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Tetsuya Ito

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A remark on the finiteness of purely cosmetic surgeries

TETSUYA ITO

By estimating the knot Floer thickness in terms of the genus and the braid index, we show that a knot K in S^3 does not admit purely cosmetic surgery whenever $g(K) \geq \frac{3}{2}b(K)$, where $g(K)$ and $b(K)$ denote the genus and the braid index, respectively. In particular, this establishes the finiteness of purely cosmetic surgeries; for a fixed b, all but finitely many knots with braid index b satisfies the cosmetic surgery conjecture.

57K10; 57K30

For a knot K in the 3–sphere S^3 and $r \in \mathbb{Q}$, let $S^3_K(r)$ be the r–surgery on K. Two Dehn surgeries $S^3_K(r)$ and $S^3_K(r')$ on the same knot K are *purely cosmetic* if $r \neq r'$ but $S^3_K(r) \cong S^3_K(r')$. Here we write $M \cong N$ if M and N are *orientation-preservingly* homeomorphic.

Conjecture 1 (cosmetic surgery conjecture) *A nontrivial knot does not admit purely cosmetic surgeries.*

One must be careful to take account of orientations; there are several examples of *chirally cosmetic surgery*, a pair of Dehn surgeries on the same knot, that yields orientation-reversingly homeomorphic 3–manifolds. For example, for the trefoil knot K, $S^3_K(9) \cong -S^3_K\left(\frac{9}{2}\right)$; see Mathieu [7]. Here $-M$ is the 3–manifold M with opposite orientation.

For a knot K in S^3, let $g(K)$ be the genus and $b(K)$ be the braid index of K. The aim of this note is to point out the following finiteness result on purely cosmetic surgeries, which gives strong supporting evidence for Conjecture 1:

Theorem 1 *If $g(K) \geq \frac{3}{2}b(K)$, then K does not admit a purely cosmetic surgery. In particular, for given $b > 0$, there are only finitely many knots with braid index b that admit purely cosmetic surgeries.*

Here, the latter finiteness assertion follows from Birman and Menasco's finiteness theorem [2]: for given $g, b > 0$ there are only finitely many knots with genus g and braid index b.

Our proof of Theorem 1 is based on a quantitative refinement of Birman and Menasco's finiteness theorem [5] and the following quite strong constraint for purely cosmetic surgeries:

Theorem 2 (Hanselman [4]) *Let K be a nontrivial knot and $\mathrm{th}(K)$ be the Heegaard Floer thickness of K. If $S^3_K(r) \cong S^3_K(r')$ for $r \neq r'$, then either*

- $\{r, r'\} = \{2, -2\}$ *and* $g(K) = 2$, *or*
- $\{r, r'\} = \{1/q, -1/q\}$ *for some* $0 < q \leq (\mathrm{th}(K) + 2g(K))/2g(K)(g(K)-1)$.

Here, $\mathrm{th}(K)$ is the thickness of the knot Floer homology.

Thus, if $g(K) \neq 2$ and $\mathrm{th}(K)$ is small compared with $g(K)$, then K does not admit purely cosmetic surgery. This motivates us to study a relation between $g(K)$ and $\mathrm{th}(K)$, in particular the (upper) bound of $\mathrm{th}(K)/g(K)$. Here, we give an upper bound of the thickness $\mathrm{th}(K)$ in terms of $g(K)$ and $b(K)$.

Although our argument applies in the cases $b(K) = 2, 3$, we restrict our attention to the case $b(K) \geq 4$.

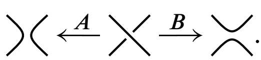
Lemma 3 *If $b(K) \geq 4$,*

$$\mathrm{th}(K) \leq \tfrac{1}{2}(2b(K) - 5)(2g(K) - 1 + b(K)).$$

Proof For a knot diagram D, the *Turaev genus* $g_T(D)$ is defined by

$$g_T(D) = \tfrac{1}{2}(c(D) + 2 - |s_A| - |s_B|),$$

where $c(D)$ is the crossing number of D and $|s_A|$ and $|s_B|$ are the number of circles obtained by A– and B–smoothing, respectively, of crossings of D given by



The Turaev genus $g_T(K)$ of a knot K is the minimum of $g_T(D)$ among diagrams D of K. In [6], Lowrance showed the inequality

$$\mathrm{th}(K) \leq g_T(K).$$

For any diagram D, $|s_A|, |s_B| \geq 1$, so $g_T(D) \leq \frac{1}{2}c(D)$. Hence, we have a canonical upper bound of the Turaev genus,

$$g_T(K) \leq \tfrac{1}{2}c(K). \tag{1}$$

Finally, by the quantitative Birman–Menasco finiteness theorem[1] [5], if $b(K) \geq 4$, we get

$$c(K) \leq (2b(K)-5)(2g(K)-1+b(K)).$$

These three inequalities prove the desired inequality. □

Proof of Theorem 1 In the following we assume that $b(K) \geq 4$ since Varvarezos [8] proved the cosmetic surgery conjecture for the case $b(K) = 3$. Also, we assume that $g(K) \neq 2$.

Assume to the contrary that K admits a purely cosmetic surgery. By Theorem 2, such a knot must satisfy

$$1 \leq \frac{\mathrm{th}(K) + 2g(K)}{2g(K)(g(K)-1)} \iff 2g(K)(g(K)-2) \leq \mathrm{th}(K),$$

so, by Lemma 3, we conclude that, when a knot K admits a purely cosmetic surgery, it satisfies

$$2g(K)(g(K)-2) \leq \tfrac{1}{2}(2b(K)-5)(2g(K)-1+b(K)).$$

That is, we get a constraint for a knot K to admit a purely cosmetic surgery:

$$4g(K)^2 + (2-4b(K))g(K) + (2b(K)-5)(1-b(K)) \leq 0. \tag{2}$$

Now the assertion of the theorem follows from an easy computation that, if $g(K) \geq \frac{3}{2}b(K)$, then (2) is never satisfied. □

As the proof indicates, our sufficient condition $g(K) \geq \frac{3}{2}b(K)$ can be improved if one can improve on the estimate of $\mathrm{th}(K)$ in Lemma 3.

Remark 4 Instead of using an obvious bound (1) of the Turaev genus, by using a different upper bound [3, Corollary 7.3]

$$g_T(K) \leq c(K) - \operatorname{span} V_K(t),$$

where $V_K(t)$ denotes the Jones polynomial, we get a different constraint: if K admits a purely cosmetic surgery, then

$$2g(K)^2 + (6-4b(K))g(K) + (2b(K)-5)(1-b(K)) + \operatorname{span} V_K(t) \leq 0. \tag{3}$$

[1] When $b(K) = 2, 3$, a similar inequality holds but the coefficient $2b(K)-5$ is 1 or $\frac{5}{3}$, respectively.

Here, we give a mild improvement of Lemma 3. For a diagram D of a knot K, the *dealternation number* $\mathrm{dalt}(D)$ of D is the minimum number of crossing change needed to make D into an alternating diagram. The *dealternation number* of a knot K is the minimum of $\mathrm{dalt}(D)$ among diagrams D of K. It is known that $g_T(K) \le \mathrm{dalt}(K)$ [1], so evaluating the dealternation number also gives an upper bound on the thickness.

We prove the following estimate of the dealternation number (and hence the Turaev genus and the thickness) in terms of the genus and braid index, which is interesting in its own right:

Theorem 5 *If $b(K) \ge 4$, then*

$$\mathrm{th}(K) \le g_T(K) \le \mathrm{dalt}(K) \le \left(b(K) - 3 + \frac{1}{b(K)}\right)(2g(K) - 1 + b(K)).$$

Proof Let $n = b(K)$ and let B_n be the braid group of n strands. We denote the standard generators of B_n by $\sigma_1, \dots, \sigma_{n-1}$. We say that a braid is *alternating* if it is a product of $\{\sigma_1, \sigma_2^{-1}, \sigma_3, \sigma_4^{-1}, \dots, \sigma_{2i-1}, \sigma_{2i}^{-1}, \dots\}$. Obviously, the closure of an alternating braid is an alternating diagram.

For $1 \le i < j \le n$, let $a_{i,j}$ be the band generator given by

$$a_{i,j} = (\sigma_i \sigma_{i+1} \cdots \sigma_{j-2}) \sigma_{j-1} (\sigma_i \sigma_{i+1} \cdots \sigma_{j-2})^{-1}.$$

A band generator $a_{i,j}$ can be seen as the boundary of a twisted band connecting the i^{th} and j^{th} strands of the braid. Thus, when K is represented as the closure of a braid $\beta \in B_n$, by giving β as a product of band generators, we get a Seifert surface F_β of K, called the *Bennequin surface* associated to the braid (word) β.

First we treat the case that K bounds a minimum genus Bennequin surface of minimum braid index. That is, K is represented by a closed n–braid β such that its Bennequin surface F_β is a minimum genus Seifert surface of K.

Thanks to the relation

$$\sigma_{j-1} \sigma_j^{\pm 1} \sigma_{j-1}^{-1} = \sigma_j^{-1} \sigma_{j-1}^{\pm 1} \sigma_j,$$

by taking suitable word representatives of the $a_{i,j}^{\pm 1}$, each band generator $a_{i,j}$ except $a_{1,n}$ can be made so that it is alternating by changing at most $n-3$ crossings. The exceptional case $a_{1,n}^{\pm 1}$ can be made so that it is alternating by changing $n-2$ crossings. Thus,

$$\mathrm{dalt}(K) \leq \sum_{\substack{1\leq i<j\leq n\\(i,j)\neq(1,n)}} (n-3)r_{i,j} + (n-2)r_{1,n} = \sum_{1\leq i<j\leq n} (n-3)r_{i,j} + r_{1,n},$$

where $r_{i,j}$ is the number of $a_{i,j}^{\pm 1}$ in the braid β.

On the other hand, since we assume that the Bennequin surface F_β associated with the n–braid β has genus $g(K)$,

$$\sum_{1\leq i<j\leq n} r_{i,j} = 2g(K)-1+n.$$

Let $\delta = \sigma_1\sigma_2\sigma_3\cdots\sigma_{n-1}$. Since $\delta a_{i,j}\delta^{-1} = a_{i+1,j+1}$ (here we regard indices modulo n; for example, $\delta a_{1,n}\delta^{-1} = a_{2,n+1}$ is understood as $a_{1,2}$), by taking conjugates of δ if necessary, we may assume that

$$r_{1,n} \leq \frac{1}{n}(r_{1,2}+r_{2,3}+r_{3,4}+\cdots+r_{n-1,n}+r_{1,n}) \leq \frac{1}{n}(2g(K)-1+n).$$

Thus, we conclude

$$\mathrm{dalt}(K) \leq \sum_{1\leq i<j\leq n} (n-3)r_{i,j} + r_{1,n} \leq \Bigl(n-3+\frac{1}{n}\Bigr)(2g(K)-1+n),$$

as desired.

Next, we assume that K does not bound a minimum genus Bennequin surface of the minimum braid index. To treat this case we quickly review a main strategy of the proof of the quantitative Birman–Menasco theorem [5], namely how to relate the genus, braid index and crossing number (although we do not need to use or know the details).

We put a minimum genus Seifert surface F of K so that it admits a braid foliation. Let R_{aa} and R_{ab} be the number of *aa* tiles and *ab* tiles of the braid foliation. What we showed in [5] is two inequalities:

$$c(K) \leq (2n-5)R_{aa} + (n-3)R_{ab} \tag{4}$$

and

$$2R_{aa} + R_{ab} \leq 2(2g(K)-1+b(K)). \tag{5}$$

More precisely, the inequality (4) is obtained by observing that the braid foliation gives rise to an explicit closed n–braid representative β such that one *aa* tile provides a braid which is a band generator,

$$a_{i,j}^{\pm 1}, \quad (i,j)\neq(1,n),$$

and that one *ab* tile provides a braid of the form

$$\gamma_{i,j}^{\pm 1}, \quad |i-j| \le n-3.$$

Here, $\gamma_{i,j}$ denotes the braid

$$\gamma_{i,j} = \begin{cases} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1} & \text{if } i < j, \\ \sigma_i \sigma_{i-1} \cdots \sigma_{j-1} & \text{if } i > j. \end{cases}$$

(when $i = j$, we regard $\gamma_{i,j}$ as the trivial braid).

If n is odd, then each braid $\gamma_{i,j}$ can be made into an alternating braid by at most $\frac{1}{2}(n-3)$ crossing changes. Each band generator $a_{i,j}$ coming from an *aa* tile can be made into an alternating braid by at most $n-3$ changes since $a_{1,n}$ does not appear. Therefore,

$$\begin{aligned} \operatorname{dalt}(K) &\le (n-3)R_{aa} + \tfrac{1}{2}(n-3)R_{ab} = \tfrac{1}{2}(n-3)(2R_{aa} + R_{ab}) \\ &\le (n-3)(2g(K) - 1 + n). \end{aligned}$$

If n is even, let M be the number of the $\gamma_{i,j}$ produced by *ab* tiles such that $\gamma_{i,j}$ is made into an alternating braid by $\frac{1}{2}(n-2)$ crossing changes. By taking the mirror image of β if necessary, we may assume that $M \le \frac{1}{2}R_{ab}$. Since other braids $\gamma_{i,j}$ from *ab* tiles can be made into an alternating braid by at most $\frac{1}{2}(n-4)$ crossing changes,

$$\begin{aligned} \operatorname{dalt}(K) &\le (n-3)R_{aa} + \tfrac{1}{2}(n-4)(R_{ab} - M) + \tfrac{1}{2}(n-2)M \\ &\le (n-3)R_{aa} + \tfrac{1}{2}(n-3)R_{ab} = \tfrac{1}{2}(n-3)(2R_{aa} + R_{ab}) \\ &\le (n-3)(2g(K) - 1 + n). \end{aligned}$$

□

Using this refinement we can improve a sufficient condition in Theorem 1. For example, for the case $b(K) = 4$, a direct computation shows that:

Corollary 6 *A knot K with braid index 4 does not admit purely cosmetic surgery if $g(K) \neq 2, 3$.*

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Department of Mathematics, Graduate School of Science, Kyoto University
Kyoto, Japan

tetitoh@math.kyoto-u.ac.jp

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Volume 23 Issue 5 (pages 1935–2414) 2023

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TAKUJI KASHIWABARA and HADI ZARE

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ANDRÉS ÁNGEL and HELLEN COLMAN

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NANCY SCHERICH

A model for configuration spaces of points 2029

RICARDO CAMPOS and THOMAS WILLWACHER

The Hurewicz theorem in homotopy type theory 2107

J DANIEL CHRISTENSEN and LUIS SCOCCOLA

A concave holomorphic filling of an overtwisted contact 3–sphere 2141

NAOHIKO KASUYA and DANIELE ZUDDAS

Modifications preserving hyperbolicity of link complements 2157

COLIN ADAMS, WILLIAM H MEEKS III and ÁLVARO K RAMOS

Golod and tight 3–manifolds 2191

KOUYEMON IRIYE and DAISUKE KISHIMOTO

A remark on the finiteness of purely cosmetic surgeries 2213

TETSUYA ITO

Geodesic complexity of homogeneous Riemannian manifolds 2221

STEPHAN MESCHER and MAXIMILIAN STEGEMEYER

Adequate links in thickened surfaces and the generalized Tait conjectures 2271

HANS U BODEN, HOMAYUN KARIMI and ADAM S SIKORA

Homotopy types of gauge groups over Riemann surfaces 2309

MASAKI KAMEKO, DAISUKE KISHIMOTO and MASAHIRO TAKEDA

Diffeomorphisms of odd-dimensional discs, glued into a manifold 2329

JOHANNES EBERT

Intrinsic symmetry groups of links 2347

CHARLES LIVINGSTON

Loop homotopy of 6–manifolds over 4–manifolds 2369

RUIZHI HUANG

Infinite families of higher torsion in the homotopy groups of Moore spaces 2389

STEVEN AMELOTTE, FREDERICK R COHEN and YUXIN LUO