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Algebraic & Geometric *Topology*

On the functoriality of $\mathfrak{sl}_2$ tangle homology

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We construct an explicit equivalence between the (bi)category of $\mathfrak{gl}_2$ webs and foams and the Bar-Natan (bi)category of Temperley–Lieb diagrams and cobordisms. With this equivalence we can fix functoriality of every link homology theory that factors through the Bar-Natan category. To achieve this, we define web versions of arc algebras and their quasihereditary covers, which provide strictly functorial tangle homologies. Furthermore, we construct explicit isomorphisms between these algebras and the original ones based on Temperley–Lieb cup diagrams. The immediate application is a strictly functorial version of the Beliakova–Putyra–Wehrli quantization of the annular link homology.

1 Introduction

In 1999 Khovanov [18] defined for any link in the 3–sphere a chain complex, whose homotopy type — hence, homology — is a link invariant and whose Euler characteristic is the Jones polynomial. It was later extended by Khovanov to tangles between even

collections of points [19] and then to all tangles by Brundan and Stroppel [6] and Chen and Khovanov [10]. The main advantage of the Khovanov homology with respect to the Jones polynomial is that link cobordisms induce chain maps between Khovanov's complexes; see Bar-Natan [2], Jacobsson [17], and Khovanov [22]. Even though the original construction is not strictly functorial — the sign of the chain map associated with a link cobordism depends on the decomposition of the cobordism into elementary pieces [17] — it was used by Rasmussen to provide a lower bound for the slice genus of a knot and a combinatorial proof of the Milnor conjecture [32].

In the last 15 years there were many attempts to fix the functoriality of Khovanov homology. In Caprau [8] and Clark, Morrison and Walker [11] this was done by modifying the Bar-Natan category [2], in which the construction of the complex can be naturally described, by taking into account orientations and by enlarging the ground ring; see also Vogel [35]. In 2014 Blanchet [5] proposed a more conceptual solution, which does not change the ring of scalars, but replaces circles and surfaces in the Bar-Natan category with *webs* and *foams*: certain planar trivalent graphs and singular cobordisms between them respectively. This construction, commonly referred to as $\mathfrak{gl}_2$ *homology*, has been widely accepted as the most natural way to fix functoriality of Khovanov homology. The resulting, a priori potentially different, $\mathfrak{gl}_2$ homology is known to coincide with Khovanov homology in case of links [5].

To obtain a computable invariant of tangles, Chen and Khovanov constructed a functor from the Bar-Natan bicategory to the bicategory of bimodules over extended Khovanov's arc algebras [10]. Because of its diagrammatic definition, it is straightforward to generalize this functor to the case of webs. However, it is no longer clear whether the new algebra or tangle invariant is isomorphic to those constructed by Chen and Khovanov. A partial solution to this problem, that considers only a special class of webs, was presented by Ehrig, Stroppel and Tubbenhauer [13; 14], but not much was known beyond this case. The special webs from [13; 14] are discussed in Example 6.6 below.

The Hochschild homology of the Chen–Khovanov invariant of an (n, n)–tangle T has been identified by Beliakova, Putyra and Wehrli in [4] with the annular Khovanov homology of the annular closure $\widehat{T}$ of the tangle. In the same paper the annular invariant has been quantized by deforming the Hochschild homology. The original goal of this paper was to make this quantized annular homology functorial, in order to construct its colored versions following Khovanov [21] and Cooper and Krushkal [12]. These quantized colored homologies are constructed in the follow up paper by the authors [3],

where we also show that both complexes coincide when the deformation parameter is generic; see also Putyra [28].

In order to obtain a strictly functorial quantized annular homology, we wanted first to understand the Ehrig–Stroppel–Tubbenhauer isomorphism between Khovanov's arc algebras and their web algebras, and then reconstruct the Chen–Khovanov functor in the framework of webs and foams. However, after a chain of simplifications of their arguments, especially replacing the foam basis used in [13] with another one, more natural from the topological perspective, we understood the real reason why all the isomorphisms popped out: *foams and cobordisms constitute equivalent bicategories*. Despite sounding as a natural thing, the result is by no means obvious — compare eg Lauda, Queffelec and Rose [23, Section 3.1.2] or [14, page 186].

In this paper we construct a functor between foams and cobordisms by taking into account orientability of foams and another beautiful topological tool called *shadings*. They allow to think of a web (resp. a foam) as two transversely intersecting flat tangles (resp. surfaces), so that each of the two tangles or surfaces can be isotoped separately (see Lemma 3.6, the bicolored isotopy lemma). This approach makes many results on foams straightforward, but also leads to a simple diagrammatic representation of a basis of the space of foams bounded by a fixed web, on which the action of foams is very easy to compute. We use this basis to construct explicitly equivalences between the two bicategories in both directions and to obtain full web versions of the TQFT functors from [6; 10; 19].

To summarize, the above mentioned equivalence between foams and cobordisms gives an ultimate solution to all functoriality issues related with Khovanov homology. Our recipe is simple: precompose any link or tangle homology theory that factors through the Bar-Natan category with our equivalence and obtain a strictly functorial theory. In the following sections we discuss the above approach in more details.

1.1 The equivalence of foams and Bar-Natan cobordisms

In order to compute the Khovanov homology of a link L, one first picks its diagram D and constructs the *cube of resolutions* of D: a commutative diagram in the shape of the c–dimensional cube, where c equals the number of crossings in D, with vertices decorated by Kauffman resolutions of D and edges by saddle cobordisms between them [18]. Applying a 2–dimensional TQFT to this cube, changing signs of some maps, and collapsing the cube along diagonals results in an actual chain complex, which —
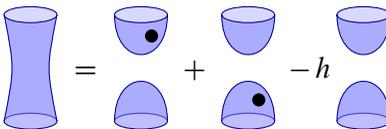
depending on the choice of the TQFT functor — computes the Khovanov homology of L or its deformation.

It was observed by Bar-Natan that most of the construction can be performed formally *before* applying a TQFT functor to get an invariant of a tangle T in the form of a formal complex $[\![T]\!]$ called the *Khovanov bracket* of T [2]. This complex is constructed in the *Bar-Natan bicategory* **BN**, the locally additive graded bicategory with objects collections of points on a line, 1–morphisms generated by flat tangles, and 2–morphisms generated by isotopy classes of surfaces embedded in a 3–space and decorated by dots, modulo the following local relations:
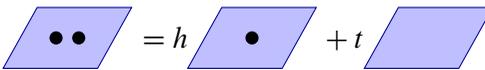
- *sphere evaluations*:

(1-1)
$$\text{} = 0 \qquad \text{} = 1$$

- *neck-cutting relation*:

(1-2)
$$\text{} = \text{} + \text{} - h\,\text{}$$

- *dot-reduction*:

(1-3)
$$\text{} = h\,\text{} + t\,\text{}$$

Here h and t are fixed elements of the ring of scalars $\Bbbk$. When $h = 0$, then the neck-cutting relation evaluates a handle attached to a plane as a dot scaled by 2. Because of that it is common to think of a dot as "half" of a handle, even when 2 is not an invertible scalar. However, this interpretation is not correct if $h \neq 0$, in particular in the universal case $\Bbbk = \mathbb{Z}[h, t]$.

The formal bracket is projectively functorial [2]. Indeed, there is a way to associate a formal chain map with each Reidemeister move as well as any cobordism with a unique critical point. One constructs a formal chain map for any tangle cobordism by decomposing the cobordism into a sequence of the above elementary pieces and composing the associated maps; choosing a different decomposition may at most change the global sign of the map.

In Blanchet's construction [5] the role of flat tangles is played by $\mathfrak{gl}_2$–*webs*, trivalent graphs with each edge colored blue or red,[1] and dotted surfaces are replaced with

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[1] When compared to [5], blue edges are those with label 1 and red edges are those with label 2.

foams, which are singular cobordisms with each facet also colored blue or red. They constitute a bicategory **Foam**, where certain local relations between foams, including (1-1)–(1-3), are imposed; see Definition 2.6. Following [2] we can construct a formal complex $[\![T]\!]_F$ in **Foam**, which we refer to as the *Blanchet–Khovanov bracket*.

The collection of blue edges of a web ω is a flat tangle ω_b, which we call the *underlying tangle* of ω. Likewise, there is an *underlying surface* S_b associated with any foam S. It is tempting to consider a 2–functor **Foam** $\to$ **BN** that forgets red edges in webs and red facets in foams. However, this operation is not compatible with relations between foams, and it is not clear at first how to solve this problem. For instance, it was observed in [23] that if such a functor exists, then it cannot be the identity on all foams with no red facets.

We resolved the above problem by taking into account the orientation of blue edges and facets. Shortly speaking, we fix an orientation for each flat tangle and surface in a canonical way, reinterpreting them as webs and foams respectively (recall that tangles and surfaces from **BN**, though orientable, come with no particular orientation). This results in a 2–functor, which, however, does not reach every object of **Foam**. In order to fix this we replace **BN** with the product **wBN** := **BN** $\times$ $\mathbb{Z}$, where $\mathbb{Z}$ is seen as a discrete bicategory. We use the extra integer to determine how many red points, edges, or facets have to be added to the right of the oriented blue points, tangle, or surface respectively.[2] This way we end up with a 2–functor $\mathcal{E}\colon$ **wBN** $\to$ **Foam**, such that every object of **Foam** is equivalent to one from the image of $\mathcal{E}$.

Theorem A   *The 2–functor* $\mathcal{E}\colon$ **wBN** $\to$ **Foam** *is an equivalence of bicategories.*

From the point of view of representation theory, $\mathcal{E}$ and its inverse can be understood as the categorification of the induction–restriction pair between representations of $\mathfrak{sl}_2$ and $\mathfrak{gl}_2$.

There is also a local version of Theorem A. Having fixed a collection Σ of oriented blue and red points on $\partial\mathbb{D}^2$, write **Foam**(Σ) for the category of webs in $\mathbb{D}^2$ bounded by Σ and foams in $\mathbb{D}^2 \times [0, 1]$ between such webs. Likewise we consider the category **BN**(Σ_b) of flat tangles bounded by Σ_b and dotted surfaces between them, where Σ_b is the collection of blue points from Σ. We construct a functor $\mathcal{E}_\Sigma\colon$ **BN**$(\Sigma_b) \to$ **Foam**(Σ) in Section 4.1 by extending coherently all flat tangles to webs bounded by Σ and surfaces to foams.

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[2]Compare this with the relation between the weight lattices of $\mathfrak{sl}_2$ and $\mathfrak{gl}_2$ — the latter is isomorphic to the product of the former with $\mathbb{Z}$.

Theorem B   *The functor $\mathcal{E}_\Sigma \colon \mathbf{BN}(\Sigma_b) \to \mathbf{Foam}(\Sigma)$ is an equivalence of categories.*

We construct the functor $\mathcal{E}_\Sigma$ explicitly as well as its inverse $\mathcal{E}_\Sigma^\vee$. The latter not only forgets red facets of foams, but also scales them by a sign when necessary; we provide an explicit way to compute these signs in terms of the Blanchet evaluation of foams. When combined with a homological argument presented in Ozsváth, Rasmussen and Szabó [26] and Putyra [27], Theorem B implies that for every tangle T the image of the Khovanov bracket $[\![T]\!]$ under $\mathcal{E}_\Sigma$ is isomorphic to the Blanchet–Khovanov bracket $[\![T]\!]_\mathrm{F}$. Hence, any TQFT functor on $\mathbf{BN}(\Sigma_b)$ that leads to a tangle or link homology can be precomposed with $\mathcal{E}_\Sigma^\vee$ to obtain a functor on $\mathbf{Foam}(\Sigma)$ that computes the same homology groups, but which is strictly functorial with respect to tangle cobordisms.

1.1.1   Main tools: shadings and bicolored isotopies   The key step in the proofs of Theorems A and B is to understand how foams with the same underlying surface are related. We achieve this by constructing foams from *shadings*. A shading is a union of two possibly intersecting surfaces: a nonoriented blue and an oriented red one, that are in general position in $\mathbb{R}^3$, together with a checkerboard black and white coloring of the connected components of their complement, called regions. Forgetting those red facets of a shading, the orientations of which disagree with the one induced from the white regions, results in a foam, and all foams can be constructed this way. The same applies to webs.

A particularly nice feature of representing foams by shadings is the flexibility of this construction, which we call the *bicolored isotopy argument*: deforming any of the two surfaces by an isotopy results in a foam that differs from the original one only up to a sign or replacing some dots with their duals; see Proposition 2.10 in Section 2.2 for a precise statement. This has a number of important consequences:

- closed foams can be evaluated (Theorem 2.14) using the bicolored isotopy argument by moving the blue and red facets away from each other;

- more generally, foams with the same boundary and underlying surfaces coincide up to a sign and types of dots (Proposition 2.10);

- a foam, the underlying surface of which is a product $\omega \times [0, 1]$, is invertible.

We then use the above to construct a basis of the space of foams bounded by a closed web ω. It is given in terms of shadings of a plane that extend ω, the blue loops of which may carry dots. The foam associated with such a picture ω^+ is given by attaching blue and red cups to the loops of ω^+ — red cups above all blue ones — and placing

a dot at the minimum of every blue cup attached to a loop that is marked by a dot. This leads to an explicit description of the tautological TQFT functor on **Foam**($\varnothing$) that associates the space $\text{Hom}_{\textbf{Foam}(\varnothing)}(\varnothing, \omega)$ with a closed web ω, presented in Section 5. When compared with [14], our basis is not only easier to visualize, but also the formula for the action of foams involves fewer signs.

1.2 Functorial tangle homology

Khovanov extended his construction first to tangles with an even number of boundary points at each side [19]. For this he constructed a 2–functor $\mathcal{F}_{Kh}^{\circ} \colon \textbf{BN}^{\circ} \to \textbf{Bimod}$, where $\textbf{BN}^{\circ}$ is the subbicategory of $\textbf{BN}$ with only even collections of points as objects. The 2–functor $\mathcal{F}_{Kh}^{\circ}$ associates with a collection of $2n$ points the *arc algebra*

$$(1\text{-}4) \qquad\qquad H^n := \bigoplus_{a,b} \text{Hom}_{\textbf{BN}}(a, b),$$

where a and b run through the set of Temperley–Lieb cup diagrams in $\mathbb{R} \times (-\infty, 0]$ with $2n$ boundary points at the top boundary line.[3] This algebra H^n is known to categorify the invariant subspace $\text{Inv}(V^{\otimes n})$ of $V^{\otimes n}$, where V is the fundamental representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. Cup diagrams parametrize indecomposable projective H^n–modules, which in turn correspond to elements of the canonical basis of $V^{\otimes n}$. Let $CKh(T)$ be the chain complex associated with an $(2n, 2n')$–tangle T, ie the result of applying $\mathcal{F}_{Kh}^{\circ}$ to $[\![T]\!]$. The functors $CKh(T) \otimes (-)$ lift the action of tangles on $\text{Inv}(V^{\otimes n})$ to the derived categories of the arc algebras [19].

In order to categorify the whole tensor power $V^{\otimes n}$, Chen and Khovanov considered a family of algebras $A^{k,n-k}$, where $0 \le k \le n$, each constructed as a subquotient of H^n. These algebras were discovered independently by Stroppel [34], who proved with Brundan [6; 7] that they are quasihereditary covers of arc algebras and Koszul. Furthermore, projective modules over $A^{k,n-k}$ categorify the weight space $V^{\otimes n}(\lambda)$ with $\lambda = n - 2k$ [7; 10]. As in the case of arc algebras, there is a family of 2–functors $\mathcal{F}_{Kh}^{\lambda} \colon \textbf{BN} \to \textbf{Bimod}$, such that $\mathcal{F}_{Kh}^{\lambda}$ assigns to a collection of n points the algebra $A^{k,n-k}$ with $\lambda = n - 2k$ [6; 10]. Write $CKh(T; \lambda)$ for the result of applying $\mathcal{F}_{Kh}^{\lambda}$ to $[\![T]\!]$. Then the functor $CKh(T; \lambda) \otimes (-)$ lifts the action of T on the weight space $V^{\otimes n}(\lambda)$.

Using Theorem A we can construct a strictly functorial version of both Khovanov and Chen–Khovanov homologies by precomposing $\mathcal{F}_{Kh}^{\circ}$ and $\mathcal{F}_{Kh}^{\lambda}$ with $\mathcal{E}^{\vee}$. We provide a direct construction of both invariants.

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[3]This presentation of H^n comes from Rozansky [33].

Following [14] we call the web version of H^n the *Blanchet–Khovanov algebra*. It is defined for any collection of oriented red and blue points Σ that is *balanced*, ie bounds a web, as the direct sum

$$\mathfrak{W}^{\mathcal{B}} := \bigoplus_{a,b \in \mathcal{B}} \mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(a,b),$$

where $\mathcal{B}$ is a *cup basis* of webs bounded by Σ; its elements play the role of cup diagrams for H^n. Although $\mathfrak{W}^{\mathcal{B}}$ depends a priori on $\mathcal{B}$, we show that different choices of basis lead to isomorphic algebras. Moreover, there is a special basis of webs — the *red-over-blue* basis — such that forgetting red facets in cup foams is compatible with multiplication. In particular, $\mathfrak{W}^{\mathcal{B}}$ admits a *positive basis*. This results immediately in an algebra isomorphism $\mathfrak{W}^{\mathcal{B}} \cong H^n$, where n is half of the blue points in Σ. We further extend this construction to a 2–functor $\mathcal{F}_w^{\circ} : \mathbf{Foam}^{\circ} \to \mathbf{Bimod}$ following the construction of $\mathcal{F}_{Kh}^{\circ}$.

Suppose that T is an oriented tangle, the input and output of which are balanced. Then all resolutions of T are in $\mathbf{Foam}^{\circ}$ and $\mathcal{F}_w^{\circ}$ can be applied to $[\![T]\!]_{\mathrm{F}}$ to produce a chain complex of bimodules $C_{\mathfrak{W}}(T)$. We call it the *Blanchet–Khovanov complex*.

Theorem C   *The 2–functor $\mathcal{F}_w^{\circ}$ is equivalent to $\mathcal{F}_{Kh}^{\circ} \circ \mathcal{E}^{\vee}$. In particular, the complexes $C_{\mathfrak{W}}(T)$ and $CKh(T)$ are isomorphic for any tangle T with balanced input and output.*

The construction of a web version of Chen–Khovanov algebras is more challenging. We first describe two extensions of a sequence Σ to a balanced one Σ° by inserting extra blue points to the left and to the right of Σ. Then we pick a basis $\mathcal{B}$ of webs bounded by Σ° and the corresponding Blanchet–Khovanov algebra $\mathfrak{W}^{\mathcal{B}}$. The *extended Blanchet–Khovanov algebra* $\mathfrak{A}^{\Sigma,\lambda}$, where $\lambda \in \mathbb{Z}$ has the same parity as the number of blue points in Σ, is a certain subquotient of $\mathfrak{W}^{\mathcal{B}}$. Following the same procedure we associate a bimodule with a web and a bimodule map with a foam for every $\lambda \in \mathbb{Z}$, obtaining a family of 2–functors $\mathcal{F}_w^{\lambda} : \mathbf{Foam} \to \mathbf{Bimod}$, each defined on the entire foam bicategory. As in the previous construction, $\mathcal{F}_w^{\lambda}$ is compatible with relations between foams, so that applying it to $[\![T]\!]_{\mathrm{F}}$ results in an invariant chain complex of bimodules $C_{\mathfrak{W}}(T; \lambda)$. We call it the *extended Blanchet–Khovanov complex* of T.

We construct an explicit isomorphism $\mathfrak{A}^{\mathcal{B},\lambda} \cong A^{k,n-k}$, where n counts blue points in Σ and $\lambda = n - 2k$. Contrary to the previous case, it is not enough to forget red facets in cup foams to get the isomorphism, because the basic webs from $\mathcal{B}$ may have too many blue arcs. This issue is resolved by *stabilization* — adding beneath webs and foams

extra blue arcs and disks respectively. We then extend this isomorphism to bimodules and prove the following fact:

Theorem D *The 2–functor $\mathcal{F}_w^\lambda$ is equivalent to $\mathcal{F}_{Kh}^\lambda \circ \mathcal{E}^\vee$. In particular, the complexes $C_{\mathfrak{W}}(T; \lambda)$ and $CKh(T; \lambda)$ are isomorphic for any tangle T.*

All the isomorphisms are constructed explicitly and — in case nice bases are used — given by very simple formulas. Furthermore, by the discussion following Theorem B, the tangle homology computed with $\mathcal{F}_w^\circ$ and $\mathcal{F}_w^\lambda$ are isomorphic to the Khovanov and Chen–Khovanov invariants respectively.

1.3 Functoriality of quantized annular Khovanov homology

The above results allow us to construct a strictly functorial version of the quantized annular Khovanov homology, which was the motivation for this paper. Combining Theorem D with [4, Proposition 6.6] we get:

Corollary E *Suppose $\Bbbk$ is flat over $\mathbb{Z}[q^{\pm 1}]$. Then the quantum Hochschild homology groups $qHH_i(\mathfrak{A}^{\mathcal{B},\lambda})$ with coefficients in $\Bbbk$ vanish for $i > 0$, whereas the Chern character map*

$$h \colon K_0(\mathfrak{A}^{\mathcal{B},\lambda}) \otimes_{\mathbb{Z}[q^{\pm 1}]} \Bbbk \to qHH_0(\mathfrak{A}^{\mathcal{B},\lambda})$$

is an isomorphism.

Choose now an oriented tangle T that is bounded at both top and bottom by the same collection of oriented points Σ. We define for its annular closure $\hat{T}$ the *quantum annular $\mathfrak{gl}_2$ complex* as

$$Kh_{\mathbb{A}_q}(\hat{T}) := \bigoplus_\lambda qHH_\bullet(\mathfrak{A}^{\mathcal{B},\lambda}, C_{\mathfrak{A}}(T; \lambda))$$

where $\mathcal{B}$ is a cup basis of webs bounded by Σ and $C_{\mathfrak{A}}(T; \lambda)$ — the chain complex of bimodules obtained by applying $\mathcal{F}_w^\lambda$ to $[\![T]\!]_F$. Corollary E together with [4, Theorem B] imply the following:

Corollary F *The quantum annular $\mathfrak{gl}_2$ homology $Kh_{\mathbb{A}_q}(L)$ is a triply graded invariant of annular links that is strictly functorial with respect to annular link cobordisms. Moreover, it admits an action of $\mathcal{U}_q(\mathfrak{gl}_2)$ that commutes with the differential and the maps induced by annular link cobordisms.*

It follows now from Theorem D and the following discussion that $Kh_{\mathbb{A}_q}(L)$ is isomorphic with the quantized annular complex as constructed in [4].

1.4 Further generalizations

The Khovanov homology has been extended by Asaeda, Przytycki, and Sikora to links in thickened surfaces [1], but the functoriality has not been addressed until the recent paper of Quefellec and Wedrich [31]. There they have defined $\mathfrak{gl}_2$ foams in thickened *oriented* surfaces, and the natural question is whether the results of this paper can be extended to show equivalence of the two constructions. This is addressed in a follow-up paper, where we also discuss foams in arbitrary 3–manifolds, including nonorientable ones.

Another natural question is about $\mathfrak{gl}_N$ foams for $N > 2$; see Khovanov [20], Mackaay, Stošić and Vaz [25], and Queffelec and Rose [30]. Again there are two (bi)categories involved: of *enhanced* and *not enhanced* foams, the latter allowing only facets of labels up to $N - 1$. We expect that a proper generalization of this paper would prove equivalence of both (bi)categories, hence also of the associated link homologies. Notice that functoriality of $\mathfrak{gl}_N$ homology has been shown by Ehrig, Tubbenhauer and Wedrich in [15] using enhanced foams.

1.5 Organization of the paper

Section 2 provides a brief exposition of webs and foams. All the results presented there are well known, except perhaps the choice of defining relations. Section 3 discusses shadings, their connection to webs and foams, and bicolored isotopies. It ends with a construction of a basis of the space of foams bounded by a given web. The equivalence of bicategories **BN** and **Foam** together with the local versions are constructed in Section 4, in which we also compare the two versions of the Khovanov bracket. Finally, Sections 5–7 provide detailed constructions of TQFT functors: a description of the tautological functor on **Foam**($\varnothing$) in terms of planar pictures, the constructions of the Blanchet–Khovanov algebras, their subquotients, and the 2–functors $\mathcal{F}_w^\circ$ and $\mathcal{F}_w^\lambda$.

1.6 Conventions and notation

Throughout the paper we fix a commutative unital ring $\Bbbk$ and linearity means $\Bbbk$–linearity. We denote by $\{d\}$ the upward degree shift by d, ie $M\{d\}_i = M_{i-d}$ for a graded module M. Hence, a homogeneous $m \in M$ has degree $\deg(m) + d$ when seen as an element of $M\{d\}$. We write $Com_{/h}(\mathbf{C})$ for the homotopy category of a linear category $\mathbf{C}$, the objects of which are formal complexes in $\mathbf{C}$ and morphisms — homotopy classes of chain maps.

Manifolds are assumed to be smooth (or at least piecewise smooth when necessary) and submanifolds are neat — that is $N \subset M$ is transverse to ∂M and $\partial N = N \cap \partial M$, see Hirsch [16]. Orientation of a surface $S \subset \mathbb{R}^3$ is often identified with the *canonical normal vector field* ν, defined by the property that for each $p \in S$ the triple (e_1, e_2, ν_p), where (e_1, e_2) is an oriented basis of $T_p S$, is an oriented basis of $T_p \mathbb{R}^3$. Such a vector field is unique up to an isotopy and can be found by the right-hand rule.

Acknowledgements

2 Main players

This section provides basic definitions and facts about webs and foams. Most of the material is well known [5; 8; 13; 23]; the exception is the choice of defining relations. The main purpose of this part is to fix notation and introduce terms used throughout the paper.

2.1 Webs

A *web* is an oriented trivalent graph with edges colored blue or red[4] in such a way, that at each vertex either two blue edges *merge* to a red one, or a red edge *splits* into two blue edges:

(2-1)

In this paper webs will be always embedded in a disk $\mathbb{D}^2$ or a sphere $\mathbb{S}^2$ with a fixed basepoint $*$ that lies on $\partial \mathbb{D}^2$ in the case of a disk. Edges of a web in a disk can be attached transversely to the boundary circle away from $*$; each boundary point inherits

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[4]Red edges are drawn as double thick lines to make the difference visible when the paper is printed black and white.

then both the color and orientation from the attached edge: outwards (resp. inwards) oriented edges terminate with positive (resp. negative) points. A web is *closed* if its boundary is empty.

Remark 2.1 By moving the basepoint $*$ to infinity, we can consider webs in $\mathbb{D}^2$ or $\mathbb{S}^2$ as embedded in a half plane $\mathbb{R} \times (-\infty, 0]$ or a full plane $\mathbb{R}^2$ respectively.

Definition 2.2 We write *Web* for the module generated by isotopy[5] classes of webs in a disk, modulo the local[6] relations

(2-2)
$$\bigcirc = q + q^{-1} \qquad\qquad \bigcirc = 1$$

(2-3)
$$\rlap{\vdash}{\,\mid\mid} = \mid\, \subset \qquad\qquad \bowtie = \rangle\langle$$

where the webs above can carry any coherent orientation unless indicated. For each collection of oriented red and blue points $\Sigma \subset \partial\mathbb{D}^2$ there is a submodule *Web*(Σ) generated by webs bounded by Σ and *Web* is the direct sum of all of them.

Exercise 2.3 Show that webs satisfy the local relations

(2-4)
$$\ovalbox{} = \mid \qquad\qquad \ovalbox{} = (q + q^{-1}) \mid\mid$$

Hint: Start with the left relation in (2-3).

Blue edges of a web ω form a crossingless tangle ω_b, which we call the *underlying tangle of ω*. In particular, it is a collection of disjoint circles when ω is closed. Write $\ell(\omega)$ for the number of blue loops in ω_b. Let r(ω) be a web, the underlying tangle of which is ω_b with closed loops removed. We call it a *reduction of ω*. We construct reductions later in Section 3.2 using the bicolored isotopy argument and show the following fact, which implies in particular that r(ω) does not depend on the placement of red edges.

Proposition 2.4 *Webs with same boundary and isotopic underlying tangles coincide in Web. In particular, $\omega = (q + q^{-1})^{\ell(\omega)} \mathrm{r}(\omega)$ for any web ω.*

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[5]Isotopies are assumed to fix points on the boundary circle.

[6]The word *local* means that two webs are identified if there is a disk outside of which the webs coincide and inside they look like in the pictures.

The second statement in Proposition 2.4 is equivalent to [31, Lemma 2.1]. Let $-\omega$ be the result of reversing orientation of all edges in a web ω. This operation preserves the relations (2-2) and (2-3), hence it induces an involution on *Web*. It does not preserve the submodules *Web*(Σ), but there is a pairing

$$(2\text{-}5) \qquad\qquad (\omega, \omega') := (q + q^{-1})^{\ell(-\omega \cup \omega')},$$

which can be visualized by placing $-\omega$ and ω' on the lower and upper hemisphere of a sphere and applying Proposition 2.4 to the resulting web (entirely red webs evaluate to 1).

Lemma 2.5 *The pairing* (2-5) *is nondegenerate.*

Proof Choose a nonzero $w \in Web(\Sigma)$ and write it as a linear combination

$$c_1\omega_1 + \cdots + c_r\omega_r$$

of pairwise nonisotopic webs $\omega_1, \ldots, \omega_r$, the underlying tangles of which contain no loops. We may further assume that the polynomial c_1 contains a term q^d with the maximal value of $|d|$ among all c_i. Because $\ell(-\omega_1 \cup \omega_i) < \ell(-\omega_1 \cup \omega_1)$ for any $i \neq 1$, the term $q^d(\omega_1, \omega_1)$ is not canceled in the expansion of (w, ω_1). Hence, $(w, \omega_1) \neq 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2 Foams

A *foam* is a collection of *facets*, oriented blue and red[7] surfaces, embedded in a 3–ball B^3 with boundary components attached transversely to ∂B^3 or glued together along singular curves called *bindings* in a way, such that locally two blue facets merge into a red one in an orientation-preserving way as shown in Figure 1. Furthermore, blue facets may carry *dots*, but not the red ones, and bindings inherit orientation from blue facets. We say that a foam is *closed* if its boundary is empty. Otherwise it is bounded by a web in ∂B^3. Notice that blue facets alone, forgetting orientation, form a surface S_b with dots, the *underlying surface of S*. As in the case of webs, we fix a basepoint $* \in \partial B^3$ away from ∂S. By moving it to infinity we can reinterpret foams as embedded in a half 3–space $\mathbb{R}^2 \times (-\infty, 0]$.

There is a canonical cyclic order of facets attached to a binding that follows the right-hand rule: point the thumb of your right hand along the binding curve and slightly bend the other fingers — they indicate the orientation of a small circle around the binding, hence a cyclic order of facets. We call a blue facet *positive* or *negative* depending on

---

[7]As in the case of webs, red facets of a foam are doubled in pictures.
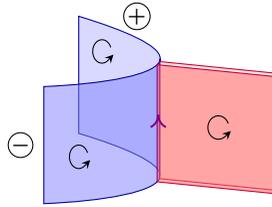
Figure 1: The local model for a foam. The orientation of the binding is coherent with the orientation of the blue facets, but opposite to the one induced from the red facet. The cyclic order is counterclockwise, when seen from above, so that the front blue facet is the negative one.

whether it succeeds or precedes the red facet respectively. For nonembedded foams this cyclic order is usually provided explicitly by drawing small arrows around the binding; see [5].

Definition 2.6 We write *Foam* for the module generated by isotopy classes of foams in B^3 with the following local relations imposed:

- *sphere evaluations*:

$$(2\text{-}6) \qquad \bigcirc = 0 \qquad \bigcirc\!\!\bullet = 1 \qquad \bigcirc = -1$$

- *neck-cutting relations*:

- *dot-reduction and dot-moving relations*:

- *red facet detachments*:

Foams bounded by a web $\omega \subset \mathbb{S}^2$ (with $* \notin \omega$) generate a submodule *Foam*(ω). As in the case of webs, *Foam* is the direct sum of all these submodules.

Remark 2.7 The sign in (2-9) and (2-10) can be read easily from the direction of the canonical normal vector at the critical point on the red surface: it is positive exactly when the normal vector is directed towards the blue plane. For example, (2-9) can be written as
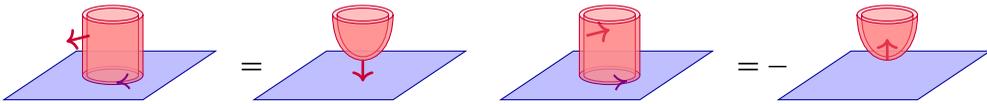
Remark 2.8 Definition 2.6 does not follow [5], where foams were defined using the universal construction, from which a sufficient set of relations has been derived. Instead, it looks more like the one in [23]. However, our set of relations is smaller and more natural from the topological point of view. The parameters h and t have already appeared in [8; 13].

When $\Bbbk$ is graded with h and t homogeneous in degree 2 and 4 respectively, then *Foam* is a graded module with a foam S being a homogeneous element in degree

$$(2\text{-}11) \qquad \qquad \deg(S) := -\chi(S_b) + 2\,dots(S).$$

Here $\chi(S_b)$ stands for the Euler characteristic of the underlying surface and $dots(S)$ counts dots carried by the foam.

The dot-moving relation — the right one in (2-8) — takes a particularly simple form for $h = 0$: it allows to move a dot on the underlying surface at a cost of a sign. To have a similar interpretation in the general case, we introduce the *dual dot* as the difference:
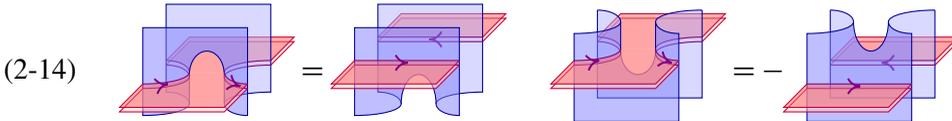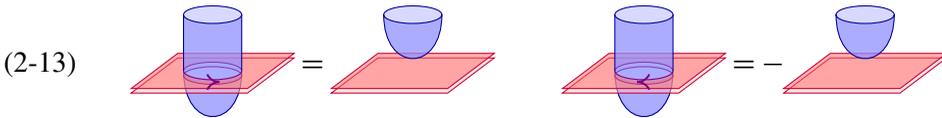
$$(2\text{-}12) \qquad \qquad$$

The following exercise lists several relations satisfied by dual dots.

Exercise 2.9 Show the following equalities between foams:
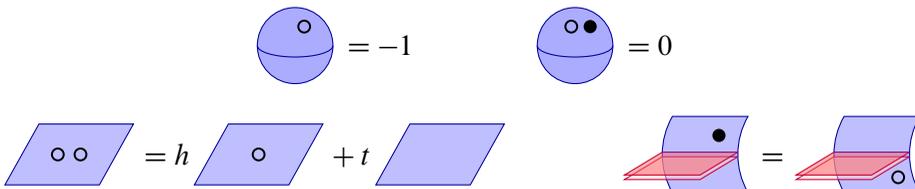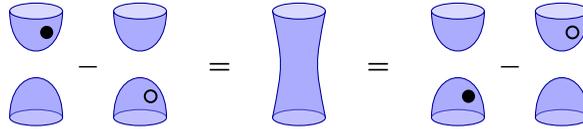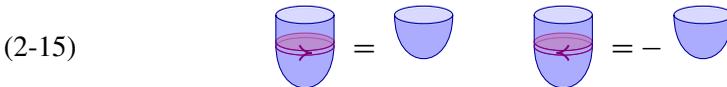
The detaching relations (2-9) and (2-10) can take many other forms. For instance, redrawing them to make red facets horizontal results in

(2-13)

(2-14)

Likewise, (2-9) together with (2-6) allow us to remove a red membrane attached to a blue cup

(2-15)

and other well-known relations arise by redrawing (2-10) and (2-15) in a way, such that blue facets form a horizontal plane and the boundary of red facets is vertical:

(2-16)

(2-17)
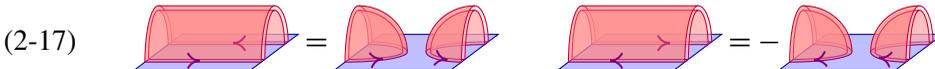
Notice that in each case the sign can be read from the direction of the normal vector as explained in Remark 2.7.

We interpret the above relations later as isotopies between two surfaces, a blue and a red one. This will be a key ingredient in the proofs of the two facts listed below. In what follows we write $S \doteq S'$ if foams S and S' differ only by a sign and dualizing dots. For instance, $S \doteq S'$ when S' is the result of moving a dot on the underlying surface of S.

Proposition 2.10 *Let S and S' be foams with isotopic underlying surfaces and same boundary. Then $S \doteq S'$ in Foam.*

We prove the above proposition at the end of Section 3.2. An important consequence of it is the uniqueness (up to a sign) of a foam $cup(\omega)$, the underlying surface of which is a collection of disjoint disks bounded by ω_b. We call it the *cup foam* associated to ω. Then for any family X of blue loops in ω_b we denote by $cup(\omega, X)$ the cup foam with a dot placed on every blue disk that bounds a curve from X. These foams constitute a linear basis of $Foam(\omega)$ as shown in Section 3.3.

Theorem 2.11 *Choose a closed web ω. The set $\{cup(\omega, X) \mid X \subset BL(\omega)\}$ is a linear basis of $Foam(\omega)$. In particular, $Foam(\omega)$ is a free graded module of rank $(q + q^{-1})^{\ell(\omega)}$.*

2.3 Decategorification

Fix a collection of red and blue oriented points $\Sigma \subset \mathbb{D}^2$. A *foam with corners in* Σ is a foam S in $\mathbb{D}^2 \times [0, 1]$ with $S \cap \partial\mathbb{D}^2 = \Sigma \times [0, 1]$. We gather them into a category **Foam**(Σ), in which

- objects are webs bounded by Σ with no relation imposed,
- morphisms from ω_0 to ω_1 are generated by foams with corners in Σ, with ω_1 at the top and $-\omega_0$ at the bottom disk of $\mathbb{D}^2 \times [0, 1]$, modulo the relations (2-6)–(2-10), and
- the composition is given by stacking foams, one on top of the other.

We further enhance it to a graded additive category by introducing formal direct sums and formal degree shifts, so that objects are of the form $\omega_1\{d_1\} \oplus \cdots \oplus \omega_r\{d_r\}$, and redefining the degree of a foam $S : \omega_0\{a\} \to \omega_1\{b\}$ as

(2-18) $$\deg(S) := (b - a) - \chi(S_b) + 2dots(S) + \tfrac{1}{2}\#\Sigma_b,$$

where, as before, $\chi(S_b)$ is the Euler characteristic of the underlying surface of S and $dots(S)$ counts dots on S, whereas $\#\Sigma_b$ is the number of blue points in Σ. The reason for the last term is to make the identity foam a morphism of degree zero; it also makes the degree additive under the composition of foams. Furthermore, reinterpreting foams with corners as foams in B^3 leads to an isomorphism of graded $\Bbbk$–modules

(2-19) $$\text{Hom}_{\textbf{Foam}(\Sigma)}(\omega, \omega') \cong Foam(-\omega \cup \omega')\{\tfrac{1}{2}\#\Sigma_b\}$$

for any webs ω and ω' bounded by Σ.

The orientation-reversing diffeomorphism of the thickened disk $(p, t) \mapsto (p, 1 - t)$ induces a contravariant involutive functor

(2-20) $$\text{Hom}_{\textbf{Foam}(\Sigma)}(\omega, \omega') \ni S \mapsto S^! \in \text{Hom}_{\textbf{Foam}(\Sigma)}(\omega', \omega)$$

that reflects a foam vertically and reverses orientation of its facets. We check directly that all the defining relations (2-6)–(2-10) are preserved.

Foams with corners categorify webs. Indeed, web relations are lifted to isomorphisms

(2-21)

(2-22)

where the sign in the bottom left corner depends on the orientation of the edges. Therefore, there is a well-defined epimorphism $\gamma : Web(\Sigma) \to K_0(\mathbf{Foam}(\Sigma)) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{k}$ that takes a web ω to its class $[\omega]$ in the Grothendieck group.

Theorem 2.12 *The linear map* $\gamma : Web(\Sigma) \to K_0(\mathbf{Foam}(\Sigma)) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{k}$ *is an isomorphism.*

Proof We have to show that γ is injective. Consider a bilinear form $\langle -, - \rangle$ on $K_0(\mathbf{Foam}(\Sigma))$ defined for webs ω and ω' as $\langle [\omega], [\omega'] \rangle := \mathrm{rk}_q \, \mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(\omega, \omega')$. It is well defined, because the rank of the morphism space depends only on the images of webs in the Grothendieck group. Theorem 2.11 and the isomorphism (2-19) imply together that $\langle [\omega], [\omega'] \rangle = (\omega, \omega')$, where the latter is the nondegenerate pairing from (2-5). Hence, $\gamma(w) = 0$ forces $(w, -) = 0$, so w must be zero. $\qquad\square$

2.4 Higher structures

It is common to consider webs embedded in a horizontal stripe $\mathbb{R} \times [0, 1]$ instead of a disk. This is equivalent to picking two basepoints on $\partial \mathbb{D}^2$, $*$ and $*'$, and placing them at the left and right infinities respectively. Such webs are morphisms of a linear category **Web**, the objects of which are finite collections of oriented red and blue points

on a line, whereas the composition is defined by stacking stripes vertically:
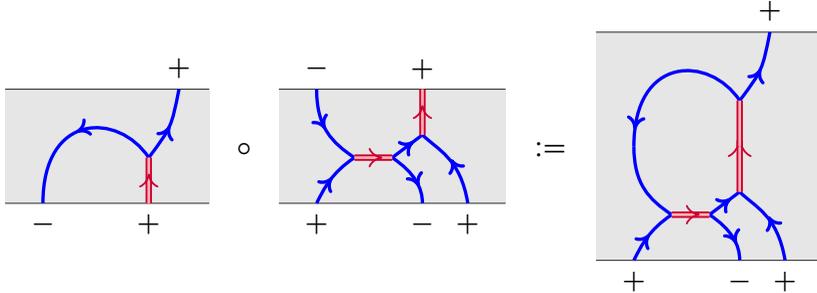
Formally, $\mathrm{Hom}_{\mathbf{Web}}(\Sigma, \Sigma') = Web(-\Sigma \cup \Sigma')$. This category is closely related to representations of $\mathcal{U}_q(\mathfrak{gl}_2)$ [9]: there is a monoidal functor $\mathcal{V}\colon \mathbf{Web} \to \mathbf{Rep}(\mathcal{U}_q(\mathfrak{gl}_2))$ such that

- a blue positive (resp. negative) point is assigned the fundamental representation V (resp. its dual V^*) and a red positive (resp. negative) point — the determinant representation $\wedge^2 V$ (resp. $\wedge^2 V^*$), whereas a sequence of such points is assigned the tensor product of the corresponding representations,

- the merge and split webs (2-1) are assigned the canonical inclusion and quotient maps between representations, and

- cups and caps represent coevaluation and evaluation maps.

The relations between webs make the above functor faithful.

Define the *weight* of a point from Σ according to:

point	$\bullet +$	$\bullet -$	$\circ +$	$\circ -$
weight	$+1$	-1	$+2$	-2

The *total weight* $w(\Sigma)$ of Σ is the sum of weights of its points. A quick analysis of the local model for webs (2-1) reveals that webs exist only between objects of the same weight. Hence, the category of webs decomposes into *weight blocks* $\mathbf{Web}^k$, each spanned by objects of weight $k \in \mathbb{Z}$. In particular, $\mathrm{Hom}_{\mathbf{Web}}(\varnothing, \Sigma) \neq 0$ only when $w(\Sigma) = 0$; such collections are called *balanced*.

In a similar manner one collects the foam categories $\mathbf{Foam}(\Sigma)$ into a bicategory $\mathbf{Foam}$, which also decomposes into blocks $\mathbf{Foam}^k$ parametrized with $k \in \mathbb{Z}$. Theorem 2.12 can be then rephrased to say that $\mathbf{Foam}^k$ categorifies $\mathbf{Web}^k$, ie the category of webs is obtained by replacing morphism categories of $\mathbf{Foam}$ with their Grothendieck groups.

2.5 Blanchet evaluation formula

We end this section recalling the evaluation formula for closed foams in a 3–ball B^3 following [5]. It requires two 2–dimensional TQFTs, one for blue and one for red facets. Each is uniquely determined by the (associative) commutative Frobenius algebra assigned to a circle. We choose the algebras

$$A_b := \mathbb{k}[X]/(X^2 - hX - t) \quad \text{and} \quad A_r := \mathbb{k}$$

for blue and red circles respectively, where $h, t \in \mathbb{k}$ are fixed parameters (the standard choice is $h = t = 0$). The comultiplications and counits are defined by the formulas

$$\Delta_b(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1, \quad \Delta_b(X) = X \otimes X + t1 \otimes 1, \quad \Delta_r(1) = -1 \otimes 1,$$

$$\epsilon_b(1) = 0, \qquad\qquad\qquad \epsilon_b(X) = 1, \qquad\qquad\qquad \epsilon_r(1) = -1.$$

A dot on a blue surface is interpreted as the multiplication with X. Notice that $h - X$, which represents a dual dot, satisfies the polynomial relation defining A_b, so that $\overline{X} := h - X$ extends to a conjugation compatible with multiplication. One checks directly that $\overline{\Delta_b(a)} = -\Delta_b(\bar{a})$ and $\epsilon_b(\bar{a}) = -\epsilon_b(a)$ for any $a \in A_b$.

When $\mathbb{k}$ is graded with h and t homogeneous in degree 2 and 4 respectively, then we make A_b a graded algebra by setting $\deg(X) = 2$; comultiplication and counit increase and decrease the degree by 2 respectively. Assigning now $A_b\{-1\}$ to a blue circle produces a graded TQFT: $\deg(1) = -1$ and $\deg(X) = +1$, in which case both multiplication and comultiplication are homogeneous in degree 1, matching the degree of a saddle. Likewise for the unit and counit. The other TQFT is upgraded by inheriting the grading on A_r from $\mathbb{k}$.

Assume that a closed foam S is obtained from a blue surface S_b and a red one S_r by identifying boundary circles $C_i^+, C_i^- \subset \partial S_b$ with $C_i^0 \subset \partial S_r$ for $1 \leq i \leq m$, such that C_i^+ and C_i^- come from the positive and negative facet respectively. Let

$$Z_b(S_b) \in (A_b \otimes A_b)^{\otimes m} \quad \text{and} \quad Z_r(S_r) \in (A_r)^{\otimes m}$$

be the elements assigned by the two TQFTs to the blue and red surface, where the first factor in $A_b \otimes A_b$ corresponds to C_i^+ and the second to C_i^-. The evaluation assigns to S the value

(2-23) $$Z(S) = \text{tr}^{\otimes m}\big(\pi^{\otimes m}(Z_b(S_b)) \otimes \eta^{\otimes m}(Z_r(S_r))\big) \in \mathbb{k},$$

where $\pi \colon A_b \otimes A_b \to A_b$ sends $x \otimes y$ to $x\bar{y}$ and $\eta \colon A_r \to A_b$ is the inclusion of algebras; the trace map $\text{tr} \colon A_b \otimes A_b \to \mathbb{k}$ is the composition of the multiplication with the counit of A_b.

Example 2.13 Let S be a blue sphere with a red disk inside and one dot, as shown below. It decomposes into three cups, two blue and a red one, where one of the blue cups carries a dot:

The orientation of the binding determines that the dotted cup is attached to the negative boundary. Hence,

$$Z_b(S_b) = 1 \otimes X, \quad Z_r(S_r) = 1,$$

resulting in $Z(S) = \text{tr}(1 \otimes \overline{X}) = \epsilon_b(h - X) = -1$.

The relations (2-15) and (2-6) evaluate the foam S from the example above to -1 as well. This is not a coincidence: the defining relations were looked up in the kernel of Z. In fact, Proposition 2.10 implies a stronger statement. It was first proven in [5].

Theorem 2.14 (cf [5]) *The evaluation* (2-23) *descends to an isomorphism*

$$Z \colon Foam(\varnothing) \to \Bbbk.$$

Proof We first check that Z is well defined, ie it preserves the relations (2-6)–(2-10). Those involving facets of one color can be checked directly, whereas moving a dot through an ith binding corresponds to taking it from a facet attached to C_i^+ (multiplication by X) and placing it on the facet attached to C_i^- (multiplication by $\overline{X} = h - X$). Hence, (2-8) is satisfied. We follow now Example 2.13 to compute

$$(2\text{-}24) \qquad Z\left(\text{} \right) = 0, \qquad Z\left(\text{} \right) = 1, \qquad Z\left(\text{} \right) = -1,$$

which immediately implies (2-9): using (2-7) cut both the red cylinder and the plane around the binding to obtain a sum of three foams, each consisting of a red cup, a blue plane, and a blue sphere with a red membrane inside. Two of these foams have an additional dot, one on the plane and the other the sphere; only the latter term survives and the sign comes from (2-24). We leave (2-10) as an exercise.

Assume now that $Z(S) = 0$. By Proposition 2.10, S coincides up to a sign with an entirely blue foam S', which is the blue surface S_b, perhaps with some dots replaced with dual dots. However, applying the blue neck-cutting relation (2-7) to any component of positive genus reduces S' further to a sum of collections of dotted spheres. These in turn can be completely evaluated with (2-8) and (2-6). Hence, $S' = Z(S') = 0$, which shows that Z is invertible. $\square$

3 Shadings and a basis of foams

This part is the backbone of the paper. We introduce here shadings of manifolds, use them to construct webs and foams, and prove the bicolored isotopy lemma: isotopic shadings encode equal webs and foams (the latter up to a sign and type of dots). Using this language we introduce then a basis of foams that is especially easy to visualize.

3.1 Shadings and trivalent manifolds

A *shading* of a manifold M^n consists of two codimension 1 submanifolds, an oriented U_r and a nonoriented U_b, that are transverse to each other and to ∂M^n, together with a *checkerboard coloring* of M^n: a choice of color, white or black, for each connected component of the complement of $U_r \cup U_b$, such that any two components with a common facet have different colors. We refer to U_r and U_b as *red* and *blue* respectively. The components of the intersection $U_r \cap U_b$ are called *bindings*; they decompose both U_r and U_b into *facets*. Finally, we refer to the components of the complement of $U_r \cup U_b$ in M^n as *regions*.

Lemma 3.1   *Assume M^n is simply connected and fix a point $* \in M^n$. Then a pair of codimension 1 submanifolds of M^n, that are transverse to each other and away from $*$, determines a unique shading of M^n with the region containing $*$ painted white.*

Proof   Given a pair of transverse codimension 1 submanifolds (U_r, U_b) we construct a desired shading as follows. Given $p \in M^n \setminus (U_r \cup U_b)$ choose a path γ from $*$ to p, transverse to both U_r and U_b, and let $d(\gamma) := \#(\gamma \cap U_r) + \#(\gamma \cap U_b)$ count the intersection points of γ with both submanifolds. Color p white or black depending on whether $d(\gamma)$ is even or odd. Because M^n is simply connected, the parity of $d(\gamma)$ does not depend on the choice of γ and the color of p is well defined.   □

Remark 3.2   It follows from Lemma 3.1 that every codimension 1 submanifold U of a simply connected manifold M^n admits a *standard orientation*: the one induced from white regions, when U is considered as a shading with $U_r = \varnothing$. In particular, every codimension 1 submanifold of M^n is orientable. When M^n is a line and U a collection of blue points, then the standard orientation on U is the alternating one. Likewise for the case $M^n = \mathbb{S}^1$, assuming the cardinality of U is even (otherwise it does not extend to a shading).

A *trivalent manifold* embedded in M^n is a generalization of webs and foams. It is a collection of *facets*, oriented codimension 1 submanifolds colored blue or red, with
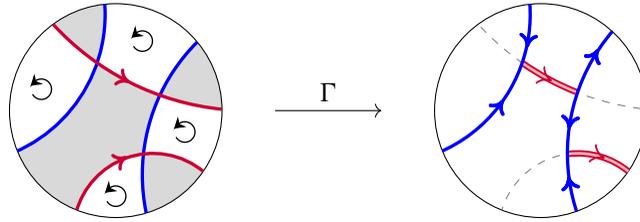
Figure 2: The construction of a web from a shading of a disk. The annihilated red edges are drawn as dashed lines on the right diagram.

boundary components attached transversely to ∂M^n or glued together along *bindings* in such a way that locally two blue facets merge into a red one. In other words, each point of a trivalent manifold has a neighborhood diffeomorphic to either $\mathbb{R}^{n-1}$ or $Y \times \mathbb{R}^{n-2}$, where Y is an oriented *merge* or a *split* from (2-1).

Given a shading (U_r, U_b) of M^n we construct a trivalent manifold $\Gamma(U_r, U_b)$ by examining the orientation on facets induced from white regions:

- Blue facets inherit the orientation.

- Red facets are preserved ("amplified") if the induced orientation agrees with the given one or annihilated otherwise.

An example is presented in Figure 2. In particular, $\Gamma(\varnothing, U_b)$ is U_b with its standard orientation as defined in Remark 3.2. It appears that every trivalent manifold arises this way, the proof of which is presented below and visualized in Figure 3. Hence, shadings can be considered as *completions* of trivalent manifolds, because of which we shall refer to shadings of $\mathbb{D}^2$ and B^3 as *completed webs* and *completed foams* respectively.
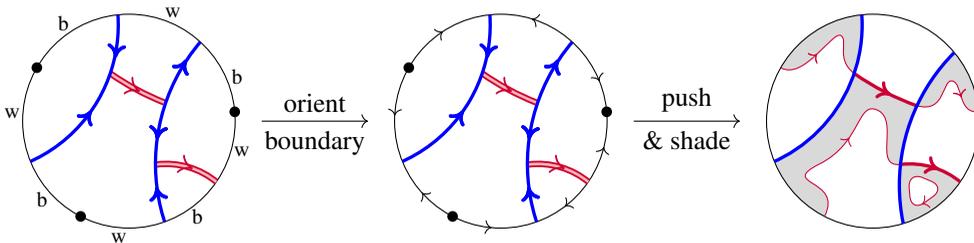

Figure 3: The construction of a shading from a planar web that extends a given shading of its boundary. The boundary of the disk is oriented in the middle picture, whereas the curves U_i' are identified and pushed inwards the corresponding regions in the third picture.

Lemma 3.3 *Choose a trivalent manifold $V \subset M^n$, such that $\partial V = \Gamma(\widetilde{U}_r, \widetilde{U}_b)$ for some shading $(\widetilde{U}_r, \widetilde{U}_b)$ of ∂M^n. Then there exists a shading (U_r, U_b) of M^n that restricts to $(\widetilde{U}_r, \widetilde{U}_b)$ on ∂M^n and satisfies $\Gamma(U_r, U_b) = V$.*

Proof Consider the orientation of ∂M^n induced from M^n and reverse it at all black edges in the given shading. Then the boundary of any region $R \subset M^n$ is a union of facets of V and regions in ∂M^n, such that oppositely oriented components meet only in two situations: when they are both contained in the boundary (so that they meet at a facet of ∂V) or both are blue facets of V adjacent to a red facet outside of R. Consider the union of those components of ∂R, the orientation of which does not match the one induced from R. They constitute certain oriented $(n-1)$–dimensional submanifolds $U_1, \ldots, U_k$. Taking a red colored copy U_i' of each U_i, push its interior inside R and paint the newly created region black. Repeating this for each region produces the desired shading.   □

A useful consequence of Lemma 3.3 is that tangles and surfaces can be extended to webs and foams with given boundary. Recall that a collection $\Sigma \subset \partial \mathbb{D}^2$ of oriented red and blue points is *balanced* if it bounds a web, which is equivalent to being of weight zero.

Proposition 3.4   (1) *Let $\Sigma \subset \partial \mathbb{D}^2$ be a balanced collection of oriented red and blue points and τ a tangle bounded by Σ_b. Then there exists a web ω bounded by Σ with $\omega_b = \tau$.*

   (2) *Let $\omega \subset \partial B^3$ be a web and W a surface bounded by ω_b. Then there is a foam S bounded by ω with $S_b = W$.*

Proof Extend Σ to a shading $\widetilde{\Sigma} = (\Sigma_r \cup \Sigma_r', \Sigma_b)$. Then $\widetilde{\Sigma}$ has an even number of points and the orientation of points from Σ matches the one induced from white regions. Let b, r, and r' be the sums of orientations of blue points in Σ, red points in Σ, and red points added to $\widetilde{\Sigma}$ respectively. Then $b + 2r = 0$, because Σ is balanced, and $b + r + r' = 0$, because the orientation of points in $\widetilde{\Sigma}$ alternate. Subtracting the two equalities reveals that $r - r' = 0$. It follows that there is an oriented collection of disjoint intervals $\tau_r \subset \mathbb{D}^2$ bounded by $\widetilde{\Sigma}_r$, the orientation of which agree with the points from Σ_r and disagree with those from Σ_r'. Hence, $\omega := \Gamma(\tau_r, \tau)$ is the desired web.

The second statement is even easier to show. Extend the web ω to a shading α. Then α_r is a collection of disjoint loops and each such collection bounds a family W_r of disjoint disks in B^3. Therefore, $S := \Gamma(W_r, W)$ is the desired foam.   □

3.2 Bicolored isotopies

Choose an isotopy Φ of M^n and a subset A. The set

$$\mathrm{Tr}_\Phi(A) = \{(\Phi_t(a), t) \mid a \in A, t \in [0, 1]\}$$

is called the *trace of $A \subset M^n$ under* Φ [16]. We say that a pair of isotopies (Φ, Ψ) of M^n is an *isotopy of a shading* (U_r, U_b) if $(\mathrm{Tr}_\Phi(U_r), \mathrm{Tr}_\Phi(U_b))$ is a shading of $M^n \times [0, 1]$ that coincides with (U_r, U_b) at the level $t = 0$. When M^n is a 2–disk, then a generic pair of isotopies can be encoded by a sequence of *bigon moves*

(3-1)

whereas in case of a 3–ball two moves are necessary:

(3-2)

 and
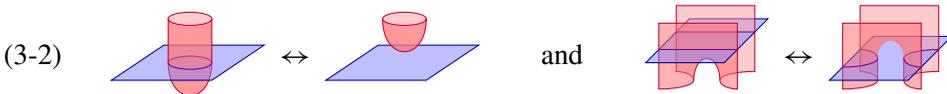
In each move a shading of one side determines a shading of the other. Hence, we obtain the following characterization of isotopies of shadings in these cases.[8]

Lemma 3.5 *Shadings (U_r, U_b) and (U_r', U_b') of $\mathbb{D}^2$ or B^3 are isotopic if and only if U_r is isotopic to U_r' and U_b is isotopic to U_b'.*

When a basepoint $* \in M^n$ is present, then one must be careful how it behaves under the isotopy. There is no problem when Ψ and Φ coincide at $*$ (and in this paper we always assume that both Ψ and Φ fix $*$). Otherwise, the basepoint should stay at the same region if possible. However, when the region disappears, then the basepoint has to reappear in a white region. For instance, when $*$ lies in the small bigon on the left-hand side of (3-1), then it reappears between the two strands on the right-hand side. Likewise, if $*$ lies in the ball bounded in the left figure in (3-2), then in the right figure it must reappear between the blue plane and the red cup.

Recall from Section 2.2 that we write $S \doteq S'$ for foams S and S' if they agree up to a sign and replacing some dots with their duals.

Lemma 3.6 (bicolored isotopy) (1) $\Gamma(\alpha_r, \alpha_b) = \Gamma(\alpha_r', \alpha_b')$ in *Web* if (α_r, α_b) and (α_r', α_b') are isotopic shadings of $\mathbb{D}^2$.

  (2) $\Gamma(W_r, W_b) \doteq \Gamma(W_r', W_b')$ in *Foam* if (W_r, W_b) and (W_r', W_b') are isotopic shadings of B^3.

---

[8]This can be extended to all manifolds by a detailed analysis of singular levels of a pair of isotopies.

Proof   It is enough to consider the case of elementary isotopies. When applied to each side of the bigon move (3-1), Γ removes red edges in both pictures from the same side of the blue line. Hence, $\Gamma(\alpha_r, \alpha_b)$ and $\Gamma(\alpha'_r, \alpha'_b)$ are related by the left relation in either (2-3) or (2-4). Likewise, the moves (3-2) correspond to the detaching relations (2-9) and (2-10).   □

The above result has far reaching consequences when paired with Lemma 3.3. The statements about comparing webs and foams with isotopic blue pieces follows, which in turn were used in the proof of Theorem 2.14 to show bijectivity of the Blanchet evaluation map Z.

Proof of Proposition 2.4   Let ω and ω' have isotopic underlying tangles and take the trace of ω_b under this isotopy as S_b; it is the underlying surface of a foam $S : \omega \to \omega'$ due to Proposition 3.4. Extend the foam to a shading $(\widetilde{S}_r, S_b)$ of $\mathbb{D}^2 \times [0, 1]$. When in generic position, it can be represented by a finite sequence of level sets, such that in between any two consecutive levels $\widetilde{S}_r$ has either no critical points (so that the level sets are related by the bicolored isotopy lemma) or a unique Morse type critical point — a cap, a cup, or a saddle — in which case the corresponding webs coincide (if the affected red edges are erased) or are identified by the right relations in (2-2) and (2-3) (if the red edges survive). Notice that S_b has no critical points.

For the second part, extend ω to a shading $(\widetilde{\omega}_r, \omega_b)$ of $\mathbb{D}^2$ and isotope closed blue loops, so that they do not intersect ω_r. Applying Γ results in a new web ω' that coincides with ω as shown above. Removing blue circles from ω' results in r(ω) and the desired equality follows from (2-2).   □

Proof of Proposition 2.10   Let foams S_1 and S_2 have isotopic blue parts. Extend them to shadings W_1 and W_2 respectively and pick a ball $\mathcal{O}$ in the interior of B^3, outside of which the red facets of the shadings coincide. Using Lemma 3.6 isotope blue facets away from $\mathcal{O}$ (this may dualize dots), reducing the problem to showing equality for foams with only red facets. In such case, use the neck-cutting relation (2-7) to reduce each red surface to a collection of disjoint disks and spheres (the existence of such a system of cuts follows from the theory of incompressible surfaces). This may change the sign of the foam. The thesis follows, because each red sphere evaluates to -1 and the disks are uniquely determined up to an isotopy by the boundary circles.   □

It follows immediately from Proposition 2.10 that the foam used in the proof of Proposition 2.4 is invertible. That would be enough to prove the latter if we knew that

Foam categorifies *Web*. However, the proof of the categorification result is based on Theorem 2.11, which is proven only in the next section.

3.3 Cup foams

We will now apply the above results to show that cup foams, as defined in Section 2.2, constitute a free basis of spaces of foams. In particular, the category of foams is nondegenerate.

Let ω be a closed web, so that ω_b is a collection of blue loops. Orient them in a standard way (see Remark 3.2) and pick a foam $I_\omega \in \mathrm{Hom}_{\mathbf{Foam}(\varnothing)}(\omega_b, \omega)$ with $\omega_b \times [0,1]$ as its underlying surface; the existence of such a foam follows from Proposition 3.4. According to Proposition 2.10, there is a sign $\mathrm{sgn}(\omega) = \pm 1$ satisfying

$$I_\omega^! I_\omega = \mathrm{sgn}(\omega)(\omega_b \times [0,1]),$$

where $I_\omega^! \in \mathrm{Hom}_{\mathbf{Foam}}(\omega, \omega_b)$ is the vertical flip of I_ω as defined in (2-20). The sign can also be computed directly as $\mathrm{sgn}(\omega) = \mathrm{sgn}(\omega)(C_\omega^\bullet C_\omega) = Z(C_\omega^\bullet I_\omega^! I_\omega C_\omega)$, where $C_\omega \in \mathrm{Hom}_{\mathbf{Foam}}(\varnothing, \omega_b)$ is a collection of disks bounded by ω_b and $C_\omega^\bullet \in \mathrm{Hom}_{\mathbf{Foam}}(\omega_b, \varnothing)$ is the same collection, except that each disk is decorated by a dot. Hence, $\mathrm{sgn}(\omega)$ is a well-defined integer, which we call the *sign of the web* ω.

Lemma 3.7 *The sign* $\mathrm{sgn}(\omega)$ *does not depend on the choice of* I_ω.

Proof Let $S \in \mathrm{Hom}_{\mathbf{Foam}}(\omega_b, \omega)$ be another foam with $S_b = \omega_b \times [0,1]$. Then $S = \pm I_\omega$ by Proposition 2.10 and $S^! S = I_\omega^! I_\omega$, because the same sign relates $S^!$ with $I_\omega^!$. □

Let $BL(\omega)$ be the collection of blue loops in ω. For each subset $X \subset BL(\omega)$ we construct the cup foam $\mathrm{cup}(\omega, X)$ by attaching blue disks to the input of I_ω and placing a dot on each disk bounded by a loop from X. Notice that red facets of $\mathrm{cup}(\omega, X)$ are above all dots and minima of blue facets. Therefore, we say that $\mathrm{cup}(\omega, X)$ is a *red-over-blue cup foam decorated by* X. We construct likewise a *cap foam* $\mathrm{cap}(\omega, X) \in \mathrm{Hom}_{\mathbf{Foam}(\varnothing)}(\omega, \varnothing)$ by reflecting $\mathrm{cup}(\omega, X)$ vertically and replacing each dot with the dual one scaled by -1. For instance, we have the following correspondence between cup and cap foams bounded by two blue loops:
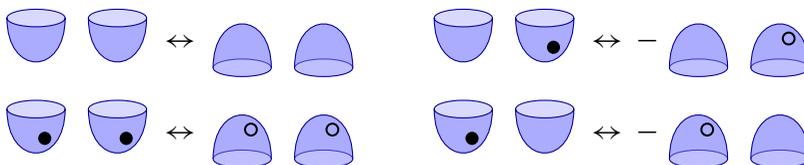
Let us now represent a foam $S \in \mathrm{Hom}_{\mathbf{Foam}(\varnothing)}(\omega, \omega')$ by a vertical cylinder labeled S, with ω and ω' at the bottom and top disk respectively. When no label is present, it is understood that $\omega = \omega'$ and the cylinder represents the identity foam $\omega \times [0, 1]$. We emphasize the cases $\omega = \varnothing$ and $\omega' = \varnothing$ by drawing a cup or a cap instead and, to simplify notation, we decorate it directly with $X \subset BL(\omega)$ when S is a cup or a cap foam:

$$\mathrm{cup}(\omega, X) = \underset{X}{\overset{\omega}{\bigcup}} \qquad \text{and} \qquad \mathrm{cap}(\omega, X) = \overset{X}{\underset{\omega}{\bigcap}}$$

Moreover, $X^c := BL(\omega) \setminus X$ stands for the complement of a subset $X \subset BL(\omega)$.

Lemma 3.8 *Foams satisfy the relations*

(3-3)
$$\begin{matrix} Y \\ \omega \\ X \end{matrix} \;=\; \begin{cases} \mathrm{sgn}(\omega) & \text{if } Y = X^c, \\ 0 & \text{otherwise,} \end{cases}$$

(3-4)
$$\left| \begin{matrix} \omega \\ \\ \omega \end{matrix} \right| \;=\; \mathrm{sgn}(\omega) \sum_{X \subset BL(\omega)} \begin{matrix} \omega \\ X \\ X^c \\ \omega \end{matrix}$$

Proof From the construction of cup and cap foams,

$$\begin{matrix} Y \\ \omega \\ X \end{matrix} \;=\; \begin{matrix} Y \\ \omega_b \\ I_\omega^! \\ \omega \\ I_\omega \\ \omega_b \\ X \end{matrix} \;=\; \mathrm{sgn}(\omega) \begin{matrix} Y \\ \omega_b \\ X \end{matrix}$$

and the right-hand side is a collection of spheres, each carrying at most one regular and one dual dot, scaled by $(-1)^{|Y|}$. Such a sphere evaluates to 1 or -1 when it carries either one regular or one dual dot respectively and vanishes otherwise (see Exercise 2.9). Hence, (3-3) follows.

The second relation follows from the equality $\omega \times [0, 1] = \mathrm{sgn}(\omega) I_\omega I_\omega^!$ and the neck-cutting relation from Exercise 2.9. $\qquad \square$

We are ready to prove that cup foams form a linear basis of foams.

Proof of Theorem 2.11 The first relation of Lemma 3.8 implies that cup foams are linearly independent. To show that they generate $Foam(\omega) \cong \mathrm{Hom}_{\mathbf{Foam}(\varnothing)}(\varnothing, \omega)$, use the second relation to write a foam S bounded by ω as a sum

which is a linear combination of cup foams, because closed foams evaluate to scalars. Finally,

$$\deg(\mathrm{cup}(\omega, X)) = 2|X| - \ell,$$

as the underlying surface of the cup foam consists of ℓ disks decorated by $|X|$ dots, so that

$$\mathrm{rk}_q \, Foam(\omega) = \sum_X q^{2|X|-\ell} = \sum_{s=0}^{\ell} \binom{\ell}{s} q^{2s-\ell} = (q + q^{-1})^{\ell}$$

as desired. □

4 Equivalences of foam and cobordism categories

In this section we prove Theorems A and B, which state that foams and Bar-Natan cobordisms constitute equivalent (bi)categories. Then we relate the complexes $[\![T]\!]$ and $[\![T]\!]_{\mathrm{F}}$ associated with a tangle T.

4.1 Embedding cobordisms into foams

Fix a balanced collection $\Sigma \subset \partial\mathbb{D}^2$ away from a fixed basepoint $* \in \partial\mathbb{D}^2$ and write Σ_b for the subset consisting of all blue points from Σ. Consider first the case when $\Sigma = \Sigma_b$ and the points are oriented in a standard way as explained in Remark 3.2. This means that, when following the orientation of the boundary circle, the first point after the basepoint is negative and then the orientation alternates. Theorem B is in this case a direct consequence of Proposition 2.4 and Theorem 2.11: each web is isomorphic to an entirely blue one (and each such web is a flat tangle equipped with the standard orientation) and for such webs ω and ω' the cup basis of $\mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(\omega, \omega')$ consists of foams with no
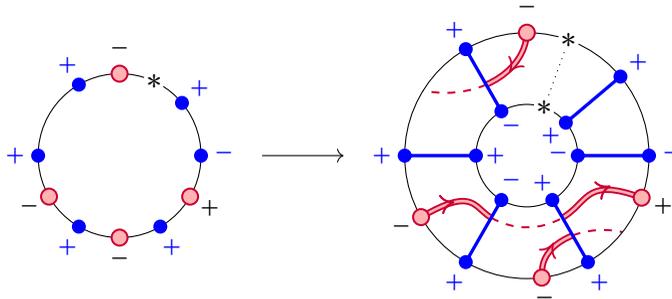
Figure 4: A collection of points Σ with four red and six blue points and an
annular web E_Σ as in Remark 4.2. Dashed red lines are not part of the web,
but they represent the additional red edges in the associated shading.

red facets. Hence, the naive map $\mathrm{Hom}_{\mathbf{BN}(\Sigma)}(\omega, \omega') \to \mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(\omega, \omega')$ that orients
a cobordism in a standard way does the job. A little more work has to be done to cover
the general case.

Lemma 4.1 *There is a web $E_\Sigma \subset \mathbb{S}^1 \times [0, 1]$ bounded by Σ at $\mathbb{S}^1 \times \{1\}$ and standardly
oriented Σ_b at $\mathbb{S}^1 \times \{0\}$, which is disjoint from $\{*\} \times [0, 1]$ and with $\Sigma_b \times [0, 1]$ as the
underlying tangle.*

Proof Let τ be a collection of radial blue intervals connecting blue points at $\mathbb{S}^1 \times \{0\}$
with those at $\mathbb{S}^1 \times \{1\}$. Cut the annulus to a disk along $\{*\} \times [0, 1]$ and apply
Proposition 3.4 to get a desired web. $\qquad\square$

Remark 4.2 The extension of a tangle to a web is constructed in Lemma 3.3 from
a shading of the disk, which is by no means unique. In case of an annulus, however,
the situation is different: there is a unique up to an isotopy family of counterclockwise
oriented arcs that bounds a given collection of oriented points at the outer boundary
circle. Some of the arc may intersect the interval $\{*\} \times [0, 1]$; moving them through the
hole results in a preferred shading and a preferred web E_Σ.

Inserting a tangle inside the web E_Σ and a surface inside the foam $E_\Sigma \times [0, 1]$ results
in a functor $\mathcal{E}_\Sigma : \mathbf{BN}(\Sigma_b) \to \mathbf{Foam}(\Sigma)$, as it preserves units and composition.

Theorem B *The functor $\mathcal{E}_\Sigma : \mathbf{BN}(\Sigma_b) \to \mathbf{Foam}(\Sigma)$ is an equivalence of categories.*

Proof It follows from Propositions 2.4 and 2.10 that $\mathcal{E}_\Sigma$ is essentially surjective
and full. Faithfulness follows from Theorem 2.11: both $\mathrm{Hom}_{\mathbf{BN}(\Sigma_b)}(\omega_b, \omega'_b)$ and

$\mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(\omega, \omega')$ are free graded modules of graded rank $(q + q^{-1})^\ell$, where ℓ counts blue loops in $-\omega \cup \omega'$. □

4.2 A coherent way to forget red facets

The inverse functor to $\mathcal{E}_\Sigma$ forgets red facets of a foam, but it may also change its sign and dualize some dots. To construct the functor explicitly, fix for each web ω an invertible foam $I_\omega \in \mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(\mathcal{E}_\Sigma(\omega_b), \omega)$. According to Proposition 2.10, for every foam $S \colon \omega \to \omega'$ there is a sign $\mathrm{sgn}(S) = \pm 1$ and a cobordism $S_{\mathrm{cob}} \colon \omega_b \to \omega'_b$, which agrees with S_b up to dualizing some dots, that fit into a commuting square

(4-1)
$$
\begin{array}{ccc}
\omega & \xrightarrow{\quad S \quad} & \omega' \\[2pt]
{\scriptstyle I_\omega} \Big\uparrow & & \Big\uparrow {\scriptstyle I_{\omega'}} \\[2pt]
\mathcal{E}_\Sigma(\omega_b) & \xrightarrow{\ \mathrm{sgn}(S)\mathcal{E}_\Sigma(S_{\mathrm{cob}})\ } & \mathcal{E}_\Sigma(\omega'_b)
\end{array}
$$

Let us explain how both $\mathrm{sgn}(S)$ and S'_b can be obtained from the given data.

To construct S_{cob} take S_b and dualize all dots that are carried by blue facets with orientation opposite to the standard orientation on S_b. Indeed, the isotopy used in the proof of Proposition 2.10 to take red facets of $I_{\omega'}^{-1} S I_\omega$ away from S_b involves an odd number of the dot migration moves (the right relation in (2-8), see also Exercise 2.9) for such dots, because this is the only move that reverses the local orientation around a dot.

The sign $\mathrm{sgn}(S)$ is uniquely defined by (4-1) in case S does not vanish and it can be computed then from the formula

$$
\mathrm{sgn}(S) = \frac{Z(C \cup S\, I_\omega)}{Z(C \cup I_{\omega'} \mathcal{E}_\Sigma(S_{\mathrm{cob}}))},
$$

where Z is the Blanchet evaluation map from Section 2.5 and C is a cup foam bounded by the web $-\mathcal{E}_\Sigma(\omega_b) \cup \omega'$, for which the two quantities being divided do not vanish.[9]

Proposition 4.3 *The assignment*

$$
\omega \mapsto \omega_b, \quad S \mapsto \mathrm{sgn}(S) S_{\mathrm{cob}}
$$

defines a functor $\mathcal{E}_\Sigma^\vee \colon \mathbf{Foam}(\Sigma) \to \mathbf{BN}(\Sigma_b)$ inverse to $\mathcal{E}_\Sigma$.

Proof Clearly, $(S'' S')_{\mathrm{cob}} = S''_{\mathrm{cob}} S'_{\mathrm{cob}}$ for composable foams $S' \colon \omega \to \omega'$ and $S'' \colon \omega' \to \omega''$. Furthermore, the equality

$$
S'' S' I_\omega = \mathrm{sgn}(S'')\, \mathrm{sgn}(S')(I_{\omega''} \mathcal{E}_\Sigma(S''_b) \mathcal{E}_\Sigma(S'_b)) = \mathrm{sgn}(S'')\, \mathrm{sgn}(S')(I_{\omega''} \mathcal{E}_\Sigma(S''_b S'_b))
$$

---

[9]Explicitly, $C = \mathrm{cup}(-\mathcal{E}_\Sigma(\omega_b) \cup \omega', X)$ where X contains exactly one boundary circle of each genus 0 component of S_b that does not carry a dot.

forces $\operatorname{sgn}(S''S') = \operatorname{sgn}(S'')\operatorname{sgn}(S')$ if the composition $S''S'$ does not vanish. Hence, $\mathcal{E}_\Sigma^\vee$ is a functor. To end the proof, we check directly that $\mathcal{E}_\Sigma^\vee \circ \mathcal{E}_\Sigma$ is the identity functor on $\mathbf{BN}(\Sigma_b)$, whereas the collection of the invertible foams I_ω constitute a natural isomorphism between $\mathcal{E}_\Sigma \circ \mathcal{E}_\Sigma^\vee$ and the identity functor on $\mathbf{Foam}(\Sigma)$. $\qquad\square$

Example 4.4 Let ω be a blue circle oriented clockwise. This is the orientation induced from the unbounded region, hence standard, so that $\omega = \omega_b$ and $\mathcal{E}_\varnothing^\vee$ simply forgets orientation:

$$\mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{a}}\;\right) = \raisebox{-1ex}{\includegraphics[height=3ex]{b}} \quad\text{and}\quad \mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{c}}\;\right) = \raisebox{-1ex}{\includegraphics[height=3ex]{d}}$$

However, when ω is oriented counterclockwise, then the invertible foam I_ω is a cylinder with a red membrane, the canonical normal vector of which is oriented upwards. The membrane can be removed with (2-15) and (2-8). This gives no difference for the cup with no dot

$$\mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{e}}\;\right) = \mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{f}}\;\right) = \mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{g}}\;\right) = \raisebox{-1ex}{\includegraphics[height=3ex]{h}}$$

but in the presence of the dot, the dot is replaced by its dual:

$$\mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{i}}\;\right) = \mathcal{E}_\varnothing^\vee\left(\;\raisebox{-1ex}{\includegraphics[height=3ex]{j}}\;\right) = \mathcal{E}_\varnothing^\vee\left(-\;\raisebox{-1ex}{\includegraphics[height=3ex]{k}}\; + h\;\raisebox{-1ex}{\includegraphics[height=3ex]{l}}\;\right) = \raisebox{-1ex}{\includegraphics[height=3ex]{m}}$$

Recall that if $h = 0$, the dual dot equals *minus* the normal dot.

Remark 4.5 Although the construction of $\mathcal{E}_\Sigma^\vee$ depends on the choice of foams I_ω, the functor is unique up to a unique natural isomorphism. To see this directly, suppose that $\widetilde{\mathcal{E}}_\Sigma^\vee$ is constructed using a different family of foams $\tilde{I}_\omega$. Then $\tilde{I}_\omega = s(\omega)I_\omega$ for a well-defined sign $s(\omega) = \pm 1$ and it follows from a direct computation that the collection of morphisms $\iota_\omega := s(\omega) \cdot \omega_b \times [0, 1]$ is a natural isomorphism from $\mathcal{E}_\Sigma^\vee$ to $\widetilde{\mathcal{E}}_\Sigma^\vee$.

4.3 An equivalence of bicategories

Recall that a 1–morphism $f : x \to y$ in a bicategory $\mathbf{C}$ is an *equivalence* if there exists $g : y \to x$ such that the compositions $f \circ g$ and $g \circ f$ are isomorphic to identity 1–morphisms. A 2–functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ is an *equivalence of bicategories* when it is

- a *local equivalence*, that is, the functor $\mathcal{F}_{x,y} : \mathbf{C}(x, y) \to \mathbf{D}(\mathcal{F}(x), \mathcal{F}(y))$ is an equivalence of categories for all objects x, y of $\mathbf{C}$, and

- *essentially surjective*: each object of $\mathbf{D}$ is equivalent to an object of the form $\mathcal{F}(x)$.

Indeed, the above conditions imply the existence of an inverse of $\mathcal{F}$ [24].

There is a 2–functor

(4-2) $\mathcal{E}^0 \colon \mathbf{BN} \to \mathbf{Foam}$

that equips points, tangles, and cobordisms with the standard orientation.[10] It is a local equivalence due to Theorem B, but not essentially surjective: objects from the image of $\mathcal{E}^0$ have weight 0 or 1, so that the whole image is contained in $\mathbf{Foam}^0 \sqcup \mathbf{Foam}^1$. We fix this by enlarging the source bicategory to $\mathbf{wBN} := \mathbf{BN} \times \mathbb{Z}$, the product of $\mathbf{BN}$ with $\mathbb{Z}$ seen as a discrete bicategory. In other words, objects of $\mathbf{wBN}$ are pairs (Σ, k) consisting of an object Σ from $\mathbf{BN}$ and a number $k \in \mathbb{Z}$, whereas morphism categories are zero or copied from $\mathbf{BN}$,

(4-3) $\mathbf{wBN}((\Sigma, k), (\Sigma', k')) := \begin{cases} \mathbf{BN}(\Sigma, \Sigma') & \text{if } k = k', \\ 0 & \text{otherwise.} \end{cases}$

We then extend (4-2) to a 2–functor

(4-4) $\mathcal{E} \colon \mathbf{wBN} \to \mathbf{Foam}$

in such a way that (Σ, k) is taken to the collection $\mathcal{E}^0(\Sigma)$ with $|k|$ red points added to the right, all positive when $k > 0$ and negative otherwise. Likewise for 1– and 2–morphisms: $\mathcal{E}$ takes a tangle τ (resp. a cobordism W), orients it in a standard way, and adds to the right $|k|$ vertical red lines (resp. vertical red squares) with the appropriate orientation.

Theorem A *The 2–functor $\mathcal{E} \colon \mathbf{wBN} \to \mathbf{Foam}$ is an equivalence of bicategories.*

Proof By Theorem B, $\mathcal{E}$ is a local equivalence. Hence, it is enough to show that it is essentially surjective. For that choose an object Σ from $\mathbf{Foam}$ and let $k = \lfloor w(\Sigma)/2 \rfloor$, where $w(\Sigma)$ is the weight of Σ. Then $\Sigma^0 := \mathcal{E}(\Sigma_b, k)$ has the same weight. Considering $\mathbb{R} \times [0, 1]$ as a disk with two boundary points removed, we can apply Lemma 3.3 to the collection of points $-\Sigma^0 \cup \Sigma$ to obtain a web $E_\Sigma \colon \Sigma^0 \to \Sigma$ with vertical lines as the underlying tangle. Another application of Lemma 3.3 combined with Proposition 2.10 shows that it is an equivalence, with its mirror image the inverse 1–morphism. $\square$

We write $\mathcal{E}^\vee$ for the 2–functor inverse to $\mathcal{E}$. It can be constructed explicitly like the functors $\mathcal{E}^\vee_\Sigma$, except that the computation of signs requires not only a choice of

---

[10]Recall the convention that the basepoint $*$ is placed at the left infinity, so that the left unbounded region is painted white. This implies in particular that the left most point of an object of $\mathbf{BN}$ receives the positive orientation and the left most vertical strand of a 1–morphism is oriented upwards.
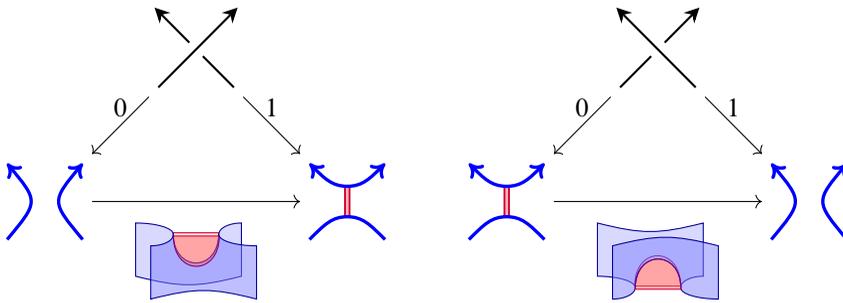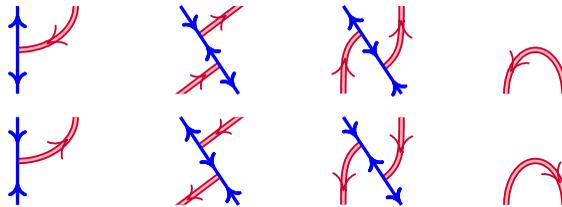
Figure 5: Web resolutions of a positive (to the left) and negative (to the right) crossing, together with the minimal foams between them.

isomorphisms between webs, but also a choice of equivalences between collections of points. For the latter one can use the webs

which are equivalences by Proposition 2.10. They can be used to construct an explicit equivalence from a collection Σ to $\Sigma^0 = \mathcal{E}(\Sigma_b, \lfloor w(\Sigma)/2 \rfloor)$ by examining the points of Σ from left to right. The details are left to the reader.

4.4 Comparison of Khovanov brackets

We finish this section by comparing two invariant complexes for a tangle T: the Khovanov bracket $[\![T]\!]$ from [2], which is a formal complex of objects from $\mathbf{BN}(\partial T)$, and the Blanchet–Khovanov bracket $[\![T]\!]_\mathrm{F}$ constructed using webs and foams instead. In what follows we recall the construction of the latter — forgetting red edges in webs and red facets in foams recovers the former.

Let c be the number of crossings in T, out of which c_+ are positive and c_- are negative. The first step to construct $[\![T]\!]_\mathrm{F}$ is to compute the *cube of resolutions* of $\mathcal{I}_\mathrm{F}(T)$: a commutative diagram with resolutions of T at vertices of the c–dimensional cube $[0,1]^c$. Namely, a vertex $\xi = (\xi_1, \ldots, \xi_c) \in \{0,1\}^c$ is decorated with the web T_ξ obtained from T by replacing each ith crossing of the tangle with its resolution of type ξ_i, as shown in Figure 5. Let ξ' be another vertex, obtained from ξ by changing

one coordinate from 0 to 1. The directed edge $\zeta \colon \xi \to \xi'$ is decorated with the minimal foam $T_\zeta \colon T_\xi \to T_{\xi'}$, which is a collection of vertical facets except over the region where the two resolutions do not match; here T_ζ is a zip or an unzip as shown in Figure 5. It is evident that $\mathcal{I}_F(T)$ commutes: directed paths between same vertices represent isotopic foams.

Pick a *sign assignment* ϵ, that is a collection of signs $\epsilon(\zeta) = \pm 1$, one sign per edge in the cube, such that the product of signs around any square in the cube is equal to -1. The standard choice is $\epsilon(\zeta) = (-1)^{s(\xi,\xi')}$, where $s(\xi,\xi')$ counts 1's left to the place at which ξ and ξ' disagree. Scaling each edge ζ by $\epsilon(\zeta)$ makes the cube anticommute and it can be shown that the isomorphism type of the cube is independent of the sign assignment — compare with [26, Lemma 2.2] or [27, Lemma 5.7]. The formal complex $[\![T]\!]_F$ is obtained by flattening the cube along diagonals and shifting degrees accordingly. Explicitly,

$$[\![T]\!]^i_F := \bigoplus_{|\xi| = i + c_-} T_\xi \{c_- - c_+ - i\}$$

where $|\xi| := \xi_1 + \cdots + \xi_c$, with the differential

$$d|_{T_\xi} = \sum_{\zeta \colon \xi \to \xi'} \epsilon(\zeta) T_\zeta.$$

The Khovanov bracket $[\![T]\!]$ is constructed following the same steps, except that webs and foams are replaced with flat tangles and cobordisms. In particular, one has to erase in Figure 5 the red edges in resolutions and red facets in foams.

Theorem 4.6 *The homotopy type of $[\![T]\!]_F$ is an invariant of the tangle T, strictly functorial with respect to tangle cobordism. Its image under $\mathcal{E}^\vee_{\partial T}$ is isomorphic to $[\![T]\!]$.*

Proof Following [2] one can show that $[\![T]\!]$ is functorial up to a sign and strict functoriality is shown in [5] in the case of links, ie when T has no endpoints. From these two facts strict functoriality follows, because every tangle can be closed to a link.

To compare $[\![T]\!]_F$ with $[\![T]\!]$ consider the cube of resolutions $\mathcal{I}_F(T)$ constructed in **Foam**(∂T) and let $\mathcal{I}(T)'$ be its image in **BN**(∂T) under the equivalence of categories $\mathcal{E}^\vee_{\partial T}$. It differs from $\mathcal{I}(T)$, the cube of resolutions in **BN**(∂T) that computes $[\![T]\!]$, only in signs at edges. Hence, the two cubes are isomorphic and the thesis follows. $\square$

Remark 4.7 The construction of $[\![T]\!]_F$ can be easily extended to an invariant of knotted webs [29] and it is conjectured to be strictly functorial with respect to foams embedded in a four-dimensional space.

5   A diagrammatic TQFT on Foam(∅)

The assignment of the module *Foam*(ω) to a closed web ω extends to a functor

$$\mathrm{Hom}_{\mathbf{Foam}(\varnothing)}(\varnothing, -) \colon \mathbf{Foam}(\varnothing) \to \Bbbk{-}\mathbf{Mod}.$$

In what follows we provide a diagrammatic description of this functor by representing red-over-blue cup foams from *Foam*(ω) using certain planar diagrams and examine how the diagrams change under the action of the linear maps associated with foams. In this section we assume that webs and foams are embedded in the plane $\mathbb{R}^2 \times \{0\}$ and the half-space $\mathbb{R}^2 \times (-\infty, 0]$ respectively.

5.1   A planar representation of cup foams

Let ω be a bounded planar web and ω^+ its completion, which is a shading (ω_r^+, ω_b^+) satisfying $\Gamma(\omega_r^+, \omega_b^+) = \omega$. It is assumed that the basepoint $*$ marks the unbounded region, so that the region is painted white. To simplify the picture and make the web ω better visible, we do not color regions and we draw red edges as double or dashed lines depending on whether they survive or disappear after Γ is applied; see Figure 6. Furthermore, we allow to mark blue loops of ω^+ with (any number of) dots. We assign to such a planar diagram a completed foam $\mathrm{cup}(\omega^+) = \mathrm{cup}_r(\omega^+) \cup \mathrm{cup}_b(\omega^+)$ bounded by ω^+ that satisfies the following conditions:

(CF1) $\mathrm{cup}_r(\omega^+) \subset \mathbb{R}^2 \times [-1, 0]$ and consists of disks that project injectively onto $\mathbb{R}^2 \times \{0\}$.

(CF2) $\mathrm{cup}_b(\omega^+)$ is a collection of disks such that

$$\mathrm{cup}_b(\omega^+) \cap (\mathbb{R}^2 \times [-1, 0]) = \omega_b^+ \times [-1, 0].$$

(CF3) Each blue disk is decorated with as many dots as its boundary loop in ω^+, all placed at heights smaller than -1 (hence, below all red facets).
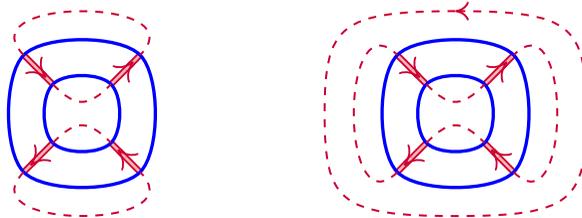
Figure 6: Two completions of the same web. The surrounding dashed circle in the right picture is required by the condition that the unbounded region is painted white.

The intersection of red and blue disks in $\mathrm{cup}(\omega^+)$ consists of intervals only; hence it is minimal among all completed cup foams bounded by ω^+. Painting the unbounded region white extends to a unique shading supported by $\mathrm{cup}(\omega^+)$. The resulting foam $\Gamma(\mathrm{cup}(\omega^+)) \in \mathit{Foam}(\omega)$ is a red-over-blue cup foam, ie its red facets are above all dots and minima of blue discs. We call it the *cup foam associated to* ω^+. The following observation is an immediate consequence of Theorem 2.11.

Lemma 5.1 *Choose a completion* ω^+ *of* ω *and consider the family of all dotted completed webs obtained from* ω^+ *by placing at most one dot on each blue loop. Then the corresponding cup foams form a linear basis of* $\mathit{Foam}(\omega)$.

Notice that dots in these pictures only mark loops. In particular, moving a dot along a loop — even passing through a crossing with a red strand — does not affect the cup foam represented by the diagram.

Example 5.2 Let ω be a blue circle. Then $\mathit{Foam}(\omega)$ is generated by two blue cups: one with and the other without a dot. These are the cup foams associated to ω^+ when ω is oriented clockwise, because this orientation is oriented from the unbounded region, hence $\omega^+ = \omega$. Otherwise, ω^+ is ω surrounded by a dashed red circle, which results in the change of the sign of the cup with a dot; see Table 1. This is consistent with the computation from Example 4.4.

5.2 Action of foams

We now provide a description of the linear maps associated to foams in terms of the dotted completed webs. In fact, it is enough to analyze the *elementary completed foams*

a pocket   a blue cup   a blue cap   a blue saddle   a dot

a reversed pocket   a red cup   a red cap   a red saddle
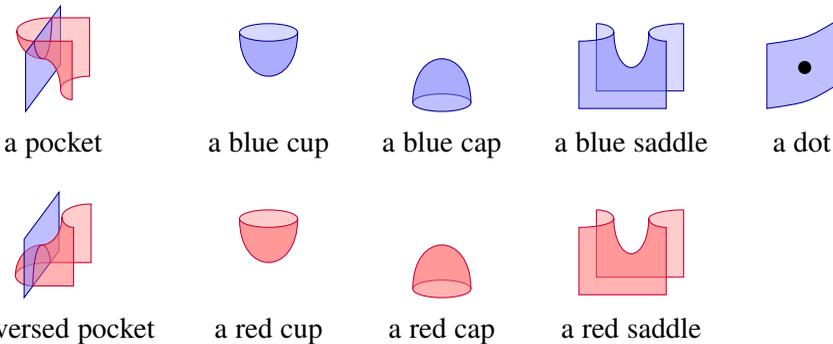
because every foam can be decomposed into these.

Pockets and bicolored isotopies A bicolored isotopy is a sequence of several bigon moves (3-1), which are realized by the pocket foams. When applied to a (completed) cup

web	completion	basis

Table 1: A basis for foams bounded by a circle, represented as planar diagrams and as foams.

foam, it results in a collection of disks that may or may not satisfy (CF1); see Figure 7. In order to describe the map by local pictures, we shade the projection of the disk bounded by the red loop involved in the bigon move, ie the region bounded by the loop.

If the projection of the red disk is pushed through a blue arc, then the resulting cup foam is minimal — no double points in the projection are created. Hence, the associated map takes a dotted web to the result of applying the bigon move:

(5-1)

(5-2)
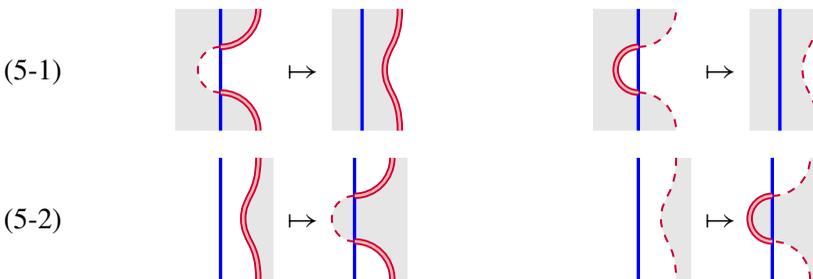
However, pulling the projection of a red disk off a blue arc creates double points, like in the right column of Figure 7. Indeed, the new red disk intersects the blue surface in a circle, so that either of (3-2) has to be applied. This may cost a sign, depending on the orientation of the edges:

(5-3)
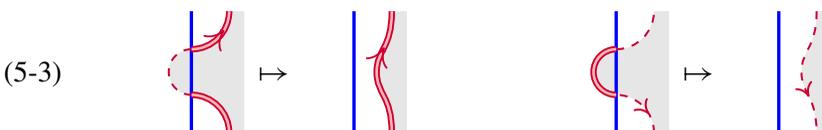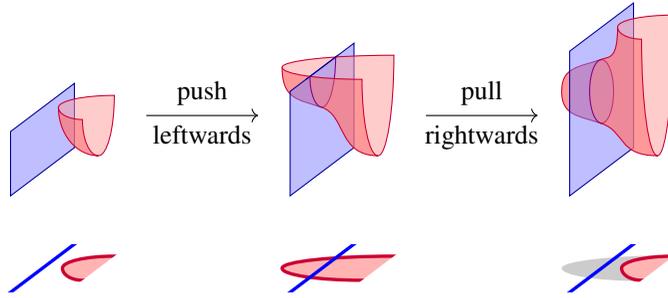
Figure 7: A cup foam with its projection on the horizontal plane (the left column) and the results of applying the bigon move twice (the middle and right columns). The middle foam is again a cup foam, but not the right one: the projection has double points — the shaded region outside of the cup — coming from the disk to the side of the vertical plane.

(5-4)

(5-5)

(5-6)
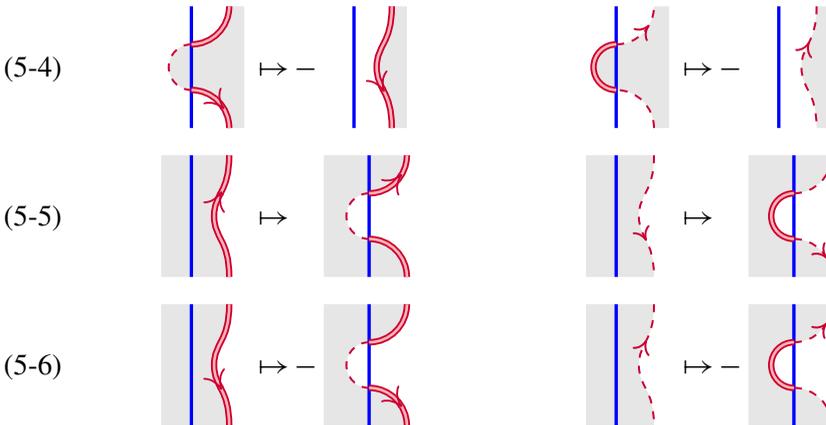
Indeed, the left moves in (5-3) and (5-4) are realized by detaching red cylinders with (2-9), whereas the right ones are by eliminating red caps with (2-16). Likewise, the relations (2-17) and (2-10) give the signs for the left and right sides respectively of both (5-5) and (5-5).

Placing a dot Placing a dot on $\mathrm{cup}(\omega^{+})$ near the boundary violates (CF3). To obtain a minimal cup foam, the dot has to be moved down.

Let p be the projection of the dot onto the horizontal plane and assume that it does not lie on a red loop. We define the *nestedness $n(p)$* as the number of red loops encircling p. It counts red facets below p in the cup foam, hence, the number of times the dot-moving relation (2-8) has to be applied to move the dot from top to the bottom of a blue disk.

Therefore, placing a dot on a blue loop results in the map

$$
(5\text{-}7) \qquad \diagup \mapsto \begin{cases} \diagup\!\!\bullet & \text{if } n(p) \text{ is even,} \\ h\,\diagup - \diagup\!\!\bullet & \text{if } n(p) \text{ is odd.} \end{cases}
$$

Blue cups, caps, and saddles   Suppose now that W is a completed foam with ω^+ at its bottom and a unique critical point that lies on the blue surface. In this case $W \cup \mathrm{cup}(\omega^+)$ is no longer a cup foam associated with the output of W: to have one, the critical point of W has to be slid downwards, below all red facets, and this may cost a sign. Moreover, a cap creates a sphere that has to be evaluated, whereas a saddle splitting one loop into two[11] creates a neck that has to be cut.

Let p be the projection of the critical point onto the horizontal plane and assume that $p \notin \omega^+$. We say that a red loop γ encircling p is *evenly distanced* if any generic path connecting p to a point q from a solid (resp. dashed) red arc of γ intersects blue circles in an even (resp. odd) number of points. Otherwise, γ is *oddly distanced*. Let $s(p)$ count *oddly distanced* counterclockwise and *evenly distanced* clockwise red loops surrounding p. This corresponds to two types of red facets below p:

(1) the facets that survive in the cup foam $\Gamma(\omega^+)$ and with the canonical normal vector (see Remark 2.7) oriented downwards, and

(2) those removed from $\Gamma(\omega^+)$ and with the canonical normal vector oriented upwards.

These are exactly the situations, in which there is a sign in the relations (2-9) and (2-10). Hence $s(p)$ determines the result of isotoping the blue critical point below all red facets. Therefore, the maps induced by critical blue points are the usual ones scaled by $(-1)^{s(p)}$:

- a cup:

$$
(5\text{-}8) \qquad\qquad \varnothing \mapsto (-1)^{s(p)} \bigcirc
$$

- a cap:

$$
(5\text{-}9) \qquad\qquad \bigcirc \mapsto (-1)^{s(p)}
$$

---

[11] Such a saddle is called a *split*. The other one, which joins two loops into one, is called a *merge*.

- a merge:

(5-10)
$$\supset \subset \ \mapsto (-1)^{s(p)} \ \asymp$$

- a split:

(5-11)
$$\asymp \ \mapsto (-1)^{s(p)} \left(\ \bullet\hspace{-3pt}\supset \subset \ + \ \supset \ \subset\hspace{-3pt}\bullet \ \right)$$

Red cups, caps, and saddles   Placing a red cup at the top of $\mathrm{cup}(\omega^+)$ results in a cup foam. Hence, no sign appears. Conversely, capping off an isolated red circle creates a red sphere, which can be removed by (2-6) at a cost of sign. Hence, we obtain the maps:

- a cup:

(5-12)
$$\varnothing \mapsto \bigcirc \qquad\qquad \varnothing \mapsto \bigcirc$$

- a cap:

(5-13)
$$\bigcirc \mapsto -1 \qquad\qquad \bigcirc \mapsto 1$$

The behavior of merging and splitting saddles depends on whether the two red circles (those being merged or the result of a split) are nested or not. In the latter case, merging two loops takes a minimal cup foam to a minimal cup foam, whereas splitting a red circle creates a neck that has to be cut with (2-7) if it survives in the foam. Therefore, the corresponding maps satisfy the rules:

- a merge:

(5-14)
$$\supset \subset \ \mapsto \ \asymp \qquad\qquad \supset \subset \ \mapsto \ \asymp$$

- a split:

(5-15)
$$\asymp \ \mapsto - \supset \subset \qquad\qquad \asymp \ \mapsto \ \supset \subset$$

Merging nested red loops of a cup foam
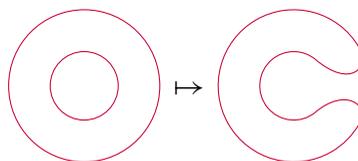
results in a red disk, which does not project injectively onto the horizontal plane, but which can be isotoped to a croissant:
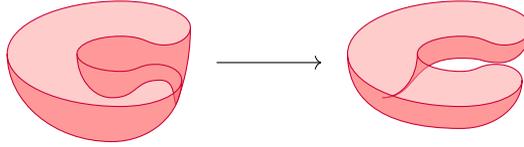
This isotopy can be described as a *finger move*: place your finger vertically near the saddle and move it inwards, pushing the red disk. The disk is then isotoped through every blue facet attached to a blue arc that cuts the inner red circle and through every cylinder attached to a blue circle surrounded by the inner red circle:
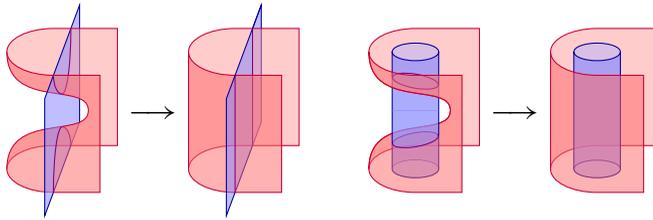
Depending on which red facets survive, each move represents two relations between foams. We leave it to the reader to check that the foams involved in the right move are always equal, whereas the left move costs a sign only in the two configurations

where the position of a saddle is marked with a cross. Let c be the number of such configurations. Then we end up with the following formula for merging nested red loops:

$$(5\text{-}16) \qquad \text{(image)} \mapsto (-1)^c \text{(image)} \qquad \text{(image)} \mapsto (-1)^c \text{(image)}$$

Dually, splitting a red loop into two nested loops is realized by the inverse of the finger move followed by stacking a red saddle over the little saddle connecting the two boundary loops. This creates a neck that has to be cut if it survives in the cup foam, so that in this case the sign is opposite to the one of the nested merge:

$$(5\text{-}17) \qquad \text{(image)} \mapsto -(-1)^c \text{(image)} \qquad \text{(image)} \mapsto (-1)^c \text{(image)}$$

Other common foams   There are other moves of interest, such as blue saddles with vertical red facets (in particular, zips and unzips) or red cups and caps that intersect vertical blue facets. All can be represented as compositions of those described above. For instance, a zip is isotopic to a pocket move followed by a saddle:
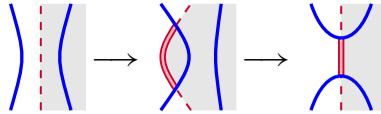
As before, the shaded regions represent a projection of a red disk, and it is clear that the first move takes a basic cup foam to a basic cup foam, so that signs are governed by the second move. Therefore, the map induced by a zip is one of

- a merging zip:

$$(5\text{-}18) \qquad \qquad \longmapsto (-1)^{s^+(p)} \qquad$$

- a splitting zip:

$$(5\text{-}19) \qquad \qquad \longmapsto (-1)^{s^+(p)} \left(\qquad + \qquad \right)$$

where $s^+(p)$ is computed like $s(p)$, except that we take into account the loop passing through the created red edge if it is oriented counterclockwise.

In the unzip the saddle precedes the pocket and to ensure that the latter does not affect the sign, we perform the saddle to the side of the red disk attached to the red edge:
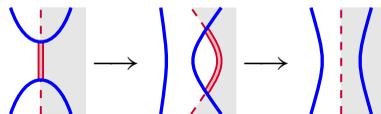
Therefore, the induced map is one of

- a merging unzip:

$$(5\text{-}20) \qquad \qquad \longmapsto (-1)^{s^-(p)} \qquad$$

- a splitting unzip:

$$(5\text{-}21) \qquad \qquad \longmapsto (-1)^{s^-(p)} \left(\qquad + \qquad \right)$$

where again $s^-(p)$ is computed like $s(p)$ without counting the loop passing through the removed red edge.

6 The Blanchet–Khovanov invariant of tangles with balanced boundaries

Let **Foam**° be the subbicategory of **Foam** generated by balanced sequences. In what follows we construct a TQFT functor $\mathcal{F}_w^\circ \colon \textbf{Foam}^\circ \to \textbf{Bimod}$. If T is an oriented tangle with balanced input and output collections of points, then its resolutions are in **Foam**°, so that applying $\mathcal{F}_w^\circ$ to $[\![T]\!]_F$ results in a chain complex of bimodules. We then show that this chain complex is isomorphic to the Khovanov's tangle invariant [19], but it admits a strictly functorial action of tangle cobordisms.

6.1 A linear basis of webs

A web $\omega \subset \mathbb{R} \times (-\infty, 0]$ is called a *cup web* if its underlying tangle is a cup diagram, ie a collection of disjoint arcs. All cup webs with the same boundary and extending the same cup diagram coincide in *Web* due to Proposition 2.4 (and are isomorphic as objects of **Foam**). Moreover, choosing a cup web for each cup diagram results in a basis of the space of webs with given boundary, which we call a *cup basis*.

Step 0: the initial collection of points.

Step 1: additional red points inserted.

Step 2: a bicolored cut with a shading.

Step 3: applying Γ produces the web E_Σ.

Step 4: connect bottom endpoints in all possible ways to obtain a web basis.
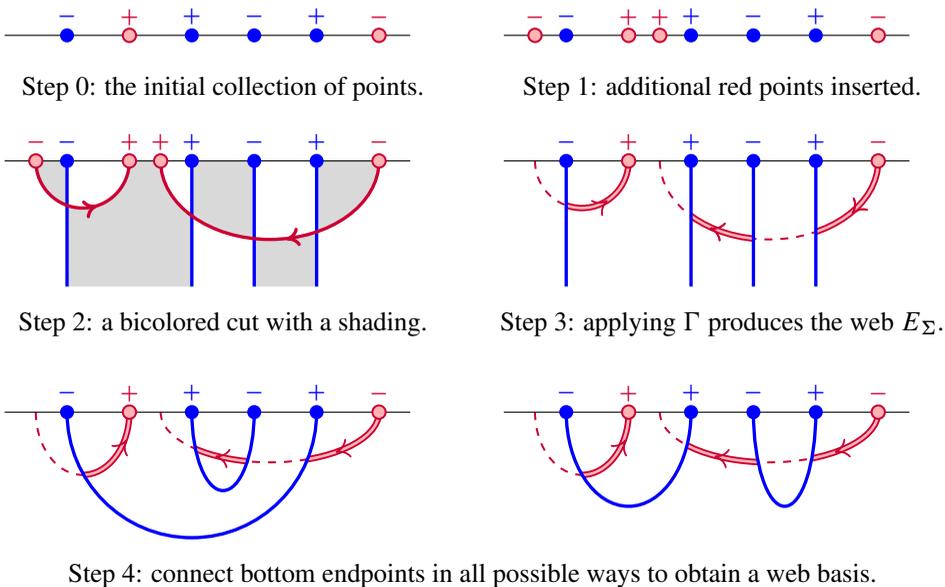
Figure 8: The construction of a cup basis for $\Sigma = $, together with completions of the cup webs (the edges erased in the third step are drawn as dashed arcs). The first three steps follow the proof of Proposition 3.4.

We describe now a particular nice cup basis of webs bounded by a balanced Σ (see also Figure 8). Let n be half of the number of blue points in Σ (being balanced, Σ has an even number of blue points). Proposition 3.4 provides an invertible web $E_\Sigma : \Sigma_b \to \Sigma$ with $2n$ vertical lines as blue edges. To obtain a cup basis, attach cup diagrams to the bottom of E_Σ. In other words, the basis is the image of cup diagrams under the equivalence $\mathcal{E}_\Sigma$ from Section 4. We call it the *red-over-blue basis of type* E_Σ, because all red edges in the webs appear above minima of blue cups.

6.2 Blanchet–Khovanov algebras

Fix a balanced collection of points Σ and let $\mathcal{B}$ be a cup basis of *Web*(Σ).

Definition 6.1 The *Blanchet–Khovanov algebra* $\mathfrak{W}^\mathcal{B}$ associated with $\mathcal{B}$ is the direct sum of spaces of foams with corners

$$\tag{6-1} \mathfrak{W}^\mathcal{B} := \bigoplus_{a,b \in \mathcal{B}} \mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(a,b)$$

with multiplication given by the composition (and zero if foams cannot be composed).

Remark 6.2 The above algebra appeared first in [14] for Σ a collection of positively oriented blue points followed by negatively oriented red points, the latter drawn in [14] at the bottom.

Choose a completion a^+ for any cup web a and write $a^!$ (resp. $(a^+)^!$) for the result of reflecting a (resp. a^+) along the horizontal line and reversing orientation of edges. Using the natural isomorphisms $\mathrm{Hom}_{\mathbf{Foam}(\Sigma)}(a,b) \cong Foam(b^!a)\{n\}$ we can represent elements of the algebra by dotted completed webs, in which case the multiplication is induced from the family of *generalized saddles*

$$\tag{6-2} (c^+)^! b^+ \sqcup (b^+)^! a^+ \xrightarrow{S_{c,b,a}} (c^+)^! a^+$$

each consisting of the identity foams $(c^+)^! \times [0,1]$ and $a^+ \times [0,1]$ glued to the half-rotation of b^+ around the boundary line. These foams take a particularly nice form when $\mathcal{B}$ is a red-over-blue basis, as they involve then only three types of moves:

- merging (5-10) and splitting (5-11) blue loops at points outside of all red circles,
- merging unnested red loops (5-14), and
- removing bigons *external* to the projection of red disks (5-1).

Hence, the product of two dotted diagrams is a positive linear combination of other diagrams.

Corollary 6.3 *The algebra $\mathfrak{W}^{\mathcal{B}}$ admits a positive basis.*

When Σ is a collection of blue points oriented in the alternating way and $\mathcal{B}$ consists of oriented cup diagrams (ie webs with no red edges), then $\mathfrak{W}^{\mathcal{B}}$ coincides with the arc algebra H^n from [19]. Indeed, $\mathfrak{W}^{\mathcal{B}}$ is the image of H^n under the embedding of bicategories $\mathcal{E} \colon \mathbf{BN} \to \mathbf{Foam}$. However, when $\mathcal{B}$ is not a red-over-blue basis, then the generalized saddles (6-2) may involve moves on red arcs that cost a sign, such as splits (5-15) or nested saddles (5-16) and (5-17). Hence, cup foams do not constitute a positive basis of the algebra in such case. Yet, it is still isomorphic to the arc algebra.

Theorem 6.4 *Let Σ be a balanced collection of points with $2n$ blue points. Then there is an algebra isomorphism $\mathfrak{W}^{\mathcal{B}} \cong H^n$ for any cup basis $\mathcal{B}$ of webs bounded by Σ. When $\mathcal{B}$ is a red-over-blue basis, then the isomorphism simply forgets red facets of basic cup foams.*

Proof   Assume first that $\mathcal{B}$ is a red-over-blue basis of type E_Σ. Then $\mathfrak{W}^{\mathcal{B}}$ is the image of H^n under the equivalence of categories $\mathcal{E}_\Sigma$, which equips a collection of dotted cups with its standard orientation. The inverse of E_Σ simply forgets red edges in webs and red facets in foams. Hence, the thesis follows.

Let now $\mathcal{B}'$ be any cup basis and pick for each cup web $a' \in \mathcal{B}'$ the isomorphic cup web $a \in \mathcal{B}$, an invertible foam $I_a \in \mathbf{Foam}(a, a')$, and $s(a) = \pm 1$ such that $I_a^! I_a = s(a)\,\omega_a \times [0, 1]$. Then the collection of linear isomorphisms

$$\varphi_{ba} \colon \mathbf{Foam}(a', b') \xrightarrow{\cong} \mathbf{Foam}(a, b), \quad S \mapsto s(b) I_b^! S I_a,$$

constitutes an isomorphism of algebras $\mathfrak{W}^{\mathcal{B}'} \cong \mathfrak{W}^{\mathcal{B}}$, where the latter is isomorphic to H^n. $\square$

Example 6.5   Let $\Sigma = $. Then the cup basis $\mathcal{B}$ consists of two elements

 and
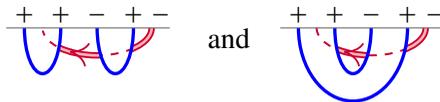
that form four pairs: two of them have two blue loops and each of the other two has one blue loop. Hence, $\dim \mathfrak{W}^{\mathcal{B}} = 12$. The multiplication of any two diagrams involves only merges and splits of blue loops (5-10)–(5-11) with $s(p) = 0$, an unnested merge of red loops (5-14) and bigon moves (5-1). None of them introduces signs, so that the product is always a positive sum of diagrams, like in H^2. For instance:
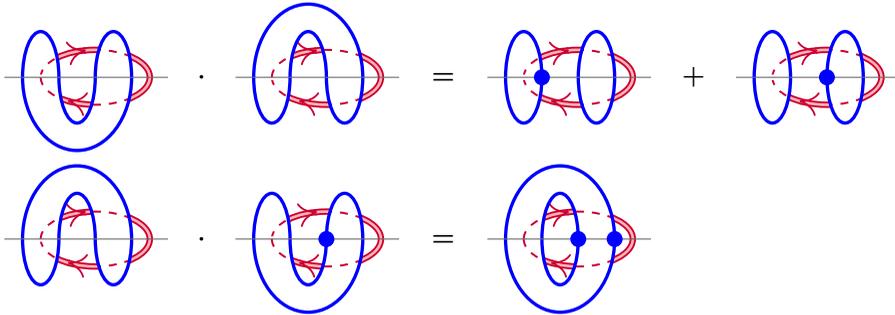
Erasing red edges recovers the usual diagrammatic calculus of H^2.

Example 6.6 Recall the Blanchet–Khovanov algebra from [14]: it is defined using webs that have only vertical red edges, $2n$ positive blue endpoints at the top and positive n red endpoints at the bottom. We call them here *EST webs*. To fit this construction into our framework, we move the bottom endpoints rightwards and to the top by appending a collection of nested red cups; see Figure 9. Contrary to the case of red-over-blue bases, minima of red cups in EST webs appear below blue cups. This is the reason why the minus sign appears in the formula for multiplication quite often, already in the case of four points. Yet Theorem 6.4 provides a direct isomorphism between this algebra and Khovanov's arc algebra. Such an isomorphism was explicitly constructed in [13] by providing a sign for each generator, then checking directly that these signs result in a homomorphism of algebras.

6.3 Blanchet–Khovanov bimodules

Pick now two balanced collections Σ and Σ' with cup bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. We assign to a web $\omega\colon \Sigma \to \Sigma'$ its *Blanchet–Khovanov bimodule* $\mathcal{F}_w^\circ(\omega)$, which is the
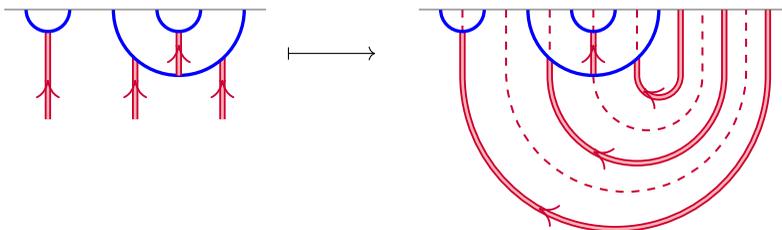
Figure 9: Turning red edges rightwards and to the top produces a cup web from an EST web. There is a natural completion, visualized by dashed arcs, with minima of red cups below all blue ones.
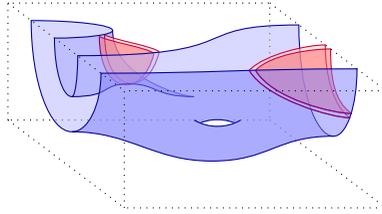
Figure 10: An element of **Foam**(a, ω, b) is a foam in a cube, bounded by the webs ω, a, and b at the top and opposite vertical facets of the cube respectively.

$(\mathfrak{W}^{\mathcal{B}}, \mathfrak{W}^{\mathcal{B}'})$–bimodule

$$(6\text{-}3) \qquad \mathcal{F}_w^{\circ}(\omega) := \bigoplus_{a \in \mathcal{B}, b \in \mathcal{B}'} \mathbf{Foam}(a, \omega, b),$$

where **Foam**(a, ω, b) is the space of foams bounded by $b^! \cup \omega \cup a$ and seen as foams in a cube with ω at the top facet, whereas a and b lie on opposite vertical facets; see Figure 10. The algebras $\mathfrak{W}^{\mathcal{B}}$ and $\mathfrak{W}^{\mathcal{B}'}$ act on the left and on the right respectively, and there is a diagrammatic presentation of this bimodule as explained in Section 5. Moreover, placing a foam $S \in \mathbf{Foam}(\omega, \omega')$ on top results in a bimodule map $\mathcal{F}_w^{\circ}(S) \colon \mathcal{F}_w^{\circ}(\omega) \to \mathcal{F}_w^{\circ}(\omega')$. The assignment $S \mapsto \mathcal{F}_w^{\circ}$ is clearly functorial in S and it preserves the foam relations (2-6)–(2-10). In particular, $\mathcal{F}_w^{\circ}(\omega) \cong \mathcal{F}_w^{\circ}(\omega')$ if S is invertible. Finally, horizontal composition of foams induces a canonical homomorphism of graded bimodules

$$(6\text{-}4) \qquad \mathcal{F}_w^{\circ}(\omega') \otimes_{\mathfrak{W}^{\Sigma'}} \mathcal{F}_w^{\circ}(\omega) \to \mathcal{F}_w^{\circ}(\omega' \omega)$$

for any pair of composable webs $\omega \colon \Sigma \to \Sigma'$ and $\omega' \colon \Sigma' \to \Sigma''$. The proof of [19, Theorem 1] can be adapted to our framework to show that (6-4) is an isomorphism.

Remark 6.7 Contrary to the case of Blanchet–Khovanov algebras, the isomorphism (6-4) may not take a pair of cup foams into a positive combination of cup foams. When using the diagrammatics of completed webs, (6-4) is induced by a collection of generalized saddles, the description of which — contrary to the case of algebras — may involve moves on red loops that cost a sign, such as (5-15)–(5-17). However, this is not the case when both webs have only blue endpoints, oriented in an alternating way — in this case all red loops lie inside the webs and are not affected when webs are composed.

As in the case of algebras, $\mathcal{F}_w^{\circ}(\omega)$ coincides with the arc bimodule $\mathcal{F}_{Kh}^{\circ}(\omega)$ defined in [19] when ω is a standardly oriented flat tangle and both $\mathcal{B}$ and $\mathcal{B}'$ are collections of

cup diagrams. Although in general $\mathcal{F}_w^\circ(\omega)$ is not a priori a bimodule over arc algebras, it can be made such through the algebra isomorphisms $H^n \cong \mathfrak{W}^{\mathcal{B}}$ and $H^{n'} \cong \mathfrak{W}^{\mathcal{B}'}$ provided by Theorem 6.4. Hence, it makes sense to compare $\mathcal{F}_w^\circ(\omega)$ with $\mathcal{F}_{Kh}^\circ(\omega_b)$.

Theorem 6.8 *Let* $\omega\colon \Sigma \to \Sigma'$ *be a web between balanced collections of points with* $2n$ *and* $2n'$ *blue points respectively. Then there is an isomorphism of* $(H^n, H^{n'})-$ *bimodules* $\mathcal{F}_w^\circ(\omega) \cong \mathcal{F}_{Kh}^\circ(\omega_b)$. *The isomorphism simply forgets red facets of basic cup foams when* $\mathcal{B}$ *and* $\mathcal{B}'$ *are red-over-blue web bases.*

Proof Assume first that $\mathcal{B}$ and $\mathcal{B}'$ are red-over-blue cup bases of types E_Σ and $E_{\Sigma'}$ respectively. Fix a foam I_ω in a cube with vertical rectangles as blue facets, bounded by ω_b and ω at the bottom and top facets, and with E_Σ and $E_{\Sigma'}$ at appropriate vertical facets. Placing it on top of an element from $\mathbf{Foam}(a, \omega_b, b)$ results in a $\Bbbk$–linear isomorphism $\mathbf{Foam}(a, \omega_b, b) \cong \mathbf{Foam}(E_\Sigma \cup a, \omega, E_{\Sigma'} \cup b)$. It is straightforward to check that these isomorphisms are compatible with the action of the arc algebras, so that they constitute an isomorphism of bimodules $\mathcal{F}_{Kh}^\circ(\omega_b) \cong \mathcal{F}_w^\circ(\omega)$; it takes a collection of dotted cups to a basic cup foam. Forgetting red facets is the inverse map.

The general case is reduced to the above as in the proof of Theorem 6.4: choose a collection of invertible foams, one per $a \in \mathcal{B}$ and one per $b \in \mathcal{B}'$, and glue them to the sides of foams generating $\mathcal{F}_{Kh}^\circ(\omega)$. $\qquad\square$

Remark 6.9 When $\mathcal{B}$ is a red-over-blue basis of type E_Σ, then the action of H^n can be understood pictorially as follows: a dotted surface $S \in H^n$ is standardly oriented and combined with $E_\Sigma \times [0, 1]$ before acting on $\mathcal{F}_{Kh}^\circ(\omega)$. The same applies to $H^{n'}$ if $\mathcal{B}'$ is a red-over-blue basis.

We say that a linear basis $\{x_1, \dots, x_d\}$ of an (A, B)–bimodule is *positive with respect to bases* $\{a_i\}$ of A and $\{b_j\}$ of B, when each $a_i x_k$ and $x_k b_j$ has positive coefficients in this basis. Because dotted cups constitute a positive basis of arc bimodules, Theorem 6.8 implies the existence of a positive basis for Blanchet–Khovanov bimodules.

Corollary 6.10 *Suppose that both* $\mathcal{B}$ *and* $\mathcal{B}'$ *are red-over-blue cup bases of webs. Then basic cup foams constitute a positive basis for* $\mathcal{F}_w^\circ(\omega)$.

Although the formulas for the actions of the algebras on a Blanchet–Khovanov bimodule involve no signs when red-over-blue bases as used, this is not the case for action of

foams: the square

$$
\begin{CD}
\mathcal{F}_w^\circ(\omega) @>{\mathcal{F}_w^\circ(S)}>> \mathcal{F}_w^\circ(\omega') \\
@A{I_\omega}A{\cong}A @A{I_{\omega'}}A{\cong}A \\
\mathcal{F}_{Kh}^\circ(\omega_b) @>{\mathcal{F}_{Kh}^\circ(S_b)}>> \mathcal{F}_{Kh}^\circ(\omega'_b)
\end{CD}
$$

(6-5)

commutes only up to a sign, where we abuse the notation and denote the isomorphism from the proof of Theorem 6.8 by the same symbol I_ω as the foam used to construct it. However, the sign does not depend on the direct summand of the bimodule: it is determined by the configuration of red loops (see Section 5) and the configuration is the same for all closures $b^!\omega a$.

6.4 A functorial homology for tangles with balanced boundaries

The previous sections describe a morphism of bicategories $\mathcal{F}_w^\circ \colon \mathbf{Foam}^\circ \to \mathbf{Bimod}$, which we extend naturally to $Com_{/h}(\mathbf{Foam}^\circ)$. As mentioned in the introduction, we can apply it to the formal bracket $[\![T]\!]_F$ of a tangle T with balanced input and output, producing a chain complex $C_{\mathfrak{W}}(T)$. Invariance and functoriality of the bracket implies that the homotopy type of $C_{\mathfrak{W}}(T)$ is an invariant of the tangle T that is functorial with respect to tangle cobordisms.

Theorem C   *The 2–functor $\mathcal{F}_w^\circ$ is equivalent to $\mathcal{F}_{Kh}^\circ \circ \mathcal{E}^\vee$. In particular, the complexes $C_{\mathfrak{W}}(T)$ and $CKh(T)$ are isomorphic for any tangle T with balanced input and output.*

Proof   The two functors coincide on objects by Theorem 6.4 and on 1–morphisms by Theorem 6.8. Furthermore, the collection of isomorphisms I_ω is natural in ω, because the square (6-5) commutes when S_b is replaced with $\mathcal{E}^\vee(S)$. Indeed, the sign relating $\mathcal{F}_w^\circ(S) \circ I_\omega$ with $I_{\omega'} \circ \mathcal{F}_{Kh}^\circ(S_b)$ is exactly the one provided by $\mathcal{E}^\vee$. The last statement is a direct consequence of Theorem 4.6. $\square$

7 Subquotient algebras and an invariant for all tangles

Inspired by [10] we use the TQFT from the previous section to define a family of 2–functors $\mathcal{F}_w^\lambda \colon \mathbf{Foam} \to \mathbf{Bimod}$ parametrized by $\lambda \in \mathbb{Z}$, which are defined on the whole bicategory of foams. As before, these 2–functors lead to invariant chain complexes for tangles that are strictly functorial versions of the Chen–Khovanov tangle invariants. Contrary to the previous sections, we assume here that $h = t = 0$. In particular, a foam vanishes when it has a blue facet with two dots.
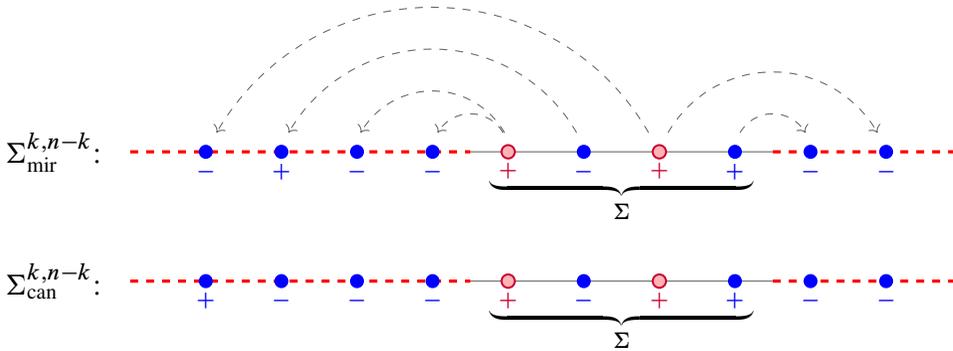
Figure 11: A visualization of two ways to balance a sequence for $k = 2$. For the mirror balancing (above) replace each red point with two blue points first, then reflect the left $k + m = 4$ points to the left and the remaining ones to the right platform, changing their orientations. In the canonical balancing (below) the points on each platform are ordered with respect to their orientation.

7.1 Balancing

Suppose that Σ has m red and n blue points and choose $0 \leqslant k \leqslant n$. We say that a sequence Σ° on a line with platforms is a *balancing of Σ of type $n - 2k$* if it is balanced and obtained from Σ by placing ℓ and r blue points to the left and right of Σ respectively, where $r - \ell = n - 2k$. We say that the extra points lie on *platforms*, which are drawn as dashed lines. In what follows we describe two methods to balance a given sequence; see Figure 11.

The *mirror balancing* $\Sigma_{\text{mir}}^{k,n-k}$ of Σ of type $\lambda = n - 2k$ is constructed as follows. First, replace each red point by two blue points oriented the same way and call the new sequence Σ'. Then $\Sigma_{\text{mir}}^{k,n-k}$ is obtained from Σ by reflecting the first $k + m$ points of Σ' on the left and the remaining ones on the right platform, so that both sequences appear on the platforms in a reversed order; we also change the orientation of these points (compare with Figure 11). It is a balanced sequence, which is an alternating sequence of blue points if Σ is. However, it depends heavily on the orientations of points of Σ. The next construction does not share this drawback.

The *canonical balancing* $\Sigma_{\text{can}}^{k,n-k}$ of type $\lambda = n - 2k$ is constructed again by placing $k + m$ points on the left and $n - k + m$ points on the right platform, except that now we order the points in such a way that, when read from left to right, positive points on each platform appear first. Moreover, we want the minimal number of negative

(resp. positive) points on the left (resp. right) platform. This leads to one of the following distributions, depending on the total weight $w = w(\Sigma)$ of the sequence Σ.

If $|w| \geq |\lambda|$:

      Left platform: place $\frac{1}{2}(|w| - \lambda)$ points of type $-\operatorname{sgn}(w)$, then fill with $+$'s.

      Right platform: place $\frac{1}{2}(|w| + \lambda)$ points of type $-\operatorname{sgn}(w)$, then fill with $-$'s.

If $|w| < \lambda$:

      Left platform: fill with $+$'s.

      Right platform: place $\frac{1}{2}(\lambda - w)$ points of type $+$, then fill with $-$'s.

If $|w| < -\lambda$:

      Left platform: place $\frac{1}{2}(w - \lambda)$ points of type $-$, then fill with $+$'s.

      Right platform: fill with $-$'s.

We check directly that in each case we obtain a balanced sequence with at most $k + m$ negative and at most $n - k + m$ positive points on the left and right platform respectively.

Remark 7.1 The distribution of points on platforms in $\Sigma_{\mathrm{can}}^{k,n-k}$ depends only on the total weight w of the sequence and the type λ of the balancing, but not directly on the number of points nor their orientation. This is why we call it *canonical*.

7.2 Webs and foams with platforms

We now allow foams to meet the side vertical facets of the ambient cube in collections of horizontal blue lines. More precisely, fix a web $\omega \colon \Sigma_0 \to \Sigma_1$ together with balanced collections Σ_0° and Σ_1°, such that the first ℓ and last r points of both Σ_1° and Σ_1° are blue, oriented the same way, and removing them recovers Σ_1 and Σ_2 respectively. Given cup webs a and b bounded by Σ_0° and Σ_1° respectively, we write $\widetilde{\mathbf{Foam}}^{(\ell,r)}(a, \omega, b)$ for the space of foams embedded in a cube with the following boundary:

- the web ω at the top facet of the cube,
- ℓ and r horizontal blue lines at the vertical facets to the left and to the right of ω respectively, and
- the cup webs a and b at the vertical facets attached to the input and output of ω.

Figure 12 provides examples of such foams. We say that such a foam is *violating* if it has a connected component that either

- meets a platform in more than one line, or
- intersects a platform and carries a dot.
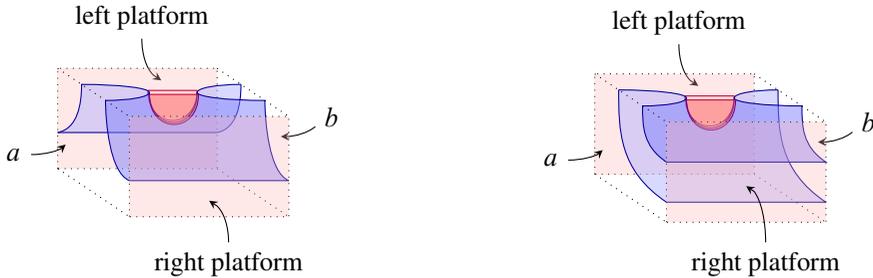
Figure 12: Examples of foams with platforms. The right foam is violating, because it is connected and intersects the right platform (the front facet in the picture) in two lines. The left foam is not violating unless it carries a dot.

It is straightforward to check that the property of being a violating foam is preserved by foam relations. Hence, violating foams generate a linear subspace of $\widetilde{\mathbf{Foam}}^{(\ell,r)}(a,\omega,b)$. We write $\mathbf{Foam}^{(\ell,r)}(a,\omega,b)$ for the quotient space, or simply $\mathbf{Foam}^{(\ell,r)}(a,b)$ when ω is the identity web.

Gluing foams horizontally results in a linear map

$$\widetilde{\mathbf{Foam}}^{(\ell,r)}(a,\omega_0,b) \otimes \widetilde{\mathbf{Foam}}^{(\ell,r)}(b,\omega_1,c) \to \widetilde{\mathbf{Foam}}^{(\ell,r)}(a,\omega_0\omega_1,c)$$

and it is straightforward to notice that a foam $S'S$ is violating when either S or S' is a violating foam. Hence, there is an induced linear map

$$\mathbf{Foam}^{(\ell,r)}(a,\omega_0,b) \otimes \mathbf{Foam}^{(\ell,r)}(b,\omega_1,c) \to \mathbf{Foam}^{(\ell,r)}(a,\omega_0\omega_1,c).$$

Consider now *webs with platforms* as discussed in Section 7.1. Their blue arcs fall into three families visualized in Figure 13:

- *inner arcs*, with at least one endpoint not on a platform,
- *outer arcs*, with each endpoint on a different platform, and
- *violating arcs*, with both endpoints on the same platform.

Webs with no violating arcs and no red endpoints on platforms are *admissible*. Outer arcs of an admissible web are nested one in another and the most nested one of them
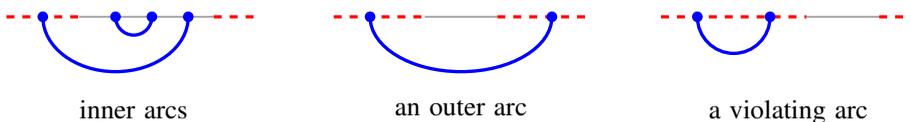
Figure 13: Three types of blue arcs in a cup diagram with platforms.
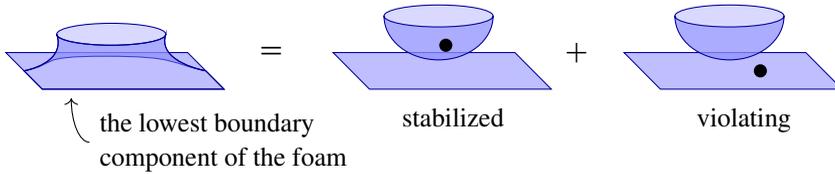
Figure 14: A way to destabilize a foam bounded by stabilized webs.

encloses all inner arcs. Notice that $\mathbf{Foam}^{(\ell,r)}(a, \omega, b) = 0$ when either a or b has a violating arc.

Lemma 7.2 *Let* Σ° *be a balancing of* Σ *with* ℓ *and* r *points on the left and on the right platform, and let* n *count the blue points of* Σ. *Then an admissible cup web bounded by* Σ° *has at least* $(\ell + r - n)/2$ *outer arcs.*

Proof An admissible web bounded by Σ° has at most n inner arcs, so that at least $(\ell + r) - n$ points from the platforms must be connected by outer arcs.   □

Given a cup basis $\mathcal{B}$ of webs bounded by Σ°, we shall write $\mathcal{B}^{\ell,r}$ for the subset of admissible webs with ℓ and r points on the left and right platform respectively.

7.3 Stabilization

We say that a foam $\hat{S}$ is a *stabilization* of a foam S, when it is obtained by placing a blue horizontal rectangle below S. Likewise, *stabilizing a web* means adding an additional outer arc. It follows that $\hat{S}$ is a violating foam if and only if the foam S is violating. Hence, there is a well-defined injection

$$(7\text{-}1) \qquad \mathbf{Foam}^{(\ell,r)}(a, \omega, b) \xrightarrow{\widehat{(-)}} \mathbf{Foam}^{(\ell+1,r+1)}(\hat{a}, \omega, \hat{b})$$

where $\hat{a}$ and $\hat{b}$ are appropriate stabilizations of the webs. It is also surjective: by applying the neck-cutting relation (2-7) we can write every foam $S \in \mathbf{Foam}^{(\ell,r)}(\hat{a}, \omega, \hat{b})$ as a sum $S_0 + S_1$, such that the lowest blue boundary curve bounds a blue disk in each S_i, see Figure 14. Furthermore, stabilization is natural with respect to placing foams on top as well as to the horizontal composition of foams, ie

$$W \cup_\omega \hat{S} = \widehat{W \cup_\omega S} \qquad \text{and} \qquad \hat{S}'\hat{S} = \widehat{S'S}$$

for any $S \in \mathbf{Foam}^{(\ell,r)}(a, \omega, b)$, $S' \in \mathbf{Foam}^{(\ell,r)}(b, \omega', c)$ and $W \colon \omega \to \omega''$.

Let $\mathcal{B}$ be a cup basis of webs bounded by Σ° and $\mathcal{B}^{\ell,r}$ the subset of admissible webs. We write $\hat{\mathcal{B}}^{\ell,r}$ for the set of stabilized basic webs; they are bounded by a bigger

collection $\widehat{\Sigma^\circ}$. It is in general only a subset of a basis of admissible webs bounded by $\widehat{\Sigma^\circ}$. However, it is a basis when platforms carry sufficiently many points.

Lemma 7.3 *Let ℓ and r count points of Σ° on the left and on the right platform, whereas n is the number of the remaining blue points. Then $\widehat{\mathcal{B}}^{\ell,r}$ is a cup basis if $\ell + r \geqslant n$.*

Proof The collection $\widehat{\Sigma^\circ}$ has $\ell + 1$ points on the left and $r + 1$ points on the right platform. Hence, by Lemma 7.2, every admissible web bounded by $\widehat{\Sigma^\circ}$ has an outer arc. □

7.4 Subquotient algebras and bimodules

We are now ready to construct a foam version of the subquotient algebras and bimodules from [10]. Let Σ be a sequence of n blue and m red points, and pick $\lambda = n - 2k$ with $0 \leqslant k \leqslant n$. Choose a balancing Σ° with $k + m$ and $n - k + m$ points on the left and right platform respectively and a cup basis $\mathcal{B}$ for webs bounded by it; the admissible webs form a subset $\mathcal{B}^{k+m,n-k+m}$.

Definition 7.4 The *extended Blanchet–Khovanov algebra* $\mathfrak{A}^{\mathcal{B},\lambda}$ is the direct sum of spaces of foams with platforms

$$\text{(7-2)} \qquad \mathfrak{A}^{\mathcal{B},\lambda} := \bigoplus_{a,b \in \mathcal{B}^{k+m,n-k+m}} \mathbf{Foam}^{(k+m,n-k+m)}(a,b)$$

with multiplication given by the composition (and zero if foams cannot be composed).

It follows from the definition that $\mathfrak{A}^{\mathcal{B},\lambda}$ is a quotient of a subalgebra of $\mathfrak{W}^{\mathcal{B}}$. In particular, it inherits the description in terms of dotted completed webs with the following modifications:

- the horizontal line, along which cup webs are glued, has platform sections on its sides;
- we allow only completions of admissible cup foams; and
- such a diagram vanishes when it contains a blue loop intersecting a platform at least twice or a blue loop intersecting a platform and carrying a dot at the same time.

In particular, $\mathfrak{A}^{\mathcal{B},\lambda}$ is isomorphic to the Chen–Khovanov algebra $A^{k,n-k}$ when Σ consists of n blue points that are oriented in an alternating way and Σ° is the mirror balancing. With the help of the stabilization map (7-1) we can find such isomorphisms for all extended Blanchet–Khovanov algebras.

Theorem 7.5 *Let Σ be a collection of m red and n blue points with a balancing Σ° of type $\lambda = n - 2k$, where $0 \leqslant k \leqslant n$. Then there is an algebra isomorphism $\mathfrak{A}^{\mathcal{B},\lambda} \cong A^{k,n-k}$ for any cup basis $\mathcal{B}$ of webs bounded by Σ°. When $\mathcal{B}$ is a red-over-blue basis, the isomorphism simply forgets red facets of basic cup foams and drops m lowest blue rectangles.*

Proof We assume that $\mathcal{B}$ is a red-over-blue basis — the general case is proven the same way as in Theorem 6.4. Let $\mathcal{B}_0$ be the collection of admissible cup diagrams with $2n$ blue endpoints, k of which are on the left and $n - k$ on the right platform. It follows from Lemma 7.2 that each cup web from $\mathcal{B}$ has m outer arcs, so that it is constructed by placing an invertible web E on top of cup diagrams from $\mathcal{B}_0$ stabilized m times. Hence, as a $\mathbb{k}$–module, $\mathfrak{A}^{\mathcal{B},\lambda}$ is isomorphic to $\mathrm{stab}^{(m)}(A^{k,n-k})$, the algebra $A^{k,n-k}$ stabilized m times. Because stabilization is compatible with horizontal composition of foams, we obtain an algebra isomorphism

$$A^{k,n-k} \xrightarrow{\widehat{(-)}^{(m)}} \mathrm{stab}^{(m)}(A^{k,n-k}) \xrightarrow{(E \times [0,1]) \cup (-)} \mathfrak{A}^{\mathcal{B},\lambda}$$

which takes a dotted surface with platforms, adds m extra horizontal rectangles below, and then glues $E \times [0,1]$ to it along the top and platforms. The inverse of this map simply forgets red facets and drops the extra m blue rectangles as desired. □

We follow the same ideas to construct a collection of bimodules for a web $\omega \colon \Sigma_0 \to \Sigma_1$. Let n_i be the number of blue points in Σ_i. Choose $0 \leqslant k_i \leqslant n_i$ for $i = 0, 1$ such that $n_0 - 2k_0 = n_1 - 2k_1 =: \lambda$, and let Σ_0° and Σ_1° be the *canonical* balancings of type λ, except that we stabilize one of them, so that both sequences have the same numbers of points on platforms: ℓ on the left and r on the right one. Notice that Σ_0 and Σ_1 have the same weight, so that their balancings agree on platforms. Choose cup bases $\mathcal{B}_0$ and $\mathcal{B}_1$ for Σ_0° and Σ_1° respectively. We assign to the web ω the $(\mathfrak{A}^{\mathcal{B}_0,\lambda}, \mathfrak{A}^{\mathcal{B}_1,\lambda})$–bimodule

$$\mathcal{F}_w^\lambda(\omega) := \bigoplus_{a \in \mathcal{B}_0^{\ell,r}, b \in \mathcal{B}_1^{\ell,r}} \mathbf{Foam}^{(\ell,r)}(a, \omega, b),$$

which we call the *extended Blanchet–Khovanov bimodule of weight λ*. The algebras $\mathfrak{A}^{\mathcal{B}_0,\lambda}$ and $\mathfrak{A}^{\mathcal{B}_1,\lambda}$ act on the left and right by composing foams horizontally, stabilized sufficiently many times when necessary. It should be also clear that placing a foam $W \colon \omega \to \omega'$ on top induces a bimodule map $\mathcal{F}_w^\lambda(W) \colon \mathcal{F}_w^\lambda(\omega) \to \mathcal{F}_w^\lambda(\omega')$, and that taking tensor products over the algebras corresponds to composing webs

$$\mathcal{F}_w^\lambda(\omega) \otimes_{\mathfrak{A}^{\mathcal{B},\lambda}} \mathcal{F}_w^\lambda(\omega') \xRightarrow{\cong} \mathcal{F}_w^\lambda(\omega'\omega).$$

As before, Theorem 7.5 allows us to think of $\mathcal{F}_w^\lambda(\omega)$ as a bimodule over the algebras $A^{k_i, n_i - k_i}$, so that we can compare it with the bimodule $C_{CK}(\omega_b; \lambda)$ assigned to the flat tangle ω_b in [10]. We leave the following as an easy exercise:

Theorem 7.6 *Let $\omega\colon \Sigma \to \Sigma'$ be a web between collections of points with n and n' blue points, respectively, and choose $0 \leqslant k \leqslant n$ and $0 \leqslant k' \leqslant n'$, such that*

$$n - 2k = n' - 2k' = \lambda.$$

Then there is an isomorphism of $(A^{k,n-k}, A^{k',n'-k'})$–bimodules $\mathcal{F}_w^\lambda(\omega) \cong C_{CK}(\omega_b, \lambda)$, which — up to stabilization — forgets red facets of cup foams when $\mathcal{F}_w^\lambda(\omega)$ is constructed using red-over-blue web bases.

7.5 A functorial homology for all tangles

The above sections describe a family of 2–functors $\mathcal{F}_w^\lambda\colon \mathbf{Foam} \to \mathbf{Bimod}$ parametrized with $\lambda \in \mathbb{Z}$, which — as before — we extend to the bicategory of formal complexes $Com_{/h}(\mathbf{Foam})$. Applying $\mathcal{F}_w^\lambda$ to the bracket $[\![T]\!]_{\mathrm{F}}$ of a tangle T results in a chain complexes of bimodules, the homotopy type of which is an invariant of T and which is functorial with respect to tangle cobordisms.

Theorem D *The 2–functor $\mathcal{F}_w^\lambda$ is equivalent to $\mathcal{F}_{Kh}^\lambda \circ \mathcal{E}^\vee$. In particular, the complexes $C_{\mathfrak{W}}(T; \lambda)$ and $CKh(T; \lambda)$ are isomorphic for any tangle T.*

Proof The equivalence of $\mathcal{F}_w^\lambda$ and $\mathcal{F}_{Kh}^\lambda \circ \mathcal{E}^\vee$ follows from Theorems 7.5 and 7.6 along the same lines as in the proof of Theorem C. The second statement is a direct consequence of Theorem 4.6. ☐

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