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A model structure for weakly horizontally invariant double categories

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We construct a model structure on the category DblCat of double categories and double functors, whose trivial fibrations are the double functors that are surjective on objects, full on horizontal and vertical morphisms, and fully faithful on squares; and whose fibrant objects are the weakly horizontally invariant double categories.

We show that the functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, a more homotopical version of the usual horizontal embedding $\mathbb{H}$, is right Quillen and homotopically fully faithful when considering Lack's model structure on 2Cat. In particular, $\mathbb{H}^{\simeq}$ exhibits a levelwise fibrant replacement of $\mathbb{H}$. Moreover, Lack's model structure on 2Cat is right-induced along $\mathbb{H}^{\simeq}$ from the model structure for weakly horizontally invariant double categories.

We also show that this model structure is monoidal with respect to Böhm's Gray tensor product. Finally, we prove a Whitehead theorem characterizing the weak equivalences with fibrant source as the double functors which admit a pseudoinverse up to horizontal pseudonatural equivalence.

18D20, 18N10, 18N40

1 Introduction

This paper aims to study and compare the homotopy theories of two related types of 2–dimensional categories: 2–*categories* and *double categories*. While 2–categories consist of objects, morphisms, and 2–morphisms, double categories admit two types of morphisms between objects — *horizontal* and *vertical* morphisms — and their 2–morphisms are given by *squares*. In particular, a 2–category $\mathcal{A}$ can always be seen as a horizontal double category $\mathbb{H}\mathcal{A}$ with only trivial vertical morphisms. This assignment $\mathbb{H}$ gives a full embedding of 2–categories into double categories.

The category 2Cat of 2–categories and 2–functors admits a model structure, constructed by Lack in [13; 14]. In this model structure, the weak equivalences are the biequivalences; the trivial fibrations are the 2–functors which are surjective on objects, full on morphisms, and fully faithful on 2–morphisms; and all 2–categories are fibrant. Moreover, Lack gives a characterization of the cofibrant objects as the 2–categories whose underlying category is free. With this well-established model structure at hand, we raise the question of whether there is a homotopy theory for double categories which contains that of 2–categories.

Several model structures for double categories were first constructed by Fiore and Paoli in [4], and by Fiore, Paoli and Pronk in [5], but the homotopy theory of 2–categories does not embed in any of these homotopy theories for double categories. The first positive answer to this question is given by the authors in [16], and further related results appear in work in progress by Campbell [2]. In [16], we construct a model structure on the category DblCat of double categories and double functors that is right-induced from two copies of Lack's model structure on 2Cat; its weak equivalences are called the *double biequivalences*, and like most of the model structure, they exhibit a strong horizontal bias.

As a consequence, this model structure is very well behaved with respect to the horizontal embedding $\mathbb{H}$: the functor $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is both left and right Quillen, and Lack's model structure is both left- and right-induced along it. In particular, this says that Lack's model structure on 2Cat is created by $\mathbb{H}$ from the model structure on DblCat of [16]. Moreover, the functor $\mathbb{H}$ is homotopically fully faithful, and it embeds the homotopy theory of 2–categories into that of double categories in a reflective and coreflective way.

As it was constructed with a pronounced horizontal bias, this model structure is unsurprisingly not well behaved with respect to the vertical direction. For example, trivial fibrations, which are full on horizontal morphisms, are only surjective on vertical morphisms, and the free double category on two composable vertical morphisms is not cofibrant, as opposed to its horizontal analogue. In particular, this prevents the model structure from being monoidal with respect to the Gray tensor product for double categories defined by Böhm in [1].

Additionally, the model structure of [16] is not compatible with the first-named author's nerve construction from double categories to double $(\infty, 1)$–categories in [15]. Since all objects of this model structure on DblCat are fibrant, while the nerve of a double

category is in general not fibrant, we see that the nerve functor fails to be right Quillen. In fact, the double categories whose nerve is fibrant are precisely the *weakly horizontally invariant* ones. This condition requires that every vertical morphism in the double category can be lifted along horizontal equivalences at its source and target; see Definition 2.10.

The aim of this paper is to provide a new model structure on DblCat, whose trivial fibrations behave symmetrically with respect to the horizontal and vertical directions, and whose fibrant objects are the weakly horizontally invariant double categories. We achieve this by adding the inclusion $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}\mathbb{2}$ of the two endpoints into the vertical morphism to the class of cofibrations of the model structure in [16]. In particular, by making this inclusion into a cofibration, the trivial fibrations will now be given by the double functors that are surjective on objects, full on horizontal *and* vertical morphisms, and fully faithful on squares. The existence of this model structure was independently noticed at roughly the same time by Campbell [2].

As a referee pointed out, this change in the generating cofibrations requires us to enlarge the class of weak equivalences, since now the class of double functors that are both cofibrations and double biequivalences is not closed under pushouts, and therefore cannot be the class of trivial cofibrations in a model structure. Instead, we find that the weak equivalences of the desired model structure can be described as the double functors which induce a double biequivalence between fibrant replacements.

Theorem A *There is a model structure on* DblCat, *in which the trivial fibrations are the double functors which are surjective on objects*, *full on horizontal and vertical morphisms*, *and fully faithful on squares*; *and the fibrant objects are the weakly horizontally invariant double categories.*

This new model structure on DblCat takes care of the issues posed above. Namely, it is compatible with the double $(\infty, 1)$–categorical nerve construction of [15], and it is moreover monoidal, as we prove in Theorem 7.8.

Theorem B *The model structure on* DblCat *of Theorem A is monoidal with respect to Böhm's Gray tensor product.*

While the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ remains a left Quillen and homotopically fully faithful functor between Lack's model structure and our new model structure, it is not right Quillen anymore. Indeed, the horizontal double category

$\mathbb{H}\mathcal{A}$ associated to a 2–category $\mathcal{A}$ is typically not weakly horizontally invariant; see Remark 6.4.

Instead, we consider a more homotopical version of the horizontal embedding given by the functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. It sends a 2–category $\mathcal{A}$ to the double category $\mathbb{H}^{\simeq}\mathcal{A}$, whose underlying horizontal 2–category is still $\mathcal{A}$, but whose vertical morphisms are given by the adjoint equivalences of $\mathcal{A}$. In particular, the inclusion $\mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$ is a weak equivalence, as shown in Theorem 6.5, and therefore exhibits $\mathbb{H}^{\simeq}\mathcal{A}$ as a fibrant replacement of $\mathbb{H}\mathcal{A}$ in the model structure for weakly horizontally invariant double categories.

In Theorem 6.6, we prove that $\mathbb{H}^{\simeq}$ is a right Quillen functor, and that the derived counit is levelwise a biequivalence in 2Cat; therefore, $\mathbb{H}^{\simeq}$ embeds the homotopy theory of 2–categories into that of weakly horizontally invariant double categories in a reflective way. Furthermore, we show in Theorem 6.8 that $\mathbb{H}^{\simeq}$ not only preserves, but also reflects weak equivalences and fibrations.

Theorem C *The adjunction*

$$2\mathrm{Cat} \underset{\mathbb{H}^{\simeq}}{\overset{L^{\simeq}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{DblCat}$$

is a Quillen pair between Lack's model structure on 2Cat *and the model structure on* DblCat *of Theorem A. Moreover, the derived counit of this adjunction is levelwise a biequivalence, and Lack's model structure on* 2Cat *is right-induced along* $\mathbb{H}^{\simeq}$ *from the model structure on* DblCat.

We also show in Theorem 6.1 that the identity functor from our new model structure on DblCat to the one of [16] is right Quillen and homotopically fully faithful. This implies that, unsurprisingly, the homotopy theory of weakly horizontally invariant double categories is embedded into the homotopy theory for double categories developed in [16].

To summarize, we have a triangle of right Quillen and homotopically fully faithful functors

$$\begin{array}{ccc} & 2\mathrm{Cat} & \\ {\scriptstyle \mathbb{H}^{\simeq}}\swarrow & \simeq \swarrow & \searrow{\scriptstyle \mathbb{H}} \\ \mathrm{DblCat}_{\mathsf{whi}} & \xrightarrow[\mathrm{id}]{} & \mathrm{DblCat} \end{array}$$

filled by a natural transformation which is levelwise a weak equivalence.

Finally, in a similar vein to Grandis's result [6, Theorem 4.4.5], we obtain a Whitehead theorem characterizing the weak equivalences with fibrant source in our model structure as the double functors which admit a pseudoinverse up to horizontal pseudonatural equivalences; see Theorem 8.1. Indeed, the weakly horizontally invariant condition on a double category is a 2–categorical analogue of the horizontally invariant condition of [6, Theorem and Definition 4.1.7].

Theorem D (Whitehead theorem for double categories) *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories such that $\mathbb{A}$ is weakly horizontally invariant. Then a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a double biequivalence if and only if there is a pseudodouble functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal pseudonatural equivalences $\mathrm{id}_{\mathbb{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathbb{B}}$.*

This result implies that the weak equivalences between fibrant objects in the model structure for weakly horizontally invariant double categories resemble the biequivalences between 2–categories.

Outline

In Section 2, we recall some notations and definitions of double category theory introduced in [16]. We also introduce weakly horizontally invariant double categories and the homotopical horizontal embedding functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$. Then, in Section 3, we give the main features of the model structure on DblCat. In particular, we describe the cofibrations, trivial fibrations, and weak equivalences. The proof of the existence of this model structure uses several technical results presented in Section 4 and is completed in Section 5. After establishing the model structure, we study in Section 6 its relation with the model structure on DblCat of [16] and with Lack's model structure on 2Cat. In Section 7, we prove that it is monoidal with respect to the Gray tensor product for double categories. The last section, Section 8, is devoted to the proof of the Whitehead theorem for double categories.

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In a first version of this paper, the authors had proposed a model structure with the same cofibrations and fibrant objects, but with double biequivalences as the class of weak equivalences. We are thankful for the careful reading of a referee, who alerted us to this mistake and provided an example showing that the proposed trivial cofibrations were not stable under pushout. Since then, the paper has gone through a thorough rewriting, and we are grateful to Jérôme Scherer and Denis-Charles Cisinski for helpful suggestions regarding the construction of our new class of weak equivalences. We are also grateful to the referee who read this second version and provided detailed feedback to improve the exposition.

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2 Double categorical preliminaries

We introduce in this section the concepts and notations that will be used throughout this paper; for a more detailed treatment of 2–categories and double categories, we refer the reader to [11] and [6], respectively. We denote by 2Cat the category of 2–categories and 2–functors, and by DblCat the category of double categories and double functors. We will use the fact that these categories are locally presentable and hence both complete and cocomplete. For 2Cat this is given as a special case of [12, Section 4]; the statement for DblCat can be found in the proof of [5, Theorem 4.1].

To fix notation, we first recall the definition of a double category.

Definition 2.1 A *double category* $\mathbb{A}$ consists of

(i) objects $A, B, C, \dots,$

(ii) horizontal morphisms $a\colon A \to B$ with composition denoted by ba,

(iii) vertical morphisms $u\colon A \multimap A'$ with composition denoted by vu,

(iv) squares (or cells) $\alpha\colon (u \begin{smallmatrix} a \\ b \end{smallmatrix} v)$ of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle u}\downarrow\bullet & \alpha & \bullet\downarrow{\scriptstyle v} \\ A' & \xrightarrow[b]{} & B' \end{array}$$

with both horizontal composition along their vertical boundaries and vertical composition along their horizontal boundaries, and

(v) horizontal identities $\mathrm{id}_A\colon A \to A$ and vertical identities $e_A\colon A \nrightarrow A$ for each object A, vertical identity squares $e_a\colon (\mathrm{id}_A \overset{a}{a} \mathrm{id}_B)$ for each horizontal morphism $a\colon A \to B$, horizontal identity squares $\mathrm{id}_u\colon (u \overset{\mathrm{id}_A}{\mathrm{id}_{A'}} u)$ for each vertical morphism $u\colon A \nrightarrow A'$, and identity squares $\square_A = \mathrm{id}_{e_A} = e_{\mathrm{id}_A}$ for each object A,

such that all compositions are unital and associative, and such that the horizontal and vertical compositions of squares satisfy the interchange law.

Let us also recall the following notation.

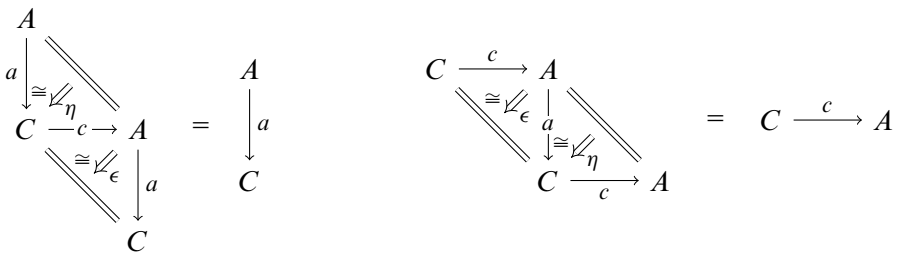
Notation 2.2 We write $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ for the horizontal embedding, which sends a 2–category $\mathcal{A}$ to the double category $\mathbb{H}\mathcal{A}$ with the same objects as $\mathcal{A}$, the morphisms of $\mathcal{A}$ as its horizontal morphisms, only trivial vertical morphisms, and the 2–morphisms of $\mathcal{A}$ as its squares. This functor has a right adjoint $\boldsymbol{H}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ that sends a double category $\mathbb{A}$ to its underlying horizontal 2–category $\boldsymbol{H}\mathbb{A}$ obtained by forgetting the vertical morphisms of $\mathbb{A}$. Note that $\boldsymbol{H}\mathbb{H} = \mathrm{id}_{2\mathrm{Cat}}$.

Similarly, there is a vertical embedding $\mathbb{V}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ which also admits a right adjoint $\boldsymbol{V}\colon \mathrm{DblCat} \to 2\mathrm{Cat}$ extracting from a double category its underlying vertical 2–category.

2.1 Weak horizontal invertibility in a double category

We recall the notions of weak horizontal invertibility for horizontal morphisms and squares introduced in [16, Section 2]. These notions were independently developed by Grandis and Paré in [8, Section 2], where the weakly horizontally invertible squares are called *equivalence cells*, and they rely on the notion of (adjoint) equivalences in a 2–category which we now recall.

Definition 2.3 A morphism $a\colon A \to C$ in a 2–category $\mathcal{A}$ is an *equivalence* if there is a morphism $c\colon C \to A$ together with 2–isomorphisms $\eta\colon \mathrm{id}_A \Rightarrow ca$ and $\epsilon\colon ac \Rightarrow \mathrm{id}_C$ in $\mathcal{A}$. It is an *adjoint equivalence* if the 2–isomorphisms η and ϵ further satisfy the following triangle identities:



Definition 2.4 A horizontal morphism $a\colon A \to B$ in a double category $\mathbb{A}$ is a *horizontal* (*adjoint*) *equivalence* if it is an (adjoint) equivalence in the underlying horizontal 2–category $\boldsymbol{H}\mathbb{A}$. We write $a\colon A \xrightarrow{\simeq} B$.

For the next definition, we remind the reader that the category DblCat is cartesian closed, and we denote its internal hom double category by $[-,-]$. In particular, we consider the functor $\boldsymbol{H}[\mathbb{V}2,-]\colon \mathrm{DblCat} \to 2\mathrm{Cat}$, where $\mathbb{V}2$ is the free double category on a vertical morphism. See [16, Definition 2.11] for an explicit description.

Definition 2.5 A square $\alpha\colon (u \begin{smallmatrix} a \\ a' \end{smallmatrix} w)$ in a double category $\mathbb{A}$ is *weakly horizontally invertible* if it is an equivalence in the 2–category $\boldsymbol{H}[\mathbb{V}2,\mathbb{A}]$. In other words, if there is a square $\gamma\colon (w \begin{smallmatrix} c \\ c' \end{smallmatrix} u)$ in $\mathbb{A}$ together with four vertically invertible squares η, η', ϵ, and ϵ' as in the following pasting equalities:

$$\begin{array}{ccc} A & = & A \\ \| & \eta\ \wr\| & \| \\ A & \xrightarrow{a} C \xrightarrow{c} & A \\ u\downarrow & \alpha\ w\downarrow\ \gamma & \downarrow u \\ A' & \xrightarrow[a']{} C' \xrightarrow[c']{} & A' \end{array} = \begin{array}{ccc} A & = & A \\ u\downarrow & \mathrm{id}_u & \downarrow u \\ A' & = & A' \\ \| & \eta'\ \wr\| & \| \\ A' & \xrightarrow[a']{} C' \xrightarrow[c']{} & A' \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{c} A \xrightarrow{a} & C \\ \| & \epsilon\ \wr\| & \| \\ C & = & C \\ w\downarrow & \mathrm{id}_w & \downarrow w \\ C' & = & C' \end{array} = \begin{array}{ccc} C & \xrightarrow{c} A \xrightarrow{a} & C \\ w\downarrow & \gamma\ u\downarrow\ \alpha & \downarrow w \\ C' & \xrightarrow[c']{} A' \xrightarrow[a']{} & C' \\ \| & \epsilon'\ \wr\| & \| \\ C' & = & C' \end{array}$$

We call γ a *weak inverse* of α, and we denote weakly horizontally invertible squares by decorating the square with the symbol $\simeq$.

Remark 2.6 In particular, the horizontal boundaries a and a' of a weakly horizontally invertible square α as above are horizontal equivalences witnessed by the data (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$. We call them the *horizontal equivalence data* of α. Moreover, if (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ are both horizontal adjoint equivalences, we call them the *horizontal adjoint equivalence data* of α.

Remark 2.7 A horizontal equivalence can always be promoted to a horizontal *adjoint* equivalence, since the corresponding result holds for 2–categories; see for example [19, Lemma 2.1.11]. Similarly, a weakly horizontally invertible square can always be promoted to one with horizontal *adjoint* equivalence data.

The next result ensures that the weak inverse of a weakly horizontally invertible square is unique with respect to fixed horizontal adjoint equivalences.

Lemma 2.8 [15, Lemma A.1.1] *Given a weakly horizontally invertible square* $\alpha\colon (u\ {}^{a}_{a'}\ w)$ *and two horizontal adjoint equivalences* (a, c, η, ϵ) *and* $(a', c', \eta', \epsilon')$ *in a double category* $\mathbb{A}$*, there is a unique weak inverse* $\gamma\colon (w\ {}^{c}_{c'}\ u)$ *of* α *with respect to these horizontal adjoint equivalences.*

2.2 Double biequivalences and weakly horizontally invariant double categories

The weak equivalences of the desired model structure for double categories rely on *double biequivalences*, which are the weak equivalences of the model structure on DblCat constructed in [16].

Definition 2.9 Let $\mathbb{A}$ and $\mathbb{B}$ be double categories. A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a (*horizontal*) *double biequivalence* if it is:

(db1) (Horizontally) biessentially surjective on objects: for every object $B \in \mathbb{B}$, there is an object $A \in \mathbb{A}$ together with a horizontal equivalence $B \xrightarrow{\simeq} FA$ in $\mathbb{B}$.

(db2) Essentially full on horizontal morphisms: for every pair of objects $A, C \in \mathbb{A}$ and every horizontal morphism $b\colon FA \to FC$ in $\mathbb{B}$, there is a horizontal morphism $a\colon A \to C$ together with a vertically invertible square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \cong & \| \\ FA & \xrightarrow[Fa]{} & FC \end{array}$$

(db3) (Horizontally) biessentially surjective on vertical morphisms: for every vertical morphism $v\colon B \twoheadrightarrow B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \twoheadrightarrow A'$ in $\mathbb{A}$ together with a weakly horizontally invertible square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} B & \xrightarrow{\simeq} & FA \\ {\scriptstyle v}\downarrow & \simeq & \downarrow{\scriptstyle Fu} \\ B' & \xrightarrow[\simeq]{} & FA' \end{array}$$

(db4) Fully faithful on squares: for every pair of horizontal morphisms $a\colon A \to C$ and $a'\colon A' \to C'$ in $\mathbb{A}$, every pair of vertical morphisms $u\colon A \twoheadrightarrow A'$ and

$w\colon C \nrightarrow C'$ in $\mathbb{A}$, and every square β in $\mathbb{B}$ as depicted below left, there is a unique square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$:

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fw} \\ FA' & \xrightarrow[Fa']{} & FC' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[a']{} & C' \end{array}$$

We also introduce the notion of *weakly horizontally invariant* double categories, which will form the class of fibrant objects in our model structure. This is a 2–categorical analogue of the notion of *horizontally invariant* double categories, introduced by Grandis and Paré in [7, Section 2.4] as the double categories whose vertical morphisms are transferable along horizontal isomorphisms.

Definition 2.10 A double category $\mathbb{A}$ is *weakly horizontally invariant* if, for every diagram in $\mathbb{A}$ as depicted below left, where a and a' are horizontal equivalences, there is a vertical morphism $u\colon A \nrightarrow A'$ together with a weakly horizontally invertible square in $\mathbb{A}$ as depicted below right:

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ & & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[a']{\simeq} & C' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ {\scriptstyle u}\downarrow & \simeq & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[a']{\simeq} & C' \end{array}$$

Example 2.11 One can easily check that the (flat) double category $\mathbb{R}\mathrm{elSet}$ of relations of sets is weakly horizontally invariant. More relevantly, this class of double categories also contains the double categories of quintets $\mathbb{Q}\mathcal{A}$ and of adjunctions $\mathbb{A}\mathrm{dj}\mathcal{A}$ built from any 2–category $\mathcal{A}$. A precise description of these double categories can be found in [6, Section 3.1]; in fact, the reader may check that all examples presented in that section are weakly horizontally invariant.

Remark 2.12 The horizontal double category $\mathbb{H}\mathcal{A}$ associated to a 2–category $\mathcal{A}$ is typically not weakly horizontally invariant. To see this, consider the horizontal double category $\mathbb{H}E_{\mathrm{adj}}$, where E_{adj} denotes the free-living adjoint equivalence. Then there is no vertical morphism in $\mathbb{H}E_{\mathrm{adj}}$ filling the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{\simeq} & 1 \\ & & \downarrow\!\!\downarrow \\ 1 & = & 1 \end{array}$$

as $\mathbb{H}E_{\mathrm{adj}}$ only contains trivial vertical morphisms, which shows that it is not weakly horizontally invariant.

Using the same reasoning, one can show that the horizontal double category $\mathbb{H}\mathcal{A}$ associated to a 2–category $\mathcal{A}$ is weakly horizontally invariant if and only if there is no adjoint equivalence in $\mathcal{A}$.

2.3 The homotopical horizontal embedding

Since all 2–categories are fibrant, Remark 2.12 implies that the functor

$$\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$$

is not right Quillen with respect to the desired model structure for weakly horizontally invariant double categories. Instead, we need to consider a more homotopical version of the horizontal embedding $\mathbb{H}$, which provides a levelwise fibrant replacement for $\mathbb{H}$.

Definition 2.13 The *homotopical horizontal embedding* is defined as the functor $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ that sends a 2–category $\mathcal{A}$ to the double category $\mathbb{H}^{\simeq}\mathcal{A}$ having the same objects as $\mathcal{A}$, the morphisms of $\mathcal{A}$ as horizontal morphisms, one vertical morphism for each adjoint equivalence $(u, u^\sharp, \eta, \epsilon)$ in $\mathcal{A}$, and squares

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle (u,u^\sharp,\eta,\epsilon)}\downarrow & \alpha & \downarrow{\scriptstyle (w,w^\sharp,\eta',\epsilon')} \\ A' & \xrightarrow[a']{} & C' \end{array}$$

given by the 2–morphisms $\alpha\colon wa \Rightarrow a'u$ in $\mathcal{A}$. Compositions are induced by compositions of morphisms and 2–morphisms in $\mathcal{A}$. Although a vertical morphism always contains the whole data of an adjoint equivalence, we often denote it by its left adjoint u.

Remark 2.14 Every vertical morphism in the double category $\mathbb{H}^{\simeq}\mathcal{A}$ is a *vertical equivalence*, ie an equivalence in the underlying vertical 2–category.

The functor $\mathbb{H}^{\simeq}$ admits a left adjoint.

Proposition 2.15 *The functor $\mathbb{H}^{\simeq}$ is part of an adjunction*

$$\mathrm{DblCat} \underset{\mathbb{H}^{\simeq}}{\overset{L^{\simeq}}{\rightleftarrows}} 2\mathrm{Cat} \quad (\perp)$$

Proof Consider the full subcategory $\Delta_{\leq 3}$ of the simplex category Δ on the objects [0], [1], [2], and [3]. Then the category DblCat can be seen as the full subcategory of $\mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}}$ on the objects $X \in \mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}}$ whose component sets $X_{i,j}$ for $2 \leq i, j \leq 3$ are obtained as certain limits over the sets $X_{0,0}$ of objects, $X_{1,0}$ of horizontal morphisms, $X_{0,1}$ of vertical morphisms, and $X_{1,1}$ of squares; eg $X_{2,1} \cong X_{1,1} \times_{X_{0,1}} X_{1,1}$.

The strategy of the proof is to show that there is an adjunction

$$\mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}} \underset{R}{\overset{L^{\simeq}}{\rightleftarrows}} 2\mathrm{Cat} \qquad (\perp)$$

such that the right adjoint R lands in $\mathrm{DblCat} \subseteq \mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}}$ and agrees with the functor $\mathbb{H}^{\simeq}$; then the adjunction above must restrict to an adjunction $L^{\simeq} \dashv \mathbb{H}^{\simeq}$, as desired. We now proceed to prove these claims.

We define a functor $\ell^{\simeq}\colon \Delta_{\leq 3} \times \Delta_{\leq 3} \to 2\mathrm{Cat}$ by giving its values on the subcategory spanned by $[0,0]$, $[1,0]$, $[0,1]$, and $[1,1]$ and setting its values on $[i,j]$ for $2 \leq i, j \leq 3$ in such a way that ℓ preserves colimits; eg $\ell^{\simeq}[2,1] = \ell^{\simeq}[1,1] \amalg_{\ell^{\simeq}[0,1]} \ell^{\simeq}[1,1]$. We set $\ell^{\simeq}[0,0] = \mathbb{1}$, $\ell^{\simeq}[1,0] = \mathbb{2}$, $\ell^{\simeq}[0,1] = E_{\mathrm{adj}}$, and $\ell^{\simeq}[1,1] = \mathcal{A}$, where E_{adj} is the 2–category containing an adjoint equivalence, and $\mathcal{A}$ is the 2–category generated by morphisms, adjoint equivalences, and 2–morphisms

$$\begin{array}{ccc} A & \longrightarrow & C \\ \wr\downarrow & \overset{\alpha}{\swarrow} & \downarrow\wr \\ A' & \longrightarrow & C' \end{array}$$

with the obvious images of the morphisms in $\Delta_{\leq 3} \times \Delta_{\leq 3}$ between these objects.

By considering the left Kan extension along the Yoneda embedding

$$\begin{array}{ccc} \Delta_{\leq 3} \times \Delta_{\leq 3} & \xrightarrow{\ell^{\simeq}} & 2\mathrm{Cat} \\ {\scriptstyle y}\downarrow & \overset{L^{\simeq}}{\nearrow} & \\ \mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}} & \overset{R}{\longleftarrow} & \end{array}$$

we obtain a functor $L^{\simeq}$, which admits a right adjoint $R\colon 2\mathrm{Cat} \to \mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}}$ given by $R(\mathcal{B})_{i,j} = 2\mathrm{Cat}(\ell^{\simeq}[i,j], \mathcal{B})$, for all $0 \leq i, j \leq 3$ and all 2–categories $\mathcal{B}$.

By definition of $\ell^{\simeq}$, the image of R lands in the subcategory DblCat. In particular, the adjunction $L^{\simeq} \dashv R$ restricts to an adjunction

$$\mathrm{DblCat} \underset{R}{\overset{L^{\simeq}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} 2\mathrm{Cat}$$

Note that the representables at $[0,0]$, $[1,0]$, $[0,1]$, and $[1,1]$ in $\mathrm{Set}^{\Delta^{\mathrm{op}}_{\leq 3}\times\Delta^{\mathrm{op}}_{\leq 3}}$ coincide with the double categories $\mathbb{1}$, $\mathbb{H}2$, $\mathbb{V}2$, and $\mathbb{H}2\times\mathbb{V}2$, so that we have

$$L^{\simeq}(\mathbb{1}) = \mathbb{1}, \quad L^{\simeq}(\mathbb{H}2) = \mathbb{2}, \quad L^{\simeq}(\mathbb{V}2) = E_{\mathrm{adj}}, \quad L^{\simeq}(\mathbb{H}2\times\mathbb{V}2) = \mathcal{A}.$$

It remains to show that $R = \mathbb{H}^{\simeq}$. For this, it is enough to show that the sets of objects, horizontal morphisms, vertical morphisms, and squares of $R\mathcal{B}$ and $\mathbb{H}^{\simeq}\mathcal{B}$ coincide, for every 2–category $\mathcal{B}$. This is indeed the case, as for $\mathbb{A} \in \{\mathbb{1}, \mathbb{H}2, \mathbb{V}2, \mathbb{H}2\times\mathbb{V}2\}$,

$$\mathrm{DblCat}(\mathbb{A}, R\mathcal{B}) \cong 2\mathrm{Cat}(L^{\simeq}\mathbb{A}, \mathcal{B}) \cong \mathrm{DblCat}(\mathbb{A}, \mathbb{H}^{\simeq}\mathcal{B}),$$

where the first isomorphism holds by the universal property of the adjunction $L^{\simeq} \dashv R$ and the second by the definition of $\mathbb{H}^{\simeq}\mathcal{B}$. □

Remark 2.16 One can show that $L^{\simeq}$ admits the following, more explicit, description. Given a double category $\mathbb{A}$, $L^{\simeq}\mathbb{A}$ is the 2–category with the same objects as $\mathbb{A}$, a morphism for each horizontal morphism in $\mathbb{A}$, and a morphism for each vertical morphism in $\mathbb{A}$ which we formally make into an adjoint equivalence; ie we also add a formal inverse morphism, and the two necessary 2–isomorphisms. Aside from these formal 2–morphisms added to create the adjoint equivalences, we also have a 2–morphism $u'a \Rightarrow cu$ for each square in $\mathbb{A}$ of the form $\alpha\colon (u \overset{a}{c} u')$.

Furthermore, the composite in $L^{\simeq}\mathbb{A}$ of two morphisms coming from horizontal morphisms in $\mathbb{A}$ is given by their composite in $\mathbb{A}$, and the composite in $L^{\simeq}\mathbb{A}$ of two adjoint equivalences coming from vertical morphisms in $\mathbb{A}$ is given by the adjoint equivalence induced by their composite in $\mathbb{A}$, while two morphisms with one coming from a horizontal morphism and one coming from a vertical morphism compose freely. Similar holds for the 2–morphisms.

Remark 2.17 The functor $\mathbb{H}^{\simeq}$ is not a left adjoint since it does not preserve colimits. To see this, consider the span of 2–categories $\mathcal{B} \leftarrow \mathcal{A} \rightarrow \mathcal{C}$. We set $\mathcal{A}$ to be the 2–category with two objects 0 and 1, and freely generated by two morphisms $f\colon 0 \to 1$ and $g\colon 1 \to 0$ and two 2–morphisms $\eta\colon \mathrm{id}_0 \Rightarrow gf$ and $\epsilon\colon fg \Rightarrow \mathrm{id}_1$. Then let $\mathcal{B}$ be

the category obtained from $\mathcal{A}$ by inverting the 2–morphism η, and $\mathcal{C}$ be the category obtained from $\mathcal{A}$ by inverting the 2–morphism ϵ. The pushout $\mathcal{B} \sqcup_{\mathcal{A}} \mathcal{C}$ contains an equivalence (f, g, η, ϵ), and hence the double category $\mathbb{H}^{\simeq}(\mathcal{B} \sqcup_{\mathcal{A}} \mathcal{C})$ contains a vertical morphism induced by the corresponding adjoint equivalence. However, the double categories $\mathbb{H}^{\simeq}\mathcal{A}$, $\mathbb{H}^{\simeq}\mathcal{B}$, and $\mathbb{H}^{\simeq}\mathcal{C}$ do not have nontrivial vertical morphisms since there are no equivalences in $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$, and hence their pushout $\mathbb{H}^{\simeq}\mathcal{A} \sqcup_{\mathbb{H}^{\simeq}\mathcal{B}} \mathbb{H}^{\simeq}\mathcal{C}$ does not contain nontrivial vertical morphisms. This shows that $\mathbb{H}^{\simeq}$ does not preserve pushouts.

3 The model structure

Just as there are nerve constructions embedding categories into $(\infty, 1)$–categories, and 2–categories into $(\infty, 2)$–categories, in [15] the first-named author constructs a double categorical nerve, embedding double categories into double $(\infty, 1)$–categories. As the latter admit a natural model structure when considered as double Segal spaces, we expect to have a model structure on DblCat making this nerve into a right Quillen functor.

Since the double categories whose nerve is fibrant are precisely the weakly horizontally invariant ones, this suggests that such a model structure should have the weakly horizontally invariant double categories as its class of fibrant objects. Moreover, since the cofibrations of the model structure for double $(\infty, 1)$–categories are the monomorphisms and the inclusion $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}2$ is the image of a monomorphism under the left adjoint of the nerve, it should be added to the class of cofibrations of the model structure on DblCat of [16]; this allows us to characterize the trivial fibrations as the double functors which are surjective on objects, full on horizontal and vertical morphisms, and fully faithful on squares.

A first attempt to keep the double biequivalences — which were shown by the authors to be the class of weak equivalences in a model structure on DblCat in [16] — as the weak equivalences of this new model structure proves unsuccessful. Indeed, the resulting class of trivial cofibrations would not be closed under pushouts; the double functor $j_{\mathbb{A}}$ of Example 3.23 is an example of a pushout of such a trivial cofibration (namely, of the inclusion $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$) that is not a double biequivalence.

Instead, we identify the weak equivalences as the double functors which induce a double biequivalence between weakly horizontally invariant replacements.

Since many technical results, presented in Section 4, are needed to prove the existence of such a model structure, its proof is delayed to Section 5.

3.1 Weak factorization systems

We first recall the definition and basic results about weak factorization systems which will be used throughout the paper.

Notation 3.1 Let $\mathcal{M}$ be a category and $\mathcal{C}$ be a class of morphisms in $\mathcal{M}$. We write ${}^{\boxslash}\mathcal{C}$ (resp. $\mathcal{C}^{\boxslash}$) for the class of morphisms in $\mathcal{M}$ that have the left (resp. right) lifting property with respect to all morphisms in $\mathcal{C}$.

Definition 3.2 A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{M}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms in $\mathcal{M}$ such that $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$, and every morphism f in $\mathcal{M}$ can be factored as $f = rl$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

Remark 3.3 Recall that given a weak factorization system $(\mathcal{L}, \mathcal{R})$, both classes contain isomorphisms and are closed under composition and retracts. Furthermore, the left class $\mathcal{L}$ is closed under coproducts, pushouts, and transfinite compositions, and the right class $\mathcal{R}$ is closed under products and pullbacks, as explained for example in [18, Lemma 11.1.4] and the comment immediately below that lemma.

The following argument will be useful when proving that a certain map belongs to the left or right class of a weak factorization system; its proof can be found, for example, in [10, Lemma 1.1.9].

Remark 3.4 (retract argument) Consider a factorization $f = rl$ of a map f in a category $\mathcal{M}$. If f has the left lifting property with respect to r, then f is a retract of l. Dually, if f has the right lifting property with respect to l, then f is a retract of r.

Weak factorization systems are often generated by a set. To introduce this notion, we recall the following terminology.

Notation 3.5 Let $\mathcal{I}$ be a set of morphisms in a cocomplete category $\mathcal{M}$. Then a morphism in $\mathcal{M}$ is:

(i) *$\mathcal{I}$–injective* if it has the right lifting property with respect to every morphism in $\mathcal{I}$. The class of all such morphisms is denoted by $\mathcal{I}\text{–inj} := \mathcal{I}^{\boxslash}$.

(ii) An *$\mathcal{I}$–cofibration* if it has the left lifting property with respect to every $\mathcal{I}$–injective morphism. The class of all such morphisms is denoted by $\mathcal{I}\text{–cof} := {}^{\boxslash}(\mathcal{I}^{\boxslash})$.

(iii) A *relative $\mathcal{I}$–cell complex* if it is a transfinite composition of pushouts of morphisms in $\mathcal{I}$. The class of all such morphisms is denoted by $\mathcal{I}\text{–cell}$.

Remark 3.6 Recall that every $\mathcal{I}$–cofibration can be obtained as a retract of a relative $\mathcal{I}$–cell complex with the same source, and that in a locally presentable category $\mathcal{M}$, the pair $(\mathcal{I}\text{–cof}, \mathcal{I}\text{–inj})$ forms a weak factorization system, for any set of morphisms $\mathcal{I}$ in $\mathcal{M}$.

Definition 3.7 A weak factorization system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{M}$ is said to be *generated by a set $\mathcal{I}$ of morphisms* if $(\mathcal{L}, \mathcal{R}) = (\mathcal{I}\text{–cof}, \mathcal{I}\text{–inj})$.

3.2 Trivial fibrations, cofibrations, and cofibrant objects

We now identify a set $\mathcal{I}_w$ of double functors such that the $\mathcal{I}_w$–injective morphisms are precisely the trivial fibrations we seek.

Notation 3.8 We denote by $\mathbb{1}$ the terminal double category, by $\mathbb{2}$ the free (2–)category on a morphism, by $\mathbb{S} = \mathbb{H}\mathbb{2} \times \mathbb{V}\mathbb{2}$ the free double category on a square, by $\delta\mathbb{S}$ its boundary, and by $\mathbb{S}_2$ the free double category on two squares with the same boundary.

Let $\mathcal{I}_w$ denote the set containing the following double functors:

(i) the unique map $I_1 : \varnothing \to \mathbb{1}$,

(ii) the inclusion $I_2 : \mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}\mathbb{2}$,

(iii) the inclusion $I_3 : \mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}\mathbb{2}$,

(iv) the inclusion $I_4 : \delta\mathbb{S} \to \mathbb{S}$,

(v) the double functor $I_5 : \mathbb{S}_2 \to \mathbb{S}$ sending the two nontrivial squares in $\mathbb{S}_2$ to the nontrivial square of $\mathbb{S}$.

Proposition 3.9 *A double functor $F : \mathbb{A} \to \mathbb{B}$ is in $\mathcal{I}_w$–inj if and only if it is surjective on objects, full on horizontal and vertical morphisms, and fully faithful on squares.*

Proof This is obtained directly from a close inspection of the right lifting properties with respect to the double functors in $\mathcal{I}_w$. □

Remark 3.10 It is straightforward to check that any double functor in $\mathcal{I}_w$–inj is a double biequivalence; see Definition 2.9.

The class of $\mathcal{I}_w$–cofibrations admits a nice characterization in terms of their underlying horizontal and vertical functors. We denote by $U\colon 2\mathrm{Cat} \to \mathrm{Cat}$ the functor sending a 2–category to its underlying category, where Cat is the category of categories and functors.

Theorem 3.11 *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{I}_w$–cof if and only if its underlying horizontal and vertical functors, $U\boldsymbol{H}F$ and $U\boldsymbol{V}F$, have the left lifting property with respect to surjective on objects and full functors.*

Proof The proof works as in [16, Proposition 4.7], with the evident modifications for the vertical direction. □

Remark 3.12 An equivalent characterization of functors which have the left lifting property with respect to surjective on objects and full functors can be found in [13, Corollary 4.12]. These are the functors $F\colon \mathcal{A} \to \mathcal{B}$ which are injective on objects, faithful, and such that there are functors $I\colon \mathcal{B} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{B}$ with $RI = \mathrm{id}_{\mathcal{B}}$, where the category $\mathcal{C}$ is obtained from the image of F by freely adjoining objects and then freely adjoining morphisms between specified objects.

In particular, we can see that a double functor in $\mathcal{I}_w$–cof is injective on objects, and faithful on horizontal and vertical morphisms.

Using the characterization mentioned in Remark 3.12, we can see that the cofibrant objects in the desired model structure are precisely the double categories whose underlying horizontal and vertical categories are free.

Corollary 3.13 *A double category $\mathbb{A}$ is such that the unique map $\varnothing \to \mathbb{A}$ is in $\mathcal{I}_w$–cof if and only if its underlying horizontal and vertical categories $U\boldsymbol{H}\mathbb{A}$ and $U\boldsymbol{V}\mathbb{A}$ are free.*

Proof The proof works as in [16, Proposition 4.9], with the evident modifications for the vertical direction. □

3.3 Weakly horizontally invariant replacements and weak equivalences

Our next goal is to introduce the class of weak equivalences; these will be the double functors that induce a double biequivalence between weakly horizontally invariant replacements. To construct a weakly horizontally invariant double category from a double category $\mathbb{A}$, we attach $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$–data freely to every horizontal adjoint equivalence

in $\mathbb{A}$, where the 2–category E_{adj} is the free-living adjoint equivalence $\{0 \xrightarrow{\simeq} 1\}$. Since this will be a key notion throughout the paper, let us first describe the double category $\mathbb{H}^{\simeq} E_{\mathrm{adj}}$.

Description 3.14 The double category $\mathbb{H}^{\simeq} E_{\mathrm{adj}}$ is generated by the data of a horizontal adjoint equivalence (f, g, η, ϵ), vertical morphisms u and v, and weakly horizontally invertible squares α and γ,

$$\begin{array}{ccc} 0 & \xrightarrow[\simeq]{f} & 1 \\ {\scriptstyle u}\downarrow & \alpha \simeq & \| \\ 1 & = & 1 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow[\simeq]{g} & 0 \\ {\scriptstyle v}\downarrow & \gamma \simeq & \| \\ 0 & = & 0 \end{array}$$

where u and v are induced by the adjoint equivalences (f, g, η, ϵ) and $(g, f, \epsilon^{-1}, \eta^{-1})$, respectively, and the squares α and γ are induced by the identity 2–morphisms at f and g, respectively.

In particular, note that $\mathbb{H}^{\simeq} E_{\mathrm{adj}}$ also contains vertically invertible squares η and ϵ given by the unit and counit of the adjoint equivalence (f, g, η, ϵ), as well as horizontally invertible squares α' and γ' which are the weak inverses of α and γ, respectively.

Furthermore, we can compose these to form weakly horizontally invertible squares β and δ,

$$\begin{array}{ccc} 0 & = & 0 \\ \| & \beta \simeq & \downarrow{\scriptstyle u} \\ 0 & \xrightarrow[\simeq]{f} & 1 \end{array} = \begin{array}{ccccc} 0 & = & 0 & = & 0 \\ \| & \eta \Rightarrow & \| & & \| \\ 0 & \xrightarrow[\simeq]{f} & 1 & \xrightarrow[\simeq]{g} & 0 \\ \| & e_f & \| & \alpha' \simeq & \downarrow{\scriptstyle u} \\ 0 & \xrightarrow[\simeq]{f} & 1 & = & 1 \end{array} \qquad \begin{array}{ccc} 1 & = & 1 \\ \| & \delta \simeq & \downarrow{\scriptstyle v} \\ 1 & \xrightarrow[\simeq]{g} & 0 \end{array} = \begin{array}{ccccc} 1 & = & 1 & = & 1 \\ \| & \epsilon^{-1} \Rightarrow & \| & & \| \\ 1 & \xrightarrow[\simeq]{g} & 0 & \xrightarrow[\simeq]{f} & 1 \\ \| & e_g & \| & \gamma' \simeq & \downarrow{\scriptstyle v} \\ 1 & \xrightarrow[\simeq]{g} & 0 & = & 0 \end{array}$$

and we can similarly construct their weak inverses β' and δ'.

Note that the horizontal composite of β with α is the vertical identity square e_f at f, and the vertical composite of β with α is the horizontal identity square id_u at u. In other words, this says that (f, u, α, β) is the data of an *orthogonal companion pair*; see [6, Section 4.1.1]. On the other hand, the horizontal composite of α' with β' is the vertical identity square e_g at g, and the vertical composite of β' with α' is the horizontal identity square id_u at u. In other words, this says that (g, u, α', β') is the data of an *orthogonal adjoint pair*; see [6, Section 4.1.2]. Similarly, (g, v, γ, δ) is the data of an orthogonal companion pair, and (f, v, γ', δ') is the data of an orthogonal adjoint pair.

Finally, one can check that the vertical morphisms (u, v) form a vertical adjoint equivalence, ie an adjoint equivalence in the underlying vertical 2–category $\boldsymbol{V}\mathbb{H}^{\simeq}E_{\mathrm{adj}}$, with unit η' given by the vertical composite of β with γ', and counit ϵ' given by the vertical composite of δ' with α. In particular, all the squares in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ are also *weakly vertically invertible* — the transposed notion of weakly horizontally invertible — with vertical weak inverses given by the obvious squares.

Notation 3.15 There is an inclusion $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ which sends the horizontal adjoint equivalence in $\mathbb{H}E_{\mathrm{adj}}$ to the horizontal adjoint equivalence (f, g, η, ϵ) in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$.

Remark 3.16 By uniqueness of weak inverses with respect to fixed horizontal adjoint equivalence data of Lemma 2.8, we can see that a double functor $G\colon \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \mathbb{A}$ is completely determined by its value on the horizontal adjoint equivalence (f, g, η, ϵ) and the squares α, γ in $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$.

We are now ready to construct a functorial weakly horizontally invariant replacement $(-)^{\mathrm{whi}}\colon \mathrm{DblCat} \to \mathrm{DblCat}^{2}$.

Construction 3.17 Let $\mathbb{A}$ be a double category and let $\mathrm{HorEq}(\mathbb{A})$ denote the set of all horizontal adjoint equivalence data in $\mathbb{A}$. Each horizontal adjoint equivalence (a, c, η, ϵ) in $\mathbb{A}$ defines a double functor $\mathbb{H}E_{\mathrm{adj}} \to \mathbb{A}$, and we define $\mathbb{A}^{\mathrm{whi}}$ as the pushout below left:

$$\begin{array}{ccc} \coprod_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}E_{\mathrm{adj}} & \longrightarrow & \mathbb{A} \\ \coprod_{\mathrm{HorEq}(\mathbb{A})} J_4 \downarrow & & \downarrow j_{\mathbb{A}} \\ \coprod_{\mathrm{HorEq}(\mathbb{A})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} & \longrightarrow & \mathbb{A}^{\mathrm{whi}} \end{array} \qquad \begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{B} \\ j_{\mathbb{A}} \downarrow & & \downarrow j_{\mathbb{B}} \\ \mathbb{A}^{\mathrm{whi}} & \xrightarrow[F^{\mathrm{whi}}]{} & \mathbb{B}^{\mathrm{whi}} \end{array}$$

This extends naturally to a functor $(-)^{\mathrm{whi}}\colon \mathrm{DblCat} \to \mathrm{DblCat}^{2}$. In particular, it sends a double category $\mathbb{A}$ to the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ and a double functor $F\colon \mathbb{A} \to \mathbb{B}$ to a commutative square in DblCat as depicted above right.

Remark 3.18 The double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is the identity on underlying horizontal categories and it is fully faithful on squares for every double category $\mathbb{A}$, since it is a pushout of coproducts of the double functor $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$. Hence a double functor $F\colon \mathbb{A} \to \mathbb{B}$ coincides with $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ on underlying horizontal categories.

Remark 3.19 The construction $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ adds $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$–data in $\mathbb{A}^{\mathrm{whi}}$ to each horizontal adjoint equivalence (a, c, η, ϵ) in $\mathbb{A}$, as detailed in Description 3.14. In particular, we can see that two vertical morphisms u and v were freely added in $\mathbb{A}^{\mathrm{whi}}$ for each equivalence (a, c, η, ϵ), as well as weakly horizontally invertible squares as in Description 3.14. We henceforth say that the morphisms u and v were *added using the horizontal adjoint equivalence data* (a, c, η, ϵ) *in* $\mathbb{A}$.

As claimed, the double category $\mathbb{A}^{\mathrm{whi}}$ is indeed weakly horizontally invariant.

Proposition 3.20 *For every double category* $\mathbb{A}$, *the double category* $\mathbb{A}^{\mathrm{whi}}$ *is weakly horizontally invariant.*

Proof Let $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ be horizontal equivalences in $\mathbb{A}$ and $w\colon C \nrightarrow C'$ be a vertical morphism in $\mathbb{A}^{\mathrm{whi}}$. By construction of $\mathbb{A}^{\mathrm{whi}}$, we have vertical morphisms $u\colon A \nrightarrow C$ and $v\colon C' \nrightarrow A'$ in $\mathbb{A}^{\mathrm{whi}}$ together with weakly horizontally invertible squares α and δ in $\mathbb{A}^{\mathrm{whi}}$:

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ {\scriptstyle u}\downarrow & \alpha \simeq & \| \\ C & = & C \end{array} \qquad\qquad \begin{array}{ccc} C' & = & C' \\ {\scriptstyle v}\downarrow & \delta \simeq & \| \\ A' & \xrightarrow[a']{\simeq} & C' \end{array}$$

Then the composite of vertical morphisms $vwu\colon A \nrightarrow A'$ together with the weakly horizontally invertible square given by the vertical composite of the squares α, id_w, and δ gives the desired lift. □

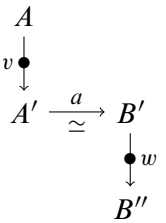
The foresight that $(-)^{\mathrm{whi}}$ will give a fibrant replacement in our desired model structure (as we show in Corollary 5.4) and that the double biequivalences will precisely be the weak equivalences between fibrant objects (proved in Proposition 5.5) motivates us to define our weak equivalences as the double functors inducing double biequivalences between weakly horizontally invariant replacements.

Definition 3.21 We define $\mathcal{W}$ to be the class of double functors $F\colon \mathbb{A} \to \mathbb{B}$ such that the induced double functor $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence.

Remark 3.22 Since double biequivalences are the weak equivalences in the model structure on DblCat of [16, Theorem 3.18], they satisfy 2-out-of-3 and are closed under retracts. As a consequence, the class $\mathcal{W}$ also has these properties, as the replacement $(-)^{\mathrm{whi}}$ is functorial.

Although double biequivalences are more tractable than our proposed weak equivalences, the passage to this bigger class is truly needed. Indeed, the class of double functors that are both in $\mathcal{I}_w$–cof and double biequivalences is not closed under pushouts, and thus cannot be the class of trivial cofibrations in a model structure.

Example 3.23 Let $\mathbb{A}$ be the double category generated by the data

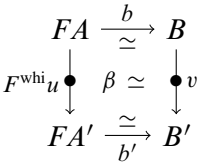


where a is a horizontal adjoint equivalence. As we will see in Corollary 5.4, the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ given by Construction 3.17 is a weak equivalence in $\mathcal{W}$. However, it is not a double biequivalence. To see this, note that a vertical morphism $u\colon A' \nrightarrow B'$ is freely added in $\mathbb{A}^{\mathrm{whi}}$. Then the composite $wuv\colon A \nrightarrow B''$ in $\mathbb{A}^{\mathrm{whi}}$ does not admit a lift along $j_{\mathbb{A}}$ as required by (db3) of Definition 2.9, as the only objects horizontally equivalent to A and B'' in $\mathbb{A}^{\mathrm{whi}}$ are themselves through the horizontal identities, and there are no vertical morphisms from A to B'' in $\mathbb{A}$.

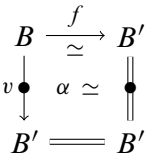
However, as we now show, double biequivalences are contained in $\mathcal{W}$. The reverse inclusion does not hold, but, as we will see in Proposition 5.5, a weak equivalence whose source is a weakly horizontally invariant double category is a double biequivalence.

We use the following technical lemma to prove that double biequivalences are contained in our class of weak equivalences.

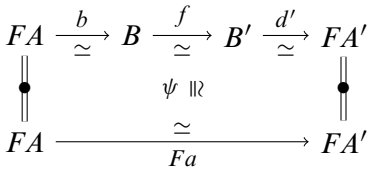
Lemma 3.24 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double biequivalence. Then for every vertical morphism $v\colon B \nrightarrow B'$ in $\mathbb{B}^{\mathrm{whi}}$ which is a composite of freely added vertical morphisms along the double functor $j_{\mathbb{B}}\colon \mathbb{B} \to \mathbb{B}^{\mathrm{whi}}$, and every pair of horizontal equivalences $b\colon FA \xrightarrow{\simeq} B$ and $b'\colon FA' \xrightarrow{\simeq} B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}^{\mathrm{whi}}$ together with a weakly horizontally invertible square in $\mathbb{B}^{\mathrm{whi}}$ of the form*



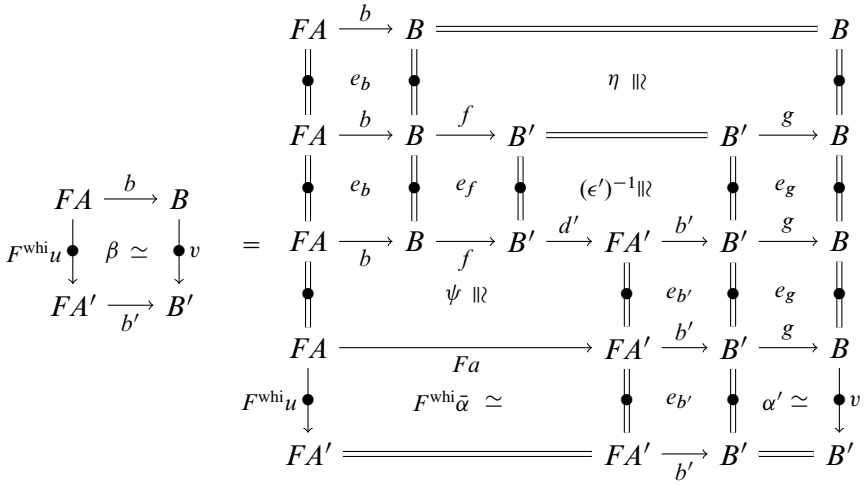
Proof First note that there is a horizontal adjoint equivalence (f, g, η, ϵ) in $\mathbb{B}$ and a weakly horizontally invertible square α in $\mathbb{B}^{\mathrm{whi}}$ of the form



obtained by composing the corresponding weakly horizontally invertible squares for each freely added vertical morphism appearing in the decomposition of v. Let $(b', d', \eta', \epsilon')$ be a choice of horizontal adjoint equivalence data for b'. Since F satisfies (db2) and (db4) of Definition 2.9, there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} A'$ in $\mathbb{A}$ together with a vertically invertible square ψ in $\mathbb{B}$ of the form



Let $u\colon A \nrightarrow A'$ be a vertical morphism in $\mathbb{A}^{\mathrm{whi}}$ freely added using horizontal adjoint equivalence data for a. We get a weakly horizontally square β, as desired,



where $\bar{\alpha}$ is the weakly horizontally invertible square in $\mathbb{A}^{\mathrm{whi}}$ that was freely added with u (see Description 3.14), and α' is the weak inverse of the square α. □

Proposition 3.25 *Every double biequivalence is in $\mathcal{W}$.*

Proof Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double biequivalence; we show that F^{whi} satisfies (db1)–(db4) of Definition 2.9. Since F and F^{whi} agree on underlying horizontal categories by Remark 3.18, and F satisfies (db1)–(db2), so does F^{whi}. Moreover, since $j_{\mathbb{A}}$, $j_{\mathbb{B}}$, and F are fully faithful on squares and $F^{\mathrm{whi}} j_{\mathbb{A}} = j_{\mathbb{B}} F$, we have that F^{whi} is also fully faithful on squares, ie it satisfies (db4). Finally, since every vertical morphism in $\mathbb{B}^{\mathrm{whi}}$ can be decomposed as an alternate composite of vertical morphisms in $\mathbb{B}$ and of composites of freely added vertical morphisms, the fact that F^{whi} satisfies (db3) follows from (db3) for F and Lemma 3.24. □

3.4 The model structure

By taking cofibrations as the $\mathcal{I}_w$–cofibrations and weak equivalences as the double functors in $\mathcal{W}$, we obtain the desired model structure on DblCat. The relevant classes of morphisms, as well as an outline of the proof with shortcuts to the corresponding results, is provided below; the technical details are deferred to Section 5.

Theorem 3.26 *There is a model structure* $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ *on* DblCat *such that*

(i) *the class* $\mathcal{C}$ *of cofibrations is given by* $\mathcal{C} := \mathcal{I}_w$–cof, *where* $\mathcal{I}_w$ *is the set described in Notation 3.8*;

(ii) *the class* $\mathcal{W}$ *of weak equivalences is as described in Definition 3.21*;

(iii) *the class* $\mathcal{F}$ *of fibrations is given by* $\mathcal{F} := (\mathcal{C} \cap \mathcal{W})^{\boxslash}$; *and*

(iv) *the fibrant objects are the weakly horizontally invariant double categories.*

Proof We follow the definition of model structure presented in [17, Definition 2.1]. By Remark 3.22, we know that the class $\mathcal{W}$ of weak equivalences satisfies the 2-out-of-3 property. Furthermore, by Proposition 5.1, we have that $\mathcal{F} \cap \mathcal{W} = \mathcal{I}_w$–inj, and hence the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W}) = (\mathcal{I}_w\text{–cof}, \mathcal{I}_w\text{–inj})$ is the weak factorization system generated by the set $\mathcal{I}_w$ of Notation 3.8. The fact that the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ forms a weak factorization system is the content of Theorem 5.6 and Corollary 5.7. We present in Theorem 5.2 the desired characterization of fibrant objects. □

4 $\mathcal{J}_w$–cofibrations and $\mathcal{J}_w$–injective double functors

As we saw in the previous section, our proposed classes of cofibrations and of trivial fibrations can be constructed from a generating set $\mathcal{I}_w$, and admit concise descriptions.

Unfortunately, a nice description of the proposed fibrations and trivial cofibrations is not available in general. To prove that these classes of double functors form a weak factorization system, we introduce an auxiliary weak factorization system ($\mathcal{J}_w$–cof, $\mathcal{J}_w$–inj) generated by a set $\mathcal{J}_w$ of double functors. Aside from admitting a simple description, the $\mathcal{J}_w$–injective double functors contain our proposed fibrations, and agree with these when we restrict to double functors with weakly horizontally invariant target; in particular, they can be used to identify our fibrant objects.

This section is largely technical, and the reader willing to trust our claims is encouraged to jump ahead to Section 5.

Let us first introduce the set $\mathcal{J}_w$.

Notation 4.1 Let $\mathcal{J}_w$ denote the set containing the following double functors:

(i) either inclusion $J_1 \colon \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$, where the 2–category E_{adj} is the free-living adjoint equivalence;

(ii) either inclusion $J_2 \colon \mathbb{H}2 \to \mathbb{H}C_{\mathrm{inv}}$, where the 2–category C_{inv} is the free-living 2–isomorphism;

(iii) the inclusion $J_3 \colon \mathbb{W}^- \to \mathbb{W}$, where the double category $\mathbb{W}$ is the free-living weakly horizontally invertible square with horizontal adjoint equivalence data, and $\mathbb{W}^-$ is its double subcategory where we remove one of the vertical morphisms:

$$\mathbb{W} = \begin{array}{ccc} 0 & \xrightarrow{\simeq} & 1 \\ \downarrow & \simeq & \downarrow \\ 0' & \xrightarrow[\simeq]{} & 1' \end{array} \qquad \mathbb{W}^- = \begin{array}{ccc} 0 & \xrightarrow{\simeq} & 1 \\ & & \downarrow \\ 0' & \xrightarrow[\simeq]{} & 1' \end{array}$$

Remark 4.2 It is straightforward from the characterization of $\mathcal{I}_w$–cofibrations given in Theorem 3.11 and using Remark 3.12 that the double functors J_1, J_2, and J_3 are in $\mathcal{I}_w$–cof, and from Definition 2.9 that they are double biequivalences. In particular, by Proposition 3.25, this implies that they are trivial cofibrations in our proposed model structure on DblCat.

4.1 $\mathcal{J}_w$–injective double functors

By studying what it means to have the right lifting property with respect to the double functors in $\mathcal{J}_w$, we can characterize the $\mathcal{J}_w$–injective double functors as follows.

Proposition 4.3 *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{J}_w$–*inj *if and only if it satisfies the following conditions*:

(df1) *For every object $C \in \mathbb{A}$ and every horizontal equivalence $b\colon B \xrightarrow{\simeq} FC$ in $\mathbb{B}$, there is a horizontal equivalence $a\colon A \xrightarrow{\simeq} C$ in $\mathbb{A}$ such that $b = Fa$.*

(df2) *For every horizontal morphism $c\colon A \to C$ in $\mathbb{A}$ and every vertically invertible square β in $\mathbb{B}$ as depicted below left, there is a vertically invertible square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$:*

$$\begin{array}{ccc} FA & \xrightarrow{b} & FC \\ \| & \beta \cong & \| \\ FA & \xrightarrow[Fc]{} & FC \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{a} & C \\ \| & \alpha \cong & \| \\ A & \xrightarrow[c]{} & C \end{array}$$

(df3) *For every diagram in $\mathbb{A}$ as depicted below left, where a and a' are horizontal equivalences, and every weakly horizontally invertible square β in $\mathbb{B}$ as depicted below middle, there is a weakly horizontally invertible square α in $\mathbb{A}$ as depicted below right such that $\beta = F\alpha$:*

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ & & \downarrow w \\ A' & \xrightarrow[a']{\simeq} & C' \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow[\simeq]{Fa} & FC \\ {\scriptstyle v}\downarrow & \beta \simeq & \downarrow{\scriptstyle Fw} \\ FA' & \xrightarrow[Fa']{\simeq} & FC' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ {\scriptstyle u}\downarrow & \alpha \simeq & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[a']{\simeq} & C' \end{array}$$

Proof This is obtained directly from a close inspection of the right lifting properties with respect to the double functors in $\mathcal{J}_w$. □

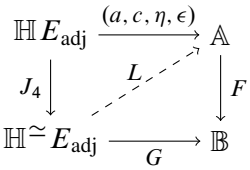
As a consequence, we can use the class $\mathcal{J}_w$–inj to identify the weakly horizontally invariant double categories; see Definition 2.10.

Corollary 4.4 *A double category $\mathbb{A}$ is weakly horizontally invariant if and only if the unique double functor $\mathbb{A} \to \mathbb{1}$ is in $\mathcal{J}_w$–*inj.

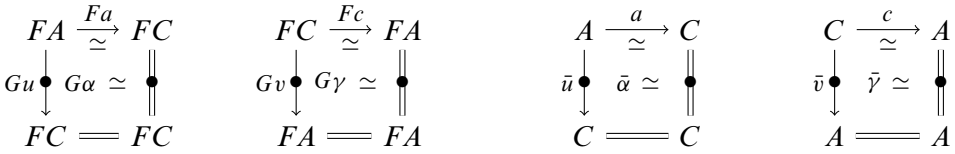
The following result tells us that every $\mathcal{J}_w$–injective double functor has the right lifting property with respect to $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$, which is useful when proving that $\mathcal{J}_w$–injective double functors with weakly horizontally invariant targets are fibrations.

Proposition 4.5 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{J}_w$–*inj*. Then F is in $\{J_4\}$–*inj*, where $J_4\colon \mathbb{H}E_{\mathrm{adj}} \to \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ is the inclusion of Notation 3.15.*

Proof Consider a commutative square in DblCat of the form



where $a\colon A \xrightarrow{\simeq} C$ is a horizontal adjoint equivalence with data (a, c, η, ϵ); we want to find a lift L as depicted. The images under G of the weakly horizontally invertible squares $\alpha, \gamma \in \mathbb{H}^{\simeq} E_{\mathrm{adj}}$ from Description 3.14 are as in the two leftmost diagrams below:



By (df3) of Proposition 4.3, there are weakly horizontally invertible squares $\bar{\alpha}$ and $\bar{\gamma}$ in $\mathbb{A}$, as in the two rightmost diagrams above, such that $F\bar{\alpha} = G\alpha$ and $F\bar{\gamma} = G\gamma$. Finally, by Remark 3.16, the data (a, c, η, ϵ), $\bar{\alpha}$, and $\bar{\gamma}$ determine a unique double functor $L\colon \mathbb{H}^{\simeq} E_{\mathrm{adj}} \to \mathbb{A}$ which gives the desired lift. Note that we indeed have $G = FL$ since their images on the generating data of $\mathbb{H}^{\simeq} E_{\mathrm{adj}}$ coincide. □

Remark 4.6 This result, together with Corollary 4.4, guarantees that for every weakly horizontally invariant double category $\mathbb{A}$, the double functor $\mathbb{A} \to \mathbb{1}$ is in $\{J_4\}$–inj.

Next, we show that the double functors which are $\mathcal{J}_w$–injective and double biequivalences are precisely the ones that are $\mathcal{I}_w$–injective.

Proposition 4.7 *A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is $\mathcal{J}_w$–injective and a double biequivalence if and only if it is $\mathcal{I}_w$–injective.*

Proof Since $\mathcal{J}_w \subseteq \mathcal{I}_w$–cof by Remark 4.2, $\mathcal{I}_w$–inj $= \mathcal{I}_w$–cof$^{\boxslash} \subseteq \mathcal{J}_w^{\boxslash} = \mathcal{J}_w$–inj. Furthermore, by Remark 3.10, a double functor in $\mathcal{I}_w$–inj is in particular a double biequivalence, which shows the converse statement.

Now suppose that F is $\mathcal{J}_w$–injective and a double biequivalence. We prove that F is $\mathcal{I}_w$–injective using Proposition 3.9. It is straightforward to see that F is surjective on objects, full on horizontal morphisms, and fully faithful on squares using (db1), (db2) and (db4) of Definition 2.9 and (df1)–(df2) of Proposition 4.3. To prove that F is full

on vertical morphisms, let A, A' be objects in $\mathbb{A}$, and $v\colon FA \nrightarrow FA'$ be a vertical morphism in $\mathbb{B}$. Since F satisfies (db3), there is a vertical morphism $w\colon C \nrightarrow C'$ in $\mathbb{A}$ together with a weakly horizontally invertible square β in $\mathbb{B}$ as depicted below left:

$$\begin{array}{ccc} FA & \xrightarrow[\simeq]{b} & FC \\ {\scriptstyle v}\downarrow & \beta \simeq & \downarrow{\scriptstyle Fw} \\ FA' & \xrightarrow[\simeq]{b'} & FC' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ {\scriptstyle u}\downarrow & \alpha \simeq & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[\simeq]{a'} & C' \end{array}$$

Since F is full on horizontal morphisms and fully faithful on squares, there are horizontal equivalences $a\colon A \xrightarrow{\simeq} C$ and $a'\colon A' \xrightarrow{\simeq} C'$ in $\mathbb{A}$ such that $b = Fa$ and $b' = Fa'$. Then, by (df3), there is a weakly horizontally invertible square α in $\mathbb{A}$ as depicted above right such that $\beta = F\alpha$; in particular, $v = Fu$. This completes the proof. □

4.2 $\mathcal{J}_w$–cofibrations and double biequivalences

We now focus on the $\mathcal{J}_w$–cofibrations. First, we show that they are cofibrations in our proposed model structure, which additionally satisfy the requirements of a double biequivalence except for condition (db3) on vertical morphisms.

Proposition 4.8 *Let $J\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{J}_w$–cell. Then the functor J*

(i) *is injective on objects, and faithful on horizontal and vertical morphisms;*

(ii) *satisfies* (db1), (db2) *and* (db4) *of Definition 2.9.*

Proof Since $\mathcal{J}_w \subseteq \mathcal{I}_w$–cof by Remark 4.2, we have that $\mathcal{J}_w$–cell $\subseteq \mathcal{I}_w$–cof; hence J is injective on objects, and faithful on horizontal and vertical morphisms, by Remark 3.12.

Now, since objects can only be added along $J_1\colon \mathbb{1} \to \mathbb{H}E_{\text{adj}}$, ie with a horizontal equivalence to an object which was already there, J satisfies (db1). Similarly, as horizontal morphisms can only be added along $J_2\colon \mathbb{H}2 \to C_{\text{inv}}$, we can check that J satisfies (db2). Finally note that J satisfies (db4), since taking pushouts along J_1, J_2, and J_3 does not create new squares within an existing boundary, nor does it identify squares. □

When the source of a $\mathcal{J}_w$–cofibration is a weakly horizontally invariant double category, we can further show that (db3) of Definition 2.9 is satisfied, and hence that every such $\mathcal{J}_w$–cofibration is a double biequivalence.

Proposition 4.9 *Let $J\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{J}_w$–cof such that $\mathbb{A}$ is weakly horizontally invariant. Then J is a double biequivalence.*

Proof We first prove the case when $J \in \mathcal{J}_w$–cell. By Proposition 4.8, we have that J satisfies (db1), (db2) and (db4) of Definition 2.9; it remains to show (db3). Let λ be an ordinal and let $\mathbb{X}\colon \lambda \to \mathrm{DblCat}$ be a transfinite composition of pushouts of double functors in $\mathcal{J}_w$ such that J is the composite

$$J\colon \mathbb{A} \cong J(\mathbb{A}) = \mathbb{X}_0 \xrightarrow{\iota_0} \operatorname*{colim}_{\mu<\lambda} \mathbb{X}_\mu = \mathbb{B}.$$

Let $v\colon B \nrightarrow B'$ be a vertical morphism in $\mathbb{B}$. We use transfinite induction to show that there is a vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square $\beta\colon (Ju \mathrel{\tilde{\simeq}} v)$ in $\mathbb{B}$. This amounts to showing that our statement holds for the base case $\lambda = 0$, for any successor ordinal $\mu + 1 < \lambda$, and for any limit ordinal $\kappa < \lambda$.

If $v \in \mathbb{X}_0 = J(\mathbb{A})$, then there is a vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$ such that $Ju = v$ and we can take $\beta = \mathrm{id}_{Ju}$.

Now suppose that $v \in \mathbb{X}_{\mu+1}$ for a successor ordinal $\mu + 1 < \lambda$. If $v \in \mathbb{X}_\mu$, then we are done by induction. Otherwise, by definition of $\mathbb{X}$, the double category $\mathbb{X}_{\mu+1}$ is a pushout along J_3 as depicted below left,

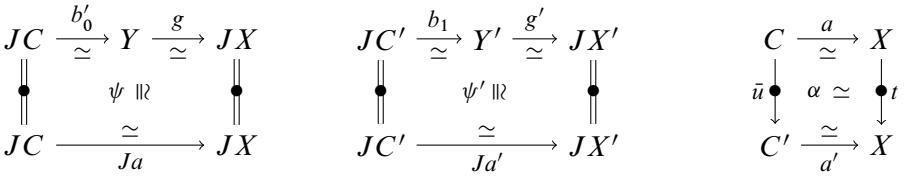
$$\begin{array}{ccc} \mathbb{W}^- & \xrightarrow{(w,d,d')} & \mathbb{X}_\mu \\ {\scriptstyle J_3}\downarrow & & \downarrow{\scriptstyle i_\mu} \\ \mathbb{W} & \xrightarrow[\delta]{} & \mathbb{X}_{\mu+1} \end{array} \qquad \begin{array}{ccc} D & \xrightarrow[\simeq]{d} & Y \\ {\scriptstyle w}\downarrow & \delta \simeq & \downarrow{\scriptstyle \overline{w}} \\ D' & \xrightarrow[d']{\simeq} & Y' \end{array}$$

where w is a vertical morphism in $\mathbb{X}_\mu$, d and d' are horizontal equivalences in $\mathbb{X}_\mu$, and δ is a weakly horizontally invertible square in $\mathbb{B}$, as depicted above right. Then the vertical morphism $v \in \mathbb{X}_{\mu+1}$ is a composite of vertical morphisms in $\mathbb{X}_\mu$ and the freely added vertical morphism $\overline{w}$. We prove that the result holds for a composite of the form $v = v_1 \overline{w} v_0$ with $v_0\colon B \nrightarrow Y$ and $v_1\colon Y' \nrightarrow B'$ two vertical morphisms in $\mathbb{X}_\mu$; the general case where $\overline{w}$ appears several times in the decomposition of v proceeds similarly.

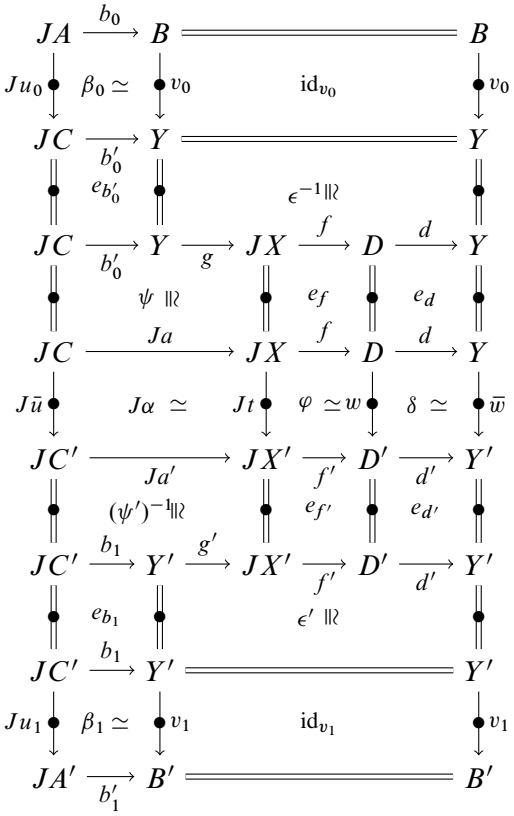
By induction, since v_0, v_1, and w are in $\mathbb{X}_\mu$, there are vertical morphisms u_0, u_1, and t in $\mathbb{A}$, and weakly horizontally invertible squares β_0, β_1 and φ in $\mathbb{B}$, as depicted below:

$$\begin{array}{ccc} JA & \xrightarrow[\simeq]{b_0} & B \\ {\scriptstyle Ju_0}\downarrow & \beta_0 \simeq & \downarrow{\scriptstyle v_0} \\ JC & \xrightarrow[b_0']{\simeq} & Y \end{array} \qquad \begin{array}{ccc} JC' & \xrightarrow[\simeq]{b_1} & Y' \\ {\scriptstyle Ju_1}\downarrow & \beta_1 \simeq & \downarrow{\scriptstyle v_1} \\ JA' & \xrightarrow[b_1']{\simeq} & B' \end{array} \qquad \begin{array}{ccc} JX & \xrightarrow[\simeq]{f} & D \\ {\scriptstyle Jt}\downarrow & \varphi \simeq & \downarrow{\scriptstyle w} \\ JX' & \xrightarrow[f']{\simeq} & D' \end{array}$$

Let (df, g, η, ϵ) and $(d'f', g', \eta', \epsilon')$ be horizontal adjoint equivalence data in $\mathbb{B}$ for the composites df and $d'f'$. Since J satisfies (db2) and (db4), there are horizontal equivalences $a\colon C \xrightarrow{\simeq} X$ and $a'\colon C' \xrightarrow{\simeq} X'$ in $\mathbb{A}$ together with vertically invertible squares ψ and ψ' in $\mathbb{B}$ as in the two leftmost squares below:



Then, as $\mathbb{A}$ is weakly horizontally invariant, there is a vertical morphism $\bar{u}\colon C \nrightarrow C'$ and a weakly horizontally invertible square α in $\mathbb{A}$ as depicted above right. Setting $u := u_1\bar{u}u_0\colon A \nrightarrow A'$ and considering the pasting of squares in $\mathbb{B}$



we obtain a weakly horizontally invertible square of the desired form between the vertical morphisms $Ju = (Ju_1)(J\bar{u})(Ju_0)$ and $v = v_1\bar{w}v_0$.

Finally, if $v \in \mathbb{X}_\kappa = \operatorname{colim}_{\mu<\kappa} \mathbb{X}_\mu$ for a limit ordinal $\kappa < \lambda$, there is an ordinal $\mu < \kappa$ such that $v \in \mathbb{X}_\mu$, and we are done by induction. This shows (db3) for J, and proves that J is a double biequivalence.

Now if $J\colon \mathbb{A} \to \mathbb{B}$ is in $\mathcal{J}_w$–cof, then it is a retract of a double functor $K\colon \mathbb{A} \to \mathbb{C}$ in $\mathcal{J}_w$–cell, whose source is also the weakly horizontally invariant double category $\mathbb{A}$. By the first part of the proof, the double functor K is a double biequivalence, and therefore so is J. □

4.3 Fibrations and $\mathcal{J}_w$–injective double functors

To conclude this section, we prove our claim that a double functor whose target is weakly horizontally invariant is a fibration precisely when it is $\mathcal{J}_w$–injective. We start by showing that the class of fibrations is included in $\mathcal{J}_w$–inj.

Lemma 4.10 *We have that $\mathcal{F} \subseteq \mathcal{J}_w$–inj.*

Proof Since every double functor in $\mathcal{J}_w$ is a double biequivalence by Remark 4.2, it is in $\mathcal{W}$ by Proposition 3.25. This, together with Remark 4.2, implies that $\mathcal{J}_w \subseteq \mathcal{C} \cap \mathcal{W}$. Therefore $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash} \subseteq \mathcal{J}_w^{\boxslash} = \mathcal{J}_w$–inj, which concludes the proof. □

For the converse inclusion, we will use the next incremental lemmas, which ultimately ensure that the weakly horizontally invariant replacement of a trivial cofibration is a $\mathcal{J}_w$–cofibration.

Lemma 4.11 *Let $I\colon \mathbb{A} \to \mathbb{B}$ be a double functor in $\mathcal{C} = \mathcal{I}_w$–cof which is fully faithful on squares. Then the induced double functor $I^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is in $\mathcal{C}$.*

Proof We show that I^{whi} is in $\mathcal{I}_w$–cof by using Theorem 3.11. Since the double functors I and I^{whi} coincide on underlying horizontal categories by Remark 3.18, and $I \in \mathcal{I}_w$–cof, the functor $U\boldsymbol{H}I = U\boldsymbol{H}I^{\mathrm{whi}}$ has the left lifting property with respect to surjective on objects and full functors. It remains to prove that $U\boldsymbol{V}I^{\mathrm{whi}}$ satisfies this lifting property.

Let $P\colon \mathcal{X} \to \mathcal{Y}$ be a surjective on objects and full functor, and consider a commutative square as below left:

$$\begin{array}{ccc} U\boldsymbol{V}\mathbb{A}^{\mathrm{whi}} & \xrightarrow{F} & \mathcal{X} \\ {\scriptstyle U\boldsymbol{V}I^{\mathrm{whi}}}\downarrow & \overset{L}{\nearrow} & \downarrow{\scriptstyle P} \\ U\boldsymbol{V}\mathbb{B}^{\mathrm{whi}} & \xrightarrow[G]{} & \mathcal{Y} \end{array} \qquad \begin{array}{ccc} U\boldsymbol{V}\mathbb{A} & \xrightarrow{F \circ U\boldsymbol{V}j_{\mathbb{A}}} & \mathcal{X} \\ {\scriptstyle U\boldsymbol{V}I}\downarrow & \overset{\hat{L}}{\nearrow} & \downarrow{\scriptstyle P} \\ U\boldsymbol{V}\mathbb{B} & \xrightarrow[G \circ U\boldsymbol{V}j_{\mathbb{B}}]{} & \mathcal{Y} \end{array}$$

Recall that the category $UV\mathbb{B}^{\mathrm{whi}}$ is obtained from $UV\mathbb{B}$ by freely adding a morphism $v_{\underline{b}}\colon B \to B'$ for each horizontal adjoint equivalence $\underline{b} = (b, d, \eta, \epsilon)$ in $\mathbb{B}$; see Construction 3.17. Hence the data of a functor $L\colon UV\mathbb{B}^{\mathrm{whi}} \to \mathcal{X}$ is equivalent to the data of a functor $\widehat{L}\colon UV\mathbb{B} \to \mathcal{X}$ together with a choice of morphism $Lv_{\underline{b}}\colon \widehat{L}B \to \widehat{L}B'$ for each $v_{\underline{b}}\colon B \to B'$. Therefore, to construct a functor $L\colon UV\mathbb{B}^{\mathrm{whi}} \to \mathcal{X}$ as depicted, it is enough to define L on the subcategory $UV\mathbb{B}$ and on each $v_{\underline{b}}\colon B \to B'$ in such a way that $PL = G$ and $L(UVI^{\mathrm{whi}}) = F$.

Since I is in $\mathcal{I}_w$–cof, Theorem 3.11 tells us that UVI has the left lifting property with respect to P, and hence there is a lift $\widehat{L}\colon UV\mathbb{B} \to \mathcal{X}$ in the diagram above right; we define L to be $\widehat{L}$ on the subcategory $UV\mathbb{B}$.

Now, consider $v_{\underline{b}}\colon B \nrightarrow B'$ for a given horizontal adjoint equivalence $\underline{b} = (b, d, \eta, \epsilon)$. Since I is injective on objects and faithful on horizontal morphisms by Remark 3.12, and fully faithful on squares by assumption, there is at most one horizontal adjoint equivalence $\underline{a} = (a, c, \eta', \epsilon')$ in $\mathbb{A}$ such that $I\underline{a} = \underline{b}$. If there is such an $\underline{a}$, then there is a unique vertical morphism $u_{\underline{a}}$ in $\mathbb{A}^{\mathrm{whi}}$ (freely added using $\underline{a}$) such that $I^{\mathrm{whi}}(u_{\underline{a}}) = v_{\underline{b}}$; set $Lv_{\underline{b}} = Fu_{\underline{a}}$. If there is no such $\underline{a}$, then $v_{\underline{b}}$ is not in the image of I^{whi}. By fullness of P, we can then choose a morphism $w\colon LB \to LB'$ in $\mathcal{X}$ such that $Pw = Gv_{\underline{b}}$, and set $Lv_{\underline{b}} = w$. □

Lemma 4.12 *If $I\colon \mathbb{A} \to \mathbb{B}$ is a double functor in $\mathcal{C} \cap \mathcal{W}$, then $I^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is in $\mathcal{J}_w$–cof.*

Proof First recall that, since $I \in \mathcal{W}$, the double functor $I^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence by definition. Next, consider a factorization $I^{\mathrm{whi}} = PJ$ with $J \in \mathcal{J}_w$–cof and $P \in \mathcal{J}_w$–inj. As $\mathbb{A}^{\mathrm{whi}}$ is weakly horizontally invariant, Proposition 4.9 ensures J is also a double biequivalence; then, by 2-out-of-3, so is P. Hence P is both $\mathcal{J}_w$–injective and a double biequivalence, and therefore it is $\mathcal{I}_w$–injective by Proposition 4.7.

Now, since I is in $\mathcal{W}$, it is fully faithful on squares, and so it follows from Lemma 4.11 that I^{whi} is in $\mathcal{I}_w$–cof. Then I^{whi} has the left lifting property with respect to $P \in \mathcal{I}_w$–inj, so, by the retract argument (see Remark 3.4), it is a retract of $J \in \mathcal{J}_w$–cell and hence is itself in $\mathcal{J}_w$–cof. □

Finally, we prove that every $\mathcal{J}_w$–injective double functor with weakly horizontally invariant target has the right lifting property with respect to every trivial cofibration I, by using its lifting property against the weakly horizontally invariant replacement I^{whi}.

Proposition 4.13 *Let $P\colon \mathbb{A} \to \mathbb{B}$ be a double functor with $\mathbb{B}$ weakly horizontally invariant. Then P is in $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}$ if and only if P is in $\mathcal{J}_w$–inj.*

Proof If P is in $\mathcal{F}$, then P is in $\mathcal{J}_w$–inj by Lemma 4.10.

Now suppose that P is in $\mathcal{J}_w$–inj. We show that P has the right lifting property with respect to every double functor in $\mathcal{C} \cap \mathcal{W}$, ie it is in $\mathcal{F}$. Let $I\colon \mathbb{C} \to \mathbb{D}$ be a double functor in $\mathcal{C} \cap \mathcal{W}$ and consider a commutative square in DblCat as below; we want to find a lift $L\colon \mathbb{D} \to \mathbb{A}$ as pictured:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{A} \\ {\scriptstyle I}\downarrow & \overset{L}{\nearrow} & \downarrow{\scriptstyle P} \\ \mathbb{D} & \xrightarrow[G]{} & \mathbb{B} \end{array}$$

Since $\mathbb{B}$ is weakly horizontally invariant, there is a lift in the diagram below left by Remark 4.6, which yields a double functor $\widehat{G}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{B}$ as in the diagram below right, given by the universal property of the pushout:

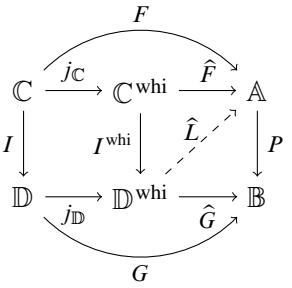
$$\begin{array}{ccccc} \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}E_{\mathrm{adj}} & \longrightarrow & \mathbb{D} & \xrightarrow{G} & \mathbb{B} \\ {\scriptstyle \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} J_4}\downarrow & & & \overset{K}{\nearrow} & \\ \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} & & & & \end{array} \qquad \begin{array}{ccccc} \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}E_{\mathrm{adj}} & \longrightarrow & \mathbb{D} & & \\ {\scriptstyle \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} J_4}\downarrow & & {\scriptstyle j_{\mathbb{D}}}\downarrow & \searrow^{G} & \\ \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} & \longrightarrow & \mathbb{D}^{\mathrm{whi}} & \overset{\widehat{G}}{\searrow} & \\ & \underset{K}{\searrow} & & & \mathbb{B} \end{array}$$

Now, since $P \in \mathcal{J}_w$–inj, by Proposition 4.5 there is a lift in the commutative diagram below left, which in turns yields a double functor $\widehat{F}\colon \mathbb{C}^{\mathrm{whi}} \to \mathbb{A}$ as in the diagram below right, given by the universal property of the pushout:

$$\begin{array}{ccccc} \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}E_{\mathrm{adj}} & \longrightarrow & \mathbb{C} & \xrightarrow{F} & \mathbb{A} \\ {\scriptstyle \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} J_4}\downarrow & & & \overset{K'}{\nearrow} & \downarrow{\scriptstyle P} \\ \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} & & \xrightarrow[K\circ(\bigsqcup_I \mathrm{id})]{} & & \mathbb{B} \end{array} \qquad \begin{array}{ccccc} \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}E_{\mathrm{adj}} & \longrightarrow & \mathbb{C} & & \\ {\scriptstyle \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} J_4}\downarrow & & {\scriptstyle j_{\mathbb{C}}}\downarrow & \searrow^{F} & \\ \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} & \longrightarrow & \mathbb{C}^{\mathrm{whi}} & \overset{\widehat{F}}{\searrow} & \\ & \underset{K'}{\searrow} & & & \mathbb{A} \end{array}$$

Here $\bigsqcup_I \mathrm{id}\colon \bigsqcup_{\mathrm{HorEq}(\mathbb{C})} \mathbb{H}^{\simeq}E_{\mathrm{adj}} \to \bigsqcup_{\mathrm{HorEq}(\mathbb{D})} \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ is the double functor induced by the action of I on $\mathrm{HorEq}(\mathbb{C})$. By construction of $\widehat{F}$ and $\widehat{G}$, and using the universal

property of the pushout for $\mathbb{C}^{\mathrm{whi}}$, we have that the following diagram commutes:



Since $I^{\mathrm{whi}} \in \mathcal{J}_w$–cof by Lemma 4.12 as $I \in \mathcal{C} \cap \mathcal{W}$, there is a lift $\widehat{L}$ in the right-hand square of the diagram above, and the composite $L := \widehat{L}\, j_{\mathbb{D}}$ gives the desired lift. □

5 Proof of Theorem 3.26

We now use the technical results of Section 4 to prove the remaining claims in Theorem 3.26. Namely, we show that the pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ form weak factorization systems, and identify the fibrant objects as the weakly horizontally invariant double categories.

Since by definition we have that $\mathcal{C} = \mathcal{I}_w$–cof, in order to prove that $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorization system it suffices to show that $\mathcal{F} \cap \mathcal{W} = \mathcal{I}_w$–inj; this is the content of the following result.

Proposition 5.1 *We have that* $\mathcal{F} \cap \mathcal{W} = \mathcal{I}_w$–inj.

Proof Since $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C} = \mathcal{I}_w$–cof, it follows that $\mathcal{I}_w$–inj $= \mathcal{I}_w$–cof$^{\boxslash} \subseteq (\mathcal{C} \cap \mathcal{W})^{\boxslash} = \mathcal{F}$. Moreover, every double functor in $\mathcal{I}_w$–inj is a double biequivalence by Remark 3.10, and these are in $\mathcal{W}$ by Proposition 3.25; hence $\mathcal{I}_w$–inj $\subseteq \mathcal{W}$.

For the inclusion $\mathcal{F} \cap \mathcal{W} \subseteq \mathcal{I}_w$–inj, note that every double functor P in $\mathcal{F} \cap \mathcal{W}$ factors as $P = QI$ with $I \in \mathcal{C} = \mathcal{I}_w$–cof and $Q \in \mathcal{I}_w$–inj. Since $Q \in \mathcal{W}$ by the above inclusion, and $P \in \mathcal{W}$ by assumption, we get that $I \in \mathcal{W}$ by 2-out-of-3; hence $I \in \mathcal{C} \cap \mathcal{W}$. Therefore, since $P \in \mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}$ has the right lifting property with respect to I, by the retract argument (see Remark 3.4) we have that P is a retract of Q and hence is also in $\mathcal{I}_w$–inj. □

Before moving on to the next factorization system, we focus on the fibrant objects. Aside from obtaining the desired characterization as the weakly horizontally invariant double

categories, we see that the weakly horizontally invariant replacements $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ of Construction 3.17 are trivial cofibrations, and hence fibrant replacements in our model structure.

Theorem 5.2 *A double category $\mathbb{A}$ is fibrant if and only if it is weakly horizontally invariant.*

Proof We recall from Corollary 4.4 that a double category $\mathbb{A}$ is weakly horizontally invariant if and only if the double functor $\mathbb{A} \to \mathbb{1}$ is in $\mathcal{J}_w$–inj. Since $\mathbb{1}$ is weakly horizontally invariant, by Proposition 4.13 this holds if and only if $\mathbb{A} \to \mathbb{1}$ is in $\mathcal{F}$, ie $\mathbb{A}$ is fibrant. □

Proposition 5.3 *Let $\mathbb{A}$ be a weakly horizontally invariant double category. Then the double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a double biequivalence.*

Proof By construction, $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a double functor in $\{J_4\}$–cof (recall Construction 3.17). Since $\mathcal{J}_w$–inj $\subseteq \{J_4\}$–inj by Proposition 4.5,

$$\{J_4\}\text{–cof} = {}^{\boxslash}\{J_4\}\text{–inj} \subseteq {}^{\boxslash}\mathcal{J}_w\text{–inj} = \mathcal{J}_w\text{–cof}.$$

Hence $j_{\mathbb{A}}$ is a $\mathcal{J}_w$–cofibration with weakly horizontally invariant source, and thus a double biequivalence by Proposition 4.9. □

Corollary 5.4 *The double functor $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is in $\mathcal{C} \cap \mathcal{W}$. In particular, this exhibits $\mathbb{A}^{\mathrm{whi}}$ as a fibrant replacement of $\mathbb{A}$.*

Proof Since $\mathcal{J}_w$–inj $\subseteq \{J_4\}$–inj by Proposition 4.5, and $\mathcal{J}_w$–cof $\subseteq \mathcal{I}_w$–cof by Remark 4.2, we have that J_4 is in $\mathcal{I}_w$–cof $= \mathcal{C}$. Hence so is $j_{\mathbb{A}}$, as it is constructed as a pushout of coproducts of J_4. The fact that $j_{\mathbb{A}}$ is in $\mathcal{W}$ follows from the relation $(j_{\mathbb{A}})^{\mathrm{whi}} = j_{\mathbb{A}^{\mathrm{whi}}}$ and the fact that the latter is a double biequivalence by Proposition 5.3. The second statement then follows from Theorem 5.2. □

We can also prove, using Proposition 5.3, that every weak equivalence with fibrant source is a double biequivalence. In particular, this implies that while our weak equivalences are more general, when restricted to the fibrant double categories they agree with the weak equivalences of the model structure on DblCat of [16]: the double biequivalences. As we will see in Section 8, the weak equivalences with fibrant source also admit a familiar description in terms of pseudoinverses.

Proposition 5.5 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double functor with $\mathbb{A}$ weakly horizontally invariant. Then F is in $\mathcal{W}$ if and only if F is a double biequivalence.*

Proof If F is a double biequivalence, then F is in $\mathcal{W}$ by Proposition 3.25.

Now suppose that F is in $\mathcal{W}$, ie that $F^{\mathrm{whi}}\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}}$ is a double biequivalence. We prove that F satisfies (db1)–(db4) of Definition 2.9. Since F and F^{whi} coincide on underlying horizontal categories by Remark 3.18 and $j_{\mathbb{B}}$ is fully faithful on squares, F satisfies (db1)–(db2) as F^{whi} does so. Moreover, since $j_{\mathbb{A}}$, $j_{\mathbb{B}}$, and F^{whi} are fully faithful on squares and $F^{\mathrm{whi}} j_{\mathbb{A}} = j_{\mathbb{B}} F$, we have that F satisfies (db4).

It remains to prove (db3). Since $\mathbb{A}$ is weakly horizontally invariant, Proposition 5.3 guarantees that $j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ is a double biequivalence; hence so is the composite $F^{\mathrm{whi}} j_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{B}^{\mathrm{whi}}$. Then, given a vertical morphism $v\colon B \nrightarrow B'$ in $\mathbb{B}$, there is a vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square β in $\mathbb{B}^{\mathrm{whi}}$ as depicted below left:

$$\begin{array}{ccc} FA & \xrightarrow[\simeq]{b} & B \\ {\scriptstyle F^{\mathrm{whi}} j_{\mathbb{A}} u = j_{\mathbb{B}} Fu}\downarrow & \beta \simeq & \downarrow{\scriptstyle j_{\mathbb{B}} v} \\ FA' & \xrightarrow[\simeq]{b'} & B' \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow[\simeq]{b} & B \\ {\scriptstyle Fu}\downarrow & \beta' \simeq & \downarrow{\scriptstyle v} \\ FA' & \xrightarrow[\simeq]{b'} & B' \end{array}$$

By fully faithfulness on squares of $j_{\mathbb{B}}$, we get a weakly horizontally invertible square β' in $\mathbb{B}$ as depicted above right, which shows (db3). □

We are now ready to finish the proof of Theorem 3.26 by showing that the classes of trivial cofibrations and fibrations form a weak factorization system. We first show that every double functor can be factored as a trivial cofibration followed by a fibration.

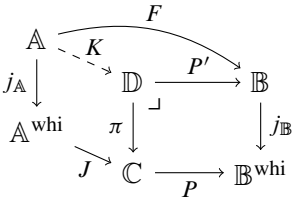
Theorem 5.6 *Every double functor $F\colon \mathbb{A} \to \mathbb{B}$ can be factored as $F = RI$ with $I \in \mathcal{C} \cap \mathcal{W}$ and $R \in \mathcal{F}$.*

Proof Given a double functor $F\colon \mathbb{A} \to \mathbb{B}$, we factor F^{whi} as

$$\begin{array}{ccc} \mathbb{A}^{\mathrm{whi}} & \xrightarrow{F^{\mathrm{whi}}} & \mathbb{B}^{\mathrm{whi}} \\ & {\scriptstyle J}\searrow \quad \nearrow{\scriptstyle P} & \\ & \mathbb{C} & \end{array}$$

where $J \in \mathcal{J}_w$–cof and $P \in \mathcal{J}_w$–inj. As $\mathbb{B}^{\mathrm{whi}}$ is weakly horizontally invariant, by Proposition 4.13 we have that $P \in \mathcal{F}$, and hence $\mathbb{C}$ is also weakly horizontally invariant.

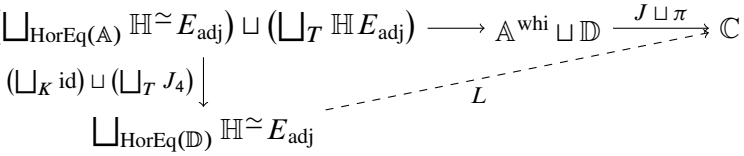
Define $\mathbb{D}$ to be the pullback of P along $j_{\mathbb{B}}$ as in the diagram



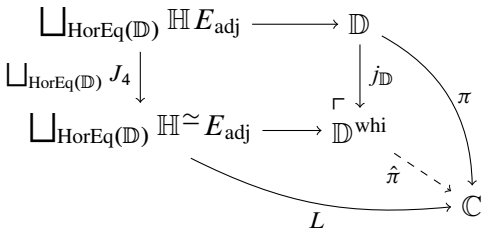
Then there is a unique double functor $K\colon \mathbb{A} \to \mathbb{D}$ making the above diagram commute. To prove the result, it suffices to show that K is in $\mathcal{W}$. Indeed, assume that this is the case and factor K as $K = QI$ with $I \in \mathcal{I}_w\text{–cof} = \mathcal{C}$ and $Q \in \mathcal{I}_w\text{–inj} = \mathcal{F} \cap \mathcal{W}$, where the latter equality holds by Proposition 5.1. As $K, Q \in \mathcal{W}$, we have $I \in \mathcal{C} \cap \mathcal{W}$ by 2-out-of-3. Hence, as $F = P'K$, this gives a factorization of F as $F = RI$ with $I \in \mathcal{C} \cap \mathcal{W}$ and $R := P'Q \in \mathcal{F}$, as desired.

As J is in $\mathcal{W}$ by Proposition 3.25, $j_{\mathbb{A}}$ is in $\mathcal{W}$ by Corollary 5.4, and $\pi K = J j_{\mathbb{A}}$, in order to prove that K is in $\mathcal{W}$, by 2-out-of-3 it is enough to show that π is in $\mathcal{W}$. For this, we construct a double functor $\hat{\pi}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{C}$ such that $\pi = \hat{\pi}\, j_{\mathbb{D}}$ and then show that $\hat{\pi}$ is a double biequivalence; this implies that $\pi \in \mathcal{W}$ by 2-out-of-3.

Let $T := \mathrm{HorEq}(\mathbb{D}) \setminus K(\mathrm{HorEq}(\mathbb{A}))$. As $\mathbb{C}$ is weakly horizontally invariant, by Remark 4.6 there is a lift L in the diagram



This yields a double functor $\hat{\pi}\colon \mathbb{D}^{\mathrm{whi}} \to \mathbb{C}$, given by the universal property of the pushout, as depicted below:



We finally show $\hat{\pi}$ satisfies (db1)–(db4). First note that π is fully faithful on squares as it is a pullback of $j_{\mathbb{B}}$ which satisfies this condition. Hence $\hat{\pi}$ also satisfies (db4), since $\hat{\pi}\, j_{\mathbb{D}} = \pi$. By Proposition 4.9, we know that $J\colon \mathbb{A}^{\mathrm{whi}} \to \mathbb{C}$ in $\mathcal{J}_w$–cof is a

double biequivalence, and so (db1)–(db3) for $\hat{\pi}$ follow from the fact that J satisfies (db1)–(db3) and that $\hat{\pi} K^{\mathrm{whi}} = J$, by construction. □

As a direct consequence of this result, we get that the trivial cofibrations are precisely the double functors which have the left lifting property with respect to all fibrations. This concludes the proof of the existence of the model structure.

Corollary 5.7 *We have that* $\mathcal{C} \cap \mathcal{W} = {}^{\boxslash}\mathcal{F}$.

Proof By definition of $\mathcal{F}$, we already know that $\mathcal{C} \cap \mathcal{W} \subseteq {}^{\boxslash}\mathcal{F}$. The reverse inclusion follows from Theorem 5.6, the retract argument (Remark 3.4), and the fact that $\mathcal{C} \cap \mathcal{W}$ is closed under retracts. □

Remark 5.8 This shows that $\mathcal{J}_w$–cof $\subseteq \mathcal{C} \cap \mathcal{W}$. Indeed, we have that $\mathcal{F} \subseteq \mathcal{J}_w$–inj by Lemma 4.10, and hence $\mathcal{J}_w$–cof $= {}^{\boxslash}\mathcal{J}_w$–inj $\subseteq {}^{\boxslash}\mathcal{F} = \mathcal{C} \cap \mathcal{W}$.

6 Quillen pairs

Having constructed a new model structure on DblCat, it is natural to wonder how it compares to the one defined by the authors in [16]. We settle this question by showing that the identity functor induces a Quillen pair between our two model structures, but not a Quillen equivalence.

We then devote the rest of the section to comparing our model structure on DblCat to Lack's model structure on 2Cat; see [13; 14] for more details. As in [16], the horizontal embedding $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ is a left Quillen and homotopically fully faithful functor, but it is no longer right Quillen as it does not preserve fibrant objects. Instead, this role is now played by its more homotopical version $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, which is also homotopically fully faithful. Furthermore, the double category $\mathbb{H}^{\simeq}\mathcal{A}$ associated to a 2–category $\mathcal{A}$ is weakly horizontally invariant and provides a fibrant replacement for $\mathbb{H}\mathcal{A}$.

First, we show that the identity adjunction embeds the homotopy theory of weakly horizontally invariant double categories into that of double categories.

Theorem 6.1 *The identity adjunction*

$$\mathrm{DblCat}_{\mathrm{whi}} \underset{\text{id}}{\overset{\text{id}}{\rightleftarrows}} \mathrm{DblCat} \quad (\perp)$$

is a Quillen pair between the model structure on DblCat *for weakly horizontally invariant double categories of Theorem 3.26 and the one of* [16, Theorem 3.18]*. Moreover, the derived counit is levelwise a weak equivalence.*

Proof The set $\mathcal{I}'$ of generating cofibrations introduced in [16, Proposition 4.3] for the model structure on DblCat constructed therein can be described as the set $\mathcal{I}_w$ where the inclusion $I_3 : \mathbb{1} \sqcup \mathbb{1} \to \mathbb{V}2$ is replaced by the unique map $\varnothing \to \mathbb{V}2$. Since the latter is also in $\mathcal{I}_w$–cof, it follows that $\mathcal{I}'$–cof $\subseteq \mathcal{I}_w$–cof, and hence $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}_{\mathrm{whi}}$ preserves cofibrations. Furthermore, by Proposition 3.25, we have that the class of double biequivalences — which is precisely the class of weak equivalences for the model structure on DblCat of [16] — is contained in the class $\mathcal{W}$ of weak equivalences in $\mathrm{DblCat}_{\mathrm{whi}}$, and hence $\mathrm{id}\colon \mathrm{DblCat} \to \mathrm{DblCat}_{\mathrm{whi}}$ also preserves weak equivalences. This shows that the identity adjunction is a Quillen pair.

It remains to show that the derived counit is levelwise a weak equivalence in $\mathrm{DblCat}_{\mathrm{whi}}$. Let $\mathbb{A}$ be a fibrant double category in $\mathrm{DblCat}_{\mathrm{whi}}$. Then the component of the derived counit at $\mathbb{A}$ is given by the cofibrant replacement $q_{\mathbb{A}} : \mathbb{A}^c \to \mathbb{A}$ in the model structure on DblCat of [16]. In particular, the double functor $q_{\mathbb{A}}$ is a double biequivalence, and hence a weak equivalence in $\mathrm{DblCat}_{\mathrm{whi}}$ by Proposition 3.25. □

However, the identity adjunction does not induce a Quillen equivalence between the two model structures on DblCat, as shown in the following remark.

Remark 6.2 The derived unit of the identity adjunction above is not a levelwise double biequivalence. To see this, recall the double category $\mathbb{A}$ described in Example 3.23. By [16, Proposition 4.9], $\mathbb{A}$ is cofibrant in the model structure on DblCat of [16]. Then the component of the derived unit at $\mathbb{A}$ is given by a fibrant replacement of $\mathbb{A}$ in $\mathrm{DblCat}_{\mathrm{whi}}$, and hence we can consider the weakly horizontally invariant replacement $j_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}^{\mathrm{whi}}$ given in Construction 3.17. In particular, as shown in Example 3.23, this is not a double biequivalence.

As a direct consequence of the above theorem, and the fact that $\mathbb{H} \dashv \boldsymbol{H}$ is a Quillen pair between Lack's model structure on 2Cat and the model structure on DblCat of [16], we get that $\mathbb{H} \dashv \boldsymbol{H}$ is also a Quillen pair between 2Cat and the model structure on DblCat introduced in this paper. Moreover, the derived unit is levelwise a biequivalence, and so the functor $\mathbb{H}$ is homotopically fully faithful.

Theorem 6.3 *The adjunction*

$$\mathrm{DblCat} \underset{\boldsymbol{H}}{\overset{\mathbb{H}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} 2\mathrm{Cat}$$

is a Quillen pair between the model structure of Theorem 3.26 and Lack's model structure. Moreover, the derived unit is levelwise a biequivalence.

Proof The fact that this is a Quillen pair follows directly from Theorem 6.1 and [16, Proposition 6.1]. To show that the derived unit is levelwise a biequivalence, let $\mathcal{A}$ be a cofibrant 2–category. The component of the derived unit at $\mathcal{A}$ is given by the underlying horizontal 2–functor of a fibrant replacement $\boldsymbol{H} j_{\mathbb{H}\mathcal{A}} \colon \mathcal{A} = \boldsymbol{H}\,\mathbb{H}\mathcal{A} \to \boldsymbol{H}\,(\mathbb{H}\mathcal{A})^f$ of the horizontal double category $\mathbb{H}\mathcal{A}$ in DblCat. In particular, if we consider the fibrant replacement given in Construction 3.17, it does not change the underlying horizontal 2–category of $\mathbb{H}\mathcal{A}$ by Remark 3.18. Hence $\boldsymbol{H} j_{\mathbb{H}\mathcal{A}}$ is an identity, and in particular a biequivalence. □

As opposed to the case where DblCat is endowed with the model structure of [16] — see [16, Theorem 6.2] — the horizontal embedding is not right Quillen when we consider our new model structure.

Remark 6.4 The functor $\mathbb{H}$ is not right Quillen as, for example, the horizontal double category $\mathbb{H}E_{\mathrm{adj}}$ is not weakly horizontally invariant, where E_{adj} denotes the free-living adjoint equivalence, as shown in Remark 2.12. Since every 2–category is fibrant, this implies that $\mathbb{H}$ does not preserve fibrant objects.

This shortcoming of the horizontal embedding $\mathbb{H}$ can be remedied by instead considering the homotopical horizontal embedding $\mathbb{H}^{\simeq} \colon 2\mathrm{Cat} \to \mathrm{DblCat}$ of Definition 2.13. As we will see, the adjunction $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ of Proposition 2.15 is compatible with the model structures considered, making the functor $\mathbb{H}^{\simeq}$ right Quillen. As a first step towards this, we show that $\mathbb{H}^{\simeq}$ provides a levelwise fibrant replacement of $\mathbb{H}$ in our model structure on DblCat.

Theorem 6.5 *Let $\mathcal{A}$ be a 2–category. Then the double category $\mathbb{H}^{\simeq}\mathcal{A}$ is weakly horizontally invariant and the inclusion $J_{\mathcal{A}} \colon \mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$ is a double biequivalence. In particular, this exhibits $\mathbb{H}^{\simeq}\mathcal{A}$ as a fibrant replacement of $\mathbb{H}\mathcal{A}$ in the model structure on* DblCat *of Theorem 3.26.*

Proof For the first statement, we have by [15, Lemma A.2.3] that a weakly horizontally invertible square $\alpha\colon (u\ {}^{a}_{a'}\ w)$ in $\mathbb{H}^{\simeq}\mathcal{A}$ corresponds to a 2–isomorphism $\alpha\colon wa \Rightarrow a'u$ in $\mathcal{A}$, where (a, c, η, ϵ) and $(a', c', \eta', \epsilon')$ are equivalences in $\mathcal{A}$. In particular, given a boundary in $\mathbb{H}^{\simeq}\mathcal{A}$ as below left,

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ & & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[a']{\simeq} & C' \end{array} \qquad \begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ {\scriptstyle u}\downarrow & \alpha\ \simeq & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[a']{\simeq} & C' \end{array}$$

there is an equivalence $u := c'wa\colon A \xrightarrow{\simeq} A'$ and a 2–isomorphism

$$\alpha := (\epsilon')^{-1}wa\colon wa \cong a'u$$

in $\mathcal{A}$, which provides a square as desired, depicted above right. This shows that $\mathbb{H}^{\simeq}\mathcal{A}$ is weakly horizontally invariant.

For the second statement, recall that $\boldsymbol{H}\,\mathbb{H}\mathcal{A} = \mathcal{A} = \boldsymbol{H}\,\mathbb{H}^{\simeq}\mathcal{A}$, and thus the inclusion $J_{\mathcal{A}}\colon \mathbb{H}\mathcal{A} \to \mathbb{H}^{\simeq}\mathcal{A}$ is the identity on underlying horizontal 2–categories; this shows that $J_{\mathcal{A}}$ satisfies (db1), (db2) and (db4) of Definition 2.9. It remains to show (db3). Let $u\colon A \nrightarrow A'$ be a vertical morphism in $\mathbb{H}^{\simeq}\mathcal{A}$, ie an adjoint equivalence $u\colon A \xrightarrow{\simeq} A'$ in $\mathcal{A}$. Then the square

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle J_{\mathcal{A}}(e_A)}\downarrow & \simeq & \downarrow{\scriptstyle u} \\ A & \xrightarrow[u]{\simeq} & A' \end{array}$$

induced by the identity at u gives a weakly horizontally invertible square in $\mathbb{H}^{\simeq}\mathcal{A}$ as required. This shows that $J_{\mathcal{A}}$ is a double biequivalence. The second statement then follows from Proposition 3.25. □

We now show that the double functor $\mathbb{H}^{\simeq}$ is right Quillen, and moreover, that it is homotopically fully faithful.

Theorem 6.6 *The adjunction*

$$\mathrm{2Cat} \underset{\mathbb{H}^{\simeq}}{\overset{L^{\simeq}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{DblCat}$$

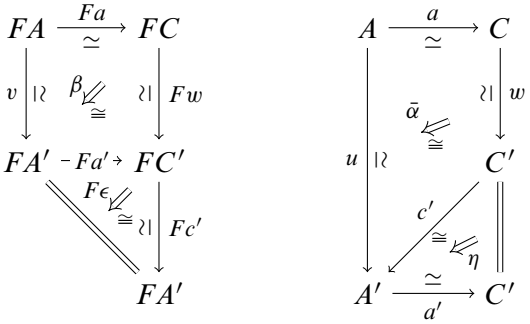
is a Quillen pair between the model structure of Theorem 3.26 and Lack's model structure. Moreover, the derived counit is levelwise a weak equivalence.

Proof We show that $\mathbb{H}^{\simeq}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$ preserves fibrations and trivial fibrations. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a Lack fibration in 2Cat. Since $\mathbb{H}^{\simeq}\mathcal{B}$ is weakly horizontally invariant (see Theorem 6.5), Proposition 4.13 guarantees that $\mathbb{H}^{\simeq}F$ is a fibration in DblCat if and only if it is $\mathcal{J}_w$–injective. Hence, we need to prove that $\mathbb{H}^{\simeq}F$ satisfies (df1)–(df3) of Proposition 4.3.

First note that $\mathbb{H}^{\simeq}F$ satisfies (df1)–(df2) by definition of F being a fibration in 2Cat. It remains to prove (df3). Consider a diagram in $\mathbb{H}^{\simeq}\mathcal{A}$ as below left, together with a weakly horizontally invertible square β in $\mathbb{H}^{\simeq}\mathcal{B}$, as depicted below right,

$$\begin{array}{ccc} A & \xrightarrow[\simeq]{a} & C \\ & & \downarrow w \\ A' & \xrightarrow[\simeq]{a'} & C' \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow[\simeq]{Fa} & FC \\ {\scriptstyle v}\downarrow & \beta\ \simeq & \downarrow{\scriptstyle Fw} \\ FA' & \xrightarrow[Fa']{\simeq} & FC' \end{array}$$

ie a 2–isomorphism $\beta\colon (Fw)(Fa) \Rightarrow (Fa')v$ in $\mathcal{B}$ by [15, Lemma A.2.3]. Let (c', a', η, ϵ) be a choice of adjoint equivalence data for a', and let δ be the 2–isomorphism in $\mathcal{B}$ given by the pasting below left:



Since F is a fibration in 2Cat, there is an equivalence $u\colon A \xrightarrow{\simeq} A'$ in $\mathcal{A}$ and a 2–isomorphism $\bar{\alpha}\colon c'wa \cong u$ in $\mathcal{A}$ such that $\delta = F\bar{\alpha}$. We set $\alpha\colon wa \cong a'u$ to be the pasting above right; by the triangle identities for (η, ϵ), we get that $\beta = F\alpha$ as desired. This proves that $\mathbb{H}^{\simeq}F$ is a fibration in DblCat.

Now suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a trivial fibration in 2Cat. By definition, we directly see that $\mathbb{H}^{\simeq}F$ is surjective on objects, full on horizontal morphisms, and fully faithful on squares. Fullness on vertical morphisms for $\mathbb{H}^{\simeq}F$ follows from the fact that a lift of an adjoint equivalence by a biequivalence is also an adjoint equivalence. Hence $\mathbb{H}^{\simeq}F$ is a trivial fibration in DblCat by Proposition 3.9, and this shows that $\mathbb{H}^{\simeq}$ is right Quillen.

It remains to show that the derived counit is levelwise a biequivalence. Let $\mathcal{A}$ be a 2–category, and let $q_{\mathbb{H}^{\simeq}\mathcal{A}}\colon (\mathbb{H}^{\simeq}\mathcal{A})^c \to \mathbb{H}^{\simeq}\mathcal{A}$ denote the cofibrant replacement of $\mathbb{H}^{\simeq}\mathcal{A}$ constructed as follows. The double category $(\mathbb{H}^{\simeq}\mathcal{A})^c$ has the same objects as $\mathcal{A}$; it has a copy $\bar{a}$ for each morphism a in $\mathcal{A}$, and horizontal morphisms in $(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by free composites of $\bar{a}$'s; it has a copy $\bar{u}$ for each adjoint equivalence u in $\mathcal{A}$, and vertical morphisms in $(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by free composites of $\bar{u}$'s; and squares in $(\mathbb{H}^{\simeq}\mathcal{A})^c$ are given by squares of $\mathbb{H}^{\simeq}\mathcal{A}$ whose boundaries are the actual composites in $\mathbb{H}^{\simeq}\mathcal{A}$ of the representative of the free composites.

Then, by studying the data of the 2–category $L^{\simeq}(\mathbb{H}^{\simeq}\mathcal{A})^c$, we can see that the derived counit at $\mathcal{A}$

$$L^{\simeq}(\mathbb{H}^{\simeq}\mathcal{A})^c \xrightarrow{L^{\simeq}q_{\mathbb{H}^{\simeq}\mathcal{A}}} L^{\simeq}\mathbb{H}^{\simeq}\mathcal{A} \xrightarrow{\epsilon_{\mathcal{A}}} \mathcal{A}$$

is a trivial fibration in 2Cat as it is surjective on objects, full on morphisms, and fully faithful on 2–morphisms. □

Remark 6.7 The components of the derived unit of the adjunction $L^{\simeq} \dashv \mathbb{H}^{\simeq}$ are not weak equivalences in DblCat. Indeed, since every 2–category is fibrant, we know that the counit and the derived counit agree on cofibrant double categories. Then, if we consider the component $\eta_{\mathbb{V}2}\colon \mathbb{V}2 \to \mathbb{H}^{\simeq}L^{\simeq}\mathbb{V}2 = \mathbb{H}^{\simeq}E_{\mathrm{adj}}$ of the unit at the cofibrant double category $\mathbb{V}2$, we see that $\mathbb{H}^{\simeq}E_{\mathrm{adj}}$ has nontrivial horizontal morphisms, given by the adjoint equivalence created by $L^{\simeq}$ from the unique vertical morphism of $\mathbb{V}2$, while $\mathbb{V}2$ does not. Therefore $\eta_{\mathbb{V}2}$ is not a double biequivalence, as it does not satisfy (db2). Then, since $\mathbb{V}2$ is weakly horizontally invariant, Proposition 5.5 implies that $\eta_{\mathbb{V}2}$ is not a weak equivalence in DblCat.

While Theorem 6.6 implies that $\mathbb{H}^{\simeq}\colon$ 2Cat $\to$ DblCat preserves weak equivalences and fibrations, the following result says that it further reflects these classes of double functors. Hence the model structure on 2Cat is completely determined from our model structure on DblCat through its image under $\mathbb{H}^{\simeq}$.

Theorem 6.8 *Lack's model structure on* 2Cat *is right-induced along the adjunction*

$$\text{2Cat} \underset{\mathbb{H}^{\simeq}}{\overset{L^{\simeq}}{\rightleftarrows}} \text{DblCat} \quad (\perp)$$

from the model structure on DblCat *of Theorem 3.26.*

Proof We need to show that a 2–functor F is a fibration (resp. biequivalence) in 2Cat if and only if $\mathbb{H}^{\simeq}F$ is a fibration (resp. weak equivalence) in DblCat. Since $\mathbb{H}^{\simeq}$ is right Quillen, we know it preserves fibrations and trivial fibrations. Moreover, since all 2–categories are fibrant, by Ken Brown's lemma — see [10, Lemma 1.1.12] — the functor $\mathbb{H}^{\simeq}$ preserves all weak equivalences. Therefore, if F is a fibration (resp. biequivalence), then $\mathbb{H}^{\simeq}F$ is a fibration (resp. weak equivalence).

If $\mathbb{H}^{\simeq}F$ is a fibration in DblCat, then by Proposition 4.13 it is $\mathcal{J}_w$–injective, since its target is weakly horizontally invariant by Theorem 6.5. Hence, conditions (df1)–(df2) of Proposition 4.3 for $\mathbb{H}^{\simeq}F$ say that F is a fibration in 2Cat.

Finally, if $\mathbb{H}^{\simeq}F$ is a weak equivalence in DblCat, then by Proposition 5.5 it is a double biequivalence, since its source is weakly horizontally invariant. By (db1) and (db2) of Definition 2.9, we have that F is biessentially surjective on objects and essentially full on morphisms. Fully faithfulness on 2–morphisms follows from applying (db4) of Definition 2.9 to squares with trivial vertical boundaries. Hence F is a biequivalence. □

Finally, we can use the above result to deduce that Lack's model structure on 2Cat is also left-induced from our model structure on DblCat along the horizontal embedding $\mathbb{H}$.

Theorem 6.9 *Lack's model structure on* 2Cat *is left-induced along the adjunction*

$$\mathrm{DblCat} \underset{\boldsymbol{H}}{\overset{\mathbb{H}}{\rightleftarrows}} \mathrm{2Cat} \quad (\perp)$$

from the model structure on DblCat *of Theorem 3.26.*

Proof We need to show that a 2–functor $F\colon \mathcal{A} \to \mathcal{B}$ is a cofibration (resp. biequivalence) in 2Cat if and only if $\mathbb{H}F$ is a cofibration (resp. weak equivalence) in DblCat.

By Theorem 3.11, the double functor $\mathbb{H}F$ is a cofibration if and only if its underlying functors $U\boldsymbol{H}\mathbb{H}F$ and $U\boldsymbol{V}\mathbb{H}F$ have the left lifting property with respect to all surjective on objects and full functors. Since $U\boldsymbol{V}\mathbb{H}F$ trivially satisfies this condition, this holds if and only if $UF = U\boldsymbol{H}\mathbb{H}F$ has the mentioned lifting property. By [13, Lemma 4.1], this is equivalent to saying that F is a cofibration.

Finally, since $\mathbb{H}^{\simeq}\mathcal{A}$ and $\mathbb{H}^{\simeq}\mathcal{B}$ are fibrant replacements of $\mathbb{H}\mathcal{A}$ and $\mathbb{H}\mathcal{B}$ in DblCat by Theorem 6.5, we have that $\mathbb{H}F$ is a weak equivalence if and only if $\mathbb{H}^{\simeq}F$ is a weak equivalence. By Theorem 6.8, this is the case if and only if F is a biequivalence. □

7 Compatibility with the Gray tensor product

We now explore the monoidality of the model structure on DblCat constructed in this paper. Although a similar argument as the one in [16, Remark 7.1] quickly shows that our model structure is not monoidal with respect to the cartesian product, in this section we prove that it is monoidal when we instead consider the Gray tensor product for double categories introduced by Böhm in [1]. This resembles the case of Lack's model structure on 2Cat, which is monoidal with respect to the Gray tensor product of 2–categories, and improves upon the model structure on DblCat of [16], which is only 2Cat–enriched.

The Gray tensor product $\otimes_{\mathrm{Gr}}\colon \mathrm{DblCat}\times\mathrm{DblCat}\to\mathrm{DblCat}$ endows the category DblCat with a symmetric monoidal structure, as shown in [1, Section 3]. We first give an explicit description of the Gray tensor product of two double categories.

Description 7.1 The Gray tensor product $\mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}$ of two double categories $\mathbb{A}$ and $\mathbb{X}$ can be described as the double category given by the following data:

(i) The objects are pairs (A, X) of objects $A\in\mathbb{A}$ and $X\in\mathbb{X}$.

(ii) Two kinds of generating horizontal morphisms: pairs $(a, X)\colon (A, X)\to(C, X)$, where $a\colon A\to C$ is a horizontal morphism in $\mathbb{A}$ and X is an object in $\mathbb{X}$, which compose as in $\mathbb{A}$; and pairs $(A, x)\colon (A, X)\to(A, Z)$, where A is an object in $\mathbb{A}$ and $x\colon X\to Z$ is a horizontal morphism in $\mathbb{X}$, which compose as in $\mathbb{X}$.

(iii) Similarly, the generating vertical morphisms are given by pairs (u, X) and (A, t) with A and X being objects of $\mathbb{A}$ and $\mathbb{X}$ respectively, and u and t being vertical morphisms of $\mathbb{A}$ and $\mathbb{X}$ respectively.

(iv) There are six kinds of generating squares: the ones determined by a square $\alpha\colon (u\ {}^{a}_{a'}\ w)$ in $\mathbb{A}$ and an object $X\in\mathbb{X}$ as shown below left, the ones given by an object $A\in\mathbb{A}$ and a square $\chi\colon (t\ {}^{x}_{x'}\ v)$ in $\mathbb{X}$ as below right,

$$\begin{array}{ccc} (A,X) & \xrightarrow{(a,X)} & (C,X) \\ {\scriptstyle (u,X)}\downarrow & (\alpha,X) & \downarrow{\scriptstyle (w,X)} \\ (A',X) & \xrightarrow[(a',X)]{} & (C',X) \end{array} \qquad \begin{array}{ccc} (A,X) & \xrightarrow{(A,x)} & (A,Z) \\ {\scriptstyle (A,t)}\downarrow & (A,\chi) & \downarrow{\scriptstyle (A,v)} \\ (A,X') & \xrightarrow[(A,x')]{} & (A,Z') \end{array}$$

the squares determined by a horizontal morphism a in $\mathbb{A}$ and a vertical morphism t in $\mathbb{X}$ as displayed below left, and the ones given by a horizontal morphism x in $\mathbb{X}$ and

a vertical morphism u in $\mathbb{A}$ as below right,

$$\begin{array}{ccc} (A,X) & \xrightarrow{(a,X)} & (C,X) \\ {\scriptstyle (A,t)}\downarrow & (a,t) & \downarrow{\scriptstyle (C,t)} \\ (A,X') & \xrightarrow[(a,X')]{} & (C,X') \end{array} \qquad \begin{array}{ccc} (A,X) & \xrightarrow{(A,x)} & (A,Z) \\ {\scriptstyle (u,X)}\downarrow & (u,x) & \downarrow{\scriptstyle (u,Z)} \\ (A',X) & \xrightarrow[(A',x)]{} & (A',Z) \end{array}$$

vertically invertible squares determined by horizontal morphisms a in $\mathbb{A}$ and x in $\mathbb{X}$,

$$\begin{array}{ccccc} (A,X) & \xrightarrow{(a,X)} & (C,X) & \xrightarrow{(C,x)} & (C,Z) \\ \| & & (a,x)\wr\| & & \| \\ (A,X) & \xrightarrow[(A,x)]{} & (A,Z) & \xrightarrow[(a,Z)]{} & (C,Z) \end{array}$$

and horizontally invertible squares given by vertical morphisms u in $\mathbb{A}$ and t in $\mathbb{X}$,

$$\begin{array}{ccc} (A,X) & = & (A,X) \\ {\scriptstyle (u,X)}\downarrow & & \downarrow{\scriptstyle (A,t)} \\ (A',X) & \underset{(u,t)}{\cong} & (A,X') \\ {\scriptstyle (A',t)}\downarrow & & \downarrow{\scriptstyle (u,X')} \\ (A',X') & = & (A',X') \end{array}$$

subject to conditions which are equivalent to requiring that the projection double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ is fully faithful on squares.

Remark 7.2 The cartesian product of two double categories is obtained by taking the product of the sets of objects, horizontal morphisms, vertical morphisms, and squares, respectively. The projection double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ sends the squares of the form (a,x) and (u,t) to the identity squares $(\mathrm{id}_a,\mathrm{id}_x)=\mathrm{id}_{(a,x)}$ and $(e_u,e_t)=e_{(u,t)}$, and acts as the identity on the remaining generators. In particular, it is straightforward from this description that $\Pi_{\mathbb{A},\mathbb{X}}$ is functorial in $\mathbb{A}$ and $\mathbb{X}$. Note that the squares of the form $(a,t):=(\mathrm{id}_a,e_t)$ and $(u,x):=(e_u,\mathrm{id}_x)$ are not identity squares in the product $\mathbb{A}\times\mathbb{X}$ even though they come from identity squares in $\mathbb{A}$ and $\mathbb{X}$.

We can show that the projection $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ is a trivial fibration in our model structure on DblCat.

Lemma 7.3 *The projection double functor $\Pi_{\mathbb{A},\mathbb{X}}\colon \mathbb{A}\otimes_{\mathrm{Gr}}\mathbb{X}\to\mathbb{A}\times\mathbb{X}$ is a trivial fibration, for all double categories $\mathbb{A}$ and $\mathbb{X}$.*

Proof We use the characterization of trivial fibrations of Proposition 3.9. Since $\Pi_{\mathbb{A},\mathbb{X}}$ is the identity on objects, it is clearly surjective on objects. Given a horizontal morphism $(a, x)\colon (A, X) \to (C, Z)$ in $\mathbb{A} \times \mathbb{X}$, the composite

$$(A, X) \xrightarrow{(a,X)} (C, X) \xrightarrow{(C,x)} (C, Z)$$

of horizontal morphisms in $\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{X}$ is sent by $\Pi_{\mathbb{A},\mathbb{X}}$ to (a, x), which shows that $\Pi_{\mathbb{A},\mathbb{X}}$ is full on horizontal morphisms. Similarly, one can show that $\Pi_{\mathbb{A},\mathbb{X}}$ is full on vertical morphisms. Fully faithfulness on squares holds by Description 7.1(iv). □

We now show that $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ gives a fibrant replacement for $\mathbb{A} \times \mathbb{X}$, where $(-)^{\mathrm{whi}}$ is the weakly horizontally invariant replacement of Construction 3.17.

Lemma 7.4 *Let $\mathbb{A}$ and $\mathbb{X}$ be double categories. Then $j_{\mathbb{A}} \times j_{\mathbb{X}}\colon \mathbb{A} \times \mathbb{X} \to \mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ provides a fibrant replacement for the double category $\mathbb{A} \times \mathbb{X}$.*

Proof First, note that $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is fibrant, as fibrant objects are closed under products. Now consider the commutative triangle

$$\begin{array}{ccc} & \mathbb{A} \times \mathbb{X} & \\ {\scriptstyle j_{\mathbb{A}\times\mathbb{X}}}\swarrow & & \searrow{\scriptstyle j_{\mathbb{A}} \times j_{\mathbb{X}}} \\ (\mathbb{A} \times \mathbb{X})^{\mathrm{whi}} & \xrightarrow[(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})]{} & \mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}} \end{array}$$

where the bottom map is induced by the projections

$$\pi_{\mathbb{A}}\colon \mathbb{A} \times \mathbb{X} \to \mathbb{A} \quad \text{and} \quad \pi_{\mathbb{X}}\colon \mathbb{A} \times \mathbb{X} \to \mathbb{X}.$$

Since $j_{\mathbb{A}\times\mathbb{X}}$ is a weak equivalence by Corollary 5.4, to prove that $j_{\mathbb{A}} \times j_{\mathbb{X}}$ is a weak equivalence it suffices to show that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is a weak equivalence; we use Proposition 3.9 to prove that it is in fact a trivial fibration.

One can see that $(\pi_{\mathbb{A}}^{\mathrm{whi}}, \pi_{\mathbb{X}}^{\mathrm{whi}})$ is the identity on underlying horizontal categories, and that it is fully faithful on squares since $j_{\mathbb{A}\times\mathbb{X}}$ and $j_{\mathbb{A}} \times j_{\mathbb{X}}$ are so. Finally, by studying the weakly horizontally invariant replacements, we can see that it is also full on vertical morphisms. Indeed, all the vertical morphisms that were freely added to $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ from the image of $\mathbb{A} \times \mathbb{X}$ were also freely added to $(\mathbb{A} \times \mathbb{X})^{\mathrm{whi}}$ from the image of $\mathbb{A} \times \mathbb{X}$. □

Mirroring the proof in [13, Section 7], we show that the cartesian product and the Gray tensor product of a weak equivalence with an identity is also a weak equivalence.

Remark 7.5 Given a double biequivalence $F\colon \mathbb{A} \to \mathbb{B}$ and a double category $\mathbb{X}$, the product $F \times \mathrm{id}_{\mathbb{X}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{B} \times \mathbb{X}$ is a double biequivalence. Indeed, it is straightforward to see that (db1)–(db4) of Definition 2.9 hold for $F \times \mathrm{id}_{\mathbb{X}}$ since they do for F.

Proposition 7.6 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a weak equivalence in the model structure on* DblCat *of Theorem 3.26. Then, for every double category $\mathbb{X}$, the induced double functors*

$$F \times \mathrm{id}_{\mathbb{X}} \colon \mathbb{A} \times \mathbb{X} \to \mathbb{B} \times \mathbb{X} \quad \textit{and} \quad F \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{X}} \colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{X} \to \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{X}$$

are also weak equivalences in DblCat.

Proof First note that the weakly horizontally invariant replacement F^{whi} is a double biequivalence, since F is a weak equivalence. Hence, by Remark 7.5, the double functor $F^{\mathrm{whi}} \times \mathrm{id}_{\mathbb{X}^{\mathrm{whi}}} \colon \mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}} \to \mathbb{B}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ is also a double biequivalence. Since $\mathbb{A}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ and $\mathbb{B}^{\mathrm{whi}} \times \mathbb{X}^{\mathrm{whi}}$ are fibrant replacements for $\mathbb{A} \times \mathbb{X}$ and $\mathbb{B} \times \mathbb{X}$ by Lemma 7.4, this shows that $F \times \mathrm{id}_{\mathbb{X}}$ is a weak equivalence by 2-out-of-3.

For the statement regarding the Gray tensor product, we know by Lemma 7.3 that the double functors $\Pi_{\mathbb{A},\mathbb{X}}$ and $\Pi_{\mathbb{B},\mathbb{X}}$ are trivial fibrations. Since the diagram

$$\begin{array}{ccc} \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{X} & \xrightarrow{F \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{X}}} & \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{X} \\ {\scriptstyle \Pi_{\mathbb{A},\mathbb{X}}}\downarrow & & \downarrow{\scriptstyle \Pi_{\mathbb{B},\mathbb{X}}} \\ \mathbb{A} \times \mathbb{X} & \xrightarrow[F \times \mathrm{id}_{\mathbb{X}}]{} & \mathbb{B} \times \mathbb{X} \end{array}$$

commutes, $F \otimes_{\mathrm{Gr}} \mathrm{id}_{\mathbb{X}}$ is also a weak equivalence by 2-out-of-3. □

This allows us to prove that our model structure on DblCat is monoidal with respect to the Gray tensor product, inspired by the proof of the monoidality of Lack's model structure on 2Cat of [13, Section 7].

Notation 7.7 Given two double functors $I\colon \mathbb{A} \to \mathbb{B}$ and $J\colon \mathbb{C} \to \mathbb{D}$, we write $I \,\square\, J$ for the pushout-product double functor

$$I \,\square\, J \colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{D} \amalg_{\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{C}} \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{C} \to \mathbb{B} \otimes_{\mathrm{Gr}} \mathbb{D}.$$

Theorem 7.8 *The model structure on* DblCat *of Theorem 3.26 is monoidal with respect to the Gray tensor product $\otimes_{\mathrm{Gr}}$.*

Proof We begin by showing that whenever I and J are cofibrations, the pushout-product $I \,\square\, J$ is also a cofibration; it is enough to consider the case when I and J are

in the set of generating cofibrations $\mathcal{I}_w = \{I_1, I_2, I_3, I_4, I_5\}$ of Notation 3.8. Moreover, since the Gray tensor product is symmetric, if we show the result for $I \square J$, then it also holds for $J \square I$. Note that $I_1 \square J \cong J$, which proves the cases involving I_1.

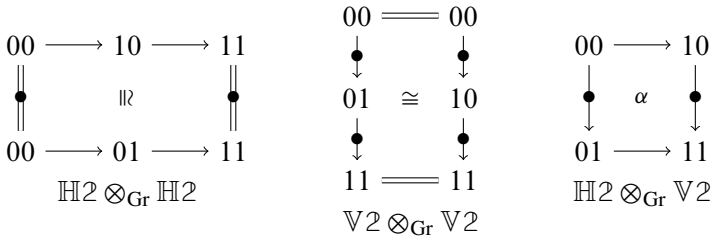
To show the cases involving I_4 or I_5, we observe the following three facts: the functors $U\boldsymbol{H}, U\boldsymbol{V} : \mathrm{DblCat} \to \mathrm{Cat}$ preserve pushouts since they are left adjoints (see Remark 4.5 of [16]); $U\boldsymbol{H}(I_4)$, $U\boldsymbol{H}(I_5)$, $U\boldsymbol{V}(I_4)$ and $U\boldsymbol{V}(I_5)$ are identities; and $U\boldsymbol{H}(\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B})$ (resp. $U\boldsymbol{V}(\mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{B})$) is completely determined by $U\boldsymbol{H}(\mathbb{A})$ and $U\boldsymbol{H}(\mathbb{B})$ (resp. $U\boldsymbol{V}(\mathbb{A})$ and $U\boldsymbol{V}(\mathbb{B})$). It then follows that $U\boldsymbol{H}(I \square J)$ and $U\boldsymbol{V}(I \square J)$ are isomorphisms, and thus $I \square J$ is a cofibration by Theorem 3.11, if either I or J is in $\{I_4, I_5\}$.

For the remaining cases, one can check that $I_2 \square I_2$ is given by the boundary inclusion $\delta(\mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{H}2) \to \mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{H}2$, where $\delta(\mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{H}2)$ is obtained by removing the nonidentity squares in $\mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{H}2$, generated by the data depicted below. Then this boundary inclusion is a cofibration by Theorem 3.11, since it is the identity on underlying horizontal and vertical categories.

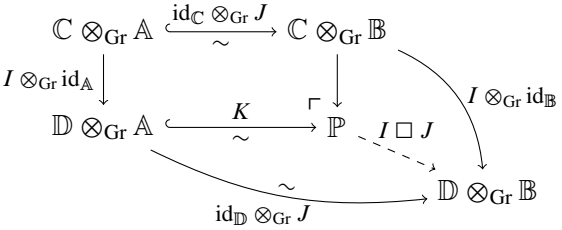
Similarly, one can show that the pushout-products $I_3 \square I_3$ and $I_2 \square I_3$ are cofibrations, as they are given by analogously defined boundary inclusions

$$\delta(\mathbb{V}2 \otimes_{\mathrm{Gr}} \mathbb{V}2) \to \mathbb{V}2 \otimes_{\mathrm{Gr}} \mathbb{V}2 \quad \text{and} \quad \delta(\mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{V}2) \to \mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{V}2,$$

respectively, where the double categories $\mathbb{H}2 \otimes_{\mathrm{Gr}} \mathbb{V}2$ and $\mathbb{V}2 \otimes_{\mathrm{Gr}} \mathbb{V}2$ are generated by the data



It remains to show that if $I \in \mathcal{I}_w$ and $J : \mathbb{A} \to \mathbb{B}$ is a trivial cofibration, then $I \square J$ is a weak equivalence. Note that I is of the form $I : \mathbb{C} \to \mathbb{D}$ with $\mathbb{C}$ cofibrant, and consider the pushout diagram



Since $\mathbb{C}$ is cofibrant and J is a cofibration, we know that $(\varnothing \to \mathbb{C}) \square J = \mathrm{id}_{\mathbb{C}} \otimes_{\mathrm{Gr}} J$ is also a cofibration by the above. Since J is a trivial cofibration by assumption, the double functor $\mathrm{id}_{\mathbb{C}} \otimes_{\mathrm{Gr}} J$ is a weak equivalence by Proposition 7.6. Then $\mathrm{id}_{\mathbb{C}} \otimes_{\mathrm{Gr}} J$ is a trivial cofibration, and therefore so is K since these are stable under pushouts. Proposition 7.6 also guarantees that $\mathrm{id}_{\mathbb{D}} \otimes_{\mathrm{Gr}} J$ is a weak equivalence, and then so is $I \square J$ by 2-out-of-3. □

Remark 7.9 Recall that, by restricting the Gray tensor product $\otimes_{\mathrm{Gr}}$ in one variable along $\mathbb{H}\colon 2\mathrm{Cat} \to \mathrm{DblCat}$, we get the tensoring functor $\otimes\colon \mathrm{DblCat} \times 2\mathrm{Cat} \to \mathrm{DblCat}$ which gives an enrichment $\boldsymbol{H}[-,-]_{\mathrm{ps}}$ of DblCat over 2Cat as in [16, Proposition 7.5]. Since the functor $\mathbb{H}$ is left Quillen by Theorem 6.3, as a corollary of Theorem 7.8 we get that the model structure on DblCat of Theorem 3.26 is also 2Cat–enriched.

8 Whitehead theorem

In this section we show a Whitehead theorem for double categories, that characterizes the weak equivalences between fibrant objects (which, by Proposition 5.5, are double biequivalences) as the double functors that admit a pseudoinverse up to horizontal pseudonatural equivalence. Such a statement is reminiscent of the Whitehead theorem for 2–categories: a 2–functor $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if there is a pseudofunctor $G\colon \mathcal{B} \to \mathcal{A}$ together with two pseudonatural equivalences $\mathrm{id}_{\mathcal{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathcal{B}}$.

Under the hypothesis that the double categories involved are *horizontally invariant* — defined analogously to the weakly horizontally invariant double categories with horizontal equivalences replaced by the stronger notion of horizontal isomorphisms; see [6, Theorem and Definition 4.1.7] — Grandis characterizes in [6, Theorem 4.4.5] the double functors F such that $U\boldsymbol{H}F$ and $U\boldsymbol{H}[\mathbb{V}2, F]$ are both equivalences of categories as the ones which admit a pseudoinverse up to *horizontal natural isomorphism*. In analogy, double biequivalences can be defined as the double functors such that $\boldsymbol{H}F$ and $\boldsymbol{H}[\mathbb{V}2, F]$ are biequivalences of 2–categories; see [16, Proposition 3.11]. Altogether our Whitehead theorem can be seen as a 2–categorical version of Grandis's result.

In the theorem below, whose proof is the content of this section, it is actually enough to require that the *source* be weakly horizontally invariant.

Theorem 8.1 (Whitehead theorem) *Let $\mathbb{A}$ and $\mathbb{B}$ be double categories such that $\mathbb{A}$ is weakly horizontally invariant. Then a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equiva-*

lence (*or equivalently*, *a double biequivalence*) *if and only if there is a pseudodouble functor* $G\colon \mathbb{B} \to \mathbb{A}$ *together with horizontal pseudonatural equivalences* $\mathrm{id}_{\mathbb{A}} \simeq GF$ *and* $FG \simeq \mathrm{id}_{\mathbb{B}}$.

Remark 8.2 If $\mathbb{A}$ and $\mathbb{B}$ are cofibrant–fibrant double categories, a double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence if and only if there is a (strict) double functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal pseudonatural equivalences $\mathrm{id}_{\mathbb{A}} \simeq GF$ and $FG \simeq \mathrm{id}_{\mathbb{B}}$. Indeed, the cofibrancy condition implies that the underlying horizontal and vertical categories of $\mathbb{A}$ and $\mathbb{B}$ are free, and therefore the weak inverse G can be chosen to be strict.

This retrieves a formulation of the usual Whitehead theorem for model categories — see [3, Lemma 4.24] — in our setting; such a result characterizes the weak equivalences between cofibrant–fibrant objects in a model structure as the homotopy equivalences. Indeed, the homotopies in our model structure are the horizontal pseudonatural equivalences, as we now show.

Given a weakly horizontally invariant double category $\mathbb{A}$, a path object for $\mathbb{A}$ is given by the double category $[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}}$ together with the double functors

$$\mathbb{A} \xrightarrow{W} [\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \xrightarrow{P} \mathbb{A} \times \mathbb{A}$$

obtained by applying the functor $[-, \mathbb{A}]_{\mathrm{ps}}$ to the composite $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}} \to \mathbb{1}$.

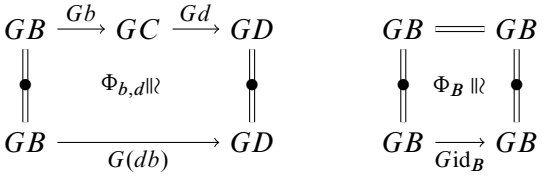
Since $\mathbb{1} \sqcup \mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ is a cofibration and the model structure on DblCat is monoidal, it follows that P is a fibration in DblCat. Similarly, since $\mathbb{1} \to \mathbb{H}E_{\mathrm{adj}}$ is a trivial cofibration in DblCat, by monoidality, the induced double functor $[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}} \to \mathbb{A}$ is a trivial cofibration in DblCat. Hence, by 2-out-of-3, we get that W is a weak equivalence in DblCat, and thus $[\mathbb{H}E_{\mathrm{adj}}, \mathbb{A}]_{\mathrm{ps}}$ is a path object for $\mathbb{A}$.

Then, by definition, a homotopy in DblCat between two double functors $F, G\colon \mathbb{A} \to \mathbb{B}$ with $\mathbb{A}$ and $\mathbb{B}$ cofibrant–fibrant is a double functor $\mathbb{A} \to [\mathbb{H}E_{\mathrm{adj}}, \mathbb{B}]_{\mathrm{ps}}$, or equivalently, a double functor $\mathbb{H}E_{\mathrm{adj}} \to [\mathbb{A}, \mathbb{B}]_{\mathrm{ps}}$, whose values on the two objects of $\mathbb{H}E_{\mathrm{adj}}$ are given by F and G. By [15, Lemma A.3.3], this corresponds to a horizontal pseudonatural equivalence from F to G.

Let us now introduce what we mean by a pseudodouble functor.

Definition 8.3 A *pseudodouble functor* $G\colon \mathbb{B} \to \mathbb{A}$ consists of maps on objects, horizontal morphisms, vertical morphisms, and squares, compatible with sources and targets, which preserve

(i) horizontal compositions and identities up to coherent vertically invertible squares



for every object $B \in \mathbb{B}$, and every pair of composable horizontal morphisms $b\colon B \to C$ and $d\colon C \to D$ in $\mathbb{B}$;

(ii) vertical compositions and identities up to coherent horizontally invertible squares $\Psi_{v,v'}$ and Ψ_B — the transposed versions of those in (i) — for every object $B \in \mathbb{B}$, and every pair of composable vertical morphisms v and v' in $\mathbb{B}$;

(iii) horizontal and vertical compositions of squares accordingly.

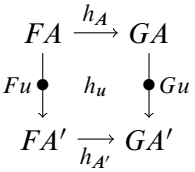
For a detailed description of the coherences, the reader can see [6, Definition 3.5.1].

The pseudodouble functor G is said to be *normal* if the squares Φ_B and Ψ_B are identities for every object $B \in \mathbb{B}$.

Definition 8.4 A *horizontal pseudonatural transformation* $h\colon F \Rightarrow G$ between pseudodouble functors $F, G\colon \mathbb{A} \to \mathbb{B}$ is a pseudodouble functor $h\colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{H}2 \to \mathbb{B}$ which restricts to F and G under the two inclusions $\mathbb{1} \to \mathbb{H}2$. More explicitly, this consists of

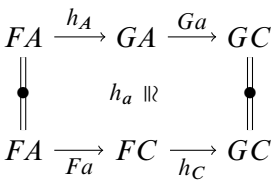
(i) a horizontal morphism $h_A\colon FA \to GA$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$;

(ii) a square h_u in $\mathbb{B}$



for each vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$;

(iii) a vertically invertible square h_a in $\mathbb{B}$



for each horizontal morphism $a\colon A \to C$ in $\mathbb{A}$.

The squares in (ii) are compatible with the coherence squares of F and G for vertical compositions and identities, and the squares in (iii) are compatible with the coherence squares of F and G for horizontal compositions and identities. Together, they satisfy a pseudonaturality condition with respect to squares in $\mathbb{A}$.

A *modification* $\mu\colon (e_F \begin{smallmatrix} h \\ k \end{smallmatrix} e_G)$ between two horizontal pseudonatural transformations $h, k\colon F \Rightarrow G$ is a pseudodouble functor $\mu\colon \mathbb{A} \otimes_{\mathrm{Gr}} \mathbb{H}\Sigma 2 \to \mathbb{B}$ which restricts to h and k under the two canonical inclusions $\mathbb{H}2 \to \mathbb{H}\Sigma 2$, where $\Sigma 2$ is the free 2–category on a 2–morphism. More explicitly, this consists of a square $\mu_A\colon (e_{FA} \begin{smallmatrix} h_A \\ k_A \end{smallmatrix} e_{GA})$ in $\mathbb{B}$, for each object $A \in \mathbb{A}$, satisfying horizontal and vertical coherence conditions with respect to the square components of the pseudonatural transformations h and k.

For more details about the coherence conditions, see [6, Section 3.8].

Remark 8.5 The pseudodouble functors from $\mathbb{A}$ to $\mathbb{B}$ together with the horizontal pseudonatural transformations and modifications between them form a 2–category. It can be seen as the sub–2–category of the 2–category of lax (double) functors of [6, Theorem 3.8.4] given by restriction to the pseudodouble functors.

The notion that we now introduce has also been independently considered by Grandis and Paré in [8, Section 3] under the name of *pointwise equivalences*.

Definition 8.6 Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be pseudodouble functors. A *horizontal pseudonatural equivalence* $\varphi\colon F \Rightarrow G$ is an equivalence in the 2–category of pseudodouble functors $\mathbb{A} \to \mathbb{B}$, horizontal pseudonatural transformations, and modifications.

Equivalently, the horizontal pseudonatural equivalences can be described as follows; see [8, Theorem 4.4] for a proof.

Lemma 8.7 *Let $F, G\colon \mathbb{A} \to \mathbb{B}$ be pseudodouble functors. A horizontal pseudonatural transformation $\varphi\colon F \Rightarrow G$ is a horizontal pseudonatural equivalence if and only if*

(i) *the horizontal morphism $\varphi_A\colon FA \xrightarrow{\simeq} GA$ is a horizontal equivalence, for every object $A \in \mathbb{A}$, and*

(ii) *the square $\varphi_u\colon (Fu \begin{smallmatrix} \varphi_A \\ \varphi_{A'} \end{smallmatrix} Gu)$ is weakly horizontally invertible, for every vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$.*

We will use the term *horizontal biequivalence* to refer to the double functors which admit a pseudoinverse up to horizontal pseudonatural equivalence.

Definition 8.8 A double functor $F\colon \mathbb{A} \to \mathbb{B}$ is a *horizontal biequivalence* if there is a pseudodouble functor $G\colon \mathbb{B} \to \mathbb{A}$ together with horizontal pseudonatural equivalences $\eta\colon \mathrm{id}_{\mathbb{A}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}}$.

Remark 8.9 If $F\colon \mathbb{A} \to \mathbb{B}$ is a horizontal biequivalence, there is a tuple $(G, \eta, \epsilon, \Theta, \Sigma)$ consisting of the following data:

(i) a *normal* pseudodouble functor $G\colon \mathbb{B} \to \mathbb{A}$;

(ii) a horizontal pseudonatural *adjoint* equivalence

$$(\eta\colon \mathrm{id}_{\mathbb{A}} \Rightarrow GF,\ \eta'\colon GF \Rightarrow \mathrm{id}_{\mathbb{A}},\ \lambda\colon \mathrm{id} \cong \eta'\eta,\ \kappa\colon \eta\eta' \cong \mathrm{id});$$

(iii) a horizontal pseudonatural *adjoint* equivalence

$$(\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}},\ \epsilon'\colon \mathrm{id}_{\mathbb{B}} \Rightarrow FG,\ \mu\colon \mathrm{id} \cong \epsilon'\epsilon,\ \nu\colon \epsilon\epsilon' \cong \mathrm{id});$$

(iv) two invertible modifications $\Theta\colon \mathrm{id}_F \cong \epsilon_F \circ F\eta$ and $\Sigma\colon \mathrm{id}_G \cong G\epsilon \circ \eta_G$, expressing the triangle (pseudo)identities for η and ϵ.

This follows from the fact that a pseudodouble functor is always pseudonaturally isomorphic to a normal one, and from a result by Gurski [9, Theorem 3.2] saying that a biequivalence can always be promoted to a biadjoint biequivalence, applied here to the tricategory of double categories, pseudodouble functors, horizontal pseudonatural transformations, and modifications.

Theorem 8.1 now amounts to showing that a double functor whose source is weakly horizontally invariant is a double biequivalence if and only if it is a horizontal biequivalence. However, it is always true that a horizontal biequivalence is a double biequivalence; no additional hypothesis is needed here. In order to prove this first result, we need the following lemma.

Lemma 8.10 *The data of Remark 8.9 induces an invertible modification $\theta\colon F\eta' \cong \epsilon_F$.*

Proof Given an object $A \in \mathbb{A}$, we define the component of θ at A to be the vertically invertible square

$$\begin{array}{ccc} FGFA & \xrightarrow{F\eta'_A} & FA \\ \| & \theta_A \wr\| & \| \\ FGFA & \xrightarrow[\epsilon_{FA}]{} & FA \end{array} \quad = \quad \begin{array}{ccccc} FGFA & \xrightarrow{F\eta'_A} & FA & = & FA \\ \| & e_{F\eta'_A} & \| & \Theta_A \wr\| & \| \\ FGFA & \xrightarrow[F\eta'_A]{} & FA & \xrightarrow{F\eta_A} FGFA & \xrightarrow{\epsilon_{FA}} FA \\ \| & & F\kappa_A \wr\| & \| & e_{\epsilon_{FA}} \ \| \\ FGFA & = & FGFA & \xrightarrow[\epsilon_{FA}]{} & FA \end{array}$$

The proof of horizontal and vertical coherences for θ is a standard check that stems from the constructions of the squares θ_A and from the horizontal and vertical coherences of the modifications $F\kappa\colon (F\eta)(F\eta') \cong \mathrm{id}$ and $\Theta\colon \mathrm{id} \cong \epsilon_F \circ F\eta$. □

Proposition 8.11 *If $F\colon \mathbb{A} \to \mathbb{B}$ is a horizontal biequivalence, then F is a double biequivalence.*

Proof We check that F satisfies (db1)–(db4) of Definition 2.9. Let $(F, G, \eta, \epsilon, \Theta, \Sigma)$ be the data of a horizontal adjoint biequivalence as in Remark 8.9. We first show (db1). For every object $B \in \mathbb{B}$, we want to find an object $A \in \mathbb{A}$ and a horizontal equivalence $B \xrightarrow{\simeq} FA$ in $\mathbb{B}$. Setting $A = GB$, we have that $\epsilon'_B\colon B \xrightarrow{\simeq} FGB = FA$ gives such a horizontal equivalence.

We now show (db2). Let A and C be objects in $\mathbb{A}$, and $b\colon FA \to FC$ be a horizontal morphism in $\mathbb{B}$. We want to find a horizontal morphism $a\colon A \to C$ in $\mathbb{A}$ and a vertically invertible square $(e_{FA}\ {}^{b}_{Fa}\ e_{FC})$ in $\mathbb{B}$. Let $a\colon A \to C$ be the composite

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gb} GFC \xrightarrow{\eta'_C} C.$$

We then have a vertically invertible square as desired,

$$\begin{array}{ccccc}
FA & = & FA & \xrightarrow{b} & FC \\
\| & \Theta_A \ \wr\| & \| & e_b & \| \\
FA & \xrightarrow{F\eta_A} FGFA & \xrightarrow{\epsilon_{FA}} FA & \xrightarrow{b} & FC \\
\| & e_{F\eta_A} \ \| & \| & \epsilon_b \ \wr\| & \| \\
FA & \xrightarrow{F\eta_A} FGFA & \xrightarrow{FGb} FGFC & \xrightarrow{\epsilon_{FC}} & FC \\
\| & e_{(FGb)(F\eta_A)} & \| & \theta_C^{-1} \ \wr\| & \| \\
FA & \xrightarrow[F\eta_A]{} FGFA & \xrightarrow[FGb]{} FGFC & \xrightarrow[F\eta'_C]{} & FC
\end{array}$$

where θ_C is the component at C of the invertible modification θ of Lemma 8.10.

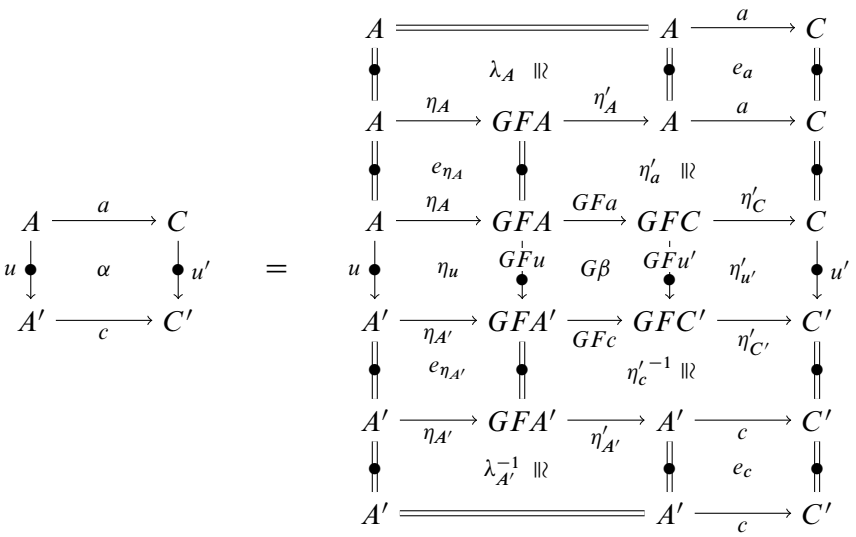
We now show (db3). Let $v\colon B \nrightarrow B'$ be a vertical morphism in $\mathbb{B}$. We want to find a vertical morphism $u\colon A \nrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square $(v \overset{\simeq}{\simeq} Fu)$ in $\mathbb{B}$. Let $u\colon A \nrightarrow A'$ be the vertical morphism $Gv\colon GB \nrightarrow GB'$. Then ϵ'_v gives the desired weakly horizontally invertible square

$$\begin{array}{ccc}
B & \xrightarrow[\simeq]{\epsilon'_B} & FGB \\
{\scriptstyle v}\downarrow & \epsilon'_v \simeq & \downarrow{\scriptstyle FGv} \\
B' & \xrightarrow[\epsilon'_{B'}]{\simeq} & FGB'
\end{array}$$

Finally, we show (db4). For this, let β be a square in $\mathbb{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{Fa} & FC \\ {\scriptstyle Fu}\downarrow & \beta & \downarrow{\scriptstyle Fw} \\ FA' & \xrightarrow[Fa']{} & FC' \end{array}$$

We want to show that there is a unique square $\alpha\colon (u\ {}^{a}_{c}\ u')$ in $\mathbb{A}$ such that $F\alpha = \beta$. Define α to be the square given by the pasting



The thorough reader might check that $F\alpha = \beta$ by completing the following steps. First, transform $F\eta'_{u'}$ by using the invertible modification $\theta\colon F\eta' \cong \epsilon_F$ of Lemma 8.10. Then apply, in order: the horizontal coherence of the modification $F\nu\colon (F\eta')(F\eta) \cong \mathrm{id}$, the horizontal coherence of the modification $\Theta\colon \mathrm{id} \cong \epsilon_F \circ F\eta$, the triangle identity for (μ, ν), the compatibility of $\epsilon_F\colon FGF \Rightarrow F$ with $FG\beta$ and β, and finally the horizontal coherence of the modification $\Theta\colon \mathrm{id} \cong \epsilon_F \circ F\eta$.

Suppose now that $\alpha'\colon (u\ {}^{a}_{c}\ u')$ is another square in $\mathbb{A}$ such that $F\alpha' = \beta$. If we replace $G\beta$ with $GF\alpha'$ in the pasting diagram above, it follows from the compatibility of $\eta'\colon GF \Rightarrow \mathrm{id}_{\mathbb{A}}$ with $GF\alpha'$ and α', and the vertical coherence of the modification $\mu\colon \mathrm{id} \cong \eta'\eta$, that this pasting is also equal to α'. Therefore, we must have $\alpha = \alpha'$. This completes the proof of (db4). □

It is not true in general that a double biequivalence is a horizontal biequivalence, unless we impose an additional condition on the source or on the target. In [16, Theorem 5.13] we provide a Whitehead theorem, where the target satisfies a condition related to

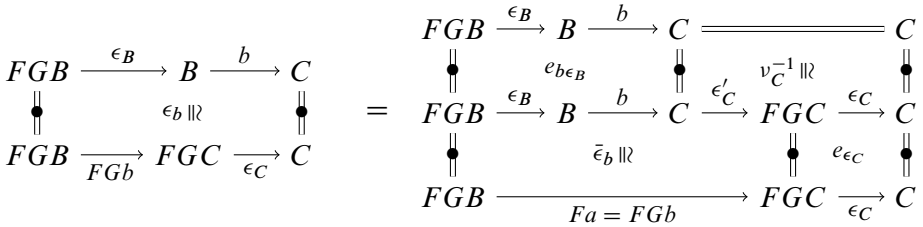
cofibrancy in the model structure of [16]. Here, we prove that such a result holds when the source of the double biequivalence is fibrant, which completes the proof of our Whitehead theorem, Theorem 8.1.

Proposition 8.12 *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a double biequivalence such that $\mathbb{A}$ is weakly horizontally invariant. Then F is a horizontal biequivalence.*

Proof We simultaneously define the pseudodouble functor $G\colon \mathbb{B} \to \mathbb{A}$ and the horizontal pseudonatural transformation $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}}$.

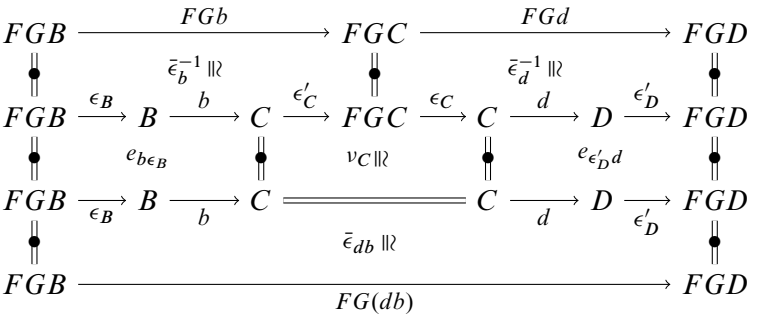
***G* and ϵ on objects** Let $B \in \mathbb{B}$ be an object. By (db1) of Definition 2.9, there is an object $A \in \mathbb{A}$ and a horizontal equivalence $b\colon FA \xrightarrow{\simeq} B$ in $\mathbb{B}$. We set $GB := A$ and $\epsilon_B := b\colon FGB \xrightarrow{\simeq} B$, and also fix horizontal equivalence data $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$.

***G* and ϵ on horizontal morphisms** Now let $b\colon B \to C$ be a horizontal morphism in $\mathbb{B}$. By (db2) applied to the horizontal morphism $\epsilon'_C b \epsilon_B\colon FGB \to FGC$, there is a horizontal morphism $a\colon GB \to GC$ in $\mathbb{A}$ and a vertically invertible square $\bar{\epsilon}_b$ as depicted inside the right-hand side of the pasting below. We set $Gb := a\colon GB \to GC$ and ϵ_b to be the square given by the pasting

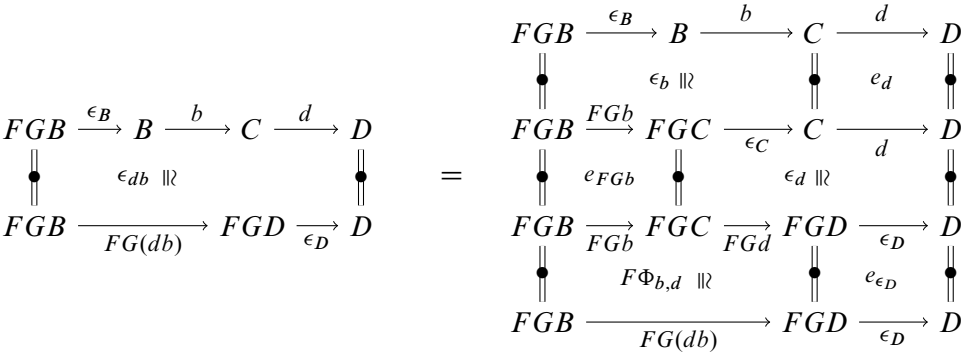


If $b = \mathrm{id}_B$, we can choose $G\mathrm{id}_B := \mathrm{id}_{GB}$ and $\bar{\epsilon}_{\mathrm{id}_B} := \mu_B^{-1}$. Then $\epsilon_{\mathrm{id}_B} = e_{\epsilon_B}$ by the triangle identities for (μ_B, ν_B).

Horizontal coherence Given horizontal morphisms $b\colon B \to C$ and $d\colon C \to D$ in $\mathbb{B}$, we define the vertically invertible comparison square between $Gd \circ Gb$ and $G(db)$ as follows. Let us denote by $\Theta_{b,d}$ the pasting



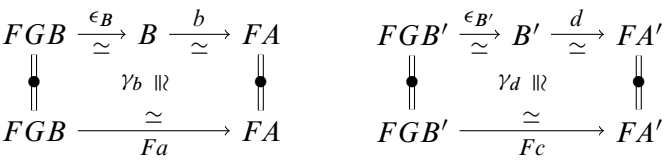
Then, by (db4), there is a unique vertically invertible square $\Phi_{b,d}$ as in Definition 8.3(i) such that $F\Phi_{b,d} = \Theta_{b,d}$. In particular, one can check that with this definition of $\Phi_{b,d}$, the squares ϵ_b, ϵ_d, and ϵ_{db} satisfy the required pasting equality



G and ϵ on vertical morphisms Now let $v\colon B \nrightarrow B'$ be a vertical morphism in $\mathbb{B}$. By (db3), there is a vertical morphism $u'\colon A \nrightarrow A'$ in $\mathbb{A}$ and a weakly horizontally invertible square γ_v in $\mathbb{B}$,

$$\begin{array}{ccc} B & \xrightarrow[\simeq]{b} & FA \\ {\scriptstyle v}\downarrow & \gamma_v \simeq & \downarrow{\scriptstyle Fu'} \\ B' & \xrightarrow[d]{\simeq} & FA' \end{array}$$

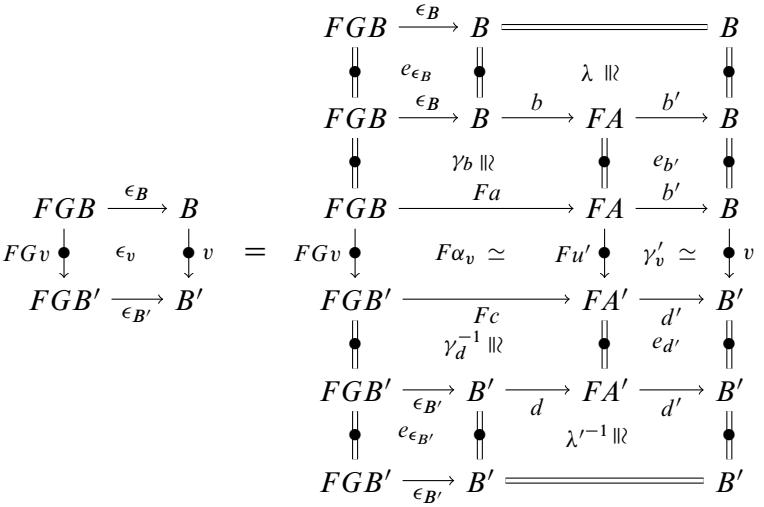
where $b\colon B \xrightarrow{\simeq} FA$ and $d\colon B' \xrightarrow{\simeq} FA'$ are horizontal equivalences. If we consider the composites of horizontal equivalences $b\epsilon_B\colon FGB \xrightarrow{\simeq} FA$ and $d\epsilon_{B'}\colon FGB' \xrightarrow{\simeq} FA'$, then by (db2) there are horizontal morphisms $a\colon GB \to A$ and $c\colon GB' \to A'$ in $\mathbb{A}$ and vertically invertible squares γ_b and γ_d:



Since lifts of horizontal equivalences by a double biequivalence are horizontal equivalences, we have that $a\colon GB \xrightarrow{\simeq} A$ and $c\colon GB' \xrightarrow{\simeq} A'$ are horizontal equivalences in $\mathbb{A}$; thus, since $\mathbb{A}$ is weakly horizontally invariant, there is a vertical morphism $u\colon GB \nrightarrow GB'$ and a weakly horizontally invertible square

$$\begin{array}{ccc} GB & \xrightarrow[\simeq]{a} & A \\ {\scriptstyle u}\downarrow & \alpha_v \simeq & \downarrow{\scriptstyle u'} \\ GB' & \xrightarrow[c]{\simeq} & A' \end{array}$$

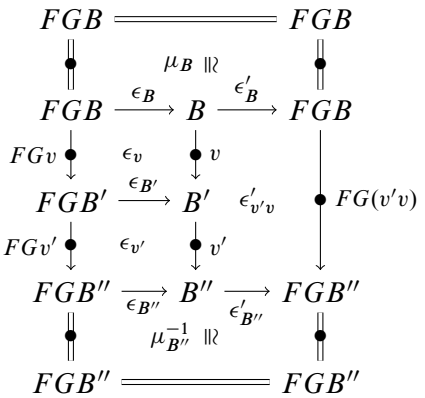
We set $Gv := u\colon GB \nrightarrow GB'$. To define the weakly horizontally invertible square ϵ_v, let us first fix a weak inverse γ'_v of γ_v with respect to some horizontal equivalences (b, b', λ, κ) and $(d, d', \lambda', \kappa')$. We set ϵ_v to be the square given by the pasting



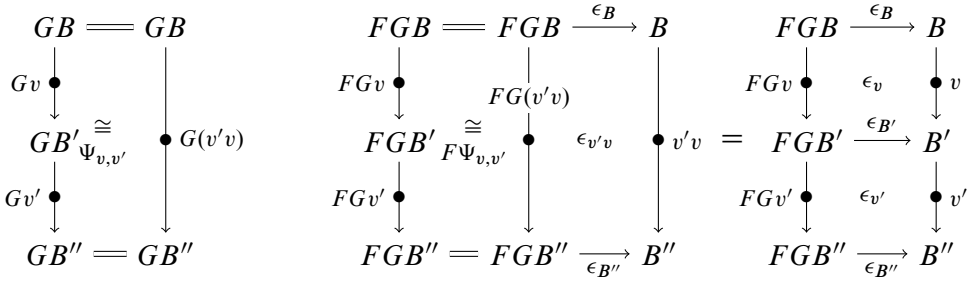
Note that all the squares in the pasting are weakly horizontally invertible by [15, Lemma A.2.1], and thus so is ϵ_v. We write ϵ'_v for its unique weak inverse with respect to the horizontal adjoint equivalences $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$ and $(\epsilon_{B'}, \epsilon'_{B'}, \mu_{B'}, \nu_{B'})$, as given by Lemma 2.8.

If $v = e_B$, we can choose $Ge_B := e_{GB}$ and $\gamma_{e_B} := e_{\epsilon_B}$. Then α_{e_B} can be chosen to be the identity square at the object GB and we get $\epsilon_{e_B} = e_{\epsilon_B}$.

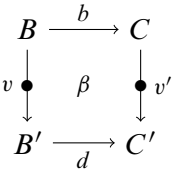
Vertical coherence Given vertical morphisms $v\colon B \nrightarrow B'$ and $v'\colon B' \nrightarrow B''$ in $\mathbb{B}$, we define the horizontally invertible comparison square between $Gv' \bullet Gv$ and $G(v'v)$ as follows. Let us denote by $\Omega_{v,v'}$ the pasting



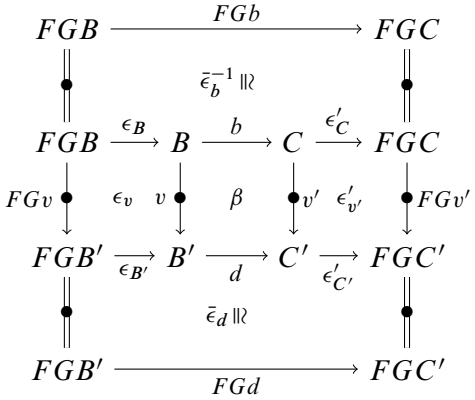
Note that this square is horizontally invertible, since it is weakly horizontally invertible and its horizontal boundaries are identities. By (db4), there is a unique horizontally invertible square $\Psi_{v,v'}$ as depicted below left such that $F\Psi_{v,v'} = \Omega_{v,v'}$. In particular, one can check that, with this definition of $\Psi_{v,v'}$, the squares ϵ_v, $\epsilon_{v'}$ and $\epsilon_{v'v}$ satisfy the pasting equality below right:



G on squares Consider a square in $\mathbb{B}$



Let us denote by δ the pasting



Then, by (db4), there is a unique square $\alpha\colon (Gv \,{}^{Gb}_{Gd}\, Gv')$ such that $F\alpha = \delta$. We set $G\beta := \alpha\colon (Gv \,{}^{Gb}_{Gd}\, Gv')$.

Let $b\colon B \to C$ be a horizontal morphism in $\mathbb{B}$, and $\beta = e_b\colon (e_B \,{}^{b}_{b}\, e_C)$. Then we have that $\delta = e_{FGb}$, since $\epsilon_{e_B} = e_{\epsilon_B}$ and $\epsilon'_{e_C} = e_{\epsilon_C}$, and the unique square $\alpha\colon (e_{GB} \,{}^{Gb}_{Gb}\, e_{GC})$ such that $F\alpha = e_{FGb}$ is given by e_{Gb}. Therefore, $Ge_b = e_{Gb}$.

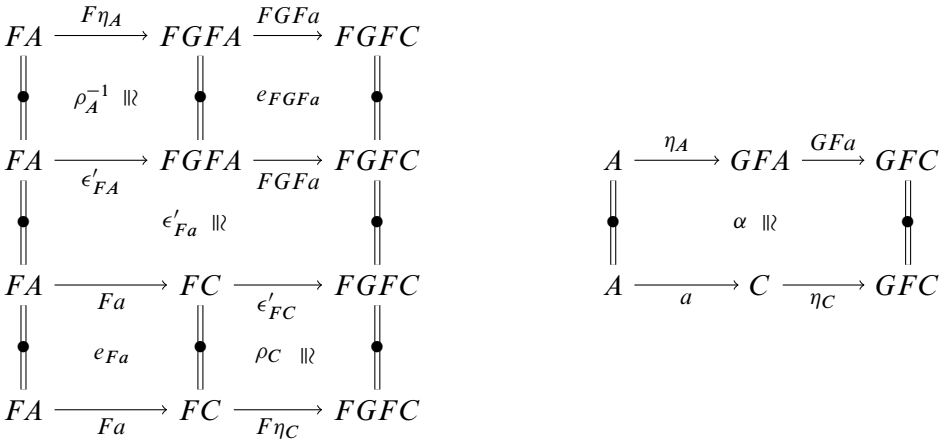
Now let $v\colon B \nrightarrow B'$ be a vertical morphism in $\mathbb{B}$, and $\beta = \mathrm{id}_v\colon (v \,{}^{\mathrm{id}_B}_{\mathrm{id}_{B'}}\, v)$. Then we have that $\delta = \mathrm{id}_{FGv}$, since $\bar{\epsilon}^{-1}_{\mathrm{id}_B} = \mu_B$ and $\bar{\epsilon}_{\mathrm{id}_{B'}} = \mu^{-1}_{B'}$, and ϵ'_v is the weak inverse of ϵ_B with respect to the horizontal adjoint equivalence data $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$ and $(\epsilon_{B'}, \epsilon'_{B'}, \mu_{B'}, \nu_{B'})$. The unique square $\alpha\colon (Gv \,{}^{\mathrm{id}_{GB}}_{\mathrm{id}_{GB'}}\, Gv)$ such that $F\alpha = \mathrm{id}_{FGv}$ is given by id_{Gv}. Therefore, $G\mathrm{id}_v = \mathrm{id}_{Gv}$.

Naturality and adjointness of ϵ and ϵ' The assignment of G on squares is natural with the data of ϵ_B, ϵ_b and ϵ_v, and therefore the latter assemble into a horizontal pseudonatural equivalence $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}}$. Moreover, since $(\epsilon_B, \epsilon'_B, \mu_B, \nu_B)$ are horizontal adjoint equivalences, the data of ϵ'_B, ϵ'_b and ϵ'_v also assemble into a horizontal pseudonatural equivalence $\epsilon'\colon \mathrm{id}_{\mathbb{B}} \Rightarrow FG$, where ϵ'_b is defined in a similar manner as ϵ_b was. In particular, $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathbb{B}}$ and $\epsilon'\colon \mathrm{id}_{\mathbb{B}} \Rightarrow FG$ are adjoint equivalences, where the invertible modifications are given by $\mu\colon \mathrm{id} \cong \epsilon'\epsilon$ and $\nu\colon \epsilon\epsilon' \cong \mathrm{id}$.

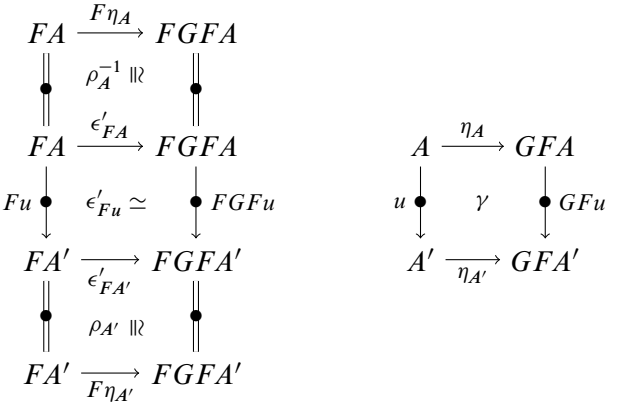
It remains to define the horizontal pseudonatural equivalence $\eta\colon \mathrm{id}_{\mathbb{A}} \Rightarrow GF$. For this purpose, we use the horizontal pseudonatural equivalence $\epsilon'\colon \mathrm{id}_{\mathbb{B}} \Rightarrow FG$.

η on objects Let $A \in \mathbb{A}$, and consider the horizontal equivalence $\epsilon'_{FA}\colon FA \xrightarrow{\simeq} FGFA$. By (db2), there is a horizontal morphism $a\colon A \to GFA$ and a vertically invertible square $\rho_A\colon (e_{FA} \,{}^{\epsilon'_{FA}}_{Fa}\, e_{FGFA})$. We set $\eta_A := a\colon A \to GFA$. Note that $\eta_A\colon A \xrightarrow{\simeq} GFA$ is a horizontal equivalence.

η on horizontal morphisms Let $a\colon A \to C$ be a horizontal morphism in $\mathbb{A}$. We denote by ψ_a the pasting below left. By (db4) there is a unique vertically invertible square α as below right such that $F\alpha = \psi_a$; let $\eta_a := \alpha$:



η on vertical morphisms Let $u\colon A \nrightarrow A'$ be a vertical morphism in $\mathbb{A}$. We denote by ψ_u the pasting below left:



Note that all the squares in ψ_u are weakly horizontally invertible by [15, Lemma A.2.1], and thus so is ψ_u. By (db4) there is a unique weakly horizontally invertible square $\gamma\colon (u \,{}^{\eta_A}_{\eta_{A'}}\, GFu)$ as above right such that $F\gamma = \psi_u$; let $\eta_u := \gamma$.

Naturality of η Since $\epsilon'\colon \mathrm{id}_{\mathbb{B}} \Rightarrow FG$ is a horizontal pseudonatural transformation, η_A, η_a, and η_u assemble into a horizontal pseudonatural transformation $\eta\colon \mathrm{id}_{\mathbb{A}} \Rightarrow GF$. Note that η is a horizontal pseudonatural equivalence, because the η_A are horizontal equivalences and the η_u are weakly horizontally invertible squares. Moreover, $\rho\colon \epsilon'_F \cong F\eta$ gives the data of an invertible modification. □

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