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Bialgebraic approach to rack cohomology

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We interpret the complexes defining rack cohomology in terms of a certain dg bialgebra. This yields elementary algebraic proofs of old and new structural results for this cohomology theory. For instance, we exhibit two explicit homotopies controlling structure defects on the cochain level: one for the commutativity defect of the cup product, and the other for the “Zinbielity” defect of the dendriform structure. We also show that, for a quandle, the cup product on rack cohomology *restricts* to, and the Zinbiel product *descends* to quandle cohomology. Finally, for rack cohomology with suitable coefficients, we complete the cup product with a compatible coproduct.

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1 Introduction

A *quandle* X is an algebraic structure whose axioms describe what survives from a group when one only looks at the conjugacy operation. Quandles have been intensively studied since the 1982 work of D Joyce [14] and S Matveev [19], who showed how to extract powerful knot invariants from them. But the history of quandles goes back much further. *Racks*, which slightly generalize quandles, can be traced back to an unpublished 1959 correspondence between J Conway and G Wraith; *keis*, or involutive quandles, appeared in 1943 in an article of M Takasaki [29], and other related structures were mentioned as early as the end of the 19th century. A thorough account, focusing on topological aspects through the rack space, can be found in R Fenn, C Rourke and B Sanderson [13]. Other viewpoints and various applications to algebra, topology, and set theory are treated, for instance, in Andruskiewitsch and Graña [1], Dehornoy [8], Elhamdadi and Nelson [9], Kinyon [15] and Przytycki [21].

Rack cohomology $H_R(X)$ was defined by Fenn, Rourke and Sanderson in 1993 [11]. It strengthened and extended the applications of racks. The *cup product* \smile on $H_R(X)$ was first described in topological terms by F Clauwens [5]. Later S Covez proposed a cubical interpretation, which allowed him to refine the cup product to a *dendriform structure* using shuffle combinatorics, and further to a *Zinbiel¹ product* $\underline{\smile}$ using acyclic models [6; 7]. This yields in particular the commutativity of \smile .

Rack cohomology is a particular case of the cohomology of set-theoretic solutions to the *Yang–Baxter equation*, as constructed by J S Carter, M Elhamdadi and M Saito [2]. This very general cohomology theory still belongs to the cubical context. It thus carries a commutative cup product, explicitly described by M Farinati and J García Galofre [10]. V Lebed [17] gave it two new interpretations: in terms of M Rosso’s quantum shuffles [26], and via graphical calculus. She gave a graphical version of an explicit homotopy on the cochain level, which controls the commutativity defect of \smile .

Our purpose is to study a *differential graded bialgebra* $B(X)$ that is attached to any rack X , and governs the algebraic structure of its cohomology. This construction was first unveiled in [10]. We show that $B(X)$ is graded cocommutative up to an explicit homotopy, which implies the commutativity of the cup product \smile on $H_R(X)$. This yields a purely algebraic version of the diagrammatic construction from [17]. On

¹The word *Zinbiel* is the mirror image of the word *Leibniz*, and the corresponding structures are Koszul-dual. The definitions will be recalled in Section 7.

a quotient $\bar{B}(X)$ of $B(X)$, we refine the coproduct to a dg codendriform structure. Even better, this codendriform structure is co-Zinbiel up to another explicit homotopy, which has not been described before. As a result, the associative commutative cup product on $H_R(X)$ stems from a Zinbiel product \smile , which coincides with the one from [7]. Both the rack cohomology $H_R(X)$ and our dg bialgebra $B(X)$ can be enriched with coefficients. For finite X and suitably chosen coefficients, we complete \smile with a compatible coassociative coproduct. Here again our bialgebraic interpretation considerably simplifies all verifications.

The rack cohomology of a quandle received particular attention, starting from the work of Carter, Jelsovsky, Kamada, Langford and Saito [3] and R A Litherland and S Nelson [18]. It is known to split into two parts, called *quandle* and *degenerate*: $H_R = H_Q \oplus H_D$. The degenerate part H_D is far from being empty, since it contains (a shifted copy of) H_Q as a direct factor: $H_D = H_Q[1] \oplus H_L$. However, as an abelian group, it does not carry any new information. As shown by J Przytycki and K Putyra [23], it is completely determined by H_Q . We recover these cohomology decompositions at the bialgebraic level, and show that the cup product on rack cohomology restricts to H_Q . This result is new, to our knowledge. Rather unexpectedly, our proof heavily uses the Zinbiel product \smile refining \smile , even though \smile does not *restrict* to H_Q . What we show is that \smile *induces* a Zinbiel product on H_Q ; for this we need to regard H_Q as a quotient rather than a subspace of H_R . Our results suggest the questions:

- (i) Does H_D allow one to reconstruct H_Q as a Zinbiel, or at least associative, algebra?
- (ii) In the opposite direction, does H_Q determine the whole rack cohomology H_R as a Zinbiel, or at least associative, algebra?

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2 Cohomology of shelves and racks

We start with recalling the classical complexes defining rack homology and cohomology. They will be given a bialgebraic interpretation in Section 3. The consequences of this interpretation will be explored in the remainder of the paper.

A *shelf* is a set X together with a binary operation $\triangleleft: X \times X \rightarrow X$ given by $(x, y) \mapsto x \triangleleft y$ (sometimes denoted by $x \triangleleft y = x^y$) satisfying the self-distributivity axiom

$$(1) \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

for all $x, y, z \in X$. In exponential notation, it reads $(x^y)^z = (x^z)^{(y^z)}$. A shelf is called a *rack* if the maps $- \triangleleft y: X \rightarrow X$ are bijective for all $y \in X$, a *spindle* if $x \triangleleft x = x$ for all $x \in X$, and a *quandle* if it is both a rack and a spindle. The fundamental family of examples of quandles is given by groups X with $x \triangleleft y = y^{-1}xy$.

Define $C_n(X) = \mathbb{Z}X^n$ to be the free abelian group with basis X^n , and put

$$C^n(X) = \mathbb{Z}X^n \cong \text{Hom}(C_n(X), \mathbb{Z}).$$

Define the differential $\partial: C_\bullet(X) \rightarrow C_{\bullet-1}(X)$ as the linearization of

$$(2) \quad \partial(x_1 \cdots x_n) = \sum_{i=1}^n (-1)^{i-1} (x_1 \cdots \hat{x}_i \cdots x_n - x_1^{x_i} \cdots x_{i-1}^{x_i} x_{i+1} \cdots x_n),$$

where \hat{x}_i means that x_i was omitted. Here and below we denote by $x_1 \cdots x_n$ the element (x_1, \dots, x_n) of X^n . For cohomology, the differential will be $\partial^*: C^\bullet(X) \rightarrow C^{\bullet+1}(X)$. These maps are of square zero (by direct computation or see Remark 11 later) and define, respectively, the *rack*² homology $H^R(X)$ and cohomology $H_R(X)$ of the shelf X .

In knot theory, a quandle Q can be used to color arcs of knot diagrams; a coloring rule involving the operation \triangleleft is imposed at each crossing. The three quandle axioms are precisely what is needed for the number of Q -colorings of a diagram to depend on the underlying knot only. These Q -coloring counting invariants can be enhanced by Boltzmann-type weights, computed using a 2-cocycle of Q . Similarly, n -cocycles of Q yield invariants of $(n-1)$ -dimensional surfaces knotted in \mathbb{R}^{n+1} ; see [4; 24]. Now, together with arcs, one can color diagram regions. The colors can be taken from a Q -set, and the weights are given by cocycles with coefficients, which we will describe next.

Given a shelf X , an X -set is a set S together with a map $\triangleleft: S \times X \rightarrow S$ satisfying

$$(3) \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

for all $x \in S$ and $y, z \in X$. The basic examples are

- (i) X itself, with \triangleleft as the action map \triangleleft ;

²The terminology is inconsistent here. Indeed rack (co)homology was originally defined for racks, and only later was it realized that it works and is interesting, more generally, for shelves. The same goes for the quandle (co)homology of spindles, considered in Section 8.

- (ii) any set with the trivial action $x \blacktriangleleft y = x$;
- (iii) the *structure monoid* $M(X)$ of X (denoted simply by M if X is understood), which is a quadratic monoid

$$M(X) = \langle X : yx^y = xy \text{ for all } x, y \in X \rangle,$$

and where the action map \blacktriangleleft is concatenation in M .

An X -set can also be seen as a set with an action of the monoid $M(X)$.

More generally, an X -module is an abelian group R together with a map $\blacktriangleleft : R \times X \rightarrow R$ (often written exponentially) which is linear in R , and obeys relation (3) for all $x \in R$ and $y, z \in X$. In other words, it is an $M(X)$ -module. The basic examples are the linearization $\mathbb{Z}S$ of an X -set S , or any abelian group with the trivial action.

Take a shelf X and an X -module R . Take the free R -module $C_n(X, R) = RX^n$ with basis X^n , and put $C^n(X, R) = \text{Hom}(C_n(X, R), \mathbb{Z})$. The differential ∂ on C_n is the linearization of

$$(4) \quad \partial(rx_1 \cdots x_n) = \sum_{i=1}^n (-1)^{i-1} (rx_1 \cdots \hat{x}_i \cdots x_n - r^{x_i} x_1^{x_i} \cdots x_{i-1}^{x_i} x_{i+1} \cdots x_n),$$

and the differential on C^n is the induced one. Again, these maps are of square zero and define, respectively, the *rack homology* $H^R(X, R)$ and *cohomology* $H_R(X, R)$ of the shelf X with coefficients in R . If R is the linearization of an X -set S , we use the notation $C_R(X, S)$, $H_R(X, S)$ etc. Choosing S to be the empty set, one recovers the previous definitions. Another interesting coefficient choice is the *structure algebra* $A(X)$ of X (often denoted simply by A), which is the monoid algebra of $M(X)$:

$$A(X) = \mathbb{Z}M(X) \cong \mathbb{Z}\langle X \rangle / \langle yx^y - xy : x, y \in X \rangle.$$

Declaring every $x \in X$ group-like, one gets an associative bialgebra structure on A . This coefficient choice is universal in the following sense. Any X -module R is a right $M(X)$ -module, hence a right $A(X)$ -module. Then one has an obvious isomorphism of chain complexes

$$C_n(X, R) \cong R \otimes_{A(X)} C_n(X, M(X)),$$

where A acts on the first factor of $C_n(X, M) \cong A \otimes_{\mathbb{Z}} \mathbb{Z}X^n$ by multiplication on the left, and the differential acts on the second factor of $R \otimes_A C_n(X, M)$.

3 A dg bialgebra associated to a shelf

The algebraic objects introduced in this section are aimed to yield a simple and explicit description of a differential graded algebra structure on the complex $(C^\bullet(X), \partial^*)$ above, which is commutative, and in fact even Zinbiel, up to explicit homotopies.

Fix a shelf X . All the (bi)algebra structures below will be over \mathbb{Z} , and will be (co)unital. Also, the tensor product $A \otimes B$ of two graded algebras will always be endowed with the product algebra structure involving the Koszul sign

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2,$$

where $b_1 \in B$ and $a_2 \in A$ are homogeneous of degree $|b_1|$ and $|a_2|$, respectively. The Koszul sign also appears when $A^* \otimes B^*$ acts on $A \otimes B$. Similarly, by a (co)derivation on a graded (co)algebra we will always mean a super(co)derivation, and by commutativity we will mean supercommutativity.

Define $B(X)$ (also denoted by B) as the algebra freely generated by two copies of X with the relations

$$B(X) = \mathbb{Z}\langle x, e_y : x, y \in X \rangle / \langle yx^y - xy, ye_{xy} - e_{xy} : x, y \in X \rangle.$$

The interest of this construction lies in the rich structure it carries:

Theorem 1 *For any shelf X , $B(X)$ is a differential graded bialgebra and a differential graded $A(X)$ -bimodule, where:*

- *The grading is given by declaring $|e_x| = 1$ and $|x| = 0$ for all $x \in X$.*
- *The differential d is the unique derivation of degree -1 determined by*

$$d(e_x) = 1 - x \quad \text{and} \quad d(x) = 0 \quad \text{for all } x \in X.$$

- *The comultiplication $\Delta: B \rightarrow B \otimes B$ and the counit $\varepsilon: B \rightarrow k$ are defined on the generators by*

$$\begin{aligned} \Delta(e_x) &= e_x \otimes x + 1 \otimes e_x, & \varepsilon(e_x) &= 0, \\ \Delta(x) &= x \otimes x, & \varepsilon(x) &= 1 \end{aligned} \quad \text{for all } x \in X,$$

and extended multiplicatively.

- *The A -actions $\lambda: A \otimes B \rightarrow B$ and $\rho: B \otimes A \rightarrow B$ are defined by*

$$x \cdot b \cdot y = xby \quad \text{for all } x, y \in X \text{ and } b \in B.$$

By differential graded bialgebra we mean that the differential is both a derivation with respect to multiplication, and a coderivation with respect to comultiplication.

Notice that B is neither commutative nor cocommutative in general.

As usual, from Theorem 1 one deduces dg algebra and dg $A(X)$ -bimodule structures on the graded dual $B^*(X)$ of $B(X)$.

Proof Since the relations are homogeneous, B is a graded algebra. In order to see that d is well defined, one must check that the relations $yx^y \sim xy$ and $ye_{xy} \sim e_x y$ are compatible with d . The first relation is easier:

$$d(yx^y - xy) = d(y)x^y + yd(x^y) - d(x)y - xd(y) = 0 + 0 - 0 - 0 = 0.$$

For the second relation, one has

$$\begin{aligned} d(ye_{xy} - e_x y) &= yd(e_{xy}) - d(e_x)y = y(1 - x^y) - (1 - x)y = y - yx^y - y + xy \\ &= xy - yx^y. \end{aligned}$$

So the ideal of relations defining B is stable by d .

Since d is a derivation and d^2 vanishes on generators, we have $d^2 = 0$, hence a structure of differential graded algebra on B .

Next, we need to check that Δ is well defined. The first relation is easy since all $x \in X$ are group-like in B :

$$\begin{aligned} \Delta(xy - yx^y) &= (x \otimes x)(y \otimes y) - (y \otimes y)(x^y \otimes x^y) = xy \otimes xy - yx^y \otimes yx^y \\ &= xy \otimes (xy - yx^y) + (xy - yx^y) \otimes yx^y. \end{aligned}$$

For the second relation, we check

$$\begin{aligned} \Delta(ye_{xy} - e_x y) &= (y \otimes y)(e_{xy} \otimes x^y + 1 \otimes e_{xy}) - (e_x \otimes x + 1 \otimes e_x)(y \otimes y) \\ &= ye_{xy} \otimes yx^y + y \otimes ye_{xy} - e_x y \otimes xy - y \otimes e_x y \\ &= (ye_{xy} - e_x y) \otimes yx^y + e_x y \otimes (yx^y - xy) + y \otimes (ye_{xy} - e_x y). \end{aligned}$$

So the ideal defining the relations is also a coideal.

Clearly, Δ respects the grading.

Let us now check that d is a coderivation. It is enough to see this on generators:

$$\begin{aligned} \Delta(d(x)) &= \Delta(0) = 0 = d(x) \otimes x + x \otimes d(x) = (d \otimes 1 + 1 \otimes d)\Delta(x), \\ \Delta(d(e_x)) &= \Delta(1 - x) = 1 \otimes 1 - x \otimes x. \end{aligned}$$

This coincides with

$$\begin{aligned} (d \otimes 1 + 1 \otimes d)\Delta(e_x) &= (d \otimes 1 + 1 \otimes d)(e_x \otimes x + 1 \otimes e_x) \\ &= (1 - x) \otimes x + 1 \otimes (1 - x) = 1 \otimes x - x \otimes x + 1 \otimes 1 - 1 \otimes x \\ &= 1 \otimes 1 - x \otimes x. \end{aligned}$$

The map ε is also well defined, since

$$\varepsilon(xy - yx^y) = \varepsilon(ye_{xy} - e_{xy}) = 0.$$

An easy verification on the generators shows that it is indeed a counit.

Finally, the formula $x \cdot b \cdot y = xby$ obviously defines commuting degree-preserving A -actions on B . By the definition of the differential d , one has $d(xby) = xd(b)y$, thus d respects this bimodule structure. \square

Example 2 Let us compute $\Delta(e_x e_y)$. By definition, $\Delta(e_x e_y) = \Delta(e_x)\Delta(e_y)$ in $B \otimes B$, and this is equal to

$$\begin{aligned} (e_x \otimes x + 1 \otimes e_x)(e_y \otimes y + 1 \otimes e_y) &= e_x e_y \otimes xy + e_x \otimes x e_y - e_y \otimes e_x y + 1 \otimes e_x e_y \\ &= e_x e_y \otimes xy + e_x \otimes x e_y - e_y \otimes y e_x + 1 \otimes e_x e_y. \end{aligned}$$

Note the Koszul sign appearing in the product $(1 \otimes e_x)(e_y \otimes y) = -e_y \otimes e_x y$.

The structure on $B(X)$ survives in homology:

Proposition 3 For any shelf X the homology $H(X)$ of $B(X)$ inherits a graded algebra structure. Moreover, the $A(X)$ -actions on $B(X)$ induce trivial actions on $H(X)$:

$$x \cdot h \cdot y = h \quad \text{for all } x, y \in X \text{ and } h \in H.$$

Dually, the cohomology $H^*(X)$ of $B(X)^*$ inherits a graded algebra structure and trivial $A(X)$ -actions.

Here and below, by $B(X)^*$ we mean the graded dual of $B(X)$.

Proof The only nonclassical statement here is the triviality of the induced actions. Take $x \in X$ and $b \in B$. By the definition of d , one has $d(e_x b) = d(e_x)b - e_x d(b)$, hence

$$(5) \quad d(e_x b) = (1 - x)b - e_x d(b).$$

If b is a cycle, this shows that $x \cdot b = b$ modulo a boundary. Hence the induced left A -action on H is trivial. The cases of the right action and the actions in cohomology are analogous. \square

The proposition implies the following remarkable property of H : if $b \in B$ is a representative of some homology class in H , and if one lets an $x \in X$ act upon all the letters from X occurring in b (where the action is $y \mapsto y^x$), then one obtains another representative of the same homology class.

One can also define a version of the bialgebra $B(X)$ with coefficients in any unital commutative ring k :

$$B(X, k) = k\langle x, e_y : x, y \in X \rangle / \langle yx^y - xy, ye_{xy} - e_{xy} : x, y \in X \rangle.$$

In particular, all the tensor products should be taken over k . Theorem 1 and its proof extend verbatim to this setting. For suitable coefficients k , one can say even more:

Proposition 4 *For any shelf X and any field k , the homology $H(X, k)$ of $B(X, k)$ inherits a graded bialgebra structure. If, moreover, X is finite, then the cohomology $H^*(X, k)$ of $B(X, k)^*$ also inherits a graded bialgebra structure.*

This results from the following general observation; it is surely known to specialists, however the authors were unable to find it in the literature:

Lemma 5 *Let k be a field.*

- (i) *If $(C = \bigoplus C_i, d, \Delta)$ is a k -linear dg coassociative coalgebra, then Δ induces a coproduct on the homology H of (C, d) .*
- (ii) *If $(C = \bigoplus C_i, d, \cdot)$ is a k -linear dg algebra of finite dimension in each degree, then \cdot induces a coproduct on the cohomology H^* of (C^*, d^*) .*

Proof (i) The relation $\Delta d = (d \otimes \text{Id} + \text{Id} \otimes d)\Delta$ implies that Δ survives in the quotient $C / \text{Im}(d)$. To restrict it further to $H = \text{Ker}(d) / \text{Im}(d)$, we shall check that

$$\Delta(\text{Ker}(d)) \subseteq \text{Ker}(d) \otimes \text{Ker}(d) + \text{Im}(d) \otimes C + C \otimes \text{Im}(d).$$

Since k is a field, the space $K := \text{Ker}(d)$ has a complement L in C , on which d is injective. Putting $I := \text{Im}(d)$, one has

$$\begin{aligned} (d \otimes \text{Id})(L \otimes L) &= I \otimes L, & (\text{Id} \otimes d)(L \otimes L) &= L \otimes I, \\ (d \otimes \text{Id} + \text{Id} \otimes d)(K \otimes L) &= K \otimes I, & (d \otimes \text{Id} + \text{Id} \otimes d)(L \otimes K) &= I \otimes K, \\ (d \otimes \text{Id} + \text{Id} \otimes d)(K \otimes K) &= 0. \end{aligned}$$

Moreover, in the first two lines all the maps are bijective. Now, from

$$(d \otimes \text{Id} + \text{Id} \otimes d)\Delta(K) = \Delta d(K) = 0$$

and from the disjointness of L and K (and hence I), one sees that $\Delta(K)$ cannot have components in $L \otimes L$, and its components from $K \otimes L$ (resp. $L \otimes K$) necessarily lie in $I \otimes L$ (resp. $L \otimes I$).

(ii) Due to the finite dimension in each degree, the product on (C, d) induces a coproduct on (C^*, d^*) , to which we apply the first statement. □

Remark 6 The dg bialgebra $B(X)$ admits a variation $B'(X)$, where one adds the inverses x^{-1} of the generators $x \in X$, with $|x^{-1}| = 0$, $d(x^{-1}) = 0$, $\Delta(x^{-1}) = x^{-1} \otimes x^{-1}$ and $\varepsilon(x^{-1}) = 1$. One obtains a dg Hopf algebra, with the antipode defined on the generators by

$$s(x) = x^{-1}, \quad s(x^{-1}) = x \quad \text{and} \quad s(e_x) = -e_x x^{-1},$$

and extended superantimultiplicatively, in the sense of $s(ab) = (-1)^{d(a)d(b)}s(b)s(a)$ for homogeneous a and b . Indeed, one easily verifies that this map

- is well defined, that is, compatible with the defining relations of $B'(X)$;
- is of degree 0, that is, $|s(b)| = |b|$ for all homogeneous b ;
- yields the inverse of Id in the convolution algebra, that is, if $\Delta(b) = \sum_i a_i \otimes c_i$ for a given $b \in B'(X)$, one has

$$\sum_i s(a_i)c_i = \sum_i a_i s(c_i) = \varepsilon(b);$$

- commutes with the differential d , that is, $ds = sd$.

For the square of the antipode, one computes $s^2(e_x) = x e_x x^{-1}$. In a spindle it equals e_x , yielding $s^2 = \text{Id}$. In general s need not be of finite order; for the rack $X = \mathbb{Z}$ with $x^y = x + 1$, one has $s^2 \cdot (e_x) = e_{x-1}$. In a rack, one simplifies $s^2(e_x) = e_{x \tilde{\triangleleft} x}$, where the operation $\tilde{\triangleleft}$ is defined by $(x \triangleleft y) \tilde{\triangleleft} y = x$ for all $x, y \in X$. The map $x \mapsto x \tilde{\triangleleft} x$ plays an important role in the study of racks; see for instance [28]. Finally, in the computation

$$\begin{aligned} s(e_{x_1} e_{x_2} \cdots e_{x_{n-1}} e_{x_n}) &= (-1)^{\binom{n}{2}} (-e_{x_n} x_n^{-1}) (-e_{x_{n-1}} x_{n-1}^{-1}) \cdots (-e_{x_2} x_2^{-1}) (-e_{x_1} x_1^{-1}) \\ &= (-1)^{\frac{1}{2}n(n+1)} e_{x_n} e_{x_{n-1}^{x_n}} \cdots e_{x_2^{x_3 \cdots x_n}} e_{x_1^{x_2 \cdots x_n}} x_n^{-1} \cdots x_1^{-1} \end{aligned}$$

one recognizes the *remarkable map*

$$(x_1, x_2, \dots, x_n) \mapsto (x_1^{x_2 \cdots x_n}, x_2^{x_3 \cdots x_n}, \dots, x_n)$$

of Przytycki [20].

4 The bialgebra encodes the cohomology

We will now show that the dg bialgebra $B(X)$ knows everything about the homology (C_\bullet, ∂) and the cohomology (C^\bullet, ∂^*) of our shelf X , and about its variations (C_\bullet^M, ∂) and $(C_M^\bullet, \partial^*)$ with coefficients in the structure monoid $M(X)$.

First, we need to modify B slightly:

Lemma 7 *The following data define a dg coalgebra and a right dg A -module:*

$$(\mathbb{Z} \otimes_A B, \text{Id}_{\mathbb{Z}} \otimes d, \text{Id}_{\mathbb{Z}} \otimes \Delta, \text{Id}_{\mathbb{Z}} \otimes \varepsilon, \text{Id}_{\mathbb{Z}} \otimes \rho).$$

Here the grading is the one induced from B , and the A -action on \mathbb{Z} is the trivial one: $\lambda \cdot x = \lambda$ for all $x \in X$ and $\lambda \in \mathbb{Z}$.

The dg coalgebra from the lemma will be denoted by $\bar{B} = \bar{B}(X)$. It has obvious variants $\bar{B}(X, k)$ with coefficients in any unital commutative ring k .

Proof On the level of abelian groups, one has

$$(6) \quad \bar{B} \simeq B / \langle xb - b : x \in X, b \in B \rangle.$$

The grading survives in this quotient since $|x| = 0$, and the degree -1 differential survives since $d(x) = 0$ and $|x| = 0$ imply $d(xb) = xd(b) \sim d(b)$. Further, we have $\Delta(xb) = (x \otimes x)\Delta(b) \sim \Delta(b)$, so Δ induces a coproduct on \bar{B} compatible with the grading and the differential. For the counit, we have $\varepsilon(xb) = \varepsilon(x)\varepsilon(b) = \varepsilon(b)$. Finally, the right A -action also descends to \bar{B} , as $(xb) \cdot y = xby = x(by) \sim by = b \cdot y$. \square

Remark 8 Observe that we lose the product in the quotient \bar{B} . Indeed, for all $x, y \in X$ we have $y \sim 1$, but $e_x \cdot y = e_x y = ye_{x^y} \sim e_{x^y} \approx e_x = e_x \cdot 1$.

Lemma 9 *As a left A -module, B can be presented as*

$$B \cong A \otimes \mathbb{Z}\langle X \rangle.$$

Proof Consider the map

$$A \otimes \mathbb{Z}\langle X \rangle \rightarrow B \quad \text{given by } x_1 \cdots x_k \otimes y_1 \cdots y_n \mapsto x_1 \cdots x_k e_{y_1} \cdots e_{y_n}.$$

It is well defined since the relations in A hold true for the corresponding generators of B .

Going in the opposite direction is trickier. A monomial b in B is a product of generators of the form x and e_y . Let $a(b)$ be what remains in b when all generators of the form e_y are erased. Further, start with a new copy of b and erase all generators of the form x one by one, starting from the left. When erasing a generator x , replace all generators of the form e_y to its left by e_{y^x} . After that replace all the e_y by y . This yields a monomial $t(b) \in \mathbb{Z}\langle X \rangle$. Analyzing the defining relations of B , and using the self-distributivity axiom (1) for X , one sees that we obtain a well-defined map

$$B \rightarrow A \otimes \mathbb{Z}\langle X \rangle \quad \text{given by } b \mapsto a(b) \otimes t(b).$$

Both maps are clearly A -equivariant, and are mutually inverse. \square

From this follows:

Proposition 10 *One has the following isomorphisms of complexes:*

$$\begin{aligned} (C_{\bullet}^M, \partial) &\cong B, & (C_{\bullet M}^{\bullet}, \partial^*) &\cong B^*, & (C_{\bullet}, \partial) &\cong \bar{B}, & (C^{\bullet}, \partial^*) &\cong \bar{B}^*, \\ (C_{\bullet}(X, kM(X)), \partial) &\cong B(X, k), & (C^{\bullet}(X, kM(X)), \partial^*) &\cong B^*(X, k), \\ (C_{\bullet}(X, k), \partial) &\cong \bar{B}(X, k), & (C^{\bullet}(X, k), \partial^*) &\cong \bar{B}^*(X, k). \end{aligned}$$

Here B^* denotes the graded dual of B with the induced differential, and similarly for \bar{B}^* . In the last isomorphisms, the ring k is considered as a trivial X -module.

Proof The preceding lemma yields isomorphisms of abelian groups

$$B \cong C_{\bullet}^M \quad \text{and} \quad \bar{B} \cong \mathbb{Z}\langle X \rangle = C_{\bullet},$$

and their k -versions.

To compute the differential induced on this by d , we use that d is a derivation:

$$\begin{aligned} d(e_{x_1} \cdots e_{x_n}) &= \sum_{i=1}^n (-1)^{i-1} e_{x_1} \cdots e_{x_{i-1}} d(e_{x_i}) e_{x_{i+1}} \cdots e_{x_n} \\ &= \sum_{i=1}^n (-1)^{i-1} e_{x_1} \cdots e_{x_{i-1}} (1 - x_i) e_{x_{i+1}} \cdots e_{x_n} \\ &= \sum_{i=1}^n (-1)^{i-1} e_{x_1} \cdots e_{x_{i-1}} e_{x_{i+1}} \cdots e_{x_n} - \sum_{i=1}^n (-1)^{i-1} e_{x_1} \cdots e_{x_{i-1}} x_i e_{x_{i+1}} \cdots e_{x_n}. \end{aligned}$$

Using the relation $e_x y = y e_{x^y}$, one gets

$$e_{x_1} \cdots e_{x_{i-1}} x_i e_{x_{i+1}} \cdots e_{x_n} = x_i e_{x_1^{x_i}} \cdots e_{x_{i-1}^{x_i}} e_{x_{i+1}} \cdots e_{x_n},$$

which in C_{\bullet}^M corresponds to $x_i x_1^{x_i} \cdots x_{i-1}^{x_i} x_{i+1} \cdots x_n$. We thus recover the differential (4). In the quotient \bar{B} , the last computation simplifies as $e_{x_1^{x_i}} \cdots e_{x_{i-1}^{x_i}} e_{x_{i+1}} \cdots e_{x_n}$, and we recover the differential (2). □

Remark 11 This proposition provides a very simple proof that $\partial^2 = 0$ in $C_{\bullet}(X, M(X))$ and its versions.

From the proof of Lemma 9 one deduces the useful fact:

Lemma 12 *The map $A \rightarrow B$ given by $x \mapsto x$ is an injective algebra morphism.*

In what follows we will often identify A with its isomorphic image in B .

The isomorphisms in Proposition 10 allow one to transport the structure from B and \bar{B} to rack (co)homology:

Theorem 13 *Take a shelf X and a field k . Then*

- (i) *the chain complex $(C_\bullet(X), \partial)$ carries a coassociative coproduct,*
- (ii) *the cochain complex $(C^\bullet(X), \partial^*)$ carries an associative product,*
- (iii) *the chain complex $(C_\bullet(X, M(X)), \partial)$ carries a bialgebra structure,*
- (iv) *the cochain complex $(C^\bullet(X, M(X)), \partial^*)$ carries an associative product, enriched to a bialgebra structure when X is finite.*

This induces

- (i) *associative products on $H^*(X)$, $H^*(X, M(X))$, $H^*(X, k)$, and $H(X, M(X))$,*
- (ii) *a coassociative coproduct on $H(X, k)$,*
- (iii) *a bialgebra structure on $H(X, kM(X))$,*
- (iv) *an associative product on $H^*(X, kM(X))$, which is completed to a bialgebra structure for finite X .*

The product in cohomology is called the *cup product*, and is denoted by \smile .

Example 14 Take $f, g \in C^2(X)$. To compute $f \smile g$, one needs to compute the summands in $\Delta(e_x e_y e_z e_t)$ with two tensors of type e_u in each factor. Using the computation from Example 2

$$\begin{aligned} \Delta(e_x e_y e_z e_t) &= \Delta(e_x e_y) \Delta(e_z e_t) \\ &= (e_x e_y \otimes xy + 1 \otimes e_x e_y + e_x \otimes x e_y - e_y \otimes y e_x) \\ &\quad \cdot (e_z e_t \otimes zt + 1 \otimes e_z e_t + e_z \otimes z e_t - e_t \otimes t e_z) \\ &= e_x e_y \otimes x y e_z e_t + e_z e_t \otimes e_x e_y z t - e_x e_z \otimes x e_y z e_t + e_x e_t \otimes x e_y t e_z \\ &\quad + e_y e_z \otimes y e_x z e_t - e_y e_t \otimes y e_x t e_z + \dots, \end{aligned}$$

where the dots hide terms on which f and g vanish. Pushing the e_u to the right and the elements of X to the left, we get

$$\begin{aligned} e_x e_y \otimes x y e_z e_t + e_z e_t \otimes z x e_x z t e_y z t - e_x e_z \otimes x z e_y z e_t \\ + e_x e_t \otimes x t e_y t e_z t + e_y e_z \otimes y z e_x z e_t - e_y e_t \otimes y t e_x t e_z t + \dots, \end{aligned}$$

so finally $(f \smile g)(e_x e_y e_z e_t)$ is equal to

$$f(e_x e_y)g(e_z e_t) + f(e_z e_t)g(e_{xzt} e_{yzt}) - f(e_x e_z)g(e_{yz} e_t) + f(e_x e_t)g(e_{yt} e_{zt}) + f(e_y e_z)g(e_{xyz} e_t) - f(e_y e_t)g(e_{xyt} e_{zt}).$$

This formula is to be compared with (23) of [5]. A full explanation of this agreement is given in the next section.

The last piece of structure to be extracted from Proposition 10 is the A -action:

Proposition 15 *For a shelf X , the complex $(C^\bullet(X), \partial^*)$ is a left $A(X)$ -module, with*

$$(x \cdot f)(x_1 \cdots x_n) := (x_1^x \cdots x_n^x),$$

where $f \in C^n(X)$ and $x, x_1, \dots, x_n \in X$. The induced $A(X)$ -action in cohomology is trivial.

Proof This directly follows from Propositions 3 and 10. □

This property of rack cohomology was first noticed by Przytycki and Putyra [22]. In our bialgebraic interpretation it becomes particularly natural.

5 An explicit expression for the cup product in cohomology

To give an explicit formula for the cup product in rack cohomology, we need to compute $\Delta(e_{x_1} \cdots e_{x_n})$ for any x_1, \dots, x_n in the rack X , generalizing Example 2. For this we will introduce some notation. First, for any $n \geq 1$ and for any $i \in \{1, \dots, n\}$ we define two maps $\delta_i^0, \delta_i^1: X^n \rightarrow X^{n-1}$ by

$$\begin{aligned} \delta_i^0(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \\ \delta_i^1(x_1, \dots, x_n) &= (x_1 \triangleleft x_i, \dots, x_{i-1} \triangleleft x_i, x_{i+1}, \dots, x_n). \end{aligned}$$

The above identification of B with $A \otimes \mathbb{Z}\langle X \rangle$ given by $ae_{x_1} \cdots e_{x_n} \leftrightarrow a \otimes x_1 \cdots x_n$ allows one to transport δ_i^0 and δ_i^1 to A -linear endomorphisms of B :

$$\begin{aligned} \delta_i^0(ae_{x_1} \cdots e_{x_n}) &= \begin{cases} ae_{x_1} \cdots e_{x_{i-1}} e_{x_{i+1}} \cdots e_{x_n} & \text{if } i \leq n, \\ 0 & \text{if } i > n, \end{cases} \\ \delta_i^1(ae_{x_1} \cdots e_{x_n}) &= \begin{cases} ax_i e_{x_1 \triangleleft x_i} \cdots e_{x_{i-1} \triangleleft x_i} e_{x_{i+1}} \cdots e_{x_n} & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases} \end{aligned}$$

A straightforward computation using self-distributivity yields

$$(7) \quad \delta_i^\varepsilon \delta_j^\eta = \delta_{j-1}^\eta \delta_i^\varepsilon$$

for any $i < j$ and any $\varepsilon, \eta \in \{0, 1\}$. Identities (7) are the defining axioms for \square -sets [27]; see also [5; 12]. Now, the boundary (2) can be rewritten as

$$(8) \quad \partial = \sum_{i \geq 1} (-1)^{i-1} (\delta_i^0 - \delta_i^1).$$

For any finite subset S of \mathbb{N} and for $\varepsilon \in \{0, 1\}$, we denote by δ_S^ε the composition, in increasing order, of the maps δ_a^ε for $a \in S$.

Proposition 16 *Given a rack X , the coproduct in $B(X)$ can be computed by the formula*

$$(9) \quad \Delta(ae_{x_1} \cdots e_{x_n}) = \sum_{S \subset \{1, \dots, n\}} \epsilon(S) a \delta_S^0(e_{x_1} \cdots e_{x_n}) \otimes a \delta_{S^c}^1(e_{x_1} \cdots e_{x_n})$$

for all $a \in \langle X \rangle$ and $x_i \in X$. Here $S^c = \{1, \dots, n\} \setminus S$, and $\epsilon(S)$ is the signature of the unshuffle permutation of $\{1, \dots, n\}$ which puts S^c on the left and S on the right.

We use the canonical form $ae_{x_1} \cdots e_{x_n}$ of a monomial in $B(X)$.

Proof Since $\Delta(a) = a \otimes a$, we can omit this part of our monomial. Let us then proceed by induction on n , the case $n = 1$ being immediate:

$$\begin{aligned} &\Delta(e_{x_1} \cdots e_{x_n}) \\ &= \Delta(e_{x_1} \cdots e_{x_{n-1}}) \Delta(e_{x_n}) \\ &= \left(\sum_{B \subset \{1, \dots, n-1\}} \epsilon(B) \delta_B^0(e_{x_1} \cdots e_{x_{n-1}}) \otimes \delta_{B^c}^1(e_{x_1} \cdots e_{x_{n-1}}) \right) (e_{x_n} \otimes x_n + 1 \otimes e_{x_n}) \\ &= \sum_{B \subset \{1, \dots, n-1\}} (-1)^{|B|} \epsilon(B) \delta_B^0(e_{x_1} \cdots e_{x_{n-1}}) e_{x_n} \otimes \delta_{B^c}^1(e_{x_1} \cdots e_{x_{n-1}}) x_n \\ &\quad + \sum_{B \subset \{1, \dots, n-1\}} \epsilon(B) \delta_B^0(e_{x_1} \cdots e_{x_{n-1}}) \otimes \delta_{B^c}^1(e_{x_1} \cdots e_{x_{n-1}}) e_{x_n} \\ &= \sum_{\substack{S \subset \{1, \dots, n\} \\ n \notin S}} \epsilon(S) \delta_S^0(e_{x_1} \cdots e_{x_n}) \otimes \delta_{S^c}^1(e_{x_1} \cdots e_{x_n}) \\ &\quad + \sum_{\substack{S \subset \{1, \dots, n\} \\ n \in S}} \epsilon(S) \delta_S^0(e_{x_1} \cdots e_{x_n}) \otimes \delta_{S^c}^1(e_{x_1} \cdots e_{x_n}) \\ &= \sum_{S \subset \{1, \dots, n\}} \epsilon(S) \delta_S^0(e_{x_1} \cdots e_{x_n}) \otimes \delta_{S^c}^1(e_{x_1} \cdots e_{x_n}). \quad \square \end{aligned}$$

Corollary 17 *The cup product in rack cohomology induced from the coproduct in B coincides with the cup product given by Clauwens in [5, (32)].*

Proof This is immediate by comparing (9) with [5, (32)]. The overall sign $(-1)^{km}$ in [5] is the Koszul sign. □

6 The cup product is commutative

In the preceding section we established that our cup product on rack cohomology coincides with Clauwens’s product. The latter comes from the cohomology of a topological space, and is thus commutative (where, as usual, we mean supercommutative). We will now give a direct algebraic proof based on an explicit homotopy argument. This homotopy is a specialization of the graphically defined map, constructed for solutions to the Yang–Baxter equation by Lebed [17].

Let us start with a low-degree illustration:

Example 18 Take $f, g \in C^1(X)$, identified with A –linear maps from B to \mathbb{Z} (also denoted by f and g) determined by the values $f(e_x) := f(x)$ and $g(e_x) := g(x)$ for $x \in X$, and vanishing in degrees other than 1. Then the cup product $f \smile g \in C^2(X)$ is defined by

$$\begin{aligned} (f \smile g)(e_x e_y) &= (f \otimes g)\Delta(e_x e_y) \\ &= (f \otimes g)(e_x e_y \otimes xy + 1 \otimes e_x e_y + e_x \otimes x e_y - e_y \otimes y e_x) \end{aligned}$$

(see Example 2). Since f and g vanish on elements of degree 0 and 2, and are left A –linear (where x and y act on \mathbb{Z} trivially), the only remaining terms are

$$-f(e_x)g(xe_y) + f(e_y)g(ye_x) = -f(e_x)g(e_y) + f(e_y)g(e_x).$$

Note the Koszul sign $(-1)^{|g||e_x|} = -1$, and similarly in the second term. So the product is in general not commutative. On the other hand, the cocycle condition $\partial^*g = 0$ means precisely $g(e_x) = g(e_{xy})$ for all x and y , so the cup product restricted to 1–cocycles is commutative.

Now, take $f, g \in C^1(X, M(X))$, identified with maps $B \rightarrow \mathbb{Z}$ vanishing in degrees other than 1. Then, for monomials $a \in A$ and $x, y \in X$, one computes

$$(f \smile g)(ae_x e_y) = -f(ae_x)g(axe_y) + f(ae_y)g(ae_x y)$$

and

$$\begin{aligned} (10) \quad (f \smile g + g \smile f)(ae_x e_y) \\ = -f(ae_x)g(axe_y) + f(ae_y)g(ae_x y) - g(ae_x)f(axe_y) + g(ae_y)f(ae_x y). \end{aligned}$$

Suppose that f and g are 1-cocycles. This yields the relation

$$\begin{aligned} 0 &= (-d^* f)(ae_x e_y) = f(d(ae_x e_y)) = f(d(a)e_x e_y) + f(ad(e_x)e_y) - f(ae_x d(e_y)) \\ &= f(a(1-x)e_y) - f(ae_x(1-y)) = f(ae_y) - f(axe_y) - f(ae_x) + f(ae_x y), \end{aligned}$$

and similarly for g . Note the Koszul sign in $-d^* f = fd$. Define a map $h: B \rightarrow \mathbb{Z}$ by

$$h(ae_x) = f(ae_x)g(ae_x)$$

for all monomials $a \in A$ and all $x \in X$. Then (10) becomes

$$\begin{aligned} -f(ae_x)g(axe_y) + f(ae_y)(g(ae_x) - g(ae_y) + g(axe_y)) \\ -g(ae_x)(f(ae_x y) - f(ae_x) + f(ae_y)) + g(ae_y)f(ae_x y) \\ = (f(ae_y) - f(ae_x))g(axe_y) - h(ae_y) + h(ae_x) + (g(ae_y) - g(ae_x))f(ae_x y) \\ = (f(axe_y) - f(ae_x y))g(axe_y) - h(ae_y) + h(ae_x) + (g(axe_y) - g(ae_x y))f(ae_x y) \\ = h(axe_y) - h(ae_y) + h(ae_x) - h(ae_x y) = (d^* h)(ae_x e_y), \end{aligned}$$

yielding the relation $f \smile g + g \smile f = d^* h$, and hence the supercommutativity of the cup product of degree 1 cohomology classes.

Lemma 19 Let $h: B \rightarrow B \otimes B$ be the degree 1 linear map defined on monomials in B written in the canonical form as follows: $h(a) = 0$, and

$$h(ae_{x_1} \cdots e_{x_n}) := \sum_{i=1}^n (-1)^{i-1} (a \otimes a)(\tau\Delta)(e_{x_1} \cdots e_{x_{i-1}})(e_{x_i} \otimes e_{x_i})\Delta(e_{x_{i+1}} \cdots e_{x_n}),$$

where $\tau: B \otimes B \rightarrow B \otimes B$ is the signed flip $\tau(a \otimes b) = (-1)^{d(a)d(b)} b \otimes a$ for homogeneous a and b . Then for any homogeneous $b_1, b_2 \in B$ we have

$$(11) \quad h(b_1 b_2) = h(b_1)\Delta(b_2) + (-1)^{|b_1|}(\tau\Delta)(b_1)h(b_2).$$

Also, h induces a map $\bar{B} \rightarrow \bar{B} \otimes \bar{B}$.

The induced map will still be denoted by h .

Proof Using the fact that both Δ and $\tau\Delta$ are algebra morphisms, one rewrites the definition of h as

$$\begin{aligned} h(ae_{x_1} \cdots e_{x_n}) \\ = \sum_{i=1}^n (-1)^{i-1} (a \otimes a)(\tau\Delta)(e_{x_1}) \cdots (\tau\Delta)(e_{x_{i-1}})(e_{x_i} \otimes e_{x_i})\Delta(e_{x_{i+1}}) \cdots \Delta(e_{x_n}). \end{aligned}$$

This immediately yields (11) on $b_1 = a_1 e_{x_1} \cdots e_{x_p}$ (ie any monomial in B) and $b_2 = e_{x_{p+1}} \cdots e_{x_{p+q}}$. To check (11) on general monomials b_1 and $b'_2 = a_2 b_2$, with

$a_2 \in \langle X \rangle$ and b_2 a product of the e_x , one observes that the maps h , Δ , and $\tau\Delta$ are X -equivariant both on the left and on the right, which gives

$$\begin{aligned} h(b_1(a_2b_2)) &= h((b_1a_2)b_2) = h(b_1a_2)\Delta(b_2) + (-1)^{|b_1a_2|}(\tau\Delta)(b_1a_2)h(b_2) \\ &= h(b_1)(a_2 \otimes a_2)\Delta(b_2) + (-1)^{|b_1|}(\tau\Delta)(b_1)(a_2 \otimes a_2)h(b_2) \\ &= h(b_1)\Delta(a_2b_2) + (-1)^{|b_1|}(\tau\Delta)(b_1)h(a_2b_2). \end{aligned}$$

Finally, h survives in the quotient \bar{B} since its X -equivariance reads

$$h(ae_{x_1} \cdots e_{x_n}) = (a \otimes a)h(e_{x_1} \cdots e_{x_n}) \sim h(e_{x_1} \cdots e_{x_n}),$$

with the usual notation. □

For example, an easy computation gives

$$h(e_x e_y) = (x e_y + e_x) \otimes e_x e_y - e_x e_y \otimes (e_x y + e_y),$$

which in \bar{B} becomes

$$h(e_x e_y) = (e_y + e_x) \otimes e_x e_y - e_x e_y \otimes (e_x y + e_y).$$

Proposition 20 *The map h is a homotopy between Δ and $\tau\Delta$:*

$$(12) \quad (d \otimes \text{Id}_B + \text{Id}_B \otimes d)h + hd = \Delta - \tau\Delta.$$

Of course, h remains a homotopy in \bar{B} as well.

Proof We will use the shorthand notation $dh := (d \otimes \text{Id}_B + \text{Id}_B \otimes d)h$.

If x is a degree zero generator of B then $(dh + hd)(x) = 0$ and $\Delta(x) = x \otimes x = (\tau\Delta)(x)$, hence (12) holds. Now, for a degree one generator e_x we have

$$\begin{aligned} (dh + hd)(e_x) &= d(e_x \otimes e_x) + 0 = (1 - x) \otimes e_x - e_x \otimes (1 - x) \\ &= -x \otimes e_x + e_x \otimes x + 1 \otimes e_x - e_x \otimes 1 = (\Delta - \tau\Delta)(e_x). \end{aligned}$$

The proof can then be carried out by induction on the degree, using (11):

$$\begin{aligned} (dh + hd)(b_1b_2) &= d(h(b_1)\Delta(b_2) + (-1)^{|b_1|}(\tau\Delta)(b_1)h(b_2)) + h(db_1 \cdot b_2 + (-1)^{|b_1|}b_1 \cdot db_2) \\ &= dh(b_1)\Delta(b_2) + (-1)^{|b_1|+1}h(b_1)d\Delta(b_2) + (-1)^{|b_1|}d(\tau\Delta)(b_1)h(b_2) \\ &\quad + (\tau\Delta)(b_1)dh(b_2) + hd(b_1)\Delta(b_2) + (-1)^{|b_1|+1}(\tau\Delta)(db_1)h(b_2) \\ &\quad \quad \quad + (-1)^{|b_1|}h(b_1)\Delta(db_2) + (\tau\Delta)(b_1)hd(b_2) \\ &= (\Delta - \tau\Delta)(b_1)\Delta(b_2) + (\tau\Delta)(b_1)(\Delta - \tau\Delta)(b_2) \\ &= \Delta(b_1)\Delta(b_2) - (\tau\Delta)(b_1)(\tau\Delta)(b_2) = (\Delta - \tau\Delta)(b_1b_2). \end{aligned} \quad \square$$

Theorem 21 For a shelf X , the map h induces a homotopy between the cup product \smile and its opposite version $\smile^{op} := \smile\tau$ on $C^\bullet(X)$ and $C^\bullet(X, M(X))$.

Using a standard argument we obtain an elementary algebraic proof of the commutativity of the cup product on the rack cohomologies $H_R(X)$ and $H_R(X, M(X))$. The same result holds for the more general cohomologies $H_R(X, k)$ and $H_R(X, kM(X))$.

Proof The cup product of two cochains f and g is given by the convolution product

$$f \smile g = \mu(f \otimes g)\Delta,$$

where the coproduct Δ is taken in \bar{B} , and μ is the multiplication in \mathbb{Z} . Hence for any homogeneous $x \in \bar{B}$ of degree $|f| + |g|$ we have

$$\begin{aligned} (f \smile g - (-1)^{|f||g|}g \smile f)(x) &= \sum_{(x)} (-1)^{|f||g|} f(x_1)g(x_2) - g(x_1)f(x_2) \\ &= \sum_{(x)} (-1)^{|x_1||x_2|} f(x_1)g(x_2) - f(x_2)g(x_1) \\ &= \mu(f \otimes g)(\Delta - \tau\Delta)(x) \\ &= \mu(f \otimes g)(hd + dh)(x). \end{aligned}$$

We use Sweedler’s notation for $\Delta(x)$. Hence $H : \text{Hom}_A(B, \mathbb{Z})^{\otimes 2} \rightarrow \text{Hom}_A(B, \mathbb{Z})$ defined by

$$H(f \otimes g) := \mu(f \otimes g)h$$

is a homotopy between \smile and \smile^{op} . The proof for the cohomology with coefficients in M is similar. □

7 Rack cohomology is a Zinbiel algebra

To better understand the coproduct Δ on $\bar{B}(X)$, we now refine it to an (almost) *dg codendriform structure*. That is, in positive degree it decomposes as $\Delta = \overleftarrow{\Delta} + \overrightarrow{\Delta}$, the two parts $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$ being compatible. Moreover, we establish the relation $\overleftarrow{\Delta} = \tau\overrightarrow{\Delta}$ (where τ is as usual the signed flip), up to an explicit homotopy \bar{h} . This homotopy is inspired by the homotopy h from Section 6, and is, to our knowledge, new. We thus recover the *Zinbiel product* on rack cohomology, first described by Covez in [7].

Coalgebras need not be unital in this section. General definitions are given over a unital commutative ring k ; in particular, all the tensor products are taken over k here.

Definition 22 A graded coalgebra $(C = \bigoplus_{i \geq 0} C_i, \Delta)$ is called *+–codendriform* if there exist two maps of degree 0 on its positive-degree part $C^+ = \bigoplus_{i \geq 1} C_i$, denoted by $\overleftarrow{\Delta}: C^+ \rightarrow C^+ \otimes C$ and $\overrightarrow{\Delta}: C^+ \rightarrow C \otimes C^+$, satisfying

$$(13) \quad (\overleftarrow{\Delta} \otimes \text{Id})\overleftarrow{\Delta} = (\text{Id} \otimes \Delta)\overleftarrow{\Delta},$$

$$(14) \quad (\text{Id} \otimes \overrightarrow{\Delta})\overrightarrow{\Delta} = (\Delta \otimes \text{Id})\overrightarrow{\Delta},$$

$$(15) \quad (\text{Id} \otimes \overleftarrow{\Delta})\overrightarrow{\Delta} = (\overrightarrow{\Delta} \otimes \text{Id})\overleftarrow{\Delta},$$

and where Δ decomposes as $\overleftarrow{\Delta} + \overrightarrow{\Delta}$ on C^+ and Δ is coassociative on C_0 . It is called *+–co-Zinbiel* if moreover $\overleftarrow{\Delta} = \tau \overrightarrow{\Delta}$, where τ is the signed flip. A *dg +–codendriform* or *+–co-Zinbiel coalgebra* carries in addition a differential d satisfying

$$(16) \quad \overleftarrow{\Delta}d = (d \otimes \text{Id})\overleftarrow{\Delta} + (\text{Id} \otimes d)\overleftarrow{\Delta} \quad \text{on } \bigoplus_{i \geq 2} C_i,$$

$$(17) \quad \overrightarrow{\Delta}d = (d \otimes \text{Id})\overrightarrow{\Delta} + (\text{Id} \otimes d)\overrightarrow{\Delta} \quad \text{on } \bigoplus_{i \geq 2} C_i,$$

$$(18) \quad \Delta d = (d \otimes \text{Id})\overleftarrow{\Delta} + (\text{Id} \otimes d)\overrightarrow{\Delta} \quad \text{on } C_1.$$

Dually, one defines (dg) *+–dendriform* and *+–Zinbiel* algebras.

In the case when the 0–degree part C_0 is empty, one recovers the familiar (co)dendriform and (co)Zinbiel structures. One can play with this idea further, and extend a positively graded codendriform coalgebra $(C^+, \overleftarrow{\Delta}^+, \overrightarrow{\Delta}^+)$ by a unit: $C := C^+ \oplus k1$, with $\Delta(1) = \overleftarrow{\Delta}(1) = \overrightarrow{\Delta}(1) = 1 \otimes 1$, and $\overleftarrow{\Delta}(c) = \overleftarrow{\Delta}^+(c) + c \otimes 1$ and $\overrightarrow{\Delta}(c) = \overrightarrow{\Delta}^+(c) + 1 \otimes c$ for all $c \in C$. One can also go in the opposite direction:

Lemma 23 Let $(C, \Delta, \overleftarrow{\Delta}, \overrightarrow{\Delta})$ be a +–codendriform coalgebra. Denote by $\varepsilon: C \rightarrow C^+$ and $\iota: C^+ \rightarrow C$ the obvious projection and inclusion, where $C^+ := \bigoplus_{i > 0} C_i$. Put $\Delta^+ := (\varepsilon \otimes \varepsilon)\Delta$, $\overleftarrow{\Delta}^+ := (\varepsilon \otimes \varepsilon)\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}^+ := (\varepsilon \otimes \varepsilon)\overrightarrow{\Delta}$. Then $(C^+, \overleftarrow{\Delta}^+, \overrightarrow{\Delta}^+)$ is a codendriform coalgebra.

The proof is straightforward. These observations explain our choice of the name. In the literature there exist alternative approaches to such “almost codendriform” structures. Finally, one easily checks that a +–codendriform structure refines a coassociative one:

Lemma 24 In a (dg) +–codendriform coalgebra, the coproduct Δ is necessarily coassociative. It is also compatible with the differential: writing $\Delta(b) = \sum_i a_i \otimes c_i$ for a given $b \in C$, with all the a_i homogeneous, one has

$$\Delta(d(b)) = \sum_i d(a_i) \otimes c_i + \sum_i (-1)^{|a_i|} a_i \otimes d(c_i).$$

Let us now return to shelves and their associated dg bialgebras.

Proposition 25 *Let X be a shelf. Define two maps $\overleftarrow{\Delta}: B(X)^+ \rightarrow B(X)^+ \otimes B(X)$ and $\overrightarrow{\Delta}: B(X)^+ \rightarrow B(X) \otimes B(X)^+$ by*

$$\begin{aligned} \overleftarrow{\Delta}(ae_{x_1} \cdots e_{x_n}) &= (ae_{x_1} \otimes ax_1)\Delta(e_{x_2} \cdots e_{x_n}), \\ \overrightarrow{\Delta}(ae_{x_1} \cdots e_{x_n}) &= (a \otimes ae_{x_1})\Delta(e_{x_2} \cdots e_{x_n}), \end{aligned}$$

where as usual we use the canonical form of monomials in $B(X)$, and extend this definition by linearity. These maps and the coproduct Δ yield a $+-$ codendriform structure on $B(X)$.

Proof Put $\Delta^2 = (\Delta \otimes \text{Id})\Delta = (\text{Id} \otimes \Delta)\Delta$. Then both sides of (13) act on a canonical monomial as follows:

$$ae_{x_1} \cdots e_{x_n} \mapsto (ae_{x_1} \otimes ax_1 \otimes ax_1)\Delta^2(e_{x_2} \cdots e_{x_n}).$$

Similarly, both sides of (14) and (15) act by

$$ae_{x_1} \cdots e_{x_n} \mapsto (a \otimes a \otimes ae_{x_1})\Delta^2(e_{x_2} \cdots e_{x_n})$$

and

$$ae_{x_1} \cdots e_{x_n} \mapsto (a \otimes ae_{x_1} \otimes ax_1)\Delta^2(e_{x_2} \cdots e_{x_n}),$$

respectively. Thus our maps satisfy relations (13)–(15). Finally, in positive degree their sum clearly yields Δ , and in degree 0 the coproduct Δ is coassociative. \square

The maps above are not compatible with the differential in general, since

$$\begin{aligned} \overleftarrow{\Delta}d(e_x e_y) &= e_y \otimes y - x e_y \otimes xy - e_x \otimes x + e_x y \otimes xy, \\ (d \otimes \text{Id})\overleftarrow{\Delta} + (\text{Id} \otimes d)\overleftarrow{\Delta}(e_x e_y) &= e_y \otimes xy - x e_y \otimes xy - e_x \otimes x + e_x y \otimes xy \\ &\quad + 1 \otimes x e_y - x \otimes x e_y. \end{aligned}$$

As usual, the solution is to work in the quotient $\overline{B}(X)$. Indeed, $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$ descend to maps $\overline{B}(X)^+ \rightarrow \overline{B}(X)^+ \otimes \overline{B}(X)$ and $\overline{B}(X)^+ \rightarrow \overline{B}(X) \otimes \overline{B}(X)^+$, still denoted by $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$, and one has:

Proposition 26 *The induced maps $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$ make $\overline{B}(X)$ a dg $+-$ codendriform coalgebra.*

Proof Recall the interpretation (6) of \overline{B} as the quotient of B by $xb \sim b$ for all $x \in X$ and $b \in B$. This yields that the maps $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$ are symmetric in \overline{B} :

$$\overleftarrow{\Delta}(e_{x_1} \cdots e_{x_n}) = (e_{x_1} \otimes 1)\Delta(e_{x_2} \cdots e_{x_n}), \quad \overrightarrow{\Delta}(e_{x_1} \cdots e_{x_n}) = (1 \otimes e_{x_1})\Delta(e_{x_2} \cdots e_{x_n}).$$

Also, it turns (5) into

$$(19) \quad d(e_{x_1} \cdots e_{x_n}) = -e_{x_1} d(e_{x_2} \cdots e_{x_n}).$$

We can now establish relation (16):

$$\begin{aligned} \tilde{\Delta}d(e_{x_1} \cdots e_{x_n}) &= \tilde{\Delta}(-e_{x_1} d(e_{x_2} \cdots e_{x_n})) = -(e_{x_1} \otimes 1)\Delta d(e_{x_2} \cdots e_{x_n}) \\ &= -(e_{x_1} \otimes 1)(d \otimes \text{Id})\Delta(e_{x_2} \cdots e_{x_n}) - (e_{x_1} \otimes 1)(\text{Id} \otimes d)\Delta(e_{x_2} \cdots e_{x_n}) \\ &= (d \otimes \text{Id})(e_{x_1} \otimes 1)\Delta(e_{x_2} \cdots e_{x_n}) + (\text{Id} \otimes d)(e_{x_1} \otimes 1)\Delta(e_{x_2} \cdots e_{x_n}) \\ &= (d \otimes \text{Id} + \text{Id} \otimes d)\tilde{\Delta}(e_{x_1} \cdots e_{x_n}). \end{aligned}$$

Relation (17) is proved similarly. Finally, relation (18) follows from

$$\Delta d = (d \otimes \text{Id} + \text{Id} \otimes d)\Delta$$

in degree 1. □

Proposition 27 Define the map $\bar{h}: B(X) \rightarrow B(X) \otimes B(X)$ by $\bar{h}(a) = 0$ and

$$\bar{h}(ae_{x_1} \cdots e_{x_n}) = -(ax_1 \otimes ae_{x_1})h(e_{x_2} \cdots e_{x_n}).$$

It induces a map $\bar{B} \rightarrow \bar{B} \otimes \bar{B}$, still denoted by \bar{h} , which is a homotopy between $\vec{\Delta}$ and $\tau\tilde{\Delta}$.

Proof The map \bar{h} clearly descends to \bar{B} . For this induced map, one has

$$\bar{h}(e_{x_1} \cdots e_{x_n}) = -(1 \otimes e_{x_1})h(e_{x_2} \cdots e_{x_n}).$$

It remains to check the relation

$$(d \otimes \text{Id}_{\bar{B}} + \text{Id}_{\bar{B}} \otimes d)\bar{h} + \bar{h}d = \tau\tilde{\Delta} - \vec{\Delta}: \bar{B}^+ \rightarrow \bar{B} \otimes \bar{B}.$$

Using (19), one computes

$$\begin{aligned} (d \otimes \text{Id})\bar{h}(e_{x_1} \cdots e_{x_n}) &= -(d \otimes \text{Id})(1 \otimes e_{x_1})h(e_{x_2} \cdots e_{x_n}) \\ &= (1 \otimes e_{x_1})(d \otimes \text{Id})h(e_{x_2} \cdots e_{x_n}), \end{aligned}$$

$$\begin{aligned} (\text{Id} \otimes d)\bar{h}(e_{x_1} \cdots e_{x_n}) &= -(\text{Id} \otimes d)(1 \otimes e_{x_1})h(e_{x_2} \cdots e_{x_n}) \\ &= (1 \otimes e_{x_1})(\text{Id} \otimes d)h(e_{x_2} \cdots e_{x_n}), \end{aligned}$$

$$\bar{h}d(e_{x_1} \cdots e_{x_n}) = -\bar{h}(e_{x_1} d(e_{x_2} \cdots e_{x_n})) = (1 \otimes e_{x_1})hd(e_{x_2} \cdots e_{x_n}).$$

The sum yields

$$\begin{aligned}
 (1 \otimes e_{x_1})((d \otimes \text{Id} + \text{Id} \otimes d)h + hd)(e_{x_2} \cdots e_{x_n}) & \\
 &= (1 \otimes e_{x_1})(\Delta - \tau\Delta)(e_{x_2} \cdots e_{x_n}) \\
 &= (1 \otimes e_{x_1})\Delta(e_{x_2} \cdots e_{x_n}) - \tau((e_{x_1} \otimes 1)\Delta(e_{x_2} \cdots e_{x_n})) \\
 &= \vec{\Delta}(e_{x_1}e_{x_2} \cdots e_{x_n}) - \tau\overleftarrow{\Delta}(e_{x_1}e_{x_2} \cdots e_{x_n}),
 \end{aligned}$$

as desired. □

As usual, using Lemma 9 one deduces from Proposition 25 a +–dendriform structure on the complex defining rack cohomology, and from Proposition 27 a +–Zinbiel product on the rack cohomology. Lemma 23 then yields dendriform and Zinbiel structures in positive degree:

Theorem 28 *For a shelf X , the complex $(\bigoplus_{n \geq 1} C^n(X), \partial^*)$ admits a dendriform structure, which is Zinbiel up to a homotopy induced by \vec{h} . The rack cohomology of X thus receives a strictly Zinbiel product.*

Remark 29 The dendriform structures above are not surprising. In [16; 17], rack cohomology is interpreted in terms of quantum shuffle algebras, which are key examples of dendriform structures. The shuffle interpretation generalizes to the cohomology of solutions to the Yang–Baxter equation, where dendriform structures reappear as well. The Zinbiel structure in cohomology is on the contrary remarkable, and does not extend to the YBE setting. Shuffles also suggest that, for $B(X)^+$, the codendriform structure and the associative product are compatible, in the sense of [25]. However, this does not seem to yield Zinbiel-coassociative structures on rack cohomologies. If we choose to work without coefficients (ie in $\bar{B}(X)^+$), the dendriform structure is compatible with the differential but the coproduct is lost. If we take coefficients $kM(X)$ (ie we work in $B(X)^+$), where k is a field and X is finite, the coproduct is preserved but the dendriform structure does not survive in cohomology.

8 Quandle cohomology inside rack cohomology

If X is a spindle (eg a quandle), then the complex $C_\bullet(X, k)$ has a *degenerate subcomplex*

$$C_\bullet^D(X, k) = \langle x^2 : x \in X \rangle.$$

In other words, it is the linear envelope of all monomials with repeating neighbors. Carter et al [3] defined the *quandle (co)homology* of X via the complexes

$$C_{\bullet}^Q(X, k) := C_{\bullet}(X, k) / C_{\bullet}^D(X, k) \quad \text{and} \quad C_{\bullet}^Q(X, k) := \text{Hom}(C_{\bullet}^Q(X, k), k).$$

Litherland and Nelson [18] showed that in this case the complex $C_{\bullet}(X, k)$ splits:

$$C = C^N \oplus C^D.$$

The quandle (co)homology is then the (co)homology of the complement C^N . We will now show that this decomposition is already visible at the level of the dg bialgebra $B(X)$. Moreover, in the bialgebraic setting it will be particularly easy to prove that

- the Zinbiel product on rack cohomology induces one on quandle cohomology but does not restrict to quandle cohomology,
- the associative cup product on rack cohomology restricts to quandle cohomology.

Proposition 30 *Let X be a spindle. In $B(X)$, consider the ideal*

$$B^D(X) := \langle e_x^2 : x \in X \rangle,$$

and the left sub- $A(X)$ -module $B^N(X)$ generated by the elements 1 and

$$(20) \quad (e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n} \quad \text{where } n \geq 1 \text{ and all } x_i \in X.$$

Then B decomposes as a dg A -bimodule:

$$(21) \quad B(X) = B^N(X) \oplus B^D(X).$$

Moreover B^D is a coideal and B^N is a left coideal and a left codendriform coideal of B .

Proof The expression (20) vanishes when $x_i = x_{i+1}$ for some i . Moreover, one has

$$e_{x_1} \cdots e_{x_n} = (e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n} + \text{terms from } B^D.$$

This implies the decomposition (21) of abelian groups.

The subspaces B^N and B^D are homogeneous, and for any $y \in X$ one has

$$(e_{x_1} - e_{x_2}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n}y = y(e_{x_1^y} - e_{x_2^y}) \cdots (e_{x_{n-1}^y} - e_{x_n^y})e_{x_n^y}.$$

So, B^N and B^D are graded sub- A -bimodules of B .

Let us now check that B^D is a differential coideal. Indeed, using the property $x^x = x$ of a spindle, one computes

$$(22) \quad d(e_x^2) = d(e_x)e_x - e_x d(e_x) = (1-x)e_x - e_x(1-x) = e_x - x e_x - e_x + x e_x = 0,$$

$$(23) \quad \Delta(e_x^2) = e_x^2 \otimes x^2 + 1 \otimes e_x^2 + e_x \otimes x e_x - e_x \otimes x e_x = e_x^2 \otimes x^2 + 1 \otimes e_x^2.$$

To check that B^N is a subcomplex of B , we need its alternative description:

Lemma 31 $B^N(X)$ is the left sub- $A(X)$ -module generated by the elements 1 and

$$(24) \quad (e_{x_1} - e_{y_1})(e_{x_2} - e_{y_2}) \cdots (e_{x_{n-1}} - e_{y_{n-1}})e_{x_n} \quad \text{where } n \geq 1 \text{ and all } x_i, y_i \in X.$$

Proof It is sufficient to represent an element of the form (24) as a linear combination of elements of the form (20). This can be done inductively using the following observation:

$$\begin{aligned} &(e_x - e_y)(e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n} \\ &= (e_x - e_{x_1})(e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n} \\ &\quad - (e_y - e_{x_1})(e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n}. \quad \square \end{aligned}$$

Now, for $a \in A$ and $x_1, \dots, x_n \in X$, we have

$$\begin{aligned} &d(a(e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n}) \\ &= ad(e_{x_1} - e_{x_2})(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n} \\ &\quad - a(e_{x_1} - e_{x_2})d((e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n}) \\ &= a(x_2 - x_1)(e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n} \\ &\quad - a(e_{x_1} - e_{x_2})d((e_{x_2} - e_{x_3}) \cdots (e_{x_{n-1}} - e_{x_n})e_{x_n}). \end{aligned}$$

An inductive argument using the lemma shows that this lies in B^N .

It remains to prove that Δ , $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$ send B^N to $B \otimes B^N$. In degree 0 everything is clear. In higher degree, from

$$\Delta(e_x - e_y) = e_x \otimes x - e_y \otimes y + 1 \otimes (e_x - e_y)$$

one sees that any of Δ , $\overleftarrow{\Delta}$ and $\overrightarrow{\Delta}$ sends an expression of the form (24) to a linear combination of tensor products, where on the right one has a product of terms of the form z , $e_x - e_y$ and possibly an e_u at the end. By Lemma 31, all these right parts lie in B^N . □

The proposition describes all the structure inherited from B by B^D and B^N . Indeed,

- B^D is not a subcoalgebra, as follows from (23),

- B^D is not a coideal in the dendriform sense, since

$$(25) \quad \overleftarrow{\Delta}(e_x^2) = e_x^2 \otimes x^2 + e_x \otimes xe_x,$$

- B^N is not a subalgebra of B , since $e_x \in B^N$ for any $x \in X$, whereas $e_x^2 \in B^D$,
- B^N is not a subcoalgebra either, since one has

$$(26) \quad \Delta((e_x - e_y)e_y) = e_y^2 \otimes (x - y)y + \text{terms from } B^N \otimes B^N,$$

and $e_y^2 \otimes (x - y)y$ is a nonzero term from $B^D \otimes B^N$ in general.

In particular, there is no natural way to define a codendriform structure on B^N . Passing to the quotient \bar{B} does not solve this problem: \bar{B}^D is still not a codendriform coideal because of (25), and \bar{B}^N is not a subcoalgebra of \bar{B} . Indeed, even if (26) implies $\Delta((e_x - e_y)e_y) \in \bar{B}^N \otimes \bar{B}^N$, things go wrong in degree 3, since

$$\Delta((e_x - e_y)(e_y - e_z)e_z) = e_z^2 \otimes (e_{xY} - e_{xZ} - e_Y + e_{YZ}) + \text{terms from } \bar{B}^N \otimes \bar{B}^N,$$

where $X = x^z$ and $Y = y^z$. One gets a term from $\bar{B}^D \otimes \bar{B}^N$ which does not vanish in general. However, since $e_{xY} = e_X = e_{xZ}$ and $e_{YZ} = e_Y$ modulo the boundary, this term disappears in homology. More generally:

Proposition 32 *Let X be a spindle. The homology $\bar{H}(X)$ of $\bar{B}(X)$ decomposes as a graded abelian group:*

$$(27) \quad \bar{H}(X) = \bar{H}^N(X) \oplus \bar{H}^D(X).$$

If k is a field, then one obtains a decomposition

$$(28) \quad \bar{H}(X, k) = \bar{H}^N(X, k) \oplus \bar{H}^D(X, k),$$

with \bar{H}^D a coassociative coideal and \bar{H}^N a co-Zinbiel (and hence coassociative) coalgebra.

Dually, the cohomology $\bar{H}^\bullet(X)$ of $\bar{B}(X)^*$ decomposes as

$$(29) \quad \bar{H}^\bullet(X) = \bar{H}_N^\bullet(X) \oplus \bar{H}_D^\bullet(X),$$

with \bar{H}_D^\bullet a Zinbiel (and hence associative) ideal, and \bar{H}_N^\bullet an associative subalgebra of \bar{H}^\bullet . The same holds for $\bar{H}^\bullet(X, k)$.

Proof Proposition 30 yields the desired decompositions, and, together with Propositions 4 and 26, shows that \bar{H}^D is a coideal and \bar{H}^N a left codendriform coideal. In particular,

$$\overleftarrow{\Delta}((\bar{H}^N)^+) \subseteq (\bar{H}^N)^+ \otimes \bar{H}^N \oplus (\bar{H}^D)^+ \otimes \bar{H}^N$$

and

$$\vec{\Delta}((\bar{H}^N)^+) \subseteq \bar{H}^N \otimes (\bar{H}^N)^+ \oplus \bar{H}^D \otimes (\bar{H}^N)^+.$$

But Proposition 27 yields the relation $\vec{\Delta} = \tau \overleftarrow{\Delta}$ in homology, hence the terms in $(\bar{H}^D)^+ \otimes \bar{H}^N$ and $\bar{H}^D \otimes (\bar{H}^N)^+$ above must be trivial. This shows that \bar{H}^N is in fact a co-Zinbiel coalgebra.

The proof for the cohomology \bar{H}^\bullet is analogous. □

Again, this proposition describes all the structure inherited by $\bar{H}^D(X, k)$: it is neither a subcoalgebra, nor a codendriform coideal. Indeed, computations (23) and (25) still yield counterexamples, since e_x^2 and e_x represent nontrivial classes in \bar{H}^D and \bar{H}^N , respectively.

Now, in order to understand what our proposition means for quandle cohomology, we need to recall Lemma 9 and observe that the construction of B^D precisely repeats that of the degenerate complex. This yields:

Lemma 33 *For any spindle X , one has isomorphisms of complexes*

$$(C_\bullet^Q, \partial) \cong \bar{B}^N \quad \text{and} \quad (C_\bullet^*, \partial^*) \cong \bar{B}_N^*.$$

Proposition 32 then translates as follows:

Theorem 34 *The rack cohomology of a spindle X decomposes into quandle and degenerate parts, so one has the isomorphism*

$$H_R(X) \simeq H_Q(X) \oplus H_D(X)$$

of graded abelian groups. Moreover,

- H_Q is an associative subalgebra of H_R and H_D is an associative ideal,
- H_D is a Zinbiel ideal, hence H_Q carries an induced Zinbiel product.

The situation is rather subtle here. The Zinbiel product on rack cohomology does not restrict to the quandle cohomology; to get a Zinbiel product on H_Q , we need to consider it as a quotient of H_R . However, the associative product induced by the Zinbiel product does restrict to H_Q .

9 Quandle cohomology vs rack cohomology

The rack cohomology of spindles and quandles shares a lot with the Hochschild cohomology of monoids and groups. This analogy suggests that the degenerate subcomplex C^D can be ignored, and that the rack cohomology H_R and the quandle cohomology H_Q carry the same information about a spindle. Litherland and Nelson [18] showed this is not as straightforward as that: the degenerate part is highly nontrivial, and in particular contains the entire quandle part:

$$C_{\bullet}^D \simeq C_{\bullet-1}^Q \oplus C_{\bullet}^L \quad \text{for } \bullet \geq 2.$$

Here $C_{\bullet}^L(X) := \mathbb{Z}X \otimes C_{\bullet-1}^D(X)$ is the *late degenerate subcomplex*, which is the linear envelope of all monomials with repetition at some place other than the beginning. This refines the rack cohomology splitting from Theorem 34:

$$(30) \quad H_R^{\bullet} \simeq H_Q^{\bullet} \oplus H_Q^{\bullet-1} \oplus H_L^{\bullet} \quad \text{for } \bullet \geq 2.$$

We will now recover this decomposition in our bialgebraic setting. However, our methods are not sufficient for coupling this decomposition with the algebraic structure on H_R :

Question 35 Do the cup product and the Zinbiel product on the rack cohomology of a spindle respect the decomposition (30) in any sense? In particular, can the quandle cohomology regarded as a *Zinbiel algebra* be reconstructed from the degenerate cohomology?

Now, even though H_D is big, it is degenerate in a certain sense. Indeed, Przytycki and Putyra [23] showed the quandle cohomology H_Q of a spindle to completely determine its rack cohomology H_R , and hence H_D , on the level of abelian groups. In light of the preceding section, the following question becomes particularly interesting:

Question 36 Can Zinbiel and associative structures on the rack cohomology of a spindle be recovered from the corresponding structures on its quandle cohomology?

Let us now return to our dg bialgebra $B(X)$:

Proposition 37 *Let X be a spindle. Put*

$$B^L(X) := B^+(X) \otimes_{A(X)} B^D(X),$$

where B^+ is the positive-degree part of B . One has the following isomorphism of graded $A(X)$ -bimodules:

$$(31) \quad B^\bullet_D(X) \simeq B^\bullet_L(X) \oplus B^{\bullet-1}_Q(X) \quad \text{for } \bullet \geq 2.$$

This results immediately from the following technical lemma:

Lemma 38 *Let X be a spindle. Define a map $s : B^+(X) \rightarrow B^D(X)$ as follows: take any element from B^+ written using the generators of the form x and e_y , and in each of its monomials replace the first letter of the form e_y by $e_y e_y$. Then*

- s is a well-defined injective A -bilinear map of degree 1,
- one has the decomposition of graded A -bimodules

$$B^D = B^L \oplus s((B^N)^+).$$

The map s yields the first degeneracy s_1 for the cubical structure underlying quandle cohomology, hence the notation.

Proof To show that s is well defined, one needs to check that it is compatible with the relation $e_x y = y e_{x^y}$ in B , that is, we should have $e_x e_x y = y e_{x^y} e_{x^y}$. This is indeed true:

$$e_x e_x y = e_x y e_{x^y} = y e_{x^y} e_{x^y}.$$

This map is A -bilinear and of degree 1 by construction. Injectivity becomes clear if one writes all the monomials in B in the canonical form $x_1 \cdots x_k e_{y_1} \cdots e_{y_n}$.

Further, the map $b \otimes b' \mapsto bb'$ identifies B^L with an A -invariant subspace of B^D , which is again clear using canonical forms.

Now, $s(B^+)$ and B^L are graded A -subbimodules of B^D , with $s(B^+) + B^L = B^D$ and $s(B^+) \cap B^L = s(B^D)$. As usual, this is clear using canonical forms. Since $s(B^+) = s(B^D \oplus (B^N)^+) = s(B^D) \oplus s((B^N)^+)$, we obtain the desired decomposition $B^D = B^L \oplus s((B^N)^+)$. □

As usual, decomposition (31) implies

$$(32) \quad \bar{B}^\bullet_D(X) \simeq \bar{B}^\bullet_L(X) \oplus \bar{B}^{\bullet-1}_Q(X) \quad \text{for } \bullet \geq 2,$$

with obvious notation. And as usual this decomposition respects more structure than (31):

Proposition 39 *Let X be a spindle. Then (32) is an isomorphism of differential graded $A(X)$ –bimodules.*

Proof Let us check that the differential preserves \bar{B}^L . An element of \bar{B}^L is a linear combination of (the \sim –equivalence classes of) terms of the form $e_x b$, with $b \in B^D$. Since $d(e_x b) = -e_x d(b)$ in \bar{B} by relation (6), and since d preserves B^D , we have $d(e_x b) \in \bar{B}^L$.

Now, the map s from Lemma 38 induces a map $s: \bar{B}^+ \rightarrow \bar{B}^D$. We need to check that the differential preserves $s((\bar{B}^N)^+) \simeq \bar{B}_{-1}^Q(X)$. Adapting the arguments from the proof of Proposition 30, we see that an element of $s((\bar{B}^N)^+)$ is a linear combination of the classes of terms of the form $(e_x^2 - e_y^2)b$, with $b \in B^N$. Then (22) yields

$$d((e_x^2 - e_y^2)b) = d(e_x^2 - e_y^2)b + (e_x^2 - e_y^2)d(b) = (e_x^2 - e_y^2)d(b).$$

Since d preserves B^N , we conclude $d((e_x^2 - e_y^2)b) \in s((B^N)^+)$. □

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