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Loop homotopy of 6–manifolds over 4–manifolds

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Let M be the 6–manifold M arising as the total space of the sphere bundle of a rank 3 vector bundle over a simply connected closed 4–manifold. We show that, after looping, M is homotopy equivalent to a product of loops on spheres in general. This particularly implies a cohomological rigidity property of M after looping. Furthermore, passing to rational homotopy we show that such an M is Koszul.

55P15, 55P35, 57R19; 55P10, 55P40, 55P62

1 Introduction

Classification of manifolds is a fundamental problem in geometry and topology. Numerous investigations have been made around this problem in both the smooth and topological categories. For instance, in the general case, Wall [33; 35] studied $(n-1)$–connected $2n$–manifolds and $(n-1)$–connected $(2n+1)$–manifolds. For concrete cases with specified dimension, Barden [2] classified simply connected 5–manifolds, and Wall [34], Jupp [23] and Zhubr [37; 38] classified simply connected 6–manifolds. More recently, Kreck and Su [25] classified certain nonsimply connected 5–manifolds, while Crowley and Nordström [15] and Kreck [24] studied the classification of various kinds of 7–manifolds.

In the literature mentioned, the homotopy classification of M was usually carried out as a byproduct of a system of invariants. However, it is almost impossible to extract nontrivial homotopy information of M directly from the classification. On the other hand, unstable homotopy theory is a powerful tool for studying the homotopy properties of manifolds preserved by suspending or looping. From the suspension viewpoint, So and Theriault [31] determined the homotopy type of the suspension of connected 4–manifolds, while Huang [19] studied the suspension of simply connected 6–manifolds. From the loop viewpoint, Beben and Theriault [6] studied the loop decompositions of $(n-1)$–connected $2n$–manifolds, while Beben and Wu [8] and Huang and Theriault [20] studied the loop decompositions of the $(n-1)$–connected $(2n+1)$–manifolds. The homotopy groups of these manifolds were also investigated by Samik Basu and Somnath Basu [3; 4] from different point of view. Moreover, a theoretical method of loop decomposition was developed by Beben and Theriault [7], which is quite useful for studying the homotopy of manifolds.

We study the loop homotopy of certain simply connected 6–manifolds constructed from 4–manifolds. Let N be a simply connected closed 4–manifold with $H^2(N;\mathbb{Z})\cong\mathbb{Z}^{\oplus d}$ for $d\geq 1$. A rank 3 vector bundle ξ over N is classified by a map $f\colon N\to B\mathrm{SO}(3)$, where $B\mathrm{SO}(3)$ is the classifying space of the special orthogonal group $\mathrm{SO}(3)$. The sphere bundle of ξ

$$S^2 \xrightarrow{i} M \xrightarrow{p} N \tag{1}$$

defines the closed 6–manifold M. Since the integral cohomologies of N and S^2 are free and concentrated in even degree, the Serre spectral sequence of (1) collapses, and $H^*(M;\mathbb{Z})\cong H^*(N;\mathbb{Z})\otimes H^*(S^2;\mathbb{Z})$. Our main result is the following theorem, which will be proved in Section 4.

Theorem 1.1 *Let N be a simply connected closed 4–manifold with $H^2(N;\mathbb{Z})\cong\mathbb{Z}^{\oplus d}$ for $d\geq 1$. Let M be the total manifold of the sphere bundle of a rank 3 vector bundle over N. Then:*

- *If $d=1$,*
$$\Omega M\simeq S^1\times\Omega S^2\times\Omega S^5.$$
- *If $d\geq 2$,*
$$\Omega M\simeq S^1\times\Omega S^2\times\Omega(S^2\times S^3)\times\Omega\big(J\vee(J\wedge\Omega(S^2\times S^3))\big),$$
where $J=\bigvee_{i=1}^{d-2}(S^2\vee S^3)$.

From Theorem 1.1 and its proof, it can be easily seen that the decompositions in Theorem 1.1 are compatible with the S^2–bundle (1) after looping. In particular, this means that though the fibre bundle (1) does not split in general, its loop does. Moreover, as discussed in [6, page 217], the term $J\vee(J\wedge\Omega(S^2\times S^3))$ in the second decomposition of Theorem 1.1 is a bouquet of spheres. Hence by the Hilton–Milnor theorem, we see that ΩM is homotopy equivalent to a product of loops on spheres with S^1. Additionally, since the decompositions of Theorem 1.1 only depend on the value of d, which is determined by and determines $H^2(M;\mathbb{Z})$, we have a rigidity property for M after looping.

Corollary 1.2 *Let M and M' be two 6–manifolds satisfying the conditions of Theorem 1.1. Then $\Omega M \simeq \Omega M'$ if and only if $H^2(M;\mathbb{Z}) \cong H^2(M';\mathbb{Z})$.* □

Theorem 1.1 can be improved if we pass from integral homotopy to rational homotopy. Indeed, by Theorem 1.1 it is straightforward to compute the homotopy groups of M in terms of those of spheres. However, there is an additional Lie algebra structure on the homotopy groups of any *CW* complex X. In rational homotopy theory, the graded Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ is called the *homotopy Lie algebra* of X, and X is called *coformal* if the rational homotopy type of X is completely determined by its homotopy Lie algebra. If X is further *formal*, that is the homotopy type of X is determined by the graded commutative algebra $H^*(X;\mathbb{Q})$, then X is *Koszul* in the sense of Berglund [9, Definition 1.1]. In the latter case, $H^*(X;\mathbb{Q})$ is a *Koszul algebra* and $\pi_*(\Omega X)\otimes\mathbb{Q}$ is a *Koszul Lie algebra* [9]. The following theorem concerns these additional structures on M of the type in Theorem 1.1.

Theorem 1.3 *Let N be a simply connected closed 4–manifold with $H^2(N;\mathbb{Z})\cong\mathbb{Z}^{\oplus d}$. Let M be the total manifold of the sphere bundle of a rank 3 vector bundle over N. Then*:

- *If $d=1$, M is not coformal.*
- *If $d\geq 2$, M is Koszul, and there is an isomorphism of graded Lie algebras*

$$\pi_*(\Omega M)\otimes\mathbb{Q}\cong H^*(M;\mathbb{Q})^{!\mathscr{L}ie},$$

  where $(-)^{!\mathscr{L}ie}$ is the Koszul dual Lie functor defined in [9, Section 2].

We turn to the remaining case, when $d=0$, that is, $N\cong S^4$. Note, we still have the 6–manifold M as constructed in (1). Though the homotopy classification of such manifolds was almost determined by Yamaguchi [36], this case is surprisingly much harder than the general one. We will explain this point after the statement of our result in this case. Let $\eta_2\colon S^3\to S^2$ be the Hopf map. For any integer n, let $S^m\{n\}$ be the homotopy fibre of the degree n map on S^m.

Theorem 1.4 *Let M be the total space of the sphere bundle of a rank 3 vector bundle over S^4. Then M has a cell structure of the form*

$$M \simeq S^2 \cup_{k\eta_2} e^4 \cup e^6,$$

where $k \in \mathbb{Z}$. Let $k = p_1^{r_1} \cdots p_\ell^{r_\ell}$ be the prime decomposition of k. Further:

- *If k is odd,*
$$\Omega M \simeq S^1 \times \prod_{j=1}^{\ell} S^3\{p_j^{r_j}\} \times \Omega S^7.$$
- *If $k = 2^r$ with $r \geq 3$,*
$$\Omega M \simeq S^1 \times S^3\{2^r\} \times \Omega S^7.$$

Note that we still have cohomological rigidity in this case, since the homotopy type of ΩM only depends on k, which is determined by the square of a generator in $H^2(M;\mathbb{Z})$. But it is less interesting since the cohomological rigidity of M without looping holds except for the case when k is even and M is Spin [36]. Further note that Theorem 1.4 is only a partial result. The difficulty in this case is due to the fact that the proof of Theorem 1.4 heavily relies on a result of Huang and Theriault [20] on the loop decomposition of 2–connected 7–manifolds. As discussed in [20, Section 6], the case when $k = 2^r m$ with m odd and greater than 1 is much more difficult. Also, since it is known that $S^3\{2\}$ is not an H–space (see Cohen [11]), we cannot expect a decomposition of the form $\Omega M \simeq S^1 \times S^3\{2\} \times \Omega S^7$ for the case when $k = 2$. In contrast, the rational homotopy of M in this case is simple. As shown in Lemma 5.2, M is rationally homotopy equivalent to $\mathbb{C}P^3$ or $S^2 \times S^4$. Moreover, it is well known that $\mathbb{C}P^3$ is not coformal (see Neisendorfer and Miller [27, Example 4.7]), while $S^2 \times S^4$ is Koszul; see Berglund [9, Examples 5.1 and 5.4].

Before we close the introduction, let us make two remarks. Firstly, our results provide further evidence on the Moore conjecture. Recall that the Moore conjecture states that a simply connected finite CW complex Z is rationally elliptic if and only if it has a finite homotopy exponent at all primes, or equivalently, Z is rationally hyperbolic if and only if it has unbound homotopy exponent at some prime. For M in our context, it is elliptic if and only if $d \leq 2$, and in any of these cases M has a finite homotopy exponent at all primes by Cohen, Moore and Neisendorfer [12; 13] and James [21]. When $d \geq 3$, M is hyperbolic such that ΩM has $\Omega(S^2 \vee S^3)$ as product summand, hence it has no bound on its homotopy exponent for any prime p; see Neisendorfer and Selick [28] or Boyde [10] for instance. Secondly, Amorós and Biswas [1] characterized simply connected rationally elliptic compact Kähler threefolds in terms of Hodge diamonds, and in particular, their second Betti numbers satisfy $b_2 \leq 3$. For M in our context, this is equivalent to $d \leq 2$, and our decompositions provide further information on the homotopy of M. For instance, the homotopy groups of M can be computed in terms of those of spheres.

The paper is organized as follows. In Section 2 we classify rank 3 bundles over the 4–manifold N. In Section 3, we prove Lemma 3.1, which implies that under Lemma 2.1 one component of the classifying map f of the bundle ξ over N is trivial in a special case. This is crucial for proving Theorem 1.1. In Section 4, we prove Theorem 1.1 by dividing it into two cases. Section 5 is devoted to the remaining case when $d = 0$ and we prove Theorem 1.4 there. We discuss the rational homotopy of 6–manifolds and prove Theorem 1.3 in Section 6.

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2 Rank 3 bundles over 4–manifolds

In this section, we discuss necessary knowledge of rank 3 vector bundles over simply connected 4–manifolds, which will be used in the subsequent sections. There are various ways to study the classification of vector bundles. Here, we adopt an approach from a homotopy theoretical point of view for later use.

Let N be a simply connected 4–manifold such that $H^2(N;\mathbb{Z}) \cong \mathbb{Z}^{\oplus d}$ with $d \geq 0$. A rank 3 vector bundle ξ over N is classified by a map $f\colon N \to B\mathrm{SO}(3)$. The sphere bundle of ξ

$$S^2 \xrightarrow{i} M \xrightarrow{p} N$$

defines the closed 6–manifold M. For N, there is the homotopy cofiber sequence

$$S^3 \xrightarrow{\phi} \bigvee_{i=1}^{d} S^2 \xrightarrow{\rho} N \xrightarrow{q} S^4 \xrightarrow{\Sigma\phi} \bigvee_{i=1}^{d} S^3, \tag{2}$$

where ϕ is the attaching map of the top cell of N, ρ is the injection of the 2–skeleton, and q is the pinch map onto the top cell. Let $s\colon S^1 \cong \mathrm{SO}(2) \to \mathrm{SO}(3)$ be the canonical inclusion of Lie groups.

Lemma 2.1 *There is a surjection*

$$\Phi\colon [S^4, B\mathrm{SO}(3)] \times [N, BS^1] \to [N, B\mathrm{SO}(3)]$$

of pointed sets that restricts to q^* on $[S^4, B\mathrm{SO}(3)]$, and to $(Bs)_*$ on $[N, BS^1]$.

Proof By (2), there is the exact sequence of pointed sets

$$0 = \Bigl[\bigvee_{i=1}^{d} S^3, B\mathrm{SO}(3)\Bigr] \to [S^4, B\mathrm{SO}(3)] \xrightarrow{q^*} [N, B\mathrm{SO}(3)] \xrightarrow{\rho^*} \Bigl[\bigvee_{i=1}^{d} S^2, B\mathrm{SO}(3)\Bigr] \to [S^3, B\mathrm{SO}(3)] = 0,$$

in a strong sense: there is an action of $[S^4, B\mathrm{SO}(3)]$ on $[N, B\mathrm{SO}(3)]$ through q^* such that the sets $\rho^{*-1}(x)$, for $x \in \bigl[\bigvee_{i=1}^{d} S^2, B\mathrm{SO}(3)\bigr]$, are precisely the orbits. It is known that $\bigl[\bigvee_{i=1}^{d} S^2, B\mathrm{SO}(3)\bigr] \cong \bigoplus_d \mathbb{Z}/2\mathbb{Z}$ and $[S^4, B\mathrm{SO}(3)] \cong \mathbb{Z}$. Moreover, there is the commutative diagram

$$\begin{tikzcd}
{[N, BS^1]} \arrow[r, "\rho^*", "\cong"'] \arrow[d, "(Bs)_*"] & \bigl[\bigvee_{i=1}^{d} S^2, BS^1\bigr] \arrow[r, "\cong"] \arrow[d, "(Bs)_*"] & \bigoplus_d \mathbb{Z} \arrow[d, "\bigoplus_d \rho_2"] \\
{[N, B\mathrm{SO}(3)]} \arrow[r, "\rho^*"] & \bigl[\bigvee_{i=1}^{d} S^2, B\mathrm{SO}(3)\bigr] \arrow[r, "\cong"] & \bigoplus_d \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

where ρ^* is an isomorphism onto $\bigl[\bigvee_{i=1}^{d} S^2, BS^1\bigr] \cong \bigoplus_d \mathbb{Z}$ and ρ_2 is the mod 2 reduction, hence $(Bs)_*$ is surjective onto $\bigl[\bigvee_{i=1}^{d} S^2, B\mathrm{SO}(3)\bigr]$. Now for any $f \in [N, B\mathrm{SO}(3)]$ we have $\rho^*(f) = (Bs)_*(x)$ for some $x \in \bigl[\bigvee_{i=1}^{d} S^2, BS^1\bigr]$. Write $\alpha = (\rho^{*-1})(x)$. Then $Bs_*(\alpha)$ and f belong to the same orbit of the action, for they have same image in $\bigl[\bigvee_{i=1}^{d} S^2, B\mathrm{SO}(3)\bigr]$ through ρ^*. Hence, there exists an $f' \in [S^4, B\mathrm{SO}(3)]$ such that $q^*(f') \cdot (Bs_*(\alpha)) = f$. □

From Lemma 2.1 and its proof, for the classifying map $f\colon N \to B\mathrm{SO}(3)$, we have associated a pair of maps

(3) $$(f', \alpha) \in [S^4, B\mathrm{SO}(3)] \times [N, BS^1],$$

such that $q^*(f')\cdot(Bs_*(\alpha)) = f$ and $\omega_2(\xi) \equiv \alpha \mod 2$. We also notice that if $\rho^*(f) \neq 0$, or equivalently ξ is non-Spin, the element α can be always chosen to be primitive, that is, α is not divisible by any integer k with $k \neq \pm 1$. This is important for our later use.

Let $\pi\colon W \to N$ be a map from a closed manifold W. The pullback of the bundle ξ along π has an associated sphere bundle

$$S^2 \xrightarrow{\iota} Z \xrightarrow{\mathfrak{p}} W,$$

which defines the closed manifold Z. The following lemma is critical for proving Proposition 4.1.

Lemma 2.2 *Suppose for W there is a homotopy cofibration*

$$W_{m-1} \xrightarrow{\varrho} W \xrightarrow{\mathfrak{q}} S^m,$$

such that $\pi \circ \varrho$ factors as

$$W_{m-1} \xrightarrow{\pi_\prime} \bigvee_{i=1}^{d} S^2 \xrightarrow{\rho} N$$

for some $\pi_\prime$, where W_{m-1} is the $(m-1)$–skeleton of W. Then if $f' \circ q \circ \pi$ and $\alpha \circ \pi$ are both nullhomotopic, the bundle $\pi^*(\xi)$ is trivial, and in particular

$$Z \cong S^2 \times W.$$

Proof By the assumption, there is a diagram of homotopy cofibrations

$$\begin{tikzcd}
W_{m-1} \arrow[r, "\varrho"] \arrow[d, "\pi_\prime"] & W \arrow[r, "\mathfrak{q}"] \arrow[d, "\pi"] & S^m \arrow[d, "\pi'"] \\
\bigvee_d S^2 \arrow[r, "\rho"] & N \arrow[r, "q"] & S^4
\end{tikzcd}$$

which defines the map π'. It follows that there is a morphism of exact sequences of pointed sets

$$\begin{tikzcd}
{[S^4, B\mathrm{SO}(3)]} \arrow[r, "q^*"] \arrow[d, "\pi'^*"] & {[N, B\mathrm{SO}(3)]} \arrow[r, "\rho^*"] \arrow[d, "\pi^*"] & {\left[\bigvee_{i=1}^d S^2, B\mathrm{SO}(3)\right]} \arrow[d, "\pi_\prime^*"] \\
{[S^m, B\mathrm{SO}(3)]} \arrow[r, "\mathfrak{q}^*"] & {[W, B\mathrm{SO}(3)]} \arrow[r, "\varrho^*"] & {[W_{m-1}, B\mathrm{SO}(3)]}
\end{tikzcd}$$

such that the action of $[S^4, B\mathrm{SO}(3)]$ on $[N, B\mathrm{SO}(3)]$ is compatible with the action of $[S^m, B\mathrm{SO}(3)]$ on $[W, B\mathrm{SO}(3)]$ through π'^*. Hence, by (3), the classifying map $f\circ\pi$ of $\pi^*(\xi)$ satisfies

$$\begin{aligned} f\circ\pi = \pi^*\big(q^*(f')\cdot(Bs_*(\alpha))\big) &= \mathfrak{q}^*(\pi'^*(f'))\cdot\pi^*(Bs_*(\alpha)) \\ &= \pi^*(q^*(f'))\cdot\pi^*((Bs_*(\alpha))) = (f'\circ q\circ\pi)\cdot Bs_*(\alpha\circ\pi), \end{aligned}$$

which is nullhomotopic by the assumption. □

Lemma 2.1 also gives a byproduct on the classification of rank 3 vector bundles over N via characteristic classes, which could also be proved by other methods, like the classical obstruction theory.

Proposition 2.3 *A rank 3 vector bundle ξ over N is completely determined by its second Stiefel–Whitney class $\omega_2(\xi)$ and its first Pontryagin class $p_1(\xi)$.*

Proof Given two rank 3 vector bundles ξ_1 and ξ_2 over N, suppose that $\omega_2(\xi_1)=\omega_2(\xi_2)$ and $p_1(\xi_1)=p_1(\xi_2)$. We want to show that $\xi_1\cong\xi_2$, or equivalently, $f_1\simeq f_2$, where $f_1, f_2\colon N\to B\mathrm{SO}(3)$ are the classifying maps of ξ_1 and ξ_2, respectively. By Lemma 2.1 and (3), $f_1=q^*(f_1')\cdot(Bs_*(\alpha))$ for a pair of maps $(f_1',\alpha)\in[S^4, B\mathrm{SO}(3)]\times[N, BS^1]$ such that $\omega_2(\xi_1)\equiv\alpha \bmod 2$. Since $\omega_2(\xi_1)=\omega_2(\xi_2)$, there exists $f_2'\in[S^4, B\mathrm{SO}(3)]$ such that $f_2=q^*(f_2')\cdot(Bs_*(\alpha))$. It follows that to show $f_1\simeq f_2$, it suffices to show $f_1'\simeq f_2'$. Indeed, for either ξ_i the expression of f_i can be explicitly described as

$$f_i\colon N\xrightarrow{\mu'} N\vee S^4\xrightarrow{\alpha\vee f_i'} BS^1\vee B\mathrm{SO}(3)\xrightarrow{Bs\vee\mathrm{id}} B\mathrm{SO}(3)\vee B\mathrm{SO}(3)\xrightarrow{\nabla} B\mathrm{SO}(3),$$

where μ' is the coaction map and ∇ is the folding map. In particular, it is easy to see that

$$p_1(\xi_i)=q^*(p_1(f_i'))+\alpha^2, \tag{4}$$

where we denote by $p_1(f_i')$ the first Pontryagin class of the bundle over S^4 determined by f_i'. Since $p_1(\xi_1)=p_1(\xi_2)$, (4) implies that $q^*(p_1(f_1'))=q^*(p_2(f_i'))$. Moreover, it is clear that $q^*\colon H^4(S^4;\mathbb{Z})\to H^4(N;\mathbb{Z})$ is an isomorphism. Hence $p_1(f_1')=p_1(f_2')$. Now since $[S^4, B\mathrm{SO}(3)]\simeq\mathbb{Z}$, and the morphism $\frac{1}{4}p_1\colon[S^4, B\mathrm{SO}(3)]\to H^4(S^4;\mathbb{Z})$ sending each map to one fourth of the first Pontryagin class of the associated bundle is an isomorphism [18], we see that $f_1'\simeq f_2'$. Then $f_1\simeq f_2$ and the proposition follows. □

3 The induced map between top cells

Let N be a simply connected closed 4–manifold such that $H^2(N;\mathbb{Z}) \cong \mathbb{Z}^{\oplus d}$ with $d \geq 1$. Consider the circle bundle

$$S^1 \xrightarrow{j} Y \xrightarrow{\pi} N$$

classified by a primitive element $\beta \in H^2(N;\mathbb{Z})$, which defines the simply connected 5–manifold Y. By [16, Lemma 1], Y has cell structure of the form

$$Y \simeq \bigvee_{d-1} (S^2 \vee S^3) \cup e^5.$$

Then, by the cellular approximation theorem, there is the diagram of homotopy cofibration

$$\begin{tikzcd} \bigvee_{d-1}(S^2 \vee S^3) \arrow[r, "\varrho"] \arrow[d] & Y \arrow[r, "\mathfrak{q}"] \arrow[d, "\pi"] & S^5 \arrow[d, "\pi'"] \\ \bigvee_d S^2 \arrow[r, "\rho"] & N \arrow[r, "q"] & S^4 \end{tikzcd} \tag{5}$$

where the bottom cofibration is part of (2), ϱ is the inclusion of the 3–skeleton of Y followed by the quotient $\mathfrak{q}$, and π' is induced from π. In this section, we prove the following key lemma for understanding rank 3–bundles over Y in a special case. Let $[N]$ be the fundamental class of N. Let $\langle x \cup y, [N]\rangle \in \mathbb{Z}$ be the canonical pairing for any cohomology classes x, $y \in H^2(N;\mathbb{Z})$.

Lemma 3.1 *The induced map π' in* (5) *is nullhomotopic when $\langle \beta^2, [N]\rangle$ is odd.*

Proof The primitive element β is represented by a map $\beta\colon N \to \mathbb{C}P^\infty \simeq K(\mathbb{Z},2)$. By the cellular approximation theorem, β factors through $\mathbb{C}P^2$,

$$\beta\colon N \xrightarrow{\widetilde{\beta}} \mathbb{C}P^2 \xrightarrow{x} \mathbb{C}P^\infty,$$

which defines the map $\widetilde{\beta}$, and x represents a generator $x \in H^2(\mathbb{C}P^2;\mathbb{Z})$. The factorization gives a diagram of circle bundles

$$\begin{tikzcd} S^1 \arrow[r, "j"] \arrow[d, equal] & Y \arrow[r, "\pi"] \arrow[d, "\widehat{\beta}"] & N \arrow[d, "\widetilde{\beta}"] \\ S^1 \arrow[r] & S^5 \arrow[r, "\pi_0"] & \mathbb{C}P^2 \end{tikzcd} \tag{6}$$

where the bundle in the second row is classified by x, and $\widehat{\beta}$ is the induced map. By the cellular approximation theorem, there is a homotopy commutative diagram

(7)
$$\begin{tikzcd}
 & Y \arrow[rr, "\pi"] \arrow[dl, "q"'] \arrow[dd, "\widehat{\beta}"] & & N \arrow[dl, "q"'] \arrow[dd, "\widetilde{\beta}"] \\
S^5 \arrow[rr, "\pi'"] \arrow[dd, "\widehat{\beta}'"'] & & S^4 \arrow[dd, "\widetilde{\beta}'"] & \\
 & S^5 \arrow[rr, "\pi_0"] \arrow[dl, equal] & & \mathbb{C}P^2 \arrow[dl, "q_0"] \\
S^5 \arrow[rr, "\pi_0'"] & & S^4 &
\end{tikzcd}$$

where the rear and top faces are the right squares in (6) and (5), respectively, q_0 is the quotient map onto the top cell of $\mathbb{C}P^2$, π_0' is defined to be $q_0 \circ \pi_0$, and $\widehat{\beta}'$ and $\widetilde{\beta}'$ are the induced maps. By the homotopy commutativity of the right face of (7), the assumption that $\langle \beta^2, [N] \rangle$ is odd is equivalent to $\widetilde{\beta}'$ having odd degree. Further, since the homotopy cofibre of π_0 is $\mathbb{C}P^3$, for which the Steenrod operation $\mathrm{Sq}^2\colon H^4(\mathbb{C}P^3; \mathbb{Z}/2\mathbb{Z}) \to H^6(\mathbb{C}P^3; \mathbb{Z}/2\mathbb{Z})$ is trivial, we obtain that $\pi_0' = q_0 \circ \pi_0$ is nullhomotopic. Now consider the front face of (7). Combining the above arguments and the fact that $\pi_5(S^4) \cong \mathbb{Z}/2\mathbb{Z}\{\eta_4\}$ [32], we see that $\pi' \simeq \widetilde{\beta}' \circ \pi' \simeq \pi_0' \circ \widehat{\beta}'$ is nullhomotopic. $\square$

4 Proof of Theorem 1.1

Let N be a simply connected 4–manifold such that $H^2(N; \mathbb{Z}) \cong \mathbb{Z}^{\oplus d}$ with $d \geq 1$. A rank 3 vector bundle ξ over N is classified by a map $f\colon N \to B\mathrm{SO}(3)$ with the associated sphere bundle
$$S^2 \xrightarrow{i} M \xrightarrow{p} N,$$
which defines the closed 6–manifold M. Recall, by Lemma 2.1 and (3), the classifying map $f\colon N \to B\mathrm{SO}(3)$ for the bundle ξ is determined by
$$(f', \alpha) \in [S^4, B\mathrm{SO}(3)] \times [N, BS^1]$$
such that $f = q^*(f') \cdot (Bs)_*(\alpha)$ and $\omega_2(\xi) \equiv \alpha \bmod 2$, where q and s are defined before Lemma 2.1. Moreover, by the discussion after Lemma 2.1, when ξ is non-Spin we suppose that α is primitive.

For the loop homotopy of M, we may study S^1–bundles over M pulled back from those over the 4–manifold N. Consider the circle bundle

(8)
$$S^1 \xrightarrow{j} Y \xrightarrow{\pi} N$$

classified by a primitive element $\beta \in H^2(N;\mathbb{Z})$, which defines the simply connected 5–manifold Y. Based on the previous remark on the choice of α, we make the following convention on the choice of β:

- $\beta = \alpha$ if ξ is non-Spin, or
- β can be any primitive element if ξ is Spin.

The remainder of this section is devoted to the proof of Theorem 1.1 by dividing it into two cases according to the parity of $\langle \beta^2, [N]\rangle$. In Section 4.1, we prove Theorem 1.1 using Lemma 3.1 under the assumption that $\langle \beta^2, [N]\rangle$ is odd. This is the case when the circle bundle (8) plays an essential role. However, when $\langle \beta^2, [N]\rangle$ is even, we have to apply a different method to prove Theorem 1.1. This is done in Section 4.2.

4.1 Case I: $\langle \beta^2, [N]\rangle$ is odd

In this case, by the choice of the circle bundle (8), consider the pullback of fibre bundles

$$\begin{tikzcd} & S^2 \arrow[r, equal] \arrow[d, "\iota"] & S^2 \arrow[d, "i"] \\ S^1 \arrow[r, "\jmath"] \arrow[d, equal] & X \arrow[r, "\psi"] \arrow[d, "\mathfrak{p}"] & M \arrow[d, "p"] \\ S^1 \arrow[r, "j"] & Y \arrow[r, "\pi"] & N \end{tikzcd} \tag{9}$$

which defines the closed 7–manifold X with bundle projections ψ and $\mathfrak{p}$ onto M and Y, respectively. We show that the induced bundle over Y in (9) is trivial in this case.

Proposition 4.1 *If $\langle \beta^2, [N]\rangle$ is odd, then the bundle $\pi^*(\xi)$ defined in (9) is trivial, and, in particular,*
$$X \cong S^2 \times Y.$$

Proof By Lemma 3.1, π' is nullhomotopic. This implies that $f' \circ q \circ \pi \simeq f' \circ \pi' \circ \mathfrak{q}$ is nullhomotopic by the homotopy commutativity of the right square in (5).

If ξ is non-Spin, then $\beta = \alpha$. We obtain the homotopy fibration $Y \xrightarrow{\pi} N \xrightarrow{\alpha} BS^1$, which implies that $\alpha \circ \pi$ is nullhomotopic, hence so is $(Bs_*)(\alpha \circ \pi)$. Then by Lemma 2.2 the classifying map $f \circ \pi$ of the bundle $\pi^*(\xi)$ is nullhomotopic, and the proposition follows in this case.

If ξ is Spin, by Lemma 2.1 the classifying map $f\colon N \to B\mathrm{SO}(3)$ of ξ is in the image of q^*, that is, there exists a map $f'\colon S^4 \to B\mathrm{SO}(3)$ such that $f' \circ q \simeq f$, and then the bundle ξ is the pullback of the bundle ξ' over S^4 classified by f'. Hence $f\circ\pi\simeq f'\circ q\circ\pi$ is nullhomotopic by the previous argument, and then the bundle $\pi^*(\xi)$ is trivial. In particular, $X\cong S^2\times Y$ and the proposition follows in this case. □

Proof of Theorem 1.1 in Case I As in the beginning of this subsection, consider the circle bundle $S^1\xrightarrow{j}Y\xrightarrow{\pi}N$ classified by the primitive element $\alpha\in H^2(N;\mathbb{Z})$. Then by Proposition 4.1, the total space X of the sphere bundle of $\pi^*(\xi)$ satisfies $X\cong S^2\times Y$. Hence, by (9),

$$\Omega M\simeq S^1\times\Omega X\simeq S^1\times\Omega S^2\times\Omega Y. \tag{10}$$

If $d=1$, then Y has to be S^5, and hence $\Omega M\simeq S^1\times\Omega S^2\times\Omega S^5$. If $d\geq 2$, by [7, Example 4.4] or [3] there is a homotopy equivalence

$$\Omega Y\simeq\Omega(S^2\times S^3)\times\Omega\big(J\vee(J\wedge\Omega(S^2\times S^3))\big) \tag{11}$$

with $J=\bigvee_{i=1}^{d-2}(S^2\vee S^3)$. Combining (10) with (11), we obtain the loop decomposition of M in the theorem. □

4.2 Case II: $\langle\beta^2,[N]\rangle$ is even

In this case, the induced bundle $\pi^*(\xi)$ defined in (9) may not be trivial, and we need to apply a different method to prove Theorem 1.1. Indeed, in this case we can work with the sphere bundle $S^2\xrightarrow{i}M\xrightarrow{p}N$ directly, and show that it splits after looping.

Proposition 4.2 *If $\langle\beta^2,[N]\rangle$ is even, the sphere bundle $S^2\xrightarrow{i}M\xrightarrow{p}N$ of ξ defined in* (1) *is homotopically trivial after looping, and in particular*

$$\Omega M\simeq\Omega S^2\times\Omega N.$$

Proof By Poincaré duality there exists a class $\alpha\in H^2(N;\mathbb{Z})$ such that $\langle\alpha\cup\beta,[N]\rangle=1$. Since by assumption $\langle\beta^2,[N]\rangle$ is even, $\alpha\neq\beta$. Hence by [6, proof of proposition 3.2 and Lemma 3.3] there is a Poincaré duality space Q such that $H^*(Q;\mathbb{Z})\cong H^*(S^2\times S^2;\mathbb{Z})$ as graded rings, $\Omega Q\simeq\Omega S^2\times\Omega S^2$, and there is a map

$$h\colon N\to Q$$

such that Ωh has a right homotopy inverse and $h^*(x)=\alpha$ with $x\in H^2(Q;\mathbb{Z})$ a generator. Let us fix a homotopy equivalence $e\colon\Omega S^2\times\Omega S^2\to\Omega Q$ defined in [6, Lemma 2.3] with its inverse denoted by e^{-1}.

Recall, ξ is determined by a pair of maps $(f', \alpha) \in [S^4, B\mathrm{SO}(3)] \times [N, BS^1]$. By Lemma 2.1, define a rank 3 vector bundle ζ over Q by $(f', x) \in [S^4, B\mathrm{SO}(3)] \times [Q, BS^1]$. It follows that $\xi = h^*(\zeta)$ and there is a pullback of sphere bundles

$$\begin{tikzcd} S^2 \arrow[r, "i"] \arrow[d, equal] & M \arrow[r, "p"] \arrow[d, "\tilde{h}"] & N \arrow[d, "h"] \\ S^2 \arrow[r, "\tilde{i}"] & \tilde{Q} \arrow[r, "\tilde{p}"] & Q \end{tikzcd} \tag{12}$$

where the second row is the sphere bundle of ζ and $\tilde{h}$ is the induced map. Since $H^*(Q;\mathbb{Z})$ and $H^*(S^2;\mathbb{Z})$ are concentrated in even degrees, the Serre spectral sequence for the fibration $S^2 \to \tilde{Q} \to Q$ collapses for degree reasons, and then

$$H^*(\tilde{Q};\mathbb{Z}) \cong H^*(S^2;\mathbb{Z}) \otimes H^*(Q;\mathbb{Z}).$$

Apply the loop functor to (12). It is clear that there is a map $i_1 \times i_2 \colon S^1 \times S^1 \to \Omega\tilde{Q}$ such that the composition

$$S^1 \times S^1 \xrightarrow{i_1 \times i_2} \Omega\tilde{Q} \xrightarrow{\Omega\tilde{p}} \Omega Q \xrightarrow{e^{-1}} \Omega S^2 \times \Omega S^2$$

is homotopic to $E \times E$ with $E \colon S^1 \to \Omega S^2$ the suspension map. By the universal property of $\Omega\Sigma$, there is a unique extension $I \colon \Omega S^2 \times \Omega S^2 \to \Omega\tilde{Q}$ of $i_1 \times i_2$ up to homotopy such that

$$\Omega S^2 \times \Omega S^2 \xrightarrow{I} \Omega\tilde{Q} \xrightarrow{\Omega\tilde{p}} \Omega Q \xrightarrow{e^{-1}} \Omega S^2 \times \Omega S^2$$

is homotopic to the identity. Therefore, the sphere bundle of ζ splits after looping to give

$$\Omega\tilde{Q} \simeq \Omega S^2 \times \Omega Q \simeq \Omega S^2 \times \Omega S^2 \times \Omega S^2.$$

In particular, $\Omega\tilde{i}$ has a left homotopy inverse $\tilde{r}$, which implies that $\tilde{r} \circ \Omega\tilde{h}$ is a left homotopy inverse of Ωi. Then the sphere bundle in the top row of (12) splits after looping, and in particular $\Omega M \simeq \Omega S^2 \times \Omega N$. □

Proof of Theorem 1.1 in Case II Since $\langle \beta^2, [N] \rangle$ is even and β is primitive, we have $d \geq 2$. By Proposition 4.2, $\Omega M \simeq \Omega S^2 \times \Omega N$. Further, by [6, Theorem 1.3] there is a homotopy equivalence

$$\Omega N \simeq S^1 \times \Omega(S^2 \times S^3) \times \Omega\big(J \vee (J \wedge \Omega(S^2 \times S^3))\big)$$

with $J = \bigvee_{i=1}^{d-2}(S^2 \vee S^3)$. Then in this case the theorem follows by combining the above decompositions. □

5 The case when $d = 0$

In this section, we study the case when $d = 0$ and prove Theorem 1.4 as an immediate corollary of Propositions 5.3 and 5.4. Indeed, we work in a slightly more general context, that is, we study the loop decomposition of the closed 6–manifold M with cell structure of the form

$$M \simeq S^2 \cup e^4 \cup e^6. \tag{13}$$

Notice that M in Theorem 1.4, as the total space of an S^2–bundle over S^4, is an example of (13). Yamaguchi [36] almost determined the homotopy classification of M in (13) with correction by [5; 29], and summarized the criterion for whether M has the same homotopy type as an S^2–bundle over S^4 in [36, Remark 4.8] based on [30].

By (13) there are generators $x \in H^2(M;\mathbb{Z})$ and $y \in H^4(M;\mathbb{Z})$ such that

$$x^2 = ky \tag{14}$$

for some $k \in \mathbb{Z}$. Consider the S^1–bundle

$$S^1 \xrightarrow{j} X \to M \tag{15}$$

classified by $x \in H^2(M;\mathbb{Z}) \cong [M, BS^1]$, which defines the closed 7–manifold X. Let $P^n(k)$ be the Moore space such that $\widetilde{H}^*(P^n(k);\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$ if $* = n$ and 0 otherwise [26].

Lemma 5.1 *If $k \neq 0$, there is a homotopy equivalence*

$$X \simeq P^4(k) \cup e^7.$$

Proof The lemma can be proved directly by analyzing the Serre spectral sequence of the fibration $X \to M \xrightarrow{x} BS^1$ induced from (15). Here we provide an alternative proof using results in geometric topology. By [22, Theorem 1.3], X is homotopy equivalent to the total space of an S^3–bundle over S^4. Then by the homotopy classification of S^3–bundles over S^4 [14; 30], X is homotopy equivalent to $P^4(k') \cup e^7$ for some $k' \in \mathbb{Z}$. Notice that $\pi_3(X) \cong \pi_3(M) \cong \pi_3(S^2 \cup_{k\eta_2} e^4) \cong \mathbb{Z}/k$, where $\eta_2 \in \pi_3(S^2)$ is the Hopf element. Then $k = k'$ because $\pi_3(P^4(k') \cup e^7) \cong \mathbb{Z}/k'$, and the lemma follows. □

Lemma 5.1 has an immediate consequence on the rational homotopy of M.

Lemma 5.2 *Let M be a closed 6–manifold with cell structure of the form* (13). *Then if $k \neq 0$ there is a rational homotopy equivalence $M \simeq_{\mathbb{Q}} \mathbb{C}P^3$, and if $k = 0$ then $M \simeq_{\mathbb{Q}} S^2 \times S^4$.*

Proof Let $x^2 = ky$ for some $k \in \mathbb{Q}$, where $x, y \in H^*(M;\mathbb{Q})$ are two generators with $\deg(x) = 2$. By Poincaré duality, it is easy to see that the cohomology algebra $H^*(M;\mathbb{Q})$ is determined by k, and is isomorphic to $H^*(\mathbb{C}P^3;\mathbb{Q})$ if $k \neq 0$ and $H^*(S^2 \times S^4;\mathbb{Q})$ if $k = 0$. Since every simply connected 6–manifold is formal [27, Proposition 4.6], the rational homotopy type of M is determined by its rational cohomology algebra $H^*(M;\mathbb{Q})$. Hence $M \simeq_{\mathbb{Q}} \mathbb{C}P^3$ or $M \simeq_{Q} S^2 \times S^4$. □

5.1 The subcase when k is odd

When k is odd, the loop decomposition of the Poincaré complex $P^4(k) \cup e^7$ was determined by Huang and Theriault [20]. For any prime p, let $S^m\{p^r\}$ be the homotopy fibre of the degree p^r map on S^m. Let $k = p_1^{r_1} \cdots p_\ell^{r_\ell}$ be the prime decomposition of k. By [20, Theorem 1.1], when k is odd there is a homotopy equivalence

$$\Omega(P^4(k) \cup e^7) \simeq \prod_{j=1}^{\ell} S^3\{p_j^{r_j}\} \times \Omega S^7. \tag{16}$$

Proposition 5.3 *Let M be a closed 6–manifold with cell structure of the form $S^2 \cup_{k\eta_2} e^4 \cup e^6$. If k is odd, then M has the same homotopy type as an S^2–bundle over S^4, and there is a homotopy equivalence*

$$\Omega M \simeq S^1 \times \prod_{j=1}^{\ell} S^3\{p_j^{r_j}\} \times \Omega S^7. \tag{17}$$

Proof The homotopy equivalence (17) follows immediately from Lemma 5.1, (15) and (16). For the first statement, recall that there is the fibre bundle [18, Section 1.1]

$$S^2 \to \mathbb{C}P^3 \to S^4,$$

classified by a generator of $\pi_4(BSO(3)) \cong \mathbb{Z}$. Pulling back this bundle along a self-map of S^4 of degree k, we obtain the 6–manifold M' in the following diagram of S^2–bundles:

$$\begin{tikzcd} S^2 \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] & S^4 \arrow[d, "k"] \\ S^2 \arrow[r] & \mathbb{C}P^3 \arrow[r] & S^4 \end{tikzcd}$$

It is easy to see that $x'^2 = ky'$, where $x' \in H^2(M';\mathbb{Z})$ and $y' \in H^4(M';\mathbb{Z})$ are two generators. By [36, Corollary 4.6], when k is odd the homotopy type of M is uniquely determined by k, and hence $M \simeq M'$. □

5.2 The subcase when k is even

In [20, Section 6], Huang and Theriault showed that for $P^4(2^r) \cup e^7$ with $r \geq 3$, there is an homotopy equivalence

$$\Omega(P^4(2^r) \cup e^7) \simeq S^3\{2^r\} \times \Omega S^7, \tag{18}$$

provided there is a map $P^4(2^r) \cup e^7 \to S^4$ inducing a surjection in mod-2 homology.

Proposition 5.4 *Let M be a closed 6–manifold with cell structure of the form $S^2 \cup_{2^r\eta_2} e^4 \cup e^6$. If $r \geq 3$, then there is a homotopy equivalence*

$$\Omega M \simeq S^1 \times S^3\{2^r\} \times \Omega S^7.$$

Proof Recall by Lemma 5.1 and its proof that $X \simeq P^4(2^r) \cup e^7$ and X is homotopy equivalent to the total space of an S^3–bundle over S^4

$$S^3 \to X \xrightarrow{q} S^4.$$

It is clear that $q_*\colon H_4(X;\mathbb{Z}/2\mathbb{Z}) \to H_4(S^4;\mathbb{Z}/2\mathbb{Z})$ is surjective. Hence, by (18), $\Omega X \simeq S^3\{2^r\} \times \Omega S^7$. The lemma then follows from (15) immediately. □

6 Coformality of 6–manifolds

In this section, we study the rational homotopy theory of 6–manifolds as an application of our decompositions in Theorem 1.1. We briefly recall some necessary terminology used in this section, and for a detailed account of rational homotopy theory one can refer to the standard literature [17].

Recall, a *CW* complex X is rationally *formal* if its rational homotopy type is determined by the graded commutative algebra $H^*(X;\mathbb{Q})$, and is rationally *coformal* if its rational homotopy type is determined by the graded Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$, which is called the *homotopy Lie algebra* of X and is denoted by L_X. Suppose $(\Lambda V_X, d)$ is a *Sullivan model* of X. The differential d equals $\sum_{i\geq 0} d_i$ with $d_i\colon V_X \to \Lambda^{i+1} V_X$, and $(\Lambda V_X, d)$ is *minimal* if the linear part d_0 equals 0. In the latter case, V_X is dual to $\pi_*(\Omega X) \otimes \mathbb{Q}$.

Moreover, X is coformal if and only if it has a *purely quadratic* Sullivan model $C^*(L_X,0)=(\Lambda(sL_X)^\#,d_1)$, where $C^*(-)$ is the *commutative cochain algebra functor*, s is the suspension, and # is the dual operation.

Proposition 6.1 *Let M be a 6–manifold as in Theorem 1.1 such that $d\geq 2$. Then M is coformal.*

Proof Consider the S^2–bundle

$$S^2 \xrightarrow{i} M \xrightarrow{p} N \tag{19}$$

in (9). By [27, Proposition 4.4] N is coformal since $d\geq 2$, and hence has a minimal Sullivan model of the form $C^*(L_N,0)=(\Lambda(sL_N)^\#,d_1)$ as the associated commutative cochain algebra of $(L_N,0)$ [17, Example 7 in Chapter 24(f)]. Let

$$\hat{p}\colon C^*(L_N,0)\to (C^*(L_N)\otimes\Lambda(a,b),d)$$

be a relative minimal Sullivan model of p whose quotient $(\Lambda(a,b),\bar{d})$ is a minimal Sullivan model of S^2 with $db=a^2$ and $\deg(a)=2$. It follows that there is the short exact sequence of the linear part of the model of (19),

$$0\to((sL_N)^\#,0)\to((sL_N)^\#\oplus\mathbb{Q}(a,b),d_0)\to(\mathbb{Q}(a,b),0)\to 0, \tag{20}$$

such that $H^*((sL_N)^\#\oplus\mathbb{Q}(a,b),d_0)$ is dual to $\pi_*(M)\otimes\mathbb{Q}$. However, since the homotopy groups of (19) split by Theorem 1.1 and its proof, we see from (20) that the linear part d_0 equals 0 for $(sL_N)^\#\oplus\mathbb{Q}(a,b)$, and hence $(C^*(L_N)\otimes\Lambda(a,b),d)$ is a minimal model of M.

To show M is coformal, it suffices to show that the differential d is quadratic on $\mathbb{Q}(a,b)$ in $(C^*(L_N)\otimes\Lambda(a,b),d)$. Since N is simply connected, $(sL_N)^\#$ concentrates in degrees larger than or equal to 2. So, by the minimality of $(C^*(L_N)\otimes\Lambda(a,b),d)$ and degree reasons,

$$da=0 \quad\text{and}\quad db=a^2+ay+\sum_i z_iw_i$$

for some degree 2 elements $y,z_i,w_i\in(sL_N)^\#$. So $d=d_1$ in $(C^*(L_N)\otimes\Lambda(a,b),d)$. This shows that M is coformal. □

Proof of Theorem 1.3 First, it is well known that $\mathbb{C}P^i$ is not coformal for $i\geq 2$ by [27, Example 4.7]. If $d=1$, then M is determined by a fibre bundle $S^2\to M\to\mathbb{C}P^2$. It has a model of the form

$$(\Lambda(c,x),dx=c^3)\to(\Lambda(c,x,a,b),\tilde{d})\to(\Lambda(a,b),db=a^2),$$

where $\deg(c) = \deg(a) = 2$. By degree reasons $\tilde{d}(a) = 0$, and $\tilde{d}(b) = a^2 + kc^2$ for some $k \in \mathbb{Q}$, which implies that $(\Lambda(c, x, a, b), \tilde{d})$ is minimal. However, $\tilde{d}$ is not quadratic as $\tilde{d}(x) = c^3$. Hence M is not coformal.

When $d \geq 2$, by Proposition 6.1 M is coformal. Moreover, Neisendorfer and Miller [27, Proposition 4.6] showed that every simply connected 6–manifold is formal. Hence, by [9, Theorem 1.2], M is Koszul. By [9, Theorem 1.3], there is an isomorphism of graded Lie algebras

$$\pi_*(\Omega M) \otimes \mathbb{Q} \cong H^*(M; \mathbb{Q})^{!\mathscr{L}ie},$$

where $(-)^{!\mathscr{L}ie}$ is the Koszul dual Lie functor. □

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