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Golod and tight 3–manifolds

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The notions Golodness and tightness for simplicial complexes come from algebra and geometry, respectively. We prove these two notions are equivalent for 3–manifold triangulations, through a topological characterization of a polyhedral product for a tight-neighborly manifold triangulation of dimension ≥ 3.

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1 Introduction

Let $\mathbb{F}$ be a field, and let $S = \mathbb{F}[x_1, \dots, x_m]$, where we assume each x_i is of degree 2. Serre [26] proved that for $R = S/I$ where I is a homogeneous ideal of S, there is a coefficientwise inequality

$$P(\operatorname{Tor}^R(\mathbb{F}, \mathbb{F}); t) \leq \frac{(1+t^2)^m}{1 - t(P(\operatorname{Tor}^S(R, \mathbb{F}); t) - 1)},$$

where $P(V; t)$ denotes the Poincaré series of a graded vector space V. In the extreme case that the equality holds, R is called *Golod*. It was Golod who proved that R is Golod if and only if all products and (higher) Massey products in the Koszul homology of R vanish, where the Koszul homology of R is isomorphic to $\operatorname{Tor}^S(R, \mathbb{F})$ as a vector space.

Let K be a simplicial complex with vertex set $[m] = \{1, 2, \dots, m\}$. Let $\mathbb{F}[K]$ denote the Stanley–Reisner ring of K over $\mathbb{F}$, where we assume generators of $\mathbb{F}[K]$ are of degree 2. Then $\mathbb{F}[K]$ expresses combinatorial properties of K, and conversely, it is of particular interest to translate a given algebraic property of the Stanley–Reisner ring $\mathbb{F}[K]$ into a combinatorial property of K. We say that K is $\mathbb{F}$–*Golod* if $\mathbb{F}[K]$ is Golod. We aim to characterize Golod complexes combinatorially.

Recently, a new approach to a combinatorial characterization of Golod complexes has been taken. We can construct a space Z_K, called the *moment–angle complex* for K, in accordance with the combinatorial information of K. Then combinatorial properties are encoded in the topology of Z_K, and in particular, Golodness can be read from a homotopical property of Z_K as follows. Baskakov, Buchstaber and Panov [6] proved that the cohomology of Z_K with coefficients in $\mathbb{F}$ is isomorphic to the Koszul homology of $\mathbb{F}[K]$, where the isomorphism respects products and (higher) Massey products. Then it follows that K is Golod over any field whenever Z_K is a suspension, and so Golod complexes have been studied also in connection with desuspension of Z_K and a more general *polyhedral product*; see Grbić, Panov, Theriault and Wu [10; 11], Grujić and Welker [12], and the authors [14; 15; 16; 17; 18; 19]. See the survey by Bahri, Bendersky and Cohen [4] for more information about moment–angle complexes and polyhedral products. Here we remark that there is a Golod complex K such that Z_K is not a suspension as shown by Yano and the first author [20].

In [15; 17; 19], the authors characterized Golod complexes of dimension one and two in terms of both combinatorial properties of K and desuspension of Z_K. Here we recall the characterization of Golodness of a closed connected surface triangulation, proved in [15]. The original statement in [15] is given in terms of polyhedral products, but here we state in terms of moment–angle complexes, which is easier, as in [17, Theorem 1.3]. Recall that a simplicial complex is called *neighborly* if every pair of vertices forms an edge.

Theorem 1.1 [15, Theorem 1.1] *Let S be a triangulation of a closed connected $\mathbb{F}$–orientable surface. Then the following statements are equivalent*:

(1) *S is $\mathbb{F}$–Golod.*

(2) *S is neighborly.*

(3) *Z_S is a suspension.*

We introduce another notion of simplicial complexes coming from geometry. S-S Chern and R K Lashof proved that the total absolute curvature of an immersion $f: M \to \mathbb{R}^n$ of a compact manifold M is bounded below by the Morse number of some Morse function on M. On the other hand, the Morse number is bounded below by the Betti number. Tightness of an immersion f is defined by the equality between the total absolute curvature of an immersion f and the Betti number of M, which is the case that the total absolute curvature is minimal. See Kühnel and Lutz [22] and Kuiper [23]. It is known that an immersion f is tight if and only if for almost every closed half-space H, the inclusion $f(M) \cap H \to f(M)$ is injective in homology.

Tightness of a simplicial complex is defined as a combinatorial analog of tightness of an immersion. See [22] for details. Let K be a simplicial complex with vertex set $[m]$.

For $\varnothing \neq I \subset [m]$, the full subcomplex of K over I is defined by

$$K_I = \{\sigma \in K \mid \sigma \subset I\}.$$

Definition 1.2 Let K be a connected simplicial complex with vertex set $[m]$. We say that K is $\mathbb{F}$*–tight* if the natural map $H_*(K_I;\mathbb{F}) \to H_*(K;\mathbb{F})$ is injective for each $\varnothing \neq I \subset [m]$.

Golodness and tightness have origins in different fields of mathematics, algebra and geometry, respectively. The aim of this paper is to prove the seemingly irrelevant these two notions are equivalent for 3–manifold triangulations through the topology of Z_K or more general polyhedral products (see Section 5). Now we state the main theorem.

Theorem 1.3 *Let M be a triangulation of a closed connected $\mathbb{F}$–orientable 3–manifold. Then the following statements are equivalent*:

(1) *M is $\mathbb{F}$–Golod.*

(2) *M is $\mathbb{F}$–tight.*

(3) *Z_M is a suspension.*

Recall that a d–manifold triangulation is called *stacked* if it is the boundary of a $(d+1)$–manifold triangulation whose interior simplices are of dimension $\geq d$. Stacked manifold triangulations have been studied in several directions, and we will use its connection to tightness (Section 2). See Bagchi, Datta, Murai and Spreer [3; 9] and [22] for more on stacked manifold triangulations. Bagchi, Datta and Spreer [3] (cf Theorem 2.3) proved that a closed connected $\mathbb{F}$–orientable 3–manifold triangulation is $\mathbb{F}$–tight if and only if it is neighborly and stacked. Then we get the following corollary of Theorem 1.3, which enables us to compare with Theorem 1.1, the 2–dimensional case.

Corollary 1.4 *Let M be a triangulation of a closed connected $\mathbb{F}$–orientable 3–manifold. Then the following statements are equivalent*:

(1) *M is $\mathbb{F}$–Golod.*

(2) *M is neighborly and stacked.*

(3) *Z_M is a suspension.*

We will investigate a relation between Golodness and tightness of d–manifold triangulations for $d \geq 3$, not only for $d = 3$, through tight-neighborliness. We will prove the following theorem, where Theorem 1.3 is its special case $d = 3$.

Theorem 1.5 *Let M be a triangulation of a closed connected $\mathbb{F}$–orientable d–manifold for $d \geq 3$, and consider the following conditions:*

(1) *M is $\mathbb{F}$–Golod.*

(2) *M is $\mathbb{F}$–tight.*

(3) *M is tight-neighborly.*

(4) *the fat-wedge filtration of $\mathbb{R}Z_M$ is trivial.*

Then there are implications

$$(1) \implies (2) \impliedby (3) \implies (4) \implies (1).$$

Moreover, for $d = 3$, the implication $(2) \implies (3)$ also holds, so all conditions are equivalent.

Remarks on Theorem 1.5 are in order. Tight-neighborly triangulations of d–manifolds for $d \geq 3$ will be defined in Section 2. To clarify a connection to Theorem 1.3 and Corollary 1.4, we need to mention that a triangulated manifold of dimension ≥ 3 is tight-neighborly if and only if it is neighborly and stacked as noted soon before Theorem 2.3 below. The space $\mathbb{R}Z_K$ is the real moment–angle complex, and properties of its fat-wedge filtration will be given in Section 5. In particular, we will see that if the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial, then Z_K is a suspension. So Theorem 1.3 is the special case of Theorem 1.5 for $d = 3$ as mentioned above. Datta and Murai [9] proved that if M is tight-neighborly and $d \geq 4$, then it is $\mathbb{F}$–tight and $\beta_i(M;\mathbb{F}) = 0$ for $2 \leq i \leq d-2$, where $\beta_i(M;\mathbb{F}) = \dim H_i(M;\mathbb{F})$ denotes the i^{th} Betti number. So if $\beta_i(M;\mathbb{F}) = 0$ for $2 \leq i \leq d-2$ and $d \geq 4$, then all conditions in Theorem 1.5 are equivalent, where the triviality of the Betti numbers is necessary because as in [2, Example 3.15], there is an $\mathbb{F}$–tight 9–vertex triangulation of $\mathbb{C}P^2$ for any field $\mathbb{F}$, which is not tight-neighborly.

The paper is organized as follows. Section 2 collects properties of tight and tight-neighborly manifold triangulations that will be needed in later sections. Section 3 introduces a weak version of Golodness and proves that weak Golodness implies tightness of orientable manifold triangulations. Section 4 investigates a simplicial complex $F(M)$ constructed from a tight-neighborly d–manifold triangulation M for $d \geq 3$, and Section 5 recalls the fat-wedge filtration technique for polyhedral products, which is the main ingredient in desuspending Z_K. Section 6 applies the results in Sections 4 and 5 to prove Theorem 1.5. Finally, Section 7 poses a problem on Golodness and tightness of d–manifold triangulations for $d \geq 4$, and shows related results.

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2 Tightness

This section collects facts about tight and tight-neighborly manifold triangulations that we will use. As mentioned in Section 1, tightness of a simplicial complex is a discrete analog of a tight space studied in differential geometry with connection to minimality of the total absolute curvature, and tight complexes have been studied mainly for manifold triangulations. First, we show:

Lemma 2.1 *Every $\mathbb{F}$–tight complex is neighborly.*

Proof Let K be an $\mathbb{F}$–tight complex. Then for two vertices v and w of K, the natural map $H_0(K_{\{v,w\}};\mathbb{F}) \to H_0(K;\mathbb{F})$ is injective. Since K is connected, $H_0(K;\mathbb{F}) \cong \mathbb{F}$, and so $H_0(K_{\{v,w\}};\mathbb{F}) \cong \mathbb{F}$. Then v and w must be joined by an edge. □

Next, we explain a conjecture on tight manifold triangulations. Let K be a simplicial complex. Let $|K|$ denote its geometric realization of K, and let

$$f(K) = (f_0(K), f_1(K), \dots, f_{\dim K}(K))$$

denote the f–vector of K. We say that K is *strongly minimal* if for any simplicial complex L with $|K| \cong |L|$, it holds that

$$f_i(K) \le f_i(L)$$

for each $i \ge 0$. Kühnel and Lutz [22] conjectured that every $\mathbb{F}$–tight triangulation of a closed connected manifold is strongly minimal. Clearly, the only $\mathbb{F}$–tight closed connected 1–manifold triangulation is the boundary of a 2–simplex, so the conjecture is true in dimension 1. Moreover, the 2–dimensional case was verified, as mentioned in [22], and the 3–dimensional case was verified by Bagchi, Datta and Spreer [3]. But the case of dimensions ≥ 4 is still open.

As for minimality of manifold triangulations, we have another notion introduced by Lutz, Sulanke and Swartz [24].

Definition 2.2 A closed connected d–manifold triangulation M with vertex set $[m]$ for $d \ge 3$ is *tight-neighborly* if

$$\binom{m-d-1}{2} = \binom{d+2}{2}\beta_1(M;\mathbb{F}).$$

Tight-neighborly manifold triangulations are known to be vertex minimal. By definition, tight-neighborliness seems to depend on the ground field $\mathbb{F}$, but it is actually independent of the ground field $\mathbb{F}$ as tight-neighborly manifold triangulations are neighborly and stacked. Tightness and tight-neighborliness have the following relation. Let $S^1 \mathbin{\tilde{\times}} S^{d-1}$ denote the nontrivial S^{d-1}–bundle over S^1.

Theorem 2.3 *Let M be a closed connected $\mathbb{F}$–orientable d–manifold triangulation for $d \geq 3$, and consider the following conditions*:

(1) *M is $\mathbb{F}$–tight.*

(2) *M is tight-neighborly.*

(3) *M is neighborly and stacked.*

(4) *M has the topological type of either*

$$S^d, \quad (S^1 \mathbin{\tilde{\times}} S^{d-1})^{\#k}, \quad (S^1 \times S^{d-1})^{\#k}.$$

Then there are implications

$$(1) \Longleftarrow (2) \Longleftrightarrow (3) \Longrightarrow (4).$$

Moreover, the implication $(1) \Longrightarrow (2)$ *also holds for* $d = 3$.

Proof The implications are shown in [9] for $d \geq 4$ and [3] for $d = 3$. □

Remark The integer k in Theorem 2.3 for $d = 3$ is known to satisfy $80k + 1$ is a perfect square. For $k = 1, 30, 99, 208, 357, 546$, tight-neighborly triangulations of $(S^1 \mathbin{\tilde{\times}} S^2)^{\#k}$ are constructed in [8], but no tight-neighborly triangulation of $(S^1 \times S^2)^{\#k}$ is known.

3 Weak Golodness

This section introduces weak Golodness and studies it for manifold triangulations. Let K be a simplicial complex with vertex set $[m]$, and let $\mathcal{H}_*(\mathbb{F}[K])$ denote the Koszul homology of the Stanley–Reisner ring $\mathbb{F}[K]$. As mentioned in Section 1, K is $\mathbb{F}$–Golod if and only if all products and (higher) Massey products in $\mathcal{H}_*(\mathbb{F}[K])$ vanish. Now we define weak Golodness.

Definition 3.1 A simplicial complex K is *weakly* $\mathbb{F}$*–Golod* if all products in $\mathcal{H}_*(\mathbb{F}[K])$ vanish.

Clearly, K is weakly $\mathbb{F}$–Golod whenever it is $\mathbb{F}$–Golod. Berglund and Jöllenbeck [7] stated that Golodness and weak Golodness of every simplicial complex are equivalent, but this was disproved by Katthän [21]. Thus defining weak Golodness makes sense.

We recall a combinatorial description of the multiplication in $\mathcal{H}_*(\mathbb{F}[K])$. For disjoint nonempty subsets $I, J \subset [m]$, there is an inclusion

$$\iota_{I,J}\colon K_{I\sqcup J} \to K_I * K_J, \quad \sigma \mapsto (\sigma\cap I, \sigma\cap J).$$

Baskakov, Buchstaber and Panov proved:

Lemma 3.2 [6, Theorem 1] *There is an isomorphism of vector spaces*

$$\mathcal{H}_i(\mathbb{F}[K]) \cong \bigoplus_{\varnothing\neq I\subset[m]} \widetilde{H}^{i-|I|-1}(K_I;\mathbb{F})$$

for $i>0$ such that for nonempty subsets $I, J\subset [m]$ the multiplication

$$\widetilde{H}^{i-|I|-1}(K_I;\mathbb{F})\otimes \widetilde{H}^{j-|J|-1}(K_J;\mathbb{F}) \to \widetilde{H}^{i+j-|I\cup J|-1}(K_{I\cup J};\mathbb{F})$$

is trivial for $I\cap J\neq\varnothing$ and the induced map of $\iota_{I,J}$ for $I\cap J=\varnothing$.

Let M be a triangulation of a closed connected $\mathbb{F}$–oriented d–manifold with vertex set $[m]$. We consider a relation between the inclusion $\iota_{I,J}$ and Poincaré duality. For any subset $I\subset[m]$, Poincaré duality [13, Proposition 3.46] holds such that the map

$$H^i(|M_I|;\mathbb{F}) \to H_{d-i}(|M|, |M|-|M_I|;\mathbb{F}), \quad \alpha\mapsto \alpha\frown[M]$$

is an isomorphism, where $[M]$ denotes the fundamental class of M. By Lemma 70.1 of [25], $|M|-|M_I| \simeq |M_J|$ for $J=[m]-I$. Then there is an isomorphism

$$D_{I,J}\colon H^i(M_I;\mathbb{F}) \xrightarrow{\cong} H_{d-i}(M, M_J;\mathbb{F}).$$

Let $\partial\colon H_*(M, M_J;\mathbb{F}) \to H_{*-1}(M_J;\mathbb{F})$ denote the boundary map of the long exact sequence

$$\cdots \to H_*(M_J;\mathbb{F}) \to H_*(M;\mathbb{F}) \to H_*(M, M_J;\mathbb{F}) \xrightarrow{\partial} H_{*-1}(M_J;\mathbb{F}) \to \cdots.$$

Lemma 3.3 *Let M be a triangulation of a closed connected $\mathbb{F}$–oriented d–manifold with vertex set $[m]$. For any partition $[m]=I\sqcup J$ and $\alpha\in H^i(M_I;\mathbb{F})$,*

$$(\partial\circ D_{I,J})(\alpha) = (-1)^{i+1}(\alpha\otimes 1)((\iota_{I,J})_*([M])) \in H_{d-i-1}(M_J;\mathbb{F}),$$

where we regard $(\iota_{I,J})_*([M])$ as an element of

$$\bigoplus_{i+j=d-1} H_i(M_I;\mathbb{F})\otimes H_j(M_J;\mathbb{F}) \cong H_d(M_I * M_J;\mathbb{F}).$$

Proof Let $\varphi \in C^i(M_I;\mathbb{F})$ be a representative of α. We define $\bar\varphi \in C^i(M;\mathbb{F})$ by

$$\bar\varphi(\sigma) = \begin{cases} \varphi(\sigma) & \text{if } \sigma \in M_I, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\alpha \frown [M]$ is represented by $\bar\varphi \frown \mu$ where μ represents $[M]$. Let $[v_0,\dots,v_i]$ denote an oriented i–simplex with vertices $v_0,\dots,v_i$. We may set

$$\mu = \sum_k a_k[v_0^k, v_1^k,\dots,v_d^k] \in C_d(M;\mathbb{F})$$

for $a_k \in \mathbb{F}$, where $v_0^k,\dots,v_{n_k}^k \in I$ and $v_{n_k+1}^k,\dots,v_d^k \in J$ for some n_k. Then $(\partial \circ D_{I,J})(\alpha)$ is represented by

$$\partial(\bar\varphi \frown \mu) = (\bar\varphi \circ \partial) \frown \mu = [(\bar\varphi \circ \partial) \frown \mu] = \sum_k a_k \bar\varphi(\partial[v_0^k,\dots,v_{i+1}^k])[v_{i+1}^k,\dots,v_d^k].$$

Since $(\bar\varphi \circ \partial)|_{C_{i+1}(M_I;\mathbb{F})} = \varphi \circ \partial = 0$, we have $\bar\varphi(\partial[v_0^k,\dots,v_{i+1}^k]) \neq 0$ only when $n_k = i$. Then $(\partial \circ D_{I,J})(\alpha)$ is represented by

$$\sum_{n_k=i} a_k \bar\varphi(\partial[v_0^k,\dots,v_{i+1}^k])[v_{i+1}^k,\dots,v_d^k] = (-1)^{i+1} \sum_{n_k=i} a_k \varphi([v_0^k,\dots,v_i^k,\widehat{v_{i+1}^k}])[v_{i+1}^k,\dots,v_d^k].$$

On the other hand, since the $C_i(M_I;\mathbb{F}) \otimes C_{d-i-1}(M_J;\mathbb{F})$ part of μ is given by $\sum_{n_k=i} a_k[v_0^k,\dots,v_d^k]$, $(\iota_{I,J})_*([M])$ is represented by

$$\sum_{n_k=i} a_k[v_0^k,\dots,v_i^k] \otimes [v_{i+1}^k,\dots,v_d^k]. \qquad \square$$

Now we are ready to prove:

Theorem 3.4 *If a triangulation of a closed connected $\mathbb{F}$–orientable d–manifold is weakly $\mathbb{F}$–Golod, then it is $\mathbb{F}$–tight.*

Proof Let M be a triangulation of a closed connected $\mathbb{F}$–oriented d–manifold with vertex set $[m]$. Let $[m] = I \sqcup J$ be a partition. Suppose that the map $\iota_{I,J}$ is trivial in cohomology with coefficients in $\mathbb{F}$. Then by the universal coefficient theorem, $\iota_{I,J}$ is trivial in homology with coefficients in $\mathbb{F}$ too. Thus, by Lemma 3.3, the boundary map

$$\partial\colon H_*(M, M_J;\mathbb{F}) \to H_{*-1}(M_J;\mathbb{F})$$

is trivial, and so the natural map $H_*(M_J;\mathbb{F}) \to H_*(M;\mathbb{F})$ is injective, completing the proof. $\square$

4 The complex $F(M)$

Throughout this section, let M be a closed connected tight-neighborly d–manifold triangulation for $d \geq 3$ with vertex set $[m]$. Let K be a simplicial complex with vertex set $[m]$. A subset $I \subset [m]$ is a *minimal nonface* of K if every proper subset of I is a simplex of K and I itself is not a simplex of K. Define a simplicial complex $F(M)$ by filling all minimal nonfaces of cardinality $d+1$ into M. This section investigates the complex $F(M)$.

We set notation. The link of a vertex v in a simplicial complex K is defined by

$$\mathrm{lk}_K(v) = \{\sigma \in K \mid v \notin \sigma \text{ and } \sigma \sqcup v \in K\}.$$

For a finite set S, let $\Delta(S)$ denote the simplex with vertex set S. Then $I \subset [m]$ is a minimal nonface of K if and only if $K_I = \partial\Delta(I)$. Let K_1 and K_2 be simplicial complexes of dimension d such that $K_1 \cap K_2$ is a single d–simplex σ. Then we write

$$K_1 \# K_2 = K_1 \cup K_2 - \sigma \quad \text{and} \quad K_1 \circ K_2 = K_1 \cup K_2.$$

The following lemma may be known, but we produce a proof for completeness of the paper; cf [1; 3; 9].

Lemma 4.1 *For each $v \in [m]$, there exist $V(v,1), \dots, V(v,n_v) \subset [m]$ such that $|V(v,k)| = d+1$ for $1 \leq k \leq n_v$ and*

$$\mathrm{lk}_M(v) = \partial\Delta(V(v,1)) \# \cdots \# \partial\Delta(V(v,n_v)).$$

Proof The case $d=3$ is proved in [3, Proof of Theorem 1.2]. For $d \geq 4$, tight-neighborliness implies local stackedness, that is, every vertex link is a stacked sphere, as in [9]. Moreover, stacked spheres are characterized by Bagchi and Datta [1] such that every stacked $(d-1)$–sphere is of the form $\partial\Delta^d \# \cdots \# \partial\Delta^d$. Then we obtain the result for $d \geq 4$. ◻

Generalizing neighborliness, we say that a simplicial complex is *k–neighborly* if every $k+1$ vertices form a simplex. So 1–neighborliness is precisely neighborliness.

Lemma 4.2 *For each $v \in [m]$ and $1 \leq k \leq n_v$, $M_{V(v,k)\sqcup v}$ is $(d-1)$–neighborly.*

Proof By Lemma 4.1, $\mathrm{lk}_M(v)_{V(v,k)}$ is $\partial\Delta^d$ with some $(d-1)$–simplices removed, implying it is $(d-2)$–neighborly. So if I is a subset of $V(v,k)$ with $|I| = d-1$, then $I \sqcup v$ is a simplex of M. It remains to show $M_{V(v,k)}$ is $(d-1)$–neighborly. Let J be any subset of $V(v,k)$ with $|J|=d$. Then $\partial\Delta(J)$ is a subcomplex of M. If $M_J=\partial\Delta(J)$, then $M_{J\sqcup v}=\partial\Delta(J)*v$, which is contractible. So the inclusion $M_J\to M_{J\sqcup v}$ is not injective in homology with coefficients in $\mathbb{F}$. By Theorem 2.3, M is $\mathbb{F}$–tight, so we get a contradiction. Thus J must be a simplex of M, completing the proof. □

We prove local properties of the complex $F(M)$.

Proposition 4.3 (1) *For each $v\in[m]$,*

$$\mathrm{lk}_{F(M)}(v)=\partial\Delta(V(v,1))\circ\cdots\circ\partial\Delta(V(v,n_v)).$$

(2) *For each $v\in[m]$ and $1\le k\le n_v$, $V(v,k)\sqcup v$ is a minimal nonface of $F(M)$.*

Proof (1) Let σ be the $(d-1)$–simplex

$$(\partial\Delta(V(v,1))\#\cdots\#\partial\Delta(V(v,k)))\cap\partial\Delta(V(v,k+1)).$$

Then by Lemma 4.2, $\partial\Delta(\sigma\sqcup v)$ is a subcomplex of M, implying $\sigma\sqcup v$ is a simplex of $F(M)$. Then by induction, we get $\partial\Delta(V(v,1))\circ\cdots\circ\partial\Delta(V(v,n_v))\subset\mathrm{lk}_{F(M)}(v)$. The reverse inclusion is obvious by the construction of $F(M)$, completing the proof.

(2) By Lemma 4.2, $V(v,k)$ is a simplex of $F(M)$, so every proper subset I of $V(v,k)\sqcup v$ is a simplex of $F(M)$. By (1), $V(v,k)\sqcup v$ is not a simplex of $F(M)$. □

We compute the homology of $F(M)$. Let

$$S(M)=\{V(v,k)\sqcup v\mid v\in[m]\text{ and }1\le k\le n_v\}.$$

Then $S(M)$ is the set of all subsets $I\subset[m]$ such that $|I|=d+2$ and $\mathrm{lk}_{M_I}(v)$ is $(d-2)$–neighborly for some $v\in I$.

Lemma 4.4

$$F(M)=\bigcup_{I\in S(M)}\partial\Delta(I).$$

Proof Let $K=\bigcup_{I\in S(M)}\partial\Delta(I)$. By Proposition 4.3, $K\subset F(M)$. For any k–simplex σ of $F(M)$ with $0\le k\le d-1$ and $v\in\sigma$, $\sigma-v$ is a simplex of $\mathrm{lk}_M(v)$ because σ is a simplex of M too. Then $\sigma-v\subset V(v,l)$ for some $1\le l\le n_v$, implying σ is a simplex of K. Thus the $(d-1)$–skeleton of $F(M)$ is included in K. Take any d–simplex σ of $F(M)$. Then σ is either a simplex or a minimal nonface of M. In both cases, $\partial\Delta(\sigma-v)$ is a subcomplex of $\mathrm{lk}_M(v)$ for $v\in\sigma$. Then $\sigma-v\subset V(v,l)$ for some $1\le l\le n_v$, implying σ is a simplex of K. Thus $F(M)\subset K$. □

By Lemma 4.4, there is an inclusion $g_I\colon\partial\Delta(I)\to F(M)$ for each $I\in S(M)$. Let $u_I\in H_d(F(M);\mathbb{Z})$ be the Hurewicz image of g_I.

Proposition 4.5 *The integral homology of $F(M)$, except for dimension* 1, *is given by*

$$\widetilde{H}_*(F(M);\mathbb{Z}) = \begin{cases} \mathbb{Z}\langle u_I \mid I \in S(M)\rangle & \text{if } * = d, \\ 0 & \text{if } * \neq 1, d. \end{cases}$$

Proof Since $F(M)$ is obtained from M by attaching d–simplices, we only need to calculate H_{d-1} and H_d by Theorem 2.3. By Proposition 4.3, each component of $\mathrm{lk}_{M_I}(v)$ is $(d-2)$–connected, where $\mathrm{lk}_{M_I}(v) = \mathrm{lk}_M(v)_{I-v}$. Then there is an exact sequence

$$(1)\quad 0 \to \widetilde{H}_d(F(M)_{I-v};\mathbb{Z}) \to H_d(F(M)_I;\mathbb{Z}) \xrightarrow{\partial} H_{d-1}(\mathrm{lk}_{F(M)_I}(v);\mathbb{Z}) \\ \to H_{d-1}(F(M)_{I-v};\mathbb{Z}) \to H_{d-1}(F(M)_I;\mathbb{Z}) \to 0.$$

By Proposition 4.3, there is an inclusion $\partial\Delta(V(v,k)) \to \mathrm{lk}_{F(M)_I}(v)$ for $V(v,k) \sqcup v \subset I$, and we write the Hurewicz image of this inclusion by $\bar{u}_{V(v,k)}$. Then we have

$$H_{d-1}(\mathrm{lk}_{F(M)_I}(v);\mathbb{Z}) = \mathbb{Z}\langle \bar{u}_{V(v,k)} \mid V(v,k) \sqcup v \subset I\rangle$$

such that $\partial(u_{V(v,k)\sqcup v}) = \bar{u}_{V(v,k)}$. Hence the map ∂ in (1) is surjective, so we get an isomorphism

$$H_{d-1}(F(M)_{I-v};\mathbb{Z}) \cong H_{d-1}(F(M)_I;\mathbb{Z}).$$

Thus we obtain $H_{d-1}(F(M)_I;\mathbb{Z}) = 0$ for any $I \subset [m]$ by induction on $|I|$, where $H_{d-1}(F(M)_I;\mathbb{Z}) = 0$ for $|I| = 1$. We also get a split exact sequence

$$0 \to H_d(F(M)_{I-v};\mathbb{Z}) \to H_d(F(M)_I;\mathbb{Z}) \xrightarrow{\partial} H_{d-1}(\mathrm{lk}_{F(M)_I}(v);\mathbb{Z}) \to 0.$$

Then by induction on $|I|$, we also obtain

$$H_d(F(M)_I;\mathbb{Z}) = \mathbb{Z}\langle u_{V(v,k)} \mid V(v,k) \sqcup v \subset I\rangle. \qquad \square$$

By Theorem 2.3, $\pi_1(|M|)$ is a free group. Since $|F(M)|$ is obtained by attaching d–cells to $|M|$, the inclusion $|M| \to |F(M)|$ is an isomorphism in π_1, so $\pi_1(|F(M)|)$ is a free group too. Then there is a map $f\colon B \to |F(M)|$ which is an isomorphism in π_1, where B is a wedge of circles. Let $\widehat{F}(M)$ be the cofiber of f. Since there is an exact sequence

$$\cdots \to H_*(B;\mathbb{Z}) \xrightarrow{f_*} H_*(F(M);\mathbb{Z}) \to \widetilde{H}_*(\widehat{F}(M);\mathbb{Z}) \to \cdots,$$

the natural map $H_*(F(M);\mathbb{Z}) \to H_*(\widehat{F}(M);\mathbb{Z})$ is an isomorphism for $* \neq 1$. Let $\hat{g}_I$ be the composite $|\partial\Delta(I)| \xrightarrow{g_I} |F(M)| \to \widehat{F}(M)$ for $I \in S(M)$, and let $\hat{u}_I$ be the Hurewicz image of $\hat{g}_I$. By Proposition 4.5, we get:

Corollary 4.6 *The reduced homology of $\widehat{F}(M)$ is given by*

$$\widetilde{H}_*(\widehat{F}(M);\mathbb{Z}) = \begin{cases} \mathbb{Z}\langle \hat{u}_I \mid I \in S(M)\rangle & \text{if } * = d, \\ 0 & \text{if } * \neq d. \end{cases}$$

Since $\widehat{F}(M)$ is path-connected, there is a map

$$g\colon \bigvee_{I\in S(M)} |\partial\Delta(I)| \to \widehat{F}(M)$$

such that $g|_{|\partial\Delta(I)|} \simeq \hat{g}_I$ for each $I \in S(M)$. Then by Corollary 4.6 and the Whitehead theorem, we obtain the following.

Corollary 4.7 *The map $g\colon \bigvee_{I\in S(M)} |\partial\Delta(I)| \to \widehat{F}(M)$ is a homotopy equivalence.*

5 Polyhedral product

Throughout this section, let K be a simplicial complex with vertex set $[m]$. Let $(\underline{X},\underline{A}) = \{(X_i,A_i)\}_{i=1}^m$ be a collection of pairs of pointed spaces indexed by vertices of K. For $I \subset [m]$, let

$$(\underline{X},\underline{A})^I = Y_1 \times \cdots \times Y_m$$

where $Y_i = X_i$ for $i \in I$ and $Y_i = A_i$ for $i \notin I$. The *polyhedral product* of $(\underline{X},\underline{A})$ over K is defined by

$$Z_K(\underline{X},\underline{A}) = \bigcup_{\sigma\in K} (\underline{X},\underline{A})^\sigma.$$

For $\varnothing \neq I \subset [m]$, let $(\underline{X}_I,\underline{A}_I) = \{(X_i,A_i)\}_{i\in I}$. Then we can define $Z_{K_I}(\underline{X}_I,\underline{A}_I)$. The following lemma is immediate from the definition of a polyhedral product.

Lemma 5.1 *For each $\varnothing \neq I \subset [m]$, $Z_{K_I}(\underline{X}_I,\underline{A}_I)$ is a retract of $Z_K(\underline{X},\underline{A})$.*

For a collection of pointed spaces $\underline{X} = \{X_i\}_{i=1}^m$, let $(C\underline{X},\underline{X}) = \{(CX_i,X_i)\}_{i=1}^m$. For $0 \le i \le m$, we define a subspace of $Z_K(C\underline{X},\underline{X})$ by

$$Z_K^i(C\underline{X},\underline{X}) = \{(x_1,\dots,x_m) \in Z_K(C\underline{X},\underline{X}) \mid \text{at least } m-i \text{ of } x_1,\dots,x_m \text{ are basepoints}\}.$$

Using the basepoint of each X_i, we regard $Z_{K_I}(C\underline{X}_I,\underline{X}_I)$ as a subspace of $Z_K(C\underline{X},\underline{X})$ so that we can alternatively write

$$Z_K^i(C\underline{X},\underline{X}) = \bigcup_{I\subset[m],\,|I|=i} Z_{K_I}(C\underline{X}_I,\underline{X}_I). \tag{2}$$

There is a filtration

$$*=Z^0_K(C\underline{X},\underline{X})\subset Z^1_K(C\underline{X},\underline{X})\subset\cdots\subset Z^m_K(C\underline{X},\underline{X})=Z_K(C\underline{X},\underline{X}),$$

which we call the *fat-wedge filtration* of $Z_K(C\underline{X},\underline{X})$. By [17, Theorem 4.1],

$$Z^i_K(C\underline{X},\underline{X})/Z^{i-1}_K(C\underline{X},\underline{X})=\bigvee_{I\subset[m],\,|I|=i}|\Sigma K_I|\wedge\widehat{X}^I,$$

where $\widehat{X}^I=\bigwedge_{i\in I}X_i$. Moreover, it is shown in [17, Corollary 4.2] that the fat-wedge filtration of $Z_K(C\underline{X},\underline{X})$ splits after a suspension, and the decomposition of Bahri, Bendersky, Cohen and Gitler [5, Theorem 2.2.1] is reproduced as:

Theorem 5.2 (BBCG decomposition) *There is a homotopy equivalence*

$$\Sigma Z_K(C\underline{X},\underline{X})\simeq\Sigma\bigvee_{\varnothing\neq I\subset[m]}|\Sigma K_I|\wedge\widehat{X}^I.$$

In particular, if the BBCG decomposition desuspends, then $Z_K(C\underline{X},\underline{X})$ itself desuspends. Moreover, if each X_i is a connected CW complex, then the BBCG decomposition desuspends whenever $Z_K(C\underline{X},\underline{X})$ desuspends [17]. Then we aim to desuspend the BBCG decomposition. Desuspension of the BBCG decomposition was studied for specific Golod complexes such as shifted complexes [11; 12; 14] by ad hoc methods, and desuspension for much broader classes of simplicial complexes, including the previous specific simplicial complexes, was proved by using the fat-wedge filtration technique [17].

The moment–angle complex Z_K introduced in Section 1 is the polyhedral product $Z_K(D^2,S^1)$. The *real moment–angle complex* $\mathbb{R}Z_K$ is defined to be the polyhedral product $Z_K(D^1,S^0)$, and we denote its fat-wedge filtration by

$$*=\mathbb{R}Z^0_K\subset\mathbb{R}Z^1_K\subset\cdots\subset\mathbb{R}Z^m_K=\mathbb{R}Z_K$$

where we choose the basepoint of $S^0=\{-1,+1\}$ to be -1. The fat-wedge filtration of $\mathbb{R}Z_K$ is proved to be a cone decomposition [17, Theorem 3.1]. For $\varnothing\neq I\subset[m]$, let $j_{K_I}:\mathbb{R}Z^{|I|-1}_{K_I}\to\mathbb{R}Z^{|I|-1}_K$ denote the inclusion.

Theorem 5.3 [17, Theorem 3.1] *For each $\varnothing\neq I\subset[m]$, there is a map*

$$\varphi_{K_I}:|K_I|\to\mathbb{R}Z^{|I|-1}_{K_I}$$

such that

$$\mathbb{R}Z^i_K=\mathbb{R}Z^{i-1}_K\bigcup_{I\subset[m],\,|I|=i}C|K_I|,$$

where the attaching maps are $j_{K_I}\circ\varphi_{K_I}$.

We say that the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial if φ_{K_I} is nullhomotopic for each $\varnothing \neq I \subset [m]$. We remark that φ_{K_I} is nullhomotopic if and only if $j_{K_I} \circ \varphi_{K_I}$ is, because $\mathbb{R}Z_{K_I}^{|I|-1}$ is a retract of $\mathbb{R}Z_K^{|I|-1}$. The fat-wedge filtration is useful for desuspending the BBCG decomposition because we have the following criterion.

Theorem 5.4 [17, Theorem 1.2] *If the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial, then for any $\underline{X}$, there is a homotopy equivalence*

$$Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\varnothing \neq I \subset [m]} |\Sigma K_I| \wedge \hat{X}^I.$$

For $\varnothing \neq I \subset [m]$, define a map $\alpha_I : \mathbb{R}Z_{K_I}^{|I|-1} \to \mathbb{R}Z_K^{m-1}$ by $\alpha_I(x_i \mid i \in I) = (y_1, \dots, y_m)$ such that

$$y_i = \begin{cases} x_i & \text{if } i \in I, \\ +1 & \text{if } i \notin I, \end{cases}$$

for $(x_i \mid i \in I) \in \mathbb{R}Z_{K_I}^{|I|-1}$. Note that α_I is not the natural inclusion because the basepoint of $S^0 = \{-1, +1\}$ is taken to be -1 as mentioned above. For $\varnothing \neq J \subset I \subset [m]$ and $|J| \le i \le |I|$, let π denote the composite of projections

$$\mathbb{R}Z_{K_I}^i \to \mathbb{R}Z_{K_J} \to \mathbb{R}Z_{K_J}/\mathbb{R}Z_{K_J}^{|J|-1} = |\Sigma K_J|.$$

By the construction of φ_K, we have:

Lemma 5.5 *For $\varnothing \neq J \subsetneq I \subset [m]$, there is a commutative diagram*

$$\begin{tikzcd} |K_I| \arrow[r, "\varphi_{K_I}"] \arrow[d] & \mathbb{R}Z_{K_I}^{|I|-1} \arrow[r, "\pi"] \arrow[d, "\alpha_I"] & |\Sigma K_J| \arrow[d, "|\Sigma j|"] \\ |K| \arrow[r, "\varphi_K"] & \mathbb{R}Z_K^{m-1} \arrow[r, "\pi"] & |\Sigma K_{J \sqcup ([m]-I)}| \end{tikzcd}$$

where $j : K_J \to K_{J \sqcup ([m]-I)}$ is the inclusion.

The following two lemmas, proved in [17, Proof of Theorem 7.2] and [17, Lemma 10.1] respectively, are quite useful in detecting the triviality of φ_K.

Lemma 5.6 *Let $\overline{K}$ be a simplicial complex obtained by filling all minimal nonfaces into K. Then φ_K factors through the inclusion $|K| \to |\overline{K}|$.*

Lemma 5.7 *If $\varphi_{K_I} \simeq *$ for each $\varnothing \neq I \subsetneq [m]$, then the composite*

$$|K| \xrightarrow{\varphi_K} \mathbb{R}Z_K^{m-1} \to \mathbb{R}Z_{K_J} \xrightarrow{\pi} |\Sigma K_J|$$

is nullhomotopic for each $\varnothing \neq J \subsetneq [m]$.

Finally, we estimate the connectivity of $\mathbb{R}Z_K$.

Lemma 5.8 *If K is k–neighborly, then $\mathbb{R}Z_K$ is k–connected.*

Proof The proof can be done by the same calculation as [17, Proposition 5.3]. Here, we give an alternative proof. By definition, $\pi_*(\mathbb{R}Z_K)$ is isomorphic to $\pi_*(\mathbb{R}Z_{K_k})$ for $* \le k$, where K_k denotes the k–skeleton of K. Since K is k–neighborly, $K_k = \Delta_k^{m-1}$. Since Δ_k^{m-1} is shifted, it follows from [14] that there is a homotopy equivalence

$$\mathbb{R}Z_{\Delta_k^{m-1}} \simeq \bigvee_{\varnothing \neq I \subset [m]} |\Sigma(\Delta_k^{m-1})_I|.$$

Since each $|\Sigma(\Delta_k^{m-1})_I|$ is k–connected, the proof is done. □

6 Proof of Theorem 1.5

Throughout this section, let M be a tight-neighborly triangulation of a closed connected $\mathbb{F}$–orientable d–manifold with vertex set $[m]$, unless otherwise is specified. We aim to prove that the fat-wedge filtration of $\mathbb{R}Z_M$ is trivial. First, we compute the fundamental group of $|F(M)_I|$ for $\varnothing \neq I \subset [m]$.

Lemma 6.1 *For each $\varnothing \neq I \subset [m]$, $\pi_1(|F(M)_I|)$ is a free group.*

Proof Since the fundamental group of a suspension is a free group, we prove $|F(M)_I|$ is a suspension by induction on I. For $|I| = 1$, $|F(M)_I|$ is obviously a suspension. Suppose that $|F(M)_{I-v}|$ is a suspension for $v \in I$. Note that

$$F(M)_I = F(M)_{I-v} \cup (\mathrm{lk}_{F(M)_I}(v) * v) \tag{3}$$

where $F(M)_{I-v} \cap (\mathrm{lk}_{F(M)_I}(v) * v) = \mathrm{lk}_{F(M)_I}(v)$. Since $\mathrm{lk}_{F(M)_I}(v) = \mathrm{lk}_{F(M)}(v)_{I-v}$, it follows from Proposition 4.3 that there are inclusions

$$\mathrm{lk}_{F(M)_I}(v) \to (\Delta(V(v,1)) \circ \cdots \circ \Delta(V(v,n_v)))_{I-v} \to F(M)_{I-v}.$$

Since M is neighborly by Theorem 2.3, so is M_{I-v}, implying $F(M)_{I-v}$ is connected. On the other hand, each component of $(\Delta(V(v,1)) \circ \cdots \circ \Delta(V(v,n_v)))_{I-v}$ is contractible. Then the inclusion $|(\Delta(V(v,1)) \circ \cdots \circ \Delta(V(v,n_v)))_{I-v}| \to |F(M)_{I-v}|$ is nullhomotopic, and so the inclusion $|\mathrm{lk}_{F(M)_I}(v)| \to |F(M)_{I-v}|$ is nullhomotopic too. Thus by (3), we get a homotopy equivalence

$$|F(M)_I| \simeq |F(M)_{I-v}| \vee |\Sigma\, \mathrm{lk}_{F(M)_I}(v)|.$$

Since $|F(M)_{I-v}|$ is a suspension by the induction hypothesis, $|F(M)_I|$ turns out to be a suspension, completing the proof. □

Let $\varnothing \neq I \subset [m]$. By Lemma 5.6, the map φ_{M_I} decomposes as

$$|M_I| \to |F(M)_I| \to \mathbb{R}Z_{M_I}^{|I|-1}. \tag{4}$$

By Lemma 6.1, there is a map $f_I \colon B_I \to |F(M)_I|$, where B_I is a wedge of circles, such that f_I is an isomorphism in π_1. Let $\widehat{F}(M)_I$ denote the cofiber of f_I, where $\widehat{F}(M)_{[m]}$ coincides with $\widehat{F}(M)$ in Section 4. On the other hand, since M is neighborly by Lemma 2.1, so is M_J for any $\varnothing \neq J \subset [m]$. Then by (2) and Lemma 5.8, we can see that $\mathbb{R}Z_{M_I}^{|I|-1}$ is simply connected. In particular, there is a commutative diagram

$$\begin{tikzcd}
|F(M)_I| \arrow[r] \arrow[d] & \widehat{F}(M)_I \arrow[d] \\
\mathbb{R}Z_{M_I}^{|I|-1} \arrow[r, equal] & \mathbb{R}Z_{M_I}^{|I|-1}
\end{tikzcd} \tag{5}$$

Then by combining (4) and (5), we get:

Lemma 6.2 *For each $\varnothing \neq I \subset [m]$, the map φ_{M_I} factors through the inclusion $|M_I| \to \widehat{F}(M)_I$.*

Proposition 6.3 *For each $\varnothing \neq I \subsetneq [m]$, the map φ_{M_I} is nullhomotopic.*

Proof As is computed in the proof of Proposition 4.5, $\widetilde{H}_*(F(M)_I;\mathbb{Z}) = 0$ unless $* = 1, d$. Thus as well as $\widehat{F}(M)$, we can see that $\widehat{F}(M)_I$ is $(d-1)$–connected. Since $I \neq [m]$, $|M_I|$ is homotopy equivalent to a CW complex of dimension $\leq d-1$. Then we obtain that the inclusion $|M_I| \to \widehat{F}(M)_I$ is nullhomotopic. Thus by Lemma 6.2, the proof is complete. □

It remains to show that φ_M is nullhomotopic. By Lemma 5.5, there is a commutative diagram

$$\begin{tikzcd}
\bigvee_{I \in S(M)} |M_I| \arrow[r] \arrow[d, "\bigvee_{I \in S(M)} \varphi_{M_I}"'] & |M| \arrow[d, "\varphi_M"] \\
\bigvee_{I \in S(M)} \mathbb{R}Z_{M_I}^{d+1} \arrow[r, "\bigvee_{I \in S(M)} \alpha_I"] & \mathbb{R}Z_M^{m-1}
\end{tikzcd}$$

Then since $F(M)_I = \partial\Delta(I)$ for $I \in S(M)$ by Proposition 4.3, we get a commutative diagram

$$\begin{tikzcd}
\bigvee_{I \in S(M)} |\partial\Delta(I)| \arrow[r, "\bigvee_{I \in S(M)} g_I"] \arrow[d] & |F(M)| \arrow[d] \\
\bigvee_{I \in S(M)} \mathbb{R}Z_{M_I}^{d+1} \arrow[r, "\bigvee_{I \in S(M)} \alpha_I"] & \mathbb{R}Z_M^{m-1}
\end{tikzcd} \tag{6}$$

Juxtaposing the commutative diagrams (5) and (6), we get a commutative diagram

$$\begin{tikzcd} \bigvee_{I\in S(M)} |\partial\Delta(I)| \arrow[r, "g"] \arrow[d] & \widehat{F}(M) \arrow[d] \\ \bigvee_{I\in S(M)} \mathbb{R}Z_{M_I}^{d+1} \arrow[r, "\bigvee_{I\in S(M)}\alpha_I"] & \mathbb{R}Z_M^{m-1} \end{tikzcd}$$

and by Corollary 4.7 and Lemma 6.2, we obtain:

Lemma 6.4 *The map $\varphi_M\colon |M|\to \mathbb{R}Z_M^{m-1}$ is homotopic to the composite*

$$|M|\to \widehat{F}(M)\xrightarrow{g^{-1}} \bigvee_{I\in S(M)} |\partial\Delta(I)| \to \bigvee_{I\in S(M)} \mathbb{R}Z_{M_I}^{d+1} \xrightarrow{\bigvee_{I\in S(M)}\alpha_I} \mathbb{R}Z_M^{m-1}.$$

We will investigate the composition of maps in Lemma 6.4 by identifying a homotopy set with a homology.

Lemma 6.5 *Let W be a finite wedge of S^d. Then there is an isomorphism of sets*

$$[|M|,W]\cong H^d(M;\mathbb{Z})\otimes H_d(W;\mathbb{Z})$$

which is natural with respect to maps among finite wedges of S^d.

Proof Since $\dim M=d$, the statement follows from the Hopf degree theorem. □

Lemma 6.6 *For each $v\in I\in S(M)$, the natural map*

$$H^d(M;\mathbb{Z})\otimes H_{d-1}(M_{I-v};\mathbb{Z})\to H^d(M;\mathbb{Z})\otimes H_{d-1}(M_{[m]-v};\mathbb{Z})$$

is injective.

Proof By Lemma 4.2, $|M_{I-v}|$ is contractible or S^{d-1}. In particular, $H_{d-1}(M_{I-v};\mathbb{Z})$ is a free abelian group, and so there is a natural isomorphism

$$H_{d-1}(M_{I-v};\mathbb{F})\cong H_{d-1}(M_{I-v};\mathbb{Z})\otimes\mathbb{F}. \tag{7}$$

By definition, $|M_{[m]-v}|$ is $|M|$ removed the open star of v, which is homotopy equivalent to $|M|-v$ by [25, Lemma 70.1]. Then by Theorem 2.3, $|M_{[m]-v}|$ is homotopy equivalent to a wedge of finitely many, possibly zero, copies of S^1 and S^{d-1}. Then $H_*(M_{[m]-v};\mathbb{Z})$ is a free abelian group, and so there is a natural isomorphism

$$H_{d-1}(M_{[m]-v};\mathbb{F})\cong H_{d-1}(M_{[m]-v};\mathbb{Z})\otimes\mathbb{F}. \tag{8}$$

Since M is $\mathbb{F}$–tight by Theorem 2.3, the natural map

$$H_{d-1}(M_{I-v};\mathbb{F}) \to H_{d-1}(M_{[m]-v};\mathbb{F})$$

is injective. Then by (7) and (8), the natural map

$$H_{d-1}(M_{I-v};\mathbb{Z}) \otimes \mathbb{F} \to H_{d-1}(M_{[m]-v};\mathbb{Z}) \otimes \mathbb{F}$$

is injective too. Since both $H_{d-1}(M_{I-v};\mathbb{Z})$ and $H_{d-1}(M_{[m]-v};\mathbb{Z})$ are free abelian groups, the case that M is orientable is proved because $H^d(M;\mathbb{Z}) \cong \mathbb{Z}$. If M is nonorientable, then $H^d(M;\mathbb{Z}) \cong \mathbb{F}_2$ and the base field $\mathbb{F}$ is of characteristic 2, where $\mathbb{F}_2$ is the field of two elements. Thus the case that M is not orientable is proved too. □

Proposition 6.7 *The map $\varphi_M : |M| \to \mathbb{R}Z_M^{m-1}$ is nullhomotopic.*

Proof Note that $m \geq d+2$. Let $\varnothing \neq J \subset I \in S(M)$. By Lemma 4.2, $|M_J|$ is contractible for $|J| \leq d$, and $|M_J|$ is contractible or S^{d-1} for $|J| = d+1$. Then by Proposition 6.3, there is a homotopy equivalence

$$\mathbb{R}Z_{M_I}^{d+1} \simeq \bigvee_{v\in I} |\Sigma M_{I-v}|, \tag{9}$$

where $|\Sigma M_{I-v}|$ is contractible or S^d as mentioned above. Let

$$A = \bigvee_{I\in S(M)} \bigvee_{v\in I} |\Sigma M_{I-v}| \quad \text{and} \quad B = \bigvee_{I\in S(M)} \bigvee_{v\in I} |\Sigma M_{[m]-v}|,$$

where $A \simeq \bigvee_{I\in S(M)} \mathbb{R}Z_{M_I}^{d+1}$ by (9). Let $f : |M| \to A$ denote the composition of the first three maps in Lemma 6.4. Then it suffices to show f is nullhomotopic. By Lemma 6.5, f is identified with some element ϕ of $H_d(M;\mathbb{Z}) \otimes H_d(A;\mathbb{Z})$, so f is nullhomotopic if and only if $\phi = 0$.

As in the proof of Lemma 6.6, $|\Sigma M_{[m]-v}|$ is a wedge of finitely many copies of S^2 and S^d for each vertex v of M. Let C_v denote the S^d–wedge part of $|\Sigma M_{[m]-v}|$. Then there is a projection $q_v : B \to C_v$. By Lemmas 5.5, 5.7 and 6.4, the composite

$$|M| \xrightarrow{f} A \to |\Sigma M_{I-v}| \to |\Sigma M_{[m]-v}| \tag{10}$$

is nullhomotopic for each $v \in I \in S(M)$. Then by Lemma 6.5, ϕ is mapped to 0 by

$$1 \otimes (q_v \circ j)_* : H^d(M;\mathbb{Z}) \otimes H_d(A;\mathbb{Z}) \to H^d(M;\mathbb{Z}) \otimes H_d(C_v;\mathbb{Z})$$

for each $v \in I \in S(M)$, where $j : A \to B$ denotes the inclusion. Since the map

$$\bigoplus_{v\in I\in S(M)} (q_v)_* : H_d(B;\mathbb{Z}) \to \bigoplus_{v\in I\in S(M)} H_d(C_v;\mathbb{Z})$$

is an isomorphism, we get $(1 \otimes j_*)(\phi) = 0$. Thus we obtain $\phi = 0$ by Lemma 6.6, completing the proof. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5 The implications (1) $\Longrightarrow$ (2) $\Longleftarrow$ (3) are proved by Theorems 2.3 and 3.4. The implication (3) $\Longrightarrow$ (4) is proved by Propositions 6.3 and 6.7. If (4) holds, then by Theorem 5.4, Z_M is a suspension. So by the fact that K is $\mathbb{F}$–Golod whenever Z_K is a suspension, as mentioned in Section 1, we obtain the implication (4) $\Longrightarrow$ (1), completing the proof. □

7 A further problem

So far, we have been studying a relationship between Golodness and tightness through tight-neighborliness which perfectly works in dimension 3. However, in dimensions ≥ 4, tight-neighborliness does not work well because it is not equivalent to tightness as mentioned in Section 1. So we pose:

Problem 7.1 What condition on closed connected d–manifold triangulations with $d \geq 4$ guarantees $\mathbb{F}$–Golodness and $\mathbb{F}$–tightness being equivalent?

One approach is to put a topological condition on manifolds. For example, the condition on the Betti number is stated in Section 1. We also have the following theorem, in which manifold triangulations are not tight-neighborly.

Theorem 7.2 *Let M be a triangulation of a closed $(d-1)$–connected $2d$–manifold for $d \geq 2$. Then the following are equivalent*:

1. *M is $\mathbb{F}$–Golod for any field $\mathbb{F}$.*
2. *M is $\mathbb{F}$–tight for any field $\mathbb{F}$.*
3. *M is d–neighborly.*
4. *the fat-wedge filtration of $\mathbb{R}Z_M$ is trivial.*

Proof The implication (1) $\Longrightarrow$ (2) holds by Theorem 3.4 because M is orientable. Suppose M has a minimal nonface I with $|I| \leq d+1$. Then $M_I = \partial\Delta(I)$, implying $H_{|I|-2}(M_I;\mathbb{F}) \neq 0$. Since M is $\mathbb{F}$–tight, the natural map

$$H_{|I|-2}(M_I;\mathbb{F}) \to H_{|I|-2}(M;\mathbb{F})$$

is injective, and since M is $(d-1)$–connected, $\widetilde{H}_*(M;\mathbb{F})=0$ for $*<d$. Then we get a contradiction, so we obtain the implication (2) $\Longrightarrow$ (3). The implication (3) $\Longrightarrow$ (4) follows from [17, Theorem 1.6]. The implication (4) $\Longrightarrow$ (1) holds by the fact that K is $\mathbb{F}$–Golod over any field $\mathbb{F}$ whenever Z_K is a suspension, as mentioned in Section 1. Therefore, the proof is complete. $\square$

In closing the paper, we consider a relation between weak $\mathbb{F}$–Golodness and $\mathbb{F}$–tightness. As proved in Theorem 3.4, weak $\mathbb{F}$–Golodness implies $\mathbb{F}$–tightness for a closed connected $\mathbb{F}$–orientable manifold triangulations. So one might ask whether or not this implication holds for simplicial complexes which are not manifolds. The answer is no. For example, if K is the join of a vertex and the boundary of a simplex, then it is $\mathbb{F}$–Golod for any field $\mathbb{F}$ as the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial but it is not $\mathbb{F}$–tight as in the proof of Lemma 4.2. However, the opposite implication always holds as follows, which shows that Theorem 3.4 is thought of as a "wrong way" implication.

Proposition 7.3 *Let K be a simplicial complex with vertex set $[m]$. If K is $\mathbb{F}$–tight, then it is weakly $\mathbb{F}$–Golod.*

Proof Take any disjoint subsets $\varnothing\neq I,J\subset[m]$. Then there is a map

$$\iota_{I,J}\colon K_{I\sqcup J}\to K_I*K_J$$

as in Section 3. By Lemma 3.2, K is weakly $\mathbb{F}$–Golod if and only if the map $\iota_{I,J}$ is trivial in homology with coefficients in $\mathbb{F}$. Now we suppose K is $\mathbb{F}$–tight. Then $K_{I\sqcup J}$ is $\mathbb{F}$–tight too, and so we only need to consider the case $I\sqcup J=[m]$. By the Künneth theorem, the map

$$(j_I*j_J)_*\colon \widetilde{H}_*(K_I*K_J;\mathbb{F})\to\widetilde{H}_*(K*K;\mathbb{F})$$

is injective, where $j_I\colon K_I\to K$ denotes the inclusion. Then it suffices to show the composite $(j_I*j_J)\circ\iota_{I,J}$ is nullhomotopic.

Now we may assume $|K|\subset\mathbb{R}^m$ by identifying a simplex $\{i_1,\dots,i_k\}\in K$ with

$$\{t_1e_{i_1}+\cdots+t_ke_{i_k}\mid t_1+\cdots+t_k=1,\ t_1,\dots,t_k\geq 0\},$$

where $e_1,\dots,e_m$ is the standard basis of $\mathbb{R}^m$. We may assume $|K*K|\subset\mathbb{R}^{2m}$ in the same way. Consider a homotopy $h_t^i\colon\mathbb{R}^{2m}\times[0,1]\to\mathbb{R}^{2m}$ defined by

$$h_t^i(x_1,\dots,x_m,y_1,\dots,y_m)$$
$$=(x_1,\dots,(1-t)x_i+ty_i,\dots,x_m,y_1,\dots,tx_i+(1-t)y_i,\dots,y_m)$$

for $(x_1,\dots,x_m,y_1,\dots,y_m)\in\mathbb{R}^{2m}$. Then h^i_t restricts to a homotopy

$$h^i_t\colon |K*K|\times[0,1]\to|K*K|$$

such that for $i\in I$,

$$(j_I*j_J)\circ\iota_{I,J}=h^i_0\circ(j_I*j_J)\circ\iota_{I,J}\simeq h^i_1\circ(j_I*j_J)\circ\iota_{I,J}=(j_{I-i}*j_{J\cup i})\circ\iota_{I-i,J\cup i}.$$

Thus for $v\in[m]$, $(j_I*j_J)\circ\iota_{I,J}\simeq(j_v*j_{[m]-v})\circ\iota_{v,[m]-v}$. Since $|v*K_{[m]-v}|$ is contractible, we get $(j_I*j_J)\circ\iota_{I,J}\simeq *$, completing the proof. □

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