A quantum invariant of links in $T^2 \times I$ with
volume conjecture behavior

JOE BONINGER

A quantum invariant of links in $T^2 \times I$ with volume conjecture behavior

JOE BONINGER

We define a polynomial invariant J_n^T of links in the thickened torus. We call J_n^T the nth toroidal colored Jones polynomial, and show it satisfies many properties of the original colored Jones polynomial. Most significantly, J_n^T exhibits volume conjecture behavior. We prove the volume conjecture for the two-by-two square weave, and provide computational evidence for other links. We also give two equivalent constructions of J_n^T, one as a generalized operator invariant we call a pseudo-operator invariant, and another using the Kauffman bracket skein module of the torus. Finally, we show J_n^T produces invariants of biperiodic and virtual links. To our knowledge, J_n^T gives the first example of volume conjecture behavior in a virtual (nonclassical) link.

57K14, 81R50

1 Introduction

A growing body of evidence supports the idea that the asymptotic growth rate of quantum invariants of links and 3–manifolds encodes geometric information. This hypothesis was initiated by the well-known volume conjecture of Kashaev, Murakami and Murakami.

Conjecture 1.1 [12; 24] *For a knot $K \subset S^3$, let $J_n(K; e^{2\pi i/n})$ be the nth colored Jones polynomial of K evaluated at $e^{2\pi i/n}$. Then*

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n(K; e^{2\pi i/n})| = \text{Vol}(S^3 \setminus K).$$

Here Vol indicates *simplicial volume*, which we define to be the sum of the hyperbolic volumes of the hyperbolic pieces in the Jaco–Shalen–Johannson decomposition of $S^3 \setminus K$; see Soma [29]. We say a quantum invariant exhibits *volume conjecture behavior* if theoretical or computational evidence supports a conjectured limit as above.
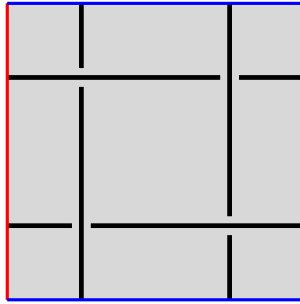
Figure 1: The two-by-two square weave (opposite sides of the diagram are identified).

Conjecture 1.1 has been generalized to other 3–manifolds in several ways. In [6], Costantino extended the colored Jones polynomial to links in $\#_k(S^2 \times S^1)$ using Turaev's theory of shadows and proved the volume conjecture for an infinite family of hyperbolic links. More recently, Chen and Yang [5] discovered volume conjecture behavior exhibited by the Witten–Reshetikhin–Turaev and Turaev–Viro invariants of 3–manifolds, two quantum invariants closely related to the colored Jones polynomial. These conjectures have been verified in many cases; see Detcherry, Kalfagianni and Yang [7] and Ohtsuki [26].

We define a polynomial invariant J_n^T, for $n \in \mathbb{N}$, of oriented links in the thickened torus, $T^2 \times I$. We call J_n^T the nth *toroidal colored Jones polynomial*, and show it satisfies many properties of the colored Jones polynomial for links in S^3. For example, we give one construction of J_n^T using the theory of operator invariants, and another using the Kauffman bracket skein module of $T^2 \times I$. Significantly, J_n^T is the first example of volume conjecture behavior in the Kauffman bracket skein module of a manifold other than S^3.

We state the volume conjecture for J_n^T precisely as follows.

Conjecture 1.2 *For any link $L \subset T^2 \times I$ such that $(T^2 \times I) \setminus L$ is hyperbolic,*

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})| = \mathrm{Vol}((T^2 \times I) \setminus L).$$

Here the simplicial volume Vol is simply the hyperbolic volume of $(T^2 \times I) \setminus L$. We prove Conjecture 1.2 for the two-by-two square weave $W \subset T^2 \times I$ shown in Figure 1.

Theorem 6.2 *We have*

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(W; e^{2\pi i/n})| = 4v_{\mathrm{oct}} = \mathrm{Vol}((T^2 \times I) \setminus W),$$

where $v_{\mathrm{oct}} \approx 3.6638$ is the volume of the regular ideal hyperbolic octahedron.

| link	$(2\pi/n) \cdot \log \lvert J_n^T(L; e^{2\pi i/n})\rvert$ at $n =$						Vol
	10	20	30	50	75	100	
2.1	5.4685	5.5004	5.4843	5.4548	5.4309	5.4215	5.3335
3.2	7.5047	7.6976	7.7393	7.7566	7.7564	7.7528	7.7069
3.5	5.9817	6.2649	6.3345	6.3733	6.3836	6.3852	6.3545
3.7	9.0885	9.3732	9.4523	9.5017	9.5182	9.5231	9.5034
B	7.1834	7.3637	7.3903	7.3953	7.3891	7.3825	$2v_{\text{oct}} \approx 7.3278$
ℓ	9.5569	9.9321	10.0405	10.1130	10.1411	10.1519	$10v_{\text{tet}} \approx 10.149$

Table 1: Computational evidence for Conjecture 1.2.

In addition to Theorem 6.2, the computations in Table 1 support our volume conjecture. Each row gives the normalized logarithm of the modulus of the toroidal colored Jones polynomial of a certain link, at the relevant root of unity, for different values of n. The first four rows are genus one virtual knots in Green's table [10] — each of these corresponds to a knot in $T^2 \times I$ (see Kuperberg [20]) with volume computed by Adams, Eisenberg, Greenberg, Kapoor, Liang, O'Connor, Pacheco-Tallaj and Wang in [1]. In the fifth and sixth rows, B and ℓ refer respectively to the virtual 2–braid and triaxial weave shown in Figure 2. (The geometry of ℓ is discussed by Champanerkar, Kofman and Purcell in [4].) Finally, $v_{\text{tet}} \approx 1.0149$ is the volume of the regular ideal hyperbolic tetrahedron.

Pseudo-operator invariant

Volume conjecture behavior is not the only interesting feature of J_n^T. Like the original colored Jones polynomial, J_n^T is defined using the theory of operator invariants and
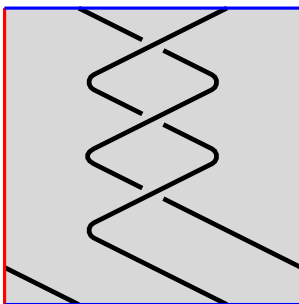
The virtual 2–braid B.
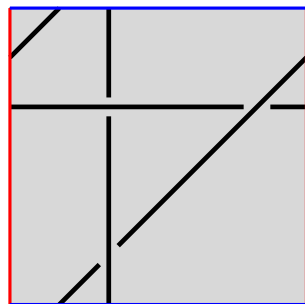
The triaxial weave ℓ.

Figure 2

the quantum group $\mathscr{A} = U_q(\mathrm{sl}(2, \mathbb{C}))$, the quantized universal enveloping algebra of $\mathrm{sl}(2, \mathbb{C})$ specialized to a root of unity q. Briefly, given a link $L \subset T^2 \times I$ with diagram $D \subset T^2$, we use the flat geometry of T^2 to label certain points of D as critical points. We then assign $\mathscr{A}$–linear operators to each critical point of D and use these local assignments to compute J_n^T as a state sum. This is similar to the construction of the colored Jones polynomial of links in S^3, with a key conceptual difference: in S^3, the local assignments of $\mathscr{A}$–linear operators to critical points (crossings and local extrema) extend to a global assignment of a single $\mathscr{A}$–linear operator to the entire link. With J_n^T no global assignment is possible, and for this reason we refer to J_n^T as a *pseudo-operator invariant*. The theory of pseudo-operator invariants, which generalizes the theory of operator invariants, may have applications beyond the invariant J_n^T. In [Section 3](), we develop this theory in detail and in the process construct another invariant $\hat{J}_{\boldsymbol{n},q}^T$ of framed, unoriented links in $T^2 \times I$. The invariant $\hat{J}_{\boldsymbol{n},q}^T$ is analogous to the invariant $J_{L,\boldsymbol{n}}$ of Kirby and Melvin [17], where $\boldsymbol{n} = (n_1, \ldots, n_k)$ is a multi-integer indicating an integer n_i assigned to each component of L.

Skein module invariant

We also consider an SU(2) toroidal colored Jones polynomial obtained by specializing to the quantum group SU(2)$_q$. We show that if $C \subset T^2$ is a contractible, simple closed curve, the level two SU(2) invariant $\hat{J}_{\boldsymbol{2},q}^T = \hat{J}_{(2,\ldots,2),q}^T$ satisfies

$$\hat{J}_{\boldsymbol{2},q}^T(C) = -q^{1/2} - q^{-1/2}.$$

If $C \subset T^2$ is a simple closed curve which is not contractible,

$$\hat{J}_{\boldsymbol{2},q}^T(C) = 2.$$

Additionally, we prove $\hat{J}_{\boldsymbol{2},q}^T$ satisfies the Kauffman bracket skein relation. These observations motivate the following definition and theorem, which characterize $\hat{J}_{\boldsymbol{2},q}^T$ skein-theoretically.

Definition 5.2 Define a Kauffman-type bracket $\langle * \rangle_\tau \in \mathbb{Z}[A^{\pm 1}, z]$ on link diagrams in T^2 (and framed links in $T^2 \times I$) by the relations:

  (a) $\langle \varnothing \rangle_\tau = 1$.

  (b) Let $C \subset T^2$ be a simple closed curve disjoint from a diagram $D \subset T^2$.

  (i) If C is contractible, $\langle C \sqcup D \rangle_\tau = (-A^2 - A^{-2})\langle D \rangle_\tau$.

  (ii) If C is not contractible, $\langle C \sqcup D \rangle_\tau = z\langle D \rangle_\tau$.

  (c) $\langle \diagdown\!\!\!\!\diagup \rangle_\tau = A\langle \,)(\, \rangle_\tau + A^{-1}\langle \asymp \rangle_\tau$.

Here A and z are indeterminates.

Theorem 5.3 *For any framed link $L \subset T^2 \times I$,*

$$\hat{J}^T_{2,q}(L) = \langle L \rangle_\tau |_{A^4 = q, z = 2}.$$

As a corollary, for any oriented, unframed link $L \subset T^2 \times I$ with diagram $D \subset T^2$,

$$J^T_2(L; q) = [-A^{-3w(D)} \langle D \rangle_\tau]|_{A^4 = q, z = 2}.$$

This gives a skein-theoretic construction of the toroidal Jones polynomial generalizing that of the usual Jones polynomial. In fact, our Theorem 5.4 and Corollary 5.6 prove much stronger statements defining $\hat{J}^T_{n,q}$ and J^T_n skein-theoretically for all n and n; to accomplish this we use the Kauffman bracket skein module of the thickened torus.

Why $z = 2$?

Relations (a), (b) and (c) in Definition 5.2 are identical to the relations defining the standard Kauffman bracket [14], with the additional stipulation in (b)(ii) that essential, simple closed curves can be removed from a diagram by multiplying by z. (A somewhat similar bracket is defined by Krushkal in [19].) To obtain Theorem 5.3, and for a geometrically motivated theory, it is necessary to fix $z = 2$. Indeed, only when $z = 2$ do we obtain an R–matrix, allowing us to do calculations as in Table 1. Proposition 3.8 below shows *any* pseudo-operator invariant takes the value 2 on essential, simple closed curves in T^2, if those curves have been colored by a 2–dimensional representation of a quantum group. In the appendix we examine this property further using rotation number and Lin and Wang's definition of the usual Jones polynomial [21] — see Proposition A.1 and the following discussion.

Comparison with J_n

For any link L in $T^2 \times I$, there exists a link $\hat{L} \subset S^3$ such that $(T^2 \times I) \setminus L$ and $S^3 \setminus \hat{L}$ are homeomorphic: $\hat{L}$ has a Hopf sublink H whose components are the cores of the tori which make up $S^3 \setminus (T^2 \times I)$; see Figure 3. We show $J^T_n(L)$ and $J_n(\hat{L})$ are fundamentally distinct invariants.

A key difference between the two is that J^T_n is unchanged by orientation-preserving homeomorphisms of the torus:

Proposition 4.5 *If the link diagram $D' \subset T^2$ is obtained from a diagram $D \subset T^2$ by an orientation-preserving homeomorphism of T^2, then for the corresponding links $L', L \subset T^2 \times I$, we have $J^T_n(L'; q) = J^T_n(L; q)$ for all n.*
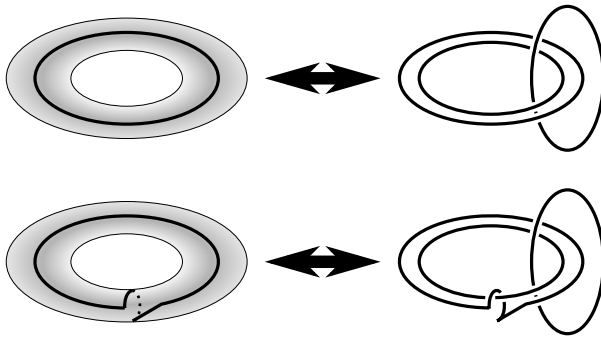
Figure 3: Nonisotopic links with homeomorphic complements in $T^2 \times I$ and S^3.

Using this proposition, we can construct infinite families of nonisotopic links in $T^2 \times I$ with identical toroidal colored Jones polynomials, whose corresponding links in S^3 all have distinct colored Jones polynomials. See Figure 3 for a simple example, where the links on the left in $T^2 \times I$ have the same toroidal colored Jones polynomials, but the corresponding links on the right in S^3 have different colored Jones polynomials.

While this makes J_n^T a less sensitive invariant than J_n, it also makes J_n^T applicable in a wider range of contexts. In Section 8, for example, we show J_n^T gives invariants of virtual links and biperiodic links. To our knowledge, J_n^T is the first invariant of virtual links to exhibit volume conjecture behavior in a nonclassical setting.

Finally, while $J_n^T(L; q)$ and $J_n(\hat{L}; q)$ are different invariants, there is an important special case when the toroidal colored Jones polynomial and usual colored Jones polynomial completely determine each other (see Figure 12):

Theorem 7.3 Let L' be a link in S^3, and consider an inclusion of L' in an embedded 2–sphere in $T^2 \times I$. Let $K \subset T^2 \times I$ be a knot projecting to an essential, simple closed curve in $T^2 \times \{0\}$, and let L be a connect sum $L = L' \# K$. Then

$$J_n^T(L; q) = n \cdot J_n(L'; q)$$

for all n.

An immediate corollary of Theorem 7.3 is that, for L and L' as in the theorem,

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})| = \lim_{n \to \infty} \frac{2\pi}{n} \log |J_n(L'; e^{2\pi i/n})|.$$

In Section 7, we use this fact to prove that a suitable generalization of our volume conjecture, Conjecture 1.2, implies the original volume conjecture, Conjecture 1.1 — see

Conjecture 7.1 and Corollary 7.5 below. It is not clear whether the reverse implication is true.

Outline

In Section 2, we review Kauffman bracket skein modules and operator invariants. In Section 3 we define a general pseudo-operator invariant Φ of framed links in $T^2 \times I$, and in Section 4 we specialize Φ to $U_q(\mathrm{sl}(2, \mathbb{C}))$ to obtain J_n^T and $\hat{J}_{n,q}^T$. In Section 5 we define these invariants skein-theoretically. In Section 6 we prove Theorem 6.2, and in Section 7 we discuss generalizations of Conjecture 1.2. In particular, we consider the case of nonhyperbolic links in $T^2 \times I$ and show that a generalization of Conjecture 1.2 implies the original volume conjecture. In Section 8 we discuss J_n^T as an invariant of biperiodic and virtual links. Finally, in the appendix, we study the behavior of J_2^T through the lens of Lin and Wang's formulation of the Jones polynomial [21].

Acknowledgements

2 Background

2.1 Kauffman bracket skein modules

For a 3–manifold M and indeterminate A, let $\mathcal{L}(M)$ be the free $\mathbb{Z}[A^{\pm 1}]$–module generated by regular isotopy classes of framed links in M. The *Kauffman bracket skein module* of M [27; 30], $\mathcal{S}(M)$, is the quotient of $\mathcal{L}(M)$ by the submodule generated by the relations

(i) $\bigcirc \sqcup L = (-A^2 - A^{-2})L$,

(ii) $\times = A\,)\,(+ A^{-1} \smile$.

The links in each expression above are identical except in a ball where they look as shown, and all diagrams are assumed to have blackboard framing. Each link $L \subset M$ is represented in $\mathcal{S}(M)$ by $\langle L \rangle$, called the *Kauffman bracket* of L. If $M = \Sigma \times I$, Σ an orientable surface, we also denote the skein module of M by $\mathcal{S}(\Sigma)$. In this case, gluing two copies of $\Sigma \times I$ together along a boundary component gives $\mathcal{S}(\Sigma)$ the structure of a $\mathbb{Z}[A^{\pm 1}]$–algebra.

As an algebra, the skein module $\mathscr{S}(\mathcal{A})$ of the thickened annulus $\mathcal{A} \times I$ is generated by a copy of its core with framing parallel to $\mathcal{A} \times \{0\}$. Sending this core to z gives an algebra isomorphism $\mathscr{S}(\mathcal{A}) \cong \mathbb{Z}[A^{\pm 1}][z]$, so that the set $\{1, z, z^2, z^3, \dots\}$ is a basis of $\mathscr{S}(\mathcal{A})$ as a $\mathbb{Z}[A^{\pm 1}]$–module. An alternate basis for $\mathscr{S}(\mathcal{A})$ is given by the Chebyshev polynomials $S_j(z)$, $j \geq 0$, defined recursively by

$$(1) \qquad S_0(z) = 1, \quad S_1(z) = z, \quad S_{j+1}(z) = z S_j - S_{j-1}.$$

If L is a link in M with k components, we can construct a multilinear map

$$(2) \qquad \langle \cdots \rangle_L : \mathscr{S}(\mathcal{A})^{\otimes k} \to \mathscr{S}(M),$$

called the *Kauffman multibracket*, as follows. For $z^{i_j} \in \mathbb{Z}[A^{\pm 1}][z] \cong \mathscr{S}(\mathcal{A})$, $i_j \geq 0$, let $L^{i_1, \dots, i_k}$ be the framed link in M obtained by cabling the jth component of L by i_j parallel copies of itself. Define

$$\langle z^{i_1}, \dots, z^{i_k} \rangle_L = \langle L^{i_1, \dots, i_k} \rangle$$

and extend $\mathbb{Z}[A^{\pm 1}]$–multilinearly to all of $\mathscr{S}(\mathcal{A})$.

Sending the empty link to 1 gives an isomorphism from $\mathscr{S}(S^3)$ to $\mathbb{Z}[A^{\pm 1}]$. Thus, for a link $L \in S^3$ with k components, the Kauffman multibracket is a map

$$\langle \cdots \rangle_L : \mathscr{S}(\mathcal{A})^{\otimes k} \to \mathbb{Z}[A^{\pm 1}].$$

Let L be an oriented, unframed link in S^3 with k components and D a diagram for L with writhe $w(D)$. The nth *colored Jones polynomial* of L, $J_n(L; q)$, is defined by

$$(3) \qquad J_n(L; q) = \left[\frac{((-1)^{n-1} A^{n^2-1})^{-w(D)}}{-A^2 - A^{-2}} \langle S_{n-1}(z), \dots, S_{n-1}(z) \rangle_D \right]\Bigg|_{A^4 = q}.$$

In [Section 5](#) we study the Kauffman bracket skein module $\mathscr{S}(T^2)$ of the thickened torus $T^2 \times I$ and its associated Kauffman multibracket. $\mathscr{S}(T^2)$ is generated as an algebra by isotopy classes of simple closed curves in T^2, which are in bijection with the set of tuples $(a, b) \in \mathbb{Z}^2$ such that either $a = b = 0$, or a and b are coprime, modulo the relation $(a, b) \sim (-a, -b)$. We think of (a, b) as the curve homotopic to a times a meridian plus b times a longitude, and write $(a, b)^m$ to indicate m parallel copies of such a curve. Additionally, to avoid ambiguity, we denote the image of a link $L \subset T^2 \times I$ in $\mathscr{S}(T^2)$ by $\langle L \rangle_T$ and use $\langle \cdots \rangle_{T, L}$ to mean the multibracket map determined by L,

$$(4) \qquad \langle \cdots \rangle_{T, L} : \mathscr{S}(\mathcal{A})^{\otimes k} \to \mathscr{S}(T^2).$$
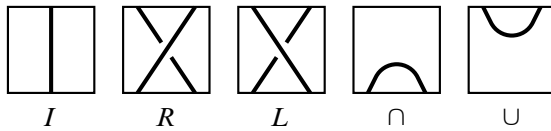
Figure 4: Elementary diagrams.

2.2 Tangle operators

An alternate definition of the colored Jones polynomial comes from the theory of *tangle operators*. The exposition here follows [17, Section 3].

Recall that a *tangle* T is a 1–manifold properly embedded (up to isotopy) in the unit cube $I^3 \subset \mathbb{R}^3$ with $\partial T \subset \{\frac{1}{2}\} \times I \times \partial I$, and define $\partial_- T = T \cap (I^2 \times \{0\})$ and $\partial_+ T = T \cap (I^2 \times \{1\})$. Choosing a regular projection onto $\{0\} \times I^2$ gives a *tangle diagram* of T.

For two tangles S and T, denote by $S \otimes T$ the tangle formed by placing S and T side by side so the boundary $I \times \{1\} \times I$ of S equals the boundary $I \times \{0\} \times I$ of T. Similarly, by $S \circ T$ we mean the tangle formed by stacking S and T vertically so $\partial_+ T = \partial_- S$; this operation can be performed only if $|\partial_+ T| = |\partial_- S|$. With these operations, the set of all tangle diagrams is generated by the five *elementary diagrams* I, R, L, $\cap$ (called a *cap*), and $\cup$ (called a *cup*) shown in Figure 4. Below, we assume tangles are equipped with orientations and framings.

Fix a quasitriangular Hopf algebra $(\mathscr{A}, \check{R})$ with R–matrix $\check{R} = \sum \alpha_i \otimes \beta_i \in \mathscr{A} \otimes \mathscr{A}$ and define a V–*coloring* of a tangle T (or one of its diagrams) to be an assignment of an $\mathscr{A}$–module to each component of T. This induces a coloring of ∂T as follows: If C is a component of color V, we assign V to each endpoint of C where C is oriented downward and the dual module V^* to each endpoint where C is oriented upward. Tensoring from left to right gives *boundary $\mathscr{A}$–modules* $T_\pm$ assigned to $\partial_\pm T$ with the empty tensor product defined to be $\mathbb{C}$.

Suppose $\mathscr{A}$ contains a unit μ with the properties

(i) $\mu \alpha \mu^{-1} = S^2(\alpha)$ for all $\alpha \in \mathscr{A}$, where S is the antipode of $\mathscr{A}$,

(ii) $\sum \alpha_i \mu^{-1} \beta_i = \sum \beta_i \mu \alpha_i$.

We call such a unit a *good unit* of $\mathscr{A}$. In this case, by the following fundamental result, any tangle T gives an $\mathscr{A}$–linear map $T_- \to T_+$.

Theorem 2.1 [17; 28] *There exist unique $\mathscr{A}$–linear operators*

$$\mathcal{F}_T = \mathcal{F}_T^{\mathscr{A},\check{R},\mu} : T_- \to T_+$$

assigned to each colored framed tangle T which satisfy

$$\mathcal{F}_{T \circ T'} = \mathcal{F}_T \circ \mathcal{F}_{T'}, \quad \mathcal{F}_{T \otimes T'} = \mathcal{F}_T \otimes \mathcal{F}_{T'},$$

and for the tangles given by the elementary diagrams with blackboard framing,

$$\mathcal{F}_I = \mathrm{Id},$$
$$\mathcal{F}_R = R \quad and \quad \mathcal{F}_L = R^{-1},$$
$$\mathcal{F}_{\curvearrowright} = E \quad and \quad \mathcal{F}_{\curvearrowleft} = E_\mu,$$
$$\mathcal{F}_{\cup} = N \quad and \quad \mathcal{F}_{\cup} = N_{\mu^{-1}},$$

where $R = \tau \circ \check{R}$, τ the transposition map $\alpha \otimes \beta \mapsto \beta \otimes \alpha$. Additionally $E(f \otimes x) = f(x)$, $E_\mu(x \otimes f) = f(\mu x)$, $N(1) = \sum e_i \otimes e^i$ and $N_{\mu^{-1}}(1) = \sum e^i \otimes (\mu^{-1} e_i)$ for any basis e_i.

The map R is also called an R–matrix, and the map $\mathcal{F}_T$ is called the *operator invariant* of T.

Remark 2.2 The quasitriangular Hopf algebra in this construction can be replaced more generally with a *ribbon category* [32].

We set $\mathscr{A} = \mathscr{A}_q = \mathcal{U}_q(\mathrm{sl}(2, \mathbb{C}))$, the quantized universal enveloping algebra of $\mathrm{sl}(2, \mathbb{C})$ specialized to $q = e^{2\pi i/r}$ (see [13]), and fix a certain good unit μ. We also limit tangle colorings to a distinguished set of $\mathscr{A}$–modules $\{V^1, \dots, V^r\}$, V^n coming from the unique n–dimensional irreducible representation of $\mathrm{sl}(2, \mathbb{C})$ [28]. If $L \subset I^2$ is a k–component link and $\boldsymbol{n} = (n_1, \dots, n_k)$ a multi-integer, let $L(\boldsymbol{n})$ be the $\mathscr{A}$–colored link L with j^{th} component colored V^{n_j}. With this setup, the colored Jones polynomial is defined to be

$$(5) \quad J_n(L; q) = (q^{(n^2-1)/4})^{-w(D)} \frac{\{1\}}{\{n\}} \mathcal{F}_{L(n,\dots,n)}^{\mathscr{A}_q} = (q^{(n^2-1)/4})^{-w(D)} \frac{1}{[n]} \mathcal{F}_{L(n,\dots,n)}^{\mathscr{A}_q},$$

where $q = e^{2\pi i/r}$, $r \in \mathbb{N}$, and $w(D)$ is the writhe of the tangle diagram of L. The terms $\{m\}$ and $[m]$ are the quantum integers defined by

$$\{m\} = \{m\}_q = q^{m/2} - q^{-m/2}, \quad [m] = [m]_q = \frac{\{m\}}{\{1\}}.$$

The boundary $\mathscr{A}$–modules of any colored link $L(\boldsymbol{n})$ are both $\mathbb{C}$, so $\mathcal{F}_{L(\boldsymbol{n})}^{\mathscr{A}_q}$ is a linear map from $\mathbb{C}$ to $\mathbb{C}$ — a scalar. This scalar is a Laurent polynomial in q.

In fact, the invariant J_n as defined in (5) is not strictly equal to J_n as defined in (3) — for example, the two definitions differ by a sign on the two-component unlink. Achieving precise equality requires a normalization of (5) equivalent to specializing $\mathcal{F}$ to the quantum group $\mathrm{SU}(2)_q$ rather than $U_q(\mathrm{sl}(2, \mathbb{C}))$ [17; 22]. For this reason, we refer to the invariant (3) as the $\mathrm{SU}(2)$ *colored Jones polynomial*.

In the following section we generalize the theory of tangle operators to links in $T^2 \times I$, leading in Section 4 to the definition of the toroidal colored Jones polynomial.

3 Pseudo-operator invariants

For a link L in the thickened torus $T^2 \times I$, we take a regular projection to $T^2 \times \{0\}$ to obtain a link diagram $D \subset T^2$. Let $\pi : \mathbb{R}^2 \to T^2$ be a smooth, orientation-preserving covering map with fundamental domain the unit square $I^2 \subset \mathbb{R}^2$ and deck transformations generated by horizontal and vertical unit shifts of $\mathbb{R}^2$. Let $\widetilde{D} = \pi^{-1}(D)$. Define $p \in D$ to be a *local extremum* of D, so that a small neighborhood of p is a cap or cup, if p has a lift $\tilde{p}$ which is a local extremum of $\widetilde{D}$ with respect to the height function $(x, y) \mapsto y$ on $\mathbb{R}^2$.

Definition 3.1 A point $p \in D$ is a *critical point* of D if it is a local extremum or crossing point. A *torus diagram* $D \subset T^2$ is a regular projection of a smooth link $L \subset T^2 \times I$ onto $T^2 \times \{0\}$ such that critical points are isolated.

Below, all diagrams in T^2 are assumed to be torus diagrams.

Fix a quasitriangular Hopf algebra $(\mathcal{A}, R)$ and good unit $\mu \in \mathcal{A}$. As in Section 2.2, a *V–coloring* (or simply *coloring*) of D is an assignment of an $\mathcal{A}$–module to each link component.

We now define an invariant $\Phi_\pi = \Phi_\pi^{\mathcal{A}, R, \mu}$ of oriented V–colored link diagrams in T^2 with framing parallel to $T^2 \times \{0\}$. Let $D \subset T^2$ be such a diagram and P the set of critical points of D. For each $p \in P$, there exists a small rectangular neighborhood $T(p)$ of p and a local section ψ of π, $\psi : T(p) \to \mathbb{R}^2$, giving $T(p)$ the structure of an oriented, blackboard-framed, elementary tangle diagram. In this way Theorem 2.1 assigns an $\mathcal{A}$–linear operator $\mathcal{F}_{T(p)}$ to each $T(p)$, for $p \in P$, with boundary $\mathcal{A}$–modules $T(p)_\pm$. We cannot generally extend these local assignments to a global assignment of an $\mathcal{A}$–linear operator to D, as Theorem 2.1 does for tangle diagrams in I^2. However, the local assignments of operators to each critical point still allow us to give D a value in $\mathbb{C}$ using the *state sum* formulation of the theory, as explained below.
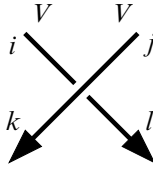
Figure 5: A V–colored tangle $T(p)$ near a crossing point p, labeled by a state σ.

For each $\mathcal{A}$–module V, fix a basis B_V of V as a $\mathbb{C}$–vector space. Removing the set of critical points P from D breaks it into components, each colored by some V and each oriented upward or downward when lifted to $\mathbb{R}^2$. A *state* σ is an assignment of a label $\sigma(S)$ to each component S of $D \setminus P$ as follows: If S is colored by the module V and oriented downward, $\sigma(S)$ is an element of B_V. If S is oriented upward, $\sigma(S)$ is an element of the dual basis B_{V^*}.

A state σ determines a *weight* $\omega_p(\sigma)$ of each critical point. For each $p \in P$, taking tensor products of the labels $\sigma(S)$ of the strands above and below p gives basis elements $\sigma(p)_\pm$ of the modules $T(p)_\pm$. Define the weight $\omega_p(\sigma) \in \mathbb{C}$ to be the coefficient of $\sigma(p)_+$ in $\mathcal{F}_{T(p)}(\sigma(p)_-)$, and define the weight of the state σ by

$$(6) \qquad \omega(\sigma) = \prod_{p \in P} \omega_p(\sigma),$$

where the empty product (if D contains no critical points) is defined to be 1. Finally, set

$$(7) \qquad \Phi_\pi(D) = \sum_\sigma \omega(\sigma),$$

where the sum is over all states of D.

For an example computation of the weight of a critical point, let p be the crossing point with neighborhood $T(p)$ shown in Figure 5. Viewing $T(p)$ as a tangle diagram, both tangle components of $T(p)$ are colored by the same $\mathcal{A}$–module V, so

$$T(p)_- = T(p)_+ = V \otimes V.$$

Additionally, since $T(p)$ is a positive crossing, $\mathcal{F}_{T(p)} = R$, viewed as a map from $V \otimes V$ to itself. In the given state σ, the diagram components of $T(p) \setminus p$ are assigned basis elements $e_i, e_j, e_k, e_l \in V$ as shown, where $\{e_0, e_1, \ldots, e_{n-1}\}$ is a basis for V. We have $\sigma(p)_- = e_k \otimes e_l$, $\sigma(p)_+ = e_i \otimes e_j$, and if R satisfies

$$R(e_k \otimes e_l) = R_{kl}^{0,0}(e_0 \otimes e_0) + \cdots + R_{kl}^{ij}(e_i \otimes e_j) + \cdots + R_{kl}^{n-1,n-1}(e_{n-1} \otimes e_{n-1}),$$

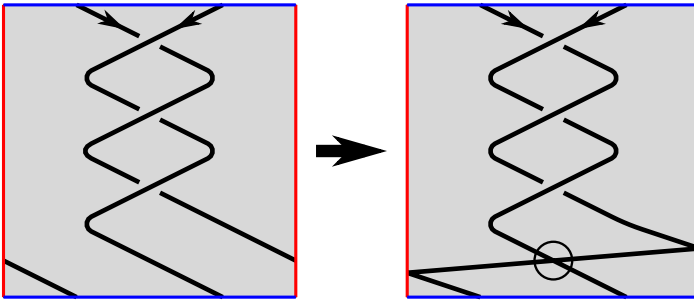where the $R_{kl}^{st} \in \mathbb{C}$ are scalars, $0 \leq s, t \leq n-1$, then $\omega_p(\sigma) = R_{kl}^{ij}$.

Figure 6: Constructing a virtual tangle diagram from a torus diagram.

Lemma 3.2 *The value $\Phi_\pi(D)$ does not depend on the choice of bases of the colors V.*

Proof To prove the lemma we give an alternate construction of Φ_π. Recall I^2 is a fundamental domain for the covering map $\pi : \mathbb{R}^2 \to T^2$; adjusting D if necessary, we assume no critical points of $\widetilde{D} = \pi^{-1}(D)$ occur in ∂I^2. Shift $\widetilde{D} \cap (\{1\} \times I)$, the points of $\widetilde{D}$ intersecting the right side of I^2, slightly upward by an isotopy of I^2 which is the identity on $\{0\} \times I$ and does not change the set of critical points of $\widetilde{D}$. Because the left and right sides of I^2 are identified in the torus, each point $q \in \widetilde{D} \cap (\{0\} \times I)$ has a corresponding point $q' \in \widetilde{D} \cap (\{1\} \times I)$, slightly higher than q as a result of the isotopy. As a final step of the construction, connect each q and q' by a curve $c_q : I \to I^2$ which satisfies $c_q(0) = q$, $c_q(1) = q'$, and is monotonically increasing in height. This produces a *virtual tangle diagram* $D' = (I^2 \cap \widetilde{D}) \cup_{q \in D \cap (\{0\} \times I)} c_q(I)$ whose classical (ie nonvirtual) critical points are the same as the critical points of D and whose virtual crossings are any point where some $c_q(I)$, for $q \in \widetilde{D} \cap (\{0\} \times I)$, intersects another point of D'. See Figure 6 for an example, where the virtual crossing on the right is circled. We assume virtual crossings are isolated from other critical points.

The coloring of D induces a coloring of D' in an obvious way. As before, let P be the set of (classical and virtual) critical points of D' with $T(p)$ a small rectangular neighborhood of $p \in P$. If $p \in P$ is a classical critical point, the functor $\mathcal{F}$ of Theorem 2.1 associates an $\mathscr{A}$–linear operator $\mathcal{F}_{T(p)}$ to $T(p)$ which agrees with the operator assigned to $T(p)$ in the construction of Φ_π. If $p \in P$ is a virtual crossing, define $\mathcal{F}_{T(p)}$ to be the transposition map $\tau : \alpha \otimes \beta \mapsto \beta \otimes \alpha$ as in [15]. (This is a $\mathbb{C}$–linear map but not generally an $\mathscr{A}$–linear one.) Because $\partial_- D'$ and $\partial_+ D'$ are identified in the torus, $D'_- = D'_+ = V$ for some $\mathscr{A}$–module V. Thus, extending the local operator assignments $\mathcal{F}_{T(p)}$, for $p \in P$, as in Theorem 2.1 associates D' with a $\mathbb{C}$–linear map $\phi : V \to V$. Define $\Phi'_\pi(D') = \mathrm{Tr}(\phi)$; we claim $\Phi'_\pi(D') = \Phi_\pi(D)$.
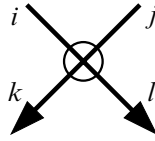
Figure 7: A state assignment near a virtual crossing.

Computing $\Phi'_\pi(D')$ as a state sum, as in [15; 17; 25], shows the two invariants agree. Fix a basis for each $\mathscr{A}$–module. As in the construction of Φ_π, a state is an assignment of basis elements to components of $D' \setminus P$ and the weight of a state is the product of the weights of the critical points. Taking the trace of ϕ ensures identified strands of D'_- and D'_+ are assigned the same basis element in any state with nonzero weight. If the strands near a virtual crossing p are assigned basis elements e_i, e_j, e_k and e_l, as in Figure 7, the weight of p is $\delta^l_i \delta^k_j$, where δ is the Kronecker delta. This ensures the identified strands on either side of p have the same state, in which case p has weight 1.

If we compute $\Phi_\pi(D)$ using the same bases, we have

$$\Phi_\pi(D) = \sum_\sigma \omega(\sigma) = \mathrm{Tr}(\phi) = \Phi'_\pi(D').$$

This shows the definition of $\Phi'_\pi(D')$ does not depend on the choice of curves c_q, for $q \in D \cap (\{0\} \times I)$. Since Φ'_π does not depend on a choice of basis, neither does Φ_π. □

We write Φ rather than Φ_π in the next definition because we will ultimately show Φ does not depend on the choice of covering map π. Before proving this, however, we give the main result of the section.

Definition 3.3 Let $L \subset T^2 \times I$ be a framed, oriented, V–colored link and D a diagram for L with framing parallel to T^2. Define the *pseudo-operator invariant* of L, depending on $(\mathscr{A}, R)$ and μ, by

$$\Phi(L) = \Phi^{\mathscr{A},R,\mu}(L) = \Phi^{\mathscr{A},R,\mu}_\pi(D).$$

Theorem 3.4 Φ *is an invariant of framed, oriented, V–colored links in $T^2 \times I$. That is, if D_1 and D_2 are two diagrams of a framed, oriented, V–colored link $L \subset T^2 \times I$ with each having framing parallel to $T^2 \times \{0\}$, then $\Phi(D_1) = \Phi(D_2)$.*

Proof Consider the lift $\tilde{D}_i$ of D_i to $\mathbb{R}^2$ for $i = 1, 2$. By construction, $\tilde{D}_i$ is the diagram of a biperiodic link $\tilde{L} \subset \mathbb{R}^2 \times I$ such that the critical points of $\tilde{D}_i$ are lifts of critical points of D_i. Let $f_t(x) \colon I \times (T^2 \times I) \to T^2 \times I$ be an ambient isotopy carrying D_1 to D_2, such that $f_0 \equiv \mathrm{Id}$ and $f_1(D_1) = D_2$. Then f_t lifts to a biperiodic isotopy $\tilde{f}_t$ of
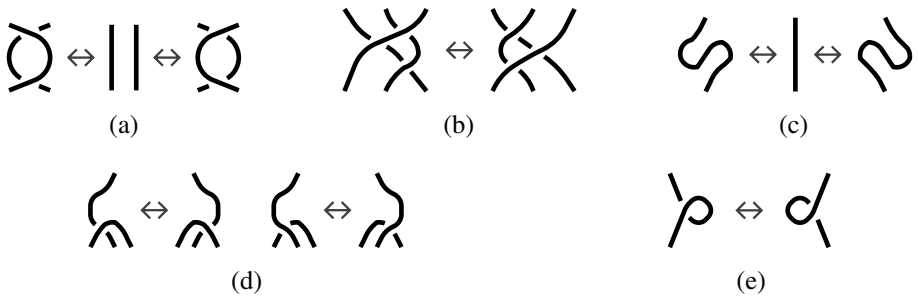
Figure 8: Local tangle moves.

$\mathbb{R}^2 \times I$ taking $\tilde{D}_1$ to $\tilde{D}_2$. Because $\tilde{D}_1$ and $\tilde{D}_2$ are locally blackboard-framed tangle diagrams, a well-known theorem [8; 28] asserts that $\tilde{f}_t$ decomposes into a sequence $\tilde{g}_t$ of diagram-preserving isotopies and the moves shown in Figure 8 (with all possible orientations). We assume the isotopies and moves are biperiodic, ie applied to each lifted copy of a region of D_1 simultaneously.

Because $\tilde{g}_t$ is biperiodic, it descends to a sequence g_t of the same moves on T^2 carrying D_1 to D_2. Hence it suffices to check invariance of Φ under each local move, which follows from properties of $\mathscr{A}$, R and μ. For example, the equation $R^{-1} \circ R = \mathrm{Id} = R \circ R^{-1}$ implies invariance under move (a). Move (b) follows from the fact that R satisfies the Yang–Baxter equation [31], and moves (c)–(e) also follow from properties of R and μ — see [17, Theorem 3.6] for details. □

Remark 3.5 The construction of Φ_π given in the proof of Lemma 3.2 is similar to Kauffman's quantum invariant for virtual links [15], in that virtual crossings are associated with the transposition map τ. However, the two invariants have significant differences. We can think of the virtual diagram D' in the proof of Lemma 3.2 as the diagram of a tangle on a cylinder $S^1 \times I$: the original diagram D sits on the "front" of the cylinder, while the added curves c_q circle around the "back". This is one difference between our invariant and Kauffman's — the use of a cylinder to create the virtual diagram rather than a torus. Another difference is that Kauffman's invariant is defined in the context of *rotational virtual knot theory* (see [16]) — it is not invariant under virtual Reidemeister I-moves. We achieve invariance under virtual I–moves by placing all classical critical points on the front of the cylinder, where the orientation of the cylinder matches the orientation of the virtual diagram. If a critical point were moved to the back of the cylinder, that point's orientation on the cylinder would not match its orientation in the virtual diagram, and the two local operator assignments in the two constructions of Φ_π would disagree.

We use the phrase "pseudo-operator invariant" because, as remarked above, a torus diagram $D \subset T^2$ cannot generally be associated with an $\mathscr{A}$–linear operator using our construction. It is interesting that the local assignments of $\mathscr{A}$–linear operators to critical points of D still allow us to define $\Phi(D)$, which seems to encode geometric information about the link L. For an example of computing Φ with a specific $\mathscr{A}$, see Section 6.

The fact that $\Phi(D) = \Phi_\pi(D)$ does not depend on π follows from the proposition below, which shows Φ_π is invariant under orientation-preserving homeomorphisms of T^2.

Proposition 3.6 *Let $D, D' \subset T^2$ be oriented, V–colored link diagrams with blackboard framing. If f is an orientation-preserving homeomorphism of T^2 satisfying $f(D) = D'$, then $\Phi_\pi(D) = \Phi_\pi(D')$.*

Proof Since Φ_π is an isotopy invariant, it suffices to prove the theorem for a set of homeomorphisms generating the mapping class group $\mathrm{Mod}(T^2)$. To this end, we consider two Dehn twists, about two curves in T^2 which lift via π to horizontal and vertical lines in $\mathbb{R}^2$. Let $l \subset T^2$ be a simple closed curve lifting to a vertical line in $\mathbb{R}^2$ such that l contains no critical points of D, and choose a bicollar neighborhood $N(l)$ of l satisfying that

(i)   no critical points of D occur within $N(l)$,

(ii)   each connected component of $D \cap N(l)$ intersects l only once, transversely.

Now suppose $f : T^2 \to T^2$ is an upward twist (from left to right) about l which is the identity outside of $N(l)$. See Figure 9 for an example. Let c be a component of $D \cap N(l)$ — then c is a curve which increases or decreases monotonically as it travels across $N(l)$ from left to right. If c is increasing, $f(c)$ is also monotonically increasing and contains no critical points. If c is decreasing, $f(c)$ contains a minimum to the left of l and a maximum to the right of l and no critical points other than these; see Figure 9. The cases of twisting downward and twisting about horizontal lines are similar. Finally, because f is injective, the crossing points of D' are the same as those of D.

It follows that the only critical points of $f(D) = D'$ which do not occur in D are max–min pairs formed as above. When $\Phi_\pi(D')$ is computed as a state sum, the weights of these max–min pairs cancel as in identity (c) of Figure 8. We conclude $\Phi_\pi(D) = \Phi_\pi(D')$. □

Corollary 3.7 *The value $\Phi_\pi(D)$ does not depend on the choice of covering map $\pi : \mathbb{R}^2 \to T^2$.*
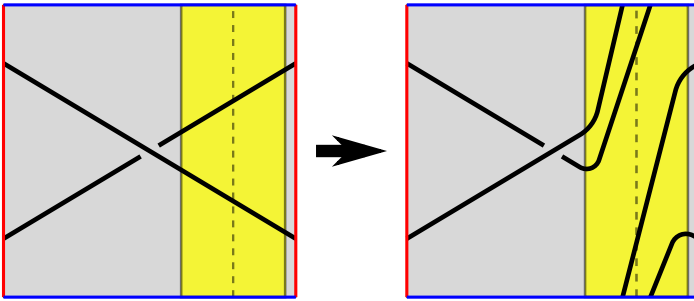
Figure 9: Twisting a diagram.

Proof Let $\pi_1, \pi_2 \colon \mathbb{R}^2 \to T^2$ be two smooth, orientation-preserving covering maps with fundamental domain I^2. The uniqueness property of covering spaces gives an orientation-preserving homeomorphism $\tilde{f} \colon \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies $\pi_2 \circ \tilde{f} = \pi_1$, and since π_1 and π_2 have the same fundamental domain and deck transformations, $\tilde{f}$ descends to an orientation-preserving homeomorphism f of T^2 satisfying $f \circ \pi_1 = \pi_2$. By Proposition 3.6, and noting the value $\Phi_{\pi_i}(D)$ is completely determined by the lift $\pi_i^{-1}(D)$,

$$\Phi_{\pi_1}(D) = \Phi_{\pi_1}(f(D)) = \Phi_{f \circ \pi_1}(D) = \Phi_{\pi_2}(D). \qquad \square$$

We conclude the section with a general property of pseudo-operator invariants. Though the proposition is a simple observation, it motivates the skein theory to come in Section 5.

Proposition 3.8 *Let $K \subset T^2 \times I$ be a knot projecting to an essential, simple closed curve in $T^2 \times I$ with framing parallel to $T^2 \times \{0\}$. If K is colored by an n–dimensional $\mathscr{A}$–module V,*

$$\Phi(K) = n.$$

Proof Applying Proposition 3.6, we may assume without loss of generality that K lifts to a vertical line in $\mathbb{R}^2$. Then the proof of Lemma 3.2 shows $\Phi(K)$ is the trace of the identity map of V, so $\Phi(K) = \dim(V) = n$. $\qquad \square$

4 Quantum invariants for sl(2, $\mathbb{C}$) and the toroidal colored Jones polynomial

4.1 An invariant of framed, unoriented links in $T^2 \times I$

As with the colored Jones polynomial, we now specialize to $\mathscr{A} = \mathscr{A}_q = \mathcal{U}_q(\mathrm{sl}(2, \mathbb{C}))$, for $q = e^{2\pi i/r}$, and limit $\mathscr{A}$–modules to the set $\{V^1, \ldots, V^r\}$ as in Section 2.2.

For a link $L \subset T^2 \times I$ with k components and multi-integer $\boldsymbol{n} = (n_1, \ldots, n_k)$, let $L(\boldsymbol{n}) = L(n_1, \ldots, n_k)$ be the V–colored link L with j^{th} component colored by V^{n_j}. By $1 \leq \boldsymbol{n} \leq r$ in the definition below we mean that $1 \leq n_j \leq r$ for all n_j.

Definition 4.1 Given a framed, unoriented link $L \subset T^2 \times I$ with k components, fix an orientation of each component. For $q = e^{2\pi i/r}$, $1 \leq \boldsymbol{n} \leq r$, define $\hat{J}^T_{\boldsymbol{n},q}(L)$ by

$$\hat{J}^T_{\boldsymbol{n},q}(L) = \Phi^{\mathscr{A}_q, R, \mu}(L(\boldsymbol{n})).$$

Theorem 4.2 $\hat{J}^T_{\boldsymbol{n},q}(L)$ *is an invariant of framed, unoriented links in* $T^2 \times I$. *That is,* $\hat{J}^T_{\boldsymbol{n},q}(L)$ *does not depend on the orientation chosen for each component of* L.

Proof Let $D, D' \subset T^2$ be oriented diagrams of $L(\boldsymbol{n})$ with framing parallel to $T^2 \times \{0\}$ and D' obtained from D by changing the orientation of a link component C. It suffices to show

$$\Phi^{\mathscr{A}_q, R, \mu}(D) = \Phi^{\mathscr{A}_q, R, \mu}(D').$$

Suppose C is colored by V^n and let C' be the corresponding component of D', also colored by V^n. Let $p \in C$ be a critical point, p' the same point of C', and $T(p)$, $T(p')$ small rectangular neighborhoods of each. Then each copy of V^n coming from C in $T(p)_\pm$ corresponds to a copy of $(V^n)^*$ in $T(p')_\pm$ and vice versa. V^n is self-dual as an $\mathscr{A}$–module via a canonical isomorphism $\varphi \colon (V^n)^* \to V^n$, and we use φ to identify the modules $T(p)_\pm$ and $T(p')_\pm$.

We apply Lemma 3.18 and Remark 3.26 of [17], which state that if n is odd, $\mathcal{F}_{T(p)} = \mathcal{F}_{T(p')}$ as maps from $T(p)_-$ to $T(p)_+$ for all p. Thus $\Phi(D) = \Phi(D')$ if n is odd. Suppose n is even. If p is a crossing then $\mathcal{F}_{T(p)} = \mathcal{F}_{T(p')}$, and if p is an extreme point then $\mathcal{F}_{T(p)} = -\mathcal{F}_{T(p')}$. Self-duality of V^n induces a bijection between the states of D and the states of D', and it follows that if σ is a state of D and σ' the corresponding state of D', $\omega(\sigma) = (-1)^j \omega(\sigma')$, where j is the total number of extreme points of C. Since $C \subset T^2$ is a closed curve, j is even and $\omega(\sigma) = \omega(\sigma')$. We conclude $\Phi(D) = \Phi(D')$ if n is even. $\square$

The invariant $\hat{J}^T_{\boldsymbol{n},q}$ should be thought of as a toroidal analogue of the invariant $J_{L,\boldsymbol{k}}$ of [17]. One might also be reminded of the Kauffman bracket skein module, another invariant of framed, unoriented links — this comparison will be made precise in the next section. Like the invariant $J_{L,\boldsymbol{k}}$ of [17] or the Kauffman bracket skein module of S^3, $\hat{J}^T_{\boldsymbol{n},q}$ can be normalized to obtain an invariant of oriented, unframed links in

$T^2 \times I$ analogous to the colored Jones polynomial.

4.2 The toroidal colored Jones polynomial

To create an invariant of unframed links, we use the fact that as an endomorphism of $V^n \otimes V^n$, $0 \le n \le r$, the R–matrix R satisfies [17; 30]

$$(\mathrm{Id} \otimes E_\mu)(R^{\pm 1} \otimes \mathrm{Id})(\mathrm{Id} \otimes N) = q^{\pm(n^2-1)/4}\,\mathrm{Id},$$

where $N\colon \mathbb{C} \to V^n \otimes (V^n)^*$ and $E_\mu\colon V^n \otimes (V^n)^* \to \mathbb{C}$ are the maps given in Theorem 2.1. Pictorially, this is equivalent to the two equations

(8) $\qquad \hat{J}^T_{\boldsymbol{n},q}(\bovpunk) = q^{(n^2-1)/4} \cdot \hat{J}^T_{\boldsymbol{n},q}(\big|), \quad \hat{J}^T_{\boldsymbol{n},q}(\bovpunk) = q^{-(n^2-1)/4} \cdot \hat{J}^T_{\boldsymbol{n},q}(\big|),$

where the diagrams represent oriented, unframed links and the component shown is colored by V^n.

If $L \subset T^2 \times I$ is oriented and unframed, any two diagrams of L are related by a sequence of the moves in Figure 8 and additions or removals of curls,

(9) $$\bovcurl \leftrightarrow | \leftrightarrow \bovcurl.$$

This fact, combined with (8), motivates Definition 4.3. Similar to above, given a diagram $D \subset T^2$ of a k–component link and multi-integer $\boldsymbol{n} = (n_1, \ldots, n_k)$, let $D(\boldsymbol{n})$ indicate D with j^{th} component colored by V^{n_j}. Define $\hat{J}^T_{\boldsymbol{n},q}(D) := \Phi^{\mathscr{A}_q, R, \mu}(D(\boldsymbol{n}))$.

Definition 4.3 Let $L \subset T^2 \times I$ be an oriented, unframed link with k components, $D \subset T^2$ a diagram of L, and $C_1, \ldots, C_k$ the link components of D. Define $J^T_{\boldsymbol{n},q}(L)$ by

$$J^T_{\boldsymbol{n},q}(L) = q^{\alpha/4}\,\hat{J}^T_{\boldsymbol{n},q}(D),$$

where $\alpha = -\sum_{j=1}^k w(C_j) \cdot (n_j{}^2 - 1)$ and $w(C_j)$ is the writhe of C_j, ie the sum of the signs of its self-crossings.

It follows from (8) and the proof of Theorem 3.4 that $J^T_{\boldsymbol{n},q}(L)$ is an invariant of oriented, unframed links in $T^2 \times I$.

If all components of L are given the same color, ie if $\boldsymbol{n} = (n, \ldots, n)$ for some $n \in \mathbb{N}$, we can define a similar invariant which agrees with $J^T_{\boldsymbol{n},q}$ if L is a knot. This next definition is our analogue of the colored Jones polynomial.

Definition 4.4 For an oriented, unframed link $L \subset T^2 \times I$ with diagram D and $n \in \mathbb{N}$, define the nth *toroidal colored Jones polynomial* $J_n^T(L; q)$ of L by

$$J_n^T(L; q) = (q^{(n^2-1)/4})^{-w(D)} \hat{J}_{(n,\dots,n),q}^T(D),$$

where $w(D)$ is the writhe of D.

Compare Definition 4.4 with (5) — the definitions are analogous except for a factor of $1/[n]$. This factor may be included in the definition of J_n because J_n is always divisible by $[n]$ as a Laurent polynomial in q; this follows from the $\mathcal{A}$–linearity of the operator in Theorem 2.1. Because there is no guarantee of global $\mathcal{A}$–linearity in the construction of J_n^T, we cannot divide by $[n]$. In particular, if $q = e^{2\pi i/n}$,

$$(10) \qquad\qquad\qquad [n] = \frac{e^{\pi i} - e^{-\pi i}}{\{1\}} = 0.$$

Additionally, the root of unity q may be replaced by an indeterminate in the definition of $J_n^T(L; q)$ without affecting calculations — see, for example, [25]. This justifies thinking of $J_n^T(L; q)$ as a Laurent polynomial in q and accounts for our slight change in notation. We prefer J_n^T to $J_{n,q}^T$ for simplicity in calculations, and don't make use of $J_{n,q}^T$ outside of this section. We also remark that, because Dehn twists do not affect the signs of crossings in a diagram, Proposition 3.6 extends to J_n^T and $J_{n,q}^T$:

Proposition 4.5 Let $L, L' \subset T^2 \times I$ be oriented links with respective diagrams $D, D' \subset T^2$. If f is an orientation-preserving homeomorphism of T^2 satisfying $f(D) = D'$, then $J_n^T(L; q) = J_n^T(L'; q)$ and $J_{n,q}^T(L) = J_{n,q}^T(L')$ for all n and $\boldsymbol{n}$.

As a final result of the subsection, we give a cabling formula for $\hat{J}_{\boldsymbol{n},q}^T$ analogous to the cabling formula for the invariant $J_{L,k}$; see [17, Theorem 4.15]. For a framed, unoriented link $L \subset T^2 \times I$, this expresses the value $\hat{J}_{\boldsymbol{n},q}^T(L)$ in terms of $\hat{J}_{\boldsymbol{2},q}^T = \hat{J}_{(2,\dots,2),q}^T$ evaluated on certain cablings of L. This will allow us to develop $\hat{J}_{\boldsymbol{n},q}^T$ from a skein-theoretic viewpoint in the next section.

Let L be a k–component link and $\boldsymbol{n} = (n_1, \dots, n_k)$ a multi-integer. As in Section 2.1, denote by $L^{\boldsymbol{n}}$ the cabling of L which replaces the jth component of L by n_j parallel pushoffs of itself, oriented compatibly if the link is oriented, with associated diagram $D^{\boldsymbol{n}}$. Below, the sum is over all $\boldsymbol{i} = (i_1, \dots, i_k)$ with $1 \leq i_j \leq \frac{1}{2}(n_j - 1)$, and

$$(-1)^{\boldsymbol{i}} \binom{\boldsymbol{n}-1-\boldsymbol{i}}{\boldsymbol{i}} = \prod_{j=1}^k (-1)^{i_j} \binom{n_j - 1 - i_j}{i_j}.$$

Theorem 4.6 (cabling formula) *Let* $L \subset (T^2 \times I)$ *be a framed, unoriented link,* $\boldsymbol{n}$ *a coloring of* L, *and* D *a torus diagram for* L *with framing parallel to* $T^2 \times \{0\}$. *Then*

$$\hat{J}^T_{\boldsymbol{n},q}(L) = \sum_{i=0}^{(\boldsymbol{n}-1)/2} (-1)^i \binom{\boldsymbol{n}-1-i}{i} \hat{J}^T_{\boldsymbol{2},q}(L^{\boldsymbol{n}-1-2i})$$

$$= \sum_{i=0}^{(\boldsymbol{n}-1)/2} (-1)^i \binom{\boldsymbol{n}-1-i}{i} q^{3w(D^{\boldsymbol{n}-1-2i})/4} J^T_2(L^{\boldsymbol{n}-1-2i}; q).$$

We sketch the proof, following closely the proof of Theorem 4.15 in [17]. We first require a lemma — cf [17, Lemma 3.10] — which gives useful properties of pseudo-operator invariants.

Lemma 4.7 *Let* $(\mathscr{A}, R)$ *be a quasitriangular Hopf algebra with good unit* μ. *Let* $D \subset T^2 \times I$ *be a colored torus diagram and* C *a link component of* D *colored by* V.

(a) *If* $V = X \oplus Y$, *or more generally* V *is an extension of* Y *by* X *(ie there is a short exact sequence* $0 \to X \to V \to Y \to 0$ *of* $\mathscr{A}$–*modules), then*

$$\Phi(D) = \Phi(D_X) + \Phi(D_Y),$$

where D_Z *denotes the torus diagram obtained by changing the color of* C *to* Z.

(b) *If* $V = X \otimes Y$, *then*
$$\Phi(D) = \Phi(D_{XY}),$$

where D_{XY} *is the diagram obtained by replacing* C *by two parallel pushoffs of itself (using the framing) colored by* X *and* Y, *respectively.*

Proof To prove (a), fix bases B_V, B_X and B_Y so that $B_X \subset B_V$ (viewing X as a subspace of V) and B_Y is the projection of $\tilde{B}_Y = B_V - B_X$. We call state labels from B_X or B_{X^*} X–*labels*, whereas those from $\tilde{B}_Y$ or $\tilde{B}_{Y^*}$ are Y–*labels*.

If σ is a state of D with nonzero weight, then the corresponding labels on the arcs of C must be either all X–labels (written $\sigma|C \subset X$) or all Y–labels (written $\sigma|C \subset Y$). This follows from the $\mathscr{A}$–invariance of $X \subset V$ (and dually of $Y^* \subset V^*$) — if the component of C on one side of a critical point has an X–label and the component on the other side has a Y–label, the weight of the critical point in that state will be zero. From this, we see

$$\Phi(D) = \sum \omega(\sigma) = \sum_{\sigma|C \subset X} \omega(\sigma) + \sum_{\sigma|C \subset Y} \omega(\sigma) = \Phi(D_X) + \Phi(D_Y),$$

as desired.

Statement (b) is a fundamental property of operator invariants — see [32, Chapter 1] or [17, Lemma 3.10]. As with statement (a), we extend to the case of pseudo-operator invariants by considering each critical point individually. □

Proof of Theorem 4.6 It is a classical result — see [17, Corollary 2.15] — that, for $0 \le n < r$, the equality

$$(11) \qquad V^{n+1} = \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} (V^2)^{n-2j}$$

holds in the representation ring of $\mathscr{A}_q$, where $q = e^{2\pi i/r}$ and the sum is over all j with $0 \le 2j \le n$. Here V^n refers to the $\mathscr{A}_q$–module V^n, while $(V^n)^j$ indicates the jth tensor product of V^n with itself. The proof of (11) uses the fact that the modules V^n satisfy the recurrence relation $V^{n+1} = V^2 V^n - V^{n-1}$, the same recurrence relation defining the Chebyshev polynomials in (1).

The first equality of Theorem 4.6 now follows from combining Lemma 4.7 and (11), and the second equality comes from Definition 4.4. □

Remark 4.8 Toroidal analogues of other link invariants can be constructed by considering quantum groups other than $U_q(\mathrm{sl}(2, \mathbb{C}))$. For example, letting $\mathscr{A} = U_q(\mathrm{sl}(m, \mathbb{C}))$ for general m gives a toroidal analogue of the specialization $P_L(q^m, q - q^{-1})$, where P_L is the two-variable HOMFLY polynomial of a link L. Letting $\mathscr{A} = U_q G$, where $G = \mathrm{so}(m)$ or $\mathrm{sp}(2m)$, leads to a toroidal analogue of a certain specialization of the two-variable Kauffman polynomial, depending on the choice of G; see [28, Section 6.1]. In Section 5, we'll construct a toroidal analogue of the SU(2) colored Jones polynomial by setting $\mathscr{A} = \mathrm{SU}(2)_q$.

5 The toroidal colored Jones polynomial and skein theory

In this section only, we consider the invariant defined by specializing Φ to the quantum group $\mathrm{SU}(2)_q$ rather than $U_q(\mathrm{sl}(2, \mathbb{C}))$. This constitutes a certain normalization of the $\mathrm{sl}(2, \mathbb{C})$ invariant and we denote the SU(2) version by the same notation, $\hat{J}^T_{n,q}$. This is consistent with literature on the colored Jones polynomial and the operators involved are discussed, for example, in [18; 22].

The goal of the section is to develop the SU(2) toroidal colored Jones polynomial skein-theoretically. This begins with an observation about the level two framed invariant $\hat{J}_{2,q}$.

Lemma 5.1 *The level two SU(2) invariant $\hat{J}_{2,q}^T$ has the following properties:*

(a) $\hat{J}_{2,q}^T(\varnothing) = 1$.

(b) *Let $C \subset T^2$ be a simple closed curve disjoint from a diagram $D \subset T^2$.*

   (i) *If C is contractible, $\hat{J}_{2,q}^T(C \sqcup D) = (-q^{1/2} - q^{-1/2})\hat{J}_{2,q}^T(D) = -[2]\hat{J}_{2,q}^T(D)$.*

   (ii) *If C is not contractible, $\hat{J}_{2,q}^T(C \sqcup D) = 2\hat{J}_{2,q}^T(D)$.*

(c) $\hat{J}_{2,q}^T(\text{\Large$\times$}) = q^{1/4}\hat{J}_{2,q}^T(\text{\Large)(}) + q^{-1/4}\hat{J}_{2,q}^T(\text{\Large$\asymp$})$.

Proof Properties (a), (b)(i) and (c) are identical to the relations defining the usual Kauffman bracket. Because they are local properties — (b)(i) is local in the sense that we can assume C exists in a coordinate neighborhood of T^2 — the proofs are the same as for the usual SU(2) Jones polynomial; see [22, Theorem 4.1]. Each property reduces to an algebraic statement about the quantum group $SU(2)_q$.

To prove property (b)(ii) holds for $\hat{J}_{2,q}^T$, suppose $C \subset T^2$ is a simple, closed essential curve. Then $\hat{J}_{2,q}^T(C) = 2$ by Proposition 3.8. The general statement follows from the multiplicativity of $\hat{J}_{n,q}$ on disjoint diagrams; that is,

$$\hat{J}_{2,q}^T(U \sqcup D) = \hat{J}_{2,q}^T(U) \cdot \hat{J}_{2,q}^T(D) = 2\hat{J}_{2,q}^T(D). \qquad \square$$

Lemma 5.1 leads to the following definition and theorem:

Definition 5.2 Define a Kauffman-type bracket $\langle * \rangle_\tau \in \mathbb{Z}[A^{\pm 1}, z]$ on link diagrams in T^2 (and framed links in $T^2 \times I$) by the relations:

(a) $\langle \varnothing \rangle_\tau = 1$.

(b) Let $C \subset T^2$ be a simple closed curve disjoint from a diagram $D \subset T^2$.

   (i) If C is contractible, $\langle C \sqcup D \rangle_\tau = (-A^2 - A^{-2})\langle D \rangle_\tau$.

   (ii) If C is not contractible, $\langle C \sqcup D \rangle_\tau = z \cdot \langle D \rangle_\tau$.

(c) $\langle \text{\Large$\times$} \rangle_\tau = A\langle \text{\Large)(} \rangle_\tau + A^{-1}\langle \text{\Large$\asymp$} \rangle_\tau$.

Theorem 5.3 *For any framed link $L \subset T^2 \times I$,*

$$\hat{J}_{2,q}^T(L) = \langle L \rangle_\tau |_{A^4 = q, z = 2}.$$

To extend Theorem 5.3 to all values of $\boldsymbol{n}$ for $\hat{J}_{\boldsymbol{n},q}$, we consider the skein module of the thickened torus, $\mathscr{S}(T^2)$. As Section 2.1 discusses, a basis for $\mathscr{S}(T^2)$ as a

$\mathbb{Z}[A^{\pm 1}]$–module is given by positive powers of the tuples (a, b) such that either $a = b = 0$ or a and b are coprime. Let $p_2 \colon \mathcal{S}(T^2) \to \mathbb{Z}[A^{\pm 1}]$ be the $\mathbb{Z}[A^{\pm 1}]$–linear map defined by

$$p_2((a, b)^m) = \begin{cases} 1 & \text{if } a = b = 0, \\ 2^m & \text{otherwise.} \end{cases}$$

Then it's clear that, for any framed link $L \subset T^2 \times I$,

$$\langle L \rangle_\tau |_{z=2} = p_2(\langle L \rangle_T),$$

where $\langle L \rangle_T$ is the class of L in $\mathcal{S}(T^2)$.

We can now state the full result using $\mathcal{S}(T^2)$.

Theorem 5.4 *For any oriented, unframed link $L \subset T^2 \times I$ with diagram $D \subset T^2$,*

$$\hat{J}_{\boldsymbol{n},q}^T(L) = p_2(\langle S_{n_1-1}(z), \ldots, S_{n_k-1}(z) \rangle_{T,D})|_{A^4=q},$$

where $\hat{J}_{\boldsymbol{n},q}^T$ is the SU(2) *invariant and $\langle * \rangle_{T,D} \colon \mathcal{S}(\mathcal{A})^{\otimes k} \to \mathcal{S}(T^2)$ is the multibracket of* (4).

Proof A closed formula for the nth Chebyshev polynomial, as defined in (1), is given by

$$(12) \qquad S_n(z) = \sum_{j=0}^{n/2} (-1)^j \binom{n-j}{j} z^{n-2j}$$

where the sum is over all integers j with $0 \le 2j \le n$. Let $L \subset T^2 \times I$ be a k–component link and $\boldsymbol{n} = (n_1, \ldots, n_k)$ a multi-integer. Applying the above formula and the multilinearity of p_2 and the Kauffman multibracket, we have

$$
\begin{aligned}
p_2(\langle S_{n_1}(z), \ldots, S_{n_k}(z) \rangle_{T,D}) &= \sum_{j=0}^{n/2} (-1)^j \binom{\boldsymbol{n}-j}{j} p_2(\langle L^{\boldsymbol{n}-2j} \rangle_T) \\
&= \sum_{j=0}^{n/2} (-1)^j \binom{\boldsymbol{n}-j}{j} \hat{J}_{2,q}^T(L^{\boldsymbol{n}-2j}) \\
&= \hat{J}_{\boldsymbol{n}-1,q}^T(L).
\end{aligned}
$$

The second equality is Theorem 5.3. The third comes from Theorem 4.6, which applies in the SU(2) theory since the same relation $V^{j+1} = V^2 V^j - V^{j-1}$ holds in the representation ring. $\qquad\square$

Having constructed the SU(2) $\hat{J}_{n,q}$ skein-theoretically, we can define a skein-theoretic toroidal colored Jones polynomial. The SU(2) version of $\hat{J}_{n,q}^T$ satisfies [23]

$$\hat{J}_{n,q}^T(\text{\reflectbox{$\circlearrowright$}}) = (-1)^{n-1} q^{(n^2-1)/4} \cdot \hat{J}_{n,q}^T(\,|\,),$$
$$\hat{J}_{n,q}^T(\circlearrowright) = (-1)^{n-1} q^{-(n^2-1)/4} \cdot \hat{J}_{n,q}^T(\,|\,),$$

where the strand shown in the diagram is colored by V^n. Subsequently:

Definition 5.5 The SU(2) *toroidal colored Jones polynomial* J_n^T of an oriented, unframed link $L \subset T^2 \times I$ with diagram D is defined by

$$J_n^T(L;q) = ((-1)^{n-1} q^{(n^2-1)/4})^{-w(D)} \hat{J}_{(n,\ldots,n),q}^T(D).$$

We immediately have:

Corollary 5.6 *The* SU(2) *toroidal colored Jones polynomial is skein-theoretically defined by*

$$J_n^T(L;q) = \left[((-1)^{n-1} A^{n^2-1})^{-w(D)} p_2(\langle S_{n-1}(z), \ldots, S_{n-1}(z) \rangle_D) \right]\Big|_{A^4=q}.$$

Compare the right side of Theorem 5.4 with (3) — the missing factor of $1/(-A^2 - A^{-2})$ is analogous to the missing $1/[n]$ factor in the $U_q(\mathrm{sl}(2,\mathbb{C}))$ case.

The skein theoretic definitions of Theorem 5.4 and Corollary 5.6 let us extend the SU(2) invariants $\hat{J}_{n,q}^T$ and J_n^T to links in orientable manifolds other than $T^2 \times I$, using the bracket $\langle * \rangle_\tau$ of Definition 5.2 (with $z = 2$) as a generalized Kauffman bracket. In S^3 the bracket $\langle * \rangle_\tau$ coincides with the usual Kauffman bracket, and thus the invariant $\hat{J}_{n,q}^T$ defined in S^3 is exactly the invariant J_L of [22]. The SU(2) toroidal colored Jones polynomial, defined skein-theoretically in S^3, satisfies

$$J_n^T(L;q) = (-A^2 - A^{-2})|_{A^4=q} \cdot J_n(L;q),$$

where J_n is the SU(2) colored Jones polynomial of (3).

6 The volume conjecture for the two-by-two square weave

We now prove the volume conjecture for the *two-by-two square weave* $W \subset T^2 \times I$, the link shown in Figure 1. More generally, a diagram for the *$2k$-by-$2l$ square weave* $W_{2k,2l}$, for $k, l \in \mathbb{N}$, is made by tiling W to form a rectangular grid with $2k$ rows and $2l$ columns of crossings. We consider only even dimensions to ensure the diagram is alternating on the torus.

The complement of $W_{2k,2l}$ in $T^2 \times I$ is geometrically simple — Champanerkar, Kofman and Purcell [3] describe a complete hyperbolic structure for $(T^2 \times I) \setminus W_{2k,2l}$ consisting of $4kl$ regular ideal hyperbolic octahedra, one for each crossing. Thus,

(13) $$\mathrm{Vol}(W_{2k,2l}) = 4kl \cdot v_{\mathrm{oct}}.$$

Separately, we have the following proposition.

Proposition 6.1 Let $q = e^{2\pi i/n}$. If $L \subset T^2 \times I$ is an oriented link which has a diagram $D \subset T^2$ with c crossings,
$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; q)| \leq c \cdot v_{\mathrm{oct}}.$$

Proof The result follows from work of Garoufalidis and Lê [9, Corollary 8.10 and Theorem 1.13]. Recall the definition of J_n^T as a state sum (Definition 4.4 and equations (6) and (7)),
$$J_n^T(L; q) = (q^{(n^2-1)/4})^{-w(D)} \sum_{\sigma} \prod_{p \in P} \omega_p(\sigma),$$

where P is the set of critical points of D and σ is a state of D. Since q is a root of unity, $|J_n^T| = \left| \sum_{\sigma} \prod_{p \in P} \omega_p(\sigma) \right|$.

Let m be the number of connected components of $D \setminus P$; then D has n^m states. If σ' is a state with maximum modulus weight, then

(14) $$\log |J_n^T| = \log \left| \sum_{\sigma} \prod_{p \in P} \omega_p(\sigma) \right|$$
$$\leq \log \left| n^m \prod_{p \in P} \omega_p(\sigma') \right| = m \log n + \sum_{p \in P} \log |\omega_p(\sigma')|.$$

Fix the standard basis for $V^n \cong \mathbb{C}^n$. In this basis, if $p \in P$ is an extremum then $|\omega_p(\sigma')| = 1$. If p is a crossing then $\omega_p(\sigma')$ is an element of the R–matrix of $\mathscr{A}_q$ in the given basis, and Garoufalidis and Lê proved

(15) $$\frac{2\pi}{n} \lim_{n \to \infty} \log |\omega_p(\sigma')| \leq v_{\mathrm{oct}}.$$

Considering $\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T|$, the summands $\log |\omega_p(\sigma')|$ with p a crossing are the only summands of (14) which don't vanish asymptotically. Thus, applying (15),
$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T| \leq c \cdot v_{\mathrm{oct}}. \qquad \square$$

Equation (13) and Proposition 6.1 together give

(16) $$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(W_{2k,2l}; e^{2\pi i/n})| \leq 4kl \cdot v_{\mathrm{oct}} = \mathrm{Vol}((T^2 \times I) \setminus W_{2k,2l}).$$

In other words, for the rectangular weave $W_{2k,2l}$, the asymptotic growth of the toroidal colored Jones polynomial is bounded above by the volume of the complement. This makes the $2k$-by-$2l$ rectangular weave a natural object of study for our volume conjecture, Conjecture 1.2 — to prove the conjecture for a link in this family, we need only show the upper bound in (16) is achieved.

Before proving Theorem 6.2, we give a formula for $J_n^T(W; q) = J_n^T(W_{2,2}; q)$. As an isomorphism of $V^n \otimes V^n$ the R–matrix R of $\mathcal{A}_q$ is defined by weights $R_{kl}^{ij} \in \mathbb{C}$,

$$R(e_k \otimes e_l) = \sum_{i,j=0}^{n-1} R_{kl}^{ij} e_i \otimes e_j,$$

where $\{e_0, \ldots, e_{n-1}\}$ is a preferred basis of V^n. We have

$$(17) \qquad R_{kl}^{ij} = \sum_{m=0}^{\min(n-1-i,j)} \delta_{i+m}^l \delta_{j-m}^k \cdot q^\alpha \cdot \frac{\{l\}!\{n-1-k\}!}{\{i\}!\{m\}!\{n-1-j\}!},$$

where

$$\alpha = \left(i - \tfrac{1}{2}(n-1)\right)\left(j - \tfrac{1}{2}(n-1)\right) - \tfrac{1}{2}m(i-j) - \tfrac{1}{4}m(m+1),$$

δ is the Kronecker delta, and $\{m\}! = \{m\}\{m-1\}\cdots\{2\}\{1\}$. Similarly,

$$R^{-1} \colon V^n \otimes V^n \to V^n \otimes V^n$$

is defined by the scalars

$$(18) \qquad (R^{-1})_{kl}^{ij} = \sum_{m=0}^{\min(n-1-i,j)} \delta_{i-m}^l \delta_{j+m}^k \cdot (-1)^m \cdot q^\beta \cdot \frac{\{k\}!\{n-1-l\}!}{\{j\}!\{m\}!\{n-1-i\}!},$$

where

$$\beta = -\left(i - \tfrac{1}{2}(n-1)\right)\left(j - \tfrac{1}{2}(n-1)\right) - \tfrac{1}{2}m(i-j) + \tfrac{1}{4}m(m+1).$$

Let D be the diagram for W in Figure 10, left, with components labeled by basis elements as shown — implicitly we've chosen a covering map where crossings are oriented downward and there are no maxima or minima. We have

$$J_n^T(W; q) = \sum_{a,b,\ldots,g,h=0}^{n-1} (R^{-1})_{gb}^{fa} R_{cf}^{be} R_{ah}^{dg} (R^{-1})_{ed}^{hc}.$$

The index m in (17) and (18) is sometimes thought of as the "label" of the associated crossing [9], with the corresponding summand its "weight". In this way a state of D becomes a labeling of both strands and crossings with integers between 0 and $n-1$,
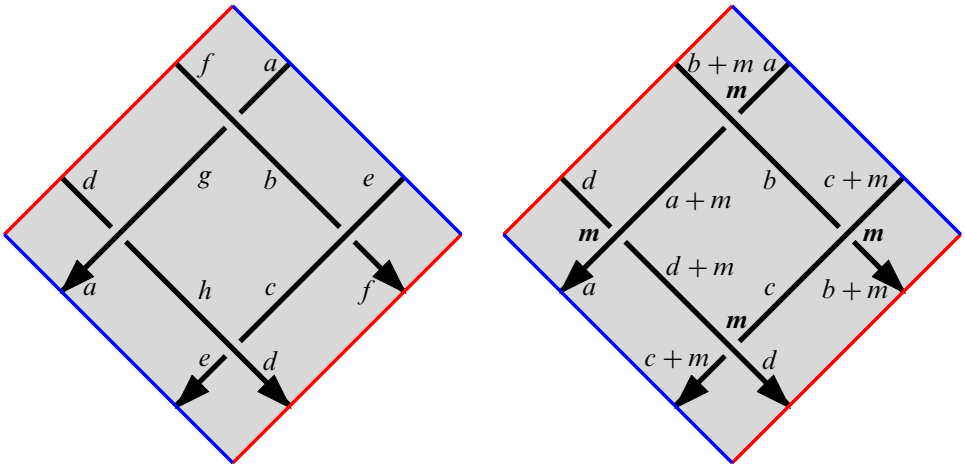
Each crossing is labeled m.

Figure 10: States of D.

and the Kronecker deltas in (17) and (18) imply we need only consider states whose crossing labels are as shown in Figure 11.

Assigning labels to strands and crossings of D according to these rules gives the diagram in Figure 10, right — the structure of W forces each crossing to have the same label in any nonzero state. We obtain the formula

$$
\begin{aligned}
J_n^T(W;q) \\
&= \sum_{m=0}^{n-1} \sum_{a,b,c,d=0}^{n-m-1} (R^{-1})_{a+m,b}^{b+m,a} R_{c,b+m}^{b,c+m} R_{a,d+m}^{d,a+m} (R^{-1})_{c+m,d}^{d+m,c} \\
&= \sum_{m=0}^{n-1} \sum_{a,b,c,d=0}^{n-m-1} q^{(a-c)(d-b)} \frac{\{a+m\}!\{n-1-a\}!}{\{a\}!\{m\}!\{n-1-a-m\}!} \cdot \frac{\{b+m\}!\{n-1-b\}!}{\{b\}!\{m\}!\{n-1-b-m\}!} \\
&\qquad \cdot \frac{\{c+m\}!\{n-1-c\}!}{\{c\}!\{m\}!\{n-1-c-m\}!} \cdot \frac{\{d+m\}!\{n-1-d\}!}{\{d\}!\{m\}!\{n-1-d-m\}!}.
\end{aligned}
$$
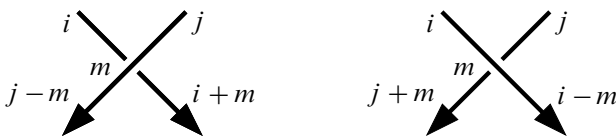
Figure 11: States of crossings with nonzero weights.

For the remainder of the section, fix $q = e^{2\pi i/n}$. This allows us to apply the identity [9, equation 38]

$$\{k\}! = (\sqrt{-1})^{n-1} \frac{n}{\{n-1-k\}!},$$

where $k \in \mathbb{N}$ and $0 \le k \le n-1$, and the formula above becomes

$$(19) \quad J_n^T(W;q)$$
$$= \sum_{m=0}^{n-1} \sum_{a,b,c,d=0}^{n-m-1} q^{(a-c)(d-b)} \frac{1}{(\{m\}!)^4} \cdot \left(\frac{\{a+m\}!\{b+m\}!\{c+m\}!\{d+m\}!}{\{a\}!\{b\}!\{c\}!\{d\}!} \right)^2.$$

Theorem 6.2 $\displaystyle\lim_{n\to\infty} \frac{2\pi}{n} \log |J_n^T(W; e^{2\pi i/n})| = 4v_{\text{oct}} = \text{Vol}((T^2 \times I) \setminus W).$

Proof By (16), we need only show

$$(20) \qquad \lim_{n\to\infty} \frac{2\pi}{n} \log |J_n^T(W; e^{2\pi i/n})| \ge 4v_{\text{oct}} = \text{Vol}((T^2 \times I) \setminus W).$$

Let $\eta(a,b,c,d) = q^{(a-c)(d-b)}$ and

$$\rho(a,b,c,d,m) = \frac{1}{(\{m\}!)^4} \cdot \left(\frac{\{a+m\}!\{b+m\}!\{c+m\}!\{d+m\}!}{\{a\}!\{b\}!\{c\}!\{d\}!} \right)^2.$$

Then (19) becomes

$$(21) \qquad J_n^T(W;q) = \sum_{m=0}^{n-1} \sum_{a,b,c,d=0}^{n-m-1} \eta(a,b,c,d) \cdot \rho(a,b,c,d,m).$$

For $k \in \mathbb{Z}$ with $0 \le k \le n-1$, $\{k\} = 2i \sin(k\pi/n) = ix$ where x is a nonnegative real number. Thus $(\{m\}!)^4$ is a nonnegative real number for all values of m. Furthermore, since

$$\frac{\{a+m\}!\{b+m\}!\{c+m\}!\{d+m\}!}{\{a\}!\{b\}!\{c\}!\{d\}!}$$

$$= \prod_{k=1}^{m} \{a+k\}\{b+k\}\{c+k\}\{d+k\}$$

$$= 16 \prod_{k=1}^{m} \sin\frac{\pi(a+k)}{n} \sin\frac{\pi(b+k)}{n} \sin\frac{\pi(c+k)}{n} \sin\frac{\pi(d+k)}{n},$$

$\rho(a,b,c,d,m)$ is a nonnegative real number for all relevant values of a, b, c, d, and m.

If $a = c$ then $\eta(a, b, c, d) = 1$ and $\eta(a, b, c, d) \cdot \rho(a, b, c, d, m)$ is a real number. If $a \neq c$, since $\rho(a, b, c, d, m) = \rho(a, c, b, d, m)$ and $\eta(a, b, c, d) = \eta(a, c, b, d)^{-1}$,

$$
\begin{aligned}
\eta(a, b, c, d)\rho(a, b, c, d, m) &+ \eta(a, c, b, d)\rho(a, c, b, d, m) \\
&= (\eta(a, b, c, d) + \eta(a, b, c, d)^{-1}) \cdot \rho(a, b, c, d, m) \\
&= 2\cos\left(\frac{2\pi(a-c)(d-b)}{n}\right) \cdot \rho(a, b, c, d, m) \in \mathbb{R}.
\end{aligned}
$$

Pairing up the summands of (21) this way we see $J_n^T(W; q)$ is a real number; in fact

$$
\begin{aligned}
(22) \quad J_n^T(W; q) &= \operatorname{Re}(J_n^T(W; q)) \\
&= \sum_{k=0}^{n-1} \sum_{a,b,c,d=0}^{n-k-1} \cos\left(\frac{2\pi(a-c)(d-b)}{n}\right) \cdot |\rho(a, b, c, d, m)|.
\end{aligned}
$$

Using the identity $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, we rewrite (22) as

$$
\begin{aligned}
(23) \quad & J_n^T(W; q) \\
&= \sum_{m=0}^{n-1} \sum_{a,b,c,d=0}^{n-m-1} \cos\left(\frac{2\pi(a-c)(d-b)}{n}\right) \left|\frac{\{a+m\}!\{b+m\}!\{c+m\}!\{d+m\}!}{\{a\}!\{b\}!\{c\}!\{d\}!}\right|^2 \left|\frac{1}{\{m\}!}\right|^4 \\
&= \sum_{m=0}^{n-1} \sum_{b,d=0}^{n-m-1} \left|\frac{1}{\{m\}!}\right|^4 \left|\frac{\{b+m\}!\{d+m\}!}{\{b\}!\{d\}!}\right|^2 \cdot \nu(b, d, m),
\end{aligned}
$$

where

$$
\begin{aligned}
\nu(b, d, m) &= \sum_{a,c=0}^{n-m-1} \cos\left(\frac{2\pi a(d-b)}{n} - \frac{2\pi c(d-b)}{n}\right) \left|\frac{\{a+m\}!\{c+m\}!}{\{a\}!\{m\}!}\right|^2 \\
&= \sum_{a,c=0}^{n-m-1} \cos\left(\frac{2\pi a(d-b)}{n}\right) \cos\left(\frac{2\pi c(d-b)}{n}\right) \left|\frac{\{a+m\}!\{c+m\}!}{\{a\}!\{m\}!}\right|^2 \\
&\quad + \sum_{a,c=0}^{n-m-1} \sin\left(\frac{2\pi a(d-b)}{n}\right) \sin\left(\frac{2\pi c(d-b)}{n}\right) \left|\frac{\{a+m\}!\{c+m\}!}{\{a\}!\{m\}!}\right|^2 \\
&= \left(\sum_{a=0}^{n-m-1} \cos\left(\frac{2\pi a(d-b)}{n}\right) \left|\frac{\{a+m\}!}{\{a\}!}\right|^2\right)^2 + \left(\sum_{a=0}^{n-m-1} \sin\left(\frac{2\pi a(d-b)}{n}\right) \left|\frac{\{a+m\}!}{\{a\}!}\right|^2\right)^2.
\end{aligned}
$$

In particular, $\nu(b, d, m)$ is a nonnegative real number for all values of b, d, and m.

Because each summand of (23) is nonnegative and real, $J_n^T(W; q)$ is bounded below for all n by the summand of (23) with $m = \lfloor n/2 \rfloor$ and $b = d = \lfloor n/4 \rfloor$. Additionally,

$v(\lfloor n/4 \rfloor, \lfloor n/4 \rfloor, \lfloor n/2 \rfloor)$ is bounded below by the summand of the equation above with $a = \lfloor n/4 \rfloor$. We have

$$
(24) \quad J_n^T(W; q) \geq \left| \frac{1}{\{\lfloor n/2 \rfloor\}!} \right|^4 \left| \frac{\{\lfloor n/4 \rfloor + \lfloor n/2 \rfloor\}! \{\lfloor n/4 \rfloor + \lfloor n/2 \rfloor\}!}{\{\lfloor n/4 \rfloor\}! \{\lfloor n/4 \rfloor\}!} \right|^2 v\left(\left\lfloor \frac{n}{4} \right\rfloor, \left\lfloor \frac{n}{4} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \right)
$$

$$
\geq \left| \frac{1}{\{\lfloor n/2 \rfloor\}!} \right|^4 \left| \frac{\{\lfloor n/4 \rfloor + \lfloor n/2 \rfloor\}!}{\{\lfloor n/4 \rfloor\}!} \right|^8
$$

$$
\geq \min\left(\left| \frac{1}{\{\lfloor n/2 \rfloor\}!} \right|^4 \left| \frac{\{\lfloor 3n/4 \rfloor\}!}{\{\lfloor n/4 \rfloor\}!} \right|^8, \left| \frac{1}{\{\lfloor n/2 \rfloor\}!} \right|^4 \left| \frac{\{\lfloor 3n/4-1 \rfloor\}!}{\{\lfloor n/4 \rfloor\}!} \right|^8 \right).
$$

Garoufalidis and Lê [9] proved that, for $\alpha \in (0, n)$,

$$
\log |\{\lfloor \alpha \rfloor\}!| = -\frac{n}{\pi} \Lambda\left(\pi \frac{\alpha}{n} \right) + O(\log n).
$$

Here Λ is the Lobachevsky function $\Lambda(z) = -\int_0^z \log |2 \sin \zeta| \, d\zeta$ and $O(\log n)$ is an expression bounded by $C \log n$ for a constant C independent of n. Applying this to (24) gives

$$
\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(W; q)|
$$

$$
\geq \lim_{n \to \infty} 8\Lambda\left(\frac{\pi}{2} \right) + 16\Lambda\left(\frac{\pi}{4} \right) - 16 \cdot \min\left(\Lambda\left(\frac{3\pi}{4} \right), \Lambda\left(\frac{(3n-4)\pi}{4n} \right) \right) + \frac{O \log(n)}{n}
$$

$$
= 4\left(2\Lambda\left(\frac{\pi}{2} \right) + 4\Lambda\left(\frac{\pi}{4} \right) - 4\Lambda\left(\frac{3\pi}{4} \right) \right) = 4v_{\text{oct}},
$$

proving the theorem. $\qquad \square$

7 Generalizing the volume conjecture for links in $T^2 \times I$

7.1 Simplicial volume

With the original volume conjecture, Conjecture 1.1, in mind, we generalize our Conjecture 1.2 to links which may not be hyperbolic.

Conjecture 7.1 *For any link $L \subset T^2 \times I$ such that $(T^2 \times I) \setminus L$ is irreducible,*

$$
(25) \qquad \lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})| = \mathrm{Vol}((T^2 \times I) \setminus L),
$$

where $n > 0$ runs over all odd integers.

As in Conjecture 1.1, Vol refers to *simplicial volume* — the sum of the volumes of the hyperbolic pieces in the JSJ decomposition of $(T^2 \times I) \setminus L$. By *irreducible*, we

mean that every smooth embedded 2–sphere in $(T^2 \times I) \setminus L$ bounds a 3–ball. The irreducibility condition and the restriction to odd n are, in fact, necessary if one wishes to generalize the original volume conjecture, Conjecture 1.1, from knots to links. For a link $L \subset S^3$, $S^3 \setminus L$ being irreducible is equivalent to L not being a split link, a class of links for which the colored Jones polynomial is known to vanish [24]. Separately, van der Veen has constructed a class of nonsplit links called *Whitehead chains* for which the colored Jones polynomial vanishes at even values of n [33]. Before discussing the necessity of these two conditions in Conjecture 7.1, we give some positive results.

Call $L' \subset S^3$ a *VC–verified link* if Conjecture 1.1 is known to hold for L'. That is, L' is VC–verified if

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n(L'; e^{2\pi i / n})| = \mathrm{Vol}(S^3 \setminus L'),$$

where the limit runs over odd $n > 0$. VC–verified links include the figure eight knot, the Borromean rings, and others — see [25, Chapter 3] for a somewhat recent, comprehensive list.

Theorem 7.2   Let $L' \subset S^3$ be a VC–verified link, and consider an inclusion of L' in an embedded 2–sphere in $T^2 \times I$. Let $K \subset T^2 \times I$ be a knot projecting to an essential, simple closed curve in $T^2 \times \{0\}$, and let L be a connect sum $L = L' \# K$. Then Conjecture 7.1 holds for L.

See Figure 12 for an example where L' is the figure eight knot and K is a meridian. The main ingredient in the proof of Theorem 7.2 is the following relationship between J_n^T and J_n.

Theorem 7.3   Let L' be a link in S^3, and consider an inclusion of L' in an embedded 2–sphere in $T^2 \times I$. Let $K \subset T^2 \times I$ be a knot projecting to an essential, simple closed curve in $T^2 \times \{0\}$, and let L be a connect sum $L = L' \# K$. Then

$$J_n^T(L; q) = n \cdot J_n(L'; q).$$

Proof   Using Proposition 4.5, we assume K is a meridian. Then we can choose a diagram $D \subset T^2$ for L and a lift, $\widetilde{D}$, of D to $\mathbb{R}^2$ such that $D' = \widetilde{D} \cap I^2$ is a diagram of L' as a $(1, 1)$–tangle. (See Figure 12, where L' is the figure eight knot.) Coloring D' by V^n, Theorem 2.1 associates an $\mathcal{A}$–linear map $\phi \colon V^n \to V^n$ to D'. The irreducibility of V^n implies ϕ is a scalar multiple of the identity, and, after accounting for writhe, this scalar is $J_n(L'; q)$ — see [17, Lemma 3.9] and [9; 25].
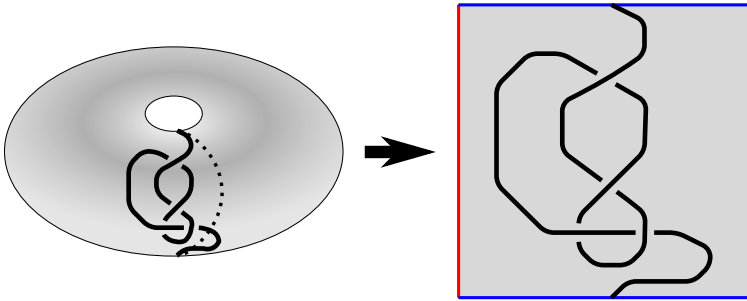
Figure 12: Composing the figure eight knot with a meridian in $T^2 \times I$.

On the other hand, by the proof of Lemma 3.2, $J_n^T(L, q)$ is the trace of ϕ (corrected for writhe). We conclude

$$J_n^T(L; q) = n \cdot J_n(L'; q). \qquad \square$$

Theorem 7.2 follows.

Proof of Theorem 7.2 By Theorem 7.3, $J_n^T(L; q) = n \cdot J_n(L'; q)$. Therefore

$$(26) \qquad \lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})| = \lim_{n \to \infty} \frac{2\pi}{n} \log |J_n(L'; e^{2\pi i/n})|.$$

The complement $(T^2 \times I) \setminus K$ is homeomorphic to $S^3 \setminus H'$, where H' is the link shown in Figure 13, left. This implies $(T^2 \times I) \setminus L$ is homeomorphic to $S^3 \setminus (L' \# H')$, where the composition is formed as in Figure 13, right. By [29],

$$\mathrm{Vol}((T^2 \times I) \setminus L) = \mathrm{Vol}(S^3 \setminus (L' \# H')) = \mathrm{Vol}(S^3 \setminus L') + \mathrm{Vol}(S^3 \setminus H') = \mathrm{Vol}(S^3 \setminus L'),$$

and combining this with (26) gives

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})| = \lim_{n \to \infty} \frac{2\pi}{n} \log |J_n(L'; e^{2\pi i/n})|$$
$$= \mathrm{Vol}(S^3 \setminus L') = \mathrm{Vol}((T^2 \times I) \setminus L). \qquad \square$$

Figure 13: The links H' and $L' \# H'$ in S^3, from the proof of Theorem 7.2, with L' the figure eight knot.

We've shown any positive result for the original Conjecture 1.1 gives a positive result for Conjecture 7.1. The proof also shows why restricting to odd n is necessary — if we let $L' \subset S^3$ be a Whitehead chain, as in [33], the link $L' \# K \subset T^2 \times I$ (defined as above) will satisfy Conjecture 7.1 but the toroidal colored Jones polynomial will vanish for even n.

Using the nice behavior of simplicial volume and the toroidal colored Jones polynomial under split unions of links, we can push the result of Theorem 7.2 further. We define a *split union* $L = L_1 \sqcup L_2$ of links $L_1, L_2 \subset T^2 \times I$ to be a union such that L admits a torus diagram which is a disjoint union of diagrams of L_1 and L_2. Additionally, define a *torus link* to be a link in $T^2 \times I$ with a diagram consisting of a set of disjoint, simple closed curves in T^2.

Corollary 7.4 *Let $L'_1, L'_2, \ldots, L'_m \subset S^3$ be VC–verified links, and define K as in Theorem 7.2. Let $L_i = L'_i \# K$ for $i = 1, \ldots, m$. Then Conjecture 7.1 holds for the split union $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$. In particular, Conjecture 7.1 holds for all torus links with no nullhomotopic components.*

Proof The result follows just as in Theorem 7.2 after checking that

$$J_n^T(L) = J_n^T(L_1) \cdot J_n^T(L_2) \cdots J_n^T(L_m)$$

and

$$\mathrm{Vol}(L) = \mathrm{Vol}(L_1) + \mathrm{Vol}(L_2) + \cdots + \mathrm{Vol}(L_m).$$

To prove the second statement, let L'_i be the unknot for all i — every torus link with no nullhomotopic components can be obtained this way. Alternatively, one could use Proposition 3.8 and a direct computation. The result also holds for torus links with nullhomotopic components (see Proposition 7.6 below), but the complement of such a link in $T^2 \times I$ is not irreducible. $\qquad\square$

If we view links in the thickened torus as generalizations of $(1, 1)$–tangles, as in the proof of Theorem 7.3 suggests, and think of the colored Jones polynomial as an invariant of $(1, 1)$–tangles, the toroidal colored Jones polynomial becomes a generalization of the colored Jones polynomial rather than a toroidal analogue. This view is supported by Corollary 7.5 below, which shows Conjecture 7.1 implies Conjecture 1.1.

Corollary 7.5 *For knots, Conjecture 7.1 implies Conjecture 1.1.*
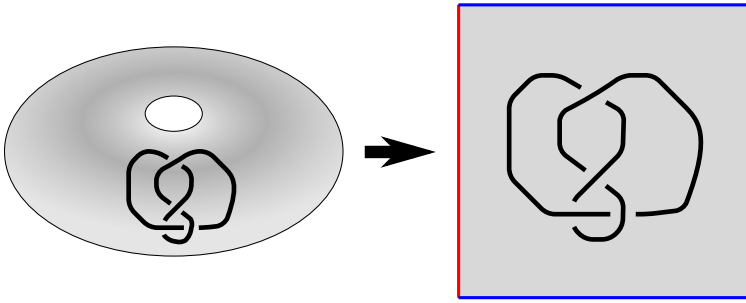
Figure 14: A nullhomotopic inclusion of the figure eight knot in $T^2 \times I$.

Proof Given a knot $K' \subset S^3$, let $L = K' \# K$ with K defined as above. Then, assuming Conjecture 7.1,

$$\lim_{n \to \infty} \frac{2\pi}{n} \log |J_n(K'; e^{2\pi i/n})| = \lim_{n \to \infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})|$$
$$= \mathrm{Vol}((T^2 \times I) \setminus L) = \mathrm{Vol}(S^3 \setminus K'). \qquad \square$$

In this sense, Conjecture 7.1 generalizes Conjecture 1.1. It is interesting to note that Conjecture 1.1 does not seem to imply Conjecture 7.1.

As we noted earlier, just as Conjecture 1.1 fails for split links [24], Conjecture 7.1 fails for links in $T^2 \times I$ which have one or more nullhomotopic split components. By a *nullhomotopic split component* of a link $L \subset T^2 \times I$, we mean a sublink $L' \subset L$ such that L' is contained in an embedded 2–sphere in $(T^2 \times I) \setminus L$, and no proper sublink of L' is contained in such a sphere. This is implied by the following:

Proposition 7.6 *If $L \subset T^2 \times I$ is a link with a nullhomotopic split component, $J_n^T(L; e^{2\pi i/n}) = 0$ for all n. In particular, if L is nullhomotopic, $J_n^T(L; e^{2\pi i/n}) = 0$ for all n.*

Proof Let $L \subset T^2 \times I$ be a link with nullhomotopic split component L_1, and let $L_2 = L \setminus L_1$. Then $J_n^T(L; q) = J_n^T(L_1; q) \cdot J_n^T(L_2; q)$, so it suffices to show $J_n^T(L_1, e^{2\pi i/n}) = 0$.

Since L_1 is nullhomotopic, it has a torus diagram D which lifts to a diagram $\widetilde{D} \subset \mathbb{R}^2$ such that $\widetilde{D} \cap I^2$ is a diagram for L_1 as a link in S^3. See Figure 14, where L_1 is the figure eight knot. A direct computation shows $J_n^T(L_1; q) = [n] J_n(L_1; q)$, and $[n] = 0$ when $q = e^{2\pi i/n}$. $\qquad \square$

Remark 7.7   In [34], van der Veen noted that Conjecture 1.1 can be changed to account for split links by choosing a different normalization of the colored Jones polynomial. Essentially, each split component adds a factor of $[n]$ to J_n — if a link L has s split components, we can divide by $[n]^s$ to obtain a nonzero value at the root $q = e^{2\pi i/n}$.

Analogously, if $L \subset T^2 \times I$ has s *nullhomotopic* split components, we can ask whether

$$\lim_{n\to\infty} \frac{2\pi}{n} \log \left| \frac{1}{[n]^s} J_n^T(L; e^{2\pi i/n}) \right| = \mathrm{Vol}((T^2 \times I) \setminus L).$$

Replacing (25) in Conjecture 7.1 with the above equation, we can remove the hypothesis that $(T^2 \times I) \setminus L$ be irreducible.

7.2   Higher-genus surfaces

Taking a different direction, one could attempt to generalize Conjecture 1.2 to links in thickened surfaces of genus greater than one. As we noted earlier, while there is no obvious way to define pseudo-operator invariants for links in these surfaces, Corollary 5.6 lets us define the SU(2) toroidal colored Jones polynomial skein-theoretically in any orientable manifold.

As defined, volume conjecture behavior is unlikely to occur in thickened surfaces of genus greater than one. To see why, let $\Sigma \times I$ be such a thickened surface containing a link L. Since $\Sigma \times I$ has boundary components which are not spheres or tori, there is not a unique way to assign a complete hyperbolic structure to the complement of L. One way to resolve this ambiguity, as in [1], is to choose the hyperbolic structure on $(\Sigma \times I) \setminus L$ which has totally geodesic boundary. If such a structure exists, $(\Sigma \times I) \setminus L$ is called *tg–hyperbolic* and it has a finite *tg–hyperbolic volume*.

Proposition 6.1 says that, in the case of a link L in the thickened torus with crossing number c,

$$(27) \qquad \lim_{n\to\infty} \frac{2\pi}{n} \log |J_n^T(L; e^{2\pi i/n})| \le c \cdot v_{\mathrm{oct}}.$$

A similar bound exists for links in S^3 — see [9, Theorem 1.13] — and we conjecture that (27) holds for J_n^T for links in any genus thickened surface. In surfaces with genus greater than one, however, there are many links whose tg–hyperbolic volume exceeds this bound. Consider, for example, the virtual link 3.1 of [10] viewed as a link in the thickened orientable surface of genus two: its crossing number is three and its tg–hyperbolic volume is $\approx 18.75 > 3v_{\mathrm{oct}}$ [1]. Thus, volume convergence as

defined above is not possible if (27) holds for genus two surfaces and we choose the tg–hyperbolic structure on the complement of L.

This does not mean *no* volume conjecture can exist for links in higher genus surfaces — just that any such conjecture would need to look different from Conjecture 1.1 and Conjecture 1.2. It may be interesting to examine what kind of relationship can exist between the SU(2) toroidal colored Jones polynomial of a link in a higher-genus surface and its tg–hyperbolic volume.

8 The toroidal colored Jones polynomial as an invariant of biperiodic and virtual links

Beyond its volume conjecture behavior, the toroidal colored Jones polynomial may be useful as an invariant of biperiodic and virtual links. A *biperiodic link* is a properly embedded 1–manifold $\tilde{L} \subset \mathbb{R}^2 \times I$, such that $\tilde{L}$ is invariant under translations by a 2–dimensional lattice Λ and $L = \tilde{L}/\Lambda$ is a link in $T^2 \times I$; see [4]. We call Λ *maximal* if it is not properly contained in another invariant lattice for $\tilde{L}$, in which case the resulting link $L \subset T^2$ is a *minimal representative* of $\tilde{L}$. For a given biperiodic link $\tilde{L}$, there are many possible choices of minimal representative. However, if $L_1, L_2 \subset T^2 \times I$ are two minimal representatives of $\tilde{L} \subset \mathbb{R}^2 \times I$ with respective diagrams $D_1, D_2 \subset T^2$, then there exists an orientation-preserving homeomorphism f of T^2 such that $f(D_1) = D_2$; see [11, Proposition 2.1]. Hence, Proposition 4.5 gives the following:

Theorem 8.1 *If $\tilde{L} \subset \mathbb{R}^2 \times I$ is a biperiodic link and $L \subset T^2 \times I$ is a minimal representative of $\tilde{L}$, define $J_n^T(\tilde{L}) = J_n^T(L)$. Then J_n^T is an invariant of biperiodic links in $\mathbb{R}^2 \times I$.*

Another nonclassical type of link, *virtual links*, are an area of extensive study — see [15] for an introduction. By [2; 20], any virtual link L' is represented uniquely by a link L in a minimal-genus thickened surface, up to an orientation-preserving homeomorphism of the surface. The $U_q(\mathrm{sl}(2, \mathbb{C}))$ toroidal colored Jones polynomial is defined only for links in $T^2 \times I$, but the SU(2) toroidal colored Jones polynomial can be defined skein-theoretically for links in any thickened surface. Similar to above, we have:

Theorem 8.2 *If L' is a virtual link and $L \subset \Sigma \times I$ is a minimal representative of L', define $J_n^T(L') = J_n^T(L)$. Then J_n^T is an invariant of virtual links.*

Here Σ is a closed, orientable surface and J_n^T is the SU(2) toroidal colored Jones polynomial, defined skein-theoretically as in Corollary 5.6. To prove Theorem 8.2, we need only show the skein-theoretic J_n^T is preserved by orientation-preserving homeomorphisms of surfaces. This is done in the lemma below.

Lemma 8.3 *Let L and L' be links in $\Sigma \times I$ with respective diagrams $D, D' \subset \Sigma$, Σ a closed, orientable surface. If f is an orientation-preserving homeomorphism of Σ satisfying $f(D) = D'$, then $J_n^T(L;q) = J_n^T(L';q)$ for all $n \in \mathbb{N}$. Here J_n^T is the SU(2) toroidal Jones polynomial, defined skein-theoretically.*

Proof Because f preserves orientation, D and D' have the same writhe. Thus it suffices to prove the result for $\hat{J}_{n,q}^T$, which follows from the case of $\hat{J}_{2,q}^T$. Equivalently, we show the bracket $\langle * \rangle_\tau$ defined in Section 5 is invariant under orientation-preserving homeomorphisms of Σ.

The claim follows by induction on crossing number, noting f induces a bijection on the crossings of D and D'. If D has no crossings, $\langle D \rangle_\tau$ is determined by whether or not D is contractible, which is preserved by f. For an arbitrary diagram D, we can "resolve" a crossing using the relation (c) of Definition 5.2. Since f commutes with both types of crossing resolution in relation (c), the claim follows inductively. □

As Remark 3.5 discusses, J_n^T is distinct from existing quantum invariants of virtual links. To our knowledge, it is the first invariant of virtual links to exhibit volume conjecture behavior for genus one virtual links, ie links in the thickened torus. Continuing our discussion from Section 7.2, it is interesting to ask what kind of volume conjecture behavior emerges in higher-genus virtual links.

Appendix   The toroidal colored Jones polynomial and rotation number

The following generalization of property (b) of Lemma 5.1 is not hard to prove, using Proposition 3.8 and a direct computation.

Proposition A.1 *Let $K \subset T^2 \times I$ be a knot projecting to a simple closed curve in T^2.*

(a) *If K is nullhomotopic, the SU(2) toroidal colored Jones polynomial J_n^T satisfies*

$$J_n^T(K;q) = -[n],$$

and the $U_q(\mathrm{sl}(2, \mathbb{C}))$ toroidal colored Jones polynomial J_n^T satisfies

$$J_n^T(K; q) = [n].$$

(b) *If K is not nullhomotopic, the* SU(2) *and $U_q(\mathrm{sl}(2, \mathbb{C}))$ toroidal colored Jones polynomials both satisfy*

$$J_n^T(K; q) = n.$$

Proposition A.1 says, in a sense, that contractible, simple closed curves in T^2 are "quantized" by the toroidal colored Jones polynomial while essential, simple closed curves are not. We would like to motivate geometrically why this striking phenomenon occurs.

To accomplish this, we recall Lin and Wang's definition of the Jones polynomial [21], adapted from work in [31]. As we will see, their construction extends in a natural way to define J_2^T and, by cabling, J_n^T for all $n > 2$. Its use of rotation number provides insight into Proposition A.1, at least for $n = 2$.

We briefly recall Lin and Wang's definition. First, fix the preferred basis $\{e_0, e_1\}$ of V^2 we used in Section 7. In this basis the R–matrix coefficients are

$$R_{0,0}^{0,0} = R_{1,1}^{1,1} = q^{1/4}, \qquad\qquad R_{0,1}^{1,0} = R_{1,0}^{0,1} = q^{-1/4},$$
$$R_{0,1}^{0,1} = q^{1/4} - q^{-3/4}, \qquad (R^{-1})_{0,0}^{0,0} = (R^{-1})_{1,1}^{1,1} = q^{-1/4},$$
$$(R^{-1})_{0,1}^{1,0} = (R^{-1})_{1,0}^{0,1} = q^{1/4}, \quad (R^{-1})_{1,0}^{1,0} = q^{-1/4} - q^{3/4},$$

and all other entries of R and R^{-1} are zero.

Given a diagram D of an oriented link $L \subset S^3$, let P_c be the set a crossing points of D. In this context, a state s of D is an assignment of 0 or 1 to each component of $D \setminus P_c$. (States are defined differently here than in Section 3 — we ignore local extrema and do not make use of $(V^2)^*$.) If a state s labels a neighborhood of a positive crossing p with $i, j, k, l \in \{0, 1\}$ as in Figure 5, the weight of the crossing is $\omega_p(s) = R_{kl}^{ij}$. If s labels a neighborhood of a negative crossing the same way, the weight of p is $\omega_p(s) = (R^{-1})_{kl}^{ij}$.

Similar to (6), we define the total weight of a state s to be

$$\omega^c(s) = \prod_{p \in P_c} \omega_p(s).$$

A state s is called *admissible* if $\omega^c(s) \neq 0$. Examining the coefficients of R and R^{-1}, we see s is admissible if and only if each crossing of D has one of the patterns of

Figure 15: Admissible states near crossings.

labels shown in Figure 15, where dashed and solid lines indicate 0– and 1–labels, respectively. If either of the two rightmost cases in Figure 15 occurs in D, we resolve the given crossing into two vertical lines. This decomposes D into a set of closed curves, each labeled entirely by 0 or entirely by 1 in s. Define $\mathrm{rot}_i(D, s)$ to be the sum of the rotation numbers (the degree of the Gauss map) of all i–labeled curves of D after these resolutions take place. Then:

Proposition A.2 [21] *We have*

$$J_2(L; q) = \frac{1}{[n]}(q^{3/4})^{-w(D)} \sum_{s \in \mathrm{Adm}_c(D)} q^{(\mathrm{rot}_1(D,s) - \mathrm{rot}_0(D,s))/2} \cdot \omega^c(s),$$

where $\mathrm{Adm}_c(D)$ *is the set of admissible states of* D.

Removing the factor of $1/[n]$, this definition extends to a torus with no trouble. It agrees with our definition of J_2^T.

Theorem A.3 *Let* $L \subset T^2 \times I$ *be an oriented link with diagram* $D \subset T^2$. *Then*

$$(28) \qquad J_2^T(L; q) = (q^{3/4})^{-w(D)} \sum_{s \in \mathrm{Adm}_c(D)} q^{(\mathrm{rot}_1(D,s) - \mathrm{rot}_0(D,s))/2} \cdot \omega^c(s),$$

where all terms are defined as in Proposition A.2.

Proof We sketch the proof. Let P denote the set of crossing points and local extrema of D, as in Section 3, and use σ to denote a state of D in the pseudo-operator invariant context (see (6) and the preceding discussion). Call a state σ *admissible* if $\omega(\sigma) \neq 0$, and let $\mathrm{Adm}(D)$ be the set of admissible states of D in this context.

In the given basis for V^2, the operator $\mu: V^2 \to V^2$ is defined by

$$(29) \qquad\qquad \mu_0^0 = q^{-1/2}, \quad \mu_1^1 = q^{1/2},$$

and all other coefficients are zero [17]. Thus, a state σ is admissible only if both sides of every extreme point of D are assigned the same number, either 0 or 1. (Here $i \in \{0, 1\}$ might refer to the basis element e_i or the dual element e^i.) It follows that $\mathrm{Adm}(D)$ is in bijection with $\mathrm{Adm}_c(D)$. Furthermore, if $\sigma \in \mathrm{Adm}(D)$, we can perform crossing
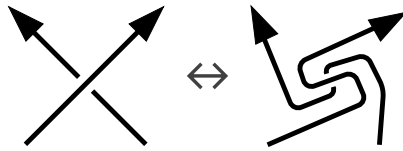
Figure 16: An isotopy which reorients a crossing downward.

resolutions like those preceding Proposition A.2 to decompose D into a set of closed curves, each of which is labeled entirely by 0 or entirely by 1. Therefore it makes sense to write $\mathrm{rot}_i(D, \sigma)$ for an admissible state σ.

Finally, we may assume all crossings of D have both strands oriented downward — otherwise, we can apply an isotopy as in Figure 16. This isotopy does not change the value of (28), since it does not change the diagram or any rotation numbers. With this assumption, if $p \in D$ is a crossing point, $\omega_p(\sigma) = \omega_p(s)$ for any state $\sigma \in \mathrm{Adm}(D)$ with corresponding state $s \in \mathrm{Adm}_c(D)$.

We now compute

$$
\begin{aligned}
J_2^T(L; q) &= (q^{3/4})^{-w(D)} \sum_{\sigma \in \mathrm{Adm}(D)} \prod_{p \in P} \omega_p(\sigma) \\
&= (q^{3/4})^{-w(D)} \sum_{\sigma \in \mathrm{Adm}(D)} \prod_{p \in (P \setminus P^c)} \omega_p(\sigma) \prod_{p \in P^c} \omega_p(\sigma) \\
&= (q^{3/4})^{-w(D)} \sum_{\sigma \in \mathrm{Adm}(D)} (\mu_0^0)^{\mathrm{rot}_0(D,\sigma)} (\mu_1^1)^{\mathrm{rot}_1(D,\sigma)} \omega^c(\sigma) \\
&= (q^{3/4})^{-w(D)} \sum_{s \in \mathrm{Adm}_c(D)} q^{(\mathrm{rot}_1(D,s) - \mathrm{rot}_0(D,s))/2} \cdot \omega^c(s).
\end{aligned}
$$

The key observation of the third equality is that μ counts rotation numbers. Examining Theorem 2.1, we see that a weight of μ_i^i is assigned to each left-oriented, i–colored cap and a weight of $(\mu^{-1})_i^i = (\mu_i^i)^{-1}$ is assigned to each left-oriented, i–colored cup. Thus, if C is a curve of D (after crossing resolution) labeled entirely by i, the exponent of the product of the μ_i^i's gives the rotation number of C. (See Figure 17.)   □

Having defined J_2^T as in Theorem A.3, the higher invariants J_n^T, for $n > 2$, can be recovered using the cabling formula, Theorem 4.6.

As promised, we only needed to normalize the formula in Proposition A.2 to define J_2^T as in Theorem A.3. From this perspective, J_n and J_n^T become two instances of the same formula, and the definition of the latter is forced by the definition of the former. In other words, from this point of view, there is no other way we could have defined J_n^T.
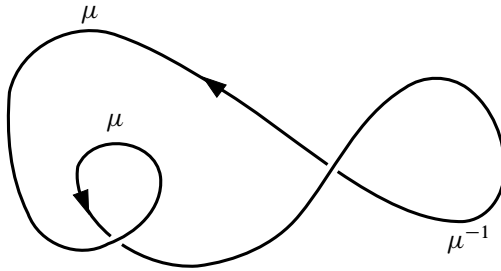
Figure 17: The exponent of the product of the μ's is the rotation number of the curve (in this case 1).

Additionally, (28) provides insight into Proposition A.1. Let $K \subset T^2 \times I$ be a knot which projects to a simple, closed curve $C \subset T^2$. Then C has no crossings, and only two state assignments as defined in (28). If C is contractible, it has rotation number ± 1 and

$$J_2^T(K;q) = q^{1/2} + q^{-1/2} = [2].$$

If C is not contractible, it has rotation number 0 and

$$J_2^T(K;q) = q^0 + q^0 = 2.$$

While we cannot fully explain why the toroidal colored Jones polynomial "quantizes" contractible curves and not essential ones, this discussion suggests a relationship with the curvature of a link.

Remark A.4 The exact R–matrix used here is slightly different than the one used in [21, Section 2.3]. To recover that matrix from ours, first multiply R by $q^{1/4}$ (and multiply R^{-1} by $q^{-1/4}$), then make the variable substitution $q' = -q^{1/2}$. We also use downward-oriented crossings rather than upward-oriented ones — these two convention changes result in a slightly different formula for J_2.

References

[1]   **C Adams**, **O Eisenberg**, **J Greenberg**, **K Kapoor**, **Z Liang**, **K O'Connor**, **N Pacheco-Tallaj**, **Y Wang**, *tg–hyperbolicity of virtual links*, J. Knot Theory Ramifications 28 (2019) art. id. 1950080   MR Zbl

[2]   **J S Carter**, **S Kamada**, **M Saito**, *Stable equivalence of knots on surfaces and virtual knot cobordisms*, J. Knot Theory Ramifications 11 (2002) 311–322   MR Zbl

[3]   **A Champanerkar**, **I Kofman**, **J S Purcell**, *Geometrically and diagrammatically maximal knots*, J. Lond. Math. Soc. 94 (2016) 883–908   MR Zbl

[4] **A Champanerkar**, **I Kofman**, **J S Purcell**, *Geometry of biperiodic alternating links*, J. Lond. Math. Soc. 99 (2019) 807–830 MR Zbl

[5] **Q Chen**, **T Yang**, *Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants*, Quantum Topol. 9 (2018) 419–460 MR Zbl

[6] **F Costantino**, *Coloured Jones invariants of links and the volume conjecture*, J. Lond. Math. Soc. 76 (2007) 1–15 MR Zbl

[7] **R Detcherry**, **E Kalfagianni**, **T Yang**, *Turaev–Viro invariants, colored Jones polynomials, and volume*, Quantum Topol. 9 (2018) 775–813 MR Zbl

[8] **P J Freyd**, **D N Yetter**, *Braided compact closed categories with applications to low-dimensional topology*, Adv. Math. 77 (1989) 156–182 MR Zbl

[9] **S Garoufalidis**, **T T Q Lê**, *Asymptotics of the colored Jones function of a knot*, Geom. Topol. 15 (2011) 2135–2180 MR Zbl

[10] **J Green**, *A table of virtual knots*, electronic resource (2004) Available at http://www.math.toronto.edu/drorbn/Students/GreenJ

[11] **S A Grishanov**, **V R Meshkov**, **A V Omelchenko**, *Kauffman-type polynomial invariants for doubly periodic structures*, J. Knot Theory Ramifications 16 (2007) 779–788 MR Zbl

[12] **R M Kashaev**, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. 39 (1997) 269–275 MR Zbl

[13] **C Kassel**, *Quantum groups*, Graduate Texts in Math. 155, Springer (1995) MR Zbl

[14] **L H Kauffman**, *State models and the Jones polynomial*, Topology 26 (1987) 395–407 MR Zbl

[15] **L H Kauffman**, *Virtual knot theory*, European J. Combin. 20 (1999) 663–690 MR Zbl

[16] **L H Kauffman**, *Rotational virtual knots and quantum link invariants*, J. Knot Theory Ramifications 24 (2015) art. id. 1541008 MR Zbl

[17] **R Kirby**, **P Melvin**, *The 3–manifold invariants of Witten and Reshetikhin–Turaev for* $\mathrm{sl}(2,\mathbf{C})$, Invent. Math. 105 (1991) 473–545 MR Zbl

[18] **A N Kirillov**, **N Y Reshetikhin**, *Representations of the algebra* $U_q(\mathrm{sl}(2))$, q–*orthogonal polynomials and invariants of links*, from "Infinite-dimensional Lie algebras and groups" (V G Kac, editor), Adv. Ser. Math. Phys. 7, World Sci., Teaneck, NJ (1989) 285–339 MR Zbl

[19] **V Krushkal**, *Graphs, links, and duality on surfaces*, Combin. Probab. Comput. 20 (2011) 267–287 MR Zbl

[20] **G Kuperberg**, *What is a virtual link?*, Algebr. Geom. Topol. 3 (2003) 587–591 MR Zbl

[21] **X‑S Lin**, **Z Wang**, *Random walk on knot diagrams, colored Jones polynomial and Ihara–Selberg zeta function*, from "Knots, braids, and mapping class groups — papers dedicated to Joan S. Birman" (J Gilman, W W Menasco, X‑S Lin, editors), AMS/IP Stud. Adv. Math. 24, Amer. Math. Soc., Providence, RI (2001) 107–121 MR Zbl

[22] **H R Morton**, *Invariants of links and 3–manifolds from skein theory and from quantum groups*, from "Topics in knot theory" (M E Bozhüyük, editor), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 399, Kluwer Acad., Dordrecht (1993) 107–155 MR Zbl

[23] **H R Morton**, **P Strickland**, *Jones polynomial invariants for knots and satellites*, Math. Proc. Cambridge Philos. Soc. 109 (1991) 83–103 MR Zbl

[24] **H Murakami**, **J Murakami**, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. 186 (2001) 85–104 MR Zbl

[25] **H Murakami**, **Y Yokota**, *Volume conjecture for knots*, SpringerBriefs in Mathematical Physics 30, Springer (2018) MR Zbl

[26] **T Ohtsuki**, *On the asymptotic expansion of the quantum* SU(2) *invariant at* $q = \exp(4\pi\sqrt{-1}/N)$ *for closed hyperbolic* 3–*manifolds obtained by integral surgery along the figure-eight knot*, Algebr. Geom. Topol. 18 (2018) 4187–4274 MR Zbl

[27] **J H Przytycki**, *Skein modules of* 3–*manifolds*, Bull. Polish Acad. Sci. Math. 39 (1991) 91–100 MR Zbl

[28] **N Y Reshetikhin**, **V G Turaev**, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990) 1–26 MR Zbl

[29] **T Soma**, *The Gromov invariant of links*, Invent. Math. 64 (1981) 445–454 MR Zbl

[30] **V G Turaev**, *The Conway and Kauffman modules of a solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. 167 (1988) 79–89, 190 MR Zbl

[31] **V G Turaev**, *The Yang–Baxter equation and invariants of links*, Invent. Math. 92 (1988) 527–553 MR Zbl

[32] **V G Turaev**, *Quantum invariants of knots and* 3–*manifolds*, De Gruyter Studies in Mathematics 18, de Gruyter, Berlin (1994) MR Zbl

[33] **R van der Veen**, *Proof of the volume conjecture for Whitehead chains*, Acta Math. Vietnam. 33 (2008) 421–431 MR Zbl

[34] **R van der Veen**, *The volume conjecture for augmented knotted trivalent graphs*, Algebr. Geom. Topol. 9 (2009) 691–722 MR Zbl

Department of Mathematics, The Graduate Center, CUNY
New York, NY, United States

jboninger@gradcenter.cuny.edu

ALGEBRAIC & GEOMETRIC TOPOLOGY