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A construction of pseudo-Anosov homeomorphisms using positive twists

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We introduce a construction of pseudo-Anosov homeomorphisms on n–times punctured spheres and surfaces of higher genus using only positive half-twists and Dehn twists. These constructions produce explicit examples of pseudo-Anosov maps with various number-theoretic properties associated to the stretch factors. For instance, we produce examples where the trace field is not totally real and the Galois conjugates of the stretch factor are on the unit circle. It follows that for these examples, no power of these maps can arise from either Thurston's or Penner's constructions.

15A18, 37E30, 57M07

1 Introduction

Pseudo-Anosov homeomorphisms play an important role in the study of homeomorphisms of orientable finite-type surfaces. The Nielsen–Thurston classification theorem states that, up to isotopy, every homeomorphism of a surface is either periodic, reducible, or pseudo-Anosov; see Thurston [14]. Reducible homeomorphisms can be reduced into pieces which are either finite-order or pseudo-Anosov. Therefore, understanding periodic and pseudo-Anosov homeomorphisms helps us understand arbitrary homeomorphisms. In this paper, we focus our attention on pseudo-Anosov homeomorphisms.

Despite the importance of pseudo-Anosov homeomorphisms, constructing explicit examples is difficult. Thurston [14] and Penner [10] each produced simple constructions which produce an infinite number of pseudo-Anosov maps on surfaces. In both constructions, the starting point is a pair of two filling multicurves which are used to produce pseudo-Anosov homeomorphisms.

We introduce a new construction of pseudo-Anosov homeomorphisms on punctured spheres. To prove that this construction produces maps which are different from the maps produced by the constructions of Penner and Thurston, we analyze the number-theoretic properties associated to these homeomorphisms. We also explain how to lift the construction to surfaces of higher genus using a branched cover.

1.1 The stretch factor of a pseudo-Anosov homeomorphism and number-theoretic properties

Let $S = S_{g,n}$ be a surface of genus g with n points removed from its interior. When convenient, we treat the punctures as marked points. A homeomorphism f of a finite type surface S is *pseudo-Anosov* if there is a representative homeomorphism ϕ, a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ such that $\phi(\mathcal{F}^u) = \lambda \mathcal{F}^u$ and $\phi(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$. The number λ by which ϕ stretches and contracts its foliations is called the *stretch factor* of f, and $\mathcal{F}^u$ and $\mathcal{F}^s$ are called the *unstable foliation* and *stable foliation*, respectively.

A considerable amount of research has probed the properties of the stretch factors of pseudo-Anosov maps. For example, Thurston [14] proved that the stretch factor of a pseudo-Anosov homeomorphism on a surface $S_{g,0}$ is an algebraic integer whose degree is bounded above by $6g-6$, where g is the genus of the surface.

Hubert and Lanneau [5] proved that for any pseudo-Anosov homeomorphism arising from Thurston's construction, the trace field $\mathbb{Q}(\lambda+\lambda^{-1})$ is always totally real. Shin and Strenner [13] proved that the Galois conjugates of the stretch factor for pseudo-Anosov homeomorphisms arising from Penner's construction are never on the unit circle in $\mathbb{C}$. This result from Shin and Strenner proved that it is not the case that every pseudo-Anosov homeomorphism has a power that arises from Penner's construction. This leads one to ask whether every pseudo-Anosov homeomorphism has a power which arises from either Penner's or Thurston's construction. To this end, one could ask whether it is possible for a pseudo-Anosov homeomorphism to have a trace field which is not totally real and have Galois conjugates of its stretch factor on the unit circle. The results of this paper allow us to construct examples giving a positive answer to this question.

Theorem 1.1 *Let S be either $S_{0,n}$ for $n \geq 8$, or $S_{g,k}$ for $g \geq 3$ and $k \geq 0$. Then there exists a pseudo-Anosov homeomorphism ϕ_S on S with stretch factor λ_{ϕ_S} such that*

(i) *the trace field $\mathbb{Q}(\lambda_{\phi_S}+\lambda_{\phi_S}^{-1})$ is not totally real, and*

(ii) *there exist Galois conjugates of λ_{ϕ_S} on the unit circle.*

In particular, no power of ϕ_S arises from either Penner's or Thurston's constructions.

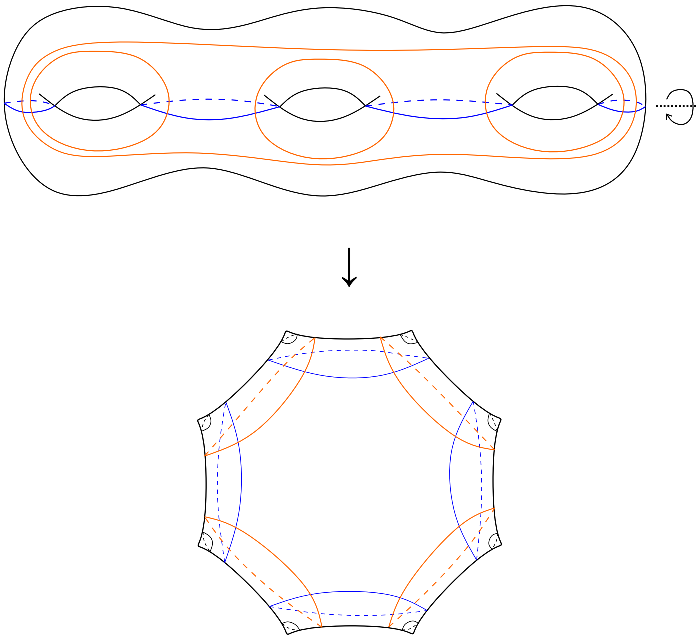

Figure 1: Two-fold branched covering map from $S_{3,0}$ to $S_{0,8}$ induced by the hyperelliptic involution.

In order to prove Theorem 1.1, we begin by constructing a family of pseudo-Anosov homeomorphisms on $S_{0,n}$ for $n \geq 6$. We then show that in this family there exists pseudo-Anosov homeomorphisms where the trace field is not totally real and where there exist Galois conjugates of its stretch factor on the unit circle. If $S_{0,n}$ is a sphere with n marked points, we can find branched covers $S_{g,m} \to S_{0,n}$. See Figure 1 for an illustration. By lifting pseudo-Anosov homeomorphisms from $S_{0,n}$ to $S_{g,m}$ through these branched covers, we obtain the other examples in Theorem 1.1. Since the lifted maps have the same stable and unstable foliations as ϕ, and the stretch factor is a power of λ, we can promote the examples from Theorem 1.4 to examples on surfaces with positive genus.

1.2 Constructing pseudo-Anosov maps on punctured spheres

Each map ϕ_S from Theorem 1.1 is a member of a new, large family of pseudo-Anosov maps on punctured spheres that we introduce. We'll see below that the stretch factors for the maps in this family exhibit various number theoretic properties, similar to the properties in Theorem 1.1.

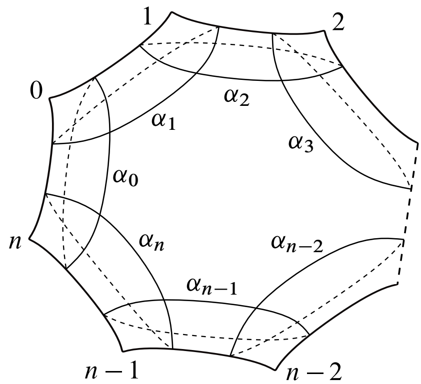

Figure 2: Labeling of the punctures on the n-times punctured sphere.

For the n–times punctured sphere $S_{0,n}$, label the punctures $0, 1, \dots, n-1$ and fix the curves $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ as shown in Figure 2 such that α_i cuts off the punctures i and $i-1 \pmod n$, and $i(\alpha_j, \alpha_{j+1}) = 2$. For each $i \in \{0, \dots, n\}$, let D_i denote the right half-twist about the curve α_i. The square of a half twist is one example of a Dehn twist.

The basis of our construction of pseudo-Anosov homeomorphisms will consist of products of the D_i's. If $\rho\colon \mathbb{Z}/n \to \mathbb{Z}/n$ is the map $j \mapsto j+1 \bmod n$ which cyclically permutes the set $\{0, \dots, n\}$, then we say a partition $\mu = \{\mu_0, \dots, \mu_k\}$ of $\{0, \dots, n-1\}$ is *evenly spaced* if $k < n$ and $\rho(\mu_i) = \mu_{i+1}$, where the indices are taken mod k. In the case of $n = 6$,

$$\mu = \{\{0,3\},\{1,4\},\{2,5\}\} \quad \text{and} \quad \bar{\mu} = \{\{0,2,4\},\{1,3,5\}\}$$

are both evenly spaced partitions of $\{0, 1, \dots, 5\}$. If $\mu = \{\mu_0, \dots, \mu_k\}$ is a partition of $\{0, \dots, n-1\}$, then for each $j \in \{0, 1, \dots, k\}$, let D_{μ_j} denote the product of half twists $\prod_{i \in \mu_j} D_i$. Note, the order of this product is irrelevant as for each $i, k \in \mu_j$, D_i commutes with D_k since the curves α_i and α_k are disjoint.

Our first construction of pseudo-Anosov homeomorphisms are products $D_{\mu_0}^{q_0} \cdots D_{\mu_k}^{q_k}$ where the set $\{\mu_0, \dots, \mu_k\}$ is an evenly spaced partition of $\{0, \dots, n-1\}$ and each $q_j \geq 2$.

Theorem 1.2 *Let $n \geq 6$, let $q_j \geq 2$ for each $j \in \{0, 1, \dots, k\}$, and let $\{\mu_0, \dots, \mu_k\}$ be an evenly spaced partition of $\{0, \dots, n-1\}$. Then*

$$\phi = \prod_{j=0}^{k} D_{\mu_j}^{q_j}$$

is a pseudo-Anosov homeomorphism of $S_{0,n}$.

Our second construction of pseudo-Anosov homeomorphisms, which will be used to produce the maps for Theorem 1.1, is an augmentation of the construction from Theorem 1.2. We say a partition $\{\mu_0, \dots, \mu_k\}$ of $\{0, 1 \dots, n-1\}$ *reduces to an evenly spaced partition* if there exists $k' \in \{0, 1, \dots, k\}$ and $n' \in \{0, 1, \dots, n-1\}$ such that $\{\mu_0, \mu_1, \dots, \mu_{k'}\}$ is an evenly spaced partition of $\{0, 1, \dots, n'-1\}$ and μ_j contains a single element for all $j > k'$.

Theorem 1.3 *Let $n \geq 7$, let $q_j \geq 2$ for each $j \in \{0, 1, \dots, k\}$, and let*

$$\{\mu_0, \dots, \mu_{k'}, \dots, \mu_k\}$$

be a partition of $\{0, 1, \dots, n-1\}$ that reduces to an evenly spaced partition. Then

$$\phi = \prod_{j=0}^{k} D_{\mu_j}^{q_j}$$

is a pseudo-Anosov homeomorphism of $S_{0,n}$.

To prove that the homeomorphisms in Theorems 1.2 and 1.3 are pseudo-Anosov, we explicitly construct the train tracks for each of the maps. For each pseudo-Anosov homeomorphism, we show that the associated train track has a specific structural form. Additionally, we show that the train track matrix associated to each map is Perron–Frobenius. Lemma 2.4 states that having both the specific form of the train track and the Perron–Frobenius train track matrix concurrently will imply that the homeomorphisms we constructed are pseudo-Anosov.

The stretch factors for the pseudo-Anosov maps arising from Theorems 1.2 and 1.3 exhibit a wide variety of number-theoretic properties. In addition to the map ϕ_S from Theorem 1.1, we produce examples of pseudo-Anosov maps with any desired combination of having totally real trace field, or not, and having Galois conjugates of the stretch factor on the unit circle, or not.

Theorem 1.4 *For any of the following four statements, there exists a pseudo-Anosov homeomorphism whose stretch factor λ satisfies the statement*:

(1) $\mathbb{Q}(\lambda + \lambda^{-1})$ *is totally real and there exists no Galois conjugates of λ on the unit circle.*

(2) $\mathbb{Q}(\lambda + \lambda^{-1})$ *is not totally real and there exist no Galois conjugates of λ on the unit circle.*

(3) $\mathbb{Q}(\lambda+\lambda^{-1})$ *is totally real and there exist Galois conjugates of* λ *on the unit circle.*

(4) $\mathbb{Q}(\lambda+\lambda^{-1})$ *is not totally real and there exist Galois conjugates of* λ *on the unit circle.*

These homeomorphisms are constructed on the surfaces $S_{0,6}$, $S_{0,7}$, $S_{0,8}$, *and* $S_{0,8}$, *respectively.*

We derive Theorem 1.1 as a corollary of Theorem 1.4. As discussed above, statement (4) of Theorem 1.4 proves Theorem 1.1 in the case of the 8–times punctured sphere. By using a branched cover to lift the homeomorphism from Theorem 1.4(4) to other surfaces, we complete the proof of Theorem 1.1.

Outline

In Section 2, we begin by proving an adaptation of the nesting lemma, which was first introduced by Masur and Minsky [8], an important lemma which allows us to determine whether a map is pseudo-Anosov. In Section 3, we will prove Theorems 1.2 and 1.3 which detail the main constructions of pseudo-Anosov homeomorphisms presented in this paper. Section 4 provides additional modifications which can be made to the main constructions in order to produce additional pseudo-Anosov homeomorphisms, and Section 5 details the various number-theoretic properties associated to the stretch factors which will allow us to show that this construction differs from the previous constructions. Lastly, Section 6 proves that the construction produces pseudo-Anosov mapping classes on surfaces of higher genus through a branched cover.

Acknowledgements

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2 The nesting lemma

In this section, we begin by covering the required background information regarding train tracks and the complex of curves. Afterwards, we prove the nesting lemma. The nesting lemma is inspired by the work of Masur and Minsky in which they show that the diameter of the cure complex is infinite. The nesting lemma in this paper allows us to determine whether a map is pseudo-Anosov by analyzing the train track associated to the map.

2.1 Train tracks

In this section, we recall some of the basic definitions for train tracks. For a thorough treatment of the topic, the author recommends *Combinatorics of train tracks* by Penner and Harer [11].

A *train track* $\tau \subset S$ is an embedded 1–complex whose vertices are called *switches* and edges are called *branches*. Branches are smooth parametrized paths, and at each switch of τ, there is a well-defined tangent space to the branches coming into the switch. The tangent vector at the switch pointing toward the edge can have two possible directions which divides the ends of edges at the switch into two sets. The end of a branch of τ which is incident on a switch is called *incoming* if the one-sided tangent vector of the branch agrees with the direction at the switch and *outgoing* otherwise. Neither the set of incoming nor the set of outgoing branches are permitted to be empty. In this paper, whether a switch is incoming or outgoing is not part of the data in the train track, ie a train track is unoriented.

The valence of each switch in τ is at least 3, except for possibly one bivalent switch in a closed curve component. Finally, we require that every complementary component of $S \setminus \tau$ has a negative generalized Euler characteristic, in particular, for a complementary component $R \in S \setminus \tau$,

$$\chi(R) - \tfrac{1}{2} V(R) < 0$$

where $\chi(R)$ is the usual Euler characteristic and $V(R)$ is the number of outward pointing cusps on $\partial(R)$.

A *train route* is a nondegenerate smooth path in τ. A train route traverses a switch only by passing from an incoming to an outgoing edge (or vice-versa). We call a train track τ *large* if every component of $S \setminus \tau$ is a polygon or a once-punctured polygon, and we call τ *generic* if all switches are trivalent.

If σ is a train track which is a subset of τ, we write $\sigma < \tau$ and say σ is a *subtrack* of τ. In this case we may also say that τ is an *extension* of σ. If there is a homotopy of S such that every train route on σ is taken to a train route on τ we say σ is *carried* on τ and write $\sigma \prec \tau$.

Let $\mathcal{B}$ denote the set of branches of τ. A nonnegative, real-valued function $\mu\colon \mathcal{B} \to \mathbb{R}_+$ is called a *transverse measure* on τ if for each switch, μ satisfies the switch condition: for any switch, the sums of μ over incoming and outgoing branches are equal.

A train track is *recurrent* if there is a transverse measure which is positive on every branch, or equivalently, if each branch is contained in a closed train route.

Let α be a simple closed curve which intersects τ. We say α intersects τ *efficiently* if $\alpha \cup \tau$ has no bigon complementary regions. A track τ is *transversely recurrent* if every branch of τ is crossed by some simple curve α intersecting τ transversely and efficiently. We call a track *birecurrent* if it is both recurrent and transversely recurrent.

Any positive scaling of a transverse measure is also a transverse measure. Therefore, the set of all transverse measures when viewed as a subset of $\mathbb{R}^{\mathcal{B}}$ is a cone over a compact polyhedron in projective space. For a recurrent train track τ, let $P(\tau)$ denote the *polyhedron of measures* supported on τ. By $\operatorname{int}(P(\tau))$ we denote the set of weights on τ which are positive on every branch. We say that σ *fills* τ if $\sigma \prec \tau$ and $\operatorname{int}(P(\sigma)) \subseteq \operatorname{int}(P(\tau))$. Similarly, a curve α *fills* τ if $\alpha \prec \tau$ and α traverses every branch of τ.

One way to obtain a transverse measure on a train track τ is as follows: Fix a reference hyperbolic metric on S. A *geodesic lamination* in S is a closed set foliated by geodesics. A geodesic lamination is *measured* if it supports a measure on arcs transverse to its leaves, which is invariant under isotopies preserving the leaves. The space of all compactly supported measured geodesic laminations on S, with suitable topology, is known as $\mathcal{ML}(S)$, and changing the reference metric on S will yield spaces which are equivalent. A geodesic lamination λ is *carried* on τ if there is a homotopy of S taking λ to a set of train routes. In such a case, λ induces a transverse measure on τ, which in turn uniquely determines λ.

In this paper, we will blur the distinction between $P(\tau)$ as a subset of $\mathcal{ML}(S)$, and as a subset of the space $\mathbb{R}_+^{\mathcal{B}}$ of nonnegative functions on the branch set $\mathcal{B}$ of τ.

Let σ be a large track. A *diagonal extension* of σ is a track κ such that $\sigma < \kappa$ and every branch of $\kappa \setminus \sigma$ is a *diagonal* of σ, ie the endpoints of each edge in $\kappa \setminus \sigma$ terminate in the corner of a complementary region of σ. Let $E(\sigma)$ denote the set of all recurrent diagonal extensions of σ. Note that it is a finite set, and let $PE(\sigma)$ denote $\bigcup_{\kappa \in E(\sigma)} P(\kappa)$. Let $\operatorname{int}(PE(\sigma))$ denote the set of measures $\mu \in PE(\sigma)$ which are positive on every branch of σ.

2.2 The linear algebra of train tracks

For each pseudo-Anosov homeomorphism ϕ, there exists a train track τ which is invariant under the action of ϕ. Under this action, ϕ changes the weights of the branches of τ in a linear way. Thus, the action of the pseudo-Anosov homeomorphism is able to be completely described by the *train track matrix*. In fact, for each pseudo-Anosov mapping class the transition matrix M is *Perron-Frobenius* and the positive eigenvector determines an invariant measure.

Definition 2.1 (Perron–Frobenius matrix) A *Perron–Frobenius matrix* is a matrix with entries $a_{i,j} \geq 0$ such that some power of the matrix has strictly positive entries.

It is also known that the eigenvalue of this eigenvector will correspond to the stretch factor of the pseudo-Anosov map [9].

Theorem 2.2 *Given a pseudo-Anosov mapping class f, there exists a train track τ invariant under the action of f such that the matrix M which determines the action on the transverse measures is Perron–Frobenius. The positive eigenvector determines an invariant measure corresponding to the invariant foliation and the eigenvalue is the stretch factor.*

2.3 Complex of curves

The complex of curves, defined by Harvey [3], is a combinatorial object which encodes the intersection patterns of simple closed curves in S_g.

Definition 2.3 (complex of curves, $C(S_{g,n})$) The *complex of curves* is an abstract simplicial complex associated to a surface S. Its 1–skeleton is given by the following data:

- **Vertices** There is one vertex of $C(S)$ for each isotopy class of essential simple closed curves in S.
- **Edges** There is an edge between any two vertices of $C(S)$ corresponding to isotopy classes a and b with $i(a,b) = 0$, ie a and b are disjoint.

We note that $C(S)$ is a *flag complex*, which means that $k+1$ vertices span a k–simplex of $C(S)$ if and only if they are pairwise connected by edges. We will only make use of the 1–skeleton of the complex of curves, and we will denote the 1–skeleton by $C(S)$. By specifying that each edge has length 1, we turn $C(S)$ into a metric space. We let $d_{C(S)}$ denote the distance function obtained by taking shortest paths.

2.4 The nesting lemma

We are now in a position where we can state and prove the nesting lemma. This lemma is one of the key steps to proving that the homeomorphisms constructed in this paper are pseudo-Anosov.

Lemma 2.4 (the nesting lemma) *Let τ be a large, generic, birecurrent train track. Let $\phi\colon S \to S$ be a map such that τ is carried by ϕ. If the matrix associated to τ is a Perron–Frobenius matrix, then ϕ is a pseudo-Anosov map.*

Before we prove this lemma, we will state some lemmas from Masur and Minsky [8]. Let μ be a measured lamination on a surface S. We define $N(\tau)$ to be the union of $E(\sigma)$ over all large, recurrent subtracks $\sigma < \tau$, and we define $PN(\tau) = \bigcup_{\kappa \in N(\tau)} P(\kappa)$. The following lemma provides a sufficient condition for when μ is contained in $\operatorname{int}(PE(\sigma))$.

Lemma 2.5 [8, Lemma 4.1] *There exists $\delta > 0$ (depending only on S) for which the following holds. Let $\sigma < \tau$ where σ is a large track. If $\mu \in P(\tau)$ and, for every branch b of $\tau \setminus \sigma$ and b' of σ, $\mu(b) < \delta\mu(b')$, then σ is recurrent and $\mu \in \operatorname{int}(PE(\sigma))$.*

Let σ and τ be two large recurrent tracks such that $\sigma \prec \tau$. In this case we say that the train tracks σ and τ are *nested*. The following two lemmas tell us that when we have train tracks which are nested, their diagonal extensions are also nested in a suitable sense. Additionally, these lemmas tell us that the way in which the diagonal branches cover each other is controlled.

Lemma 2.6 [8, Lemma 4.2] *Let σ and τ be large recurrent tracks, and suppose $\sigma \prec \tau$. If σ fills τ, then $PE(\sigma) \subseteq PE(\tau)$. Even if σ does not fill τ, we have $PN(\sigma) \subseteq PN(\tau)$.*

Lemma 2.7 [8, Lemma 4.3] *Let $\sigma \prec \tau$ where σ is a large recurrent track, and let $\sigma' \in E(\sigma)$ and $\tau' \in E(\tau)$ be such that $\sigma' \prec \tau'$. Then any branch b of $\tau' \setminus \tau$ is traversed by branches of σ' with degree at most m_0, a number which depends only on S.*

The final lemma we require from Masur and Minsky is one which gives us a relation between nesting train tracks and their distance in the complex of curves.

Lemma 2.8 [8, Lemma 4.4] *Let α and β be simple, nonperipheral closed curves in a surface S. If σ is a large birecurrent train track and $\alpha \in \operatorname{int}(PE(\sigma))$ (ie α is carried on the maximal train track σ so that it is carried on every branch), then*

$$d_{\mathcal{C}(S)}(\alpha, \beta) \leq 1 \implies \beta \in PE(\sigma).$$

In other words,

$$\mathcal{N}_1(\operatorname{int}(PE(\sigma))) \subset PE(\sigma),$$

where $\mathcal{N}_1$ denotes a radius 1 neighborhood in $\mathcal{C}(S)$.

Notice that since there exist simple closed curves on the surface S which are also in $\operatorname{int}(PE(\sigma))$, it means that we can consider a radius 1 neighborhood of $\operatorname{int}(PE(\sigma))$ in $C(S)$.

Before we prove Lemma 2.4, we give a brief outline. Given the train track τ, we consider a measure μ which is positive on each branch. Since $\phi(\tau)$ fills τ, it will follow that there exists some $k \in \mathbb{N}$ such that $\phi^k(PE(\tau)) \subset \operatorname{int}(PE(\tau))$. From this, it follows that $\phi^{jk}(\tau)$ fills $\phi^{(j-1)k}(\tau)$, which will imply that $PE(\phi^{jk}(\tau)) \subset \operatorname{int}(PE(\phi^{(j-1)k}(\tau)))$ for any j. At this point, we will suppose for the sake of contradiction that ϕ is not a pseudo-Anosov map. This would imply that there exists a curve α on the surface S disjoint from τ such that there exists an $m \in \mathbb{N}$ such that $\phi^m(\alpha) = \alpha$. Using Lemma 2.8 and the fact that $PE(\phi^{jk}(\tau)) \subset \operatorname{int}(PE(\phi^{(j-1)k}(\tau)))$ for any j, we will find that $d_{C(S)}(\alpha, \phi^{jk}(\alpha)) \to \infty$, which contradicts that there exists some m such that $\phi^m(\alpha) = \alpha$. Therefore ϕ is a pseudo-Anosov map.

Proof of Lemma 2.4 Let μ be a measure that is positive on every branch of $\phi(\tau)$, ie $\mu \in \operatorname{int}(P(\phi(\tau)))$. Since $\phi(\tau)$ preserves τ, μ is a measure that is positive on any branch of τ, and we have $\operatorname{int}(P(\phi(\tau))) \subseteq \operatorname{int}(P(\tau))$. This implies that $\phi(\tau)$ fills τ.

The next paragraph follows the argument found in [8, Theorem 4.6] which shows that if $\phi(\tau)$ fills τ, then there exists some $k \in \mathbb{N}$ such that $\phi^k(PE(\tau)) \subset \operatorname{int}(PE(\tau))$.

Suppose that $\tau' \in E(\tau)$ is a diagonal extension of τ. By Lemma 2.6, $\phi(\tau')$ is carried by some $\tilde{\tau} \in E(\tau)$. There exists a constant $c_0 = c_0(S)$ such that for some $c \leq c_0$ the power $\phi' = \phi^c$ takes τ' to a train track carried by τ', since the number of train tracks in $E(\tau_0)$ is bounded in terms of the topology of S. Let $\mathcal{B}$ represent the branch set of τ', and $\mathcal{B}_\tau \subset \mathcal{B}$ represent the branch set of τ. In the coordinates of $\mathbb{R}^{\mathcal{B}}$ we may represent ϕ' as an integer matrix M, with a submatrix M_τ which gives us the restriction to $\mathbb{R}^{\mathcal{B}_\tau}$. Penner shows in [10] that M_τ^n has all positive entries, where n is the dimension $|\mathcal{B}_\tau|$.

In fact, Penner shows that $|M_\tau^n(x_\tau)| \geq 2|x_\tau|$ for any vector x_τ which represents a measure on τ. Indeed, M_τ has a unique eigenvector in the positive cone of $\mathbb{R}^{\mathcal{B}_\tau}$, which corresponds to $[(\tau,\mu)]$. On the other hand, for a diagonal branch $b \in \mathcal{B} \setminus \mathcal{B}_\tau$, Lemma 2.7 shows that $|M^i(x)| \leq m_\tau |x|$ for all $x \in \mathbb{R}^{\mathcal{B}}$ and all powers $i > 0$. Since τ is generic, we have that any transverse measure x on τ' must put a positive measure on a branch of $\mathcal{B}_\tau$. This implies that given $\delta > 0$ there exists k_1, depending only on δ and S, such that for some $k \leq k_1$ we have $\max_{b \in \mathcal{B}\setminus\mathcal{B}_\tau} \phi^k(x)(b) \leq \delta \min_{b\in\mathcal{B}_\tau} h^k(x)(b)$, for any $x \in P(\tau')$. We apply this to each $\tau' \in E(\tau)$, and by applying Lemma 2.5, we see that, for an appropriate choice of δ,

$$\phi^k(PE(\tau)) \subset \operatorname{int}(PE(\tau)).$$

Using the above argument, we find that $\phi^{jk}(\tau)$ fills $\phi^{(j-1)k}(\tau)$, from which it follows that

$$PE(\phi^{jk}(\tau)) \subset \operatorname{int}(PE(\phi^{(j-1)k}(\tau))) \tag{1}$$

for any j.

By way of contradiction, suppose that ϕ is not a pseudo-Anosov map. Then there exists a curve α on the surface S disjoint from τ such that $\phi^m(\alpha) = \alpha$ for some $m \in \mathbb{N}$.

Since there are elements of $PE(\tau)$ which are not in $\phi^k(PE(\tau))$, it is possible to find $\gamma \in C(S)$ such that $\gamma \notin PE(\tau)$ and $\phi^k(\gamma) \in PE(\tau)$. Then $\phi^{2k}(\gamma) \in \operatorname{int}(PE(\tau))$, which implies that $d_{C(S)}(\gamma, \phi^{2k}(\gamma)) \geq 1$ by Lemma 2.8.

Since $\phi^{jk}(\gamma) \in PE(\phi^{(j-1)k}(\tau))$ for $j \geq 1$, we use (1) to find

$$\begin{aligned}
\phi^{3k}(\gamma) &\in PE(\phi^{2k}(\tau)) \subset \operatorname{int}(PE(\phi^k(\tau))),\\
\phi^{3k}(\gamma) &\in PE(\phi^{k}(\tau)) \subset \operatorname{int}(PE(\tau)),\\
\phi^{3k}(\gamma) &\in PE(\tau).
\end{aligned}$$

By Lemma 2.8, we have that for any k,

$$\mathcal{N}_1(\operatorname{int}(PE(\phi^k(\tau)))) \subset PE(\phi^k(\tau)).$$

Therefore, we find that

$$\begin{aligned}
\phi^{3k}(\gamma) \in PE(\phi^{2k}(\tau)) \subset \mathcal{N}_1(\operatorname{int}(PE(\phi^k(\tau)))) &\subset PE(\phi^k(\tau))\\
&\subset \mathcal{N}_1(\operatorname{int}(PE(\tau))) \subset PE(\tau).
\end{aligned}$$

Therefore, since $\gamma \notin PE(\tau)$, we have that $d_{C(S)}(\gamma, \phi^{3k}(\gamma)) \geq 2$.

We continue inductively to show that $d_{C(S)}(\gamma, \phi^{jk}(\gamma)) \geq j-1$, which implies that as $j \to \infty$, $d_{C(S)}(\gamma, \phi^{jk}(\gamma)) \to \infty$. Since

$$\begin{aligned} d_{C(S)}(\alpha, \phi^{jk}(\alpha)) &\geq d_{C(S)}(\gamma, \phi^{jk}(\gamma)) - d_{C(S)}(\alpha, \gamma) - d_{C(S)}(h^{jk}(\alpha), h^{jk}(\gamma)) \\ &= d_{C(S)}(\gamma, \phi^{jk}(\gamma)) - 2d_{C(S)}(\alpha, \gamma), \end{aligned}$$

we have $d_{C(S)}(\alpha, \phi^{jk}(\alpha)) \to \infty$, which contradicts that there exists some m such that $\phi^m(\alpha) = \alpha$. Therefore ϕ is a pseudo-Anosov map. □

3 Main construction on *n*–times punctured spheres

In this section, we will prove Theorem 1.2, that the homeomorphisms induced by evenly spaced partitions are pseudo-Anosov, and Theorem 1.3, that the homeomorphisms induced by partitions which reduce to an evenly spaced partition are pseudo-Anosov.

We begin with the proof of Theorem 1.2.

Theorem 1.2 *Let $n \geq 6$, let $q_j \geq 2$ for each $j \in \{0, 1, \dots, k\}$, and let $\{\mu_0, \dots, \mu_k\}$ be an evenly spaced partition of $\{0, \dots, n-1\}$. Then*

$$\phi = \prod_{j=0}^{k} D_{\mu_j}^{q_j}$$

is a pseudo-Anosov homeomorphism of $S_{0,n}$.

Proof Consider the surface $S_{0,n}$ for some fixed $n \in \mathbb{N}$. Fix $k > 1$, $k \in \mathbb{N}$, and fix a partition $\mu = \{\mu_1, \dots, \mu_k\}$ of the n punctures of $S_{0,n}$ such that $\rho(\mu_{i-1}) = \rho(\mu_{(i \bmod k)})$. We will prove that

$$\phi = \prod_{i=1}^{k} D_{\mu_i}^{q_i} = D_{\mu_k}^{q_k} \cdots D_{\mu_2}^{q_2} D_{\mu_1}^{q_1}$$

is a pseudo-Anosov mapping class.

We first construct the train track τ so that $\phi(\tau)$ is carried by τ. Consider the partition $\mu = \{\mu_1, \dots, \mu_k\}$. Construct a k–valent pretrack by having a branch loop around each puncture in the set μ_k, with each of these branches meeting in the center where they are smoothly connected by a k–gon. See Figures 3, left, and 4, left, for examples which correspond to pretracks from Example A.1 of the appendix. For the remaining labeled punctures in μ_i, loop a branch around each puncture and have this branch turn left

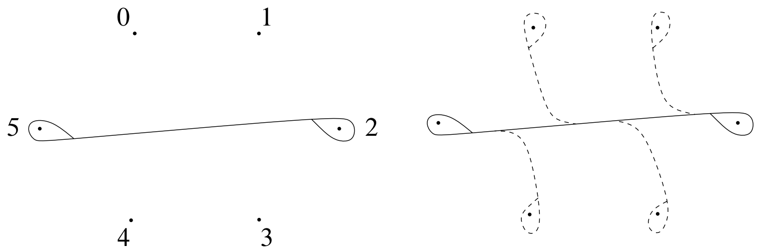

Figure 3: Constructing the train track for the map ϕ_μ from Example A.1.

towards the k–valent pretrack meeting the branch of the pretrack whose label is next in the ordering. See Figures 3, right, and 4, right, for examples which correspond to the train tracks from Example A.1. We notice that the train track has rotational symmetry of order k.

For each k, $D_{\mu_i}^{q_i}$ acts locally the same, by which we mean the following: Each half-twist in $D_{\mu_i}^{q_i}$ involves a branch located around puncture b on the k–valent pretrack and the branch located around puncture b' which is directly next to puncture b in the clockwise direction. As we consider a right half-twist to be positive, we notice that the branch at puncture b will begin to turn into the branch at puncture b', see Figures 11 and 13 for examples. Therefore, after the twist, the branch around puncture b' is now on the k–valent pretrack, and the branch around puncture b is directly next to the branch at puncture b' in the counterclockwise direction. Branches which are neither on the k–valent pretrack nor directly clockwise to the k–valent pretrack are unaffected by $D_{\mu_i}^{q_i}$. Thus, after each application of $D_{\mu_i}^{q_i}$, we arrive at a train track which is able to be obtained by a rotation of $2\pi/n$ of our starting train track. Since τ has a rotational symmetry of order k, we notice that $\phi(\tau)$ is carried by τ.

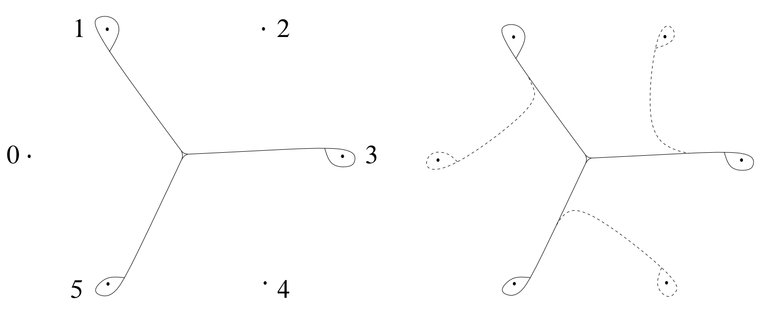

Figure 4: Constructing the train track for the map $\phi_{\bar{\mu}}$ from Example A.1.

Let M_τ denote the matrix representing the induced action of the space of weights on τ. To prove that M_τ is Perron–Frobenius, fix an initial weight on each branch. For each application of $D_{\mu_i}^{q_i}$, the labels on the k–valent pretrack and directly next to the k–valent pretrack in the clockwise direction will become a linear combination of the labels associated to these two branches. In particular, let w be the weight of a branch on the k–valent pretrack, and let w' be the weight of the branch directly next to this branch in the clockwise direction. After applying l half-twists, we see that the weight of branch w is $lw' + (l-1)w$ and the weight of branch w' is $(l+1)w' + lw$. Since τ rotates clockwise by $2\pi/n$ after each application of $D_{\mu_i}^{q_i}$, we know that after k applications of ϕ, the weight of each branch will be a linear combination of the initial weights of each branch and the constants of this linear combination are strictly positive integers. Equivalently, this implies that each entry in M_τ^k is a strictly positive integer value. This implies that the matrix M_τ is Perron–Frobenius.

To finish the proof, we can see by inspection that each of the train tracks which were constructed above are large, generic, and birecurrent. Therefore, we apply Lemma 2.4 which completes the proof that ϕ is a pseudo-Anosov mapping class. □

Using a similar argument as the proof of Theorem 1.2, we now provide a proof for Theorem 1.3.

Theorem 1.3 *Let $n \geq 7$, let $q_j \geq 2$ for each $j \in \{0, 1, \dots, k\}$, and let*

$$\{\mu_0, \dots, \mu_{k'}, \dots, \mu_k\}$$

be a partition of $\{0, 1, \dots, n-1\}$ that reduces to an evenly spaced partition. Then

$$\phi = \prod_{j=0}^{k} D_{\mu_j}^{q_j}$$

is a pseudo-Anosov homeomorphism of $S_{0,n}$.

Proof Fix some value of $n \in \mathbb{N}$, some $k > 1$, $k \in \mathbb{N}$, and a partition $\mu = \{\mu_1, \dots, \mu_k\}$ of the n punctures of $S_{0,n}$ such that $\rho(\mu_{i-1}) = \rho(\mu_{(i \bmod k)})$. We perform the modification outlined in the statement of the theorem to obtain a partition $\mu' = \{\mu'_1, \dots, \mu'_k, \mu'_{k+1}\}$, on the $(n+1)$–times punctured sphere $S_{0,n+1}$, which defines the map

$$\phi' = \prod_{i=1}^{k+1} D_{\mu_i}^{q'_i} = D_{\mu'_{k+1}}^{q'_{k+1}} D_{\mu'_k}^{q'_k} \cdots D_{\mu'_2}^{q'_2} D_{\mu'_1}^{q'_1}.$$

We prove that ϕ' is a pseudo-Anosov mapping class.

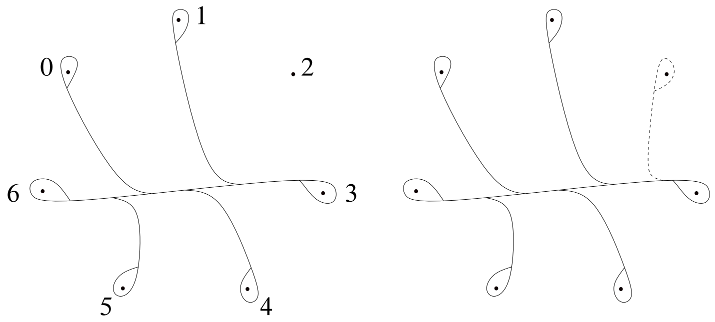

Figure 5: Constructing the train track for the map ψ.

We first construct the train track τ' so that $\phi'(\tau')$ is carried by τ'. We begin by considering the train track τ associated to the map $\phi = \prod_{i=1}^{k} D_{\mu_i}^{q_i}$ defined by the partition μ. We then add in a new puncture onto the sphere between punctures $k-1$ and k, and relabel the punctures. See Figures 5, left, and 6, left, for examples which correspond to the pretracks from Example A.2. Add a branch from puncture $k+1$ so that it turns tangentially into the k–valent pretrack meeting the same branch on the pretrack as the branches associated to punctures $1, \dots, k-1$. See Figures 5, right, and 6, right, for examples which correspond to the train tracks from Example A.2. We denote this modified train track by τ'.

To show that $\phi'(\tau')$ is carried by τ', we notice that by the same reasoning in the proof of Theorem 1.2 that for each $1 \le i < k+1$, the application of $D_{\mu_i'}^{q_i'}$ will result in a train track which is obtained through a rotation by $2\pi/n$ of our starting train track. After the first k applications of $D_{\mu_i'}^{q_i'}$, we have a resulting train track which is obtained by a

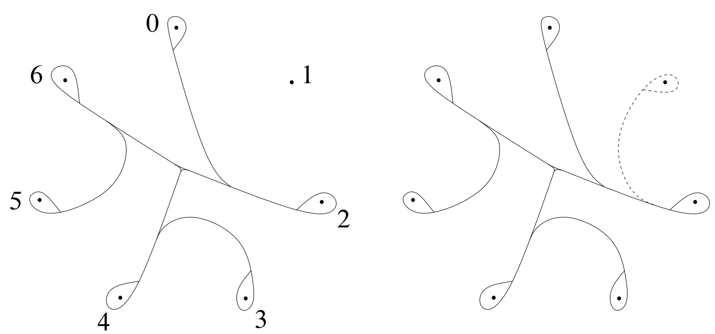

Figure 6: Constructing the train track for the map $\phi_{\bar{\mu}'}$.

rotation by $2\pi k/n$ of our starting train track, which is not quite τ'. By applying the final twist $D_{\mu'_{k+1}}^{q'_{k+1}}$, we find $\phi'(\tau') = \tau'$ and thus $\phi'(\tau')$ is carried by τ'. See Figures 15 and 17 for examples.

By the same reasoning as in the proof of Theorem 1.2, the matrix representing the induced action on the space of weights on τ' will be Perron–Frobenius. To finish the proof, we note that each of the train tracks that we have constructed are large, generic, and birecurrent. Therefore, we can apply Lemma 2.4 which completes the proof that the map is a pseudo-Anosov mapping class. □

4 Modifications of construction

By considering the constructions described in Section 3, we notice that there are additional modifications one can make to the construction to obtain more pseudo-Anosov mapping classes.

Recall that in Theorem 1.3, we added extra branches to the same branch on the $|\mu_k|$–valent pretrack. The first modification in this section is obtained by allowing additional branches to be added to any of the branches of the $|\mu_k|$–valent pretrack. To make this precise, we say a partition $\{\mu_0, \dots, \mu_k\}$ of $\{0, 1, \dots, n-1\}$ 2*–reduces to an evenly spaced partition* if there exists an $n' \in \{0, 1, \dots, n-1\}$ such that $\{\mu'_0, \mu'_1, \dots, \mu'_k\}$ is the partition of $\{0, 1, \dots, n'-1\} \cup \{(n'+1)-1, (n'+2)-1, \dots (n-1)-1\}$ defined by

$$\mu'_j = \{i \mid i \in \mu_j \text{ and } i < n'\} \cup \{i-1 \mid i \in \mu_k \text{ and } i > n'\}, \tag{2}$$

and $\{\mu'_0, \mu'_1, \dots, \mu'_k\}$ reduces to an evenly spaced partition. Similarly, we can iteratively define a partition which k*–reduces to an evenly spaced partition*. Theorem 1.3 holds for any partition which k-reduces to an evenly spaced partition, and the proof follows by considering a modification of the train track analogous to the modification found in the proof of Theorem 1.3.

To obtain a second modification, consider a map ϕ from any of the possible maps found in Theorem 1.3 or any of the modifications outlined above. It is possible to find an additional pseudo-Anosov map which has the same train track as ϕ. Since the train track rotates by $2\pi/n$ for the first k applications of $D_{\mu'_i}^{q'_i}$, we can define a map that will continue to rotate the train track by $2\pi/n$ in place of doing the final twist(s) $D_{\mu'_{k+1}}^{q'_{k+1}}$.

For example, consider the first map from Example A.2. After applying $D^2_{\mu'_3} D^2_{\mu'_2} D^2_{\mu'_1}$, where $\mu' = \{\{1,5\},\{2,6\},\{3,7\}\} = \{\mu'_1, \mu'_2, \mu'_3\}$, the train track has rotated by $6\pi/7$. We can apply the rotations associated to punctures 4 and 1 next, then around puncture 5 and 2, around punctures 6 and 3, and finally around punctures 7 and 4, which will have rotated our train track by a full rotation. In other words, you will obtain a new "partition" containing the sets $\mu''_i = \{i, i + \lceil 7/3 \rceil\}$ for $1 \le i \le 7$. More precisely, we obtain the following additional construction:

Theorem 4.1 *Consider the surface $S_{0,n}$. Consider one of the maps from Theorem 1.2, in particular, consider a partition of the n punctures into $1 < k < n$ sets $\{\mu_i\}_{i=1}^k$ such that the partition is evenly spaced. Apply any number of applications of Theorem 1.3 to obtain a new partition $\mu' = \{\mu'_1, \dots, \mu'_k, \mu'_{k+1}, \dots, \mu'_{k+l}\}$, where $|\mu_{k+j}| < |\mu_1|$ for all $1 \le j \le l$ which defines a map on the p–sphere, where $p = \sum_{i=1}^{k+l} |\mu_i|$. Consider the train track τ' associated to this map. Define the partition μ'' to be the partition containing the sets $\mu''_i = \{i, i + \lceil p/k \rceil, \dots, i + (|\mu_1| - 1)\lceil p/k \rceil\}$, where $1 \le i \le p$. Then μ'' defines the pseudo-Anosov mapping class*

$$\phi' = \prod_{i=1}^{p} D^{q''_i}_{\mu''_i} = D^{q''_p}_{\mu''_p} \cdots D^{q''_2}_{\mu''_2} D^{q''_1}_{\mu''_1}$$

on $S_{0,p}$, where $q''_j = \{q''_{j_1}, \dots, q''_{j_l}\}$ is the set of powers associated to each μ''_i.

Remark 4.2 The proofs for the modifications found in this section follow the same format as the proofs for the theorems found in Section 3.

5 Number-theoretic properties

In this section, we prove Theorem 1.4 which describes the number-theoretic properties associated to the maps arising from Theorems 1.2 and 1.3. To prove Theorem 1.4, we provide explicit examples of pseudo-Anosov mapping classes resulting from the constructions outlined in Theorems 1.2 and 1.3 which have the specific number-theoretic properties we are looking for. An important consequence of Theorem 1.4 is that the construction outlined in this paper differs from the constructions of both Penner and Thurston.

5.1 Number-theoretic properties

We begin by introducing two properties of pseudo-Anosov homeomorphisms: the placement of the Galois conjugates of the stretch factor, and the trace field.

5.1.1 Galois theory We recall that if K is a field containing the subfield F, then K is said to be an *extension field* (or simply an *extension*) of F. We denote this by K/F. Let $\mathrm{Aut}(K/F)$ be the collection of automorphisms of K which fix F. If K/F is a field extension, then K is said to be *Galois* over F and K/F is a *Galois extension* if $|\mathrm{Aut}(K/F)| = [K : F]$. In a Galois extension K/F, the other roots of the minimal polynomial over F of any element $\alpha \in K$ are precisely the distinct conjugates of α under the Galois group K/F. Therefore, the *Galois conjugates* are precisely the other roots of the minimal polynomial over F of an element $\alpha \in K$.

Let ϕ be a pseudo-Anosov homeomorphism with stretch factor λ. Let $L/\mathbb{Q}$ be the Galois extension where L is the splitting field of the minimal polynomial of λ. Whether there exist Galois conjugates of the stretch factor on the unit circle is a number-theoretic property of ϕ.

5.1.2 Trace fields The *trace field* of a linear group is the field generated by the traces of its elements. In particular the trace field of a group $\Gamma \subset SL_2(\mathbb{R})$ is the subfield of $\mathbb{R}$,

$$\{\mathrm{tr}(A) \mid A \in \Gamma\}.$$

Work by Hubbard and Masur shows that for each pseudo-Anosov homeomorphism, one can obtain a corresponding flat structure [4]. An *affine diffeomorphism* is a diffeomorphism which preserves the flat structure, and they form a group which we call the *affine diffeomorphism group*. See [15] for a discussion of flat surfaces and the affine diffeomorphism group. Kenyon and Smillie [6] proved that if the affine diffeomorphism group of a surface contains an orientation preserving pseudo-Anosov element f with largest eigenvalue λ, then the trace field is $\mathbb{Q}(\lambda + \lambda^{-1})$.

5.2 Proof of Theorem 1.4

We now provide a proof of Theorem 1.4 by providing explicit examples of maps satisfying each of the number-theoretic properties.

Theorem 1.4 *For any of the following four statements, there exists a pseudo-Anosov homeomorphism whose stretch factor λ satisfies the statement*:

(1) $\mathbb{Q}(\lambda + \lambda^{-1})$ *is totally real and there exists no Galois conjugates of λ on the unit circle.*

(2) $\mathbb{Q}(\lambda + \lambda^{-1})$ *is not totally real and there exist no Galois conjugates of λ on the unit circle.*

(3) $\mathbb{Q}(\lambda+\lambda^{-1})$ *is totally real and there exist Galois conjugates of* λ *on the unit circle.*

(4) $\mathbb{Q}(\lambda+\lambda^{-1})$ *is not totally real and there exist Galois conjugates of* λ *on the unit circle.*

These homeomorphisms are constructed on the surfaces $S_{0,6}$*,* $S_{0,7}$*,* $S_{0,8}$*, and* $S_{0,8}$*, respectively.*

Proof First, we provide an example where the Galois conjugates are never on the unit circle, and that the field $\mathbb{Q}(\lambda+1/\lambda)$ is totally real. This proves case (1).

Case (1) Consider the pseudo-Anosov map $\phi_{2,2,2}$ on $S_{0,6}$ which is induced by the partition $\mu_{2,2,2}=\{\{0,3\},\{1,4\},\{2,5\}\}$ as studied in Example A.1,

$$\phi_{2,2,2}=D_5^2D_2^2D_4^2D_1^2D_3^2D_0^2.$$

See Example A.1 for details. We find that the induced action on the space of weights on $\tau_{2,2,2}$ is given by the matrix

$$M=\begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 2\\ 6 & 3 & 2 & 4 & 0 & 4\\ 12 & 6 & 3 & 6 & 0 & 8\\ 0 & 0 & 2 & 3 & 2 & 0\\ 4 & 0 & 4 & 6 & 3 & 2\\ 6 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$p_{2,2,2}=(x-1)^2(x+1)^2(x^2-18x+1).$$

The polynomial $p_{2,2,2}$ is the characteristic polynomial associated to the action of this map on the train track $\tau_{2,2,2}$. We notice that the leading eigenvalue $\lambda_{2,2,2}$ is a root of the factor

$$p_{\phi_{2,2,2},\lambda_{2,2,2}}(x)=x^2-18x+1,$$

which is an irreducible polynomial with real roots. Since $\lambda_{2,2,2}\in\mathbb{R}$ is not on the unit circle, as $\lambda_{2,2,2}>1$, $\lambda_{2,2,2}^{-1}$ is also not on the unit circle. Therefore, the Galois conjugates of the stretch factor are not on the unit circle.

To show that $\mathbb{Q}(\lambda_{2,2,2}+\lambda_{2,2,2}^{-1})$ is totally real, we notice that we can write

$$\frac{p_{2,2,2}}{x}=\left(x+\frac{1}{x}\right)-18=q\left(x+\frac{1}{x}\right).$$

By considering the roots of $q(y) = y - 18$, we notice that the only root is $y = 18$ which implies that the field $\mathbb{Q}(\lambda_{2,2,2} + \lambda_{2,2,2}^{-1})$ is totally real. This completes the proof of case (1).

Next, we will provide an example where the Galois conjugates are never on the unit circle, and that the field $\mathbb{Q}(\lambda + \lambda^{-1})$ is not totally real. This will prove case (2).

Case (2) We consider the map $\phi_{3,3,1} = D_2^2 D_6^2 D_4^2 D_1^2 D_5^2 D_3^2 D_0^2$ on $S_{0,7}$ induced by the partition $\mu_{3,3,1} = \{\{0,3,5\},\{1,4,6\},\{2\}\}$ from Example A.2. We find that the induced action on the space of weights on $\tau_{3,3,1}$ is given by the matrix

$$M = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 4 & 0 & 0 & 4 \\ 12 & 6 & 3 & 6 & 0 & 0 & 8 \\ 0 & 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 0 & 4 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$p_{3,3,1} = (x+1)(x^3 - 15x^2 + 7x - 1)(x^3 - 7x^2 + 15x - 1).$$

The polynomial $p_{3,3,1}$ is the characteristic polynomial associated to the action of this map on the train track We notice that the leading eigenvalue, $\lambda_{3,3,1}$, is a root of the polynomial

$$p_{\phi_{3,3,1},\lambda_{3,3,1}}(x) = x^3 - 15x^2 + 7x - 1.$$

The roots of this polynomial are

$$5 + \tfrac{1}{3}\sqrt[3]{2916 - 12\sqrt{19}} + \left(\tfrac{2}{3}\right)^{2/3}\sqrt[3]{243 + \sqrt{93}},$$

$$5 + \tfrac{1}{3}(-1 + i\sqrt{3})\sqrt[3]{2916 - 12\sqrt{19}} - \frac{(1 + i\sqrt{3})\left(\sqrt[3]{\tfrac{1}{2}(243 + \sqrt{93})}\right)}{3^{2/3}},$$

$$5 - \tfrac{1}{3}(-1 + i\sqrt{3})\sqrt[3]{2916 - 12\sqrt{19}} + \frac{(1 + i\sqrt{3})\left(\sqrt[3]{\tfrac{1}{2}(243 + \sqrt{93})}\right)}{3^{2/3}}.$$

By the rational root theorem,

$$p_{\phi_{\bar{\mu}'},\lambda}(x) = x^3 - 15x^2 + 7x - 1$$

is irreducible over $\mathbb{Q}$. None of the roots of $p_{\phi_{3,3,1},\lambda_{3,3,1}}(x)$ are on the unit circle, so we have that there are no Galois conjugates of the stretch factor on the unit circle. We now consider the polynomial

$$\left(\frac{1}{x^3}\right)(x^3-15x^2+7x-1)(x^3-7x^2+15x-1)=x^3-22x^2+127x-276.$$

We rewrite this polynomial as

$$q\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)^3-22\left(x+\frac{1}{x}\right)^2+124\left(x+\frac{1}{x}\right)-232.$$

We calculate that the roots of the polynomial $q(y)$ are

$$\begin{gathered}\tfrac{1}{3}\left(22+\sqrt[3]{1801-9\sqrt{26554}}+\sqrt[3]{1801+9\sqrt{26554}}\right),\\ \tfrac{1}{6}\left(44+i(\sqrt{3}+i)\sqrt[3]{1801-9\sqrt{26554}}+(-1-i\sqrt{3})\sqrt[3]{1801+9\sqrt{26554}}\right),\\ \tfrac{1}{6}\left(44+(-1-i\sqrt{3})\sqrt[3]{1801-9\sqrt{26554}}+i(\sqrt{3}+i)\sqrt[3]{1801+9\sqrt{26554}}\right).\end{gathered}$$

By unique factorization, $q(y)$ is irreducible. Additionally, since two of the roots are imaginary, the field $\mathbb{Q}(\lambda_{3,3,1}+\lambda_{3,3,1}^{-1})$ is not totally real. This completes the proof of case (2).

We now provide an example where there are Galois conjugates of the stretch factor on the unit circle, and that the field $\mathbb{Q}(\lambda+\lambda^{-1})$ is totally real. This will prove case (3).

Case (3) We now consider the pseudo-Anosov map

$$\phi_{2,2,2,1,1}=D_4^2D_3^2D_7^2D_2^2D_6^2D_1^2D_5^2D_0^2$$

on $S_{0,8}$, which is induced by the partition $\mu_{2,2,2,1,1}=\{\{0,5\},\{1,6\},\{2,7\},\{3\},\{4\}\}$. This is first map from Example A.1 with the modification from Theorem 1.3 applied twice so that there are two partitions with one element each. The train track $\tau_{2,2,2,1,1}$, where $\phi_{2,2,2,1,1}(\tau_{2,2,2,1,1})$ is carried by $\phi_{2,2,2,1,1}$, is depicted in Figure 7.

The matrix associated to this map is

$$M=\begin{pmatrix}3&2&0&0&0&0&0&2\\6&3&2&4&0&0&0&4\\12&6&3&6&0&0&0&8\\0&0&2&3&2&0&0&0\\0&0&4&6&3&2&0&0\\0&0&8&12&6&3&2&0\\4&0&16&24&12&6&3&2\\6&0&32&48&24&12&6&3\end{pmatrix}$$

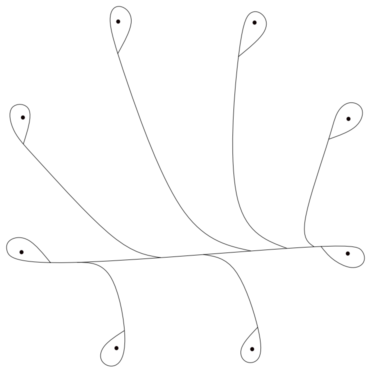

Figure 7: The train track associated to $\phi_{2,2,2,1,1}$.

which has the characteristic polynomial

$$p_{2,2,2,1,1}(x) = (x+1)^4(x^4 - 28x^3 + 6x^2 - 28x + 1).$$

The polynomial $p_{2,2,2,1,1}(x)$ is the characteristic polynomial associated to the action of this map on the train track $\tau_{2,2,2,1,1}$. Our leading eigenvalue $\lambda_{2,2,2,1,1}$ is a root of

$$p_{\lambda_{2,2,2,1,1},\phi_{2,2,2,1,1}}(x) = x^4 - 28x^3 + 6x^2 - 28x + 1.$$

The roots of this polynomial are

$$\lambda^{-1} = 7 + 4\sqrt{3} - 2\sqrt{24 + 14\sqrt{3}}, \quad \lambda = 7 + 4\sqrt{3} + 2\sqrt{24 + 14\sqrt{3}},$$
$$x_1 = 7 - 4\sqrt{3} - 2i\sqrt{14\sqrt{3} - 24}, \quad x_2 = 7 - 4\sqrt{3} + 2i\sqrt{14\sqrt{3} - 24}.$$

To prove that $p_{\lambda_{2,2,2,1,1},\phi_{2,2,2,1,1}}(x)$ is irreducible we use the following fact, a proof of which is found in [1].

Fact 5.1 *If $f(x) \in \mathbb{Z}[x]$ is primitive of degree $d \geq 1$ and there are at least $2d+1$ different integers a such that $|f(a)|$ is 1 or a prime number, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.*

Thus, it suffices find 9 values of x such that $p_{\lambda,\psi}(x)$ is prime. Indeed, the following tuples $(x, p_{\lambda_{2,2,2,1,1},\phi_{2,2,2,1,1}}(x))$ contain 9 x–values such that $p_{\lambda_{2,2,2,1,1},\phi_{2,2,2,1,1}}(x)$ is prime:

$$(-24, 722977), \quad (-16, 182209), \quad (-12, 70321), \quad (0, 1), \quad (2, -239),$$
$$(6, -4703), \quad (8, -10079), \quad (24, -52511), \quad (38, 556321).$$

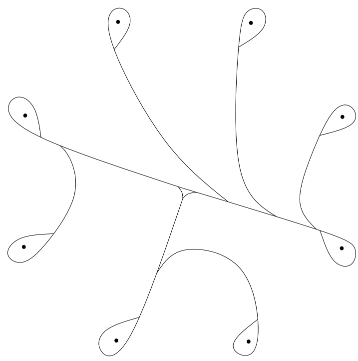

Figure 8: The train track associated to $\phi_{3,3,1,1}$.

Therefore, $p_{\lambda_{2,2,2,1,1},\phi_{2,2,2,1,1}}(x)$ is irreducible over $\mathbb{Q}$. Notice that $|x_1| = 1$ and $|x_2| = 1$, which we can verify by direct computation or by applying [7, Theorem 1]. This implies that there are Galois conjugates of the stretch factor on the unit circle. We now show that the field $\mathbb{Q}(\lambda_{2,2,2,1,1} + \lambda_{2,2,2,1,1}^{-1})$ is totally real by writing

$$\frac{p_{\lambda,\psi}}{x} = \left(x + \frac{1}{x}\right)^2 - 28\left(x + \frac{1}{x}\right) + 4 = q\left(x + \frac{1}{x}\right).$$

We notice that the roots of $q(y) = y^2 - 28y + 4$ are

$$14 - 8\sqrt{3} \quad \text{and} \quad 14 + 8\sqrt{3},$$

which implies that $q(y)$ is irreducible by unique factorization. Additionally, since both roots are real we find that the field $\mathbb{Q}(\lambda_{2,2,2,1,1} + \lambda_{2,2,2,1,1}^{-1})$ is totally real.

Lastly, we provide two examples where there are Galois conjugates of the stretch factor on the unit circle, and where the field $\mathbb{Q}(\lambda + \lambda^{-1})$ is not totally real. This will prove case (4).

Case (4) We begin with an example on $S_{0,8}$ where there are Galois conjugates of the stretch factor on the unit circle, and where the field $\mathbb{Q}(\lambda + 1/\lambda)$ is not totally real. We consider the second map from Example A.1 and apply the modification from Theorem 1.3 twice to obtain the partition $\mu_{3,3,1,1} = \{\{0, 4, 6\}, \{1, 5, 7\}, \{2\}, \{3\}\}$. This induces the map

$$\phi_{3,3,1,1} = D_3^2 D_2^2 D_7^2 D_5^2 D_1^2 D_6^2 D_4^2 D_0^2.$$

The train track $\tau_{3,3,1,1}$, where $\phi_{3,3,1,1}(\tau_{3,3,1,1})$ is carried by $\phi_{3,3,1,1}$, is depicted in Figure 8.

The matrix associated to this map is

$$M = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 0 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 2 & 4 & 0 & 0 & 8 \\ 24 & 12 & 6 & 3 & 6 & 0 & 0 & 16 \\ 0 & 0 & 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 0 & 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 0 & 0 & 4 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$p_{\phi_{3,3,1,1}}(x) = x^8 - 24x^7 + 156x^6 - 424x^5 - 186x^4 - 424x^3 + 156x^2 - 24x + 1.$$

The polynomial $p_{\phi_{3,3,1,1}}(x)$ is the characteristic polynomial associated to the action of this map on the train track $\tau_{3,3,1,1}$. To show that $p_{\phi_{3,3,1,1}}(x)$ is irreducible, it suffices find 17 values of x such that $p_{\phi_{3,3,1,1}}(x)$ is prime by Fact 5.1. Indeed, the following tuples $(x, p_{\phi_{3,3,1,1}}(x))$ contain 17 x–values such that $p_{\phi_{3,3,1,1}}(x)$ is prime:

$$(-160, 496582824202141441), \quad (-102, 14653782370731169),$$
$$(-90, 5537981240501761), \quad (-76, 1495649690458849),$$
$$(-52, 81377571089569), \quad (-46, 32071763417569), \quad (-40, 11167704826561),$$
$$(-22, 134573887009), \quad (0, 1), \quad (8, -7522751), \quad (26, 59124433057),$$
$$(72, 502376857985089), \quad (86, 2218259932983937),$$
$$(90, 3237148147105441), \quad (120, 34853759407811521),$$
$$(158, 331770565001360449), \quad (164, 449704327465370209).$$

Therefore, we have that $p_{\phi_{3,3,1,1}}(x)$ is irreducible, and the leading eigenvalue $\lambda_{3,3,1,1}$ is a root of $\phi_{3,3,1,1}$. By applying [7, Theorem 1] we find that there are roots of this polynomial which are on the unit circle, and thus, there exist Galois conjugates of $\lambda_{3,3,1,1}$ on the unit circle. We now write

$$\frac{p_{\phi_{3,3,1,1}}}{x^4} = \left(x + \frac{1}{x}\right)^4 - 24\left(x + \frac{1}{x}\right)^3 + 152\left(x + \frac{1}{x}\right)^2 - 352\left(x + \frac{1}{x}\right) - 496.$$

We prove that $q(y) = y^4 - 24y^3 + 152y^2 - 352y - 496$ is an irreducible polynomial as follows. By Gauss's lemma, a primitive polynomial is irreducible over the integers if and only if it is irreducible over the rational numbers. Since $q(y)$ is primitive, it suffices to show that $q(y)$ is irreducible over the integers. The rational root theorem gives us that $q(y)$ has no roots, so if it is reducible then $q(y) = (y^2+ay+b)(y^2+cy+d)$. Therefore, suppose that $q(y) = (y^2+ay+b)(y^2+cy+d)$. Expanding gives rise to the system of equations

$$(3)\qquad a+c=-24, \quad ac+b+d=152, \quad ad+bc=-352, \quad bd=-496.$$

Substituting $a=-24-c$ and $b=-496/d$ into the second and third equations give

$$(4)\qquad -24c-c^2-\frac{496}{d}+d=152, \quad (-24-c)d-\frac{496c}{d}=-352.$$

We solve for c in the second equation to find

$$c=\frac{24d^2-352d}{-d^2-496}.$$

Substituting this into the first equation gives

$$-24\left(\frac{24d^2-352d}{-d^2-496}\right)d-\left(\frac{24d^2-352d}{-d^2-496}\right)^2 d+d^2-152d-496=0,$$

which has no integer roots. Therefore, there is no d satisfying the conditions we require, so $q(y)$ is irreducible. Finally, by using the formulas for the roots of a quartic equation, we see that $q(y)$ has two imaginary roots. Therefore the field $\mathbb{Q}(\lambda_{3,3,1,1}+\lambda_{3,3,1,1}^{-1})$ is not totally real.

We end with an example on $S_{0,10}$ where there are Galois conjugates of the stretch factor on the unit circle, and where the field $\mathbb{Q}(\lambda+1/\lambda)$ is not totally real. For this example, we will apply the modification from Theorem 1.3 to the map associated to the partition

$$\mu_{3,3,3}=\{\{1,4,7\},\{2,5,8\},\{3,6,9\}\}$$

to obtain the partition

$$\mu_{3,3,3,1}=\{\{1,5,8\},\{2,6,9\},\{3,7,10\},\{4\}\}.$$

This induces the map

$$\phi_{3,3,3,1}=D_4D_{10}D_7D_3D_9D_6D_2D_8D_5D_1.$$

The train track $\tau_{3,3,3,1}$, where $\phi_{3,3,3,1}(\tau_{3,3,3,1})$ is carried by $\phi_{3,3,3,1}$, is depicted in Figure 9.

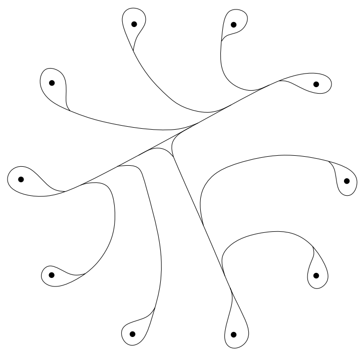

Figure 9: The train track associated to $\phi_{3,3,3,1}$.

The matrix associated to this map is

$$M = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 2 & 4 & 0 & 0 & 0 & 0 & 8 \\ 24 & 12 & 6 & 3 & 6 & 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 2 & 3 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 6 & 3 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 12 & 6 & 3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 6 & 3 & 2 \\ 6 & 0 & 0 & 0 & 0 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}$$

which has the characteristic polynomial

$$\begin{aligned} &p_{\psi_{3,3,3,1}}(x) \\ &\quad = x^{10}-30x^9+285x^8-1864x^7-30x^6+204x^5-30x^4-1864x^3+285x^2-30x+1. \end{aligned}$$

The polynomial $p_{\phi_{3,3,3,1}}(x)$ is the characteristic polynomial associated to the action of this map on the train track $\tau_{3,3,3,1}$. To show that $p_{\phi_{3,3,3,1}}(x)$ is irreducible, it suffices to find 21 values of x such that $p_{\phi_{3,3,3,1}}(x)$ is prime by Fact 5.1. Indeed, the following tuples $(x, p_{\phi_{3,3,3,1}}(x))$ contain 21 x–values such that $p_{\phi_{3,3,3,1}}(x)$ is prime:

$$(-224, 3624925501673023779835553), \quad (-208, 1744689867642402680825529),$$
$$(-168, 212959698910125401213299), \quad (-160, 131841102599563910448801),$$
$$(-76, 9311115506417745721), \quad (-72, 5528387844285055921),$$
$$(-54, 351062674729634953), \quad (-52, 245103960106710121),$$
$$(-12, 405908347321), \quad (-10, 87091192801), \quad (0, 1), \quad (2, -189671),$$

$$(6,-285219647),\quad (24,6967292292721),\quad (74,31613941134619237 21),$$
$$(186,4196018761052156350 1353),\quad (204,10729584062649689003 1721),$$
$$(216,19167875390287223870 1553),\quad (234,43160454224060394260 0521),$$
$$(258,11602676139063590660 71321),\quad (264,14642012363050069874 79121).$$

Therefore, we have that $p_{\phi_{3,3,3,1}}(x)$ is irreducible; thus the leading eigenvalue $\lambda_{3,3,3,1}$ is a root of $p_{\phi_{3,3,3,1}}(x)$. Applying [7, Theorem 1] we find that there are roots of this polynomial which are on the unit circle. We now write

$$\frac{p_{\psi_{3,3,3,1}}}{x^5}=\Big(x+\frac{1}{x}\Big)^5-30\Big(x+\frac{1}{x}\Big)^4+280\Big(x+\frac{1}{x}\Big)^3+1744\Big(x+\frac{1}{x}\Big)^2-880\Big(x+\frac{1}{x}\Big)-3307.$$

We rewrite the above as $q(y)=y^5-30y^4+280y^3+1744y^2-880y-3307$. To see that $q(y)$ is irreducible over $\mathbb{Q}$, it suffices to show that there are 11 values of y such that $q(y)$ is prime. Indeed, the following tuples $(y,q(y))$ contain 11 y–values such that $q(y)$ is prime:

$$(-12,-1096363),\quad (-10,-500107),\quad (-6,-42379),$$
$$(-4,1493),\quad (-2.2677),\quad (0,-3307),\quad (2,3701),$$
$$(4,32341),\quad (14,479861),\quad (16,658453),\quad (22,1928821).$$

Finally, we notice that the discriminant of the polynomial $q(y)$ is calculated to be

$$-103017075043340205442 19.$$

As the discriminant is negative, we know that there must exist nonreal roots; therefore the field $\mathbb{Q}(\lambda_{3,3,3,1}+\lambda_{3,3,3,1}^{-1})$ is not totally real.

As we have provided explicit examples of pseudo-Anosov homeomorphisms for each case of Theorem 1.4, we have completed the proof. $\square$

6 Construction on surfaces of higher genus

In this section, we prove Theorem 1.1. To prove this theorem, we begin by taking the map ϕ on $S_{0,8}$ found in Theorem 1.4(4) and show that by further puncturing $S_{0,8}$, we obtain pseudo-Anosov homeomorphisms which differ from the Penner and Thurston constructions on surfaces $S_{0,n}$ for $n\geq 8$. Furthermore, we lift the constructed pseudo-Anosov mapping classes on $S_{0,2g+2}$ to pseudo-Anosov mapping classes on surfaces of genus $g>0$ through a *branched cover* by treating the marked points as punctures; see Figure 1 for an example. A branched cover $S_{g,0}\to S_{0,2g+2}$ is a true covering map in the complement of a finite set of points of $S_{0,2g+2}$. These points are called the *branch points*. We now provide a proof for Theorem 1.1.

Theorem 1.1 *Let S be either $S_{0,n}$ for $n \geq 8$, or $S_{g,k}$ for $g \geq 3$ and $k \geq 0$. Then there exists a pseudo-Anosov homeomorphism ϕ_S on S with stretch factor λ_{ϕ_S} such that*

(i) *the trace field $\mathbb{Q}(\lambda_{\phi_S} + \lambda_{\phi_S}^{-1})$ is not totally real, and*

(ii) *there exist Galois conjugates of λ_{ϕ_S} on the unit circle.*

In particular, no power of ϕ_S arises from either Penner's or Thurston's constructions.

Proof In the proof of Theorem 1.4(4), we have already shown this theorem to be true for the surfaces $S_{0,8}$ and $S_{0,10}$. To show that there exists a pseudo-Anosov homeomorphism on punctured spheres $S_{0,n}$ where $n \geq 9$, we consider the map from Theorem 1.4(4) on $S_{0,8}$.

Let ϕ be the pseudo-Anosov homeomorphism of $S_{0,8}$ found in Theorem 1.4(4). For any pseudo-Anosov homeomorphism of a surface S, the set of periodic points is dense in S [2]. Since dense sets in a surface contain an infinite number of elements, this implies that there must be at least $n-8$ periodic points. There exists a power of the pseudo-Anosov map, ϕ^k, which fixes the $n-8$ periodic points. If we delete these $n-8$ fixed points, ϕ^k restricts to a pseudo-Anosov map on an n–times punctured sphere. Additionally, the foliations from the map ϕ on $S_{0,8}$ are the same foliations as for the map ϕ^k on $S_{0,n}$, but now the stretch factor is λ^k. Since the algebraic properties are invariant under powers of the pseudo-Anosov homeomorphism, we see that we have proven the claim for surfaces $S_{0,n}$ for $n \geq 9$.

From the above, we have the desired pseudo-Anosov mapping classes for all spheres with at least 8 punctures. The hyperelliptic involution of S_g induces a branched double cover $S_g \to S_{0,2g+2}$, where $S_{0,2g+2}$ is a sphere with $2g+2$ marked points. The pseudo-Anosov maps we constructed above for $S_{0,2g+2}$ lift to pseudo-Anosov maps on S_g with the same stretch factor. Therefore, we have that the same number-theoretic properties hold for the pseudo-Anosov maps on surfaces $S_{g,0}$, where $g \geq 3$. Since we have pseudo-Anosov maps on a closed surface of genus g, we once again have that periodic points are dense. Using a similar argument as above, this implies that we may puncture the surface of genus g any number of times, to obtain the desired pseudo-Anosov mapping classes on surfaces $S_{g,k}$ for $g \geq 3$ and $k \geq 0$. □

Appendix Introductory examples

In this section, we present two detailed examples on how to apply Theorems 1.2 and 1.3. Consider the six-times punctured sphere. We will begin by constructing two pseudo-Anosov maps on the six-times punctured sphere using Theorem 1.2.

Example A.1 Consider the six-times punctured sphere and label the punctures of the sphere as introduced in Figure 2. Up to spherical symmetry, there are two unique partitions of the six punctures so that the labels of the punctures are evenly spaced, namely

$$\mu = \{\{0,3\},\{1,4\},\{2,5\}\} = \{\mu_1,\mu_2,\mu_3\}$$

and

$$\bar{\mu} = \{\{0,2,4\},\{1,3,5\}\} = \{\bar{\mu}_1,\bar{\mu}_2\}.$$

Recall that we define the half-twist associated to puncture j as the half-twist around the curve separating punctures j and $j-1$. Therefore, these partitions can define the two maps,

$$\phi = D_5^2 D_2^2 D_4^2 D_1^2 D_3^2 D_0^2 = D_{\mu_3}^2 D_{\mu_2}^2 D_{\mu_1}^2,$$

and

$$\phi_{\bar{\mu}} = D_5^2 D_3^2 D_1^2 D_4^2 D_2^2 D_0^2 = D_{\bar{\mu}_2}^2 D_{\bar{\mu}_1}^2,$$

respectively. We prove that both maps are pseudo-Anosov.

We first prove that $\phi_\mu = D_{\mu_3}^2 D_{\mu_2}^2 D_{\mu_1}^2$ is a pseudo-Anosov map on $S_{0,6}$. To prove that ϕ_μ is a pseudo-Anosov map, we find a train track τ_μ on $S_{0,6}$ so that $\phi_\mu(\tau_\mu)$ is carried by τ_μ and show that the matrix presentation of ϕ_μ in the coordinates given by τ_μ is a Perron–Frobenius matrix.

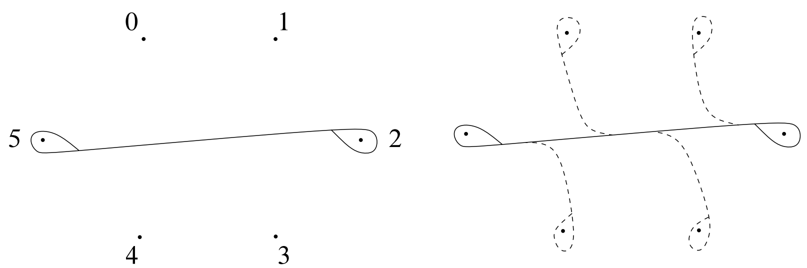

Figure 10: Constructing the train track for the map ϕ_μ.

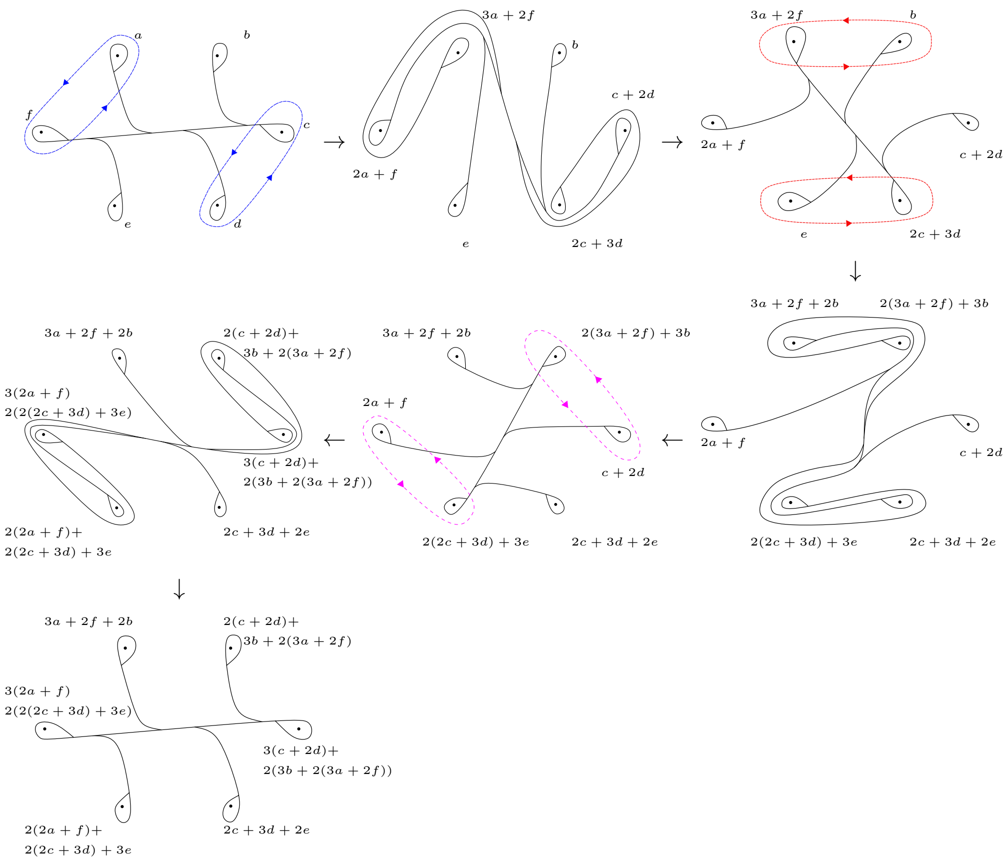

Figure 11: The train track $\phi_\mu(\tau_\mu)$ is carried by τ_μ.

First, we describe how we construct the train track for the map ϕ_μ based on the partition μ. Notice that μ has three subsets containing two punctures. As the punctures in each partition are such that $|i-j| \geq 2 \mod 6$, the twists associated to the punctures in each subset are disjoint. Since there are two twists in each subset and the partition is evenly spaced, the train track has rotational symmetry of order two. Therefore, we construct a two-valent pretrack around the punctures labeled 2 and 5, pictured in Figure 10, left. Since there are three subsets, there are two branches turning tangentially into each of the two nodes on the two-valent pretrack, where these branches will turn left towards the pretrack, pictured in Figure 10, right.

The series of images in Figure 11 depict the train track τ_μ and its images under successive applications of the Dehn twists associated to ϕ_μ. These images prove that $\phi(\tau)$ is carried by τ_μ, and for every application of $D^2_{\mu_i}$, the train track τ_μ rotates

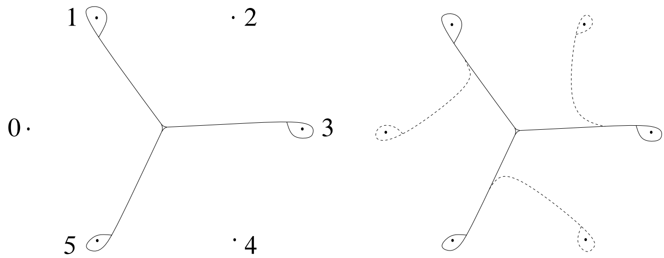

Figure 12: Constructing the train track for the map $\phi_{\bar{\mu}}$.

clockwise by $2\pi/6$. By keeping track of the weights on τ_μ, we calculate that the induced action on the space of weights on τ_μ is given by the matrix

$$A = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 4 & 0 & 4 \\ 12 & 6 & 3 & 6 & 0 & 8 \\ 0 & 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 4 & 6 & 3 & 2 \\ 6 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}.$$

Note that the space of admissible weights on τ_μ is the subset of $\mathbb{R}^6$ given by positive real numbers a, b, c, d, e and f such that $a+b+f=c+d+e$. The linear map described above preserves this subset. The square of the matrix A is strictly positive, which implies that the matrix is Perron–Frobenius. In fact, the top eigenvalue is $9+4\sqrt{5}$, which is associated to a unique irrational measured lamination F carried by τ_μ that is fixed by ϕ_μ. Lastly, since the train track τ_μ is large, generic, and birecurrent, we can apply Lemma 2.4 which finishes the proof that this map is pseudo-Anosov.

Notice that we can perform each of the half twists to any power and still have the exact same train track constructed above. However, the labels associated to the branches will subsequently increase or decrease in value according to how many twists are applied to each curve. Since all the twists are positive, we still have that all values in the resulting matrix will be positive and will be Perron–Frobenius. An application of Lemma 2.4 will give our desired result.

We will now show that $\phi_{\bar{\mu}} = D^2_{\bar{\mu}_2} D^2_{\bar{\mu}_1}$ is a pseudo-Anosov on $S_{0,6}$, which follows a similar argument as above.

We again analyze the partition $\bar{\mu}$ as it determines the construction of our train track $\tau_{\bar{\mu}}$. Notice that $\bar{\mu}$ has two subsets containing three twists each. Since there are three

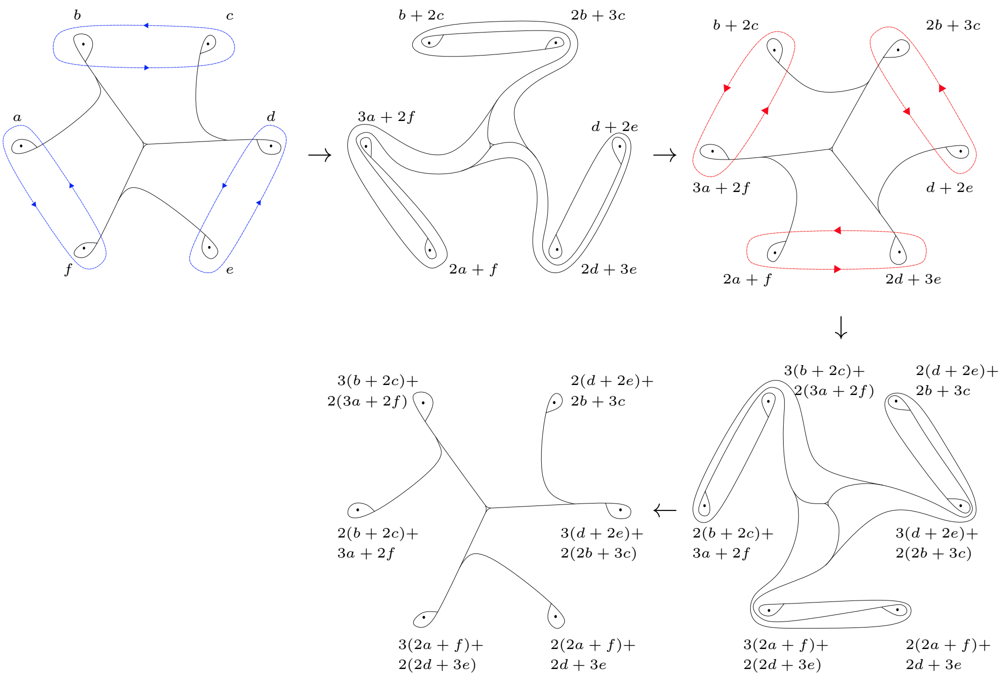

Figure 13: The train track $\phi_{\bar{\mu}}(\tau_{\bar{\mu}})$ is carried by $\tau_{\bar{\mu}}$.

punctures in each subset and the partition is evenly spaced, the train track has rotational symmetry of order three. Therefore, we will construct a three-valent pretrack around the punctures labeled 1, 3 and 5, pictured in Figure 12, left. Since there are two subsets, there is one branch turning tangentially towards each of the three nodes on the three-valent pretrack, where these branches will be turning left towards the pretrack, pictured in Figure 12, right.

The series of images in Figure 13 depict the train track $\tau_{\bar{\mu}}$ and its images under successive applications of the Dehn twists associated to $\phi_{\bar{\mu}}$. These images prove that $\phi_{\bar{\mu}}(\tau_{\bar{\mu}})$ is indeed carried by $\tau_{\bar{\mu}}$. We again notice that for every application of $D^2_{\bar{\mu}_i}$, the train track $\tau_{\bar{\mu}}$ rotates clockwise by $2\pi/6$. By keeping track of the weights on $\tau_{\bar{\mu}}$, we calculate that the induced action on the space of weights on τ is given by the matrix

$$B = \begin{pmatrix} 3 & 2 & 4 & 0 & 0 & 2 \\ 6 & 3 & 6 & 0 & 0 & 4 \\ 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 4 & 6 & 3 \end{pmatrix}.$$

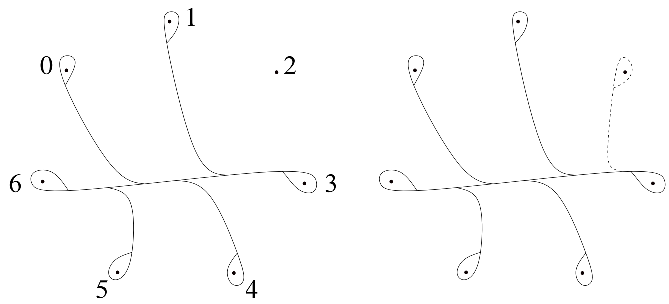

Figure 14: Constructing the train track for the map ψ.

The space of admissible weights on $\tau_{\bar{\mu}}$ is the subset of $\mathbb{R}^6$ given by positive real numbers a, b, c, d, e and f such that $b-a$, $d-c$ and $f-e$ are all positive and satisfy the triangle inequalities. The linear map described above preserves this subset. The square of the matrix B is strictly positive, which implies that the matrix is Perron–Frobenius. The top eigenvalue is $7+4\sqrt{3}$, which is associated to a unique irrational measured lamination F carried by $\tau_{\bar{\mu}}$ that is fixed by $\phi_{\bar{\mu}}$. As the train track is large, generic, and birecurrent, we may apply Lemma 2.4 to finish the proof that this map is pseudo-Anosov.

We now modify the pseudo-Anosov maps from Example A.1 to find two pseudo-Anosov maps on the seven-times punctured sphere. To achieve this, we apply Theorem 1.3 once to each of the maps found in Example A.1. For each of these maps, we note that we can apply the modification more than once to obtain additional pseudo-Anosov maps defined on spheres with more punctures.

Example A.2 We consider the seven-times punctured sphere with the labeling as introduced in Theorem 1.2. After applying Theorem 1.3 to the two partitions found in Example A.1, we obtain two partitions

$$\mu' = \{\{0,4\},\{1,5\},\{2,6\},\{3\}\} = \{\mu'_1, \mu'_2, \mu'_3, \mu'_4\},$$

and

$$\bar{\mu}' = \{\{0,3,5\},\{1,4,6\},\{2\}\} = \{\bar{\mu}'_1, \bar{\mu}'_2, \bar{\mu}'_3\}.$$

We begin by proving that the $\phi_{\mu'} = D^2_{\mu'_4} D^2_{\mu'_3} D^2_{\mu'_2} D^2_{\mu'_1}$ induced by the partition μ' is pseudo-Anosov. First, we describe how to construct the train track associated to $\phi_{\mu'}$, denoted by $\tau_{\mu'}$, from the train track τ_μ associated to the map ϕ_μ from the previous example. Consider the train track τ and place an extra puncture between the punctures

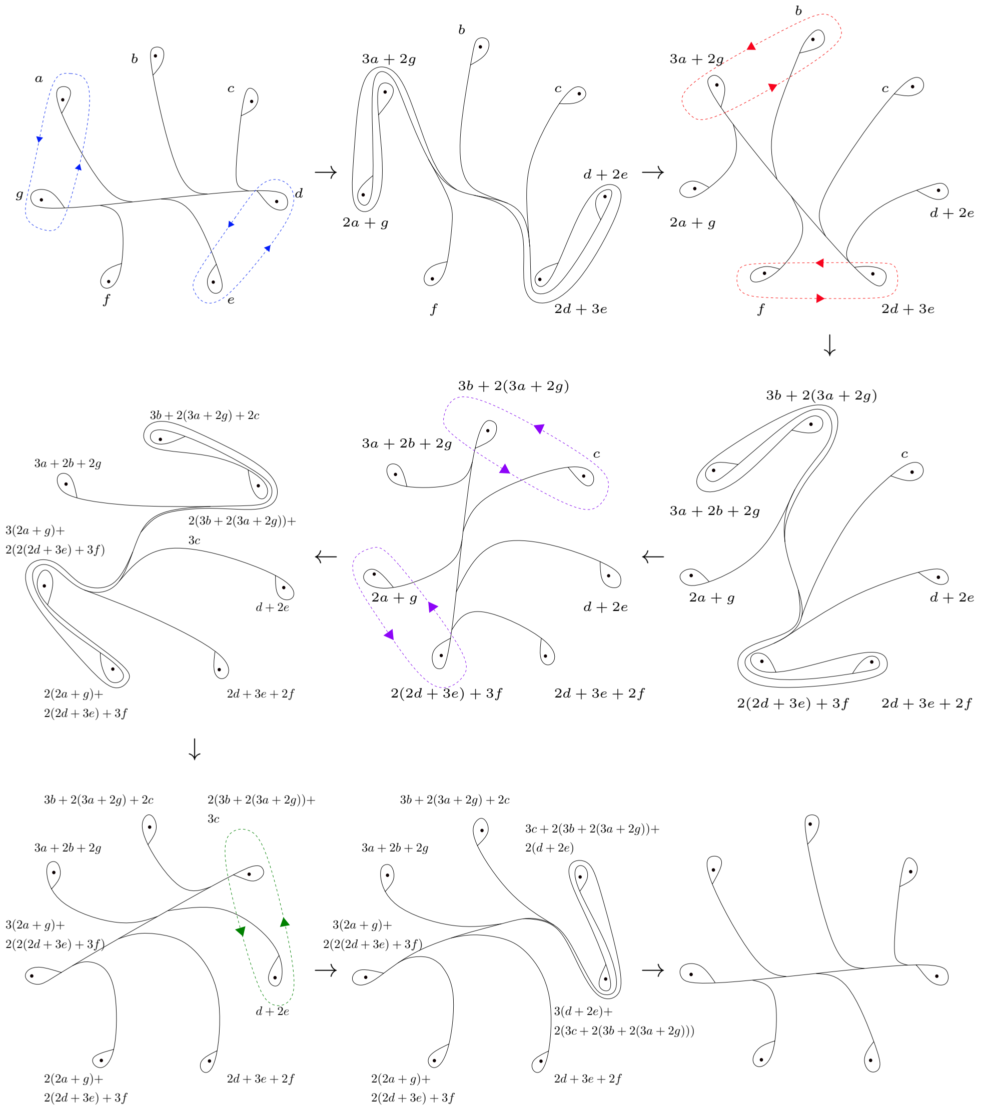

Figure 15: The train track $\phi_{\mu'}(\tau_{\mu'})$ is carried by $\tau_{\mu'}$.

labeled 1 and 2 in the previous example. Relabel the punctures so that the labeling is as in Theorem 1.2; see Figure 14, left. Therefore, we have a train track without a branch around the puncture labeled 2, but the rest of the train track is as in Example A.1 (up to relabeling). We construct a branch around the puncture labeled 2 which will turn tangentially towards the two valent pretrack, turning left towards the puncture labeled 3; see Figure 14, right.

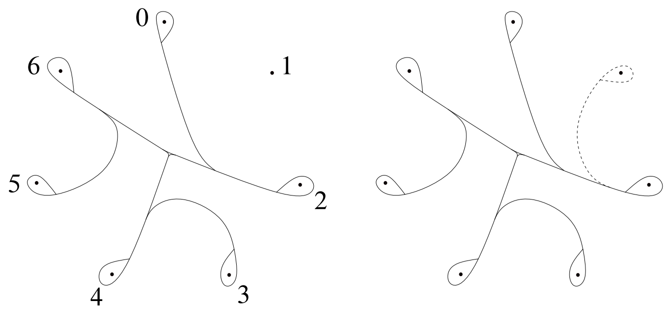

Figure 16: Constructing the train track for the map $\phi_{\bar{\mu}'}$.

The series of images in Figure 15 depict the train track $\tau_{\mu'}$ and its images under successive applications of the Dehn twists associated to $\phi_{\mu'}$. These images prove that $\phi_{\mu'}(\tau_{\mu'})$ is carried by $\tau_{\mu'}$. By keeping track of the weights on $\tau_{\mu'}$, we calculate that the induced action on the space of weights on $\tau_{\mu'}$ is given by the matrix

$$C = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 0 & 0 & 0 & 4 \\ 12 & 6 & 3 & 2 & 0 & 0 & 8 \\ 24 & 12 & 6 & 3 & 6 & 0 & 16 \\ 0 & 0 & 0 & 2 & 3 & 2 & 0 \\ 4 & 0 & 0 & 4 & 6 & 3 & 2 \\ 6 & 0 & 0 & 8 & 12 & 6 & 3 \end{pmatrix}.$$

The space of admissible weights on τ' is the subset of $\mathbb{R}^7$ given by the positive real numbers a, b, c, d, e, f and g such that $a+b+d+f=c+e+g$. The linear map described above preserves this subset. The square of the matrix C is strictly positive, which implies that the matrix is Perron–Frobenius. Additionally, the top eigenvalue is approximately 22.08646, which is associated to a unique irrational measured lamination F carried by $\tau_{\mu'}$ which is fixed by $\phi_{\mu'}$. As the train track is large, generic, and birecurrent, we may apply Lemma 2.4 to finish the proof that this map is pseudo-Anosov.

We now show that the map $\phi_{\bar{\mu}'} = D^2_{\bar{\mu}_3'} D^2_{\bar{\mu}_2'} D^2_{\bar{\mu}_1'}$ induced by the partition $\bar{\mu}'$ is pseudo-Anosov. To construct the train track, we will consider the train track $\tau_{\bar{\mu}}$ associated to the map $\phi_{\bar{\mu}}$ from the previous example. Consider the train track $\tau_{\bar{\mu}}$ and place an extra puncture between the punctures labeled 0 and 1 in the previous example. Relabel the punctures so that the labeling is as in Theorem 1.2; see Figure 16, left. Therefore, we have a train track on $S_{0,7}$ which does not have a branch around the puncture labeled 1,

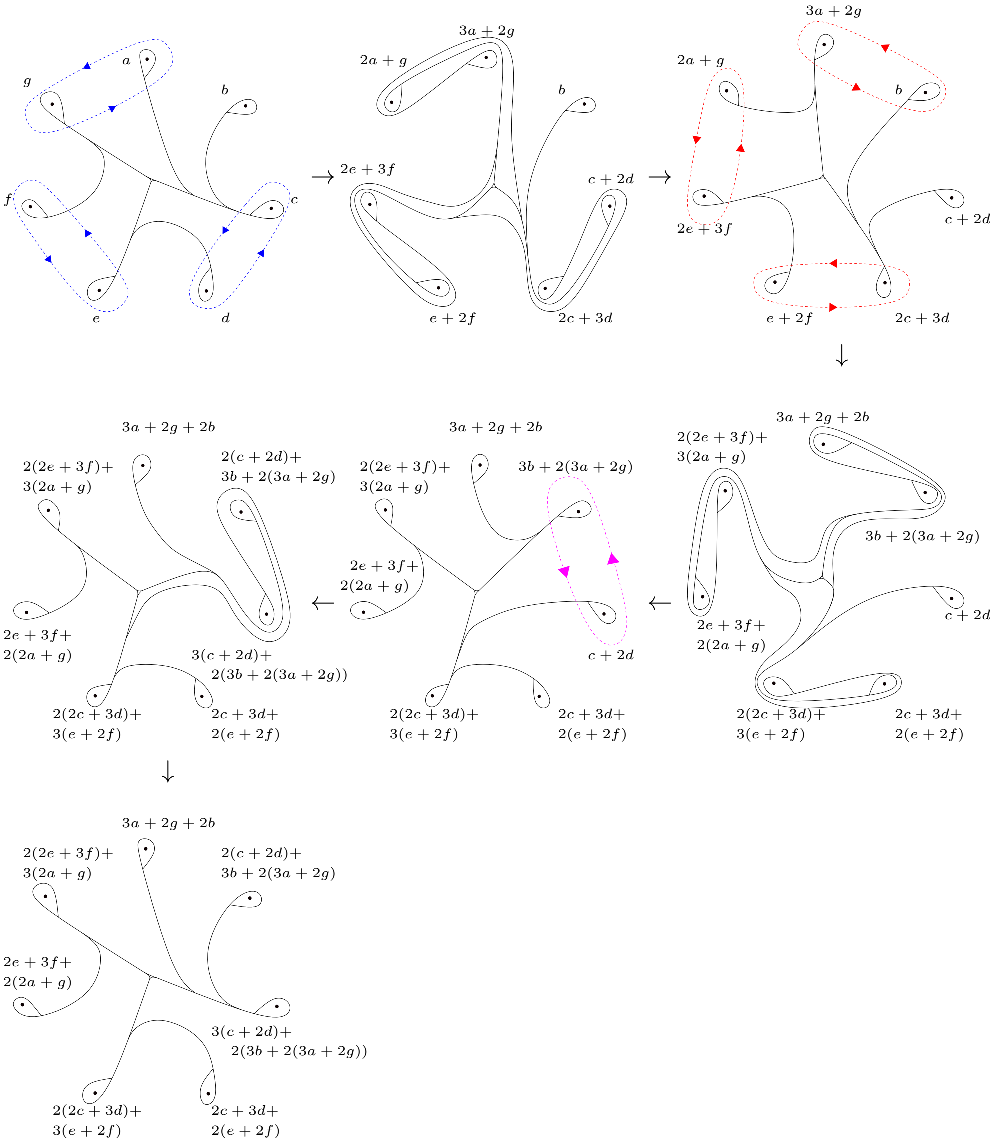

Figure 17: The train track $\phi'_{\bar{\mu}}(\tau'_{\bar{\mu}})$ is carried by $\tau'_{\bar{\mu}}$.

and the rest of the train track is as found in Example A.1 (up to relabeling). We then construct a branch around the puncture labeled 1 which will turn tangentially into the three valent pretrack, turning left towards the puncture labeled 2; see Figure 16, right.

The series of images in Figure 17 depict the train track $\tau_{\bar{\mu}'}$ and its images under successive applications of the Dehn twists associated to $\phi_{\bar{\mu}'}$.

Figure 17 shows that $\phi_{\bar{\mu}'}(\tau_{\bar{\mu}'})$ is indeed carried by $\tau_{\bar{\mu}'}$. By keeping track of the weights on $\tau_{\bar{\mu}'}$, we calculate that the induced action on the space of weights on $\tau_{\bar{\mu}'}$ is given by the matrix

$$D = \begin{pmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 2 \\ 6 & 3 & 2 & 4 & 0 & 0 & 4 \\ 12 & 6 & 3 & 6 & 0 & 0 & 8 \\ 0 & 0 & 2 & 3 & 2 & 4 & 0 \\ 0 & 0 & 4 & 6 & 3 & 6 & 0 \\ 4 & 0 & 0 & 0 & 2 & 3 & 2 \\ 6 & 0 & 0 & 0 & 4 & 6 & 3 \end{pmatrix}.$$

The space of admissible weights on $\tau_{\bar{\mu}'}$ is the subset of $\mathbb{R}^7$ given by the positive real numbers a, b, c, d, e, f and g such that $c-b-a$, $e-d$ and $g-f$ are all positive and satisfy the triangle inequalities. The linear map described above preserves this subset. The square of the matrix D is strictly positive, which implies that the matrix is Perron–Frobenius. The top eigenvalue of this matrix is

$$5+\tfrac{1}{3}\sqrt[3]{2916-12\sqrt{93}}+\left(\tfrac{2}{3}\right)^{2/3}\sqrt[3]{243+\sqrt{93}},$$

which is associated to a unique irrational measured lamination F carried by $\tau'_{\bar{\mu}}$ which is fixed by $\phi'_{\bar{\mu}}$. As the train track is large, generic, and birecurrent, we may apply Lemma 2.4 to finish the proof that this map is pseudo-Anosov.

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