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Intrinsic symmetry groups of links

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The set of isotopy classes of ordered n–component links in S^3 is acted on by the symmetric group $\mathbb{S}_n$ via permutation of the components. The subgroup $\mathbb{S}(L) \subset \mathbb{S}_n$ is defined to be the set of elements in the symmetric group that preserve the ordered isotopy type of L as an unoriented link. The study of these groups was initiated in 1969, but the question of whether or not every subgroup of $\mathbb{S}_n$ arises as such an intrinsic symmetry group of some link has remained open. We provide counterexamples; in particular, if $n \geq 6$, then there does not exist an n–component link L for which $\mathbb{S}(L)$ is the alternating group $\mathbb{A}_n$.

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1 Introduction

The *oriented diffeomorphism group* of an ordered link $L = \{L_1, \dots, L_n\} \subset S^3$ consists of all orientation-preserving diffeomorphisms of S^3 that preserve the link setwise. We denote this group by $\mathcal{D}(L)$. The action of $\mathcal{D}(L)$ on the components of L defines a homomorphism from $\mathcal{D}(L)$ to the symmetric group $\mathbb{S}_n$; its image is denoted by $\mathbb{S}(L)$. A basic question asks whether every subgroup $H \subset \mathbb{S}_n$ arises as $\mathbb{S}(L)$ for some n–component link. We provide obstructions. Our examples of groups that do not arise are the alternating groups $\mathbb{A}_n$ for $n \geq 6$.

Theorem *If $n \geq 6$, then there does not exist an ordered n–component link L that satisfies $\mathbb{S}(L) = \mathbb{A}_n$.*

The study of symmetries of links is usually placed in the context of an extension of the symmetric group, called the *Whitten group*,

$$\Gamma_n = \mathbb{Z}_2 \oplus ((\mathbb{Z}_2)^n \rtimes \mathbb{S}_n).$$

In the semidirect product, $\mathbb{S}_n$ acts on $(\mathbb{Z}_2)^n$ by permuting the coordinates. If we let $\mathcal{D}^*(L)$ denote the set of diffeomorphisms of an n–component link L, including those that reverse the orientation of S^3, then there is a natural map of $\mathcal{D}^*(L)$ to Γ_n. The first

$\mathbb{Z}_2$ factor keeps track of the orientation of S^3 and the remaining $\mathbb{Z}_2$ factors track the orientations of the components of L. The image of this map is denoted by $\Sigma(L)$. The question of which subgroups of Γ_n arise as $\Sigma(L)$ for some n–component link L was considered by Fox and Whitten in the mid-1960s, first appearing in print in 1969 [17].

There is a quotient map $\Phi\colon \Gamma_n \to \mathbb{S}_n$ which carries $\Sigma(L) \cap \big(0 \oplus ((\mathbb{Z}_2)^n \rtimes \mathbb{S}_n)\big)$ to $\mathbb{S}(L)$. Thus, we have the following corollary:

Corollary *If $n \geq 6$ and $H \subset \Gamma_n$ has the property that $\Phi\big(H \cap \big(0 \oplus ((\mathbb{Z}_2)^n \rtimes \mathbb{S}_n)\big)\big) = \mathbb{A}_n$, then there is no link L with $\Sigma(L) = H$. In particular, if $n \geq 6$, then the subgroup $0 \oplus (0 \oplus \mathbb{A}_n) \subset \Gamma_n$ is not of the form $\Sigma(L)$ for any n–component link.*

Summary of proof The basic idea of our approach is as follows. For a given link L there is a Jaco–Shalen–Johannson (JSJ) decomposition of the complement of L into hyperbolic and Seifert fibered components $\{C_i\}$. This decomposition is unique up to isotopy. We first observe that, if $\mathbb{S}(L)$ does not contain an index two subgroup, then one of the C_i (say C_1) is invariant under the action of $\mathcal{D}(L)$ up to isotopy.

If C_1 is hyperbolic, we can replace the action of $\mathcal{D}(L)$ restricted to C_1 with a finite group of isometries of C_1. We then use a reembedding of C_1 into S^3 (as first described by Budney in [3]) to extend that action to S^3. It follows from results such as Boileau, Leeb and Porti [2] that the action on S^3 is conjugate to a linear action. We then find that $\mathbb{S}(L)$ is a quotient of a finite subgroup of SO(4). Finally, a group-theoretic analysis reduces the problem to the simpler one of considering quotients of finite subgroups of SO(3), which are enumerated.

In contrast to the hyperbolic case, if C_1 is Seifert fibered, then the diffeomorphism group of C_1 itself is large, sufficiently so that we can construct enough symmetries of L to show that $\mathbb{S}(L) = \mathbb{S}_n$.

Outline Section 2 describes the general theory of intrinsic symmetry groups of oriented links, as first considered by Fox and Whitten [17]. Sections 3 and 4 describe the classical case of knots, $n = 1$, and results for the case of $n = 2$. Section 5 presents prime, nonsplit links, with full symmetry group for all n.

In Section 6 we describe JSJ decompositions, the associated tree diagrams, and a proof that, in the case of $\mathbb{S}(L) = \mathbb{A}_n$, some component of the decomposition is fixed (up to isotopy) by the action of the diffeomorphism group. Section 7 explains how that distinguished component can be reembedded into S^3 as the complement of a link. The reembedding is used in Section 8 to show that if the fixed component is hyperbolic, then $\mathbb{S}(L)$ is a subgroup of a quotient of a finite subgroup of SO(4). Finally, in Section 9 we present the Seifert fibered case. In the concluding Section 10, we present a few questions and include an example of a four-component link L with $\mathbb{S}(L) = \mathbb{A}_4$.

Notational comment We are calling the groups studied here the *intrinsic symmetry groups* of links. The *symmetry group* of a link consists of the group of diffeomorphisms of S^3 that leave the link invariant, modulo isotopy. Even for knots, these symmetry groups include, for instance, all dihedral groups.

Acknowledgments I have benefited from comments from Ryan Budney, whose work provides a backdrop for our approach. Nathan Dunfield provided an example of a four-component link L with $\mathbb{S}(L) = \mathbb{A}_4$. I was also helped by discussions with Jim Davis, Allan Edmonds, Charlie Frohman, Michael Larsen, Swatee Naik and Dylan Thurston. Suggestions from the referee led to several improvements in the exposition.

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2 The general setting of oriented links

We now describe the general theory of *intrinsic symmetry groups* of links. This theory was initially developed by Fox and was first presented by Whitten in [17]. To be precise, we will momentarily consider links in three-manifolds that are diffeomorphic to S^3, rather than work specifically with S^3. In this setting we have the following definition: an *n–component link* is an ordered $(n+1)$–tuple of oriented manifolds, $L = (S, L_1, L_2, \dots, L_n)$, where S is diffeomorphic to S^3 and the L_i are disjoint submanifolds of S, each diffeomorphic to S^1. The set of n–component links will be denoted by $\mathcal{L}_n$.

Given a second link $L' = (S', L'_1, L'_2, \dots, L'_n)$, an *orientation-preserving diffeomorphism* from L to L' is an orientation-preserving diffeomorphism $F\colon S \to S'$ such that $F(L_i) = L'_i$ as oriented manifolds for all i.

For any oriented manifold M, $-M$ denotes its orientation reverse. Let $\mathbb{Z}_2$ be the cyclic group of order two written multiplicatively: $\mathbb{Z}_2 = \{1, -1\}$. If $\epsilon = -1 \in \mathbb{Z}_2$, we will let $\epsilon M = -M$, and if $\epsilon = 1 \in \mathbb{Z}_2$, we will let $\epsilon M = M$. The group $\mathbb{Z}_2 \oplus (\mathbb{Z}_2)^n$ acts on $\mathcal{L}_n$ by changing the orientations of the factors. The symmetric group $\mathbb{S}_n$ acts on $\mathcal{L}_n$ by permuting the component knots. These actions do not commute, but together define an action on the set of knots by the *Whitten group*

$$\Gamma_n = \mathbb{Z}_2 \oplus ((\mathbb{Z}_2)^n \rtimes \mathbb{S}_n).$$

In this semidirect product, $\mathbb{S}_n$ acts on the n–fold product by permuting the coordinates. To be precise, given an element $s = (\eta, (\epsilon_1, \ldots, \epsilon_n), \rho) \in \Gamma_n$ and an n–component link L, we let

$$sL = (\eta S, \epsilon_1 L_{\rho(1)}, \cdots, \epsilon_n L_{\rho(n)}).$$

Notice that these group actions are defined to be on the left. Thus, elements in $\mathbb{S}_n$ are multiplied right to left.

Definition 2.1 For a link $L \in \mathcal{L}_n$, the *intrinsic symmetry group* of L is the subgroup $\Sigma(L) = \{s \in \Gamma_n \mid sL \cong L\} \subset \Gamma_n$. Note that "$\cong$" indicates the existence of an orientation- and order-preserving diffeomorphism.

There are two fundamental questions regarding such link symmetries:

Problem 1 Given an n–component link L, determine $\Sigma(L)$.

Problem 2 For each subgroup $H \subset \Gamma_n$, does there exist an n–component link L such that $\Sigma(L) = H$?

The first can be effectively answered for low crossing number links with programs such as SnapPy [6]. The second is the focus of this paper; we present the first examples of groups that cannot arise as the symmetry group of a link.

2.1 Restricting to the oriented category and basic observations

There is a canonical index two subgroup $\overline{\Gamma}_n \subset \Gamma_n$ consisting of elements of the form

$$(1, (\epsilon_1, \ldots, \epsilon_n), \rho).$$

This subgroup maps onto $\mathbb{S}_n$. We leave it to the reader to verify the following, which implies that any constraint on what groups occur as $\mathbb{S}(L)$ places a constraint on what groups can arise as $\Sigma(L)$:

Theorem 2.2 *The image of $\Sigma(L) \cap \overline{\Gamma}_n$ in $\mathbb{S}_n$ is precisely $\mathbb{S}(L)$.*

After the initial sections of this paper, we will be restricting our work to orientation-preserving diffeomorphisms of S^3 and will work with unoriented links. We will use the following conventions, which were summarized in the introduction:

(1) Links will all be of the form $L = (S^3, L_1, L_2, \dots, L_n)$, where S^3 has some fixed orientation and the L_i are disjoint unoriented submanifolds, each diffeomorphic to S^1.

(2) We will consider diffeomorphisms of the link that are orientation-preserving on S^3 and that possibly permute the set of L_i.

(3) The set of such diffeomorphisms will be denoted by $\mathcal{D}(L)$.

(4) Given $F \in \mathcal{D}(L)$, we have
$$(S^3, F(L_1), F(L_2), \dots, F(L_n)) = (S^3, L_{\rho(1)}, L_{\rho(2)} \dots, L_{\rho(n)})$$
for some $\rho \in \mathbb{S}_n$. This defines a homomorphism $\Phi\colon \mathcal{D}(L) \to \mathbb{S}_n$.

(5) The image Φ in $\mathbb{S}_n$ is denoted by $\mathbb{S}(L)$.

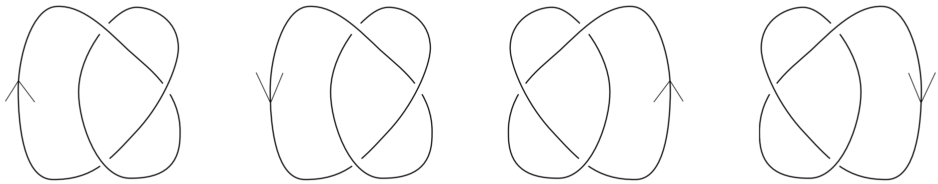
3 Examples: knots

Before restricting to the orientation-preserving diffeomorphism group, in this section and the next we will summarize what is known in general for links of one and of two components. Then, in Section 5, we show that for all n there is a prime, nonsplittable n–component link L with $\Sigma(L) = \Gamma_n$.

Let $n = 1$. The symmetric group $\mathbb{S}_1$ is trivial and thus the first Whitten group is $\Gamma_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The knots $(1,-1)K$, $(-1,1)K$ and $(-1,-1)K$ have been called the *reverse*, K^r, the *mirror image*, $m(K)$, and the *reversed mirror image*, $m(K)^r$, respectively. (Older references have called the reverse of K the *inverse*. The name "reverse" is used to distinguish it from the concordance inverse, which is represented by the reversed mirror image.) Figure 1 illustrates the possibilities. A detailed account of the key results in the study of knot symmetries is contained in [8]. Here is a brief summary.

The group Γ_1 has five subgroups: the entire group, the trivial subgroup and the three subgroups containing exactly one of the nontrivial elements of Γ_1. Each is realized as $\Sigma(K)$ for some knot K.

- The unknot and the figure eight knot, 4_1, have full symmetry group. They are called *fully amphicheiral*.
- The trefoil knot is reversible. Dehn showed that it does not equal its mirror image, a fact that can now be proved using such invariants as the signature or the Jones polynomial. Thus, 3_1 is *reversible*.



K $\quad$ $(1,-1)K = K^r$ $\quad$ $(-1,1)K = m(K)$ $\quad$ $(-1,1)K = m(K)^r$

Figure 1: Symmetries of knots.

- Trotter [14] proved the existence of nonreversible knots. His examples in [14] have nonzero signature and thus have trivial symmetry group. We say that such knots are *chiral*. Hartley [7] proved that 9_{32} is nonreversible and, since it has nonzero signature, it too is chiral.
- Kawauchi [9] proved that $K = 8_{17}$ is nonreversible. It is easily seen that $K = m(K)^r$, and thus 8_{17} is *negative amphicheiral*.
- The simplest example of a low crossing number knot that is nonreversible and for which $K = m(K)$ is $12a_{147}$, which was detected by the program SnapPy. (Presumably the general techniques developed by Hartley in [7] would also show that this knot is not reversible.) More complicated examples of such *positive amphicheiral* knots were first discovered by Trotter.

4 Two-component links

Here we summarize the results of [1; 5] concerning two-component links. We have that $\Gamma_2 = \mathbb{Z}_2 \oplus ((\mathbb{Z}_2)^2 \rtimes \mathbb{S}_2)$ is of order 16. In [1; 5], the authors describe the 27 conjugacy classes of subgroups of Γ_2. They then show that tables of prime, nonsplittable links provide examples of links realizing 21 of these subgroups. One of the missing subgroups is Γ_2 itself. This is clearly the symmetry group of the unlink; in a note on MathOverflow [4], Budney showed that Γ_2 is the symmetry group of a nonsplittable Brunnian link. We will expand on that example in the next section.

To conclude this section, we list the subgroups that are currently not known to be the symmetry groups of two-component links, where τ denotes the transposition in $\mathbb{S}_2$:

- $\langle(1,(-1,1)\tau)\rangle \cong \mathbb{Z}_4$.
- $\langle(1,(-1,1)),(-1,(1,1))\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- $\langle(1,(1,-1)),(-1,(-1,1))\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

- $\langle(-1,(-1,1)),(1,(-1,1)\tau)\rangle \cong D_4$, the dihedral group with four elements.
- $\langle(1,(1,-1)),(1,(-1,1)),(-1,(1,1))\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

5 Fully amphicheiral links for all *n*

In Figure 2, we illustrate a knot K in a solid torus D. Two parallel strands of K are tied in a knot J, where J is chosen to be fully amphicheiral; the figure eight knot would be sufficient. As oriented pairs, we have $(D,K) \cong (-D,K) \cong (D,-K) \cong (-D,-K)$.

Budney's example [4] of a two-component link L with full symmetry group $\Sigma(L)=\Gamma_2$ is formed from the Hopf link by replacing neighborhoods of each component with copies of (D,K). An example of a three-component link with full symmetry group is built in the same way, starting with the Borromean link. Notice that, in both these examples, the links are Brunnian. Problem (5) in Section 10 asks: Does there exist a Brunnian link with four or more components with full symmetry group?

We conclude this section with an elementary observation:

Theorem 5.1 *For every n, there exists a prime, nonsplittable link L for which $\Sigma(L) \cong \Gamma_n$.*

Proof To form an n–component link with full symmetry group, proceed as follows: Starting with any nontrivial fully amphicheiral knot J', form a link by replacing J' with n parallel copies of J'; formally, form the $(0,n)$–companion of J'. (Again, the simplest example would be to let J' be the figure eight knot.) Next, replace a neighborhood of each component of that link with a copy of (D,K) as illustrated in Figure 2, built

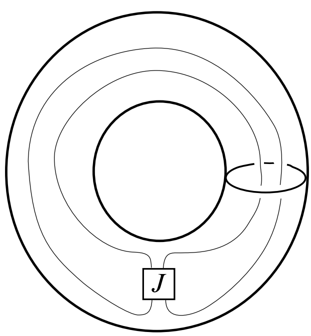


Figure 2: Companion.

using the fully amphicheiral knot J. Innermost circle arguments, dating to the work of Schubert [13], can be used to show that this link L is prime and nonsplittable. The fact that we used parallel copies of J' implies that the components can be freely permuted. By replacing the components with (D, K), we have ensured that the components can be independently reversed. The fact that J and J' are fully amphicheiral ensures that there is an orientation-reversing diffeomorphism of S^3 that preserves the link. □

6 Torus decompositions and tree diagrams

A principal tool in understanding knot and link complements is the Jaco–Shalen–Johannson torus decomposition, which we refer to as the JSJ decomposition. An excellent resource is [3], which contains details for the results we summarize here.

Let X be the complement of a nonsplittable link L in S^3. The JSJ decomposition of X is given by a finite family of disjoint incompressible embedded tori, $\{T_i\}$, with the property that each component of the complement of $\bigcup T_i$ has either a complete hyperbolic structure or is Seifert fibered. There is the additional condition that no T_i is boundary parallel and that no two of the T_i are parallel. Up to isotopy, there is a unique minimal set $\{T_i\}$ with these properties; this set provides the JSJ decomposition. No two T_i in the decomposition are isotopic.

We can associate a finite tree $\mathrm{Tr}(L)$ to this decomposition, as follows: Let the components of $X \setminus \bigcup T_i$ be denoted by $\{C_i\}$. The vertices of the $\mathrm{Tr}(L)$ correspond to the C_j. Two vertices are joined by an edge if the closures of the corresponding C_i intersect; there is one edge for each T_i. When possible, we will use the names C_i and T_i to denote the vertices and edges. We will say that a component C_i *contains a component* $L_j \in L$ if L_j is in the closure of C_i.

6.1 The subtrees $\mathrm{Tr}_L(K)$ and $\widehat{\mathrm{Tr}}(L)$

Let K be a component of L. Its orbit under the action of $\mathcal{D}(L)$ is a sublink of L, $\{K_1, \dots, K_l\}$ for some $l \geq 1$ with $K_1 = K$. Each K_i is contained in a vertex of $\mathrm{Tr}(L)$. The set of such vertices is denoted by $\{D_1, \dots, D_k\}$. Since the action of $\mathcal{D}(L)$ on the set of K_i is transitive, each D_j contains the same number of components of L. In particular, k divides l. Later we will expand on this observation.

The vertices $\{D_1, \dots, D_k\}$ in $\mathrm{Tr}(L)$ span a unique minimal subtree, which we denote by $\mathrm{Tr}_L(K)$. In the case that the action of $\mathcal{D}(L)$ is transitive on L, the orbit of K is all of L, and we write $\widehat{\mathrm{Tr}}(L) = \mathrm{Tr}_L(K)$. (Notice that $\widehat{\mathrm{Tr}}(L)$ need not equal $T(L)$; for instance, vertices of $T(L)$ of valence one that do not contain components of L are not included in $\widehat{\mathrm{Tr}}(L)$.)

Theorem 6.1 *If $\mathcal{D}(L)$ acts transitively on L, then the tree $\widehat{\mathrm{Tr}}(L)$ either contains exactly one vertex, or its valence one vertices are precisely the set $\{D_1, \dots, D_k\}$.*

Proof It is an elementary observation that, in the subtree of a tree spanned by the set of vertices $\{D_j\}$, the only vertices of valence one correspond to elements in the set $\{D_j\}$, and that, if there is more than one D_j, then at least one of them is a vertex of valence one. We need to see that each D_j has valence one.

Suppose that the vertex D_1 is of valence one in $\widehat{\mathrm{Tr}}(L)$ and that it contains L_1. Let D_2 be another vertex and suppose it contains L_2. There is an element $F \in \mathcal{D}(L)$ such that $F(L_1) = L_2$. The map F is isotopic relative to L to a diffeomorphism F' that preserves the JSJ decomposition. This F' induces an automorphism of $\mathrm{Tr}(L)$ that leaves $\widehat{\mathrm{Tr}}(L)$ invariant. Thus, there is an automorphism of $\widehat{\mathrm{Tr}}(L)$ that carries D_1 to D_2. It follows that D_2 is of valence one in $\widehat{\mathrm{Tr}}(L)$. □

6.2 The group $\mathcal{D}^*(L)$

Fix a JSJ decomposition of $S^3 \setminus L$.

Definition 6.2 We let $\mathcal{D}^*(L) \subset \mathcal{D}(L)$ be the subgroup consisting of elements that leave the JSJ decomposition invariant.

Theorem 6.3 *The image of $\mathcal{D}^*(L)$ in $\mathbb{S}_n$ equals $\mathbb{S}(L)$.*

Proof Given an element in $\mathbb{S}(L)$, there is a diffeomorphism $F \in \mathcal{D}(L)$ that maps to it. We have that F is isotopic relative to L to an element $F' \in \mathcal{D}^*(L)$. The map F' induces the same permutation of the components of L as does F. □

Theorem 6.4 *In the case that $\mathcal{D}^*(L)$ acts transitively on the components of L, the action of $\mathcal{D}^*(L)$ on $\widehat{\mathrm{Tr}}(L)$ factors through an action of $\mathbb{S}(L)$ on $\widehat{\mathrm{Tr}}(L)$.*

Proof An automorphism of a tree is completely determined by its action on the valence one vertices of the tree. We leave this elementary observation to the reader. □

6.3 The structure of $\widehat{\mathrm{Tr}}(L)$ when $\mathbb{S}(L) = \mathbb{A}_n$

In Figure 3 we provide an example of a labeled tree to serve as a model for the discussion that follows.

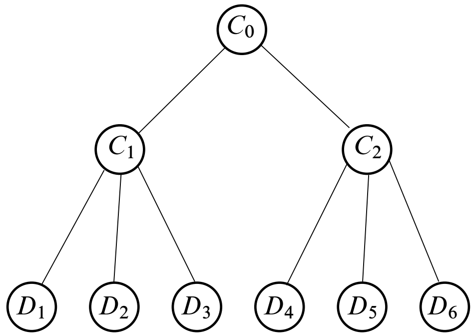


Figure 3: Tree diagram for a sublink K of L on which $\mathcal{D}(L)$ acts transitively.

Lemma 6.5 *If $\mathbb{S}(L) = \mathbb{A}_n$ and $\widehat{\mathrm{Tr}}(L)$ contains more than one vertex, then each D_i contains exactly one L_1 and the number of vertices in the set $\{D_i\}$ is n.*

Proof Suppose that D_1 contains L_1 and L_2 and that D_2 contains L_3 and L_4. Then the permutation $(123) \in \mathbb{A}_n$ does not induce an action on $\widehat{\mathrm{Tr}}(L)$. □

Theorem 6.6 *If $\mathbb{S}(L) = \mathbb{A}_n$ with $n \geq 3$, then $\widehat{\mathrm{Tr}}(L)$ is a rooted tree with either exactly one vertex, C, or with n vertices of valence one. In the second case, there is a unique vertex with valence greater than two; the tree $\widehat{\mathrm{Tr}}(L)$ is built from that high valence vertex C by attaching n linear branches, all of the same length. The vertex C is invariant under the action of $\mathbb{A}_n$ on $\widehat{\mathrm{Tr}}(L)$.*

Proof Figure 4 is a schematic of a tree. We are asserting that $\widehat{\mathrm{Tr}}(L)$ is of this form.

We have seen that each D_i contains precisely one L_i and these are the valence one vertices of $\widehat{\mathrm{Tr}}(L)$. A tree with more than two valence one vertices always contains

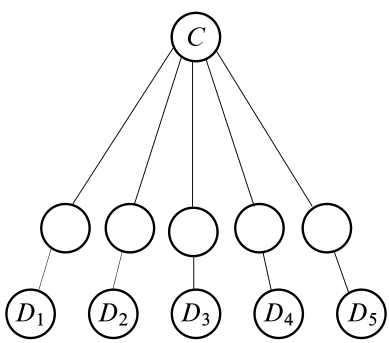


Figure 4: Possible tree diagram $\widehat{\mathrm{Tr}}(L)$ for a five-component link L on which $\mathbb{S}(L) = \mathbb{A}_5$.

some vertex with valence greater than 2. It remains to show that there is a unique such vertex of valence greater than two. (For an example of the sort of tree we need to rule out, build a tree from two copies of the graph illustrated in Figure 4 by joining the roots with single edge.)

An elementary exercise shows that, for any tree on which $\mathbb{A}_n$ acts with this action transitive on the vertices of valence one, there is an invariant vertex or edge: proceed by induction, removing all valence one vertices and their adjacent edges from the tree.

We next observe that, in the case that the symmetry group is $\mathbb{A}_n$, there must be an invariant vertex. The action of the symmetry group of the tree is transitive on its valence one vertices, so, if there is an invariant edge, some elements must reverse that edge. It follows that the subgroup of the symmetry group that does not reverse the edge is index two. But $\mathbb{A}_n$ does not contain an index two subgroup for $n \geq 3$. $\square$

6.4 The structure of the core C in the case that $\mathbb{S}(L) = \mathbb{A}_n$

Suppose that $\mathbb{S}(L) = \mathbb{A}_n$. Then, by Theorem 6.6, there is a core C in the JSJ decomposition of L. This core is acted on by $\mathcal{D}^*(L)$. The boundary of C is the union of two sets of tori, $\{T_1, \dots, T_n\} \cup \{S_1, \dots, S_m\}$. Each T_i bounds a submanifold $W_i \subset S^3$ that contains the link component L_i and does not contain C. A schematic appears in Figure 5. In this diagram we have included extra edges showing $\widehat{\mathrm{Tr}}(L)$ might be a proper subtree of $\mathrm{Tr}(L)$ and that C might have more than n boundary components.

Let $\mathcal{D}(C)$ be the diffeomorphism group of the core C. It contains a subgroup $\mathcal{D}(C, T)$ that leaves invariant the set of T_i. This group maps to $\mathbb{S}_n$ via its action on $\{T_i\}$.

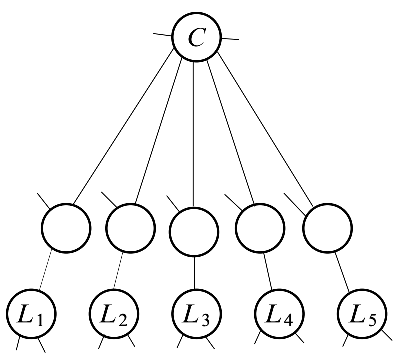


Figure 5: Possible tree diagram $\widehat{\mathrm{Tr}}(L)$ for a five-component link L on which $\mathcal{D}^*(L)$ acts transitively.

Theorem 6.7 *In the case that $\mathbb{S}(L) = \mathbb{A}_n$ with $n \geq 5$, with core C, the group $\mathcal{D}(C, T)$ acts on $\{T_i\}$ as either $\mathbb{A}_n$ or $\mathbb{S}_n$.*

Proof It is clear that the action contains $\mathbb{A}_n$. The only subgroups of $\mathbb{S}_n$ that contain $\mathbb{A}_n$ are $\mathbb{A}_n$ and $\mathbb{S}_n$. □

Notice that it might happen that there are elements of $\mathcal{D}(C, T)$ that do not map to elements of $\mathbb{A}_n$; it is possible that not every action on C extends to S^3.

7 Reembeddings

Reembeddings appear in two different ways in our proof. In the case of C hyperbolic, we embed C in S^3 as a link complement. In the Seifert fibered case, we embed C into a closed Seifert fibered space as the complement of a set of regular fibers. In this section, we describe the embedding into S^3.

In the previous section, some of the (torus) boundary components of the core C were denoted by T_i. We will now see that by using reembeddings we can view these T_i, along with the other boundary components S_i of C, as peripheral tori for a link in S^3. This is presented in [3], where Budney gave a reembedding theorem for submanifolds of S^3. Here we present a slightly enhanced version of that result, keeping track of boundary curves. First we set up some notation.

Let $X \subset S^3$ be a compact, connected submanifold with one of its boundary components a torus T. The complement of T consists of two spaces, Y_1 and Y_2. We have $H_1(Y_1) \cong \mathbb{Z} \cong H_1(Y_2)$. We assume $X \subset Y_1$. When needed, we will write these as $Y_1(X, T)$ and $Y_2(X, T)$.

We have that $\ker(H_1(T) \to H_1(Y_1)) \cong \mathbb{Z}$. The generator can be represented by a simple closed curve we denote by l. Similarly, a representative of $\ker(H_1(T) \to H_1(Y_2)) \cong \mathbb{Z}$ is denoted by m. There is no natural orientation for these choices. However, we can assume that they are oriented so that the intersection number of m and l is 1 with respect to the orientation of T viewed as the boundary of Y_1. We can also assume that m and l intersect transversely in exactly one point. With this setup, we have the following:

Theorem 7.1 *There exists an orientation-preserving embedding $F: X \to S^3$ such that $F(T)$ is the boundary of a tubular neighborhood of a knot in S^3 having meridian $F(m)$ and longitude $F(l)$.*

Proof An embedded torus in S^3 bounds (on one side or the other) a solid torus, which we denote by W. If $Y_2 = W$, then m is the meridian of W and F can be taken to be the identity.

If $Y_1 = W$, then form the boundary union $Z = Y_1 \cup W'$, where W' is a solid torus, attached so that its meridian is identified with m and its longitude is identified with l. Then Z is the union of two solid tori and the choice of identification ensures that $H_1(Z) = 0$. Thus, $Z \cong S^3$. □

Corollary 7.2 *Suppose that $X \subset S^3$ is a compact manifold with boundary a union of tori $\{T_1, \dots, T_k\}$. There exists a link $L = \{L_1, \dots, L_k\}$ and an orientation-preserving homeomorphism $F\colon X \to S^3 \setminus \nu(L)$, where $\nu(L)$ is an open tubular neighborhood. Furthermore, it can be assumed that F preserves meridians and longitudes.*

Corollary 7.3 *With $X \subset S^3$ and L as in Corollary 7.2, suppose that a diffeomorphism $g\colon S^3 \to S^3$ satisfies $g(X) = X$. Then the diffeomorphism of $F(X)$ given as the composition $F \circ g \circ F^{-1}$ extends to a diffeomorphism of (S^3, L).*

Note Not every diffeomorphism of X determines a diffeomorphism of L. It is essential here that the diffeomorphism of X extends to S^3.

7.1 Summary theorem

Theorem 7.4 *Suppose that $\mathbb{S}(L) = \mathbb{A}_n$. Then there is a link $(L'_1, \dots, L'_n, J_1, \dots J_m)$ with complement diffeomorphic to C and that is either hyperbolic or Seifert fibered. The mapping class group of this link has a subgroup that preserves $(L'_1, \dots L'_n)$. The image of this subgroup in $\mathbb{S}_n$ is either $\mathbb{A}_n$ or $\mathbb{S}_n$.*

Proof To prove this using the previous results, we need to show that a JSJ decomposition exists — that is, that L is nonsplittable. If L does split, it splits as a union of nonsplit sublinks, say $D_1, \dots, D_k$, where each D_i is contained in a ball that does not intersect the other D_j. The transitivity of the $\mathbb{A}_n$–actions implies that the D_i are identical links. Thus, we can write $D_i = \{D_i^1, \dots, D_i^m\}$ for some m that is independent of i.

If $m = 1$, then L is consists of n copies of a knot J, each copy in a separate ball. In this case, the symmetry group would be $\mathbb{S}_n$. If $m = n$, then we are in the nonsplit case, as desired.

Finally, if $1 < m < n$, then any element of $\mathcal{M}$ that carries D_1^1 to D_2^1 must carry D_1^2 to some D_2^i. But not every element of $\mathbb{A}_n$ behaves in this way. □

To complete the proof that $\mathbb{A}_n$ for $n \geq 6$ is not the intrinsic symmetry group of any link, we consider the hyperbolic and Seifert fibered cases separately.

8 The case of C hyperbolic

We use the notion of *core* as in the previous section.

Theorem 8.1 *If $\mathbb{A}_n \subset \mathbb{S}(L)$ and the core C is hyperbolic, then some finite subgroup of* SO(4) *contains a finite subgroup having $\mathbb{A}_n$ as a quotient.*

Proof For each element $\phi \in \mathcal{D}(C, \partial C)$ that extends to S^3, let ϕ' denote an isometry that is isotopic to ϕ relative to the boundary. Note that the actions of ϕ and ϕ' on the finite set of components $\{\partial C\}$ are the same. The set of ϕ' generates a subgroup of $\mathrm{Isom}(C)$. This is necessarily a finite group, H. The group H contains the subgroup $H' \subset H$ that leaves invariant the set $\{L'_1, \dots, L'_n\}$. The image of H' in $\mathbb{S}_n$ contains $\mathbb{A}_n$. By restricting to a further subgroup H'', we can assume the image is precisely $\mathbb{A}_n$.

By results such as [12; 2], any finite subgroup of $\mathrm{Diff}(S^3)$, such as H'', is isomorphic to a subgroup of SO(4). □

Corollary 8.2 *If $\mathbb{A}_n \subset \mathbb{S}(L)$ then $n \leq 5$.*

Proof This follows from the results of the next subsection. □

8.1 The only subgroup of SO(4) that maps onto a noncyclic simple group is isomorphic to $\mathbb{A}_5$

We prove somewhat more than this.

Theorem 8.3 *If A is a nonabelian simple group and a subgroup $H \subset \mathrm{SO}(4)$ surjects onto A, then $A \cong \mathbb{A}_5$.*

Denote the surjection from H to A by $\phi\colon H \to A$. We begin by recalling the structure of SO(4).

The set of unit quaternions is homeomorphic to S^3 and as a Lie group is isomorphic to SU(2). Quotienting by ± 1 yields a two-fold cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

Let x and y be unit quaternions and view elements $v \in \mathbb{R}^4$ as quaternions. Then x and y define a homomorphism $\psi_{x,y}\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ by $\psi_{x,y}(v) = xvy^{-1}$. This yields a two-fold covering of $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$. Hence,

$$\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\langle(-1, -1)\rangle.$$

There is a two-fold covering space $q\colon (\mathrm{SU}(2)\times\mathrm{SU}(2))/\langle(-1,-1)\rangle \to \mathrm{SO}(3)\times\mathrm{SO}(3)$. We thus have the diagram

$$\begin{tikzcd}
(\mathrm{SU}(2)\times\mathrm{SU}(2))/\langle(-1,-1)\rangle \arrow[r, "\cong"] \arrow[d, "\text{two-fold cover } q"'] & \mathrm{SO}(4) \\
\mathrm{SO}(3)\times\mathrm{SO}(3) &
\end{tikzcd}$$

We will write elements of $\mathrm{SO}(4)$ and of $\mathrm{SO}(3)\times\mathrm{SO}(3)$ as equivalence classes of pairs of unit quaternions.

Lemma 8.4 *The map ϕ induces a surjection $\phi'\colon q(H)\subset \mathrm{SO}(3)\times\mathrm{SO}(3)\to A$.*

Proof If the map $q\colon H\to q(H)$ is an isomorphism, then this is trivially true. It is possible that $q\colon H\to q(H)$ is two-to-one, which can occur if and only if the central element $(1,-1)\in H$. In this case, $q(H)\cong H/\langle(1,-1)\rangle$. Since A is nonabelian and simple, the image of $(1,-1)$ in A is trivial. □

Lemma 8.5 *Let $G\subset \mathrm{SO}(3)\times\mathrm{SO}(3)$. Let G_1 and G_2 be the images of the projections of G onto the first and second factors of the product. If $\phi'\colon G\to A$ where A is nonabelian and simple, then a subgroup of G_1 or G_2 maps onto A. In particular, A is a quotient of a finite subgroup of* $\mathrm{SO}(3)$.

Proof Let $F=G\cap(\mathrm{SO}(3)\times\{1\})$. We have that F is a normal subgroup of G and, thus, $\phi'(F)=A$ or $\phi'(F)=\{1\}$. In the first case, we are done, so assume that $\phi'(F)=\{1\}$.

We now define a surjective homomorphism $\phi''\colon G_2\to A$. Given $y\in G_2$, there exists an element $x\in G_1$ such that $(x,y)\in G$. Set $\phi''(y)=\phi'((x,y))$. To see that this is well defined, notice that, if $(x_1,y)\in G$ and $(x_2,y)\in G$, then $x_1x_2^{-1}\in F$. Thus, $\phi'((x_1,y))=\phi'((x_2,y))$. It is easily checked that ϕ' is surjective and is a homomorphism. □

Lemma 8.6 *The group $\mathbb{A}_5$ is the only finite noncyclic simple group contained in* $\mathrm{SO}(3)$.

Proof The finite subgroups of $\mathrm{SO}(3)$ are classified. Here is the list of possibilities:

- Cyclic groups $A_n\cong\mathbb{Z}_n$.
- Dihedral groups D_n.
- Tetrahedral group $E_6\cong\mathbb{A}_4$.

- Octahedral group $E_7 \cong \mathbb{S}_4$.
- Icosahedral group $E_8 \cong \mathbb{A}_5$.

The subgroups of the dihedral group are either dihedral, and thus not simple, or cyclic. The smallest nonabelian simple group is $\mathbb{A}_5$. □

9 The case of C Seifert fibered

We begin with a basic example.

Example 9.1 Consider the $(n+2)$–component link L formed as follows. Let T be a standardly embedded torus in S^3 and form the (np, nq)–torus link on T with $q > p > 1$ relatively prime. Add to this the cores of two solid tori bounded by T. There is a Seifert fibration of S^3 with the torus link represented by regular fibers and the two cores being neighborhoods of singular fibers of type p/q and q/p.

We leave it to the reader to confirm that, for this link, $\Sigma(L) \cong \mathbb{Z}_2 \oplus \mathbb{S}_n$. It should be clear how the components of the (np, nq)–torus link can be freely permuted. The $\mathbb{Z}_2$ arises from a diffeomorphism that reverses all the components.

Two exercises arise here. The first is to show that every symmetry fixes the two core circles. The second is to show that the complement of this link is homeomorphic to the complement of $n+2$ fibers of the Hopf fibration of S^3.

More examples can be built from this one. Let $J \subset S^1 \times B^2$ be a knot for which $\partial(S^1 \times B^2)$ is incompressible in the complement of J. A new link can be formed by replacing neighborhoods of the components of L with copies of $S^1 \times B^2$. Then the symmetry group of this new link will be isomorphic to either $\mathbb{Z}_2 \oplus \mathbb{S}_{n-2}$ or $\mathbb{S}_{n-2}$, depending on the symmetry type of J.

9.1 C is the complement of regular fibers in a closed Seifert manifold

Example 9.2 Figure 6 provides a schematic of one possible case in which the core C is Seifert fibered. Some of the labels in the diagram will be explained later. A link L can be formed by filling each T_i with pairs $(S^1 \times B^2, J_i)$ and the S_i are filled with either solid tori or nontrivial knot complements. There are constraints required for this to produce a link in S^3 and we do not assert that in all cases in which C is Seifert fibered it will be of this form. We illustrate it to provide a good model to have in mind

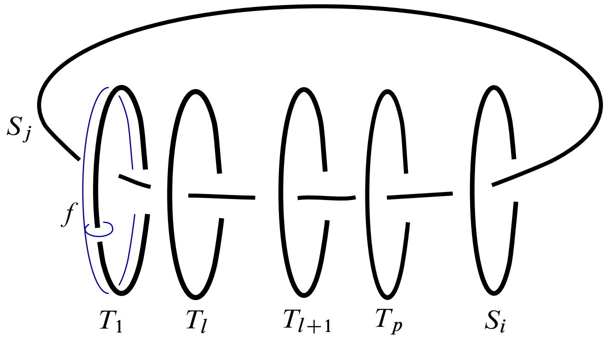


Figure 6: Possible Seifert fibered core C.

as we develop the notation and arguments that follow. Another good model is provided by Example 9.1.

Notice the complement of this link is homeomorphic to the complement of the link formed by giving the parallel strands a full twist. In this case, all the components, including the horizontal one, are fibers of the Hopf fibration of S^3. More generally we have the following:

Theorem 9.3 *The core C is diffeomorphic to the complement of a set of regular fibers in a closed Seifert manifold.*

Proof Build a manifold M by attaching solid tori to the boundary components of C so that each longitude is identified with the fiber of the fibration of C. Then the Seifert fibration of C extends to M and the cores of the solid tori are regular fibers. □

We now fix the choice of that M and its Seifert fibration.

9.2 Notation and a basis for $H_1(T_i)$

For each T_i, there is a basis of $H_1(T_i)$ represented by a pair of curves, $\{f_i, g_i\}$; since T_i bounds the solid torus neighborhood of a regular fiber, we let f_i denote the fiber and let g_i denote the meridian of the solid torus.

Each torus T_i bounds a submanifold of S^3 that contains the component L_i; denote it by W_i. All the pairs (W_i, L_i) are diffeomorphic, so we choose one and denote it by (W, K) with boundary T. We have that T contains a canonical longitude that is null-homologous in W, which we denote by λ; choose a second curve intersecting it once and denote it by μ.

We now see that (S^3, L) is built from C by attaching copies of W to the T_i using attaching maps we denote by G_i. (Other manifolds have to be attached along the other boundary components of C, which we have denoted by S_i.) Denote the images of $\{\lambda, \mu\}$ under G_i by $\{\lambda_i, \mu_i\}$.

Theorem 9.4 *The intersection number of λ_i with f_i is nonzero.*

Proof Our proof depends on the uniqueness of the fibrations of Seifert fibered manifolds, up to isotopy. This does not hold for all Seifert manifolds (eg $S^1 \times B^2$), but Waldhausen [15; 16] proved that, if the Seifert fibered manifold M is sufficiently large, that is, if it contains an incompressible surface that is not boundary parallel, then the fibration is unique. (See also [10].) In the case that the three-manifold has four or more boundary components, it is clearly sufficiently large. The preimage of a circle in the base space that bounds two of the boundary components is an incompressible torus and is not boundary parallel.

We now claim that the λ_i are not fibers of the fibration. Consider $i \neq j$ and the pair λ_i and λ_j. Any element of $\mathcal{D}(L)$ that maps L_i to L_j carries λ_i to $\pm\lambda_j$. Self-homeomorphisms of Seifert fibered spaces with more than three boundary components preserve fibers up to isotopy, so, if λ_i is a fiber, then λ_j is also a fiber.

Suppose that are λ_i and λ_j are fibers. Then there is a vertical annulus A in C joining λ_i to λ_j. There are also surfaces B_i and B_j in W_i and W_j with boundaries λ_i and λ_j. The union of A with $B_i \cup B_j$ is a closed surface in S^3. There is also a curve on T_i meeting this surface in exactly one point. This is impossible in S^3. □

9.3 Maps between the T_i

Without loss of generality, we will focus on T_1 and T_2. We denote a chosen element in $\mathcal{D}(L)$ that carries L_1 to L_2 by F. Note that we can assume $F(f_1) = f_2$, $F(\lambda_1) = \lambda_2$ and $F(\mu_1) = \mu_2$. However, maps of C do not necessarily preserve the g_i. We can assume that $F(g_1) = g_2 + wf_2$ for some w.

For both values of i we have constants such that

$$\lambda_i = \alpha_i f_i + \beta_i g_i, \quad \mu_i = \delta_i f_i + \gamma_i g_i.$$

Applying F to the set with $i = 1$ and renaming variables, we have

$$\lambda_1 = \alpha f_1 + \beta g_1, \quad \mu_1 = \delta f_1 + \gamma g_1$$

and

$$\lambda_2 = (\alpha f_2 + \beta g_2) + \beta w f_2, \quad \mu_2 = (\delta f_2 + \gamma g_2) + \gamma w f_2.$$

9.4 Constructing the transposition

Theorem 9.5 *There is a diffeomorphism G of C that interchanges T_1 and T_2 and is the identity on all other boundary components of C. The map G can be chosen so that it preserves the f_1 and satisfies $G(g_1) = g_2 + wf_2$ and $G(g_2) = g_1 - wf_2$.*

Proof Using the fact that T_i are boundaries of regular fibers, there is a diffeomorphism G of C that interchanges T_1 and T_2 that also preserves the pairs $\{f_i, g_i\}$. This map can be assumed to be the identity on the other components.

There is a vertical annulus in C joining f_1 and f_2. We can perform a w–fold twist along this annulus. This is the identity map on all boundary components other than T_1 and T_2. On T_1 and T_2, it preserves the f_1 and f_2, it maps g_1 to $g_1 - wf_1$ and it maps g_2 to $g_2 + wf_2$. □

9.5 Main theorem in the Seifert fibered case

Theorem 9.6 *If $\mathbb{A}_n \subset \mathbb{S}(L)$ and the associated core C is Seifert fibered, then there is an element $H \in \mathcal{D}(L)$ which transposes L_1 and L_2. Equivalently, $\mathbb{S}(L) = \mathbb{S}_n$.*

Proof The map G given in Theorem 9.5 satisfies

$$G(\lambda_1) = \alpha f_2 + \beta(g_2 + wf_2) \quad \text{and} \quad G(m_1) = \delta f_2 + \gamma(g_2 + wf_2).$$

It also satisfies

$$G(\lambda_2) = \alpha f_1 + \beta(g_1 - wf_1) + \beta w f_1 \quad \text{and} \quad G(\mu_1) = \delta f_1 + \gamma(g_1 - wf_1) + \gamma w f_1.$$

Simplifying shows that this interchanges the attaching maps of W to T_1 and T_2, and thus extends as desired. □

10 Questions

(1) For the four-component link illustrated in Figure 7, the group of symmetries that preserve string orientations is isomorphic to $\mathbb{A}_4$. This example was found by Nathan Dunfield using the program SnapPy [6], where it is listed as $L12a2007$. We have illustrated the link so that each component is in a regular neighborhood of a face of the standard projection of the tetrahedron to $S^2 = \mathbb{R}^2 \cup \infty$. Recall that the orientation-preserving symmetry group of the tetrahedron is isomorphic to $\mathbb{A}_4$.

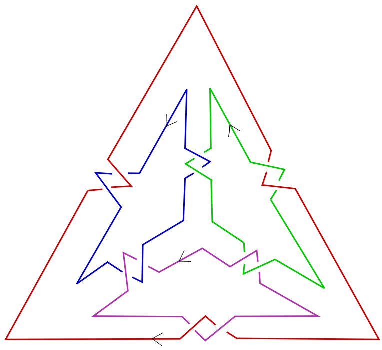


Figure 7: The link $L12a2007$.

Notice that rotation about the vertical axis interchanges two components, so the unoriented symmetry group is $\mathbb{S}(L) = \mathbb{S}_4$. However, if we build a new link by forming the connected sum of each component of $L12a2007$ with the same nonreversible knot J, then any symmetry of this new link, L', would have to preserve the orientations of the components. Thus, $\mathbb{S}(L') = \mathbb{A}_4$.

Similar constructions will likely produce links with symmetry groups that are isomorphic to polyhedral groups. For instance, using the dodecahedron would yield a 12–component link with symmetry group A_5. It is not clear how to reduce the number of components without changing the symmetry group, and we are left with the following question:

Does there exist a five-component link L with $\mathbb{S}(L) = \mathbb{A}_5$?

(2) The Fox–Whitten group Γ_n maps onto $\mathbb{S}_n$, and thus the obstructions we have developed here provide obstructions to groups $G \subset \Gamma_n$ from being oriented intrinsic symmetry groups of links. Can the techniques used here provide finer obstructions in the oriented case?

(3) As a particular example of (2), can any of the unknown cases for two-component links described in Section 4 be eliminated as possible intrinsic symmetry groups?

(4) If a subgroup $H \subset \mathbb{S}_n$ or $H \subset \Gamma_n$ is the intrinsic symmetry group for a link, is it the intrinsic symmetry group of a nonsplit link or of an irreducible link?

(5) A natural class of links consists of Brunnian links; these are nonsplittable but become the unlink upon removing any one of the components. The links produced in Theorem 5.1 having symmetry group $\mathbb{S}_n$ are not Brunnian. The examples of two-component and three-component links with $\mathbb{S}(L) = \mathbb{S}_n$ that precede the proof of that

theorem are Brunnian. Hence, we ask: For all $n \geq 3$, does there exist a Brunnian link L with $\mathbb{S}(L) = \mathbb{S}_n$?

(6) Another class to consider is alternating links, and presumably there are strong constraints on $\mathbb{S}(L)$ for these.

(7) Let M be a compact three-manifold with n torus boundary components, $\partial_i(M)$. Choose a basis of $H_1(\partial_i(M))$ for each i. One can form a Whitten-like group $\Omega_n = \mathbb{Z}_2 \oplus (G^n \rtimes \mathbb{S}_n)$, where G is the automorphism group of $\mathbb{Z} \oplus \mathbb{Z}$. Each manifold M gives rise to a subgroup of Ω_n. What subgroups arise in this way? This is particularly interesting in the case that the interior of M has a complete hyperbolic structure.

(8) The previous question can be modified. Given a subgroup $H \subset \mathbb{S}_n$, is there a complete hyperbolic three-manifold with n cusps such that the H represents the permutations of the cusps that are realized by isometries of M? In relation to this, Paoluzzi and Porti [11] proved that every finite group is the isometry group of the complement of a hyperbolic link in S^3. Notice that their isometries need not extend to S^3. Applying their construction to a subgroup of $\mathbb{S}_n$ does not produce an n–component link.

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