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A concave holomorphic filling of an overtwisted contact 3–sphere

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We prove that the closed 4–ball admits non-Kähler complex structures with strongly pseudoconcave boundary. Moreover, the induced contact structure on the boundary 3–sphere is overtwisted.

32V40; 32Q55, 57R17

1 Introduction

In [4], Antonio J Di Scala and the authors constructed a family of pairwise inequivalent complex surfaces $E = E(\rho_1, \rho_2)$ together with a holomorphic map $f : E \to \mathbb{CP}^1$ admitting compact fibers (the parameters ρ_1 and ρ_2 are such that $1 < \rho_2 < \rho_1^{-1}$). A relevant property of E is that it is diffeomorphic to $\mathbb{R}^4$, giving an extension to real dimension four of a result of Calabi and Eckmann [2].

The compact fibers of f were shown to be smooth elliptic curves and a singular rational curve with one node, and these are the only compact complex curves of E. The existence of embedded compact holomorphic curves implies the nonexistence of a compatible symplectic structure on E. Thus, the complex surface E is non-Kähler.

Further, in [5] we proved that E cannot be realized as a complex domain in any smooth compact complex surface.

In the present paper, we study the structure of E away from a compact subset by providing an exhausting family of embedded strongly pseudoconcave 3–spheres; see Proposition 4.1. From this we derive our main theorem. In order to state our results, we recall the notion of *Calabi–Eckmann type complex manifold* introduced in [4], which was inspired by the results of [2].

Definition 1.1 A complex manifold W is said to be of Calabi–Eckmann type if there exists a compact complex manifold X of positive dimension, and a holomorphic immersion $k\colon X \to W$ which is nullhomotopic as a continuous map.

We also recall the definition of strong pseudoconvexity and pseudoconcavity. Let (W, J) be a complex manifold with complex structure J and complex dimension ≥ 2, and let $M \subset W$ be a smooth real oriented hypersurface. Then, near every point $p \in M$ we can consider a local defining function for M, namely a smooth function $u\colon U \to \mathbb{R}$ defined in a certain open neighborhood U of p in W, such that u has no critical points and $M \cap U = u^{-1}(0)$ is the oriented boundary of the sublevel $u^{-1}(-\infty, 0]$. Moreover, M carries the complex tangencies distribution $\xi = TM \cap J(TM)$, which we assume to be endowed with the canonical complex orientation induced by J.

Definition 1.2 We say that a real oriented hypersurface $M \subset W$ is *strongly pseudoconvex* in W if there exists a strictly plurisubharmonic local defining function for M near every point $p \in M$, namely a defining function u whose complex Hessian Hu is positive definite. The oriented hypersurface M is said to be *strongly pseudoconcave* if it becomes strongly pseudoconvex by reversing its orientation.

In particular, we can consider complex manifolds with strongly pseudoconvex or pseudoconcave boundary. It is a standard fact that when $\dim_{\mathbb{C}} W$ is even, an oriented real hypersurface $M \subset W$ is strongly pseudoconvex (resp. pseudoconcave) if and only if the complex tangencies distribution on M is a positive (resp. negative) contact structure. Since we consider real 3–manifolds embedded in complex surfaces, we mainly refer to strong pseudoconvexity or pseudoconcavity by means of this characterizing property.

Main Theorem *The closed ball B^4 admits a Calabi–Eckmann type complex structure J with strongly pseudoconcave boundary. Moreover, the (negative) contact structure ξ determined on $\partial B^4 = S^3$ by the complex tangencies is overtwisted and homotopic as a plane field to the standard positive tight contact structure on S^3.*

In other words, (B^4, J) is a concave holomorphic filling of the overtwisted contact sphere (S^3, ξ). As far as the authors know, this is the first example of this sort in the literature.

This 4–ball arises as a smooth submanifold of E containing certain compact fibers of the map $f\colon E \to \mathbb{CP}^1$, and so it is evidently of Calabi–Eckmann type.

Our strategy for proving the theorem relies on finding a closed piecewise smooth 3–manifold $M \subset E$ supporting an open book decomposition whose pages are holomorphic annuli and whose monodromy is a left-handed (negative) Dehn twist about the core of the annulus; hence the underlying manifold M is homeomorphic to S^3.

Moreover, we prove that M can be approximated by a 1–parameter family of strongly pseudoconcave smoothly embedded 3–spheres $M_\tau \subset E$, for a suitable parameter $\tau \in (0,1)$. Namely, the complex domain outside the embedded 4–ball with corners $D \subset E$ bounded by M is foliated by strongly pseudoconcave 3–spheres. This implies the existence of a strictly plurisubharmonic function on $E - D$.

As a consequence, the open book decomposition of M is compatible with the contact structure of M_τ given by complex tangencies, which is then overtwisted. For the basics of the three-dimensional contact topology we use throughout the paper, the reader is referred, for example, to the book of Ozbagci and Stipsicz [15, Chapters 4 and 9].

Remark By Eliashberg's classification of overtwisted contact structures on closed oriented 3–manifolds [7], the negative contact structure in the main theorem is uniquely determined up to isotopy.

We point out that in all (odd) dimensions greater than three, a closed co-oriented overtwisted contact manifold (see Borman, Eliashberg and Murphy [1] for the definition) cannot be the strongly pseudoconcave boundary of a complex manifold. Indeed, such a holomorphic filling would give a strongly pseudoconvex CR structure on the contact manifold with reversed orientation. Thus, it can be filled by a Stein space — Rossi's theorem [17] — and therefore it can be filled by a Kähler manifold — Hironaka's theorem [9; 10] — which is impossible for an overtwisted contact manifold. In this sense, our result is particular to dimension three.

Lisca and Matić [13, Theorem 3.2] proved that any Stein filling W of a contact 3–manifold can be realized as a domain in a smooth complex projective surface S. Hence $S - \operatorname{Int} W$ is a concave holomorphic filling of a Stein fillable contact 3–manifold.

On the other hand, Eliashberg in [6] proved that for any closed contact 3–manifold (N, ξ), the 4–manifold $N \times [0,1]$ admits a complex structure such that the height function is strictly plurisubharmonic, providing a holomorphic cobordism of (N, ξ) with itself. However, its proof is not constructive.

Our result gives a rather explicit complex cobordism of an overtwisted contact 3–sphere with itself, by taking $\bigcup_{\tau \in [1/3, 1/2]} M_\tau \cong S^3 \times [0,1]$ as a complex domain in E.

Remark In [11] the authors prove that every closed contact 3–manifold can be filled as the strongly pseudoconcave boundary of a compact complex surface of Calabi–Eckmann type. We point out that this generalization depends on our main theorem.

The paper is organized as follows. In Section 2, we recall the construction of the complex surface E given in [4] and present a holomorphic model of the complement $C = E - \operatorname{Int} D$, which will be helpful for the proof of the main theorem, with D the 4–ball mentioned above. In Section 3, we construct a holomorphic open book decomposition embedded in E. Finally, in Section 4, we prove the main theorem by showing the existence of a strictly plurisubharmonic function near the embedded open book decomposition based on contact topology.

2 The complex surface *E*

In this section, we recall the construction of E, by sketching the original one in [4]. This will be helpful for the proof of our main theorem.

Throughout this paper we make use of the following notation:

$$\Delta(a,b) = \{z \in \mathbb{C} \mid a < |z| < b\},$$
$$\Delta[a,b] = \{z \in \mathbb{C} \mid a \le |z| \le b\},$$
$$\Delta(a) = \{z \in \mathbb{C} \mid |z| < a\},$$

and similarly with mixed brackets. We also denote the closed disk and the circle of radius a in $\mathbb{C}$ by $B^2(a)$ and $S^1(a)$, respectively. When $a = 1$, we drop it from the notation.

According to [4], the construction of $E = E(\rho_1, \rho_2)$ proceeds as follows. Let ρ_1 and ρ_2 be positive numbers such that $1 < \rho_2 < \rho_1^{-1}$, and choose ρ_0 such that $\rho_1\rho_2^{-1} < \rho_0 < \rho_1$.

We want to realize E as the union of two pieces. One of them is the product

$$V = \Delta(1, \rho_2) \times \Delta(\rho_0^{-1}),$$

and the other one is the total space W of a genus-1 holomorphic Lefschetz fibration $h\colon W \to \Delta(\rho_1)$ with only one singular fiber Σ.

In order to define the analytical gluing between V and W, we make use of the following Kodaira model [12]. Consider the elliptic fibration

$$(\mathbb{C}^* \times \Delta(0, \rho_1))/\mathbb{Z} \to \Delta(0, \rho_1),$$

defined by the canonical projection on the quotient space of $\mathbb{C}^* \times \Delta(0,\rho_1)$ with respect to the $\mathbb{Z}$–action given by $n \cdot (w_1, w_2) = (w_1 w_2^n, w_2)$. Then, it canonically extends to a singular elliptic fibration $h\colon W \to \Delta(\rho_1)$, and so we have an identification $W - \Sigma = (\mathbb{C}^* \times \Delta(0,\rho_1))/\mathbb{Z}$. The critical point of h is nondegenerate, namely the complex Hessian is of maximal rank, and so h is a genus-1 holomorphic Lefschetz fibration. In what follows, we shall keep the convention of denoting by (w_1, w_2) the usual complex coordinates of $\mathbb{C}^* \times \Delta(\rho_1) \subset \mathbb{C}^2$ when referring to W (up to the above identification), and by (z_1, z_2) the usual coordinates of $\mathbb{C}^2$ when referring to $V \subset \mathbb{C}^2$.

Now, let us consider the multivalued holomorphic function $\varphi\colon \Delta(0,\rho_1) \to \mathbb{C}^*$ defined by

$$\varphi(w) = \exp\Big(\frac{1}{4\pi i}(\log w)^2 - \frac{1}{2}\log w\Big).$$

We denote by $\Phi\colon U \to W$ the holomorphic map defined by

$$\Phi(z_1, z_2) = [(z_1\varphi(z_2^{-1}), z_2^{-1})],$$

where $U \subset \mathbb{C}^* \times \Delta(\rho_1^{-1}, \rho_0^{-1})$ is a certain open subset that will be specified later. Notice that Φ is single-valued. This depends on the fact that any two branches φ_1 and φ_2 of φ are related by the formula $\varphi_2(w) = w^k \varphi_1(w)$ for some $k \in \mathbb{Z}$, which is compatible with the above $\mathbb{Z}$–action. For the purpose of this section, we take $U = \Delta(1,\rho_2) \times \Delta(\rho_1^{-1}, \rho_0^{-1}) \subset V$.

It follows that Φ is a biholomorphism between $U \subset V$ and $\Phi(U) \subset W$.

We are now ready to holomorphically glue V and W by identifying the open subsets $U \subset V$ and $\Phi(U) \subset W$ by means of Φ. That is, we define the complex surface

$$E = E(\rho_1, \rho_2) = V \cup_\Phi W.$$

We consider V and W as open subsets of E via the quotient map.

By construction, there is a holomorphic map $f\colon E \to \mathbb{CP}^1$ defined by the canonical projection onto the second factor on V and by the elliptic fibration h on W, where $\mathbb{CP}^1$ is regarded as the result of gluing the disks $\Delta(\rho_0^{-1})$ and $\Delta(\rho_1)$ by identifying $\Delta(\rho_1^{-1}, \rho_0^{-1}) \subset \Delta(\rho_0^{-1})$ with $\Delta(\rho_0, \rho_1) \subset \Delta(\rho_1)$ by means of the inversion map $z \mapsto z^{-1}$.

Notice that the resulting complex surface E does not depend on ρ_0, since this parameter determines only the size of the gluing region.

Remark By taking ρ_1' and ρ_2' such that $\rho_2 < \rho_2' < (\rho_1')^{-1} < \rho_1^{-1}$, our construction yields an obvious holomorphic embedding of E in $E' = E(\rho_1', \rho_2')$ as a relatively compact complex domain. The closure $\widehat{E} = \mathrm{Cl}\, E$ in E' has Levi flat piecewise smooth boundary, and $\partial \widehat{E}$ is homeomorphic to S^3. This agrees with the interpretation of the map $f\colon E \to \mathbb{CP}^1$ given in [4] as the restriction of the Matsumoto–Fukaya fibration $S^4 \to S^2$ [14] to the complement of a neighborhood of the negative critical point in S^4. This also relates to the embedded open book decomposition that we construct in Proposition 3.1.

Let $V' = \Delta(1, s) \times \Delta(\rho_1^{-1}, \rho_0^{-1})$, where the additional parameter s is chosen so that $\rho_0^{-1} < s < \rho_1^{-1}\rho_2$. Let U' be the subset of V' defined by $U' = \{(z_1, z_2) \in V' \mid |z_2| < |z_1|\}$. We put $V'' = V \cup V' \subset \mathbb{C}^2$ and identify a point $(z_1, z_2) \in U'$ with $\psi(z_1, z_2)$, where $\psi\colon U' \to V'$ is the holomorphic embedding defined by $\psi(z_1, z_2) = (z_1 z_2^{-1}, z_2)$. Let $Y = V''/{\sim}$ be the quotient.

Proposition 2.1 *The manifold $Y = V''/{\sim}$ is biholomorphic to the preimage of the disk $\Delta(\rho_0^{-1}) \subset \mathbb{CP}^1$ by the holomorphic fibration $f\colon E \to \mathbb{CP}^1$.*

Proof The preimage $f^{-1}(\Delta(\rho_0^{-1}))$ is described as follows. Let $W(\rho_0, \rho_1)$ be the subset of W given, in the Kodaira model above, by

$$W(\rho_0, \rho_1) = (\mathbb{C}^* \times \Delta(\rho_0, \rho_1))/\mathbb{Z} = f^{-1}(\Delta(\rho_0, \rho_1)),$$

being $f = h$ in $W(\rho_0, \rho_1)$. Then, we have $U' \subset W(\rho_0, \rho_1)$, and so

$$f^{-1}(\Delta(\rho_0^{-1})) = V \cup_{U \sim U'} W(\rho_0, \rho_1).$$

Now, we define a map $\Psi\colon Y \to f^{-1}(\Delta(\rho_0^{-1}))$ by putting $\Psi([(z_1, z_2)]) = (z_1, z_2)$ on $V/{\sim}$ and $\Psi([(z_1, z_2)]) = \Phi(z_1, z_2)$ on $V'/{\sim}$. It is easy to check that Ψ is well defined and is a biholomorphism. □

In order to obtain the complement $C \subset E$ of a 4–ball D containing the singular fiber of f, we remove from Y the subset

$$Z = \{(z_1, z_2) \mid c_1 < |z_1| < c_2\} \subset V,$$

where $s\rho_1 < c_1 < c_2 < \rho_2$. Then, by Proposition 2.1, it is enough to set $C = Y - Z$.

3 The holomorphic open book decomposition

We briefly recall the notion of open book decomposition of a 3–manifold. For a more thorough treatment, the reader is referred to Ozbagci and Stipsicz [15, Chapter 9] and to Rolfsen [16, Chapter 10K].

By an *open book decomposition* of a closed, connected, oriented, manifold M of real dimension three, we mean a smooth map $f: M \to B^2$ such that

(1) the restriction $f|_{\mathrm{Cl}(f^{-1}(\mathrm{Int}\,B^2))}: \mathrm{Cl}(f^{-1}(\mathrm{Int}\,B^2)) \to B^2$ is a (trivial) fiber bundle with fiber a link $L = f^{-1}(0)$, called the *binding* of the open book;

(2) the map $\varphi: M - L \to S^1 = \partial B^2$ defined by $\varphi(x) = f(x)/|f(x)|$ is a fiber bundle.

The closure of every fiber $F_\theta = \mathrm{Cl}(\varphi^{-1}(\theta))$, for $\theta \in S^1$, is a compact surface in M, called a *page* of the open book, and $\partial F_\theta = L$. By a little abuse of terminology, we also call the surfaces $f^{-1}(\theta)$, for all $\theta \in S^1 = \partial B^2$, pages of f. The two kinds of pages are ambient isotopic in M to each other.

Given an open book decomposition $f: M \to B^2$, the orientations of M and of B^2 induce an orientation on the pages, and hence on the binding $L = \partial F_\theta$.

For an open book decomposition $f: M \to B^2$, there is an associated *monodromy* ω_f of the bundle φ, which is a diffeomorphism of a page F_* that fixes the boundary pointwise, and it is well defined up to isotopy fixing the boundary.

On the other hand, given an element ω of the mapping class group $\mathrm{Mod}_{g,b}$ of a compact, connected, oriented surface $F_{g,b}$ of genus $g \geq 0$ and with $b \geq 1$ boundary components, there is an open book decomposition $f_\omega: M_\omega \to B^2$ with monodromy ω and page $F = F_{g,b}$, and this is uniquely determined up to orientation-preserving diffeomorphisms. The construction goes as follows. Take a representative $\psi: F \to F$ of the isotopy class ω and consider the mapping torus $T_\omega = (F \times \mathbb{R})/\mathbb{Z}$, where the $\mathbb{Z}$–action is generated by the diffeomorphism $\tau: F \times \mathbb{R} \to F \times \mathbb{R}$ defined by $\tau(x,t) = (\psi(x), t-1)$.

Let M_ω be the result of gluing $\partial F \times B^2$ to T_ω along the boundary, by means of the obvious identifications $\partial(\partial F \times B^2) \cong \partial F \times S^1 \cong \partial F \times (\mathbb{R}/\mathbb{Z}) \cong \partial T_\omega$, where the last identification comes from the fact that ψ is the identity on ∂F. Then, let $f: M_\omega \to B^2$ be the canonical projection $\partial F \times B^2 \to B^2$ on $\partial F \times B^2 \subset M_\omega$, while it is the projection $T_\omega \to \mathbb{R}/\mathbb{Z} \cong \partial B^2$ on $T_\omega \subset M_\omega$.

Consider an oriented surface F and let $\gamma \subset \mathrm{Int}\,F$ be a connected simple closed curve. A *Dehn twist* $\delta_\gamma: F \to F$ about the curve γ is a diffeomorphism of F such that away from a tubular neighborhood T of γ in F, δ_γ is the identity, while in $T \cong S^1 \times [0,1]$ the diffeomorphism δ_γ either corresponds to the map $\delta_-: S^1 \times [0,1] \to S^1 \times [0,1]$ defined by

$$\delta_-(z,t) = (ze^{-2\pi i t}, t),$$

or to the map $\delta_+ = \delta_-^{-1}$, where $S^1 \times [0, 1]$ is endowed with the product orientation and its identification with $T \subset F$ is orientation-preserving. In the former case, δ_γ is called a *left-handed* (*or negative*) *Dehn twist*, while in the latter it is called a *right-handed* (*or positive*) *Dehn twist*. By changing the orientation of F, the two types of Dehn twists are swapped.

The 3–sphere admits an open book decomposition $h_-\colon S^3 \to B^2$ with binding the negative Hopf link H_-, and with page the annulus $S^1 \times [0, 1]$. The monodromy is the left-handed Dehn twist about the core circle $\gamma = S^1 \times \{\frac{1}{2}\}$ of the annulus (there is also the positive version $h_+\colon S^3 \to B^2$ of this). This is the well-known realization of the (negative) Hopf link in S^3 as a fibered link, with page the Hopf band [8].

The following proposition will be helpful in the proof of the main theorem. We keep the notation of Section 2.

Proposition 3.1 *There is a piecewise smooth embedded* 3*–sphere* $M \subset E$ *such that the restriction* $f|_M\colon M \to B^2$ *of the holomorphic map* $f\colon E \to \mathbb{CP}^1$, *is diffeomorphic to the open book decomposition* h_- *of* S^3 *described above, with* B^2 *a suitable closed disk in* $\Delta(\rho_0^{-1}) \subset \mathbb{CP}^1$. *Every page of* $f|_M$ *is a holomorphic annulus in an elliptic fiber of* f. *Moreover,* M *is not globally smooth, since it has corners along the two linked tori given by* $\partial f|_M^{-1}(\partial B^2)$, *on the complement of which* M *is foliated by holomorphic curves. Thus,* M *is Levi flat in* E.

We endow M with the orientation determined by the open book decomposition, where the pages are oriented by the induced complex structure, and the base disk B^2 takes the orientation from $\mathbb{CP}^1$. By construction, this disk is in the part of $\mathbb{CP}^1$ that corresponds, via the map f, to the Stein open subset $V \subset E$, with the boundary in the gluing region.

Fix two numbers c and ϵ such that $\rho_0 < c < \rho_1$ and

$$0 < \epsilon < \tfrac{1}{2}\min(\rho_1 - \rho_0, \rho_0 - \rho_1\rho_2^{-1}).$$

We put $a = \rho_2 - \epsilon$ and $b = c^{-1} + \epsilon$, and let $A = \Delta[a, b]$. It is then straightforward to check that $(\lambda^k A) \cap A = \varnothing$ for all $\lambda \in \Delta[c, \rho_1]$ and for all $k \in \mathbb{Z} - \{0\}$, with $\lambda^k A = \Delta[\,|\lambda|^k a, |\lambda|^k b\,]$. Moreover, by taking into account the inequalities among the ρ_i's at the beginning of Section 2, we can easily obtain

$$bc < 1 + \frac{c(\rho_1 - \rho_0)}{\rho_0\rho_1} < \rho_2. \tag{1}$$

Proof Consider the set

$$G = f^{-1}(S^1(c)) - \Phi(\Delta(bc, a) \times S^1(c^{-1})) \subset E,$$

with $S^1(c) \subset \Delta(\rho_1) \subset \mathbb{CP}^1$. The map $f_G = f|_G \colon G \to S^1(c) \cong S^1$ is a compact annulus bundle over the circle $S^1(c) \subset \Delta(\rho_1) \subset \mathbb{CP}^1$ of radius c. Here $S^1(c)$ has the clockwise orientation in the disk $\Delta(\rho_1)$, namely it is oriented as the boundary of the disk it bounds in $\Delta(\rho_0^{-1}) \subset \mathbb{CP}^1$. This choice depends on the inversion in the map Φ' below.

This bundle is trivial, and a trivialization is provided by the map $\Phi' \colon A \times S^1 \to G$ defined by

$$\Phi'(w_1, w_2) = \Phi(w_1, c^{-1} w_2) = [(w_1 \varphi(c w_2^{-1}), c w_2^{-1})].$$

Notice that Φ' is holomorphic on every fiber.

Now, we construct an open book decomposition of S^3 embedded in E. We begin with an abstract description of this open book, and then we see how it is embedded in E.

Let ψ_1 be the identity map of $S^1(a) \times S^1$, and let

$$\psi_2 \colon S^1(b) \times S^1 \to S^1(b) \times S^1$$

be defined by $\psi_2(w_1, w_2) = (w_1 w_2, w_2)$.

We use the diffeomorphism $\psi = \psi_1 \cup \psi_2 \colon \partial(\partial A \times B^2) \to \partial(A \times S^1)$ to construct the oriented 3–manifold

$$M = (\partial A \times B^2) \cup_\psi (A \times S^1)$$

obtained by gluing $\partial A \times B^2$ to $A \times S^1$ along the boundary (these two pieces are oriented in the canonical way).

Let $p \colon M \to B^2$ be defined by $p(w_1, w_2) = w_2$, for (w_1, w_2) in $\partial A \times B^2$ or $A \times S^1$. It is clear that (M, p) is an open book decomposition of M with binding

$$L = \partial A \times \{0\} \subset \partial A \times B^2 \subset M$$

and the annulus A as the page.

Now, we show that the monodromy of p is the diffeomorphism $\delta \colon A \to A$ defined by

$$\delta(z) = z e^{2\pi i \tau(|z|)},$$

where $\tau \colon [a, b] \to [0, 1]$ is an increasing diffeomorphism (for example, the affine one). Thus, δ is the identity on ∂A. Let

$$T(\delta) = \frac{A \times [0, 1]}{(z, 1) \sim (\delta(z), 0)}$$

be the mapping torus of δ.

The open book decomposition with page A and monodromy δ represents a 3–manifold $B(\delta)$ obtained by capping off $T(\delta)$ with $\partial A \times B^2$ glued along the boundary by the identity, up to the obvious identification $\partial B^2 = S^1 \cong [0,1]/(0 \sim 1)$.

Define the map $k\colon T(\delta) \to A \times S^1$ by setting

$$k([(z,t)]) = (z e^{2\pi i \tau(|z|)(t-1)}, e^{2\pi i t}).$$

Then, k is an orientation-preserving fibered diffeomorphism.

The gluing maps ψ_1 and ψ_2 used for building M correspond, by means of k, to the identity of $\partial(T(\delta)) = \partial A \times S^1$. This implies that there is a diffeomorphism $M \cong B(\delta)$, with respect to which the open book p corresponds to that of $B(\delta)$, and so δ is the monodromy of p.

In order to understand δ, we consider the diffeomorphism $q\colon A \to S^1 \times [0,1]$ defined by

$$q(z) = \left(\frac{\bar{z}}{|z|}, \tau(|z|)\right).$$

This is orientation-preserving, as it can be easily shown by writing q in polar coordinates. Moreover, $q^{-1}(w,t) = \tau^{-1}(t)\bar{w}$.

It is now straightforward to prove the identity $\delta_- = q \circ \delta \circ q^{-1}$, where δ_- is the left-handed Dehn twist defined above. Therefore, δ is a left-handed Dehn twist of A about the curve $\gamma \subset A$ of equation $\tau(|z|) = \frac{1}{2}$ (that is, the core of A). It follows that $p\colon M \to B^2$ is equivalent to the open book h_- of S^3, and in particular $M \cong S^3$.

Next, we define an embedding $g\colon M \to E$ by

$$g(z_1,z_2) = \begin{cases} \Phi'(z_1,z_2) & \text{for } (z_1,z_2) \in A \times S^1, \\ j(z_1, c^{-1}z_2) & \text{for } (z_1,z_2) \in S^1(a) \times B^2, \\ j(cz_1, c^{-1}z_2) & \text{for } (z_1,z_2) \in S^1(b) \times B^2, \end{cases}$$

where $j: V \hookrightarrow E$ is the inclusion map.

We show that g is well defined. For $(z_1,z_2) \in S^1(a) \times S^1$,

$$g(z_1,z_2) = j(z_1, c^{-1}z_2) = \Phi(z_1, c^{-1}z_2) = [(z_1\varphi(cz_2^{-1}), cz_2^{-1})] = (\Phi' \circ \psi_1)(z_1,z_2).$$

Finally, we check consistency at $(z_1,z_2) \in S^1(b) \times S^1$. First, $(z_1,z_2) \in S^1(b) \times B^2$ implies $(cz_1, c^{-1}z_2) \in V$ by inequality (1) above, so we can compute $j(cz_1, c^{-1}z_2)$.

We have

$$\begin{aligned} g(z_1, z_2) &= j(cz_1, c^{-1}z_2) \\ &= \Phi(cz_1, c^{-1}z_2) \\ &= [(cz_1\varphi(cz_2^{-1}), cz_2^{-1})] \\ &= [(z_1z_2\varphi(cz_2^{-1}), cz_2^{-1})] \\ &= (\Phi' \circ \psi_2)(z_1, z_2), \end{aligned}$$

where we are using the $\mathbb{Z}$–action considered in Section 2.

By abusing notation, we still denote by $M \subset E$ the image of g. Therefore, M is a piecewise smooth embedded submanifold of E, although it is not globally smooth. Indeed, the two codimension-0 submanifolds of $E \cong \mathbb{R}^4$ bounded by M have corners along $\partial A \times S^1 \subset M$. Away from the corners, M is foliated by holomorphic curves, and hence it is Levi flat. These holomorphic curves are the images of the disks $\{z_1\} \times B^2$ and the images of the annuli $A \times \{z_2\}$ by the embedding g, with $(z_1, z_2) \in \partial A \times S^1$. □

Let $D \subset E$ be the compact submanifold bounded by M, and let C be the noncompact one. Hence, $E = D \cup_M C$.

The argument based on Kirby calculus in [4] proves the following proposition.

Proposition 3.2 *Up to smoothing the corners, D is diffeomorphic to B^4 and C is diffeomorphic to $S^3 \times (0, 1]$.*

The same conclusion follows from the existence of a proper continuous function $u: C \to (0, 1]$, which is smooth, regular (namely, with no critical points) and strictly plurisubharmonic in $\operatorname{Int} C$. In the next section, we show the existence of such a function to prove our main theorem.

4 The proof of the main theorem

In this section we prove the following proposition and then prove our main theorem.

Proposition 4.1 *There exists a smooth 3–sphere $M_1 \subset E$ such that*

(1) *the noncompact submanifold $C_1 \subset E \cong \mathbb{R}^4$ bounded by M_1 admits a proper smooth regular strictly plurisubharmonic function $u: C_1 \to (0, 1]$;*

(2) *the complement $D_1 = E - \operatorname{Int} C_1$ is of Calabi–Eckmann type;*

(3) *M_1 is piecewise smoothly isotopic to M in E.*

Remark Property (3) of the above proposition and Proposition 3.2 imply that M_1 is smoothly standard in E, meaning that there exists a diffeomorphism $E \to \mathbb{R}^4$ mapping M_1 to the standard unit sphere S^3. Thus, $C_1 \cong S^3 \times (0,1]$ and $D_1 = E - \operatorname{Int} C_1 \cong B^4$.

Proposition 4.1 follows from the construction of C in Section 2 and the following well-known facts.

Lemma 4.2 *Let $U \subset \mathbb{C}$ be a nonempty open subset, and let $\psi\colon U \to \mathbb{R}$ be a smooth function. Let $\Omega = \{(z_1,z_2) \in U \times \mathbb{C} \mid |z_2| \le \exp(-\psi(z_1))\} \subset \mathbb{C}^2$. Then the following two conditions are equivalent*:

(1) *$\partial\Omega$ is strongly pseudoconvex (resp. pseudoconcave)*;

(2) *ψ (resp. $-\psi$) is a strictly subharmonic function.*

Lemma 4.3 *Let c be a smooth regular curve in $\mathbb{R}^2$. Then the hypersurface*

$$M_c = \{(z_1,z_2) \mid (\log|z_1|, \log|z_2|) \in c\} \subset (\mathbb{C}^*)^2$$

is strongly pseudoconvex if and only if the plane curve c is strictly convex.

Now we construct a strongly pseudoconcave hypersurface M_1 which is a perturbation of the holomorphic open book M. We make use of Proposition 2.1 and of the notation established in Section 2.

Proof of Proposition 4.1 We construct a family $\{M_t\}_{t\in(0,1]}$ of smooth closed hypersurfaces in C as follows. First, for any $t \in (0,1]$ and a sufficiently small positive number δ, we take the two functions $f_t, g_t\colon [0,\rho_1^{-1}] \to (1,\rho_2)$ given by

$$f_t(x) = \log a + t\delta(1+x^2), \quad g_t(x) = \log(bc) - t\delta(1+x^2).$$

Recall that (z_1,z_2) are the coordinates on $V = \Delta(1,\rho_2) \times \Delta(\rho_0^{-1}) \subset \mathbb{C}^2$. We then define the hypersurfaces Q_t and R_t in V by $|z_1| = \exp(f_t(|z_2|))$ and $|z_1| = \exp(g_t(|z_2|))$, respectively. By orienting Q_t and R_t as the boundary components of the manifold

$$\{(z_1,z_2) \in V \mid \exp(g_t(|z_2|)) \le |z_1| \le \exp(f_t(|z_2|))\},$$

it turns out that they are both strongly pseudoconcave by Lemma 4.2. Now we retake the coordinates (w_1,w_2) on V' so that $(w_1,w_2) = (z_1,z_2^{-1})$. Then, near Q_t, the coordinate transformation between V and V' is $(w_1,w_2) = (z_1,z_2^{-1})$, and near R_t, it is $(w_1,w_2) = (z_1z_2, z_2^{-1})$, by taking the embedding $\psi\colon U' \to V'$ into account;

see Section 2. Putting $u_j = \log|z_j|$ and $v_j = \log|w_j|$ for $j = 1, 2$, the coordinate transformation is given by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{near } Q_t, \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{near } R_t.$$

Then the defining equations of Q_t and R_t are

$$\begin{aligned} u_1 = f_t(e^{u_2}) \text{ for } u_2 \le -\log\rho_1 &\iff v_1 = \log a + t\delta(1+e^{-2v_2}) && \text{for } v_2 \ge \log\rho_1, \\ u_1 = g_t(e^{u_2}) \text{ for } u_2 \le -\log\rho_1 &\iff v_1 = \log(bc) - v_2 - t\delta(1+e^{-2v_2}) && \text{for } v_2 \ge \log\rho_1, \end{aligned}$$

respectively. Hence, they give plane curves in the $(v_1 v_2)$–plane, say c_{Q_t} and c_{R_t}. Then there exists a smooth family of strictly convex curves c_t satisfying:

(a) each curve c_t is contained in the trapezoid

$$\{(v_1, v_2) \mid v_1 > \log a,\ \log c < v_2 < \log\rho_1,\ v_1 + v_2 < \log(bc)\};$$

(b) c_t, c_{Q_t} and c_{R_t} are smoothly connected to be a regular curve;

(c) the family of curves foliates a subdomain of the trapezoid;

(d) as t goes to 0, the curve c_t piecewise smoothly converges to the polygonal line

$$\{(\log a, v_2) \mid \log c < v_2 < \log\rho_1\} \cup \{(v_1, \log c) \mid \log a \le v_1 \le \log b\} \\ \cup \{(v_1, v_2) \mid v_1 + v_2 = \log(bc),\ \log c < v_2 < \log\rho_1\}.$$

Now we define the hypersurface $S_t \subset V'$ by $S_t = \{(w_1, w_2) \mid (v_1, v_2) \in c_t\}$. Then it is strongly pseudoconvex with one orientation by Lemma 4.3, but with the natural orientation respecting those of Q_t and R_t, it is strongly pseudoconcave. Hence, the three pieces Q_t, R_t and S_t form a smooth closed strongly pseudoconcave hypersurface in Y, which we denote by M_t. Strictly speaking, R_t and the union $H_t = Q_t \cup S_t$ are hypersurfaces in $V'' = V \cup V'$. In the quotient $Y = V''/\sim$, they are glued together to form a smooth closed hypersurface M_t in Y. Since each piece is strongly pseudoconcave, so is M_t. Thus, M_t is a smooth closed strongly pseudoconcave hypersurface in Y. The equations defining Q_t and R_t above and condition (d) of c_t imply that M_t piecewise smoothly converges to M when t goes to 0. In particular, M_1 is a smooth strongly pseudoconcave 3–sphere and satisfies condition (3) of the statement.

Moreover, the smooth 3–sphere M_1 divides the complex manifold E into the two submanifolds, the compact one D_1, which is a closed 4–ball, and the noncompact one C_1. Then condition (2) is automatically fulfilled because D_1 is contractible and contains the singular rational curve of E.

By a similar construction as that of the family $\{M_t\}_{t\in(0,1]}$, we can easily prove that $\operatorname{Int} C_1$ is foliated by a family of strongly pseudoconcave 3–spheres $\{M_t\}_{t\in(1,2)}$. Therefore, the following lemma, which proves the existence of a strictly plurisubharmonic function, concludes the proof. □

Lemma 4.4 *Let $\gamma\colon X \to \mathbb{R}$ be a proper smooth regular function on a complex manifold X such that the complex tangencies define a contact structure on the level sets $\gamma^{-1}(c)$ for all $c \in \gamma(X)$. Then there exists a smooth convex and increasing function $g\colon \gamma(X) \to \mathbb{R}$ such that $g \circ \gamma$ is strictly plurisubharmonic on X.*

Proof See for example [3, Lemma 2.7]. □

Proof of Main Theorem Endow B^4 with the complex structure J induced by an orientation-preserving diffeomorphism $B^4 \cong D_1$, the 4–ball in E bounded by M_1. Then (B^4, J) is of Calabi–Eckmann type and with strongly pseudoconcave boundary (S^3, ξ), where ξ is the induced contact structure.

Since J is homotopic, through almost complex structures, to the standard complex structure of $B^4 \subset \mathbb{C}^2$, the boundary contact structure ξ is homotopic as a plane field to the standard positive tight contact structure of S^3.

We are left to show the compatibility of the contact structure on $M_1 \cong S^3$ with the open book decomposition inherited from M by a suitable diffeomorphism $\varphi\colon M \to M_1$ compatible with the splitting $M = (\partial A \times B^2) \cup_\psi (A \times S^1)$ of the definition of M in Section 3, and the splitting $M_1 = Q_1 \cup R_1 \cup S_1$ above; that is, $\varphi(\partial A \times B^2) = Q_1 \cup R_1$ and $\varphi(A \times S^1) = S_1$. We want to prove that the contact form α is positive on the binding (oriented as the boundary of a page) and that $d\alpha$ is a volume form on the pages (oriented as holomorphic curves of E) of the open book decomposition; see [15, Section 9.2].

Since u is strictly plurisubharmonic on C_1, the 1–form $\alpha = -d^{\mathbb{C}}u$ is a contact form on each level set of u, and the 2–form $d\alpha$ defines a symplectic structure compatible with the complex structure J. The contactness of M_1 is equivalent to the fact that the restriction $(\alpha \wedge d\alpha)|_{TM_1}$ is a volume form. On the other hand, the open book decomposition of M_1 is given by the function

$$\varphi\colon M_1 - L \to S^1, \quad \varphi(z_1, z_2) = \frac{z_2}{|z_2|},$$

where $L \subset M_1$ is the link of equation $z_2 = 0$. The vector $\partial/\partial\theta_1$ is tangent to the binding and the tangent space of the page is spanned by $\partial/\partial\theta_1$ and V, where V is the tangent vector of the curve $\{(e^{v_1}, e^{v_2}) \mid (v_1, v_2) \in c_1\}$. Notice that the binding consists of two components $L_1 = \{(z_1, z_2) \in Q_1 \mid z_2 = 0\}$ and $L_2 = \{(z_1, z_2) \in R_1 \mid z_2 = 0\}$, which are naturally oriented by $-\partial/\partial\theta_1$ and $\partial/\partial\theta_1$, respectively.

Now, we check the compatibility. Since the partial derivative $\partial u/\partial r_1$ is negative near L_1 and positive near L_2,

$$\begin{aligned}
\alpha\Big(-\frac{\partial}{\partial\theta_1}\Big)_{r_1=d_1,z_2=0} &= d^{\mathbb{C}}u\Big(\frac{\partial}{\partial\theta_1}\Big)_{r_1=d_1,z_2=0} &&= -r_1\Big(\frac{\partial u}{\partial r_1}\Big)_{r_1=d_1,z_2=0} > 0,\\
\alpha\Big(\frac{\partial}{\partial\theta_1}\Big)_{r_1=d_2,z_2=0} &= -d^{\mathbb{C}}u\Big(\frac{\partial}{\partial\theta_1}\Big)_{r_1=d_2,z_2=0} &&= r_1\Big(\frac{\partial u}{\partial r_1}\Big)_{r_1=d_2,z_2=0} > 0,
\end{aligned}$$

which imply the positivity of α along the binding.

In order to see that $d\alpha$ is a volume form on the pages, it is enough to show that the vectors $\partial/\partial\theta_1$, V and R span the tangent space of M_1, where

$$R = J\left(\frac{\nabla u}{\|\nabla u\|}\right)$$

is the Reeb vector field of the contact form $\alpha|_{TM_1}$. Since the r_2 component of the gradient vector is positive except on the binding, so is the θ_2 component of R. Therefore, the three vectors indeed span the tangent space except on the binding. □

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