ can be easily shown using the cyclic bar construction defining topological Hochschild homology [10, IX.2.1].

When E is a cofibrant commutative A–algebra, the equivalences given in (14) are those of commutative A–algebras. This is because $\mathrm{THH}^A(-)$ is a symmetric monoidal functor and the base change formula provides an equivalence of commutative A–algebras. □

The following is the special case of the proposition above corresponding to the map of commutative $\mathbb{S}$–algebras $\mathbb{S} \to HR$. Note that for an R–DGA X, we let $\mathrm{THH}(X)$ denote $\mathrm{THH}(HX)$ and $\mathrm{HH}^R(X)$ denote $\mathrm{THH}^{HR}(HX)$. For an E_∞ R–DGA X, we let $\mathrm{THH}(X)$ denote $\mathrm{THH}(H_{E_\infty} X)$ and $\mathrm{HH}^R(X)$ denote $\mathrm{THH}^{HR}(H_{E_\infty} X)$.

Proposition 1.3 *If X is an R–extension R–DGA, then there is an equivalence of spectra*

$$\mathrm{THH}(X) \simeq \mathrm{THH}(HR) \wedge_{HR} \mathrm{HH}^R(X).$$

If X is an R–extension E_∞ R–DGA, then the equivalence above is an equivalence of commutative $\mathbb{S}$–algebras.

Proof For an R–extension R–DGA X, we have that HX satisfies the first hypothesis of Proposition B.1 for the map of commutative $\mathbb{S}$–algebras $\varphi\colon \mathbb{S} \to HR$. This provides the equivalence in the proposition. Similarly, for an R–extension E_∞ R–DGA X, $H_{E_\infty} X$ satisfies the last hypothesis of Proposition B.1. This provides the second statement of the proposition. □

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Bounded cohomology of classifying spaces for families of subgroups

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We introduce a bounded version of Bredon cohomology for groups relative to a family of subgroups. Our theory generalizes bounded cohomology and differs from Mineyev and Yaman's relative bounded cohomology for pairs. We obtain cohomological characterizations of relative amenability and relative hyperbolicity, analogous to the results of Johnson and Mineyev for bounded cohomology.

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1 Introduction

Bounded cohomology is a homotopy invariant of topological spaces with deep connections to Riemannian geometry via the simplicial volume of manifolds; see Gromov [7]. An astonishing phenomenon known as Gromov's mapping theorem is that for every CW–complex X, the classifying map $X \to B\pi_1(X)$ induces an isometric isomorphism on bounded cohomology. This emphasizes the importance of the corresponding theory of bounded cohomology for groups, which is also of independent interest due to its plentiful applications in geometric group theory; see Frigerio [6] and Monod [20; 21]. The bounded cohomology $H_b^n(G;V)$ of a (discrete) group G with coefficients in a normed G–module V is the cohomology of the cochain complex of bounded G–maps $G^{n+1} \to V$. The inclusion of bounded G–maps into (not necessarily bounded) G–maps induces the so-called *comparison map* $H_b^n(G;V) \to H^n(G;V)$. On the one hand, the bounded cohomology groups are very difficult to compute in general. On the other hand, they characterize interesting group-theoretic properties such as amenability — Johnson [11] — and hyperbolicity — Mineyev [17; 18].

Theorem 1.1 (Johnson) *Let G be a group. The following are equivalent*:

(i) *G is amenable.*

(ii) *$H_b^n(G;V^\#) = 0$ for all dual normed $\mathbb{R}G$–modules $V^\#$ and all $n \geq 1$.*

(iii) *$H_b^1(G;V^\#) = 0$ for all dual normed $\mathbb{R}G$–modules $V^\#$.*

Theorem 1.2 (Mineyev) *Let G be a finitely presented group. The following are equivalent*:

(i) *G is hyperbolic.*

(ii) *The comparison map $H_b^n(G;V) \to H^n(G;V)$ is surjective for all normed $\mathbb{Q}G$–modules V and all $n \geq 2$.*

(iii) *The comparison map $H_b^2(G;V) \to H^2(G;V)$ is surjective for all normed $\mathbb{R}G$–modules V.*

There are well-studied notions of relative amenability and relative hyperbolicity in the literature; see Hruska [8] and Ji, Ogle and Ramsey [10]. In the present article we introduce a new "relative bounded cohomology theory" characterizing these relative group-theoretic properties as a bounded version of Bredon cohomology. For a group G, a family of subgroups $\mathcal{F}$ is a nonempty set of subgroups which is closed under conjugation and taking subgroups. For a set of subgroups $\mathcal{H}$ of G, we denote by $\mathcal{F}\langle\mathcal{H}\rangle$ the smallest family containing $\mathcal{H}$. The Bredon cohomology $H_{\mathcal{F}}^n(G;V)$ with coefficients in a G–module V (or more general coefficient systems) is a generalization of group cohomology, which is recovered when $\mathcal{F}$ consists only of the trivial subgroup. A fundamental feature of Bredon cohomology is that for a normal subgroup N of G there is an isomorphism $H_{\mathcal{F}\langle N\rangle}^n(G;V) \cong H^n(G/N;V^N)$. From a topological point of view, the Bredon cohomology of G can be identified with the equivariant cohomology of the classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$, which is a terminal object in the G–homotopy category of G–CW–complexes with stabilizers in $\mathcal{F}$. Especially the classifying spaces $E_{\mathcal{FIN}}G$ and $E_{\mathcal{VCY}}G$ for the family of finite groups and virtually cyclic groups have received a lot of attention in recent years due to their prominent role in the isomorphism conjectures of Baum–Connes and Farrell–Jones, respectively.

We introduce the *bounded Bredon cohomology* $H^n_{\mathcal{F},b}(G;V)$ of G with coefficients in a normed G–module V, which generalizes bounded cohomology (Definition 3.1). Our theory still is well-behaved with respect to normal subgroups (Corollary 3.11) and admits a topological interpretation in terms of classifying spaces for families (Theorem 3.10). We obtain the following generalizations of Theorems 1.1 and 1.2. A group G is called amenable relative to a set of subgroups $\mathcal{H}$ if there exists a G–invariant mean on the G–set $\coprod_{H\in\mathcal{H}} G/H$.

Theorem 1.3 *Let G be a group and $\mathcal{H}$ be a set of subgroups. The following are equivalent*:

(i) *G is amenable relative to $\mathcal{H}$.*

(ii) *$H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V^\#)=0$ for all dual normed $\mathbb{R}G$–modules $V^\#$ and all $n\geq 1$.*

(iii) *$H^1_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V^\#)=0$ for all dual normed $\mathbb{R}G$–modules $V^\#$.*

Theorem 1.3 is a special case of the more general Theorem 4.5. We also provide a characterization of relative amenability in terms of relatively injective modules (Proposition 4.8). Recall that a finite set of subgroups $\mathcal{H}$ is called a malnormal (resp. almost malnormal) collection if for all $H_i, H_j\in\mathcal{H}$ and $g\in G$ we have that $H_i\cap gH_jg^{-1}$ is trivial (resp. finite), unless $i=j$ and $g\in H_i$. A group G is said to be of type $F_{n,\mathcal{F}}$ for a family of subgroups $\mathcal{F}$, if there exists a model for the classifying space $E_{\mathcal{F}}G$ with cocompact n–skeleton.

Theorem 1.4 (Theorem 5.4) *Let G be a finitely generated torsionfree group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Suppose that G is of type $F_{2,\mathcal{F}\langle\mathcal{H}\rangle}$ (eg G and all subgroups in $\mathcal{H}$ are finitely presented). Then the following are equivalent*:

(i) *G is hyperbolic relative to $\mathcal{H}$.*

(ii) *The comparison map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V)\to H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V)$ is surjective for all normed $\mathbb{Q}G$–modules V and all $n\geq 2$.*

(iii) *The comparison map $H^2_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V)\to H^2_{\mathcal{F}\langle\mathcal{H}\rangle}(G;V)$ is surjective for all normed $\mathbb{R}G$–modules V.*

In Theorem 1.4 the equivalence of (i) and (iii) still holds if the group G contains torsion and $\mathcal{H}$ is almost malnormal, see Remark 5.5. Note that condition (iii) is trivially satisfied for groups of Bredon cohomological dimension $\mathrm{cd}_{\mathcal{F}\langle\mathcal{H}\rangle}$ equal to 1.

The topological interpretation of bounded Bredon cohomology via classifying spaces for families was used by Löh and Sauer [12] to give a new proof of the nerve theorem and vanishing theorem for amenable covers. We prove a converse of [12, Proposition 5.2], generalizing a recent result of Moraschini and Raptis [23, Theorem 3.1.3], where the case of a normal subgroup is treated.

Theorem 1.5 *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent*:

(i) *All subgroups in $\mathcal{F}$ are amenable.*

(ii) *The canonical map $H^n_{\mathcal{F},b}(G; V^\#) \to H^n_b(G; V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^\#$ and all $n \geq 0$.*

(iii) *The canonical map $H^1_{\mathcal{F},b}(G; V^\#) \to H^1_b(G; V^\#)$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^\#$.*

Theorem 1.5 is a special case of the more general Theorem 4.5. As an application of Theorem 1.5, the comparison map vanishes for groups which admit a "small" model for $E_{\mathcal{F}}G$, where $\mathcal{F}$ is any family consisting of amenable subgroups (Corollary 4.6). Examples are graph products of amenable groups (eg right-angled Artin groups) and fundamental groups of graphs of amenable groups.

There is another natural relative cohomology theory given by the relative cohomology of a pair of spaces. For a set of subgroups $\mathcal{H}$, it gives rise to the cohomology $H^n(G, \mathcal{H}; V)$ of the group pair $(G, \mathcal{H})$ introduced by Bieri and Eckmann [2]. A bounded version $H^n_b(G, \mathcal{H}; V)$ was defined by Mineyev and Yaman [19] to give a characterization of relative hyperbolicity; see also Franceschini [5]. A characterization of relative amenability in terms of this relative theory was obtained in [10]. There is a canonical map $H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G; V) \to H^n(G, \mathcal{H}; V)$ for $n \geq 2$ which is an isomorphism if $\mathcal{H}$ is malnormal; see Remark 2.1. Similarly, there is a map for the bounded versions but we do not know when it is an isomorphism due to the failure of the excision axiom for bounded cohomology; see Remark 3.12. We also mention that Mineyev and Yaman's relative bounded cohomology was extended to pairs of groupoids by Blank in [3].

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2 Preliminaries on Bredon cohomology and classifying spaces

In this section we briefly recall the notion of Bredon cohomology for groups and its topological interpretation as the equivariant cohomology of classifying spaces for families of subgroups. For an introduction to Bredon cohomology we refer to [4] and for a survey on classifying spaces to [13].

Let G be a group, which shall always mean a discrete group. A *family of subgroups* $\mathcal{F}$ is a nonempty set of subgroups of G that is closed under conjugation by elements of G and under taking subgroups. Typical examples are

$$\begin{aligned}\mathcal{TR} &= \{1\},\\ \mathcal{FIN} &= \{\text{finite subgroups}\},\\ \mathcal{VCY} &= \{\text{virtually cyclic subgroups}\},\\ \mathcal{ALL} &= \{\text{all subgroups}\}.\end{aligned}$$

We will moreover be interested in $\mathcal{AME} = \{\text{amenable subgroups}\}$. For a subgroup H of G, we denote by $\mathcal{F}|_H$ the family $\{L \cap H \mid L \in \mathcal{F}\}$ of subgroups of H. (In the literature this family is sometimes denoted by $\mathcal{F} \cap H$ instead.) For a set of subgroups $\mathcal{H}$, one can consider the smallest family containing $\mathcal{H}$ which is defined by $\mathcal{F}\langle\mathcal{H}\rangle = \{\text{conjugates of elements in } \mathcal{H} \text{ and their subgroups}\}$ and called the *family generated by* $\mathcal{H}$. When $\mathcal{H}$ consists of a single subgroup H, we denote $\mathcal{F}\langle\mathcal{H}\rangle$ instead by $\mathcal{F}\langle H\rangle$ and call it the *family generated by* H. We denote by $G/\mathcal{H}$ the G–set $\coprod_{H\in\mathcal{H}} G/H$.

Let R be a ring and Mod_R denote the category of R–modules. We will often suppress the ring R, so that G–modules are understood to be RG–modules. The (*$\mathcal{F}$–restricted*) *orbit category* $\mathcal{O}_{\mathcal{F}}G$ has as objects G–sets of the form G/H with $H \in \mathcal{F}$ and as morphisms G–maps. An *$\mathcal{O}_{\mathcal{F}}G$–module* is a contravariant functor $M : \mathcal{O}_{\mathcal{F}}G \to \mathrm{Mod}_R$, the category of which is denoted by $\mathcal{O}_{\mathcal{F}}G$–$\mathrm{Mod}_R$. Note that $\mathcal{O}_{\mathcal{TR}}G$–$\mathrm{Mod}_R$ can be identified with the category of G–modules (see eg [4, Chapter 1, Section 4]). For a G–module V, there is a coinduced $\mathcal{O}_{\mathcal{F}}G$–module $V^?$ given by $V^?(G/H) = V^H$. (In the literature this is sometimes called a *fixed-point functor*.) Observe that $(\cdot)^?$ is right-adjoint to the restriction $\mathcal{O}_{\mathcal{F}}G\text{–}\mathrm{Mod}_R \to \mathcal{O}_{\mathcal{TR}}G\text{–}\mathrm{Mod}_R$, $M \mapsto M(G/1)$; see eg [4, Proposition 1.31]. That is, for every $\mathcal{O}_{\mathcal{F}}G$–module M and G–module V there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{–}\mathrm{Mod}_R}(M, V^?) \cong \mathrm{Hom}_{RG}(M(G/1), V). \tag{2-1}$$

For a G–space X and a G–CW–complex Y with stabilizers in $\mathcal{F}$, there are singular and cellular $\mathcal{O}_{\mathcal{F}}G$–chain complexes

$$C_*(X^?)(G/H) = C_*(X^H) \quad \text{and} \quad C_*^{\mathrm{cell}}(Y^?)(G/H) = C_*^{\mathrm{cell}}(Y^H),$$

where $C_*(X^H)$ and $C_*^{\mathrm{cell}}(Y^H)$ denote the usual singular and cellular chain complexes, respectively.

The *Bredon cohomology* of G with coefficients in an $\mathcal{O}_{\mathcal{F}}G$–module M is defined as the R–module

$$H^n_{\mathcal{F}}(G; M) := \mathrm{Ext}^n_{\mathcal{O}_{\mathcal{F}}G\text{–}\mathrm{Mod}_R}(R, M)$$

for $n \geq 0$, where R is regarded as a constant $\mathcal{O}_{\mathcal{F}}G$–module. It can be computed as the cohomology of the cochain complex $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{–}\mathrm{Mod}_R}(R[((G/\mathcal{F})^{*+1})^?], M)$; see eg [4, Proposition 3.5]. We define the G–chain complex $C_*^{\mathcal{F}}(G)$ given by G–modules

$$C_n^{\mathcal{F}}(G) := R[(G/\mathcal{F})^{n+1}]$$

with the diagonal G–action and differentials $\partial_n \colon C_n^{\mathcal{F}}(G) \to C_{n-1}^{\mathcal{F}}(G)$,

$$\partial_n(g_0H_0, \dots, g_nH_n) = \sum_{i=0}^{n} (-1)^i (g_0H_0, \dots, \widehat{g_iH_i}, \dots, g_nH_n).$$

For a G–module V, the G–cochain complex $C^*_{\mathcal{F}}(G; V)$ is given by

$$C^n_{\mathcal{F}}(G; V) := \mathrm{Hom}_R(C_n^{\mathcal{F}}(G), V)$$

so that by adjunction (2-1),

$$H^n_{\mathcal{F}}(G; V) := H^n_{\mathcal{F}}(G; V^?) \cong H^n(C^*_{\mathcal{F}}(G; V)^G).$$

For a G–space X with stabilizers in $\mathcal{F}$, the *Bredon cohomology* of X with coefficients in an $\mathcal{O}_{\mathcal{F}}G$–module M is defined as

$$H^n_G(X; M) := H^n(\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{–}\mathrm{Mod}_R}(C_*(X^?), M))$$

for $n \geq 0$. If X is a G–CW–complex, then $H^n_G(X; M)$ can be computed using $C_*^{\mathrm{cell}}(X^?)$ instead of $C_*(X^?)$.

A *classifying space* $E_{\mathcal{F}}G$ *for the family* $\mathcal{F}$ is a terminal object in the G–homotopy category of G–CW–complexes with stabilizers in $\mathcal{F}$. It can be shown that a G–CW–complex X is a model for $E_{\mathcal{F}}G$ if and only if the fixed-point set X^H is contractible for $H \in \mathcal{F}$ and empty otherwise; see eg [13, Theorem 1.9]. An explicit model is given by the geometric realization Y of the semisimplicial set $\{(G/\mathcal{F})^{n+1} \mid n \geq 0\}$ with the usual face maps. Then Y has (nonequivariant) n–cells corresponding to $(G/\mathcal{F})^{n+1}$ and we refer to Y as the *simplicial model* for $E_{\mathcal{F}}G$. Note that a model for $E_{\mathcal{TR}}G$ is given by EG and a model for $E_{\mathcal{ALL}}G$ is the point G/G. The cellular $\mathcal{O}_{\mathcal{F}}G$–chain complex of any model for $E_{\mathcal{F}}G$ is a projective resolution of the constant $\mathcal{O}_{\mathcal{F}}G$–module R (see eg [4, Proposition 2.9]) and thus we have

$$H^n_{\mathcal{F}}(G; M) \cong H^n_G(E_{\mathcal{F}}G; M) \tag{2-2}$$

for all $\mathcal{O}_{\mathcal{F}}G$–modules M. If N is a normal subgroup of G, then a model for $E_{\mathcal{F}\langle N\rangle}G$ is given by $E(G/N)$ regarded as a G–CW–complex and we find

$$H^n_{\mathcal{F}\langle N\rangle}(G; M) \cong H^n(G/N; M(G/N)) \tag{2-3}$$

(see eg [1, Corollary 4.11]).

For a subgroup H of G, when viewed as an H–space $E_{\mathcal{F}}G$ is a model for $E_{\mathcal{F}|_H}H$ which induces the restriction map

$$\mathrm{res}^n_{H\subset G} \colon H^n_{\mathcal{F}}(G; M) \to H^n_{\mathcal{F}|_H}(H; M) \tag{2-4}$$

for all $\mathcal{O}_{\mathcal{F}}G$–modules M. For two families of subgroups $\mathcal{F} \subset \mathcal{G}$, the up to G–homotopy unique G–map $E_{\mathcal{F}}G \to E_{\mathcal{G}}G$ induces the canonical map

$$\mathrm{can}^n_{\mathcal{F}\subset\mathcal{G}} \colon H^n_{\mathcal{G}}(G; M) \to H^n_{\mathcal{F}}(G; M) \tag{2-5}$$

for all $\mathcal{O}_{\mathcal{G}}G$–modules M.

Remark 2.1 (Bieri and Eckmann's relative cohomology) For a group G and a set of subgroups $\mathcal{H}$, Bieri and Eckmann [2] have introduced the *relative cohomology* $H^n(G, \mathcal{H}; V)$ *of the pair* $(G, \mathcal{H})$ with coefficients in a G–module V. It can be identified with the relative cohomology $H^n_G\big(EG, \coprod_{H\in\mathcal{H}} G \times_H EH; V\big)$ of the pair of G–spaces $\big(EG, \coprod_{H\in\mathcal{H}} G \times_H EH\big)$. Here a model for EG is chosen that contains $\coprod_{H\in\mathcal{H}} G \times_H EH$ as a subcomplex by taking mapping cylinders. Hence there is a long exact sequence

$$\cdots \to H^n(G, \mathcal{H}; V) \to H^n(G; V) \to \prod_{H\in\mathcal{H}} H^n(H; V) \to \cdots,$$

which is one of the main features of the relative cohomology groups.

There is a relation between Bredon cohomology and Bieri and Eckmann's relative cohomology as follows. Consider the G–space X obtained as the G–pushout

$$\begin{tikzcd} \coprod_{H\in\mathcal{H}} G\times_H EH \arrow[r] \arrow[d] & EG \arrow[d] \\ \coprod_{H\in\mathcal{H}} G/H \arrow[r] & X \end{tikzcd}$$

where the left vertical map is induced by collapsing each EH to a point. Then the G–space X has stabilizers in $\mathcal{F}\langle\mathcal{H}\rangle$ and hence admits a G–map $X\to E_{\mathcal{F}\langle\mathcal{H}\rangle}G$. For an $\mathcal{O}_{\mathcal{F}}G$–module M, we have maps

$$\begin{tikzcd} H^n_G(X;M) & H^n_G(X,\coprod_{H\in\mathcal{H}} G/H;M) \arrow[l] \arrow[d, "\cong"] \\ H^n_G(E_{\mathcal{F}\langle\mathcal{H}\rangle}G;M) \arrow[u] & H^n_G(EG,\coprod_{H\in\mathcal{H}} G\times_H EH;M) \end{tikzcd}$$

where the right vertical map is an isomorphism by excision. Now, if $\mathcal{H}$ is a malnormal collection, then X is a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ and

$$H^n_{\mathcal{F}\langle\mathcal{H}\rangle}(G;M)\cong H^n(G,\mathcal{H};M(G/1))$$

for $n\geq 2$. This was shown in [1, Theorem 4.16] for the special case when $\mathcal{H}$ consists of a single subgroup.

3 Bounded Bredon cohomology

In this section we introduce a bounded version of Bredon cohomology and develop some of its basic properties. We follow the exposition in [6] for bounded cohomology. Throughout, let G be a group and $\mathcal{F}$ be a family of subgroups.

From now on, let the ring R be one of $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. A *normed G–module V* is a G–module equipped with a G–invariant norm $\|\cdot\|\colon V\to\mathbb{R}$. (That is, for all $v,u\in V$, $r\in R$, and $g\in G$ we have $\|v\|=0$ if and only if $v=0$, $\|rv\|\leq|r|\cdot\|v\|$, $\|v+u\|\leq\|v\|+\|u\|$, and $\|g\cdot v\|=\|v\|$.) A morphism $f\colon V\to W$ of normed G–modules is a morphism of G–modules with finite operator-norm $\|f\|_\infty$. We denote by $\mathrm{bHom}_R(V,W)$ the G–module of R–linear maps $f\colon V\to W$ with finite operator-norm, where the G–action is given by $(g\cdot f)(v)=g\cdot f(g^{-1}v)$. We denote the topological dual $\mathrm{bHom}_R(V,\mathbb{R})$ of V by $V^\#$. For a set S and a normed module V, we denote by $\mathrm{bMap}(S,V)$ the module of functions $S\to V$ with bounded image. Instead of $\mathrm{bMap}(S,\mathbb{R})$ we also write $\ell^\infty(S)$.

The following is our key definition. Recall the notation $G/\mathcal{F} = \coprod_{H\in\mathcal{F}} G/H$ and consider $C_n^{\mathcal{F}}(G) = R[(G/\mathcal{F})^{n+1}]$ as a normed G–module equipped with the ℓ^1–norm with respect to the R–basis $(G/\mathcal{F})^{n+1}$. For a normed G–module V, we define the cochain complex $C^*_{\mathcal{F},b}(G;V)$ of normed G–modules by

$$C^n_{\mathcal{F},b}(G;V) := \mathrm{bHom}_R(C_n^{\mathcal{F}}(G), V)$$

together with the differentials $\delta^n\colon C^n_{\mathcal{F},b}(G;V) \to C^{n+1}_{\mathcal{F},b}(G;V)$,

$$\delta^n(f)(g_0H_0,\dots,g_{n+1}H_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0H_0,\dots,\widehat{g_iH_i},\dots,g_{n+1}H_{n+1}).$$

Definition 3.1 (bounded Bredon cohomology of groups) The *bounded Bredon cohomology* of G with coefficients in a normed G–module V is defined as

$$H^n_{\mathcal{F},b}(G;V) := H^n(C^*_{\mathcal{F},b}(G;V)^G)$$

for $n \geq 0$. The inclusion $C^n_{\mathcal{F},b}(G;V) \subset C^n_{\mathcal{F}}(G;V)$ induces a map

$$c^n_{\mathcal{F}}\colon H^n_{\mathcal{F},b}(G;V) \to H^n_{\mathcal{F}}(G;V),$$

called the *comparison map*.

Note that for $\mathcal{F} = \mathcal{TR}$, Definition 3.1 recovers the usual definition of bounded cohomology.

Remark 3.2 (coefficient modules) We only consider normed G–modules as coefficients, rather than more general $\mathcal{O}_{\mathcal{F}}G$–modules equipped with a "compatible norm". Hence strictly speaking our theory is a bounded version of Nucinkis' cohomology relative to the G–set $G/\mathcal{F}$ [24], rather than a bounded version of Bredon cohomology.

Remark 3.3 (canonical seminorm) The ℓ^∞–norm on $C^n_{\mathcal{F},b}(G;V)$ descends to a *canonical seminorm* on $H^n_{\mathcal{F},b}(G;V)$. However, we do not consider seminorms anywhere in this article and regard $H^n_{\mathcal{F},b}(G;V)$ merely as an R–module.

Bounded Bredon cohomology satisfies the following basic properties.

Lemma 3.4 *The following hold*:

(i) *Let $0 \to V_0 \to V_1 \to V_2 \to 0$ be a short exact sequence of normed G–modules such that $0 \to V_0^H \to V_1^H \to V_2^H \to 0$ is exact for each $H \in \mathcal{F}$. Then there exists a long exact sequence*

$$0 \to H^0_{\mathcal{F},b}(G;V_0) \to H^0_{\mathcal{F},b}(G;V_1) \to H^0_{\mathcal{F},b}(G;V_2) \to H^1_{\mathcal{F},b}(G;V_0) \to \cdots.$$

(ii) $H^0_{\mathcal{F},b}(G;V) \cong V^G$ *for all normed G–modules V.*

(iii) $H^1_{\mathcal{F},b}(G;\mathbb{R}) = 0$.

Proof (i) For a G–set $S = \coprod_{i\in I} G/H_i$ and a normed G–module V, we can identify the module $\mathrm{bMap}_G(S,V)$ with the submodule of $\prod_{i\in I} V^{H_i}$ consisting of the elements $(v_i)_{i\in I}$ satisfying $\sup_{i\in I}\|v_i\| < \infty$. It follows that for a G–set S with stabilizers in $\mathcal{F}$, the sequence of modules

$$0 \to \mathrm{bMap}_G(S,V_0) \to \mathrm{bMap}_G(S,V_1) \to \mathrm{bMap}_G(S,V_2) \to 0$$

is exact. Applying the above to the G–sets $(G/\mathcal{F})^{n+1}$ for $n \geq 0$, we obtain that the sequence of cochain complexes

$$0 \to C^*_{\mathcal{F},b}(G;V_0)^G \to C^*_{\mathcal{F},b}(G;V_1)^G \to C^*_{\mathcal{F},b}(G;V_2)^G \to 0$$

is exact. Then the associated long exact sequence on cohomology is as desired.

(ii) We have $H^0_{\mathcal{F},b}(G;V) = \ker(\delta^0)$, where

$$\delta^0 \colon \mathrm{bHom}_{RG}(R[G/\mathcal{F}],V) \to \mathrm{bHom}_{RG}(R[(G/\mathcal{F})^2],V)$$

is given by $\delta^0(f)(g_0H_0,g_1H_1) = f(g_1H_1) - f(g_0H_0)$. Hence $\ker(\delta^0)$ consists precisely of the constant G–maps $G/\mathcal{F} \to V$, which are in correspondence to V^G.

(iii) We identify

$$C^n_{\mathcal{F},b}(G;\mathbb{R})^G \cong \mathrm{bMap}\Bigl(\coprod_{H_0,\dots,H_n\in\mathcal{F}} H_0\backslash(G/H_1\times\cdots\times G/H_n),\mathbb{R}\Bigr)$$

for $n \geq 1$ and $C^0_{\mathcal{F},b}(G;\mathbb{R})^G \cong \mathrm{bMap}\bigl(\coprod_{H_0\in\mathcal{F}} *_{H_0},\mathbb{R}\bigr)$. The differentials of this "inhomogeneous" complex in low degrees are given by

$$\delta^0(f)(H_0g_1H_1) = f(*_{H_1}) - f(*_{H_0}),$$
$$\delta^1(\varphi)(H_0(g_1H_1,g_2H_2)) = \varphi(H_1g_1^{-1}g_2H_2) - \varphi(H_0g_2H_2) + \varphi(H_0g_1H_1).$$

Then it is not difficult to check that $\ker(\delta^1) = \mathrm{im}(\delta^0)$. □

We also define the bounded cohomology of a G–space X as follows. Denote by $S_n(X)$ the set of singular n–simplices in X and consider $C_n(X) = R[S_n(X)]$ equipped with the ℓ^1–norm as a normed G–module. For a normed G–module V, we define the cochain complex $C^*_b(X;V)$ of normed G–modules by

$$C^n_b(X;V) := \mathrm{bHom}_R(C_n(X),V)$$

together with the usual differentials.

Definition 3.5 (bounded cohomology of G–spaces) The (G–*equivariant*) *bounded cohomology* of a G–space X with coefficients in a normed G–module V is defined as

$$H^n_{G,b}(X;V) := H^n(C^*_b(X;V)^G)$$

for $n \geq 0$. The inclusion $C^n_b(X;V) \subset C^n(X;V)$ induces a map

$$c^n_X\colon H^n_{G,b}(X;V) \to H^n_G(X;V)$$

called the *comparison map*.

Note that the functors $H^*_{G,b}$ are G–homotopy invariant and that $H^n_{G,b}(G/H;V)$ is isomorphic to V^H for $n=0$ and trivial otherwise. However, beware that $H^*_{G,b}$ is neither a G–cohomology theory, nor can it be computed cellularly for G–CW–complexes, as is the case already when G is the trivial group; see eg [6, Remark 5.6].

Relative homological algebra We develop the relative homological algebra that will allow us to compute bounded Bredon cohomology via resolutions, analogous to Ivanov's approach for bounded cohomology [9].

A map $p\colon A \to B$ of G–modules is called *$\mathcal{F}$–strongly surjective* if for each $H \in \mathcal{F}$ there exists a map $\tau_H : B \to A$ of H–modules such that $p \circ \tau_H = \mathrm{id}_B$. A G–module P is called *relatively $\mathcal{F}$–projective* if for every $\mathcal{F}$–strongly surjective G–map $p\colon A \to B$ and every G–map $\phi\colon P \to B$, there exists a G–map $\Phi\colon P \to A$ such that $p \circ \Phi = \phi$. A chain complex of G–modules is called *relatively $\mathcal{F}$–projective* if each chain module is relatively $\mathcal{F}$–projective. A resolution (C_*, ∂_*) of G–modules is called *$\mathcal{F}$–strong* if it is contractible as a resolution of H–modules for each $H \in \mathcal{F}$. (That is, there exist H–maps $k^H_* : C_* \to C_{*+1}$ such that $\partial_{n+1} \circ k^H_n + k^H_{n-1} \circ \partial_n = \mathrm{id}_{C_n}$.)

Lemma 3.6 *The following hold*:

(i) *If S is a G–set with stabilizers in $\mathcal{F}$, then the G–module $R[S]$ is relatively $\mathcal{F}$–projective.*

(ii) *If S is a G–set with $S^H \neq \varnothing$ for all $H \in \mathcal{F}$, then the resolution $R[S^{*+1}] \to R$ of G–modules is $\mathcal{F}$–strong.*

(iii) *If X is a G–space with contractible fixed-point set X^H for each $H \in \mathcal{F}$, then the resolution $C_*(X) \to R$ of G–modules is $\mathcal{F}$–strong.*

Proof (i) Given a lifting problem as in the definition of relative $\mathcal{F}$–projectivity,

$$\begin{tikzcd} & R[S] \arrow[ld, dashed, "\Phi"'] \arrow[d, "\phi"] & \\ A \arrow[r, "p"] & B \arrow[r] \arrow[l, bend left, "\tau_H"] & 0 \end{tikzcd}$$

we construct a lift Φ as follows. Let T be a set of representatives of $G\backslash S$ and denote the stabilizer of an element $t \in T$ by G_t. Then for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1} s = t_s$. Define $\Phi\colon R[S] \to A$ on generators by

$$\Phi(s) = g_s \cdot \tau_{G_{t_s}}(\phi(g_s^{-1} s))$$

which is independent of the choice of g_s, since the map $\tau_{G_{t_s}}$ is G_{t_s}–equivariant. Then Φ is a G–equivariant lift of ϕ.

(ii) For $H \in \mathcal{F}$, fix an element $s_H \in S^H$ and define $k_*^H\colon R[S^{*+1}] \to R[S^{*+2}]$ on generators by

$$k_n^H(s_0, \dots, s_n) = (s_H, s_0, \dots, s_n).$$

Then k_*^H is an H–equivariant contraction.

(iii) For $H \in \mathcal{F}$, fix a point $x_H \in X^H$ and define a contraction $k_*^H : C_*(X) \to C_{*+1}(X)$ of H–chain complexes inductively as follows. Starting with $k_{-1}^H\colon R \to C_0(X)$, given by $r \mapsto r \cdot x_H$, we may assume that k_{n-1}^H has been constructed. Let s be a singular n–simplex in X and denote its stabilizer by H_s. Then there exists a singular $(n+1)$–simplex s' with 0^{th} vertex x_H and opposite face s satisfying $\partial_{n+1}(s') + k_{n-1}^H(\partial_n(s)) = s$. Moreover, since X^{H_s} is contractible we may choose s' such that its image is contained in X^{H_s}. Now, for each H–orbit of singular n–simplices in X choose a representative s, define $k_n^H(s)$ to be s', and then extend H–equivariantly. $\square$

The proof of the following proposition is standard and omitted.

Proposition 3.7 *Let $f\colon V \to W$ be a map of G–modules, $P_* \to V$ be a G–chain complex with P_n relatively $\mathcal{F}$–projective for all $n \geq 0$, and $C_* \to W$ be an $\mathcal{F}$–strong resolution of G–modules. Then there exists a G–chain map $f_*\colon P_* \to C_*$ extending f, which is unique up to G–chain homotopy.*

While relatively $\mathcal{F}$–projective $\mathcal{F}$–strong resolutions are useful to compute Bredon homology, the following dual approach will compute bounded Bredon cohomology.

A map $i\colon A \to B$ of normed G–modules is called *$\mathcal{F}$–strongly injective* if for each $H \in \mathcal{F}$ there exists a map $\sigma_H\colon B \to A$ of normed H–modules with $\|\sigma_H\|_\infty \le K$ such that $\sigma_H \circ i = \mathrm{id}_A$, for a uniform constant $K \ge 0$. A normed G–module I is called *relatively $\mathcal{F}$–injective* if for every $\mathcal{F}$–strongly injective G–map $i\colon A \to B$ and every map $\psi\colon A \to I$ of normed G–modules, there exists a map $\Psi\colon B \to I$ of normed G–modules such that $\Psi \circ i = \psi$. A chain complex of normed G–modules is called *relatively $\mathcal{F}$–injective* if each chain module is relatively $\mathcal{F}$–injective. A resolution of normed G–modules is called *$\mathcal{F}$–strong* if it is contractible as a resolution of normed H–modules for each $H \in \mathcal{F}$.

Dually to Lemma 3.6 and Proposition 3.7 we obtain the following.

Lemma 3.8 *Let V be a normed G–module. The following hold*:

(i) *If S is a G–set with stabilizers in $\mathcal{F}$, then $\mathrm{bHom}_R(R[S], V)$ is a relatively $\mathcal{F}$–injective normed G–module.*

(ii) *If S is a G–set with $S^H \neq \varnothing$ for all $H \in \mathcal{F}$, then the resolution*

$$V \to \mathrm{bHom}_R(R[S^{*+1}], V)$$

of normed G–modules is $\mathcal{F}$–strong.

(iii) *If X is a G–space with contractible fixed-point set X^H for each $H \in \mathcal{F}$, then the resolution $V \to C_b^*(X; V)$ of normed G–modules is $\mathcal{F}$–strong.*

Proof Given an extension problem as in the definition of relative $\mathcal{F}$–injectivity,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"'] \arrow[d, "\psi"'] & B \arrow[l, bend right, "\sigma_H"'] \arrow[dl, dashed, "\Psi"] \\ & \mathrm{bHom}_R(R[S], V) & \end{tikzcd}$$

we construct an extension Ψ as follows. Let T be a set of representatives of $G\backslash S$ and denote the stabilizer of an element $t \in T$ by G_t. Then for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1} s = t_s$. Define

$$\Psi\colon B \to \mathrm{bHom}_R(R[S], V)$$

for $b \in B$ and $s \in S$ by

$$\Psi(b)(s) = \psi(g_s \cdot \sigma_{G_{t_s}}(g_s^{-1} b))(s).$$

One checks that Ψ is a well-defined map of normed RG–modules extending ψ.

The proofs of (ii) and (iii) are dual to those of Lemma 3.6(ii) and (iii), respectively, and are left to the reader. □

Proposition 3.9 *Let $f\colon V \to W$ be a map of normed G–modules, $V \to C^*$ be an $\mathcal{F}$–strong resolution of normed G–modules, and $W \to I^*$ be a G–chain complex with I^n relatively $\mathcal{F}$–injective for all $n \geq 0$. Then there exists a G–chain map $f^*\colon C^* \to I^*$ extending f, which is unique up to G–chain homotopy.*

As a consequence of Proposition 3.9, we may use any relatively $\mathcal{F}$–injective $\mathcal{F}$–strong resolution to compute bounded Bredon cohomology. We obtain the isomorphisms analogous to (2-2) and (2-3) for Bredon cohomology.

Theorem 3.10 *Let G be a group, $\mathcal{F}$ be a family of subgroups, and V be a normed G–module. For all $n \geq 0$ there is an isomorphism*

$$H^n_{\mathcal{F},b}(G;V) \cong H^n_{G,b}(E_{\mathcal{F}}G;V).$$

Proof Both $C^*_{\mathcal{F},b}(G;V)$ and $C^*_b(E_{\mathcal{F}}G;V)$ are relatively $\mathcal{F}$–injective $\mathcal{F}$–strong resolutions of V by Lemma 3.8; hence G–chain homotopy equivalent by Proposition 3.9. □

Corollary 3.11 *Let G be a group, N be a normal subgroup of G, and V be a normed G–module. For all $n \geq 0$ there is an isomorphism*

$$H^n_{\mathcal{F}\langle N\rangle,b}(G;V) \cong H^n_b(G/N;V^N).$$

Proof As a model for $E_{\mathcal{F}\langle N\rangle}G$ we take $E(G/N)$ regarded as a G–space. Then it suffices to observe that

$$\mathrm{bHom}_{RG}(R[S_n(E(G/N))],V) \cong \mathrm{bHom}_{R[G/N]}(R[S_n(E(G/N))],V^N)$$

and to apply Theorem 3.10 twice. □

Analogous to (2-4) and (2-5) for Bredon cohomology, for a subgroup H of G and two families of subgroups $\mathcal{F} \subset \mathcal{G}$, we have the maps

$$\mathrm{res}^n_{H\subset G,b}\colon H^n_{\mathcal{F},b}(G;V) \to H^n_{\mathcal{F}|_H,b}(H;V), \quad \mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}\colon H^n_{\mathcal{G},b}(G;V) \to H^n_{\mathcal{F},b}(G;V)$$

for all normed G–modules V.

Remark 3.12 (Mineyev and Yaman's relative bounded cohomology) Mineyev and Yaman have introduced the bounded analogue of Bieri and Eckmann's relative cohomology for pairs (Remark 2.1) in [19]. For a group G, a finite set of subgroups $\mathcal{H}$, and a normed G–module V, their relative bounded cohomology groups $H^n_b(G,\mathcal{H};V)$ can be identified with $H^n_{G,b}\big(EG,\coprod_{H\in\mathcal{H}} G\times_H EH;V\big)$ and therefore fit in a long exact sequence

$$\cdots\to H^n_b(G,\mathcal{H};V)\to H^n_b(G;V)\to\prod_{H\in\mathcal{H}} H^n_b(H;V)\to\cdots.$$

As in Remark 2.1, we denote by X the G–space obtained as a G–pushout from EG by collapsing $G\times_H EH$ to G/H for each $H\in\mathcal{H}$. Then we have maps

$$\begin{tikzcd} H^n_{G,b}(X;V) & H^n_{G,b}(X,\coprod_{H\in\mathcal{H}} G/H;V) \arrow[l] \arrow[d] \\ H^n_{G,b}(E_{\mathcal{F}\langle\mathcal{H}\rangle}G;V) \arrow[u] & H^n_{G,b}(EG,\coprod_{H\in\mathcal{H}} G\times_H EH;V) \end{tikzcd}$$

where, for $n\geq 2$, the horizontal map is an isomorphism by the long exact sequence of a pair, using the fact that $H^*_{G,b}(G/H;V)=0$ for $*\geq 1$. Hence for $n\geq 2$ we obtain a map

$$H^n_{\mathcal{F}\langle\mathcal{H}\rangle,b}(G;V)\to H^n_b(G,\mathcal{H};V).$$

However, even if $\mathcal{H}$ is a malnormal collection in which case X is a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$, this map need not be an isomorphism due to the failure of the excision axiom for bounded cohomology.

4 Characterization of relative amenability

In this section we prove a characterization of relatively amenable groups in terms of bounded Bredon cohomology analogous to Theorem 1.1.

Recall that a *G–invariant mean* on a G–set S is an $\mathbb{R}$–linear map $m\colon \ell^\infty(S)\to\mathbb{R}$ which is normalized, nonnegative, and G–invariant. (That is, for the constant function $1\in\ell^\infty(S)$, $f\in\ell^\infty(S)$, and $g\in G$ we have $m(1)=1$, $m(f)\geq 0$ if $f\geq 0$, and $m(g\cdot f)=m(f)$.) Note that for a G–map $S_1\to S_2$ of G–sets, a G–invariant mean on S_1 is pushed forward to a G–invariant mean on S_2.

Definition 4.1 (relative amenability) A group G is *amenable relative* to a set of subgroups $\mathcal{H}$ if the G–set $G/\mathcal{H}$ admits a G–invariant mean. When G is amenable relative to $\mathcal{H}$ consisting of a single subgroup H, we say that H is *coamenable* in G.

When $\mathcal{H}$ is a finite set of subgroups, we recover the notion of relative amenability studied in [10]; see also [22].

Example 4.2 Let G be a group, H be a subgroup, and $\mathcal{H}$ be a set of subgroups.

(i) If G is amenable, then G is amenable relative to $\mathcal{H}$.

(ii) If H is a normal subgroup, then H is coamenable in G if and only if the quotient group G/H is amenable.

(iii) If H has finite index in G or contains the commutator subgroup $[G,G]$, then H is coamenable in G.

(iv) If $\mathcal{H}$ is finite and G is amenable relative to $\mathcal{H}$, then $\mathcal{H}$ contains an element that is coamenable in G.

(v) G is amenable relative to $\mathcal{H}$ if and only if G is amenable relative to $\mathcal{F}\langle\mathcal{H}\rangle$.

The following lemma is proved analogously to [6, Lemma 3.2]; see also [20, Corollary 5.3.8].

Lemma 4.3 *Let G be a group and $\mathcal{H}$ be a set of subgroups. Then G is amenable relative to $\mathcal{H}$ if and only if there exists a nontrivial G–invariant element in $\ell^\infty(G/\mathcal{H})^\#$.*

By Proposition 3.9 bounded Bredon cohomology can be computed using relatively $\mathcal{F}$–injective $\mathcal{F}$–strong resolutions. If one considers coefficients in dual normed $\mathbb{R}G$–modules, then such resolutions can be obtained from G–sets whose stabilizers are amenable relative to $\mathcal{F}$.

Lemma 4.4 *Let G be a group, $\mathcal{F}$ be a family of subgroups, and $V^\#$ be a dual normed $\mathbb{R}G$–module. If S is a G–set such that every stabilizer G_s is amenable relative to $\mathcal{F}|_{G_s}$, then the normed $\mathbb{R}G$–module* $\mathrm{bHom}_{\mathbb{R}}(\mathbb{R}[S^{n+1}], V^\#)$ *is relatively $\mathcal{F}$–injective for all $n \geq 0$.*

Proof Since the stabilizers of S^{n+1} are intersections of stabilizers of S, and relative amenability passes to subgroups, it is enough to consider the case $n=0$. Let an extension problem as in the definition of relative $\mathcal{F}$–injectivity

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"'] \arrow[d, "\psi"'] & B \arrow[l, bend right, "\sigma_H"'] \arrow[dl, dashed, "\Psi"] \\ & \mathrm{bHom}_{\mathbb{R}}(\mathbb{R}[S], V^\#) & \end{tikzcd}$$

be given. Let T be a set of representatives of $G\backslash S$. We denote the stabilizer of an element $t \in T$ by G_t and by assumption there exists a G_t–invariant mean m_t on $G_t/\mathcal{F}|_{G_t}$. Note that any subgroup $L \in \mathcal{F}|_{G_t}$ can also be viewed as an element in $\mathcal{F}$. Now, for every $s \in S$ there exist unique elements $t_s \in T$ and $g_s G_{t_s} \in G/G_{t_s}$ such that $g_s^{-1} s = t_s$. Define $\Psi\colon B \to \mathrm{bHom}_{\mathbb{R}}(\mathbb{R}[S], V^{\#})$ for $b \in B$, $s \in S$, and $v \in V$ by

$$\Psi(b)(s)(v) = m_{t_s}\big(gL \mapsto (g_s g \cdot \psi(\sigma_L(g^{-1} g_s^{-1} b)))(s)(v)\big).$$

One checks that Ψ is a well-defined map of normed $\mathbb{R}G$–modules extending ψ. $\square$

For a family of subgroups $\mathcal{F}$, consider the short exact sequence of normed $\mathbb{R}G$–modules

$$0 \to \mathbb{R} \to \ell^\infty(G/\mathcal{F}) \to \ell^\infty(G/\mathcal{F})/\mathbb{R} \to 0,$$

where $\mathbb{R}$ is regarded as the constant functions. Then the sequence of topological duals

$$0 \to (\ell^\infty(G/\mathcal{F})/\mathbb{R})^{\#} \to \ell^\infty(G/\mathcal{F})^{\#} \to \mathbb{R} \to 0$$

is exact, since an $\mathbb{R}$–linear split $\mathbb{R} \to \ell^\infty(G/\mathcal{F})^{\#}$ is given by evaluation at the trivial coset of the trivial subgroup in $G/\mathcal{F}$. We define the *relative Johnson class*

$$[J_{\mathcal{F}}] \in H^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^{\#})$$

as the cohomology class of the 1–cocycle $J_{\mathcal{F}} \in C^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^{\#})$ given by

$$J_{\mathcal{F}}(g_0 H_0, g_1 H_1) = \epsilon_{g_1 H_1} - \epsilon_{g_0 H_0},$$

where $\epsilon_{g_i H_i}$ is the evaluation map at $g_i H_i$ for $i = 0, 1$.

Theorem 4.5 *Let G be a group and $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups. The following are equivalent*:

(i) *Every subgroup $H \in \mathcal{G}$ is amenable relative to $\mathcal{F}|_H$.*

(ii) *The canonical map $H^n_{\mathcal{G},b}(G; V^{\#}) \to H^n_{\mathcal{F},b}(G; V^{\#})$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^{\#}$ and all $n \geq 0$.*

(iii) *The canonical map $H^1_{\mathcal{G},b}(G; V^{\#}) \to H^1_{\mathcal{F},b}(G; V^{\#})$ is an isomorphism for all dual normed $\mathbb{R}G$–modules $V^{\#}$.*

(iv) *The relative Johnson class $[J_{\mathcal{F}}] \in H^1_{\mathcal{F},b}(G; (\ell^\infty(G/\mathcal{F})/\mathbb{R})^{\#})$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$.*

Proof Suppose that every subgroup $H \in \mathcal{G}$ is amenable relative to $\mathcal{F}|_H$. Then the resolution of normed $\mathbb{R}G$–modules $V^\# \to C^*_{\mathcal{G}}(G; V^\#)$ is $\mathcal{F}$–strong and relatively $\mathcal{F}$–injective by Lemmas 3.8(ii) and 4.4 applied to the G–set $G/\mathcal{G}$. Hence the canonical map $\mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}$ is an isomorphism for all $n \geq 0$ by Proposition 3.9.

The implications (ii) $\implies$ (iii) $\implies$ (iv) are obvious. Suppose that the relative Johnson class $[J_{\mathcal{F}}]$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$ and let $V := \ell^\infty(G/\mathcal{F})/\mathbb{R}$. We claim that for every subgroup $H \in \mathcal{G}$, the image of $[J_{\mathcal{F}}]$ under the restriction map

$$\mathrm{res}^1_{H\subset G,b} \colon H^1_{\mathcal{F},b}(G; V^\#) \to H^1_{\mathcal{F}|_H,b}(H; V^\#)$$

is trivial. Indeed, there is a commutative diagram

$$\begin{tikzcd}
H^1_{G,b}(E_{\mathcal{G}}G; V^\#) \arrow[r, "\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}"] \arrow[d] & H^1_{G,b}(E_{\mathcal{F}}G; V^\#) \arrow[d] \arrow[dr, "\mathrm{res}^1_{H\subset G,b}"] & \\
H^1_{H,b}(E_{\mathcal{G}}G; V^\#) \arrow[r] & H^1_{H,b}(E_{\mathcal{F}}G; V^\#) \arrow[r, "\cong"] & H^1_{H,b}(E_{\mathcal{F}|_H}H; V^\#)
\end{tikzcd}$$

where the vertical maps are induced by viewing a G–space as an H–space. Observe that the lower left corner $H^1_{H,b}(E_{\mathcal{G}}G; V^\#)$ is trivial, since when viewed as an H–space $E_{\mathcal{G}}G$ is a model for $E_{\mathcal{ALL}|_H}H$ and hence H–equivariantly contractible. This proves the claim.

Now, fix a subgroup $H \in \mathcal{G}$ and denote $W := \ell^\infty(H/\mathcal{F}|_H)/\mathbb{R}$. Consider the commutative diagram of normed $\mathbb{R}H$–modules

$$\begin{tikzcd}
0 \arrow[r] & V^\# \arrow[r] \arrow[d] & \ell^\infty(G/\mathcal{F})^\# \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & W^\# \arrow[r] & \ell^\infty(H/\mathcal{F}|_H)^\# \arrow[r] & \mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact, and remain exact when restricted to L–fixed-points for every $L \in \mathcal{F}|_H$. By Lemma 3.4 there are associated long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (V^\#)^H \arrow[r] \arrow[d] & (\ell^\infty(G/\mathcal{F})^\#)^H \arrow[r] \arrow[d] & \mathbb{R} \arrow[r, "\partial^0_{V^\#}"] \arrow[d, equal] & H^1_{\mathcal{F}|_H,b}(H; V^\#) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & (W^\#)^H \arrow[r] & (\ell^\infty(H/\mathcal{F}|_H)^\#)^H \arrow[r] & \mathbb{R} \arrow[r, "\partial^0_{W^\#}"] & H^1_{\mathcal{F}|_H,b}(H; W^\#) \arrow[r] & \cdots
\end{tikzcd}$$

on bounded cohomology. Observe that the image of $\partial^0_{V^\#}$ is precisely $\mathbb{R} \cdot \mathrm{res}^1_{H\subset G,b}[J_{\mathcal{F}}]$ and hence trivial by the claim above. This implies that the map $\partial^0_{W^\#}$ is trivial and hence there exists a nontrivial H–invariant element in $\ell^\infty(H/\mathcal{F}|_H)^\#$. Thus H is amenable relative to $\mathcal{F}|_H$ by Lemma 4.3. This finishes the proof. □

As special cases of Theorem 4.5 we obtain Theorem 1.3 by taking $\mathcal{G} = \mathcal{ALL}$ and Theorem 1.5 by taking $\mathcal{F} = \mathcal{TR}$. The case when $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G} = \mathcal{ALL}$ recovers Theorem 1.1.

Corollary 4.6 *Let X be a CW–complex with fundamental group G and $\mathcal{F}$ be a family consisting of amenable subgroups of G. Suppose that there exists a model for $E_{\mathcal{F}}G$ whose orbit space $G\backslash E_{\mathcal{F}}G$ is homotopy equivalent to a k–dimensional CW–complex. Then the comparison map $c_X^n\colon H_b^n(X;\mathbb{R}) \to H^n(X;\mathbb{R})$ vanishes for all $n > k$.*

Proof By Gromov's mapping theorem [7, page 40] — see also [6, Theorem 5.9] — the comparison map c_X^n vanishes if the comparison map

$$c_{EG}^n\colon H_{G,b}^n(EG;\mathbb{R}) \to H_G^n(EG;\mathbb{R})$$

vanishes. The G–map $EG \to E_{\mathcal{F}}G$ induces a commutative square

$$\begin{tikzcd}
H_{G,b}^n(EG;\mathbb{R}) \arrow[r, "c_{EG}^n"] & H_G^n(EG;\mathbb{R}) \\
H_{G,b}^n(E_{\mathcal{F}}G;\mathbb{R}) \arrow[r, "c_{E_{\mathcal{F}}G}^n"] \arrow[u, "\mathrm{can}_{\mathcal{TR}\subset\mathcal{F},b}^n", "\cong"'] & H_G^n(E_{\mathcal{F}}G;\mathbb{R}) \arrow[u, "\mathrm{can}_{\mathcal{TR}\subset\mathcal{F}}^n"']
\end{tikzcd}$$

where the canonical map $\mathrm{can}_{\mathcal{TR}\subset\mathcal{F},b}^n$ is an isomorphism by Theorem 1.5. Since we are considering trivial coefficients, the lower right corner can be identified with the (nonequivariant) cohomology of the orbit space

$$H_G^n(E_{\mathcal{F}}G;\mathbb{R}) \cong H^n(G\backslash E_{\mathcal{F}}G;\mathbb{R})$$

(see eg [4, Theorem 4.2]). □

As an application of Corollary 4.6 we obtain the following well-known examples.

Example 4.7 The comparison map vanishes in all positive degrees for CW–complexes whose fundamental groups are

(i) graph products of amenable groups (eg right-angled Artin groups);

(ii) fundamental groups of graphs of groups with amenable vertex groups.

Indeed, if G_Γ is a graph product of amenable groups, we consider the family $\mathcal{F}$ generated by the vertex groups and direct products of vertex groups whenever the corresponding vertices form a clique in the underlying graph Γ. We claim that there exists a model for $E_{\mathcal{F}}(G_\Gamma)$ with contractible orbit space. If Γ is a complete graph, then a model for $E_{\mathcal{F}}(G_\Gamma)$ is given by the point. Otherwise, Γ can be written as $\Gamma_1 \cup_{\Gamma_0} \Gamma_2$, where Γ_i is a proper full subgraph of Γ for $i = 0, 1, 2$, and we have $G_\Gamma \cong G_{\Gamma_1} *_{G_{\Gamma_0}} G_{\Gamma_2}$. Let $\mathcal{F}_i$ be the corresponding family of subgroups of G_{Γ_i} for $i = 0, 1, 2$. Then a model for the classifying space $E_{\mathcal{F}}(G_\Gamma)$ can be constructed as the following G–pushout:

$$\begin{tikzcd}
G_\Gamma \times_{G_{\Gamma_0}} E_{\mathcal{F}_0}(G_{\Gamma_0}) \times S^0 \arrow[r] \arrow[d] & G_\Gamma \times_{G_{\Gamma_1}} E_{\mathcal{F}_1}(G_{\Gamma_1}) \coprod G_\Gamma \times_{G_{\Gamma_2}} E_{\mathcal{F}_2}(G_{\Gamma_2}) \arrow[d] \\
G_\Gamma \times_{G_{\Gamma_0}} E_{\mathcal{F}_0}(G_{\Gamma_0}) \times D^1 \arrow[r] & E_{\mathcal{F}}(G_\Gamma)
\end{tikzcd}$$

By induction on the number of vertices of Γ, the classifying spaces $E_{\mathcal{F}_i}(G_{\Gamma_i})$ have contractible orbit spaces for $i = 0, 1, 2$, and hence so does $E_{\mathcal{F}}(G_\Gamma)$.

If G is the fundamental group of a graph of groups with amenable vertex groups, we consider the family $\mathcal{F}$ generated by the vertex groups. Then the Bass–Serre tree is a 1–dimensional model for $E_{\mathcal{F}}G$. Recall that the comparison map always vanishes in degree 1, since $H_b^1(G; \mathbb{R})$ is trivial for every group G.

We also obtain a characterization of relative amenability via relatively $\mathcal{F}$–injective modules, analogous to [6, Proposition 4.18]; see also [20, Theorem 5.7.1].

Proposition 4.8 *Let G be a group and $\mathcal{F}$ be a family of subgroups. The following are equivalent*:

(i) *G is amenable relative to $\mathcal{F}$.*

(ii) *Every dual normed $\mathbb{R}G$–module $V^\#$ is relatively $\mathcal{F}$–injective.*

(iii) *The trivial normed $\mathbb{R}G$–module $\mathbb{R}$ is relatively $\mathcal{F}$–injective.*

Proof Suppose that G is amenable relative to $\mathcal{F}$ and let $m_{\mathcal{F}}$ be a G–invariant mean on $G/\mathcal{F}$. The inclusion $V^\# \to C^0_{\mathcal{F},b}(G; V^\#)$ of normed G–modules admits a right inverse r given by

$$r(f)(v) = m_{\mathcal{F}}(gH \mapsto f(gH)(v))$$

for $f \in C^0_{\mathcal{F},b}(G; V^\#)$ and $v \in V$. Then the relative $\mathcal{F}$–injectivity of $V^\#$ follows from the relative $\mathcal{F}$–injectivity of $C^0_{\mathcal{F},b}(G; V^\#)$.

Clearly, condition (ii) implies (iii). Suppose that $\mathbb{R}$ is relatively $\mathcal{F}$–injective. Consider the strongly $\mathcal{F}$–injective map $i\colon \mathbb{R} \to \ell^\infty(G/\mathcal{F})$ of normed G–modules that has an H–section τ_H given by $\tau_H(f) = f(eH)$ for each $H \in \mathcal{F}$. Then the identity $\mathrm{id}_\mathbb{R}$ admits an extension along i which yields a nontrivial G–invariant element in $\ell^\infty(G/\mathcal{F})^\#$. By Lemma 4.3 this finishes the proof. □

Characterization of relative finiteness Analogously to Theorem 4.5, when instead considering all (not necessarily dual) normed $\mathbb{R}G$–modules, one obtains the theorem below. Let G be a group and $\mathcal{F}$ be a family of subgroups.

Let $\ell^1(G/\mathcal{F})$ denote the normed $\mathbb{R}G$–module of summable functions $f\colon G/\mathcal{F} \to \mathbb{R}$ with $\|f\|_1 = \sum_{gH \in G/\mathcal{F}} |f(gH)|$. Let $\ell^1_0(G/\mathcal{F})$ be the kernel of the map $\ell^1(G/\mathcal{F}) \to \mathbb{R}$, given by $f \mapsto \sum_{gH \in G/\mathcal{F}} f(gH)$. We define the class $[K_\mathcal{F}] \in H^1_{\mathcal{F},b}(G;\ell^1_0(G/\mathcal{F}))$ as the cohomology class of the 1–cocycle $K_\mathcal{F} \in C^1_{\mathcal{F},b}(G;\ell^1_0(G/\mathcal{F}))$ given by

$$K_\mathcal{F}(g_0H_0, g_1H_1) = \chi_{g_1H_1} - \chi_{g_0H_0},$$

where $\chi_{g_iH_i}$ is the characteristic function supported at g_iH_i for $i = 0, 1$.

We say that G is *finite relative* to $\mathcal{F}$, if $\mathcal{F}$ contains a finite index subgroup of G.

Theorem 4.9 *Let G be a group and $\mathcal{F} \subset \mathcal{G}$ be two families of subgroups. The following are equivalent*:

(i) *Every subgroup $H \in \mathcal{G}$ is finite relative to $\mathcal{F}|_H$.*

(ii) *The canonical map $H^n_{\mathcal{G},b}(G;V) \to H^n_{\mathcal{F},b}(G;V)$ is an isomorphism for all normed $\mathbb{R}G$–modules V and all $n \geq 0$.*

(iii) *The canonical map $H^1_{\mathcal{G},b}(G;V) \to H^1_{\mathcal{F},b}(G;V)$ is an isomorphism for all normed $\mathbb{R}G$–modules V.*

(iv) *The class $[K_\mathcal{F}] \in H^1_{\mathcal{F},b}(G;\ell^1_0(G/\mathcal{F}))$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$.*

Proof We only give a sketch of the proof which is entirely analogous to that of Theorem 4.5. Suppose that every subgroup $H \in \mathcal{G}$ is finite relative to $\mathcal{F}|_H$. One shows that the resolution of normed $\mathbb{R}G$–modules $V \to C^*_\mathcal{G}(G;V)$ is relatively $\mathcal{F}$–injective by taking averages over finite sets of cosets. Moreover, the resolution is $\mathcal{F}$–strong by Lemma 3.8(ii) and hence the canonical map $\mathrm{can}^n_{\mathcal{F}\subset\mathcal{G},b}$ is an isomorphism for all $n \geq 0$ by Proposition 3.9.

The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious. Suppose that the class $[K_{\mathcal{F}}]$ lies in the image of the canonical map $\mathrm{can}^1_{\mathcal{F}\subset\mathcal{G},b}$. Fix a subgroup $H \in \mathcal{G}$ and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ell^1_0(G/\mathcal{F}) \arrow[r] \arrow[d] & \ell^1(G/\mathcal{F}) \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ell^1_0(H/\mathcal{F}|_H) \arrow[r] & \ell^1(H/\mathcal{F}|_H) \arrow[r] & \mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

of normed $\mathbb{R}H$–modules. Following the proof of Theorem 4.5, one obtains a nontrivial H–invariant element $f \in \ell^1(H/\mathcal{F}|_H)$. Since f is constant on H–orbits, nontrivial, and summable, there exists a finite H–orbit in $H/\mathcal{F}|_H$. Thus H is finite relative to $\mathcal{F}|_H$. □

Theorem 4.9 has the following interesting special cases. If $\mathcal{F}$ is arbitrary and $\mathcal{G} = \mathcal{ALL}$, we characterize that $\mathcal{F}$ contains a finite index subgroup of G. If $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G}$ is arbitrary, we characterize that all subgroups in $\mathcal{G}$ are finite, generalizing [23, Theorem B]. We recover the characterization of finite groups [6, Theorem 3.12] for $\mathcal{F} = \mathcal{TR}$ and $\mathcal{G} = \mathcal{ALL}$.

5 Characterization of relative hyperbolicity

In this section we prove a characterization of relatively hyperbolic groups in terms of bounded Bredon cohomology analogous to Theorem 1.2.

Let G be a finitely generated group and $\mathcal{H}$ be a finite set of subgroups. Recall that G is *hyperbolic relative to* $\mathcal{H}$ if the coned–off Cayley graph is hyperbolic and fine; see eg [8]. For example, hyperbolic groups are hyperbolic relative to the trivial subgroup, free products $G_1 * G_2$ are hyperbolic relative to $\{G_1, G_2\}$, and fundamental groups of finite volume hyperbolic manifolds are hyperbolic relative to the cusp subgroups. If G is hyperbolic relative to $\mathcal{H}$, then $\mathcal{H}$ is almost malnormal and hence malnormal if G is torsionfree.

From now on, let the ring R be either $\mathbb{Q}$ or $\mathbb{R}$. A map $f\colon C \to B$ of normed RG–modules is called *undistorted* if there exists a constant $K \geq 0$ such that for all $b \in \mathrm{im}(f)$ there exists $c \in C$ with $f(c) = b$ such that $\|c\|_C \leq K \cdot \|b\|_B$. A normed RG–module P is called *boundedly projective* if for every undistorted epimorphism $f\colon C \to B$ and every map $\phi\colon P \to B$ of normed RG–modules, there exists a map $\Phi\colon P \to C$ of normed RG–modules such that $f \circ \Phi = \phi$.

The following lemma [19, Lemma 52] is useful to construct G–equivariant maps.

Lemma 5.1 (Mineyev and Yaman) *Let G be a group and S be a G–set with finite stabilizers. Then $\mathbb{Q}[S]$ is projective as a $\mathbb{Q}G$–module and boundedly projective as a normed $\mathbb{Q}G$–module when equipped with the ℓ^1–norm.*

Let X be a G–CW–complex with cocompact $(n+1)$–skeleton and consider for $k \geq 0$ the cellular chains $C_k^{\mathrm{cell}}(X; R)$ as a normed RG–module equipped with the ℓ^1–norm. We say that X satisfies a *linear homological isoperimetric inequality over R in degree n* if the boundary map

$$\partial_{n+1} \colon C_{n+1}^{\mathrm{cell}}(X; R) \to C_n^{\mathrm{cell}}(X; R)$$

is undistorted. Equivalently, there exists a constant $K \geq 0$ such that for every cellular n–boundary $b \in B_n^{\mathrm{cell}}(X; R)$ we have $\|b\|_\partial \leq K \cdot \|b\|_1$, where

$$\|b\|_\partial := \inf\{\|c\|_1 \mid c \in C_{n+1}^{\mathrm{cell}}(X; R), \partial_{n+1}(c) = b\}$$

(which is sometimes called the *filling norm*). (In [16], the terminology of the *uniform boundary condition* is used for a linear homological isoperimetric inequality.)

If G is hyperbolic relative to $\mathcal{H}$, Mineyev and Yaman [19, Theorem 41] have constructed the so-called "ideal complex" X. It is in particular a cocompact G–CW–complex with precisely one equivariant 0–cell G/H for each $H \in \mathcal{H}$ and finite edge-stabilizers. Moreover, X is (nonequivariantly) contractible and hence a model for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ provided that G is torsionfree. We summarize some of its properties [19, Theorems 47 and 51].

Theorem 5.2 (Mineyev and Yaman) *Let G be a finitely generated torsionfree group and $\mathcal{H}$ be a finite set of subgroups. If G is hyperbolic relative to $\mathcal{H}$, then there exists a cocompact model X for $E_{\mathcal{F}\langle\mathcal{H}\rangle}G$ such that*

(i) *X satisfies linear homological isoperimetric inequalities over $\mathbb{Q}$ in degree n for all $n \geq 1$;*

(ii) *there exists a map $q \colon X^{(0)} \times X^{(0)} \to C_1^{\mathrm{cell}}(X; \mathbb{Q})$ with $\partial_1(q(a,b)) = b - a$, called a homological $\mathbb{Q}$–bicombing, that is G–equivariant and satisfies*

$$\|q(a,b) + q(b,c) - q(a,c)\|_1 \leq K$$

for all $a, b, c \in X^{(0)}$ and a uniform constant $K \geq 0$.

The following criterion for relative hyperbolicity is a combination of [5, Proposition 8.3 and Theorem 8.5]; see also [14, Theorems 1.6 and 1.10].

Theorem 5.3 (Franceschini [5] and Martínez-Pedroza [14]) *Let G be a group and $\mathcal{H}$ be a finite set of subgroups. Then G is hyperbolic relative to $\mathcal{H}$ if there exists a G–CW–complex Z such that*

(i) *Z is simply connected*;

(ii) *the 2–skeleton $Z^{(2)}$ is cocompact*;

(iii) *$\mathcal{H}$ is a set of representatives of distinct conjugacy classes of vertex-stabilizers such that each infinite stabilizer is represented*;

(iv) *the edge-stabilizers of Z are finite*;

(v) *Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1.*

We prove the following characterization of relative hyperbolicity closely following Mineyev's original proof of Theorem 1.2 — [17, Theorem 11] and [18, Theorem 9].

Theorem 5.4 *Let G be a finitely generated torsionfree group and $\mathcal{H}$ be a finite malnormal collection of subgroups. Let $\mathcal{F}$ be the family $\mathcal{F}\langle\mathcal{H}\rangle$ and suppose that G is of type $F_{2,\mathcal{F}}$. Then the following are equivalent*:

(i) *G is hyperbolic relative to $\mathcal{H}$.*

(ii) *The comparison map $H^n_{\mathcal{F},b}(G;V) \to H^n_{\mathcal{F}}(G;V)$ is surjective for all normed $\mathbb{Q}G$–modules V and all $n \geq 2$.*

(iii) *The comparison map $H^2_{\mathcal{F},b}(G;V) \to H^2_{\mathcal{F}}(G;V)$ is surjective for all normed $\mathbb{R}G$–modules V.*

Proof Suppose that G is hyperbolic relative to $\mathcal{H}$. Let X be the model for $E_{\mathcal{F}}G$ that is given by Mineyev and Yaman's ideal complex (Theorem 5.2) and Y be the simplicial model for $E_{\mathcal{F}}G$ with (nonequivariant) n–cells corresponding to $(G/\mathcal{F})^{n+1}$ for all $n \geq 0$. We claim that there is a G–chain map

$$\varphi_*\colon C_*^{\mathrm{cell}}(Y;\mathbb{Q}) \to C_*^{\mathrm{cell}}(X;\mathbb{Q})$$

with φ_n bounded for all $n \geq 2$, admitting a G–homotopy left inverse. We construct φ_* inductively as follows. In degree 0, we define

$$\varphi_0\colon C_0^{\mathrm{cell}}(Y;\mathbb{Q}) = \mathbb{Q}[G/\mathcal{F}] \to C_0^{\mathrm{cell}}(X;\mathbb{Q})$$

to map a generator of the form eH to the vertex of X with stabilizer containing H. Then extend G–equivariantly and $\mathbb{Q}$–linearly to all of $\mathbb{Q}[G/\mathcal{F}]$. In degree 1, we define $\varphi_1\colon C_1^{\mathrm{cell}}(Y;\mathbb{Q}) \to C_1^{\mathrm{cell}}(X;\mathbb{Q})$ on generators by

$$\varphi_1(g_0H_0, g_1H_1) = q(\varphi_0(g_0H_0), \varphi_0(g_1H_1)),$$

where q is the homological $\mathbb{Q}$–bicombing on X from Theorem 5.2(ii). Since both q and φ_0 are G–equivariant, so is φ_1. In degree 2, we consider the maps

$$\begin{tikzcd} C_2^{\mathrm{cell}}(Y;\mathbb{Q}) \arrow[r, "\partial_2^Y"] & C_1^{\mathrm{cell}}(Y;\mathbb{Q}) \arrow[d, "\varphi_1"] \\ C_2^{\mathrm{cell}}(X;\mathbb{Q}) \arrow[r, "\partial_2^X"] & C_1^{\mathrm{cell}}(X;\mathbb{Q}) \end{tikzcd} \tag{5-1}$$

and observe that the composition $\varphi_1 \circ \partial_2^Y$ is bounded by properties of q and that ∂_2^X is undistorted by Theorem 5.2(i). There is a G–invariant decomposition

$$C_2^{\mathrm{cell}}(Y;\mathbb{Q}) \cong \mathbb{Q}[S_1] \oplus \mathbb{Q}[S_2],$$

where S_1 and S_2 denote the sets of 2–cells of Y with trivial and nontrivial stabilizers, respectively. We obtain a bounded G–map $\varphi_2\colon C_2^{\mathrm{cell}}(Y;\mathbb{Q}) \to C_2^{\mathrm{cell}}(X;\mathbb{Q})$ by using the bounded projectivity of $\mathbb{Q}[S_1]$ (Lemma 5.1) and by setting φ_2 to be zero on $\mathbb{Q}[S_2]$. This renders the square (5-1) commutative because the edge-stabilizers of X are trivial.

Assuming that φ_n has been constructed, one analogously defines a bounded G–map φ_{n+1} using that ∂_{n+1}^X is undistorted by Theorem 5.2(i). Thus one obtains a G–chain map φ_* with φ_n bounded for $n \geq 2$. To conclude the claim, we note that $C_*^{\mathrm{cell}}(Y;\mathbb{Q})$ is a relatively $\mathcal{F}$–projective $\mathcal{F}$–strong resolution of $\mathbb{Q}$ by Lemma 3.6. Hence by Proposition 3.7 any G–chain map $\psi_*\colon C_*^{\mathrm{cell}}(X;\mathbb{Q}) \to C_*^{\mathrm{cell}}(Y;\mathbb{Q})$ extending $\mathrm{id}_{\mathbb{Q}}$ is a G–homotopy left inverse of φ_*.

Now, let V be a normed $\mathbb{Q}G$–module. Applying $\mathrm{Hom}_{\mathbb{Q}G}(\,\cdot\,, V)$ yields a cochain map

$$\varphi^*\colon C^*_{\mathrm{cell}}(X;V)^G \to C^*_{\mathrm{cell}}(Y;V)^G$$

with homotopy right inverse ψ^*. In particular, the composition $\varphi^* \circ \psi^*$ induces the identity on $H^*(C^*_{\mathrm{cell}}(Y;V)^G) \cong H^*_{\mathcal{F}}(G;V)$. Finally, for $n \geq 2$ let $c \in C^n_{\mathrm{cell}}(Y;V)^G$ be a cocycle. Then $\varphi^n(\psi^n(c))$ and c represent the same cohomology class in $H^n_{\mathcal{F}}(G;V)$. We have

$$\|\varphi^n(\psi^n(c))\|_\infty = \|\psi^n(c) \circ \varphi_n\|_\infty \leq \|\psi^n(c)\|_\infty \cdot \|\varphi_n\|_\infty,$$

where φ_n is bounded by construction and so is $\psi^n(c) \in C^n_{\mathrm{cell}}(X;V)^G$ because X has only finitely many orbits of n–cells. Thus we have shown that for $n \geq 2$ every cohomology class in $H^n_{\mathcal{F}}(G;V)$ admits a bounded representative.

Obviously condition (ii) implies (iii). Suppose that the comparison map is surjective in degree 2 for coefficients in every normed $\mathbb{R}G$–module. Let Z be a model for $E_{\mathcal{F}}G$ with cocompact 2–skeleton. Since $\mathcal{H}$ is malnormal, by collapsing fixed-point sets of Z we may assume that for every nontrivial subgroup $H \in \mathcal{F}$ the fixed-point set Z^H consists of precisely one point. In other words, Z has one equivariant 0–cell of the form G/H for each $H \in \mathcal{H}$ and all other cells have trivial stabilizers. In order to apply Theorem 5.3 and conclude that G is hyperbolic relative to $\mathcal{H}$, it remains to verify that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1.

We take as coefficients the cellular 1–boundaries $V := B^{\mathrm{cell}}_1(Z;\mathbb{R})$ equipped with the norm $\|\cdot\|_\partial$. Let Y be the simplicial model for $E_{\mathcal{F}}G$. Then there is a G–chain homotopy equivalence

$$\psi_*\colon C^{\mathrm{cell}}_*(Z;\mathbb{R}) \to C^{\mathrm{cell}}_*(Y;\mathbb{R})$$

with G–homotopy inverse φ_*. Applying $\mathrm{Hom}_{\mathbb{R}G}(\,\cdot\,,V)$ yields a cochain homotopy equivalence

$$\psi^*\colon C^*_{\mathrm{cell}}(Y;V)^G \to C^*_{\mathrm{cell}}(Z;V)^G$$

with homotopy inverse φ^*. In particular, the composition $\psi^* \circ \varphi^*$ induces the identity on $H^*(C^*_{\mathrm{cell}}(Z;V)^G) \cong H^*_{\mathcal{F}}(G;V)$. Consider the 2–cocycle $u \in C^2_{\mathrm{cell}}(Z;V)^G$ given by the boundary map

$$u = \partial_2\colon C^{\mathrm{cell}}_2(Z;\mathbb{R}) \to B^{\mathrm{cell}}_1(Z;\mathbb{R}) = V.$$

Then we can write

$$u = (\psi^2 \circ \varphi^2)(u) + \delta^1_Z(v) \tag{5-2}$$

for some $v \in C^1_{\mathrm{cell}}(Z;V)^G$. Since the comparison map $H^2_{\mathcal{F},b}(G;V) \to H^2_{\mathcal{F}}(G;V)$ is surjective by hypothesis, we can write

$$\varphi^2(u) = u' + \delta^1_Y(v') \tag{5-3}$$

for a *bounded* 2–cocycle $u' \in C^2_{\mathrm{cell}}(Y;V)^G$ and some $v' \in C^1_{\mathrm{cell}}(Y;V)^G$. For a fixed vertex $y \in Y^{(0)} = G/\mathcal{F}$, let $\mathrm{Cone}_y\colon C^{\mathrm{cell}}_1(Y;\mathbb{R}) \to C^{\mathrm{cell}}_2(Y;\mathbb{R})$ be defined on generators by

$$\mathrm{Cone}_y((g_0H_0, g_1H_1)) = (y, g_0H_0, g_1, H_1).$$

Obviously Cone_y preserves the ℓ^1–norms. For a G–CW–complex W, we denote the evaluation pairing by

$$\langle\cdot,\cdot\rangle_W\colon C^*_{\mathrm{cell}}(W;V)^G \times C_*^{\mathrm{cell}}(W;\mathbb{R}) \to V.$$

Now, for $b \in C_1^{\mathrm{cell}}(Z;\mathbb{R})$ and $c \in C_2^{\mathrm{cell}}(Z;\mathbb{R})$ with $\partial_2(c) = b$, we find by (5-2) that

$$\begin{aligned} b = \langle u, c\rangle_Z &= \langle (\psi^2 \circ \varphi^2)(u) + \delta_Z^1(v), c\rangle_Z \\ &= \langle (\psi^2 \circ \varphi^2)(u), c\rangle_Z + \langle v, \partial_2^Y(c)\rangle_Z \\ &= \langle \varphi^2(u), \psi_2(c)\rangle_Y + \langle v, b\rangle_Z. \end{aligned}$$

Since $\varphi^2(u)$ is a cocycle and $\psi_2(c) - \mathrm{Cone}_y(\partial_2^Y(\psi_2(c)))$ is a cycle and hence a boundary,

$$\begin{aligned} \langle \varphi^2(u), \psi_2(c)\rangle_Y &= \langle \varphi^2(u), \mathrm{Cone}_y(\partial_2^Y(\psi_2(c)))\rangle_Y \\ &= \langle \varphi^2(u), \mathrm{Cone}_y(\psi_1(b))\rangle_Y \\ &= \langle u' + \delta_Y^1(v'), \mathrm{Cone}_y(\psi_1(b))\rangle_Y \\ &= \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle v', \partial_2^Y(\mathrm{Cone}_y(\psi_1(b)))\rangle_Y \\ &= \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle v', \psi_1(b)\rangle_Y \\ &= \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle \psi^1(v'), b\rangle_Z, \end{aligned}$$

where we used (5-3). Together, we have

$$b = \langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y + \langle \psi^1(v') + v, b\rangle_Z.$$

We claim that $\|\langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y\|_\partial \le \|u'\|_\infty \cdot \|\mathrm{Cone}_y(\psi_1(b))\|_1$. Indeed, consider the map induced by ∂_2 on coefficients

$$(\partial_2)_*\colon C^2_{\mathrm{cell}}(Y; C_2^{\mathrm{cell}}(Z;\mathbb{R}))^G \to C^2_{\mathrm{cell}}(Y;V)^G.$$

Since ∂_2 is surjective, there exists a preimage $\tilde{u}' \in C^2_{\mathrm{cell}}(Y; C_2^{\mathrm{cell}}(Z;\mathbb{R}))^G$ of u' under $(\partial_2)_*$ with $\|\tilde{u}'\|_\infty \le \|u'\|_\infty$. Then $\langle \tilde{u}', \mathrm{Cone}_y(\psi_1(b))\rangle_Y \in C_2^{\mathrm{cell}}(Z;\mathbb{R})$ is a preimage of $\langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y$ under ∂_2 witnessing the desired inequality. Similarly, one shows that $\|\langle \psi^1(v') + v, b\rangle_Z\|_\partial \le \|\psi^1(v') + v\|_\infty \cdot \|b\|_1$. It follows that

$$\begin{aligned} \|b\|_\partial &\le \|\langle u', \mathrm{Cone}_y(\psi_1(b))\rangle_Y\|_\partial + \|\langle \psi^1(v') + v, b\rangle_Z\|_\partial \\ &\le \|u'\|_\infty \cdot \|\mathrm{Cone}_y(\psi_1(b))\|_1 + \|\psi^1(v') + v\|_\infty \cdot \|b\|_1 \\ &= \|u'\|_\infty \cdot \|\psi_1(b)\|_1 + \|\psi^1(v') + v\|_\infty \cdot \|b\|_1 \\ &\le (\|u'\|_\infty \cdot \|\psi_1\|_\infty + \|\psi^1(v') + v\|_\infty) \cdot \|b\|_1. \end{aligned}$$

Finally, u' is bounded by construction and so are ψ_1 and $\psi^1(v') + v$ because they are G–maps with domain $C_1^{\mathrm{cell}}(Z;\mathbb{R})$ and Z has only finitely many orbits of 1–cells. Thus we have shown that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1. This finishes the proof. □

Remark 5.5 (groups with torsion) In Theorem 5.4, if the group G is not assumed to be torsionfree and $\mathcal{H}$ is instead assumed to be almost malnormal, one can still prove the equivalence of (i) and (iii). However, a few modifications are necessary which we shall only outline.

Assuming that G is hyperbolic relative to $\mathcal{H}$, Mineyev and Yaman's ideal complex has to be replaced by a Rips type construction X due to Martínez-Pedroza and Przytycki that is a model for $E_{\mathcal{F}\cup\mathcal{FIN}}G$. This complex X satisfies a linear homological isoperimetric inequality over $\mathbb{Z}$ in degree 1 [15, Corollary 1.5]. It is part of a hyperbolic tuple in the sense of [19, Definition 38] and hence admits a homological $\mathbb{Q}$–bicombing by [19, Theorem 47]. Then one can construct a G–chain map φ_* with φ_2 bounded similarly as before and conclude surjectivity of the comparison map in degree 2 for the family $\mathcal{F}\cup\mathcal{FIN}$. This implies the same for the family $\mathcal{F}$ over the ring $\mathbb{R}$.

For the converse implication, since $\mathcal{H}$ is almost malnormal, there exists a model Z for $E_{\mathcal{F}}G$ such that for every infinite subgroup $H \in \mathcal{F}$ the fixed-point set Z^H consists of precisely one point. Then one shows as before that Z satisfies a linear homological isoperimetric inequality over $\mathbb{R}$ in degree 1 and concludes by Theorem 5.3.

We do not know whether condition (ii) is equivalent to (i) and (iii) in this case.

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