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# Splitting Madsen–Tillmann spectra II: The Steinberg idempotents and Whitehead conjecture

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We show that there is a splitting of the spectrum $\Sigma^{-n}D(n)$ off the Madsen–Tillmann spectrum $\mathrm{MTO}(n)=\mathrm{BO}(n)^{-\gamma_n}$ compatible with the classic splitting of $M(n)$ off $\mathrm{BO}(n)_+$, localized at the prime $p=2$. For $n=2$, together with our previous splitting result on Madsen–Tillmann spectra, this shows that MTO(2) is homotopy equivalent to $\mathrm{BSO}(3)_+\vee\Sigma^{-2}D(2)$. We also discuss its implication for characteristic classes.

55P42, 55P47, 55R40, 57R20; 55R35, 55S12, 55S15, 57N70

*Dedicated to the memory of Stephen A Mitchell*



## 1 Introduction

The Madsen–Tillmann spectrum $\mathrm{MTO}(n)$ is defined to be the Thom spectrum of the virtual bundle $-\gamma_n$ over $\mathrm{BO}(n)$, where $\gamma_n$ is the universal $n$–plane bundle; see Galatius, Tillmann, Madsen and Weiss [4] — see also Galatius and Randal-Williams [3, Section 1.1.2] for the general construction of Madsen–Tillman spectra. It is known that these spectra filter the spectrum MO; i.e. there is a sequence

$$S^0=\mathrm{MTO}(0)\to\Sigma\mathrm{MTO}(1)\to\cdots\to\Sigma^{n-1}\mathrm{MTO}(n-1)\xrightarrow{\iota_n}\Sigma^n\mathrm{MTO}(n)\to\cdots,\tag{1}$$

where $\iota_n$ is induced by the inclusion $O(n-1) \subset O(n)$, with the property that

$$\operatorname{hocolim} \Sigma^n \mathrm{MTO}(n) \cong \mathrm{MO}$$

[4, remark after (3.4)].[1] Furthermore, the cofiber of the successive stages is homotopy equivalent to $\mathrm{BO}(n)_+$; i.e. we have a cofibration sequence

$$(2) \qquad \cdots \to \Sigma^{-1}\mathrm{MTO}(n-1) \to \mathrm{MTO}(n) \xrightarrow{\omega_{O(n)}} \mathrm{BO}(n)_+ \xrightarrow{\tau} \mathrm{MTO}(n-1) \to \cdots$$

[4, (3.3)], where $\omega_{O(n)}$ is the map induced by the "embedding" of $-\gamma_n$ into the 0–dimensional trivial bundle, $X_+$ is the union of $X$ with a disjoint base point, $\tau$ is the Becker–Schultz–Mann–Miller–Miller transfer [1, Section 2; 10, 3.7] — see also [6, Section 2.3] — associated to the inclusion $O(n-1) \subset O(n)$. In other words, the spectrum MO can be built up from pieces $\mathrm{BO}(n)_+$.

We have shown in our previous work that localized away from 2, $\mathrm{MTO}(2n) \simeq \mathrm{BO}(2n)_+$ and $\mathrm{MTO}(2n+1) \simeq *$ for all $n \geq 0$ [6, Theorem 1.1.B], reducing essentially the study of $\mathrm{MTO}(n)$'s to 2–local problems. Thus we will work at the prime $p = 2$. So throughout the paper homology and cohomology are taken with $\mathbb{Z}/2$ coefficients unless otherwise stated. We work most of the time in the 2–local stable homotopy category whose objects are 2–local spectra and morphisms are homotopy classes of maps of spectra; consequently, by commutative we mean homotopy commutative. We identify a spectrum with its 2–localization. We note that when both sides of a morphism in this category are of finite type then inducing an isomorphism in $\mathbb{Z}/2$–cohomology implies an isomorphism of 2–local spectra; we shall use this reasoning freely throughout the paper. We identify a (pointed) space $X$ with its suspension spectrum $\Sigma^\infty X$ unless otherwise stated. In the literature, sometimes a space $X$ is identified with $\Sigma^\infty X_+$, which explains notational discrepancies the reader may find between the current paper and results we quote. We use the same letter to denote a map $f\colon X \to Y$ and its suspensions $f\colon \Sigma^k X \to \Sigma^k Y$ with $k \in \mathbb{Z}$. For a spectrum $E$, we shall write $\Omega^\infty E = \operatorname{colim} \Omega^i E_i$ for the infinite loop space associated to $E$ and $\Omega_0^\infty E$ denotes it base point component corresponding to $0 \in \pi_0 E$, noting that if $E$ is 0–connected then $\Omega^\infty E = \Omega_0^\infty E$. For a pointed space $X$ the standard notations $QX = \Omega^\infty(\Sigma^\infty X)$ and $Q_0 X = \Omega_0^\infty(\Sigma^\infty X)$ will be used.

[1] In [4] this is said to be the colimit. The use of the word colimit can be justified by the fact that this is actually the colimit on the level of underlying point set at each $n$ if one considers spectrum $X$ as a collection of spaces $X_n$ and structure maps $\Sigma X_n \to X_{n+1}$. However, this is clearly not the colimit in the category of spectra, so we avoid the use of this term.

At the prime 2, Randal-Williams computed $H_*(\Omega_0^\infty \mathrm{MTO}(i))$ for $i=1$ and 2 [16, Theorems A and B]. Combining the two theorems, we get an exact sequence of Hopf algebras

$$H_*(Q_0\mathrm{BO}(2)_+) \to H_*(Q_0\mathrm{BO}(1)_+) \to H_*(Q_0\mathrm{BO}(0)_+) \to \mathbb{Z}/2, \tag{3}$$

where the (Hopf) kernel of the first two maps are isomorphic to $H_*(\Omega_0^\infty \mathrm{MTO}(i))$ for $i=2$ and 1, respectively. Thus a natural question to ask was whether this exact sequence could be extended further to the left with $H_*(\Omega_0^\infty \mathrm{MTO}(i))$ isomorphic to the kernel of each stage. We showed that this was impossible in [6, Proposition 1.11]. So a new question to ask, then, is to what extent we can generalize [16, Theorems A and B]. This question leads to a search for another sequence of spectra with the beginning as in (1). It turns out that there indeed is such a sequence, well known to stable homotopy theorists. For a *space* $X$, denote by $\mathrm{Sp}^k(X)$ the $k^{\text{th}}$ symmetric product of $X$, that is the quotient of $X^k$ by the obvious action of the symmetric group $\Sigma_k$. It is easy to show that this induces a functor in the stable category which we still denote by $\mathrm{Sp}^k$. Define the spectrum $D(n)$ as the cofiber of the diagonal map $\mathrm{Sp}^{2^{n-1}} S^0 \to \mathrm{Sp}^{2^n} S^0$; see Mitchell and Priddy [14, Section 4.2]. We have $D(0)=S^0$ and $D(1)\cong \Sigma\mathrm{MTO}(1)$ [14, Proposition 4.4]. Furthermore, Mitchell and Priddy defined a map $\iota_n\colon D(n-1)\to D(n)$ [14, Proposition 4.3]; thus we get a sequence

$$S^0 = D(0) \to D(1) \to \cdots \to D(n-1) \xrightarrow{\iota_n} D(n) \to \cdots . \tag{4}$$

Taking the cohomology, this sequence realizes the length filtration of the Steenrod algebra $\mathcal{A}$ [14, Proposition 4.3]. That is, we have isomorphisms

$$H^*(D(n)) \cong \mathcal{A}/G_n, \quad \text{where } G_n \text{ is the span of } \mathrm{Sq}^I,\ I \text{ is admissible and } l(I) > n. \tag{5}$$

We note that $G_n$ happens to be a left $\mathcal{A}$–ideal, so that this isomorphism is as $\mathcal{A}$ modules. It happens that this cohomological property characterizes the sequence of spectra $D(n)$ [5, Corollary 1.4.1]. Of course, as an immediate consequence of (5), we see that $\operatorname{hocolim} D(n) \cong H\mathbb{Z}/2$.

On the other hand, the spectrum $\mathrm{BO}(1)_+^{\times n}$ admits a natural (left) $\mathrm{Gl}_n(\mathbb{Z}/2)$–action. Thus the Steinberg idempotent $e_n \in \mathbb{Z}/2[\mathrm{Gl}_n(\mathbb{Z}/2)]$ [14, Definition 2.2] and its conjugate $e_n'$ [14, the sentence above Proposition 2.6] give rise to a splitting of $\mathrm{BO}(1)_+^{\times n}$ and we have $M(n) \simeq e_n\mathrm{BO}(1)_+^{\times n} \simeq e_n'\mathrm{BO}(1)_+^{\times n}$ [14, Theorem 5.1]. Moreover, through the Becker–Gottlieb transfer map, this splitting gives rise to a splitting of $M(n)$ off $\mathrm{BO}(n)_+$. We will review this splitting in more details in Section 2. The spectra $M(n)$'s and $D(n)$'s are related by the cofibration sequences [14]

$$\cdots \to \Sigma^{n-1} M(n) \to D(n-1) \to D(n) \to \Sigma^n M(n) \to \cdots. \tag{6}$$

Thus one can say that MO can be constructed with $\mathrm{BO}(n)_+$'s as building blocks, whereas $H\mathbb{Z}/2$ can be constructed with $M(n)$'s as building blocks. Furthermore, $H\mathbb{Z}/2$ and $M(n)$'s split off MO and $\mathrm{BO}(n)_+$'s, respectively. It is then natural to ask whether one can split intermediate stages as well. The purpose of this paper is to answer affirmatively to this question, and discuss some consequences, including an answer to the question on generalization of the exact sequence (3). We have the following, the main results of this paper.

**Theorem 1.1** *For each $n$, the spectrum $D(n)$ splits off $\Sigma^n \mathrm{MTO}(n)$.*

An immediate consequence of Theorem 1.1 is the following.

**Corollary 1.2** *$H_*(\Omega^\infty \Sigma^{-n} D(n))$ splits off $H_*(\Omega^\infty \mathrm{MTO}(n))$ as a Hopf algebra.*

Thus the "correct way to extend" the exact sequence (3) is just the following standard fact.

**Proposition 1.3** (Kuhn and Priddy [8]) *The sequence of Hopf algebras*

$$\cdots \to H_*(\Omega_0^\infty M(n)) \to H_*(\Omega_0^\infty M(n-1)) \to \cdots$$
$$\cdots \to H_*(\Omega_0^\infty M(2)) \to H_*(Q_0 B\mathbb{Z}/2_+) \to H_*(Q_0 S^0) \to \mathbb{Z}/2$$

*is exact. Furthermore, the image of $H_*(\Omega_0^\infty M(n)) \to H_*(\Omega_0^\infty M(n-1))$ is isomorphic to $H_*(\Omega_0^\infty \Sigma^{-n+1} D(n-1))$.*

As $D(0) \cong S^0$, $\Sigma^{-1} D(1) \cong \mathrm{MTO}(1)$, and $M(1) \cong \mathrm{BO}(1)_+$, combined with the $n = 2$ case of Theorem 1.1, we recover Theorems A and B of [16]. Of course, the cohomology being dual of homology, the exact sequences above give some information on certain characteristic classes. More precisely, recall from [6; 16] (with correction from Randal-Williams, via personal communication):

**Definition 1.4** A universally defined characteristic class in $H^*(\Omega^\infty \mathrm{MTO}(n))$ is an element in the subalgebra generated by the image of

$$H^*(\mathrm{BO}(n)) \xrightarrow{\sigma^{\infty *}} H^*(Q\mathrm{BO}(n)_+) \xrightarrow{(\Omega^\infty \omega_{O(n)})^*} H^*(\Omega^\infty \mathrm{MTO}(n)).$$

We denote by $\mu_{i_1,\dots,i_n} = (\Omega^\infty \omega_{O(n)})^*(\sigma^{\infty *}(\sigma_1^{i_1}, \dots, \sigma_n^{i_n}))$, where

$$H^*(\mathrm{BO}(n)) \cong \mathbb{Z}/2[\sigma_1, \dots, \sigma_n]$$

and $\sigma^{\infty *}$ denotes the cohomology suspension.

We note that in the definitions in [16] and [6], only basepoint components of the infinite loop spaces was considered. However, this has the effect of missing out nontrivial 0–dimensional classes as also confirmed by Randal-Williams (personal communication). Therefore, we have removed the restriction to the basepoint component in our definition. We note that [6, Theorem 1.9] remains valid as is stated.[2]

In [6], we used the summand $\mathrm{BSO}(2n+1)_+$ that split off $\mathrm{MTO}(2n)$ to show that some of these classes remain algebraically independent. Here we use the splitting of $D(n)$ off $\mathrm{MTO}(n)$ to show that there are "linear" relations corresponding to elements of $H^*(M(n))$, and that in the case of dimension 2, these relations together with the ones derived from the action of top Steenrod squares are the only relations. More precisely, we will show:

**Theorem 1.5** (i) *In $H^*(\Omega^\infty \mathrm{MTO}(n))$, we have relations*

$$(\Omega^\infty \omega_{O(n)})^*(\sigma^{\infty *}(x)) = 0 \quad \text{for } x \in H^*(M(n)) \subset H^*(\mathrm{BO}(n)).$$

(ii) *For $n = 2$, the only relations among the $\mu_{i,j}$ are those above, and $\mu_{2i,2j} = \mu_{i,j}^2$.*

(iii) *Again for $n = 2$, the subalgebra of universally defined characteristic classes in $H^*(\Omega^\infty \mathrm{MTO}(2))$ is the polynomial algebra generated by $\nu_{i,j}$'s with $i$ and $j$ odd, where $\nu_{i,j}$ is defined in [6], tensored with the boolean algebra $\mathbb{Z}/2[\mu_{0,0}]/(\mu_{0,0}^2 - \mu_{0,0})$.*

We will give a more precise description of the inclusion $H^*(M(n)) \subset H^*(\mathrm{BO}(n))$ in Proposition 5.7.

The paper is organized as follows. In Section 2 we recall the splitting related to the Steinberg idempotents and construct a map from $D(n)$ to $\Sigma^n \mathrm{MTO}(n)$ for each $n$. In Section 3, we recall relevant results from [8] and construct a map going the other way around. In Section 4 we study the composition and show that we indeed have a splitting. In Section 5 we discuss the consequences in homology of infinite loop spaces.

[2] As a matter of fact, it was assumed implicitly that the sequence $I$ was nonzero, due to the obvious relation $\mu_{0,0} = 0$ with the "old" definition. This relation holds no longer. One can easily adapt the proof of [6, Theorem 1.9] to the "new" definition.

Most of the current paper is independent of the results from the previous one, except for Theorem 1.5(ii), (iii) and the contents of Section 4.2. Thus, the current paper can be read separately from [6]. A word is due on the way some of proofs are written. In some places, the reader familiar with works we quote may find that our proofs are somewhat going backward. For example, we deduce Proposition 5.1 from Theorem 3.7, but as a matter of fact in [8, Section 5], a large part of the latter was proved as a main ingredient of the proof of the former. This is our deliberate choice; we preferred referring the readers to statements that are ready available to be quoted, rather than letting them look for details of proofs, or reproducing them ourselves.

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## 2 Some splitting derived from Steinberg idempotents

In this section we recall from [14] and [17] the splitting related to Steinberg idempotents.

Let $X$ be a spectrum, $e \in [X, X]$ an idempotent, i.e. a map such that $e \circ e = e \in [X, X]$. Note that $[X, X]$ has a natural ring structure where the multiplication is given by the composition, and $e$ is an idempotent in terms of ring theory. Denote by $eX$ the homotopy colimit $X \xrightarrow{e} X \xrightarrow{e} \cdots$. Then we have a splitting

$$X \simeq eX \vee (1-e)X.$$

Furthermore, if we still denote by $e$ the induced map in (co)homology, we get

$$H_*(eX) \cong eH_*(X), \quad H^*(eX) \cong H^*(X)e.$$

We are particularly interested in the case of idempotents arising from a group action on spectra. That is, let $G$ be a group acting on the spectrum $X$ from the left. There are several different notions of group action on spectra, here we can take any of them: all we need is a group homomorphism $G \to \mathrm{Aut}(X)$ where $\mathrm{Aut}(X)$ is the group consisting of invertible elements in $[X, X]$. This group homomorphism extends to a ring homomorphism $\mathbb{Z}_{(2)}[G] \to [X, X]$, thus sending an idempotent to an idempotent. We see that an idempotent in the group ring $\mathbb{Z}_{(2)}[G]$ gives rise to a splitting of spectra on which $G$ acts. Actually the theory of lifting idempotents allows us to settle for something less, which is one of the reasons why completion is crucial in the theory of splitting, but we will not need this for our purpose.

Now, let $G = \mathrm{Gl}_n(\mathbb{Z}/2)$. Its group-ring $\mathbb{Z}_{(2)}[\mathrm{Gl}_n(\mathbb{Z}/2)]$ contains well-known Steinberg idempotents $e_n$ and $e'_n$ defined by

$$e_n = \frac{1}{q_n} \sum_{g \in B_n} g \sum_{\sigma \in \Sigma_n} (-1)^{\mathrm{sgn}(\sigma)} \sigma, \quad e'_n = \frac{1}{q_n} \sum_{\sigma \in \Sigma_n} (-1)^{\mathrm{sgn}(\sigma)} \sigma \sum_{g \in B_n} g, \tag{7}$$

where $B_n$ denotes the subgroup consisting of upper triangular matrices, $\Sigma_n$ denotes the subgroup of permutation matrices, and $q_n$ is the index of $B_n$ in $G$.

**Remark 2.1** (i) Traditionally we consider the above elements as idempotents modulo 2, and use the lifting theory. However, as was noticed in [14, proof of Proposition 2.6] (see also [8, page 462]), $e_n$ and $e'_n$ actually are conjugate idempotents, and they can even be defined in $\mathbb{Z}_{(2)}[\mathrm{Gl}_n(\mathbb{Z}/2)]$. Let's note that working with spectra completed at 2 has some advantages, e.g. we get a better control over maps among spectra [13, Corollary 1.4(b)]. However, as far as our current work is concerned, localization is sufficient.

(ii) We use the additive structure in $[X, X]$ to extend the $G$–action on $X$. Thus even in the case when $G$ acts on the space $X$ (via maps of spaces, not just maps of spectra), the idempotents are not necessarily maps of spaces. However, in this case they can be realized as self maps of the space $\Sigma X$. In other words, the spectrum $\Sigma eX$ is a suspension spectrum.

Write $\Delta_n$ for $O(1)^n$. The identification of $O(1)$ with $\mathbb{Z}/2$ gives a natural action of $\mathrm{Gl}_n(\mathbb{Z}/2)$ on $B\Delta_n$, thus on $B\Delta_{n+}$, and we have:

**Definition 2.2** We define the spectra $M(n)$ by

$$M(n) \cong e_n B\Delta_{n+}.$$

**Remark 2.3** Originally $M(n)$ was defined as $\Sigma^{-n} D(n)/D(n-1)$, but in terms of [14, Theorem A] this is equivalent, and in recent literature we encounter this definition more often.

Now, results in representation theory imply that for any $\mathbb{Z}/2[\mathrm{Gl}_n(\mathbb{Z}/2)]$–module $W$, we have an isomorphism $We'_n \cong We_n$ induced by $\sum_{\sigma\in\Sigma_n}\sigma$ [14, Proposition 2.6(b)]. On the other hand, the composition

$$B\Delta_{n+} \xrightarrow{Bi} \mathrm{BO}(n)_+ \xrightarrow{\mathrm{Tr}_{Bi}} B\Delta_{n+}$$

induces $\sum_{\sigma\in\Sigma_n}\sigma$ in $H^*(B\Delta_n)$; that is, the composition

$$B\Delta_{n+} \xrightarrow{e_n} B\Delta_{n+} \to \mathrm{BO}(n)_+ \to B\Delta_{n+} \tag{8}$$

induces in the cohomology $e'_n$. Therefore

$$e_n B\Delta_{n+} \to B\Delta_{n+} \to \mathrm{BO}(n)_+ \to B\Delta_{n+} \to e'_n B\Delta_{n+}$$

induces an isomorphism in mod 2 cohomology. In other words:

**Theorem 2.4** [14, Theorem C] *$M(n)$ splits off* $\mathrm{BO}(n)_+$.

Of course, cohomology of a space is related to that of Thom spectra of bundles over it via Thom isomorphisms, so we can "Thomify" all of the above. More precisely, let $\rho_n$ be the reduced regular representation of $\Delta_n$ and $\gamma=\rho_1^n$ its canonical representation. The canonical representation is the direct sum of $n$ distinct projections, while the regular representation is the direct sum of all possible 1–forms. As these 1–forms are tensor products of projections, we get an isomorphism of representations

$$\bigoplus_{i>0}\Lambda^i(\gamma)\cong\rho_n,$$

where $\Lambda^i(-)$ is the $i^{\text{th}}$ exterior power functor. Therefore, if we define a representation $\bar\rho_n$ of $O(n)$ by

$$\bar\rho_n=\bigoplus_{i>0}\Lambda^i(\gamma_n),$$

it restricts to $\rho_n$ over $\Delta_n\subset O(n)$. Now, if $k$ denotes an integer, $k\rho_n$ is invariant under the action of $\mathrm{Gl}_n(\mathbb{Z}/2)$; thus if $g\in\mathrm{Gl}_n(\mathbb{Z}/2)$, we have $g^*(k\rho_n)=k\rho_n$, giving rise to a Thomified map $B\Delta_n^{k\rho_n}=B\Delta_n^{g^*(k\rho_n)}\xrightarrow{\mathrm{Th}(g)}B\Delta_n^{k\rho_n}$. Here, and throughout the paper, given a (virtual) vector bundle $\xi\to X$, we shall write $X^\xi$ for its Thom spectrum. This furnishes the Thom spectrum $B\Delta_n^{k\rho_n}$ with a $\mathrm{Gl}_n(\mathbb{Z}/2)$–action. When $k$ is negative, slightly more careful arguments are needed, but this is taken care of by [17]. Thus we can split it using the Steinberg idempotents $e_n$ and $e'_n$. Then we get a sequence of maps

$${B\Delta_n}^{k\rho_n}\xrightarrow{e_n}B\Delta_n^{k\rho_n}\to\mathrm{BO}(n)^{k\bar\rho_n}\to B\Delta_n^{k\rho_n},$$

where the last map is the twisted Becker–Gottlieb transfer [6, Theorem 1.1(1)]. As everything in sight is compatible with the Thom isomorphism, the effect of these maps in the cohomology can be deduced from those in the sequence (8). Noting that $e'_n$ is also a sum of Thomified maps, we see that this composition induces $e'_n$ in cohomology. Thus, as in Theorem 2.4:

**Theorem 2.5** *$e_n B\Delta_n^{k\rho_n}$ splits off $\mathrm{BO}(n)^{k\bar{\rho}_n}$.*

The spectra $e_n B\Delta_n^{k\rho_n}$'s are studied notably in [17] where it is called $M(n)_k$; when $k = 0$, we recover Theorem 2.4. The case $k = -1$ also interests us for the following result, which is implicit in [17]:

**Theorem 2.6**
$$e_n B\Delta_n^{-\rho_n} \cong \Sigma^{-n} D(n).$$

**Proof** This seems to be well known, but as we haven't found it spelled out in literature, for the sake of reference we give a proof here. It suffices to note that $\mathbf{R}(n)e_n$ in [17, Theorem 4.1.1(1)] is same as $\mathbf{M}(n)_{-1}$ in [17, Proposition 4.1.6], which is the cohomology of $M(n)_{-1}$ (cf. [17, page 386], whereas by Theorem 5.8 and Lemma 5.6 of [14] it is isomorphic to the cohomology of $\Sigma^{-n} D(n)$. □

Combining the theorems above shows that $\Sigma^{-n} D(n)$ splits off $\mathrm{BO}(n)^{-\bar{\rho}_n}$. As the inclusion of the representation $\gamma_n \subset \bar{\rho}_n$ induces a map of Thom spectra

$$\mathrm{BO}(n)^{-\bar{\rho}_n} \to \mathrm{BO}(n)^{-\gamma_n} = \mathrm{MTO}(n),$$

we get a map $\beta_n \colon \Sigma^{-n} D(n) \to \mathrm{MTO}(n)$. Or, equivalently, we can construct the map as the composition

$$\Sigma^{-n} D(n) \to B\Delta_n^{-\rho_n} \to B\Delta_n^{-\gamma} \to \mathrm{BO}(n)^{-\gamma_n} = \mathrm{MTO}(n).$$

We will denote the resulting map by $\beta_n$. Here, the map $B\Delta_n^{-\rho_n} \to B\Delta_n^{-\gamma}$ is induced by the inclusion of bundles $\gamma \subset \rho_n$ and the map $B\Delta_n^{-\gamma} \to \mathrm{BO}(n)^{-\gamma_n}$ is the twisted Becker–Gottlieb transfer [6, Theorem 1.1(1)], noting that $\gamma_n|_{\Delta_n} = \gamma$.

# 3 Maps from $\mathrm{MTO}(n)$ to $\Sigma^{-n} D(n)$

## 3.1 Exact sequences of spectra and the Whitehead conjecture

In this section we use results from [8] to construct maps from $\Sigma^n\mathrm{MTO}(n)$'s to $D(n)$'s. We start by fixing terminology.

**Definition 3.1** (i) A filtered spectrum $(X, F_*X, \iota_*)$ is a sequence of spectra $F_*X$

(9) $$F_0X \xrightarrow{\iota_0} F_1X \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{n-1}} F_nX \xrightarrow{\iota_n} F_{n+1}X \xrightarrow{\iota_{n+1}} \cdots$$

with a homotopy equivalence hocolim $F_nX \simeq X$. Usually $\iota_*$ is clear from the context, and $X$ is determined by $F_*X$'s, so we simply refer to it as $F_*X$. To distinguish with individual spectra, we also write $(F_*X, * \geq 0)$

(ii) A map of filtered spectra $f_*$ from $F_*X$ to $F_*Y$ is a collection of maps $f_n\colon F_nX \to F_nY$ that makes the squares

$$\begin{array}{ccc} F_nX & \longrightarrow & F_{n+1}X \\ \downarrow & & \downarrow \\ F_nY & \longrightarrow & F_{n+1}Y \end{array}$$

commutative.

Note that we don't require any condition that would be a counterpart of the injectivity on $\iota_n$'s here.

**Definition 3.2** (i) By a chain complex of spectra $(C_n, d_n)$ we understand a sequence of spectra $C_n$ with maps $d_{n-1}\colon C_n \to C_{n-1}$ such that the composition $C_{n+1} \to C_n \to C_{n-1}$ is null for all $n$. By a map $f$ of chain complexes of spectra $(C_n, d_n^C) \to (C'_n, d_n^{C'})$ we mean a collection of maps $f_n\colon C_n \to C'_n$ such that $f_n \circ d_n^C = d_n^{C'} \circ f_{n+1}$. Furthermore, if we have a map $\epsilon\colon C_0 \to E_{-1}$ with $\epsilon \circ d_0 = 0$, we say that the complex is augmented over $E_{-1}$.

(ii) Let $F_*X$ be a filtered spectrum. Define its *associated graded* complex $\mathrm{Gr}_\bullet(F_*X)$ by $\mathrm{Gr}_0(F_*X) = F_0X$ and $\mathrm{Gr}_i(F_*X) = \Sigma^{-i}\,\mathrm{cofib}(F_{i-1}X \to F_iX)$. Then we can compose the canonical maps $\mathrm{Gr}_i(F_*(X)) \to \Sigma^{-i}F_{i-1}X \to \mathrm{Gr}_{i-1}(F_*(X))$ to define a map $d_{i-1}^{\mathrm{Gr}_\bullet(F_*X)}$. As a matter of the fact, the composition $d_{i-1}^{\mathrm{Gr}_\bullet(F_*X)} \circ d_i^{\mathrm{Gr}_\bullet(F_*X)}$ factors through the composition $F_iX \to \mathrm{Gr}_i F_*X \to \Sigma F_{i-1}X$, which is trivial.

**Remark 3.3** Our notion of complex is more general than that in [7]. The complexes dealt with in [7] are the ones that arise as associated graded complexes of filtered spectra.

**Example 3.4** (i) Let $F_nX = D(n)$. Then the associated graded complex $\mathrm{Gr}_\bullet(F_*X)$ is

$$\cdots \to M(n+1) \xrightarrow{\delta_n} M(n) \to \cdots \to M(0)$$

considered in [8, Corollary 1.2].

(ii) Let $F_nY = \Sigma^n \mathrm{MTO}(n)$. Then the associated graded complex $\mathrm{Gr}_\bullet(F_*X)$ is given by $(\mathrm{BO}(n)_+, \mathrm{tr})$, where tr is the Becker–Gottlieb transfer associated to the inclusion

$O(n-1) \subset O(n)$, as the Becker–Gottlieb transfer $\mathrm{BO}(n)_+ \to \mathrm{BO}(n-1)_+$ factors as $\mathrm{BO}(n)_+ \xrightarrow{\tau} \mathrm{MTO}(n-1) \xrightarrow{\omega_{O(n-1)}} \mathrm{BO}(n-1)_+$ [6, Proposition 2.3]. We can also see that $(\mathrm{BO}(n)_+, \mathrm{tr})$ is a complex directly as follows: $O(n) \subset O(n) \times O(2) \subset O(n+2)$; thus $O(2) \subset N_{O(n+2)}(O(n))$. So the composition of the transfer associated to $O(n) \subset O(n+1)$ and that associated to $O(n+1) \subset O(n+2)$, which is the transfer associated to $O(n) \subset O(n+2)$, is trivial by [9, Chapter 4, Lemma 2.12]. Moreover, the complex of free spectra $(\mathrm{BO}(n)_+, \mathrm{tr})$ is augmented over $H\mathbb{Z}/2$ since the composition

$$\mathrm{BO}(1)_+ \to \mathrm{BO}(0)_+ \to H\mathbb{Z}/2$$

is trivial. This is just another way of saying that the transfer in $\mathbb{Z}/2$–cohomology $H^*(\mathrm{BO}(0)_+; \mathbb{Z}/2) \to H^*(\mathrm{BO}(1)_+; \mathbb{Z}/2)$ is trivial.

**Definition 3.5** [8] (i) A fibration sequence of spectra $F \to X \xrightarrow{f} Y$ is called exact if there exists a map of spaces (i.e. not a map between their suspension spectra) $g\colon \Omega^\infty Y \to \Omega^\infty X$ such that $\Omega^\infty f \circ g \simeq \mathrm{id}$.

(ii) A chain complex of spectra $\cdots \to X_n \to \cdots X_1 \to X_0 \to E_{-1}$ augmented over $E_{-1}$ is called exact if for each $n \geq 0$, $E_n \to X_n \to E_{n-1}$ is exact, where $E_n$ is inductively defined as the fiber of the map $X_n \to E_{n-1}$. Note that by the exactness of $E_{n-2} \to X_{n-2} \to E_{n-3}$, $[X_n, E_{n-2}]$ injects to $[X_n, X_{n-2}]$, so the triviality of the composition $X_n \to X_{n-1} \to X_{n-2}$ implies that of the composition $X_n \to X_{n-1} \to E_{n-2}$.

(iii) A spectrum is said to be projective if it is a summand of a suspension spectrum.

The category of spectra being a triangulated category instead of an abelian category, we have some complication here. The notion of exactness with three terms is more or less a counterpart of a split short exactness in abelian categories. The use of this seemingly too strong condition is motivated by the following fact. By definition, an exact sequence of spectra yields an exact sequence of abelian groups upon applying $[Y, -]$ for a suspension spectrum $Y$, or a spectrum which is a summand of a suspension spectrum. Thus one can regard suspension spectra as free objects, summands of suspension spectra as projective objects, and carry out homological algebra in the category of spectra. This idea was developed further in [7]. For example, we get the following:

**Proposition 3.6** *Let $(P_\bullet, d_\bullet)$ be a chain complex of projective $R$–modules with an augmentation $P_0 \to A$, and $(A_\bullet, d_\bullet)$ be a projective resolution of $A$. Then we get a chain map from $(P_\bullet, d_\bullet)$ to $(A_\bullet, d_\bullet)$.*

**Proof** This is just [7, Proposition 2.11] applied to $\mathrm{id}\colon A \to A$. □

Note that the proof of [7, Proposition 2.11] is still valid with our broader notion of complexes. However, for readers who would rather not go through the proof, we also remark that we will be using this later only when $(P_\bullet, d_\bullet)$ is of the form $(\mathrm{Gr}_n(X), d_n)$ for a filtered spectrum $F_n(X)$, which is also a complex in the sense of [7].

Now, we are ready to quote from [8]:

**Theorem 3.7** (mod 2 Whitehead conjecture [8, Corollary 1.2]) (i) *The sequence of Example 3.4(i),*

$$\cdots \xrightarrow{\delta_{k+1}} M(k+1) \xrightarrow{\delta_k} M(k) \to \cdots \to M(1) \xrightarrow{\delta_0} M(0) \xrightarrow{\epsilon} H\mathbb{Z}/2, \tag{10}$$

*is exact.*

(ii) *Denote by $E_k$ the fiber of the map $\Sigma^{-k} D(k) \to \Sigma^{-k} H\mathbb{Z}/2$. Then the above sequence can be obtained splicing together short exact sequences $E_k \to M(k) \to E_{k-1}$.*

**Remark 3.8** It is easy to see that our definition of $E_k$ agrees with that in [8].

## 3.2 Maps into $(D(n), n \geq 0)$

With the above preparation, we are ready to prove the following.

**Theorem 3.9** *Let $(X, F_* X, \iota)$ be a filtered spectrum such that*

(i) *$H_*(\iota_n)$ is injective for all $n$, and*

(ii) *$\mathrm{Gr}_n(F_* X)$ is a suspension spectrum.*

*Then any map of spectra $F_0(X) \to S^0$ extends to a map of filtered spectra $F_*(X)$ to $D(*)$.*

**Proof** First note that condition (i) implies that in the associated graded complex, the differential induces trivial map in cohomology. In particular, one can augment it by any map from $F_0(X) \to H\mathbb{Z}/2$. Let's do so by using the composition of the given map $F_0(X) \to S^0$ and the augmentation in the $(M(n), \delta_n)$, $S^0 \to H\mathbb{Z}/2$. Since $\mathrm{Gr}_0(F_* X) = F_0 X$, this yields a map $\mathrm{Gr}_0(F_* X) \to H\mathbb{Z}/2$. By Theorem 3.7, the augmented complex $(M(n), \delta_n)$ is a projective resolution of $H\mathbb{Z}/2$, so we can apply Proposition 3.6 to obtain a map of complex of spectra $f$ from $(\mathrm{Gr}_n(X), d_n)$ to $(M(n), \delta_n)$. From the proof of [7, Proposition 2.11], we see that we can choose $f_0$ to be the prescribed map in the statement of the theorem.

Thus we have found maps $f_n$ making the square

(11)
$$\begin{array}{ccc} \mathrm{Gr}_n(X) & \xrightarrow{d_{n-1}} & \mathrm{Gr}_{n-1}(X) \\ \downarrow f_n & & \downarrow f_{n-1} \\ M(n) & \xrightarrow{d_{n-1}} & M(n-1) \end{array}$$

commutative. Next we will show that there exists a map $\alpha_n \colon \Sigma^{-n} F_n(X) \to \Sigma^{-n} D(n)$ which makes the diagram

(12)
$$\begin{array}{ccc} \Sigma^{-n} F_n(X) & \longrightarrow & \mathrm{Gr}_n(X) \\ \downarrow \alpha_n & & \downarrow f_n \\ \Sigma^{-n} D(n) & \longrightarrow & M(n) \end{array}$$

commutative for each $n$. We proceed by induction on $n$. The case $n = 0$ is trivial. Suppose that we have constructed such $\alpha_{n-1}$. Consider the diagram

(13)
$$\begin{array}{ccc} \mathrm{Gr}_n(X) & \longrightarrow & \Sigma^{1-n} F_{n-1}(X) \\ \downarrow f_n & & \downarrow \alpha_{n-1} \\ M(n) & \longrightarrow & \Sigma^{1-n} D(n-1) \end{array}$$

By the definition of associated graded complex, the fiber of the top row is $\Sigma^{-n} F_n(X)$ whereas by the cofibration (6), that of the bottom row is $\Sigma^{-n} D(n)$. Thus if we can show the commutativity of the diagram (13), then we can define the map $\alpha_n$ making the diagram (12) commute. Note that the two horizontal maps induce trivial maps in cohomology, which implies that the two compositions from the top left corner to the bottom right corner factor through $E_{n-1}$ where $E_i$ is the same as in Theorem 3.7. Thus it suffices to show that the lifts in $[\mathrm{Gr}_n(X), E_{n-1}]$ of the two maps agree. However, by Theorem 3.7(ii), $[\mathrm{Gr}_n(X), E_{n-1}]$ injects to $[\mathrm{Gr}_n(X), M(n-1)]$. Thus it suffices to show that the two maps agree after composition with the map $\Sigma^{1-n} D(n-1) \to M(n-1)$. Now, consider the diagram

$$\begin{array}{ccccc} \mathrm{Gr}_n(X) & \longrightarrow & \Sigma^{1-n} F_{n-1}(X) & \longrightarrow & \mathrm{Gr}_{n-1}(X) \\ \downarrow f_n & & \downarrow \alpha_{n-1} & & \downarrow f_{n-1} \\ M(n) & \longrightarrow & \Sigma^{1-n} D(n-1) & \longrightarrow & M(n-1) \end{array}$$

The right square is commutative by the inductive hypothesis. But we chose our maps $f_n$ so that the big square commutes.

The proof is complete now, noting that, by considering the cofibers of the rows in the diagram (12), we see that the family $\{\alpha_n\}$ forms a map of filtered spectra. □

**Corollary 3.10** *There exists a map of filtered spectra*

$$\alpha_* \colon (\mathrm{MO}, \Sigma^n \mathrm{MTO}(n)) \to (H\mathbb{Z}/2, D(n))$$

*which extends the identity* $\mathrm{MTO}(0) = S^0 \to S^0$.

**Proof** It follows from Thom isomorphism that the map

$$\iota_{n-1} \colon \Sigma^{n-1}\mathrm{MTO}(n-1) \to \Sigma^n \mathrm{MTO}(n)$$

induces a monomorphism in $\mathbb{Z}/2$–homology. Moreover, for the associated graded spectrum of $(Y, F_*Y) = (\mathrm{MO}, \Sigma^n\mathrm{MTO}(n))$ we have $\mathrm{Gr}_n(\mathrm{MO}) = \mathrm{BO}(n)_+$ (Example 3.4(ii)). We also have $F_0 Y = \mathrm{MTO}(0) = S^0$ and we take the identity $S^0 \to S^0$ as our map $F(0) \to D(0) = S^0$. The result now follows from Theorem 3.9. □

# 4 The splitting

## 4.1 Proof of Theorem 1.1

We have constructed the maps $\beta_n$ in Section 2, and the maps $\alpha_n$ in Section 3. All that remains is to show that the composition $\alpha_n \circ \beta_n$ induces an isomorphism in 2–local cohomology. As $D(n)$ is of finite type, it is enough to show that it induces an isomorphism in mod 2 cohomology. Since a map of spectra induces a map of modules over Steenrod algebra in cohomology, and $H^*(D(n))$ is generated by the bottom class as a module over Steenrod algebra (5), it suffices to show that

$$H^{-n}(\alpha_n \circ \beta_n) = H^{-n}(\beta_n) \circ H^{-n}(\alpha_n)$$

is an isomorphism. Since $\alpha_0$ is just the equivalence $\mathrm{MTS}(0) \cong S^0 \cong D(0)$, $H^0(\alpha^0)$ is an isomorphism. As the family $\{\alpha_n\}$ forms a map of filtered spectra, we see that $H^{-n}(\alpha_n)$ is an isomorphism for all $n \geq 0$.

Unfortunately we have been unable to prove the fact that the family of maps going the other way, $\{\beta_n\}$, forms a map of filtered spectra.[3] So we honestly compute $H^{-n}(\beta_n)$ for all $n$. We have

$$H^*(\mathrm{BO}(n)) \cong \mathbb{Z}/2[\sigma_1, \dots, \sigma_n] \subset H^*(B\Delta_n) \cong \mathbb{Z}/2[x_1, \dots, x_n],$$

[3]The claim we made in earlier versions available online on arXiv is erroneous: one of the errors is the fact that the fiber of $j_{-2}$ has positive-dimensional cells if $n > 1$.

where $\sigma_i$ denotes the $i^{\text{th}}$ elementary symmetric polynomial in $x_j$'s. Of course, the identification is made through $Bi^*$ where $i\colon \Delta_n \cong O(1)^n \subset O(n)$ is the standard inclusion. Thus, the map $B\Delta_n^{-\rho_n} \to \mathrm{MTO}(n)$ induces an inclusion

$$H^*(\mathrm{MTO}(n)) \cong \mathbb{Z}/2[\sigma_1,\dots,\sigma_n]\cdot(\sigma_n)^{-1} \subset H^*B(\Delta_n^{-\rho_n}) \cong \mathbb{Z}/2[x_1,\dots,x_n]\cdot e(\rho_n)^{-1}$$

where

$$e(\rho) = \prod_{\epsilon_i\in 0,1,\,\prod_i \epsilon_i\neq 0} \Sigma\epsilon_i x_i.$$

Here and later, for a ring $R$ and $a \in R$ nondivisor of 0, we denote $R\cdot a^{-1}$ the free $R$–module generated by $a^{-1}$ in an appropriate localization of $R$. Now, we see that the only nontrivial element of $H^{-n}(\mathrm{MTO}(n))$, $\sigma_n^{-1}$, maps to $x_1^{-1}\cdots x_n^{-1} \in H^{-n}(B\Delta_n^{-\rho_n})$. But, this class is invariant under the $e_n$–action, so it survives in $H^{-n}(\Sigma^{-n}D(n))$ by [14, the first sentence of Remark 5.12]. Thus $H^{-n}(\beta_n)$ is also an isomorphism for all $n$. This concludes the proof of Theorem 1.1.

## 4.2 Further refinements

We have shown in [6, Theorem 1.1.A] that $\mathrm{BSO}(2n+1)_+$ splits off $\mathrm{MTO}(2n)$. More precisely, we show that the composition $Bf_{2n}\circ\omega_{O(2n)}\circ\mathrm{Tr}_{Bf_{2n}}$ is a homotopy equivalence, where $f_{2n}\colon O(2n)\to \mathrm{SO}(2n+1)$ is given by $X \mapsto (\det X)(X\oplus 1)$, $\omega_{O(2n)}$ is the map of Thom spectra induced by the embedding of $-\gamma_n$ in 0, and $\mathrm{Tr}_{Bf_{2n}}$ is the associated Becker–Schultz–Mann–Miller–Miller transfer $\mathrm{BSO}(2n+1)_+ \to \mathrm{MTO}(2n)$ [10, Section 2]; see also [1, Section 4]. One may ask how this splitting interacts with the splitting of the current paper. We show that they are complementary.

**Corollary 4.1** *$\Sigma^{-2n}D(2n)\vee \mathrm{BSO}(2n+1)_+$ splits off $\mathrm{MTO}(2n)$. When $n=1$, we have a homotopy equivalence $\mathrm{MTO}(2)\cong \Sigma^{-2}D(2)\vee\mathrm{BSO}(3)_+$.*

**Proof** Consider the composition

$$H^*(\mathrm{BSO}(2n+1))\oplus H^*(\Sigma^{-2n}D(2n)) \xrightarrow{(\alpha_{2n}\vee Bf_{2n}\circ\omega_{O(2n)})^*\circ(\beta_{2n}\vee \mathrm{Tr}_{Bf_{2n}})^*} H^*(\mathrm{BSO}(2n+1))\oplus H^*(\Sigma^{-2n}D(2n)).$$

The components $H^*(\mathrm{BSO}(2n+1))\to H^*(\mathrm{BSO}(2n+1))$ and $H^*(\Sigma^{-2n}D(2n))\to H^*(\Sigma^{-2n}D(2n))$ are automorphisms by [6, Theorem 1.1.A] and Theorem 1.1, respectively. Consider now the component $H^*(\Sigma^{-2n}D(2n))\to H^*(\mathrm{BSO}(2n+1))$. This is trivial since the source is generated over the Steenrod algebra by a negative-degree element, and the target is concentrated in nonnegative degrees by (5). Thus the map $(\alpha_{2n} \vee Bf_{2n} \circ \omega_{O(2n)})^* \circ (\beta_{2n} \vee \mathrm{Tr}_{Bf_{2n}})^*$ is an automorphism. This proves the splitting for general $n$. When $n = 1$, it suffices to compare the cohomology of both sides, or, alternatively, to compare the fibrations $\mathrm{MTO}(2) \to \mathrm{BO}(2)_+ \to \mathrm{MTO}(1)$ and $\Sigma^{-2} D(2) \to M(2) \to D(1)$. Noting that $\mathrm{BO}(2)_+ \cong M(2) \vee \mathrm{BSO}(3)_+$ (cf. [15, Theorem C]), we see that $(\alpha_2 \vee Bf_2 \circ \omega_{O(2)})^*$ induces mod 2 cohomology equivalence, which implies 2–local homotopy equivalence as everything is of finite type. □

# 5 Homology of the associated infinite loop spaces

In this section, we discuss the consequences of our splitting theorem to the homology of associated infinite loop spaces.

## 5.1 Exact sequences

We start with the following refinement of Proposition 1.3.

**Proposition 5.1** *The sequence of Hopf algebras*

$$\cdots \to H_*(\Omega_0^\infty M(n)) \to H_*(\Omega_0^\infty M(n-1)) \to \cdots$$
$$\cdots \to H_*(\Omega_0^\infty M(2)) \to H_*(Q_0 B\mathbb{Z}/2_+) \to H_*(Q_0 S^0) \to \mathbb{Z}/2$$

*is exact. It gives rise to an exact sequence of graded vector spaces after taking the module of indecomposables. Moreover, the image of $H_*(\Omega_0^\infty M(n)) \to H_*(\Omega_0^\infty M(n-1))$ is isomorphic to $H_*(\Omega_0^\infty \Sigma^{1-n} D(n-1))$.*

**Proof** Suppose we have a short exact sequence of spectra $F \to X \to Y$. By the definition of the exactness, Definition 3.5, we see that the map $H_*(\Omega^\infty X) \to H_*(\Omega^\infty Y)$ is surjective. Thus by standard arguments (see e.g. [16, Section 2.6])

$$H_*(\Omega^\infty F) \to H_*(\Omega^\infty X) \to H_*(\Omega^\infty Y)$$

is short exact. Furthermore, it is clear that this short exact sequence splits as

$$H_*(\Omega_0^\infty F) \otimes k[\pi_0(F)] \to H_*(\Omega_0^\infty X) \otimes k[\pi_0(X)] \to H_*(\Omega_0^\infty Y) \otimes k[\pi_0(Y)],$$

where $k = \mathbb{Z}/2$. Noting that both in abelian categories and in the category of spectra, an exact sequence can be decomposed into a series of short exact sequences, we see that an exact sequence of spectra leads to an exact sequence of Hopf algebras by applying the functor $H_*(\Omega^\infty)$ or $H_*(\Omega_0^\infty)$.

Now, note that the $\mathrm{Gl}_n(\mathbb{Z}/2)$–action on $B\Delta_{n+}$ extends that on $B\Delta_n$. Thus it is easy to see from (7) that we have $e'_n B\Delta_n = e'_n B\Delta_{n+}$ for $n > 1$. As a matter of fact, $\mathrm{Gl}_n(\mathbb{Z}/2)$ acts trivially on $S^0 \subset B\Delta_{n+}$, so $q_n e'_n$ restricted to $S^0$ is the signed sum of 1's and $(-1)$'s which is zero. Thus for $n \geq 2$, $M(n)$ is a summand of $B\Delta_n$ (and not just a summand of $B\Delta_{n+}$), so $\Omega^\infty M(n)$ splits off $QB\Delta_n$ as infinite loop spaces. Of course, this also implies that $M(n)$ is connected for $n \geq 2$, so $\Omega^\infty_0 M(n) = \Omega^\infty M(n)$. Therefore $H_*(\Omega^\infty_0 M(n))$ splits off $H_*(QB\Delta_n)$ as Hopf algebras; in particular, the former is isomorphic to a Hopf subalgebra of the latter, which is a polynomial algebra. It is known that any Hopf subalgebra of a polynomial algebra is polynomial by the structure theorem of Hopf algebras over $\mathbb{Z}/2$ ([2, Theorem 6.1] or [11, Theorem 7.11]). So $H_*(\Omega^\infty_0 M(n))$ is also a polynomial algebra. Thus everything in the exact sequence is polynomial. As any surjective map of algebras to a polynomial algebra splits, we see that a short exact sequence of Hopf algebras involving only polynomial algebras remain exact after passing to the modules of indecomposables. Noting that an exact sequence of polynomial Hopf algebras can be obtained by splicing together short exact sequences of polynomial Hopf algebras, we can say the same about an exact sequence of Hopf algebras, not necessarily short exact.

It remains to identify the image of each map. But this follows from Theorem 3.7(ii) and the fact that the map $E_n \to \Sigma^{-n} D(n)$ induces homotopy equivalence $\Omega^\infty_0 E_n \to \Omega^\infty_0 \Sigma^{-n} D(n)$. □

**Remark 5.2** By the comments in the first paragraph of the above proof we also have an exact sequence of Hopf algebras even if we don't restrict to the base point components; that is we also have an exact sequence of Hopf algebras

$$\cdots \to H_*(\Omega^\infty M(n)) \to H_*(\Omega^\infty M(n-1)) \to \cdots$$
$$\cdots \to H_*(\Omega^\infty M(2)) \to H_*(QB\mathbb{Z}/2_+) \to H_*(QS^0) \to \mathbb{Z}/2.$$

An immediate consequence of Proposition 5.1 is:

**Corollary 5.3** *$H^*(\Omega^\infty_0 \mathrm{MTO}(2))$ is a polynomial algebra.*

**Proof** By Corollary 4.1 we have $\Omega^\infty_0 \mathrm{MTO}(2) \cong Q_0 \mathrm{BSO}(3)_+ \times \Omega^\infty E_2$, noting that $\pi_0(E_2) = 0$ since it is a direct summand of $\pi_0(M(2))$. The short exact sequence above, dualized, implies that $H^*(\Omega^\infty E_2)$ injects to $H^*(\Omega^\infty M(3))$. Since $M(3)$ is a stable summand of $\mathrm{BO}(3)$, we see that $H^*(\Omega^\infty E_2)$ injects to $H^*(Q_0\mathrm{BO}(3))$ which is polynomial [19, Theorem 3.11]. Since $H^*(\Omega^\infty E_2)$ is a connected Hopf algebra, as in the above, by the structure theorem of Hopf algebras over $\mathbb{Z}/2$, this implies that $H^*(\Omega^\infty E_2)$ itself is a polynomial algebra. Now the corollary follows as the other tensor factor $H^*(Q_0\mathrm{BO}(3))$ is polynomial again by [19, Theorem 3.11]. □

## 5.2 Relations among $\mu$–classes

We now prove Theorem 1.5 as an application. We start with the following definitions. For a $\mathbb{Z}/2$–algebra $R$, denote by $Q(R)$ its module of indecomposables, i.e. $I(R)/(I(R)^2)$ where $I(R)$ denotes the augmentation ideal. We will write often $QR$ instead of $Q(R)$ to avoid heavy notations.

**Lemma 5.4** *Let $X$ be a pointed space, $u_X\colon X \to QX$ be the unit map, and*

$$\sigma_*^\infty\colon QH_*(QX) \to H_*(X)$$

*the homology suspension. Write $W(QH_*(QX))$ for the image of $\widetilde{H}_*(X)$ in $QH_*(QX)$ by the composition of $H_*(u_X)$ and the projection $\widetilde{H}_*(QX) \to QH_*(QX)$, and write $F(QH_*(QX)) = \mathrm{Ker}(\sigma_*^\infty)$.*[4] *Then we have*

$$QH_*(QX) \cong W(QH_*(QX)) \oplus F(QH_*(QX)).$$

*This direct sum decomposition is natural with respect to maps of spaces (and not map of suspension spectra). We will refer to it as the $WF$ decomposition.*

**Proof** The direct sum decomposition is an immediate consequence of the standard fact that the homology suspension surjects to $\widetilde{H}_*(X)$ (e.g. [6, Lemmas 4.4 and 4.5]). Since $\sigma_*^\infty$ is natural with respect to maps of spectra, and $u_X$ is natural with respect to maps of spaces, the decomposition is natural with respect to maps of spaces. □

We will extend this to slightly wider category of infinite loop spaces including the $\Omega^\infty M(n)$'s.

**Lemma 5.5** *Let $X$ be a space on which a group $G$ acts, $e \in \mathbb{Z}_{(2)}[G]$ an idempotent. Denote by $\pi\colon X \to eX$ the projection and by $i\colon eX \to X$ the section associated to the splitting of $X$ by $e$ such that $e = i \circ \pi$. Then one can decompose $QH_*(\Omega^\infty eX)$ as*

$$QH_*(\Omega^\infty eX) \cong W(QH_*(\Omega^\infty eX)) \oplus F(QH_*(\Omega^\infty eX))$$

*so that the direct sum decomposition is compatible with that of $H_*(QX)$ via $H_*(\Omega^\infty\pi)$ as well as $H_*(\Omega^\infty i)$.*

[4] The notation is voluntarily reminiscent of what we used in earlier versions. $W$ here corresponds to $W_1$, $F$ to $F_2$.

**Proof** This is equivalent to saying that $QH_*(\Omega^\infty e)$ respects the $WF$–decomposition. Let $g \in G$. Then $g$ acts on $X$ via a map of spaces, so $QH_*(\Omega^\infty g)$ respects the $WF$–decomposition. On the other hand, for $x \in H_*(QX)$,

$$H_*(\Omega^\infty(g_1+g_2))(x) = \Sigma H_*(\Omega^\infty(g_1))(x')H_*(\Omega^\infty(g_2))(x''),$$
$$H_*(\Delta_{QX})(x) = \Sigma x' \times x'',$$

but we have $H_*(\Delta_{QX})(x) = 1 \otimes x + x \otimes 1$ modulo $I \otimes I$ where $I$ is the augmentation ideal of $H_*(QX)$. Thus,

$$QH_*(\Omega^\infty(g_1+g_2)) = QH_*(\Omega^\infty(g_1)) + QH_*(\Omega^\infty(g_2)),$$

so $QH_*(\Omega^\infty e)$ also respects the $WF$ decomposition. □

As noted above, maps of spectra don't necessarily respect the $WF$ decomposition. However, as the summand $F$ is defined in terms of stable information only, some maps of spectra have nice behavior with respect to this decomposition. For example, we can prove:

**Lemma 5.6** *The map*

$$QH_*(\Omega^\infty\delta_{n-1})\colon QH_*(\Omega^\infty M(n)) \to QH_*(\Omega^\infty M(n-1))$$

*induces an inclusion*

$$W(QH_*(\Omega^\infty M(n))) \to F(QH_*(\Omega^\infty M(n-1))).$$

**Proof** The long exact sequence for the homology of the cofibration (6) implies that $H_*(\delta_k)$ is trivial for all $k$. Thus by naturality of the homology suspension, we see that the image of $QH_*(\Omega^\infty\delta_{n-1})$ is included in $F(QH_*(\Omega^\infty M(n-1)))$. By Remark 5.2 we have $\mathrm{Ker}(QH_*(\Omega^\infty\delta_{n-1})) = \mathrm{Im}(QH_*(\Omega^\infty\delta_n))$, but as before this is included in $F(QH_*(\Omega^\infty M(n)))$. So the restriction of $QH_*(\Omega^\infty\delta_{n-1})$ to $W(QH_*(\Omega^\infty M(n)))$ is injective. □

Now we are ready to prove Theorem 1.5. The inclusion $H^*(M(n)) \subset H^*(\mathrm{BO}(n))$ is given by $H^*(f_n)$, and this is determined uniquely by its compatibility with $H^*(\alpha_n)$, which in turn is determined uniquely by the fact that $H^{-n}(\mathrm{MTO}(n))$ contains only one nontrivial element, and the fact that $H^*(D(n))$ is generated by the bottom class as a module over the Steenrod algebra (5). The cofibration sequence (2) implies that

$(\Omega^\infty \omega_{O(n)})^*(\sigma^{\infty *}(x)) = 0$ if $\sigma^{\infty *}(x) \in H^*(Q(\mathrm{BO}(n)_+))$ belongs to the image of $H^*(\Omega^\infty \mathrm{MTO}(n-1))$. Now, Theorem 3.9 implies that we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{BO}(n)_+ & \xrightarrow{f_n} & M(n) & & \\
\downarrow & & \downarrow & \searrow^{\delta_{n-1}} & \\
\mathrm{MTO}(n-1) & \xrightarrow{\alpha_{n-1}} & \Sigma^{1-n} D(n-1) & \longrightarrow & M(n-1)
\end{array}$$

Thus we get

$$\begin{array}{ccccc}
H^*(Q(\mathrm{BO}(n)_+)) & \xleftarrow{H^*(\Omega^\infty f_n)} & H^*(\Omega^\infty M(n)) & & \\
\uparrow & & \uparrow & \nwarrow^{H^*(\Omega^\infty \delta^{n-1})} & \\
H^*(\Omega^\infty \mathrm{MTO}(n-1)) & \longleftarrow & H^*(\Omega^\infty(\Sigma^{1-n} D(n-1)) & \longleftarrow & H^*(\Omega^\infty M(n-1))
\end{array}$$

On the other hand, dualizing Lemma 5.6, we see that the dual of $F\big(QH_*(\Omega^\infty M(n-1))\big)$ surjects to the dual of $W(QH_*(\Omega^\infty M(n)))$, which is precisely the image of $\sigma^{\infty *}_{M(n)}$. Thus we have inclusions

$$\mathrm{Im}(\sigma^{\infty *}_{M(n)}) \subset \mathrm{Im}(PH^*(\Omega^\infty \delta^{n-1})) \subset \mathrm{Im}(H^*(\Omega^\infty \delta^{n-1})).$$

Therefore by the commutativity of the diagram above, we see that the image of the composition

$$H^*(M(n)) \xrightarrow{\sigma^{\infty *}} H^*(\Omega^\infty M(n)) \to H^*(Q(\mathrm{BO}(n)_+))$$

is contained in the image of $H^*(\Omega^\infty \mathrm{MTO}(n-1))$. This concludes the proof of (i).

Now, notice that the splitting $\mathrm{BO}(2)_+ \simeq \mathrm{BSO}(3)_+ \vee M(2)$ combined with part (i) shows that the only nontrivially characteristic classes may arise from the restriction of $(\Omega^\infty \omega_{O(2)})^* \circ \sigma^{\infty *}$ to the $H^*\mathrm{BSO}(3)$ summand of $H^*\mathrm{BO}(2)$, which was studied in [6, Theorem 1.9]. Noting that [6, Remark 4.7] allows us to talk about $\mu$–classes and $\nu$–classes interchangeably, we get (ii) and (iii).

To conclude, we give some explicit examples of those relations. First of all, we have [12, Corollary 3.11].

**Proposition 5.7** *The image of $H^*(M(n))$ in $H^*(\mathrm{BO}(n))$ is the free-module over $H^*(B\Delta_n)^{\mathrm{Gl}_n(\mathbb{Z}/2)}$ generated by a basis of $A(n-2)\,\mathrm{Sq}^{2^{n-1},\dots,2,1}(x_1^{-1}\cdots x_n^{-1})$, where $A(k)$ is the subalgebra of the Steenrod algebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^k}$. Here we identify $H^*(\mathrm{BO}(n))$ with its image in*

$$H^*(B\Delta_n) \subset H^*(B\Delta_n)^{-\gamma_n} \cong H^*(B\Delta_n)\cdot(x_1\cdots x_n)^{-1}$$

*via $Bi^*$ where $i : \Delta_n \subset O(n)$. In terms of cohomology classes, we identify $H^*(\mathrm{BO}(n))$ with the subalgebra of $H^*(B\Delta_n)$ generated by the elementary symmetric polynomials $\sigma_1 = \sigma_1(x_1, \dots, x_n), \dots, \sigma_n = \sigma_n(x_1, \dots, x_n)$.*

The action of the Steenrod algebra on $H^*(B\Delta_n) \cdot (x_1 \cdots x_n)^{-1}$ is determined by the action of the total Steenrod square (see e.g. [18]) $\mathrm{Sq}^T(x_i) = x_i + x_i^2$ for $1 \le i \le n$, and the Cartan formula $\mathrm{Sq}^T(yz) = \mathrm{Sq}^T(y)\,\mathrm{Sq}^T(z)$ for any $y, z$. Thus

$$\mathrm{Sq}^T(x_i^{-1}) = x_i^{-1}(1 + x_i)^{-1} = x_i^{-1}(1 + x_i + x_i^2 + x_i^3 + \cdots).$$

When $n = 2$, $A(0)$ is just the exterior algebra generated by $\mathrm{Sq}^1$, that is, a graded vector space spanned by 1 and $\mathrm{Sq}^1$. Furthermore, by the above, we see that

$$\mathrm{Sq}^{2,1}(x_1^{-1}x_2^{-1}) = x_1 + x_2 = \sigma_1, \mathrm{Sq}^1\,\mathrm{Sq}^{2,1}(x_1^{-1}x_2^{-1}) = \mathrm{Sq}^1(\sigma^1) = x_1^2 + x_2^2 = \sigma_1^2.$$

Since the Dickson invariant algebra $H^*(B\Delta_n)^{\mathrm{Gl}_n(\mathbb{Z}/2)}$ is generated by

$$w_2 = x_1^2 + x_1x_2 + x_2^2 = \sigma_1^2 + \sigma_2, \quad w_3 = x_1x_2(x_1 + x_2) = \sigma_1\sigma_2$$

[12, Theorem A1], we derive:

**Corollary 5.8** *The set*

$$\{(\sigma_1^2 + \sigma_2)^i(\sigma_1\sigma_2)^j\sigma_1^\epsilon \mid i \ge 0,\ j \ge 0, \epsilon \in \{1, 2\}\}$$

*forms a basis of the image of $H^*(M(2))$ in $H^*(\mathrm{BO}(2))$.*

Combined with Theorem 1.5, we get a table of these relations in low dimensions,

$$\begin{aligned}
\mu_{1,0} &= 0 \quad (i = 0, j = 0, \epsilon = 1),\\
\mu_{3,0} + \mu_{1,1} &= 0 \quad (i = 1, j = 0, \epsilon = 1),\\
\mu_{2,1} &= 0 \quad (i = 0, j = 1, \epsilon = 1),\\
\mu_{5,0} + \mu_{3,1} + \mu_{1,2} &= 0 \quad (i = 2, j = 0, \epsilon = 1),\\
\mu_{3,1} &= 0 \quad (i = 0, j = 1, \epsilon = 2),\\
\mu_{4,1} + \mu_{2,2} &= 0 \quad (i = 1, j = 1, \epsilon = 1).
\end{aligned}$$

Here we have omitted the relations that follow from lower degree relations and the general relation $\mu_{2i,2j} = \mu_{i,j}^2$. For example, setting $\epsilon = 2, i = 1$ and $j = 0$ gives $\mu_{4,0} + \mu_{2,1} = 0$; however, we have already listed $\mu_{2,1} = 0$, and we can deduce $\mu_{4,0} = 0$ from $\mu_{4,0} = \mu_{1,0}^4$ and $\mu_{1,0} = 0$.

# References


[1] **J C Becker**, **R E Schultz**, *The real semicharacteristic of a fibered manifold*, Quart. J. Math. Oxford Ser. 33 (1982) 385–403 MR Zbl

[2] **A Borel**, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. 57 (1953) 115–207 MR Zbl

[3] **S Galatius**, **O Randal-Williams**, *Stable moduli spaces of high-dimensional manifolds*, Acta Math. 212 (2014) 257–377 MR Zbl

[4] **S Galatius**, **U Tillmann**, **I Madsen**, **M Weiss**, *The homotopy type of the cobordism category*, Acta Math. 202 (2009) 195–239 MR Zbl

[5] **D J Hunter**, **N J Kuhn**, *Characterizations of spectra with $\mathcal{U}$–injective cohomology which satisfy the Brown–Gitler property*, Trans. Amer. Math. Soc. 352 (2000) 1171–1190 MR Zbl

[6] **T Kashiwabara**, **H Zare**, *Splitting Madsen–Tillmann spectra, I: Twisted transfer maps*, Bull. Belg. Math. Soc. Simon Stevin 25 (2018) 263–304 MR Zbl

[7] **N J Kuhn**, *Spacelike resolutions of spectra*, from "Proceedings of the northwestern homotopy theory conference" (H R Miller, S B Priddy, editors), Contemp. Math. 19, Amer. Math. Soc., Providence, RI (1983) 153–165 MR Zbl

[8] **N J Kuhn**, **S B Priddy**, *The transfer and Whitehead's conjecture*, Math. Proc. Cambridge Philos. Soc. 98 (1985) 459–480 MR Zbl

[9] **L G Lewis, Jr**, **J P May**, **M Steinberger**, **J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer (1986) MR Zbl

[10] **B M Mann**, **E Y Miller**, **H R Miller**, *$S^1$–equivariant function spaces and characteristic classes*, Trans. Amer. Math. Soc. 295 (1986) 233–256 MR Zbl

[11] **J W Milnor**, **J C Moore**, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965) 211–264 MR Zbl

[12] **S A Mitchell**, *Finite complexes with $A(n)$–free cohomology*, Topology 24 (1985) 227–246 MR Zbl

[13] **S A Mitchell**, *Splitting $B(\mathbb{Z}/p)^n$ and $BT^n$ via modular representation theory*, Math. Z. 189 (1985) 1–9 MR Zbl

[14] **S A Mitchell**, **S B Priddy**, *Stable splittings derived from the Steinberg module*, Topology 22 (1983) 285–298 MR Zbl

[15] **S A Mitchell**, **S B Priddy**, *Symmetric product spectra and splittings of classifying spaces*, Amer. J. Math. 106 (1984) 219–232 MR Zbl

[16] **O Randal-Williams**, *The homology of the stable nonorientable mapping class group*, Algebr. Geom. Topol. 8 (2008) 1811–1832 MR Zbl

[17] **S Takayasu**, *On stable summands of Thom spectra of $B(\mathbb{Z}/2)^n$ associated to Steinberg modules*, J. Math. Kyoto Univ. 39 (1999) 377–398 MR Zbl

[18] **G Walker**, **R M W Wood**, *Polynomials and the* mod 2 *Steenrod algebra, II: Representations of* $\mathrm{GL}(n, \mathbb{F}_2)$, London Mathematical Society Lecture Note Series 442, Cambridge Univ. Press (2018) MR Zbl

[19] **R J Wellington**, *The unstable Adams spectral sequence for free iterated loop spaces*, Mem. Amer. Math. Soc. 258, Amer. Math. Soc., Providence, RI (1982) MR Zbl

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