AGT

Algebraic & Geometric Topology

Volume 23 (2023)

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Algebraic & Geometric Topology **23**:6 (2023) 2449–2518
DOI: 10.2140/agt.2023.23.2449
Published: 7 September 2023

Classification of torus bundles that bound rational homology circles

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We completely classify orientable torus bundles over the circle that bound smooth 4–manifolds with the rational homology of the circle. Along the way, we classify certain integral surgeries along chain links that bound rational homology 4–balls and explore a connection to 3–braid closures whose double branched covers bound rational homology 4–balls.

57K30, 57K40, 57K41

1 Introduction

In [13], we showed that two infinite families of T^2–bundles over S^1 bound (smooth) rational homology circles ($\mathbb{Q}S^1 \times B^3$'s). As an application, the $\mathbb{Q}S^1 \times B^3$'s were used to construct infinite families of rational homology 3–spheres ($\mathbb{Q}S^3$'s) that bound rational homology 4–balls ($\mathbb{Q}B^4$'s). The main purpose of this article is to show that the two families of torus bundles used in [13] are the only torus bundles that bound smooth $\mathbb{Q}S^1 \times B^3$'s.

After endowing $T^2 \times [0,1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0,1]$ with the coordinates $(\boldsymbol{x}, t) = (x, y, t)$, any orientable torus bundle over S^1 is of the form $T^2 \times [0,1]/(\boldsymbol{x}, 1) \sim (\pm A\boldsymbol{x}, 0)$, where $A \in \mathrm{SL}(2, \mathbb{Z})$. The matrix A is called the *monodromy* of the torus bundle and is defined up to conjugation. Throughout, we will express the monodromy in terms of the generators $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. A torus bundle is called *elliptic* if $|\mathrm{tr}\, A| < 2$, *parabolic* if $|\mathrm{tr}\, A| = 2$, and *hyperbolic* if $|\mathrm{tr}\, A| > 2$. Moreover, a torus bundle is called *positive* if $\mathrm{tr} A > 0$ and *negative* if $\mathrm{tr} A < 0$. Torus bundles naturally arise as the boundaries of plumbings of D^2–bundles over S^2 (see Neumann [11, Section 6] for details). Using these plumbing descriptions, it is easy to draw surgery diagrams for torus bundles. Table 1 gives a complete list of torus bundles over S^1, along with their monodromies (up to conjugation) and surgery diagrams. To simplify notation, $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ will always denote the hyperbolic torus bundle with monodromy $\pm A(\boldsymbol{a}) = \pm T^{-a_1} S \cdots T^{-a_n} S$, where $\boldsymbol{a} = (a_1, \ldots, a_n)$, $a_1 \geq 3$, and $a_i \geq 2$ for all i.

Theorem 1.1 *A torus bundle over S^1 bounds a $\mathbb{Q}S^1 \times B^3$ if and only if*

- *it is negative parabolic, or*
- *it is positive hyperbolic of the form $\boldsymbol{T}_{A(\boldsymbol{a})}$, where*

  $$\boldsymbol{a} = (3 + x_1, 2^{[x_2]}, \ldots, 3 + x_{2m+1}, 2^{[x_1]}, 3 + x_2, 2^{[x_3]}, \ldots, 3 + x_{2m}, 2^{[x_{2m+1}]}),$$

  $m \geq 0$, and $x_i \geq 0$ for all i.

Elliptic torus bundles and parabolic torus bundles that bound $\mathbb{Q}S^1 \times B^3$'s are rather simple to classify. Classifying hyperbolic torus bundles, which make up the "generic" class of torus bundles, is much more involved and includes the bulk of the technical work. In [13], it is shown that $\boldsymbol{T}_{A(\boldsymbol{a})}$ indeed bounds a $\mathbb{Q}S^1 \times B^3$ when $\boldsymbol{a} = (3 + x_1, 2^{[x_2]}, \ldots, 3 + x_{2m+1}, 2^{[x_1]}, 3 + x_2, 2^{[x_3]}, \ldots, 3 + x_{2m}, 2^{[x_{2m+1}]})$. To obstruct all other hyperbolic torus bundles from bounding $\mathbb{Q}S^1 \times B^3$'s, we first consider a related class of $\mathbb{Q}S^3$'s.

Let L_n^t denote the n–component link shown in Figure 1, where t denotes the number of half-twists. We call L_n^t the *n–component, t–half-twisted chain link*. If $t = 0$, we call the chain link *untwisted*. Consider the surgery diagram for the hyperbolic torus bundle $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ given in Table 1. Now perform m–surgery along a meridian of the 0–framed unknot as in the left side of each of the four diagrams in Figure 2. Next, slide the unknot with framing $-a_1$ (or $-a_1 \pm 2$) twice over the blue m–framed unknot so that it no longer passes through the 0–framed unknot. Then cancel the 0–framed and m–framed unknots. When $n \geq 2$, the resulting 3–manifolds are obtained by

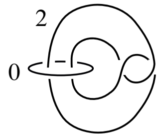
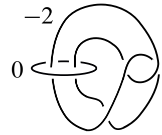
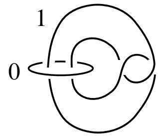
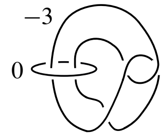
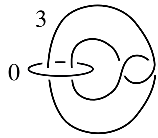
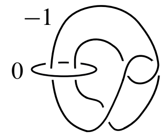
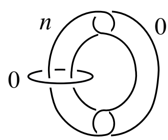
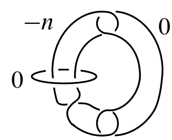
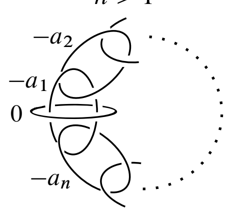
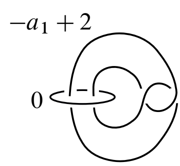
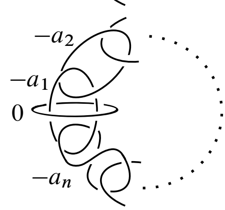
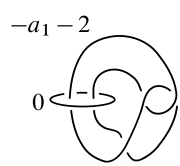
<table>
<tr><th colspan="4">elliptic torus bundles</th></tr>
<tr><th>monodromy</th><th>surgery diagram</th><th>monodromy</th><th>surgery diagram</th></tr>
<tr><td>S</td><td></td><td>$-S$</td><td></td></tr>
<tr><td>$T^{-1}S$</td><td></td><td>$-T^{-1}S$</td><td></td></tr>
<tr><td>$(T^{-1}S)^2$</td><td></td><td>$-(T^{-1}S)^2$</td><td></td></tr>
<tr><th colspan="4">parabolic torus bundles</th></tr>
<tr><th>monodromy</th><th>surgery diagram</th><th>monodromy</th><th>surgery diagram</th></tr>
<tr><td>T^n
$(n \in \mathbb{Z})$</td><td></td><td>$-T^n$
$(n \in \mathbb{Z})$</td><td></td></tr>
<tr><th colspan="4">hyperbolic torus bundles $T_{\pm A(a_1,\ldots,a_n)}$</th></tr>
<tr><th colspan="2">monodromy</th><th colspan="2">surgery diagram</th></tr>
<tr><td colspan="2"></td><td>$n>1$</td><td>$n=1$</td></tr>
<tr><td colspan="2">$T^{-a_1}S\cdots T^{-a_n}S$
$(a_1 \geq 3, a_i \geq 2 \text{ for all } i)$</td><td></td><td></td></tr>
<tr><td colspan="2">$-T^{-a_1}S\cdots T^{-a_n}S$
$(a_1 \geq 3, a_i \geq 2 \text{ for all } i)$</td><td></td><td></td></tr>
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Table 1: Monodromy and surgery diagrams of parabolic, elliptic and hyperbolic T^2–bundles over S^1.

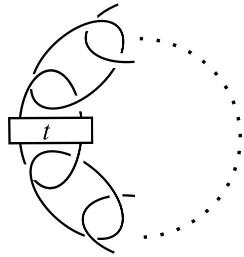

Figure 1: The n–component, t–half-twisted chain link, L^t_n. The box labeled t denotes t half-twists.

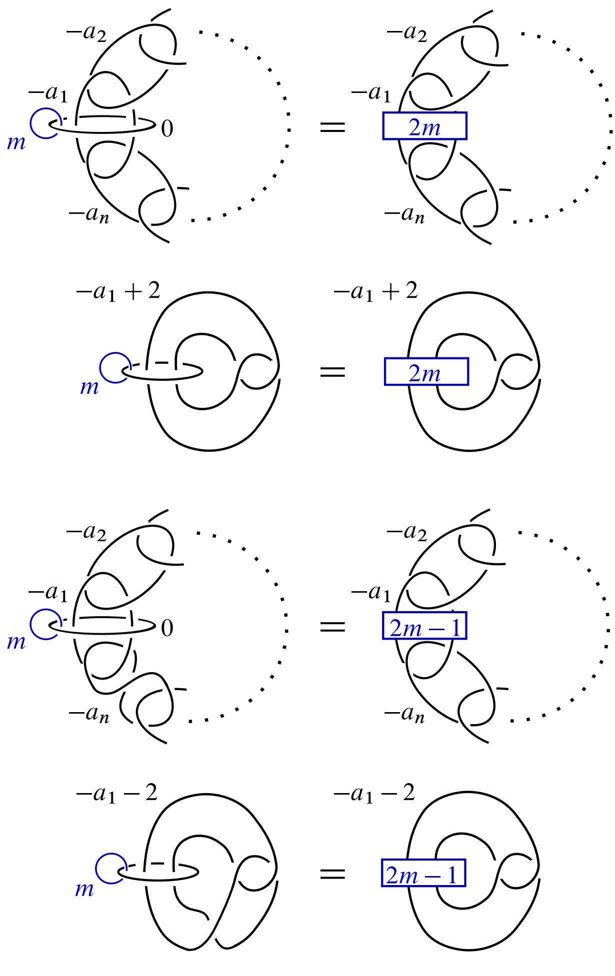

Figure 2: Surgering the hyperbolic torus bundle $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$, where $\boldsymbol{a} = (a_1, \dots, a_n)$, to obtain the rational homology sphere $Y^t_{\boldsymbol{a}}$. The blue boxes labeled $2m$ and $2m-1$ indicate the number of half-twists.

$(-a_1,\dots,-a_n)$–surgery along the chain link L^t_n, where $t=2m$ or $2m-1$. We denote these 3–manifolds by $Y^t_{\boldsymbol{a}}=S^3_{(-a_1,\dots,-a_n)}(L^t_n)$, where $\boldsymbol{a}=(a_1,\dots,a_n)$, $a_1\ge 3$, and $a_i\ge 2$ for all i. Note that, by cyclically reordering or reversing the surgery coefficients, we obtain the same 3–manifold. When $n=1$, the resulting 3–manifolds are obtained by $-(a_1\pm 2)$–surgery along L^t_1, where $t=2m+(1\pm 1)$; we denote them by $Y^t_{\boldsymbol{a}}=Y^t_{(a_1)}$. Note that $Y^t_{(a_1)}=S^3_{-a_1+2}(L^t_1)$ when t is even, and $Y^t_{(a_1)}=S^3_{-a_1-2}(L^t_1)$ when t is odd. Finally note that $Y^t_{\boldsymbol{a}}$ is a $\mathbb{Q}S^3$ for all $\boldsymbol{a}$ and t; this follows from the fact that $|H_1(Y^t_{\boldsymbol{a}})|=|\mathrm{Tor}(H_1(\boldsymbol{T}_{\pm A(\boldsymbol{a})}))|$ is finite (see Lemma A.1).

Lemma 1.2 [13] *Let Y be a $\mathbb{Q}S^1\times S^2$ that bounds a $\mathbb{Q}S^1\times B^3$ and let K be a knot in Y such that $[K]$ has infinite order in $H_1(Y;\mathbb{Z})$. Then any integer surgery on Y along K yields a $\mathbb{Q}S^3$ that bounds a $\mathbb{Q}B^4$.*

By Lemma 1.2, if $\boldsymbol{T}_{A(\boldsymbol{a})}$ bounds a $\mathbb{Q}S^1\times B^3$, then $Y^t_{\boldsymbol{a}}$ bounds a $\mathbb{Q}B^4$ for all even t, and if $\boldsymbol{T}_{-A(\boldsymbol{a})}$ bounds a $\mathbb{Q}S^1\times B^3$, then $Y^t_{\boldsymbol{a}}$ bounds a $\mathbb{Q}B^4$ for all odd t. Thus, if $Y^t_{\boldsymbol{a}}$ does not bound a $\mathbb{Q}B^4$ for some even (or odd) t, then $\boldsymbol{T}_{A(\boldsymbol{a})}$ (or $\boldsymbol{T}_{-A(\boldsymbol{a})}$) does not bound a $\mathbb{Q}S^1\times B^3$. Using this fact, we will obstruct most hyperbolic torus bundles from bounding $\mathbb{Q}S^1\times B^3$'s by identifying the strings $\boldsymbol{a}$ for which $Y^0_{\boldsymbol{a}}$ and $Y^{-1}_{\boldsymbol{a}}$ do not bound $\mathbb{Q}B^4$'s. Before writing down the result, we first recall and introduce some useful terminology.

Let $(b_1,\dots,b_k)$ be a string of integers such that $b_i\ge 2$ for all i. If $b_j\ge 3$ for some j, then we can write this string in the form $(2^{[m_1]},3+n_1,\dots,2^{[m_j]},2+n_j)$, where $m_i,n_i\ge 0$ for all i and $2^{[t]}$ denotes a string $2,\dots,2$ of t 2's. The string $(c_1,\dots,c_l)=(2+m_1,2^{[n_1]},3+m_2,\dots,3+m_j,2^{[n_j]})$ is called the *linear-dual string* of $(b_1,\dots,b_k)$. If $b_i=2$ for all $1\le i\le k$, then we define its linear-dual string to be $(k+1)$. Linear-dual strings have a topological interpretation. If Y is obtained by $(-b_1,\dots,-b_k)$–surgery along a linear chain of unknots, then the reversed-orientation manifold $\overline{Y}$ can be obtained by $(-c_1,\dots,-c_l)$–surgery along a linear chain of unknots (see Neumann [11, Theorem 7.3]). Finally, we define the linear-dual string of (1) to be the empty string.

Suppose $\boldsymbol{a}=(a_1,\dots,a_n)$ is of the form $(2^{[m_1]},3+n_1,\dots,2^{[m_j]},3+n_j)$, where $m_i,n_i\ge 0$ for all i; we define its *cyclic-dual* to be the string $\boldsymbol{d}=(d_1,\dots,d_m)=(3+m_1,2^{[n_1]},\dots,3+m_j,2^{[n_j]})$. In particular, a string of the form (x) with $x\ge 3$ has cyclic-dual $(2^{[x-3]},3)$. Notice that this definition only slightly differs from the definition of the linear-dual string. As a topological interpretation of cyclic-dual strings, the reversed-orientation of $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ is given by $\overline{\boldsymbol{T}}_{\pm A(\boldsymbol{a})}=\boldsymbol{T}_{\pm A(\boldsymbol{d})}$ (see Neumann [11, Theorem 7.3]). Finally, $(a_n,\dots,a_1)$ is called the *reverse* of $(a_1,\dots,a_n)$.

Example 1.3 Consider the strings in Theorem 1.7,

$$\boldsymbol{a} = (3 + x_1, 2^{[x_2]}, \dots, 3 + x_{2m+1}, 2^{[x_1]}, 3 + x_2, 2^{[x_3]}, \dots, 3 + x_{2m}, 2^{[x_{2m+1}]}).$$

It is easy to see that the cyclic-dual of $\boldsymbol{a}$ is simply $\boldsymbol{a}$. Moreover, $\boldsymbol{a}$ is of the above form if and only if it can be expressed in the form $\boldsymbol{a} = (b_1+1, b_2, \dots, b_{k-1}, b_k+1, c_1, \dots, c_l)$ if $k \geq 2$, where $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear-dual strings, or $\boldsymbol{a} = (b_1+2, 2^{[b_1-1]})$ if $k = 1$.

To remove the necessity of multiple cases, from now on, if $\boldsymbol{a}$ contains a substring of the form $(b_1 + 1, b_2, \dots, b_{k-1}, b_k + 1)$ and $k = 1$, then we will understand this substring to simply be $(b_1 + 2)$, as in Example 1.3.

Definition 1.4 Two strings are considered to be equivalent if one is a cyclic reordering and/or reverse of the other. Each string in the following sets is defined up to this equivalence. Moreover, strings of the form $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are assumed to be linear-dual. We define

$$\begin{aligned}
\mathcal{S}_{1a} &= \{(b_1, \dots, b_k, 2, c_l, \dots, c_1, 2) \mid k+l \geq 3\},\\
\mathcal{S}_{1b} &= \{(b_1, \dots, b_k, 2, c_l, \dots, c_1, 5) \mid k+l \geq 2\},\\
\mathcal{S}_{1c} &= \{(b_1, \dots, b_k, 3, c_l, \dots, c_1, 3) \mid k+l \geq 2\},\\
\mathcal{S}_{1d} &= \{(2, b_1+1, b_2, \dots, b_{k-1}, b_k+1, 2, 2, c_l+1, c_{l-1}, \dots, c_2, c_1+1, 2) \mid k+l \geq 2\},\\
\mathcal{S}_{1e} &= \{(2, 3+x, 2, 3, 3, 2^{[x-1]}, 3, 3) \mid x \geq 0 \text{ and } (3, 2^{[-1]}, 3) := (4)\},\\
\mathcal{S}_{2a} &= \{(b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1)\},\\
\mathcal{S}_{2b} &= \{(3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1) \mid x \geq 0 \text{ and } k+l \geq 2\},\\
\mathcal{S}_{2c} &= \{(b_1+1, b_2, \dots, b_{k-1}, b_k+1, c_1, \dots, c_l) \mid k+l \geq 2\},\\
\mathcal{S}_{2d} &= \{(2, 2+x, 2, 3, 2^{[x-1]}, 3, 4) \mid x \geq 0 \text{ and } (3, 2^{[-1]}, 3) := (4)\},\\
\mathcal{S}_{2e} &= \{(2, b_1+1, b_2, \dots, b_k, 2, c_l, \dots, c_2, c_1+1, 2), (2, 2, 2, 3) \mid k+l \geq 2\},\\
\mathcal{O} &= \{(6, 2, 2, 2, 6, 2, 2, 2), (4, 2, 4, 2, 4, 2, 4, 2), (3, 3, 3, 3, 3, 3)\},\\
\mathcal{S}_1 &= \mathcal{S}_{1a} \cup \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e},\\
\mathcal{S}_2 &= \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e},\\
\mathcal{S} &= \mathcal{S}_1 \cup \mathcal{S}_2.
\end{aligned}$$

Definition 1.5 Let $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i \geq 2$ for all i. Define $I(\boldsymbol{a})$ to be the integer $I(\boldsymbol{a}) = \sum_{i=1}^{n} (a_i - 3)$.

Remark 1.6 If $\boldsymbol{b}$ and $\boldsymbol{c}$ are linear-dual strings, it is easy to see that $I(\boldsymbol{b})+I(\boldsymbol{c})=-2$. Using this observation, it easy to check that, if $\boldsymbol{a}\in\mathcal{S}_1$, then $-4\le I(\boldsymbol{a})\le -1$, and if $\boldsymbol{a}\in\mathcal{S}_2$, then $-3\le I(\boldsymbol{a})\le 0$. In the same vein, if $\boldsymbol{a}$ and $\boldsymbol{d}$ are cyclic-dual strings, then $I(\boldsymbol{a})+I(\boldsymbol{d})=0$. Consequently, if $\boldsymbol{a},\boldsymbol{d}\in\mathcal{S}$, then $I(\boldsymbol{a})=I(\boldsymbol{d})=0$. Moreover, $\boldsymbol{a}\in\mathcal{S}$ and $I(\boldsymbol{a})=0$ if and only if $\boldsymbol{a}\in\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$.

Theorem 1.7 *Let $\boldsymbol{a}=(a_1,\dots,a_n)$, where $n\ge 1$, $a_i\ge 2$ for all i, and $a_j\ge 3$ for some j, and let $\boldsymbol{d}$ be the cyclic-dual of $\boldsymbol{a}$.*

1. *Suppose $\boldsymbol{d}\notin\mathcal{S}_{1a}\cup\mathcal{O}$. Then $Y_{\boldsymbol{a}}^{-1}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{a}\in\mathcal{S}_1$ or $\boldsymbol{d}\in\mathcal{S}_{1b}\cup\mathcal{S}_{1c}\cup\mathcal{S}_{1d}\cup\mathcal{S}_{1e}$.*
2. *Suppose $\boldsymbol{a}\notin\mathcal{S}_{1a}\cup\mathcal{O}$. Then $Y_{\boldsymbol{a}}^{1}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{d}\in\mathcal{S}_1$ or $\boldsymbol{a}\in\mathcal{S}_{1b}\cup\mathcal{S}_{1c}\cup\mathcal{S}_{1d}\cup\mathcal{S}_{1e}$.*
3. *$Y_{\boldsymbol{a}}^{0}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{a}\in\mathcal{S}_2$ or $\boldsymbol{d}\in\mathcal{S}_2$.*

Remark 1.8 The hypothesis "$a_j\ge 3$ for some j" in Theorem 1.7 ensures that $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ is a hyperbolic torus bundle. If we remove this condition from the theorem, then we would have an additional case: $a_i=2$ for all i. In this case, $Y_{\boldsymbol{a}}^{-1}$ bounds a $\mathbb{Q}B^4$ and $Y_{\boldsymbol{a}}^{0}$ does not bound a $\mathbb{Q}B^4$. This follows from Lemma 1.2 and Theorem 1.1 and the fact that the corresponding torus bundles are the parabolic torus bundles with respective monodromies $-T^n$ and T^n (see [13]).

Remark 1.9 We will see in Lemma 4.2 that, for certain strings $\boldsymbol{d}$ that are the cyclic-duals of $(b_1,\dots,b_k,2,c_l,\dots,c_1,2)$, $Y_{\boldsymbol{d}}^{-1}$ does not bound a $\mathbb{Q}B^4$ (see Theorem 1.7(1)). However, we are unable to prove this fact for all such strings. Moreover, for each $\boldsymbol{a}\in\mathcal{O}$, we are unable to obstruct $Y_{\boldsymbol{a}}^{\pm 1}$ from bounding a $\mathbb{Q}B^4$ or show that it indeed bounds a $\mathbb{Q}B^4$. These strings are outliers that are unobstructed by the analysis we present here.

Combined with Lemma 1.2, Theorem 1.7 obstructs most hyperbolic torus bundles from bounding $\mathbb{Q}S^1\times B^3$'s. In Section 3, we will obstruct the rest by considering certain cyclic covers of $\mathbb{Q}S^1\times B^3$'s. The proof of Theorem 1.7 relies on Donaldson's diagonalization theorem [6] and lattice analysis. From this analysis, it follows that, if $\boldsymbol{a}\notin\mathcal{S}_1\cup\mathcal{O}$, then $Y_{\boldsymbol{a}}^{t}$ does not bound a $\mathbb{Q}B^4$ for all odd t, and if $\boldsymbol{a}\notin\mathcal{S}_2$, then $Y_{\boldsymbol{a}}^{t}$ does not bound a $\mathbb{Q}B^4$ for all even t. Moreover, by Lemma 1.2 and Theorem 1.1, if $\boldsymbol{a}\in\mathcal{S}_{2c}$, then $Y_{\boldsymbol{a}}^{t}$ bounds a $\mathbb{Q}B^4$ for all even t. This leads to the following question:

Question 1.10 For what values of t and for which strings $\boldsymbol{a}\in\mathcal{S}\setminus\mathcal{S}_{2c}$ does $Y_{\boldsymbol{a}}^{t}$ bound a $\mathbb{Q}B^4$?

1.1 Connection to 3–braids

There is an intimate connection between the rational homology 3–spheres $Y_{\boldsymbol{a}}^t$ and 3–braid closures; we will show that $Y_{\boldsymbol{a}}^t$ is the double cover of S^3 branched over the link given by the closure of the 3–braid word $(\sigma_1\sigma_2)^{3t}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$, where σ_1 and σ_2 are the standard generators of the braid group on three strands.

Let $\boldsymbol{a}=(a_1,\dots,a_n)$ and consider $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{0}$, as shown in the top of Figure 3. Using the techniques of Akbulut and Kirby [2], it is clear that $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{0}$ are the double covers of S^3 branched over the links shown in the middle of Figure 3. The $\mathbb{Z}_2$–action inducing these covers are the 180° rotations shown in Figure 3. By isotoping these

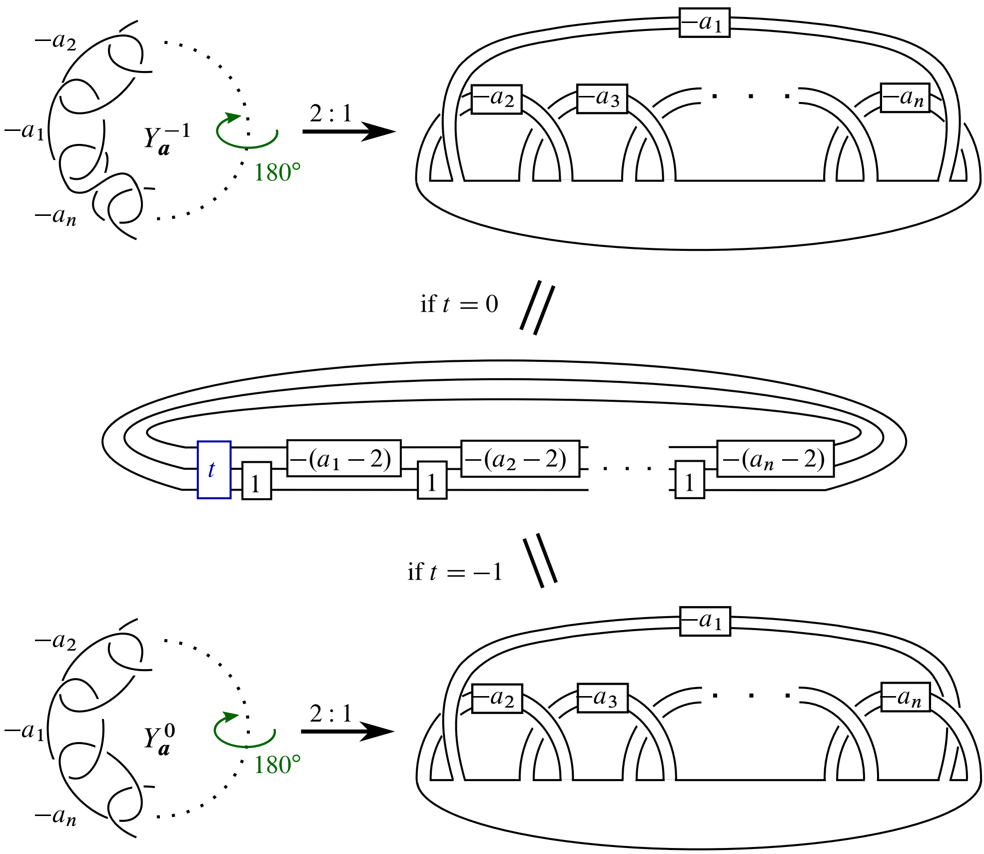

Figure 3: $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{0}$ are the double covers of S^3 branched over the closure of the 3–braid word $(\sigma_1\sigma_2)^{3t}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$, where $t=-1$ and $t=0$, respectively. The blue box labeled t indicates the number of full-twists, while all other boxes in all other diagrams indicated the number of half-twists.

links, we obtain the closures of the 3–braid words $(\sigma_1\sigma_2)^{-3}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$ and $\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$, respectively, as shown in Figure 3. Note that, in the figure, the blue box labeled t indicates the number of full-twists, while all other boxes indicate the number of half-twists.

Using Kirby calculus, we can argue that, for any t, $Y_{\boldsymbol{a}}^t$ is the double cover of S^3 branched over the closure of the 3–braid word $(\sigma_1\sigma_2)^{3t}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$. Notice that, if $t=2m-1\geq -1$ is odd, then $Y_{\boldsymbol{a}}^t$ can be realized as $(-1^{[m]})$–surgery along a link in $Y_{\boldsymbol{a}}^{-1}$, as shown in the top left of Figure 4, top, and if $t=2m\geq 0$ is even, then $Y_{\boldsymbol{a}}^t$ can be realized as $(-1^{[m]})$–surgery along a link in $Y_{\boldsymbol{a}}^{0}$, as shown in the top left of Figure 4, bottom. Under the $\mathbb{Z}_2$–action, each of these surgery curves double covers a curve isotopic to the braid axis of the 3–braid. Thus each -1–surgery curve maps to a $-\frac{1}{2}$–surgery curve isotopic to the braid axis, as shown in the intermediate stages in Figure 4. By blowing down these curves, we obtain the desired 3–braid closures at the bottom of the figures. Note that the same argument can be used when $t<-1$; the only difference is that the surgery curves would all have positive coefficients.

Coupling this characterization with Theorems 1.7 and 1.1 and Lemma 1.2, we can classify certain families of 3–braid closures admitting double branched covers bounding $\mathbb{Q}B^4$'s.

Corollary 1.11 *Let $\boldsymbol{a}=(a_1,\dots,a_n)$, where $n\geq 1$, $a_i\geq 2$ for all i, and $a_j\geq 3$ for some j, and let $\boldsymbol{d}$ be the cyclic-dual of $\boldsymbol{a}$.*

- *Suppose $\boldsymbol{d}\notin\mathcal{S}_{1a}\cup\mathcal{O}$. Then the double cover of S^3 branched over the closure of the 3–braid word $(\sigma_1\sigma_2)^{-3}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{a}\in\mathcal{S}_1$ or $\boldsymbol{d}\in\mathcal{S}_{1b}\cup\mathcal{S}_{1c}\cup\mathcal{S}_{1d}\cup\mathcal{S}_{1e}$.*
- *Suppose $\boldsymbol{a}\notin\mathcal{S}_{1a}\cup\mathcal{O}$. Then the double cover of S^3 branched over the closure of the 3–braid word $(\sigma_1\sigma_2)^{3}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{d}\in\mathcal{S}_1$ or $\boldsymbol{a}\in\mathcal{S}_{1b}\cup\mathcal{S}_{1c}\cup\mathcal{S}_{1d}\cup\mathcal{S}_{1e}$.*
- *The double cover of S^3 branched over the closure of the 3–braid word*
$$\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$$
bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{a}\in\mathcal{S}_2$.
- *If $\boldsymbol{a}\in\mathcal{S}_{2c}$, then the double cover of S^3 branched over the closure of the 3–braid word $(\sigma_1\sigma_2)^{3t}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$ bounds a $\mathbb{Q}B^4$ for all even t.*

The 3–braid knots corresponding to strings in $\mathcal{S}_{1a}\cup\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$ (and their mirrors) were shown by Lisca [10] to be 3–braid knots of finite concordance order. Moreover,

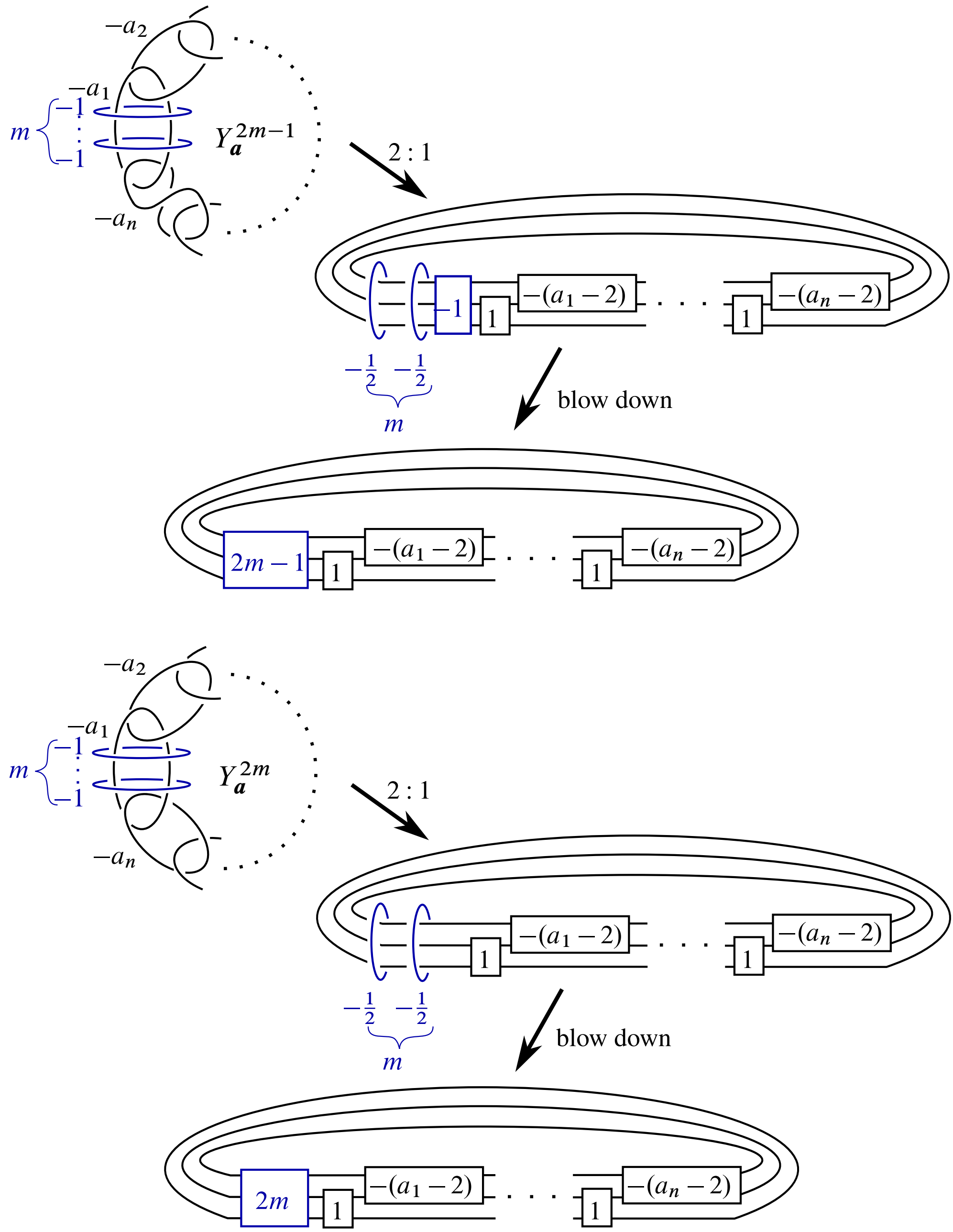

Figure 4: When $t \geq -1$, $Y^t_{\boldsymbol{a}}$ is the double cover of S^3 branched over the closure of the 3–braid word $(\sigma_1\sigma_2)^{3t}\sigma_1\sigma_2^{-(a_1-2)}\cdots\sigma_1\sigma_2^{-(a_n-2)}$. The same is true when $t < -1$.

some of them were shown be slice knots and so for these the corresponding double branched covers are already known to bound $\mathbb{Q}B^4$'s. Furthermore, by the classification in [10], many of the remaining strings in $\mathcal{S}$ correspond to infinite concordance order 3–braid knots. Thus, these give examples of infinite concordance order knots whose double branched covers bound $\mathbb{Q}B^4$'s. Rewording Question 1.10 in terms of 3–braids, a natural question is the following:

Question 1.12 Which other 3–braid closures admit double branched covers bounding $\mathbb{Q}B^4$'s?

Organization

In Section 2, we will highlight some simple obstructions to $\mathbb{Q}S^1 \times S^2$'s bounding $\mathbb{Q}S^1 \times B^3$'s, recall Heegaard Floer homology calculations of 3–braid closures due to Baldwin, and use these calculations to explore the orientation reversal of the 3–manifold $Y^t_{\boldsymbol{a}}$. These obstructions and calculations will be used in Sections 3 and 4. In particular, in Section 3, we will use the obstructions and other techniques to prove Theorem 1.1, and in Section 4, we will show that the $\mathbb{Q}S^3$'s of Theorem 1.7 do indeed bound $\mathbb{Q}B^4$'s by explicitly constructing them. In Sections 5–7, we will use lattice analysis to prove that the $\mathbb{Q}S^3$'s of Theorem 1.7 are the only such $\mathbb{Q}S^3$'s that bound $\mathbb{Q}B^4$'s. Finally, the appendix provides some continued fraction calculations that are used in Sections 2 and 4.

Acknowledgements

Thanks to Vitalijs Brejevs for pointing out a missing case in Lemma 6.1 and thanks to the referee for carefully reading through the technical aspects of the paper and suggesting ways to greatly improve the flow of the paper.

2 Obstructions

In this section, we highlight some simple ways to obstruct a $\mathbb{Q}S^1 \times S^2$ from bounding a $\mathbb{Q}S^1 \times B^3$, recall Baldwin's calculations of the Heegaard Floer homology of double covers of S^3 branched over certain 3–braid closures [3] (ie the rational homology 3–spheres $Y^t_{\boldsymbol{a}}$), and show that reversing the orientation of the rational homology sphere $Y^t_{\boldsymbol{a}}$ yields $Y^{-t}_{\boldsymbol{d}}$, where $\boldsymbol{d}$ is the cyclic-dual of $\boldsymbol{a}$. The first obstruction is a consequence of [5, Proposition 1.5 and Corollary 1.6].

Lemma 2.1 [5] *If $K \subset S^3$ is an alternating knot and $S^3_0(K)$ bounds a $\mathbb{Q}S^1 \times B^3$, then $\sigma(K) = 0$.*

The next obstruction is akin to a well-known homology obstruction of $\mathbb{Q}S^3$'s bounding $\mathbb{Q}B^4$'s [4, Lemma 3].

Lemma 2.2 *If Y bounds a $\mathbb{Q}S^1 \times B^3$, then the torsion part of $H_1(Y)$ has square order.*

Proof It is well known that, if a $\mathbb{Q}S^3$ bounds a $\mathbb{Q}B^4$, then its first homology group has square order [4, Lemma 3]. A similar but more complicated argument will prove the lemma.

Let $A = \mathrm{Tor}(H_1(Y))$. We aim to show that $|A|$ is a perfect square. Let W be a $\mathbb{Q}S^1 \times B^3$ bounded by Y. Then

$$H_i(W) \cong \begin{cases} T_2 & \text{if } i = 2, \\ \mathbb{Z} \oplus T_1 & \text{if } i = 1, \\ \mathbb{Z} & \text{if } i = 0, \end{cases}$$

where T_1 and T_2 are torsion groups. By duality and the universal coefficient theorem,

$$H_i(W, Y) \cong \begin{cases} \mathbb{Z} & \text{if } i = 3, \\ T_1 & \text{if } i = 2, \\ T_2 & \text{if } i = 1. \end{cases}$$

Consider the long exact sequence

$$\begin{array}{ccccccccccccc} H_3(W,Y) & \xrightarrow{f} & H_2(Y) & \longrightarrow & H_2(W) & \longrightarrow & H_2(W,Y) & \longrightarrow & H_1(Y) & \xrightarrow{g} & H_1(W) & \xrightarrow{h} & H_1(W,Y) \\ \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| \\ \mathbb{Z} & & \mathbb{Z} & & T_2 & & T_1 & & \mathbb{Z} \oplus A & & \mathbb{Z} \oplus T_1 & & T_2 \end{array}$$

Since $H_3(W)$ and $H_1(W, Y)$ are torsion groups, and $H_3(W, Y) \cong H_0(Y) \cong \mathbb{Z}$, the maps $H_3(W) \to H_3(W, Y)$ and $H_1(W, Y) \to H_0(Y)$ in the long exact sequence of the pair (W, Y) are trivial; hence, f is injective and g is surjective. Express the map g as $g = g_1 + g_2$, where $g_1\colon \mathbb{Z} \to \mathbb{Z} \oplus T_1$ and $g_2\colon A \to \{0\} \oplus T_1$. Notice that $\mathrm{Im}\, g \cong \mathrm{Im}\, g_1 \oplus \mathrm{Im}\, g_2$ and g_1 is injective. Thus $\mathrm{Im}\, g_2$ can be identified with a subgroup of $\mathrm{coker}\, g_1$ and $T_2 \cong \mathrm{coker}\, g \cong \mathrm{coker}\, g_1 / \mathrm{Im}\, g_2$. Moreover, it follows from duality that, if f is given by multiplication by n, then g_1 is of the form $g_1(x) = \pm nz + \sum \lambda_i b_i$, where x is a generator of the domain of g_1 and $\{z, b_i\}$ is a basis for $\mathbb{Z} \oplus T_1$ such that z is an infinite order element and the b_i are torsion elements. Thus $|\mathrm{coker}\, g_1| = n|T_1| = |\mathrm{coker}\, f||T_1|$.

By exactness, we can reduce the above sequence to the short exact sequence

$$0 \to T_1/(T_2/\operatorname{coker} f) \xrightarrow{i} \mathbb{Z} \oplus A \xrightarrow{g} \operatorname{Im} g \to 0,$$

where we identify $\operatorname{coker} f$ with its image in T_2 and $T_2/\operatorname{coker} f$ with its image in T_1. Since $g_1 \colon \mathbb{Z} \to \operatorname{Im} g_1$ is an isomorphism, we have the short exact sequence of finite groups

$$0 \to T_1/(T_2/\operatorname{coker} f) \xrightarrow{i} A \xrightarrow{g_2} \operatorname{Im} g_2 \to 0.$$

Consequently, $|A| = |T_1/(T_2/\operatorname{coker} f)| \cdot |\operatorname{Im} g_2|$.

Moreover,

$$\left|\frac{T_1}{T_2/\operatorname{coker} f}\right| = \frac{|T_1||\operatorname{coker} f|}{|T_2|} = \frac{|\operatorname{coker} g_1|}{|\operatorname{coker} g_1|/|\operatorname{Im} g_2|} = |\operatorname{Im} g_2|.$$

Thus, $|A| = |\operatorname{Im} g_2|^2$ is a square. □

2.1 Heegaard Floer homology calculations

Let $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i \geq 2$ for all $1 \leq i \leq n$ and $a_j \geq 3$ for some j. As mentioned in Section 1.1, the rational sphere $Y_{\boldsymbol{a}}^t$ is the double cover of S^3 branched over the closure of the 3–braid represented by the word $(\sigma_1\sigma_2)^{3t}\sigma_1\sigma_2^{-(a_1-2)} \cdots \sigma_1\sigma_2^{-(a_n-2)}$. In [3], Baldwin calculated the Heegaard Floer homology of these 3–manifolds equipped with a canonical spin^c structure $\mathfrak{s}_0$. In particular, he showed that

$$HF^+(Y_{\boldsymbol{a}}^{2m}, \mathfrak{s}_0) = \begin{cases} (\mathcal{T}_0^+ \oplus \mathbb{Z}_0^m)\{\frac{1}{4}(3n - \sum a_i)\} & \text{if } m \geq 0, \\ (\mathcal{T}_0^+ \oplus \mathbb{Z}_{-1}^{-m})\{\frac{1}{4}(3n - \sum a_i)\} & \text{if } m < 0, \end{cases}$$

$$HF^+(Y_{\boldsymbol{a}}^{2m+1}, \mathfrak{s}_0) = \begin{cases} (\mathcal{T}_0^+ \oplus \mathbb{Z}_{-1}^m)\{\frac{1}{4}(3n + 4 - \sum a_i)\} & \text{if } m \geq 0, \\ (\mathcal{T}_{-2}^+ \oplus \mathbb{Z}_{-2}^{-(m+1)})\{\frac{1}{4}(3n + 4 - \sum a_i)\} & \text{if } m < 0, \end{cases}$$

and

$$\{d(Y_{\boldsymbol{a}}^t, \mathfrak{s}) \mid \mathfrak{s} \neq \mathfrak{s}_0\} = \{d(Y_{\boldsymbol{a}}^s, \mathfrak{s}) \mid \mathfrak{s} \neq \mathfrak{s}_0\} \quad \text{for all } s, t \in \mathbb{Z}.$$

2.2 Reversing orientation

Let $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i \geq 2$ for all $1 \leq i \leq n$ and $a_j \geq 3$ for some j. As discussed in the introduction, reversing the orientation of the hyperbolic torus bundle $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ yields the hyperbolic torus bundle $\overline{\boldsymbol{T}}_{\pm A(\boldsymbol{a})} = \boldsymbol{T}_{\pm A(\boldsymbol{d})}$, where $\boldsymbol{d} = (d_1, \dots, d_m)$ is the cyclic-dual of $\boldsymbol{a}$ [11]. Therefore, by construction, reversing the orientation on $Y_{\boldsymbol{a}}^t$ yields $\overline{Y}_{\boldsymbol{a}}^t = Y_{\boldsymbol{d}}^s$ for some integer s. The following lemma shows that $s = -t$:

Lemma 2.3 *Let $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{d} = (d_1, \dots, d_m)$ be cyclic-dual. Then $\overline{Y}_{\boldsymbol{a}}^t = Y_{\boldsymbol{d}}^{-t}$.*

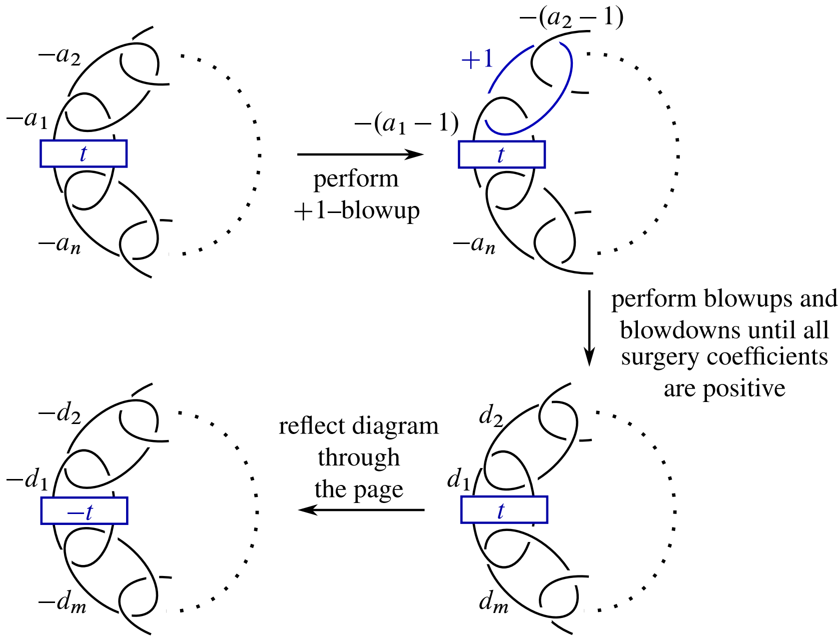

Figure 5: Proving that $\overline{Y}^t_{\boldsymbol{a}} = Y^{-t}_{\boldsymbol{d}}$, where $(d_1, \dots, d_m)$ is the cyclic-dual of $\boldsymbol{a} = (a_1, \dots, a_n)$ and $n > 1$.

Proof This is an exercise in Kirby calculus. We will focus on the case $n > 1$. The case $n = 1$ is similar, but much simpler. Start with the surgery diagram of $Y^t_{\boldsymbol{a}}$ that is made up of a t–half-twisted chain link with surgery coefficients $(-a_1, \dots, -a_n)$, as in the top left of Figure 5. We will produce a different surgery diagram for $Y^t_{\boldsymbol{a}}$ using blowups and blowdowns. Without loss of generality, assume that $a_1 \geq 3$. Let $i > 1$ be the smallest integer such that $a_i \geq 3$ and let K_i denote the unknot with surgery coefficient $-a_i$. If $a_i = 2$ for all $2 \leq i \leq n$, then set $i = n + 1$, with the understanding that $a_{n+1} = a_1$ and $K_{n+1} = K_1$. We will prove the lemma in the case $i \leq n$. The case of $i = n + 1$ is similar and requires fewer steps. Blow up the linking of the $-a_1$– and $-a_2$–framed unknots with a $+1$–framed unknot to obtain the second diagram in Figure 5. We can now perform $i - 2$ successive blowdowns of -1–framed unknots (with $i - 2 = 0$ a possibility). Next, perform $a_i - 2$ successive $+1$–blowups of the linking between K_i and the adjacent positively framed unknot; the resulting framing on K_i is -1. Continue to perform blowdowns and blowups in this way until every surgery coefficient is a positive number; we obtain the surgery diagram for $Y^t_{\boldsymbol{a}}$ made up of a chain link with positive surgery coefficients $(d_1, \dots, d_m)$, as in the third diagram of Figure 5, where $\boldsymbol{d} = (d_1, \dots, d_m)$ is the cyclic-dual of $\boldsymbol{a}$. Now we can change the orientation of $Y_{\boldsymbol{a}}^t$ by reflecting this new surgery diagram through the page. This yields a surgery diagram of $\overline{Y}_{\boldsymbol{a}}^t$ that is made up of a $-t$–half-twisted chain link with surgery coefficients $(-d_1, \dots, -d_n)$, as shown in the final diagram of Figure 5. Thus $\overline{Y}_{\boldsymbol{a}}^t = Y_{\boldsymbol{d}}^{-t}$. □

3 Torus bundles over S^1 that bound rational homology circles

In this section, we will prove Theorem 1.1. By considering the obvious handlebody diagrams of the plumbings shown in Table 1, it is rather straightforward to classify elliptic and parabolic torus bundles over S^1 that bound $\mathbb{Q}S^1 \times B^3$'s. In fact, through Kirby calculus, we will explicitly construct $\mathbb{Q}S^1 \times B^3$'s bounded by negative parabolic torus bundles and use the obstructions in Section 2 to obstruct positive parabolic torus bundles and elliptic torus bundles from bounding $\mathbb{Q}S^1 \times B^3$'s.

Proposition 3.1 *No elliptic torus bundle bounds a $\mathbb{Q}S^1 \times B^3$.*

Proof According to Table 1, there are only six elliptic torus bundles; they have monodromies $\pm S$, $\pm T^{-1}S$, and $\pm(T^{-1}S)^2$. By Lemma 2.2, if one of these torus bundles bounds a $\mathbb{Q}B^4$, then the torsion part of its first homology group must be a square. By considering the surgery diagrams in Table 1, it is easy to see that the only elliptic torus bundles that have the correct first homology are those with monodromy $T^{-1}S$ or $-(T^{-1}S)^2$. Moreover, note that, by reversing the orientation on the torus bundle with monodromy $T^{-1}S$, we obtain the torus bundle with monodromy $-(T^{-1}S)^2$. Thus we need only show that one of these torus bundles does not bound a $\mathbb{Q}S^1 \times B^3$. Consider the leftmost surgery diagram of the elliptic torus bundle with monodromy $T^{-1}S$ in Figure 6. By blowing down the 1–framed unknot, we obtain 0–surgery on the right-handed trefoil. Since the signature of the right-handed trefoil is 2, by Lemma 2.1, the elliptic torus bundle does not bound a $\mathbb{Q}S^1 \times B^3$. □

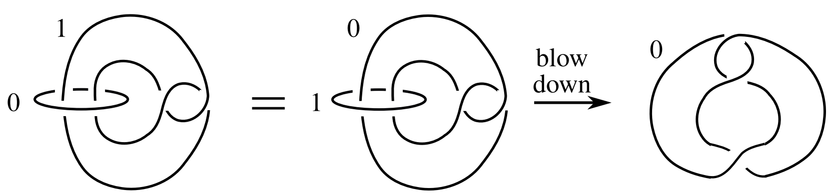

Figure 6: The elliptic torus bundle with monodromy $T^{-1}S$ does not bound a rational homology circle.

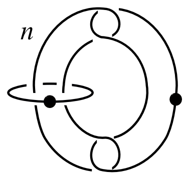

Figure 7: A $\mathbb{Q}S^1 \times B^3$ bounded by the negative parabolic torus bundle with monodromy $-T^n$.

Proposition 3.2 *Every negative parabolic torus bundle bounds a $\mathbb{Q}S^1 \times B^3$. No positive parabolic torus bundle bounds a $\mathbb{Q}S^1 \times B^3$.*

Proof By considering the surgery diagrams of the parabolic torus bundles in Table 1, it is easy to see that positive parabolic torus bundles, which have monodromy T^n, satisfy $b_1 = 2$. Thus, by the homology long exact sequence of the pair, it is easy to see that no such torus bundle can bound a $\mathbb{Q}S^1 \times B^3$. On the other hand, the negative parabolic torus bundles with monodromy $-T^n$ bound obvious $\mathbb{Q}S^1 \times B^3$'s, as shown in Figure 7. $\square$

Classifying hyperbolic torus bundles that bound $\mathbb{Q}S^1 \times B^3$'s is not as simple as the elliptic and parabolic cases. The hyperbolic torus bundles listed in Theorem 1.1 were shown to bound $\mathbb{Q}S^1 \times B^3$'s in [13].

Proposition 3.3 [13] *Let*

$$\boldsymbol{a} = (3 + x_1, 2^{[x_2]}, \dots, 3 + x_{2m+1}, 2^{[x_1]}, 3 + x_2, 2^{[x_3]}, \dots, 3 + x_{2m}, 2^{[x_{2m+1}]}) \in \mathcal{S}_{2c},$$

where $m \geq 0$ and $x_i \geq 0$ for all i. Then $\boldsymbol{T}_{A(\boldsymbol{a})}$ bounds a $\mathbb{Q}S^1 \times B^3$.

It remains to obstruct all other hyperbolic torus bundles from bounding $\mathbb{Q}S^1 \times B^3$'s. A major ingredient towards proving this fact is Theorem 1.7, which we assume to be true throughout the remainder of this section. The proof of Theorem 1.7 will be covered in Sections 4–7. Note that "most" hyperbolic torus bundles are obstructed by Theorem 1.7. In particular, by Theorem 1.7, if $\boldsymbol{a}, \boldsymbol{d} \notin \mathcal{S}_1 \cup \mathcal{O}$, then $\boldsymbol{T}_{-A(\boldsymbol{a})}$ does not bound a $\mathbb{Q}S^1 \times B^3$, and if $\boldsymbol{a}, \boldsymbol{d} \notin \mathcal{S}_2$, then $\boldsymbol{T}_{A(\boldsymbol{a})}$ does not bound a $\mathbb{Q}S^1 \times B^3$ (where $\boldsymbol{d}$ is the cyclic-dual of $\boldsymbol{a}$). Thus, it remains to prove that, if $\boldsymbol{a}$ or $\boldsymbol{d} \in \mathcal{S}_1 \cup \mathcal{O}$, then $\boldsymbol{T}_{-A(\boldsymbol{a})}$ does not bound a $\mathbb{Q}S^1 \times B^3$, and if $\boldsymbol{a}$ or $\boldsymbol{d} \in \mathcal{S}_2 \setminus \mathcal{S}_{2c}$, then $\boldsymbol{T}_{A(\boldsymbol{a})}$ does not bound a $\mathbb{Q}S^1 \times B^3$ (recall that $\boldsymbol{a} \in \mathcal{S}_{2c}$ if and only if $\boldsymbol{d} \in \mathcal{S}_{2c}$ by Example 1.3). We will prove this by considering cyclic covers of these torus bundles. But first we need to better understand the set $\mathcal{S}$. In the upcoming subsection, we will round up some necessary technical results regarding $\mathcal{S}$, and in the subsequent subsection, we will explore cyclic covers and finish the proof of Theorem 1.1.

3.1 Analyzing $\mathcal{S}$

The first technical lemma shows that the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint.

Lemma 3.4 *For a fixed string $\boldsymbol{a}$, $Y_{\boldsymbol{a}}^0$ and $Y_{\boldsymbol{a}}^{-1}$ do not both bound $\mathbb{Q}B^4$'s (and consequently $\boldsymbol{T}_{A(\boldsymbol{a})}$ and $\boldsymbol{T}_{-A(\boldsymbol{a})}$ do not both bound $\mathbb{Q}S^1\times B^3$'s). It follows that $\mathcal{S}_1\cap\mathcal{S}_2=\varnothing$.*

Proof By construction,

$$|H_1(Y_{\boldsymbol{a}}^0)| = |\mathrm{Tor}(H_1(\boldsymbol{T}_{A(\boldsymbol{a})}))| \quad \text{and} \quad |H_1(Y_{\boldsymbol{a}}^{-1})| = |\mathrm{Tor}(H_1(\boldsymbol{T}_{-A(\boldsymbol{a})}))|.$$

By Lemma A.1, $|\mathrm{Tor}(H_1(\boldsymbol{T}_{A(\boldsymbol{a})}))| = |\mathrm{Tor}(H_1(\boldsymbol{T}_{-A(\boldsymbol{a})}))| - 4$. Thus $|H_1(Y_{\boldsymbol{a}}^0)|$ and $|H_1(Y_{\boldsymbol{a}}^{-1})|$ cannot simultaneously be squares and so, by [4, Lemma 3], $Y_{\boldsymbol{a}}^0$ and $Y_{\boldsymbol{a}}^{-1}$ do not both bound $\mathbb{Q}B^4$'s. Now suppose $\boldsymbol{a}\in\mathcal{S}_1\cap\mathcal{S}_2$. Then, by Theorem 1.7, $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^0$ both bound $\mathbb{Q}B^4$'s, which is not possible. Therefore, $\mathcal{S}_1\cap\mathcal{S}_2=\varnothing$. □

Recall from Example 1.3 that a string $\boldsymbol{a}\in\mathcal{S}_{2c}$ can be expressed in two different, but equivalent, ways, namely

(1) $\boldsymbol{a} = (3+x_1, 2^{[x_2]}, \dots, 3+x_{2m+1}, 2^{[x_1]}, 3+x_2, 2^{[x_3]}, \dots, 3+x_{2m}, 2^{[x_{2m+1}]})$,

(2) $\boldsymbol{a} = (b_1+1, b_2, \dots, b_{k-1}, b_k+1, c_1, \dots, c_l)$,

where $m\geq 0$, $x_i\geq 0$ for all i, and $(b_1,\dots,b_k)$ and $(c_1,\dots,c_l)$ are linear-dual strings with $k+l\geq 2$. This relationship is easy to see:

$$(b_1+1, b_2, \dots, b_{k-1}, b_k+1) = (3+x_1, 2^{[x_2]}, \dots, 3+x_{2m+1}),$$
$$(c_1,\dots,c_l) = (2^{[x_1]}, 3+x_2, 2^{[x_3]}, \dots, 3+x_{2m}, 2^{[x_{2m+1}]}).$$

Also recall that $\mathcal{S}$ is defined up to cyclic reordering and reversing strings. Thus a string $\boldsymbol{a}=(a_1,\dots,a_n)\in\mathcal{S}_{2c}$ may not be of the form (1) written above. However, by a cyclic reordering of $\boldsymbol{a}$, we can put $\boldsymbol{a}$ in the form (1), which is equivalent to (2). Moreover, it is clear that, if $a_1\geq 3$, then $\boldsymbol{a}$ is already in the form (1) and thus already in the form (2). This simple observation will be used throughout the rest of this subsection.

Definition 3.5 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be strings. Then $\boldsymbol{ab}$ denotes the string obtained by concatenating $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{a}^p$ denotes the string obtained by concatenating $\boldsymbol{a}$ with itself p times.

The next lemma follows directly from the definitions of linear-dual and cyclic-dual strings. We leave the proof to the reader.

Lemma 3.6 (a) *Suppose $\boldsymbol{a}$ has linear-dual $\boldsymbol{x} = (x_1, \dots, x_p)$ and $\boldsymbol{b}$ has linear-dual $\boldsymbol{y} = (y_1, \dots, y_q)$. Then*

(i) *$\boldsymbol{ab}$ has linear-dual $(x_1, \dots, x_{p-1}, x_p - 1 + y_1, y_2, \dots, y_q)$, and*

(ii) *$\boldsymbol{ab}$ has cyclic-dual $(x_2, \dots, x_{p-1}, x_p - 1 + y_1, y_2, \dots, y_{q-1}, y_q - 1 + x_1)$ (up to cyclic reordering).*

(b) *If $\boldsymbol{a}$ has cyclic-dual $\boldsymbol{d}$, then $\boldsymbol{a}^p$ has cyclic-dual $\boldsymbol{d}^p$.*

Definition 3.7 We call a string $(a_1, \dots, a_n)$ a *palindrome* if $a_i = a_{n-(i-1)}$ for all $1 \le i \le n$.

Lemma 3.8 *Consider the strings $\boldsymbol{a} = (b_1 + 3, b_2, \dots, b_k, 2, c_l, \dots, c_1) \in \mathcal{S}_{2a}$ and $\boldsymbol{b} = (3 + x, b_1, \dots, b_{k-1}, b_k + 1, 2^{[x]}, c_l + 1, c_{l-1}, \dots, c_1) \in \mathcal{S}_{2b}$.*

(a) *$\boldsymbol{a} \in \mathcal{S}_{2c}$ if and only if $(b_1 + 1, b_2, \dots, b_k)$ is a palindrome.*

(b) *$\boldsymbol{b} \in \mathcal{S}_{2c}$ if and only if $(b_1, \dots, b_k)$ is a palindrome.*

Proof (a) Since $(c_1, \dots, c_l)$ is the linear-dual of $(b_1, \dots, b_k)$, $(2, c_1, \dots, c_l)$ is the linear-dual of $(b_1 + 1, b_2, \dots, b_k)$. Consequently, $(b_1 + 1, b_2, \dots, b_k)$ is a palindrome if and only if $(2, c_1, \dots, c_l)$ is a palindrome if and only if $c_l = 2$ and $c_i = c_{l-i}$ for all $1 \le i \le l - 1$.

Assume that $(b_1 + 1, b_2, \dots, b_k)$ is a palindrome. Then $b_k = b_1 + 1 \ge 3$ and, consequently, $c_l = 2$. Let $d_1 = b_1 + 2$, $d_k = b_k - 1$, and $d_i = b_i$ for all $2 \le i \le k-1$, so that $\boldsymbol{a} = (d_1 + 1, d_2, \dots, d_{k-1}, d_k + 1, 2, c_l, \dots, c_1)$. By Lemma 3.6, $(2, 2, c_1, c_2, \dots, c_{l-1})$ has linear-dual $(b_1 + 2, b_2, \dots, b_{k-1}, b_k - 1) = (d_1, \dots, d_k)$. On the other hand, since $(2, c_1, \dots, c_l)$ is a palindrome, $(2, 2, c_1, c_2, \dots, c_{l-1}) = (2, c_l, c_{l-1}, c_{l-2}, \dots, c_1)$. Set $e_1 = e_2 = 2$ and $e_i = c_{i-2}$ for all $3 \le i \le l + 1$. Then $(d_1, \dots, d_k)$ has linear-dual $(e_1, \dots, e_{l+1})$ and thus

$$(b_1 + 3, b_2, \dots, b_k, 2, c_l, \dots, c_1) = (d_1 + 1, d_2, \dots, d_{k-1}, d_k + 1, e_1, \dots, e_{l+1}) \in \mathcal{S}_{2c}.$$

Now assume $\boldsymbol{a} \in \mathcal{S}_{2c}$. Since $b_1 + 3 > 3$, $\boldsymbol{a}$ is of the form

$$\boldsymbol{a} = (d_1 + 1, d_2, \dots, d_{p-1}, d_p + 1, e_1, \dots, e_q),$$

where $(d_1, \dots, d_p)$ and $(e_1, \dots, e_q)$ are linear-dual. Thus $d_1 = b_1 + 2$ and $e_q = c_1$. Note that the length of $\boldsymbol{a}$ is $k + l + 1 = p + q$. We claim that $p = k$. Indeed, if $p > k$, then $(d_1, \dots, d_k) = (b_1 + 2, b_2, \dots, b_k)$ has linear-dual $(2, 2, c_1, \dots, c_l)$, implying that the length of $\boldsymbol{a}$ is greater than $k + l + 1$, a contradiction; if $p < k$, we arrive at a similar contradiction. Therefore $p = k$ and $q = l + 1$; consequently, $e_1 = 2$ and $e_i = c_{l-i+2}$ for all $2 \le i \le l + 1$. On the other hand, by Lemma 3.6, the linear-dual of $(d_1, \dots, d_p) = (b_1 + 2, b_2, \dots, b_k - 1)$ is $(e_1, \dots, e_q) = (2, 2, c_1, \dots, c_{l-1})$. Thus $c_l = e_2 = 2$ and $c_i = c_{l-i}$ for all $1 \le i \le l - 1$. As mentioned above, this implies that $(b_1 + 1, b_2, \dots, b_k)$ is a palindrome.

(b) Note that $(b_1, \dots, b_k)$ is a palindrome if and only if $(c_1, \dots, c_l)$ is a palindrome.

Assume $(b_1, \dots, b_k)$ is a palindrome. Let $d_1 = 2 + x$ and $d_i = b_{i-1}$ for all $2 \le i \le k + 1$. By Lemma 3.6, the linear-dual of $(d_1, \dots, d_{k+1}) = (2 + x, b_1, \dots, b_{k-1}, b_k)$ is $(2^{[x]}, c_1 + 1, c_2, \dots, c_l) = (2^{[x]}, c_l + 1, c_{l-1}, \dots, c_1)$ since $(c_1, \dots, c_l)$ is a palindrome. Relabel this string as $(e_1, \dots, e_q)$. Then

$$\boldsymbol{b} = (d_1 + 1, d_2, \dots, d_k, d_{k+1} + 1, e_1, \dots, e_q) \in \mathcal{S}_{2c}.$$

Now assume $\boldsymbol{b} \in \mathcal{S}_{2c}$. Since $3 + x \ge 3$, $\boldsymbol{b}$ is of the form

$$\boldsymbol{b} = (d_1 + 1, d_2, \dots, d_{p-1}, d_p + 1, e_1, \dots, e_q),$$

where $(d_1, \dots, d_p)$ and $(e_1, \dots, e_q)$ are linear-dual. Thus $d_1 + 1 = 3 + x$ and $e_q = c_1$. Following as in the proof of the first part, $p = k + 1$ and $q = l + x$. Consequently, $e_{x+1} = c_l + 1$ and $e_{x+j} = c_{l-j+1}$ for all $l \le j \le l$. On the other hand, the linear-dual of $(d_1, \dots, d_p) = (2 + x, b_1, \dots, b_k)$ is $(e_1, \dots, e_q) = (2^{[x]}, c_1 + 1, c_2, \dots, c_l)$. Thus $c_1 = e_{x+1} - 1 = c_l$ and $c_j = e_{x+j} = c_{l-j+1}$ for all $2 \le j \le l$. That is, $(c_1, \dots, c_l)$ is a palindrome and thus so is $(b_1, \dots, b_k)$. □

Lemma 3.9 *Let $\boldsymbol{b} \in \mathcal{S}_{2a} \cup \mathcal{S}_{2b}$ and $p \ge 4$. Then there does not exist some proper substring $\boldsymbol{a}$ of $\boldsymbol{b}$ such that $\boldsymbol{a}^p = \boldsymbol{b}$.*

Proof Let $\boldsymbol{b} = (3 + x, b_1, \dots, b_{k-1}, b_k + 1, 2^{[x]}, c_l + 1, c_{l-1}, \dots, c_1) \in \mathcal{S}_{2b}$. Suppose that $\boldsymbol{a}$ is a proper substring of $\boldsymbol{b}$ satisfying $\boldsymbol{a}^p = \boldsymbol{b}$ for some $p \ge 4$. Then $\boldsymbol{a} = (3 + x, b_1, \dots, b_m)$ for some m. If $m = 0$, then $\boldsymbol{a} = (3 + x)$ and every entry of $\boldsymbol{b}$ equals $3 + x$. The only such string satisfies $x = 0$ and $(b_1, \dots, b_k) = (2) = (c_1, \dots, c_l)$; that is, $\boldsymbol{b} = (3, 3, 3)$. But then $p = 3$, a contradiction.

Assume $m \ge 1$. Since $\boldsymbol{a}^p = \boldsymbol{b}$, we have that $b_{m+1} = 3 + x \ge 3$; consequently, either $m \le k$ or $m \ge k + x$. If $m \ge k + x$, then $m \le l$. Thus, up to switching the roles of $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$, we may assume without loss of generality that $m \le k$.

By Lemma 3.6, the linear-dual of $(b_1,\dots,b_m)$ is of the form $(c_1,\dots,c_{n-1},c_n')$, where $n \le l$ and $c_n' \le c_n$. We claim that $m=n$. First suppose $m<n$. Then, since $\boldsymbol{a}^p=\boldsymbol{b}$, we have $b_m=c_1, b_{m-1}=c_2,\dots,b_2=c_{m-1}, b_1=c_m$; that is, $(b_1,\dots,b_m)$ is a proper substring of $(c_1,\dots,c_{n-1},c_n')$. But then the linear-dual of $(b_1,\dots,b_m)$ (ie $(c_1,\dots,c_{n-1},c_n')$) is a proper substring of the linear-dual of $(c_1,\dots,c_{n-1},c_n')$ (ie $(b_1,\dots,b_m)$), which is a contradiction. A similar argument shows that $n<m$ is also not possible. Thus $m=n$.

Since $m=n$ and $\boldsymbol{a}^p=\boldsymbol{b}$, we have that $b_m=c_1$, $b_{m-1}=c_2,\dots,b_2=c_{m-1}, b_1=c_m$, and $c_{m+1}=3+x\ge 3$. If $m=k$, then, since $c_{m+1}\ge 3$, we necessarily have that $x=0$ and $p=2$, a contradiction. If $m=k-1$, then $b_k+1=b_{m+1}=3+x$ and, by Lemma 3.6, $(c_1,\dots,c_l)=(c_1,\dots,c_m'+1,2^{[x]})$; since $c_{m+1}\ge 3$, we once again have $x=0$ and $p=2$, a contradiction. Thus either $x=0$ or $m\le k-2$. In the latter case, since $(b_1,\dots,b_k)$ has linear-dual $(c_1,\dots,c_{m-1},c_m')$, by Lemma 3.6, $(b_1,\dots,b_m,3+x,b_1)$ has linear-dual $(c_1,\dots,c_{m-1},c_m'+1,2^{[x]},3,2^{[b_1-2]})$; since $c_{m+1}=3+x\ge 3$, we necessarily have that $x=0$. Thus $c_{m+1}=b_{m+1}=3$. Moreover, since $(b_1,\dots,b_m)$ has linear-dual $(c_1,\dots,c_{m-1},c_m')$, by Lemma 3.6, $(b_1,\dots,b_m,3)$ has linear-dual $(c_1,\dots,c_{m-1},c_m'+1,2)$. Therefore, $c_m=c_m'+1$.

Since $p\ge 4$, it follows that either $2m+2\le k$ or $2m+2\le l$. Without loss of generality, assume $2m+2\le k$. Then $(b_1,\dots,b_m,3,b_1,\dots,b_m,3)$ is a substring of $(b_1,\dots,b_k)$ and its linear-dual is a substring of $(c_1,\dots,c_l)$. By Lemma 3.6, $(b_1,\dots,b_m,3)$ has linear-dual $(c_1,\dots,c_m,2)$ and consequently $(b_1,\dots,b_m,3,b_1,\dots,b_m,3)$ has linear-dual $(c_1,\dots,c_m,c_1+1,c_2,\dots,c_m,2)$. But, since $\boldsymbol{a}^p=\boldsymbol{b}$, the latter string is also of the form $(b_m,\dots,b_1,3,b_m,\dots,b_2,b_1)$. Thus $c_1=2$ and $b_1=2$. But, since $(b_1,\dots,b_m)$ and $(c_1,\dots,c_m')$ are linear-dual and $c_1=b_1=2$, we necessarily have $(b_1,\dots,b_k)=(2)=(c_1,\dots,c_l)$; therefore, $\boldsymbol{b}=(3,3,3)$ and $p=3$, a contradiction. We have thus shown that there does not exist a proper substring $\boldsymbol{a}$ of $\boldsymbol{b}$ such that $\boldsymbol{b}=\boldsymbol{a}^p$ for some $p\ge 4$.

Next suppose $\boldsymbol{b}=(b_1+3,b_2,\dots,b_k,2,c_l,\dots,c_1)\in\mathcal{S}_{2a}$. Let $\boldsymbol{a}=(b_1+3,b_2,\dots,b_m)$ be a substring of $\boldsymbol{b}$ such that $\boldsymbol{a}^p=\boldsymbol{b}$, where $p\ge 4$. We first claim that $m<k$. Assume otherwise. Then $m\le l$ and since $\boldsymbol{a}^p=\boldsymbol{b}$, $(b_1+3,b_2,\dots,b_k)$ is a substring of $(c_1,\dots,c_l)$. Consequently, the linear-dual of $(b_1+3,b_2,\dots,b_k)$ (ie $(2,2,2,c_1,\dots,c_l)$) is a substring of the linear-dual of $(c_1,\dots,c_l)$ (ie $(b_1,\dots,b_k)$), implying that $l<k<m$, a contradiction. Thus $m\le k$. If $m=k$, then $b_{m+1}=b_1+3\ge 3$; on the other hand, $b_{m+1}=b_{k+1}=2$, a contradiction. Thus $k<m$. Now, following the same argument as in the first part of the proof, we see that the linear-dual of $(b_1+3,b_2,\dots,b_m)$ is of the

form $(c_1,\dots,c'_m)$, where $c'_m \le c_m$ and $m \le l$. Thus $b_{m+1} = c_{m+1} = b_1+3 \ge 5$. But, by Lemma 3.6, $(b_1+3, b_2, \dots, b_m, b_{m+1}) = (b_1+3, b_2, \dots, b_m, b_1+3)$ has linear-dual $(c_1,\dots,c_m,2^{[b_1+1]})$, implying that $c_{m+1} \ge 5$, which is another contradiction. □

Lemma 3.10 *Suppose $\boldsymbol{a} \in \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c}$ and $\boldsymbol{a}^p \in \mathcal{S}_{2c}$ for some p. Then $\boldsymbol{a} \in \mathcal{S}_{2c}$.*

Proof It suffices to show that, if $\boldsymbol{a} \in \mathcal{S}_{2a}$ or $\boldsymbol{a} \in \mathcal{S}_{2b}$, then $\boldsymbol{a} \in \mathcal{S}_{2c}$. Let $\boldsymbol{a} \in \mathcal{S}_{2a}$, so that $\boldsymbol{a}^p$ is of the form

$$\begin{aligned}\boldsymbol{a}^p = \big(&b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1,\\ &\qquad\qquad\quad \vdots\, l\\ &b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1,\\ &b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1,\\ &b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1,\\ &\qquad\qquad\quad \vdots\, p-l-1\\ &b_1+3, b_2, \dots, b_k, 2, c_l, \dots, c_1\big).\end{aligned}$$

Since $\boldsymbol{a}^p \in \mathcal{S}_{2c}$ and $b_1+3 > 3$, $\boldsymbol{a}^p = (d_1+1, d_2, \dots, d_{q-1}, d_q+1, e_1, \dots, e_r)$, where $(d_1,\dots,d_q)$ and $(e_1,\dots,e_r)$ are linear-dual strings. Following as in the proof of Lemma 3.8 and appealing to Lemma 3.6, p is odd, $l = \frac{1}{2}(p-1)$ and $q = \frac{1}{2}(p-1)(k+l+1)+k$, which is the length of the blue substring above. Thus, $(e_1,\dots,e_r)$ is the black substring of $\boldsymbol{a}^p$ above. Comparing the end of both strings, it is clear that $c_l = 2$ and $c_i = c_{l-i}$ for all $1 \le i \le l-1$. As mentioned in the first paragraph of the proof of Lemma 3.8, this implies that $(b_1+1, b_2, \dots, b_k)$ is a palindrome. By Lemma 3.8, $\boldsymbol{a} \in \mathcal{S}_{2c}$.

Now assume $\boldsymbol{a} \in \mathcal{S}_{2b}$. Then $\boldsymbol{a}^p$ is of the form

$$\begin{aligned}\boldsymbol{a}^p = \big(&3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1,\\ &\qquad\qquad\qquad\quad \vdots\, l\\ &3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1,\\ &3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1,\\ &3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1,\\ &\qquad\qquad\qquad\quad \vdots\, p-l-1\\ &3+x, b_1, \dots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \dots, c_1\big).\end{aligned}$$

Since $\boldsymbol{a}^p \in \mathcal{S}_{2c}$, $\boldsymbol{a}^p = (d_1+1, d_2, \dots, d_{q-1}, d_q+1, e_1, \dots, e_r)$, where $(d_1,\dots,d_q)$ and $(e_1,\dots,e_r)$ are linear-dual strings. Following as above, we have that p is odd,

$l = \frac{1}{2}(p-1)$ and $q = \frac{1}{2}(p-1)(k+l+x+1)+k+1$, which is the length of the blue substring above. Thus, on the one hand, $(e_1,\dots,e_r)$ is the black substring of $\boldsymbol{a}^p$ above. On the other hand, by computing the linear-dual of $(d_1,\dots,d_q)$ from the blue string above, $(e_1,\dots,e_r)$ ends in the substring $(c_1+1,\dots,c_l)$. Comparing the end of both strings, it is clear that $(c_1,\dots,c_l)=(c_l,\dots,c_1)$ and thus $(b_1,\dots,b_k)$ is also a palindrome. By Lemma 3.8, $\boldsymbol{a}\in\mathcal{S}_{2c}$. □

Corollary 3.11 *If $\boldsymbol{a},\boldsymbol{a}^p\in\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$, where $p\geq 4$, then $\boldsymbol{a}\in\mathcal{S}_{2c}$.*

Proof It follows from Lemma 3.9 that $\boldsymbol{a}^p\in\mathcal{S}_{2c}$; thus, $\boldsymbol{a}^p\in\mathcal{S}_{2c}$. By Lemma 3.10, $\boldsymbol{a}\in\mathcal{S}_{2c}$. □

The final technical lemma shows that the cyclic-duals of strings in $\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$ are also in $\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$. Although this result is implicit in the proof of Theorem 1.7, it is also relatively simple to prove directly, with the help of Lemma 3.6.

Lemma 3.12 *Let $\boldsymbol{d}$ be the cyclic-dual of $\boldsymbol{a}$. If $\boldsymbol{a}\in\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$, then $\boldsymbol{d}\in\mathcal{S}_{2a}\cup\mathcal{S}_{2b}\cup\mathcal{S}_{2c}$.*

Proof Let $\boldsymbol{a}\in\mathcal{S}_{2c}$. Using the description of $\boldsymbol{a}$ as in (1) on page 2465, it is easy to see that $\boldsymbol{d}\in\mathcal{S}_{2c}$. Next let $\boldsymbol{a}=(3+x,b_1,\dots,b_k+1,2^{[x]},c_l+1,c_{l-1},\dots,c_1)\in\mathcal{S}_{2b}$. Notice that $(3+x,b_1,\dots,b_k+1)$ has linear-dual $(2^{[x+1]},c_1+1,\dots,c_l,2)$ and $(2^{[x]},c_l+1,c_{l-1},\dots,c_1)$ has linear-dual $(2+x,b_k,\dots,b_1)$. Thus, by Lemma 3.6, $\boldsymbol{d}=(2^{[x]},c_1+1,\dots,c_l,3+x,b_k,\dots,b_1+1)\in\mathcal{S}_{2b}$.

Finally, let $\boldsymbol{a}=(b_1+3,b_2,\dots,b_k,2,c_l,\dots,c_1)\in\mathcal{S}_{2a}$. If $k+l=1$, then $\boldsymbol{a}=(4,2)$ and $\boldsymbol{d}=(2,4)\in\mathcal{S}_{2a}$. If $k+l=2$, then $\boldsymbol{a}=(5,2,2)$ and $\boldsymbol{d}=(2,2,5)\in\mathcal{S}_{2a}$. Now let $k+l\geq 3$. Then either $b_k\geq 3$ and $c_l=2$ or vice versa. Assume the former. Since $(b_1+3,b_2,\dots,b_k)$ has linear-dual $(2,2,2,c_1,\dots,c_l)$ and $(2,c_l,\dots,c_1)$ has linear-dual $(b_k+1,b_{k-1},\dots,b_1)$, by Lemma 3.6,

$$\boldsymbol{d}=(2,2,c_1,\dots,c_{l-1},c_l+b_k,b_{k-1},\dots,b_2,b_1+1).$$

Let $d_1=c_l+b_k-3$, $d_k=b_1+1$, and $d_i=b_{k-i+1}$ for all $2\leq i\leq k-1$. Also let $e_1=c_{l-1}$, $e_l=2$, and $e_i=c_{l-i}$ for all $2\leq i\leq l-1$. Then

$$\boldsymbol{d}=(2,e_l,\dots,e_1,d_1+3,d_2,\dots,d_k)$$

and $(d_1,\dots,d_k)=(b_k-1,b_{k-1},\dots,b_2,b_1+1)$ and $(e_1,\dots,e_l)=(c_{l-1},\dots,c_1,2)$ are linear-dual; thus $\boldsymbol{d}\in\mathcal{S}_{2a}$. Now assume $b_k=2$ and $c_l\geq 3$. Set $d_1=c_l+b_k-3$, $d_{l+1}=2$, $d_i=c_{l-i+1}$ for all $2\leq i\leq l$, $e_1=b_{k-1}$, $e_{k-1}=b_1+1$, and $e_i=b_{k-i}$ for all $2\leq i\leq k-2$. Proceeding as above, we see that $\boldsymbol{d}\in\mathcal{S}_{2a}$. □

3.2 Cyclic covers and proving Theorem 1.1

We are now ready to finish the proof of Theorem 1.1. The next two results explore cyclic covers of $\mathbb{Q}S^1 \times B^3$'s and cyclic covers of hyperbolic torus bundles over S^1. Coupling these results with the results in Section 3.1, we complete the proof of Theorem 1.1 in the subsequent corollaries.

Lemma 3.13 *Let W be a $\mathbb{Q}S^1 \times B^3$ and let $\widetilde{W}$ be a p–fold cyclic cover of W, where p is prime and not a divisor of $|\mathrm{Tor}(H_2(W;\mathbb{Z}))|$. If $\partial\widetilde{W}$ is a $\mathbb{Q}S^1 \times S^2$, then $\widetilde{W}$ is a $\mathbb{Q}S^1 \times B^3$.*

Proof Let $Y = \partial W$ and $\widetilde{Y} = \partial\widetilde{W}$. Since W is a $\mathbb{Q}S^1 \times B^3$ and $H_3(W;\mathbb{Z})$ has no torsion, it follows that $H_3(W;\mathbb{Z}) = 0$. Thus, by Poincaré duality and the universal coefficient theorem, we have the isomorphisms

$$H_1(W,Y;\mathbb{Z}_p) \cong H^3(W;\mathbb{Z}_p) \cong \mathrm{Ext}(H_2(W;\mathbb{Z}),\mathbb{Z}_p).$$

Since p is relatively prime to $|\mathrm{Tor}(H_2(W;\mathbb{Z}))|$, we have

$$H_1(W,Y;\mathbb{Z}_p) \cong \mathrm{Ext}(H_2(W;\mathbb{Z}),\mathbb{Z}_p) = 0.$$

By the proof of [7, Theorem 1.2], since p is prime, it follows that $H_1(\widetilde{W},\widetilde{Y};\mathbb{Z}_p) = 0$. Once again applying Poincaré duality and the universal coefficient theorem, we have the isomorphisms

$$0 = H_1(\widetilde{W},\widetilde{Y};\mathbb{Z}_p) \cong H^3(\widetilde{W};\mathbb{Z}_p) \cong \mathrm{Hom}(H_3(\widetilde{W};\mathbb{Z}),\mathbb{Z}_p) \oplus \mathrm{Ext}(H_2(\widetilde{W};\mathbb{Z}),\mathbb{Z}_p).$$

Thus $H_3(\widetilde{W};\mathbb{Z})$ is a torsion group. Thus, if we apply Poincaré duality and the universal coefficient theorem as above, but with $\mathbb{Q}$–coefficients, we obtain

$$H_1(\widetilde{W},\widetilde{Y};\mathbb{Q}) \cong H^3(\widetilde{W};\mathbb{Q}) \cong \mathrm{Hom}(H_3(\widetilde{W};\mathbb{Z}),\mathbb{Q}) \oplus \mathrm{Ext}(H_2(\widetilde{W};\mathbb{Z}),\mathbb{Q}) = 0.$$

Thus the map $H_1(\widetilde{Y};\mathbb{Q}) \to H_1(\widetilde{W};\mathbb{Q})$ induced by inclusion is surjective. Since $\widetilde{Y}$ is a $\mathbb{Q}S^1 \times S^2$, it follows that $\mathrm{rank}(H_1(\widetilde{W};\mathbb{Q})) \le 1$. Finally, since $\chi(\widetilde{W}) = p\chi(W) = 0$ and $H_3(\widetilde{W};\mathbb{Q}) = 0$, we necessarily have that $H_1(\widetilde{W};\mathbb{Q}) = \mathbb{Q}$ and $H_2(\widetilde{W};\mathbb{Q}) = 0$, proving that $\widetilde{W}$ is indeed a $\mathbb{Q}S^1 \times B^3$. □

Proposition 3.14 *Let $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ be a hyperbolic torus bundle that bounds a $\mathbb{Q}S^1 \times B^3$, say W. If p is an odd prime that does not divide $|\mathrm{Tor}(H_2(W;\mathbb{Z}))|$, then $\boldsymbol{T}_{\pm A(\boldsymbol{a}^p)}$ bounds a $\mathbb{Q}S^1 \times B^3$.*

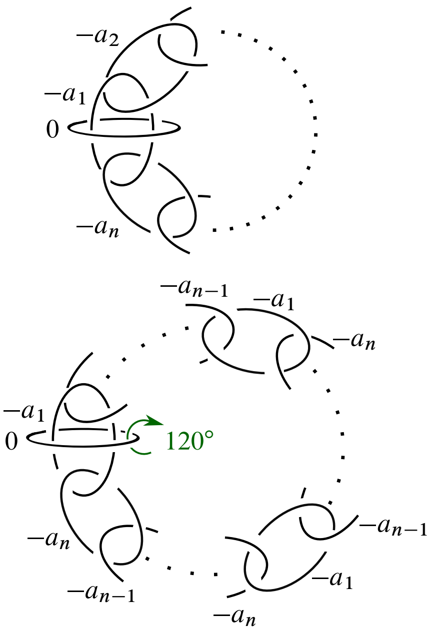

Figure 8: Surgery diagrams for $\boldsymbol{T}_{A(\boldsymbol{a})}$ (top left), $\boldsymbol{T}_{-A(\boldsymbol{a})}$ (top right), $\boldsymbol{T}_{A(\boldsymbol{a}^3)}$ (bottom left) and $\boldsymbol{T}_{-A(\boldsymbol{a}^3)}$ (bottom right). $\boldsymbol{T}_{\pm A(\boldsymbol{a}^3)}$ is a 3–fold cyclic cover of $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$. There is an obvious $\mathbb{Z}_3$–action on $\boldsymbol{T}_{\pm A(\boldsymbol{a}^3)}$ given by a rotation of 120° through the 0–framed unknot. The quotient of $\boldsymbol{T}_{\pm A(\boldsymbol{a}^3)}$ by this action is $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$.

Proof Let W be a $\mathbb{Q}S^1 \times B^3$ bounded by some negative hyperbolic torus bundle $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$, where $\boldsymbol{a} = (a_1, \dots, a_n)$. Let p be an odd prime number that is not a factor of $|\mathrm{Tor}(H_2(W;\mathbb{Z}))|$. Consider the obvious surgery diagrams of $\boldsymbol{T}_{A(\boldsymbol{a})}$ and $\boldsymbol{T}_{-A(\boldsymbol{a})}$ as in Figure 8, top. In both diagrams, let μ_i denote the homology class of the meridian of the $-a_i$–framed surgery curve and let μ_0 denote the homology class of the meridian of the 0–framed surgery curve. Then $H_1(\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z})$ is generated by $\mu_0, \dots, \mu_n$.

Consider the torus bundle $\boldsymbol{T}_{-A(\boldsymbol{a}^p)}$, which has monodromy $-(T^{-a_1}S \cdots T^{-a_n}S)^p$. The standard surgery diagram of this torus bundle includes a -1–half-twisted chain link (as in Table 1). Note that, by sliding the chain link over the 0–framed unknot $\frac{1}{2}(p-1)$ times, we may arrange that the chain link has $-p$ half-twists, as in Figure 8, bottom right (for the case $p = 3$). For the torus bundle $\boldsymbol{T}_{A(\boldsymbol{a}^p)}$, which has monodromy $(T^{-a_1}S \cdots T^{-a_n}S)^p$, consider the standard surgery diagram shown in Figure 8, bottom left (for the case $p = 3$). There is an obvious $\mathbb{Z}_p$–action on $\boldsymbol{T}_{\pm A(\boldsymbol{a}^p)}$ obtained by rotating the chain link through the 0–framed unknot by an angle of $2\pi/p$, as indicated in Figure 8, bottom. The quotient of $\boldsymbol{T}_{\pm A(\boldsymbol{a}^p)}$ by this action is clearly $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ and the induced map $f\colon H_1(\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z}) \to \mathbb{Z}_p$ satisfies $f(\mu_0)=1$ and $f(\mu_i)=0$ for all $1 \le i \le n$. Consider the long exact sequence of the pair $(W, \boldsymbol{T}_{\pm A(\boldsymbol{a})})$,

$$H_1(\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z}) \xrightarrow{i_*} H_1(W;\mathbb{Z}) \to H_1(W,\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z}) \to 0.$$

Choose a basis $\{m_0, m_1, \dots, m_k\}$ for $H_1(W;\mathbb{Z})$ such that m_0 has infinite order and m_i is a torsion element for all $1 \le i \le k$. Since $H_1(W,\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z})$ is a torsion group, $i_*(\mu_0) = \alpha m_0 + \sum_{i=1}^k \beta_i m_i$ for some $\alpha, \beta_i \in \mathbb{Z}$, where $\alpha \neq 0$. Since p is not relatively prime to $|\mathrm{Tor}(H_2(W;\mathbb{Z}))| = |H_1(W,\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z})|$ and α divides $|H_1(W,\boldsymbol{T}_{\pm A(\boldsymbol{a})};\mathbb{Z})|$, it follows that α and p are relatively prime; thus there exists an integer t such that $t\alpha \equiv 1 \bmod p$. Define a map $g\colon H_1(W;\mathbb{Z}) \to \mathbb{Z}_p$ by $g(m_0)=t$ and $g(m_i)=0$ for all $1 \le i \le k$. Then g is a surjective homomorphism satisfying $f = g \circ i_*$. Let $\widetilde{W}$ be the p–fold cyclic cover of W induced by g. Then $\partial\widetilde{W} = \boldsymbol{T}_{\pm A(\boldsymbol{a}^p)}$ and, by Lemma 3.13, $\widetilde{W}$ is a $\mathbb{Q}S^1 \times B^3$. □

The two following corollaries conclude the proof of Theorem 1.1.

Corollary 3.15 *No negative hyperbolic torus bundle bounds a $\mathbb{Q}S^1 \times B^3$.*

Proof Let $\boldsymbol{T}_{-A(\boldsymbol{a})}$ be a negative hyperbolic torus bundle that bounds a $\mathbb{Q}S^1 \times B^3$, say W. Let $p > 3$ be an odd prime number that is not a factor of $|\mathrm{Tor}(H_2(W;\mathbb{Z}))|$. By Proposition 3.14, $\boldsymbol{T}_{-A(\boldsymbol{a}^p)}$ also bounds a $\mathbb{Q}S^1 \times B^3$. Let $\boldsymbol{d}$ be the cyclic-dual of $\boldsymbol{a}$; by Lemma 3.6, $\boldsymbol{d}^p$ is the linear-dual of $\boldsymbol{a}^p$. By Lemma 1.2, $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}^p}^{-1}$ bound $\mathbb{Q}B^4$'s and so, by Theorem 1.7, $\boldsymbol{a}$ or $\boldsymbol{d}$ belongs to $\mathcal{S}_1 \cup \mathcal{O}$ and $\boldsymbol{a}^p$ or $\boldsymbol{d}^p$ belongs to $\mathcal{S}_1 \cup \mathcal{O}$.

First assume $\boldsymbol{a}, \boldsymbol{a}^p \in \mathcal{S}_1 \cup \mathcal{O}$. By Remark 1.6, $-4 \le I(\boldsymbol{a}), I(\boldsymbol{a}^p) \le 0$. Moreover, $I(\boldsymbol{a}^p) = pI(\boldsymbol{a})$. If $I(\boldsymbol{a}) < 0$, then, since $p > 3$, we have $I(\boldsymbol{a}^p) < -4$, which is a contradiction. Thus $I(\boldsymbol{a}^p) = I(\boldsymbol{a}) = 0$. By Remark 1.6, $\boldsymbol{a}, \boldsymbol{a}^p \in \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c} \cup \mathcal{O}$. Since $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$, by Lemma 3.4, we necessarily have that $\boldsymbol{a}, \boldsymbol{a}^p \in \mathcal{O}$, which is not possible since $p \neq 1$.

Next assume $\boldsymbol{a}, \boldsymbol{d}^p \in \mathcal{S}_1 \cup \mathcal{O}$. By Remark 1.6, $-4 \le I(\boldsymbol{a}), I(\boldsymbol{d}^p) \le 0$. Since $I(\boldsymbol{d}^p) = pI(\boldsymbol{d}) = -pI(\boldsymbol{a})$, we necessarily have that $I(\boldsymbol{a}) = I(\boldsymbol{d}^p) = 0$. As above, this implies that $\boldsymbol{a}, \boldsymbol{d}^p \in \mathcal{O}$. But, since $\boldsymbol{a} \in \mathcal{O}$, it is clear that $\boldsymbol{a} = \boldsymbol{d}$ and thus $\boldsymbol{d} \in \mathcal{O}$. As above, it is clear that $\boldsymbol{d}$ and $\boldsymbol{d}^p$ cannot both be contained in $\mathcal{O}$.

Finally, if $\boldsymbol{d}, \boldsymbol{d}^p \in \mathcal{S}_1 \cup \mathcal{O}$ or $\boldsymbol{d}, \boldsymbol{a}^p \in \mathcal{S}_1 \cup \mathcal{O}$, similar arguments provide similar contradictions. Therefore, ∂W cannot be a negative hyperbolic torus bundle. □

Corollary 3.16 *If a positive hyperbolic torus bundle $T_{A(\boldsymbol{a})}$ bounds a $\mathbb{Q}S^1 \times B^3$, then $\boldsymbol{a} \in \mathcal{S}_{2c}$.*

Proof Let $T_{A(\boldsymbol{a})}$ be a positive hyperbolic torus bundle that bounds a $\mathbb{Q}S^1 \times B^3$, say W, and let $p > 3$ be an odd prime number that is not a factor of $|\mathrm{Tor}(H_2(W;\mathbb{Z}))|$. Following as in the proof of Corollary 3.15, $\boldsymbol{a}$ or $\boldsymbol{d}$ belongs to $\mathcal{S}_2$ and $\boldsymbol{a}^p$ or $\boldsymbol{d}^p$ belongs to $\mathcal{S}_2$, where $\boldsymbol{d}$ is the cyclic-dual of $\boldsymbol{a}$. Suppose $\boldsymbol{a}, \boldsymbol{a}^p \in \mathcal{S}_2$. As in the proof of Corollary 3.15, $I(\boldsymbol{a}) = I(\boldsymbol{a}^p) = 0$ and so, by Remark 1.6, $\boldsymbol{a}, \boldsymbol{a}^p \in \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c}$. By Corollary 3.11, $\boldsymbol{a} \in \mathcal{S}_{2c}$. Next suppose $\boldsymbol{a}, \boldsymbol{d}^p \in \mathcal{S}_2$. Once again, following the argument in Corollary 3.15, $I(\boldsymbol{a}) = I(\boldsymbol{d}^p) = 0$ and so, by Remark 1.6, $\boldsymbol{a}, \boldsymbol{d}^p \in \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c}$. By Lemma 3.12, we necessarily have that $\boldsymbol{a}^p \in \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c}$; proceeding as in the previous case, we find $\boldsymbol{a} \in \mathcal{S}_{2c}$. Finally, if $\boldsymbol{d}, \boldsymbol{a}^p \in \mathcal{S}_2$ or $\boldsymbol{d}, \boldsymbol{d}^p \in \mathcal{S}_2$, we can similarly deduce that $\boldsymbol{a} \in \mathcal{S}_{2c}$. □

4 Surgeries on chain links bounding rational homology 4–balls

In this section, we will prove the necessary conditions of Theorem 1.7. Namely, we will show that the $\mathbb{Q}S^3$'s of Theorem 1.7 bound $\mathbb{Q}B^4$'s by explicitly constructing such $\mathbb{Q}B^4$'s via Kirby calculus. Notice that the necessary condition of Theorem 1.7(2) follows from the necessary condition of Theorem 1.7(1) in light of Lemma 2.3. Therefore, we need only show the following three cases (where $\boldsymbol{a}$ and $\boldsymbol{d}$ are cyclic-duals):

- If $\boldsymbol{a} \in \mathcal{S}_{1a}$, then $Y_{\boldsymbol{a}}^{-1}$ bounds a $\mathbb{Q}B^4$.
- If $\boldsymbol{a} \in \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e}$, then $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{d}}^{-1}$ bound $\mathbb{Q}B^4$'s.
- If $\boldsymbol{a} \in \mathcal{S}_2$, then $Y_{\boldsymbol{a}}^{0}$ and $Y_{\boldsymbol{d}}^{0}$ bound $\mathbb{Q}B^4$'s.

Figures 9–15 exhibit the Kirby calculus needed to produce these $\mathbb{Q}B^4$'s. We will describe in detail the $\mathbb{Q}B^4$ constructed in Figure 9, top. The constructions in the other cases are similar. Notice that the top figure of Figure 9, top (without the -1–framed blue unknot) is a surgery diagram for $Y_{\boldsymbol{a}}^{-1}$, where $\boldsymbol{a} = (b_1, \dots, b_k, 2, c_l, \dots, c_1, 2) \in \mathcal{S}_{1a}$. Thicken $Y_{\boldsymbol{a}}^{-1}$ to the 4–manifold $Y_{\boldsymbol{a}}^{-1} \times [0,1]$. By attaching a -1–framed 2–handle to $Y_{\boldsymbol{a}}^{-1} \times \{1\}$ along the blue unknot in Figure 9, top, we obtain a 2–handle cobordism from $Y_{\boldsymbol{a}}^{-1}$ to a new 3–manifold, which we will show is $S^1 \times S^2$. By performing a blowdown, we obtain the middle surgery diagram. Blowing down a second time, the surgery curves with framings $-b_1$ and $-c_1$ link each other once and have framings $-(b_1 - 1)$

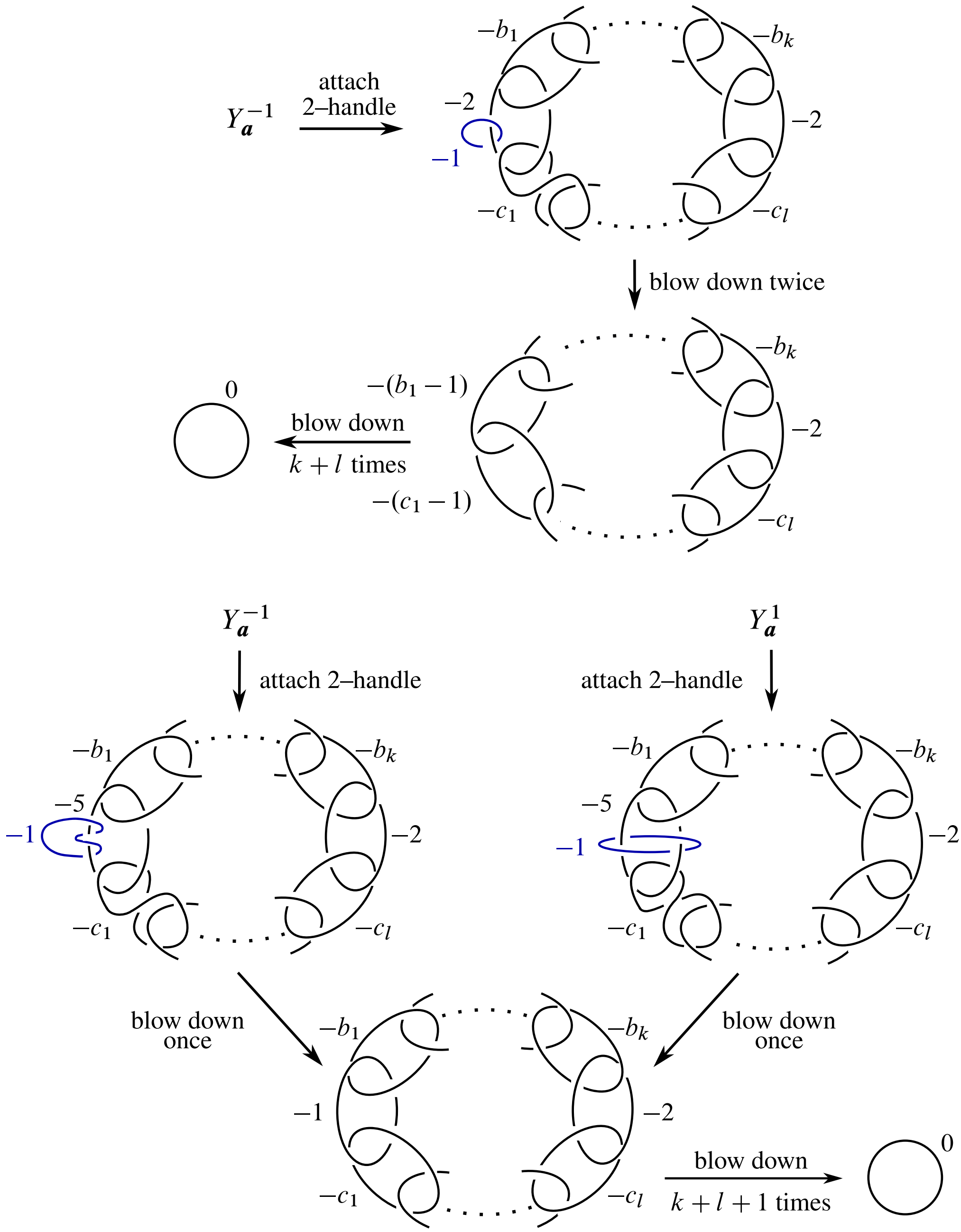

Figure 9: With Figures 10–12, we show the 3–manifolds in Theorem 1.7(1)–(2) bound rational balls. Top: if $\boldsymbol{a} \in \mathcal{S}_{1a}$, then $Y_{\boldsymbol{a}}^{-1}$ bounds a $\mathbb{Q}B^4$. Bottom: if $\boldsymbol{a} \in \mathcal{S}_{1b}$, then $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{1}$ bound $\mathbb{Q}B^4$'s.

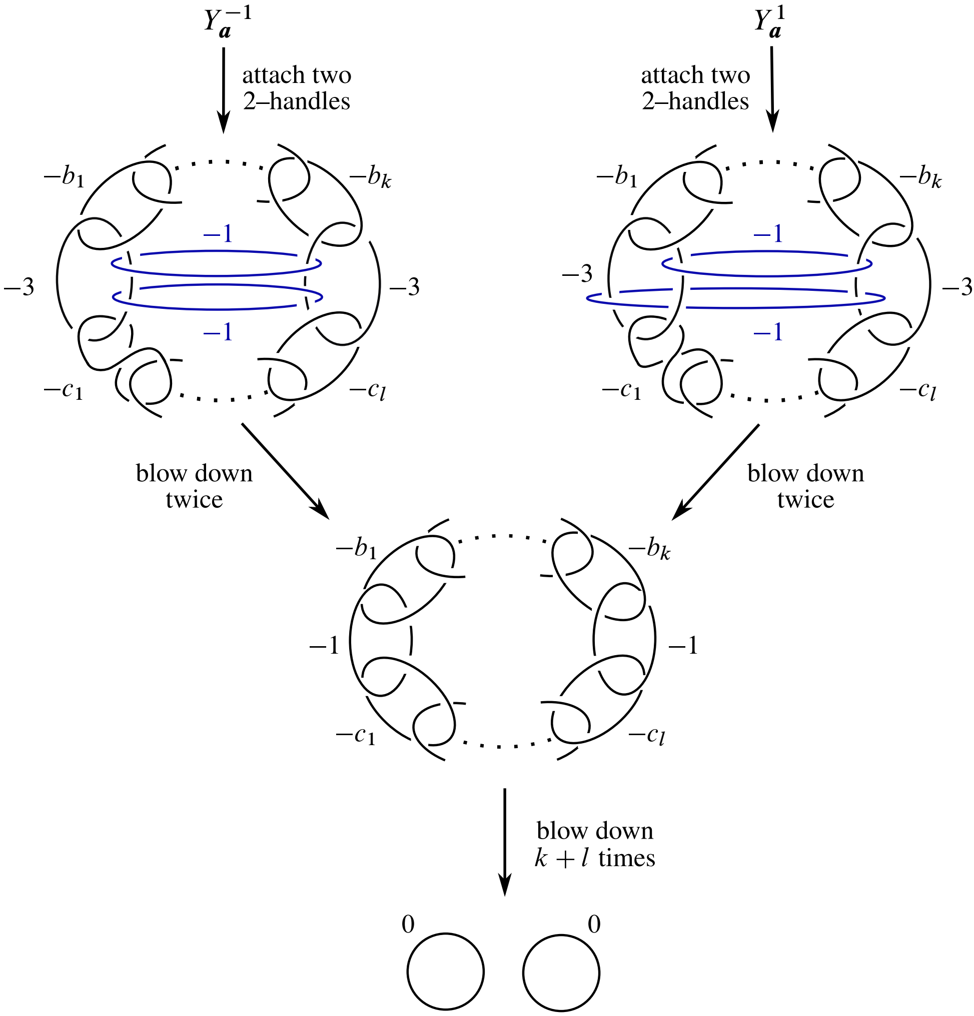

Figure 10: If $\boldsymbol{a} \in \mathcal{S}_{1c}$, then $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{1}$ bound $\mathbb{Q}B^4$'s.

and $-(c_1-1)$, respectively. Since $(b_1,\dots,b_k)$ and $(c_1,\dots,c_l)$ are linear-dual, either $-(b_1-1)$ or $-(c_1-1)$ is equal to -1. We can thus blow down again. Continuing in this way, we can continue to blow down -1–framed unknots until we obtain 0–surgery on the unknot, which is shown on the right side of the figure. Thus we have a 2–handle cobordism from $Y_{\boldsymbol{a}}^{-1}$ to $S^1\times S^2$. By gluing this cobordism to $S^1\times B^3$, we obtain the desired $\mathbb{Q}B^4$ bounded by $Y_{\boldsymbol{a}}^{-1}$.

Suppose $\boldsymbol{a} \in \mathcal{S}_{1b}\cup\mathcal{S}_{1c}\cup\mathcal{S}_{1d}\cup\mathcal{S}_{1e}$ and let $\boldsymbol{d}$ be its cyclic-dual. Then, by Lemma 2.3, $\overline{Y}_{\boldsymbol{d}}^{-1} = Y_{\boldsymbol{a}}^{1}$. To show that $Y_{\boldsymbol{d}}^{-1}$ bounds a $\mathbb{Q}B^4$, we will show that $Y_{\boldsymbol{a}}^{1}$ bounds a $\mathbb{Q}B^4$.

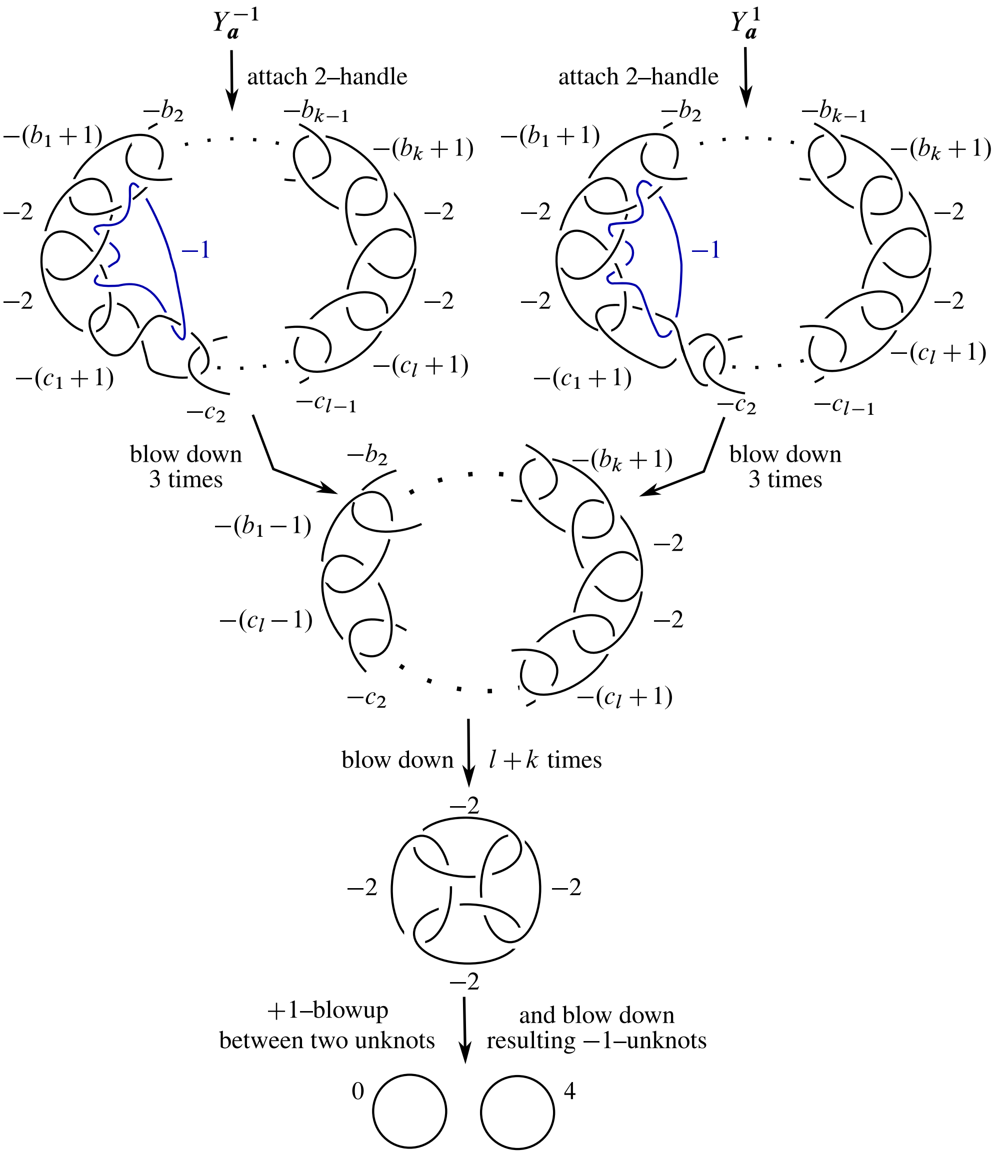

Figure 11: If $\boldsymbol{a} \in \mathcal{S}_{1d}$, then $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{1}$ bound $\mathbb{Q}B^4$'s.

Figures 9–12 show that, if $\boldsymbol{a} \in \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{2e}$, then $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{1}$ bound $\mathbb{Q}B^4$'s. Note that Figure 9, bottom, depicts a cobordism similar to the one constructed in Figure 9, top, which was described in the previous paragraph. However, the cobordisms constructed in Figures 10–12 are slightly different. In Figure 11, we have a 2–handle cobordism from $Y_{\boldsymbol{a}}^{\pm 1}$ to $S^1 \times S^2 \# L(-4,1)$, which bounds a $\mathbb{Q}S^1 \times B^3$,

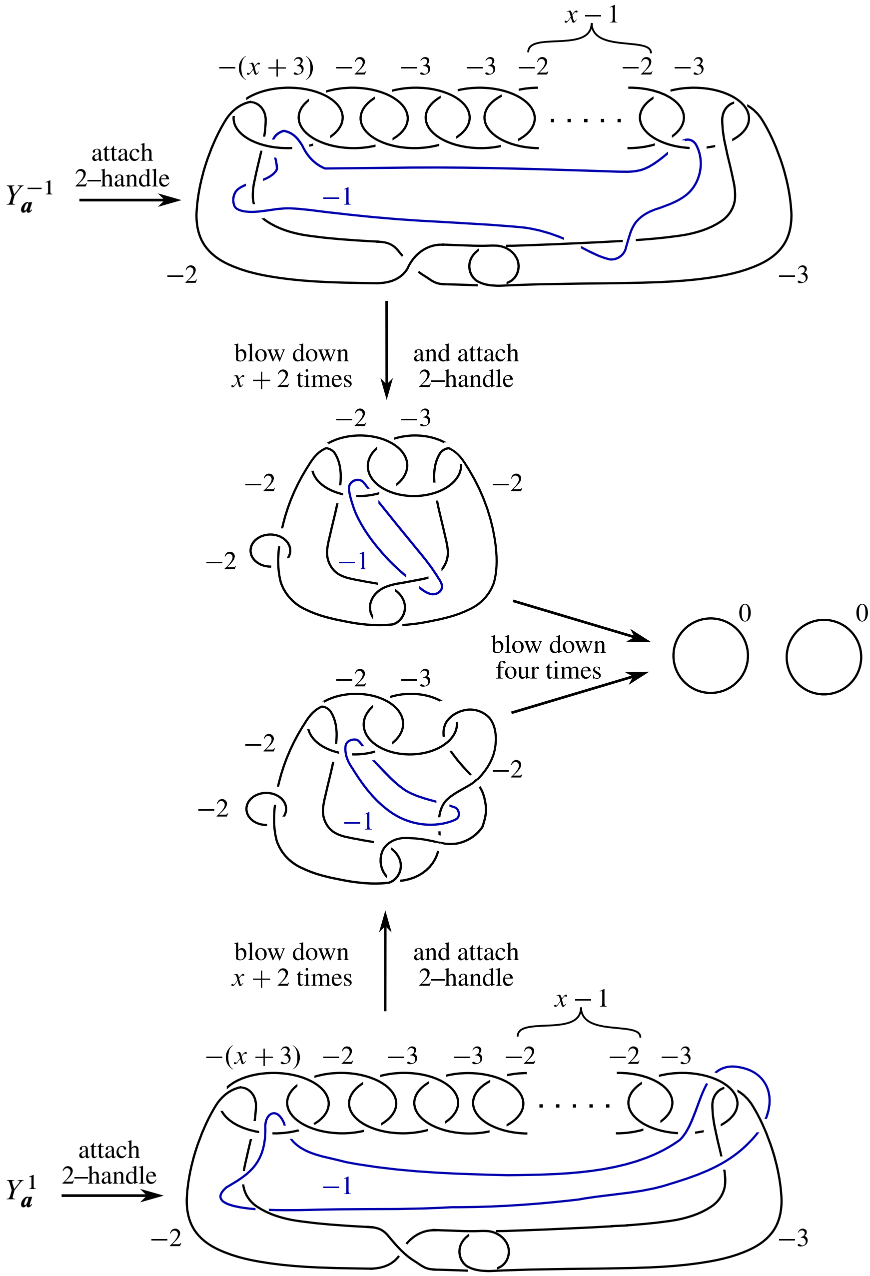

Figure 12: If $\boldsymbol{a} \in \mathcal{S}_{1e}$, then $Y_{\boldsymbol{a}}^{-1}$ and $Y_{\boldsymbol{a}}^{1}$ bound $\mathbb{Q}B^4$'s.

since $L(-4,1)$ bounds a $\mathbb{Q}B^4$ [8]. Gluing this $\mathbb{Q}S^1 \times B^3$ to the cobordism yields the desired $\mathbb{Q}B^4$. The cobordisms depicted in Figures 10 and 12 are built out of two

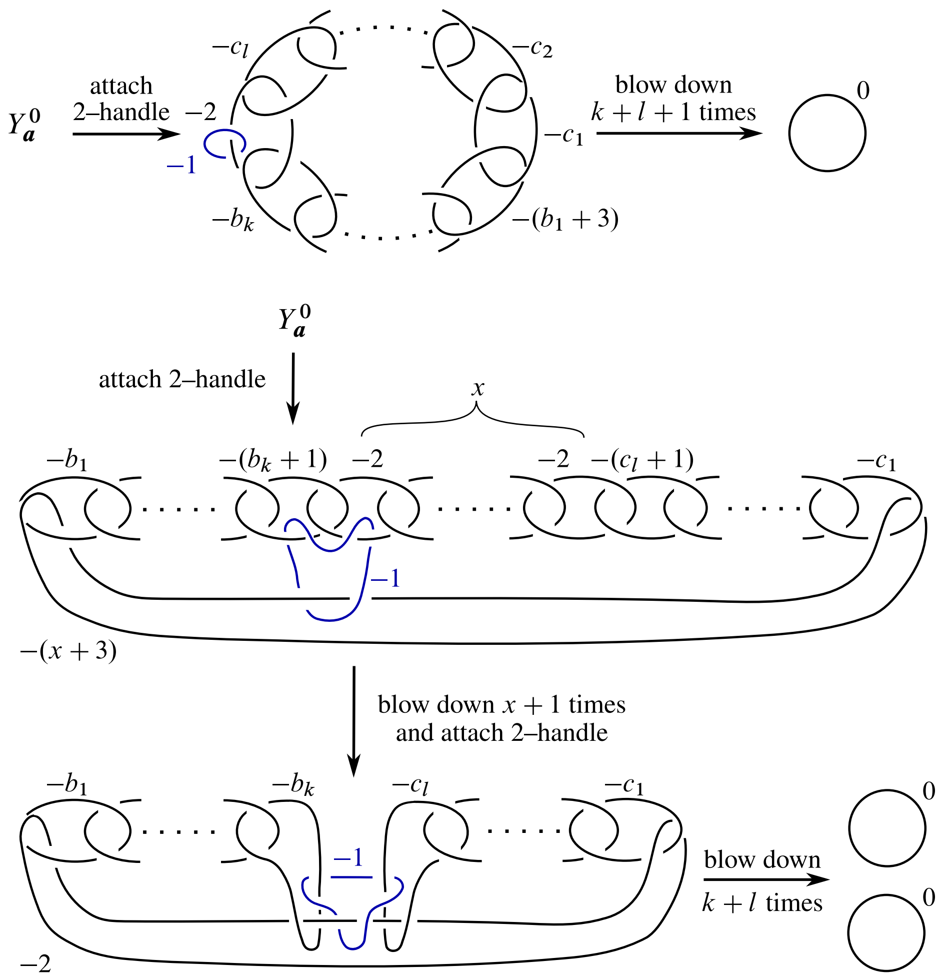

Figure 13: With Figures 14–15, we show the 3–manifolds in Theorem 1.7(3) bound rational balls Top: if $\boldsymbol{a} \in \mathcal{S}_{2a}$, then $Y_{\boldsymbol{a}}^{0}$ bounds a $\mathbb{Q}B^4$. Bottom: if $\boldsymbol{a} \in \mathcal{S}_{2b}$, then $Y_{\boldsymbol{a}}^{0}$ bounds a $\mathbb{Q}B^4$.

2–handles. These cobordisms are from $Y_{\boldsymbol{a}}^{\pm 1}$ to $S^1 \times S^2 \# S^1 \times S^2$. Gluing these cobordisms to $S^1 \times B^3 \natural S^1 \times B^3$ yields the desired $\mathbb{Q}B^4$'s.

Lastly, suppose $\boldsymbol{a} \in \mathcal{S}_2$. By Lemma 2.3, $\overline{Y}_{\boldsymbol{a}}^{0} = Y_{\boldsymbol{d}}^{0}$. Thus, once we show that $Y_{\boldsymbol{a}}^{0}$ bounds a $\mathbb{Q}B^4$, it will follow that $Y_{\boldsymbol{d}}^{0}$ also bounds a $\mathbb{Q}B^4$. Figures 13–15 show that, if $\boldsymbol{a} \in \mathcal{S}_2$, then $Y_{\boldsymbol{a}}^{0}$ bounds a $\mathbb{Q}B^4$. The $\mathbb{Q}B^4$'s in almost all of the cases are constructed in very similar ways as in the negative cases. The last case, $Y_{(2,2,2,3)}^{0}$, is much simpler; Figure 15, bottom, shows that $Y_{(2,2,2,3)}^{0} = L(-4,1)$, which bounds a $\mathbb{Q}B^4$.

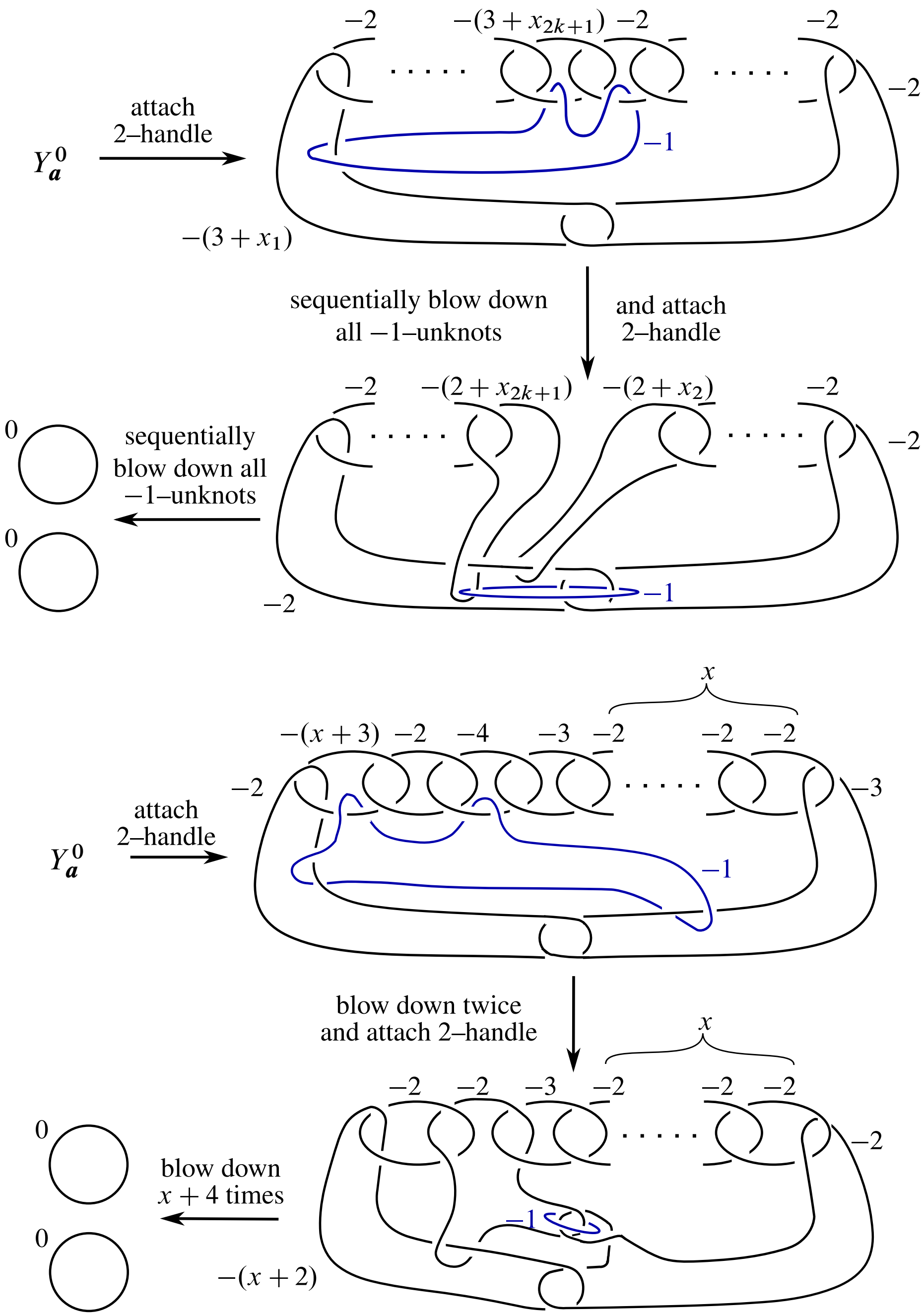

Figure 14: Top: if $\boldsymbol{a} \in \mathcal{S}_{2c}$, then $Y_{\boldsymbol{a}}^0$ bounds a $\mathbb{Q}B^4$. Bottom: if $\boldsymbol{a} \in \mathcal{S}_{2d}$, then $Y_{\boldsymbol{a}}^0$ bounds a $\mathbb{Q}B^4$.

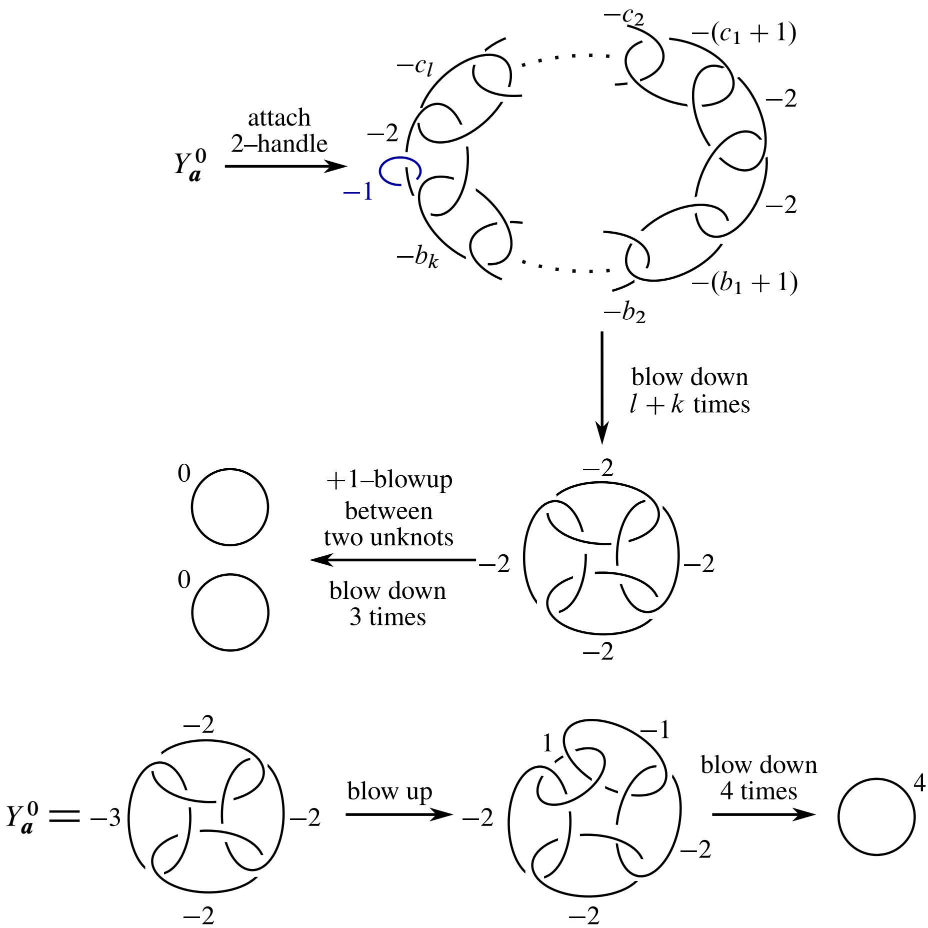

Figure 15: Top: if $\boldsymbol{a} \neq (3,2,2,2) \in \mathcal{S}_{2e}$, then $Y_{\boldsymbol{a}}^0$ bounds a $\mathbb{Q}B^4$. Bottom: if $\boldsymbol{a} = (3,2,2,2) \in \mathcal{S}_{2e}$, then $Y_{\boldsymbol{a}}^0$ bounds a $\mathbb{Q}B^4$.

As shown above, if $\boldsymbol{a} \in \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e}$, then $Y_{\boldsymbol{d}}^{-1}$ bounds a $\mathbb{Q}B^4$. However, as the next results will show, if $\boldsymbol{a} \in \mathcal{S}_{1a}$, then $Y_{\boldsymbol{d}}^{-1}$ does not necessarily bound a $\mathbb{Q}B^4$. The key is that $|H_1(Y_{\boldsymbol{a}}^{-1})|$ can be either even or odd when $\boldsymbol{a} \in \mathcal{S}_{1a}$, but, in all other cases, $H_1(Y_{\boldsymbol{a}}^{-1})$ has even order. Recall that $[b_1, \dots, b_k]$ represents the Hirzebruch–Jung continued fraction (see the appendix for details).

Proposition 4.1 *Let $\boldsymbol{a} = (b_1, \dots, b_k, 2, c_l, \dots, c_1, 2) \in \mathcal{S}_{1a}$, where $[b_1, \dots, b_k] = p/q$. Then $|H_1(Y_{\boldsymbol{a}}^{-1})| = |\mathrm{Tor}(H_1(\boldsymbol{T}_{-A(\boldsymbol{a})}))| = p^2$.*

Proof See Proposition A.3. □

Lemma 4.2 *Let $\boldsymbol{a} = (2, b_1, \dots, b_k, 2, c_l, \dots, c_1) \in \mathcal{S}_{1a}$, where $[b_1, \dots, b_k] = p/q$, and let $\boldsymbol{d} = (d_1, \dots, d_m)$ be the cyclic-dual of $\boldsymbol{a}$. If p is odd, then $Y_{\boldsymbol{d}}^{-1}$ and $Y_{\boldsymbol{a}}^1$ do not bound $\mathbb{Q}B^4$'s.*

Proof By Lemma 2.3, $\overline{Y}_{\boldsymbol{d}}^{-1} = Y_{\boldsymbol{a}}^1$, so it suffices to show that $Y_{\boldsymbol{a}}^1$ does not bound a $\mathbb{Q}B^4$. Since $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear-dual strings, it is clear that $\frac{1}{4}I(\boldsymbol{a}) = -1$ (see Remark 1.6). By the calculations in Section 2.1, $d(Y_{\boldsymbol{a}}^1, \mathfrak{s}_0) = 1 - \frac{1}{4}I(\boldsymbol{a}) = 2$. Since p is odd, by Proposition 4.1, $|H_1(Y_{\boldsymbol{a}}^1)| = |H_1(Y_{\boldsymbol{a}}^{-1})|$ has odd order and so $\mathfrak{s}_0$ extends over any $\mathbb{Q}B^4$ bounded by $Y_{\boldsymbol{a}}^1$. Thus, if $Y_{\boldsymbol{a}}^1$ bounds a $\mathbb{Q}B^4$, then $d(Y_{\boldsymbol{a}}^1, \mathfrak{s}_0) = 0$, which is not possible. □

Remark 4.3 By Lemma 1.2 and Theorem 1.1, we already know that, if $\boldsymbol{a} \in \mathcal{S}_{2c}$, then $Y_{\boldsymbol{a}}^0$ bounds a $\mathbb{Q}B^4$. However, by [13], the $\mathbb{Q}B^4$'s constructed via Theorem 1.1 necessarily admit handlebody decompositions with 3–handles. On the other hand, the $\mathbb{Q}B^4$'s constructed in this section do not contain 3–handles. Thus $Y_{\boldsymbol{a}}^0$ bounds a $\mathbb{Q}B^4$ without 3–handles, even though $\boldsymbol{T}_{A(\boldsymbol{a})}$ only bounds $\mathbb{Q}S^1 \times B^3$'s containing 3–handles.

5 Cyclic subsets

The remainder of the sections are dedicated to proving the sufficient conditions of Theorem 1.7. In fact, we will prove something more general. We will show that if t is odd and $Y_{\boldsymbol{a}}^t$ bounds a $\mathbb{Q}B^4$, then $\boldsymbol{a} \in \mathcal{S}_1 \cup \mathcal{O}$ or $\boldsymbol{d} \in \mathcal{S}_1 \cup \mathcal{O}$, and if t is even and $Y_{\boldsymbol{a}}^t$ bounds a $\mathbb{Q}B^4$, then $\boldsymbol{a} \in \mathcal{S}_2$ or $\boldsymbol{d} \in \mathcal{S}_2$. For convenience, we recall the definition of these sets.

Definition 1.4 Two strings are considered to be equivalent if one is a cyclic reordering and/or reverse of the other. Each string in the following sets is defined up to this equivalence. Moreover, strings of the form $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are assumed to be linear-dual. We define

$\mathcal{S}_{1a} = \{(b_1, \dots, b_k, 2, c_l, \dots, c_1, 2) \mid k + l \geq 3\}$,

$\mathcal{S}_{1b} = \{(b_1, \dots, b_k, 2, c_l, \dots, c_1, 5) \mid k + l \geq 2\}$,

$\mathcal{S}_{1c} = \{(b_1, \dots, b_k, 3, c_l, \dots, c_1, 3) \mid k + l \geq 2\}$,

$\mathcal{S}_{1d} = \{(2, b_1 + 1, b_2, \dots, b_{k-1}, b_k + 1, 2, 2, c_l + 1, c_{l-1}, \dots, c_2, c_1 + 1, 2) \mid k + l \geq 2\}$,

$\mathcal{S}_{1e} = \{(2, 3 + x, 2, 3, 3, 2^{[x-1]}, 3, 3) \mid x \geq 0 \text{ and } (3, 2^{[-1]}, 3) := (4)\}$,

$\mathcal{S}_{2a} = \{(b_1 + 3, b_2, \dots, b_k, 2, c_l, \dots, c_1)\}$,

$\mathcal{S}_{2b} = \{(3 + x, b_1, \dots, b_{k-1}, b_k + 1, 2^{[x]}, c_l + 1, c_{l-1}, \dots, c_1) \mid x \geq 0 \text{ and } k + l \geq 2\}$,

$$\begin{aligned}
\mathcal{S}_{2c} &= \{(b_1+1, b_2, \dots, b_{k-1}, b_k+1, c_1, \dots, c_l) \mid k+l \geq 2\},\\
\mathcal{S}_{2d} &= \{(2, 2+x, 2, 3, 2^{[x-1]}, 3, 4) \mid x \geq 0 \text{ and } (3, 2^{[-1]}, 3) := (4)\},\\
\mathcal{S}_{2e} &= \{(2, b_1+1, b_2, \dots, b_k, 2, c_l, \dots, c_2, c_1+1, 2), (2,2,2,3) \mid k+l \geq 2\},\\
\mathcal{O} &= \{(6,2,2,2,6,2,2,2), (4,2,4,2,4,2,4,2), (3,3,3,3,3,3)\},\\
\mathcal{S}_1 &= \mathcal{S}_{1a} \cup \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e},\\
\mathcal{S}_2 &= \mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2c} \cup \mathcal{S}_{2d} \cup \mathcal{S}_{2e},\\
\mathcal{S} &= \mathcal{S}_1 \cup \mathcal{S}_2,
\end{aligned}$$

Also recall, to remove the necessity of different cases, if $\boldsymbol{a} \in \mathcal{S}_{1d} \cup \mathcal{S}_{2c}$ and $k = 1$, then the substring $(b_1+1, b_2, \dots, b_{k-1}, b_k+1)$ is understood to be the substring (b_1+2).

First suppose $n = 1$ and let $\boldsymbol{a} = (a_1)$, where $a_1 \geq 3$. Then L_1^0 and L_1^{-1} are both the unknot and so $Y^0_{(a_1)} = L(a_1-2, 1)$ and $Y^{-1}_{(a_1)} = L(a_1+2, 1)$ (see Figure 2). By Lisca's classification of lens spaces that bound $\mathbb{Q}B^4$'s [8], the only such lens spaces that bound $\mathbb{Q}B^4$'s are $L(1,1) = S^3$ and $L(4,1)$. Thus $Y^{-1}_{(a_1)}$ does not bound a $\mathbb{Q}B^4$ for all $a_1 \geq 3$ and $Y^0_{(a_1)}$ bounds a $\mathbb{Q}B^4$'s if and only if $a_1 = 3$ or $a_1 = 6$. In the former case, $\boldsymbol{a} = (3) \in \mathcal{S}_{2c}$, and in the latter case, $\boldsymbol{d} = (2,2,2,3) \in \mathcal{S}_{2e}$.

We now assume the length of $\boldsymbol{a}$ is at least 2. Throughout, we will consider the standard negative definite intersection lattice $(\mathbb{Z}^n, -I_n)$. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{Z}^n$. Then, with respect to the product $\cdot$ given by $-I_n$, we have $e_i \cdot e_j = -\delta_{ij}$ for all i and j. We begin by recalling definitions and results from [8] and introducing new terminology for our purposes.

We consider two subsets $S_1, S_2 \subset \mathbb{Z}^n$ to be the same if S_2 can be obtained by applying an element of $\mathrm{Aut}(\mathbb{Z}^n)$ to S_1. Let $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ be a subset. We call each element $v_i \in S$ a *vector* and we call the string of integers $(a_1, \dots, a_n)$ defined by $a_i = -v_i \cdot v_i$ the *string associated* to S. Two vectors $z, w \in S$ are called *linked* if there exists $e \in \mathbb{Z}^n$ such that $e \cdot e = -1$ and $z \cdot e, w \cdot e \neq 0$. A subset S is called *irreducible* if, for every pair of vectors $v, w \in S$, there exists a finite sequence of vectors $v_1 = v, v_2, \dots, v_k = w \in S$ such that v_i and v_{i+1} are linked for all $1 \leq i \leq k-1$.

Definition 5.1 A subset $S = \{v_1, \dots, v_n\} \in \mathbb{Z}^n$ is

- *good* if it is irreducible and

$$v_i \cdot v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j,\\ 0 \text{ or } 1 & \text{if } |i-j| = 1,\\ 0 & \text{otherwise;} \end{cases}$$

- *standard* if
$$v_i \cdot v_j = \begin{cases} -a_i \le -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, by definition, standard subsets are good. If S is a good subset, then a vertex $v \in S$ is called *isolated* if $v \cdot w = 0$ for all $w \in S \setminus \{v\}$, *final* if there exists exactly one vertex $w \in S \setminus \{v\}$ such that $v \cdot w = 1$, and *internal* otherwise. A *component* of a good subset G is a subset of G corresponding to a connected component of the intersection graph of G (which is the graph consisting of vertices $v_1, \dots, v_n$ and an edge between two vertices v_i and v_j if and only if $v_i \cdot v_j = 1$).

Definition 5.2 A subset $S = \{v_1, \dots, v_n\} \in \mathbb{Z}^n$ is

- *negative cyclic* if either
  (1) $n = 2$ and
  $$v_i \cdot v_j = \begin{cases} -a_i \le -2 & \text{if } i = j, \\ 0 & \text{if } i \ne j, \end{cases}$$
  or
  (2) $n \ge 3$ and there is a cyclic reordering of S such that
  $$v_i \cdot v_j = \begin{cases} -a_i \le -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ -1 & \text{if } i \ne j \in \{1, n\}, \\ 0 & \text{otherwise;} \end{cases}$$
- *positive cyclic* if $-a_i \le -3$ for some i and either
  (1) $n = 2$ and
  $$v_i \cdot v_j = \begin{cases} -a_i \le -2 & \text{if } i = j, \\ 2 & \text{if } i \ne j, \end{cases}$$
  or
  (2) $n \ge 3$ and there is a cyclic reordering of S such that
  $$v_i \cdot v_j = \begin{cases} -a_i \le -2 & \text{if } i = j, \\ 1 & \text{if } |i-j| = 1, \\ 1 & \text{if } i \ne j \in \{1, n\}, \\ 0 & \text{otherwise;} \end{cases}$$
- *cyclic* if S is negative or positive cyclic.

If S is cyclic, then the indices of each vertex are understood to be defined modulo n (eg $v_{n+1} = v_1$). If $v_i \cdot v_{i+1} = \pm 1$, then we say that v_i and v_j have a *positive/negative* *intersection*. Moreover, if S is cyclic and S' is obtained from S by reversal and/or cyclic reordering, then we consider S and S' to be the same subset. In this way, associated strings of cyclic subsets are well defined up to reversal and cyclic-reordering.

Remark 5.3 By standard linear algebra, it is easy to see that, if S is good, cyclic, or the union of a good subset and a cyclic subset, then S forms a linearly independent set in $\mathbb{Z}^n$ (see [8, Remark 2.1]).

Remark 5.4 Suppose $S = \{v_1, \dots, v_n\}$ is a cyclic subset. Then, by replacing v_k with $v_k' = -v_k$, we obtain a new subset $\widehat{S} = \{v_1, \dots, v_{k-1}, v_k', v_{k+1}, \dots, v_n\}$ such that $v_{k-1} \cdot v_k' = -v_{k-1} \cdot v_k$ and $v_k' \cdot v_{k+1} = -v_k \cdot v_{k+1}$. Notice that S and $\widehat{S}$ have the same associated strings. Thus we can change the number of positive and negative intersections of S without changing the associated string. Conversely, any subset of the form $S = \{v_1, \dots, v_n\}$, where $n \geq 3$ and

$$v_i \cdot v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j, \\ \pm 1 & \text{if } |i-j| = 1, \\ \pm 1 & \text{if } i \neq j \in \{1, n\}, \\ 0 & \text{otherwise}, \end{cases}$$

can modified into a positive or negative cyclic subset by changing the signs of select vertices. In particular, for any negative cyclic subset, the negative intersection can be moved at will by negating select vertices.

Similarly, any irreducible subset of the form $G = \{v_1, \dots, v_n\}$, where

$$v_i \cdot v_j = \begin{cases} -a_i \leq -2 & \text{if } i = j, \\ \pm 1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise}, \end{cases}$$

can be modified into a good subset by changing the signs of select vertices. In Section 7, we will often create such subsets and assume that they are good, without specifying the need to possibly negate select vertices first.

Definition 5.5 Let $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ be a subset with $v_i \cdot v_i = -a_i$. We define

$$I(S) := \sum_{i=1}^{n} (a_i - 3), \qquad E_i^S := \{j : v_j \cdot e_i \neq 0\},$$
$$p_i(S) := \big|\{j : |E_j^S| = i\}\big|, \quad V_i^S := \{j : v_i \cdot e_j \neq 0\}.$$

In some cases we will drop the superscript S from the above notation if the subset being considered is understood.

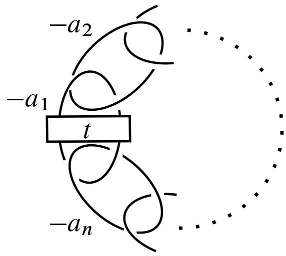

Figure 16: A 4–manifold P^t with boundary $Y^t_{\boldsymbol{a}}$.

Remark 5.6 Lisca [8] classified all standard subsets of $\mathbb{Z}^n$ with $I(S) < 0$. The results in the next three sections rely in part on his classification of standard subsets. We will review his classification in Section 5.1.

Example 5.7 The subset $S = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n + e_1\} \subset \mathbb{Z}^n$ for $n \geq 2$ is a negative cyclic subset with associated string $(2^{[n]})$. Moreover, $I(S) = -n$, $p_2(S) = n$, and $p_j(S) = 0$ for all $j \neq 2$. When $n = 4$, there is an alternative subset with associated string $(2, 2, 2, 2)$, namely $S' = \{e_1 - e_2, e_2 - e_3, -e_2 - e_1, e_1 + e_4\}$, which satisfies $p_1(S') = p_3(S') = 2$. This latter subset will be used to construct the family strings in $\mathcal{S}_{1a}$.

Let $\boldsymbol{a} = (a_1, \dots, a_n)$. The rational sphere $Y^t_{\boldsymbol{a}}$ is the boundary of the negative definite 2–handlebody P^t whose handlebody diagram is given in Figure 16. Let Q_{P^t} denote the intersection form of P^t. Note that Q_{P^t} depends only on the parity of t. Further suppose $Y^t_{\boldsymbol{a}}$ bounds a rational homology ball B. Then the closed 4–manifold $X^t = P^t \cup B$ is negative definite. By Donaldson's diagonalization theorem [6], the intersection lattice $(H_2(X^t), Q_{X^t})$ is isomorphic to the standard negative definite lattice $(\mathbb{Z}^n, -I_n)$. Thus the intersection lattice $(H_2(P^t), Q_{P^t})$ must embed in $(\mathbb{Z}^n, -I_n)$. The existence of such an embedding implies the existence of a cyclic subset $S \subset \mathbb{Z}^n$ with associated string $(a_1, \dots, a_n)$. Thus our goal is to classify all cyclic subsets of $\mathbb{Z}^n$, where $n \geq 2$.

Recall that, by reversing the orientation of $Y^t_{\boldsymbol{a}}$, we obtain the $\overline{Y}^t_{\boldsymbol{a}} = Y^{-t}_{\boldsymbol{d}}$, where $\boldsymbol{d} = (d_1, \dots, d_m)$ is the cyclic-dual of $(a_1, \dots, a_n)$ (Section 2.2). In particular, $(a_1, \dots, a_n)$ is of the form $(2^{[m_1]}, 3 + n_1, \dots, 2^{[m_k]}, 3 + n_k)$ if and only if $(d_1, \dots, d_m)$ is of the form $(3 + m_1, 2^{[n_1]}, \dots, 3 + m_k, 2^{[n_k]})$. If S and $\overline{S}$ denote the cyclic subsets associated to $(a_1, \dots, a_n)$ and $(d_1, \dots, d_m)$, respectively, then $I(S) + I(\overline{S}) = 0$. Now, since $Y^t_{\boldsymbol{a}}$ bounds a $\mathbb{Q}B^4$ if and only if $Y^{-t}_{\boldsymbol{d}}$ bounds a $\mathbb{Q}B^4$, we will focus our attention on subsets satisfying $I(S) \leq 0$. The following theorem is the main result of our lattice analysis:

Theorem 5.8 *Let S be a cyclic subset such that $I(S) \leq 0$. Then S is either negative with associated string in $\mathcal{S}_1 \cup \mathcal{O} \cup \{(2^{[n]}) \mid n \geq 2\}$ or positive with associated string in $\mathcal{S}_2$.*

Proof The theorem follows from Example 5.7 and Propositions 6.5, 7.5 and 7.14, which will be proven in Sections 6 and 7. □

We can now prove Theorem 1.7, which we recall here for convenience.

Theorem 1.7 *Let $\boldsymbol{a} = (a_1, \dots, a_n)$, where $n \geq 1$, $a_i \geq 2$ for all i, and $a_j \geq 3$ for some j, and let $\boldsymbol{d}$ be the cyclic-dual of $\boldsymbol{a}$.*

(1) *Suppose $\boldsymbol{d} \notin \mathcal{S}_{1a} \cup \mathcal{O}$. Then $Y_{\boldsymbol{a}}^{-1}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{a} \in \mathcal{S}_1$ or $\boldsymbol{d} \in \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e}$.*

(2) *Suppose $\boldsymbol{a} \notin \mathcal{S}_{1a} \cup \mathcal{O}$. Then $Y_{\boldsymbol{a}}^{1}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{d} \in \mathcal{S}_1$ or $\boldsymbol{a} \in \mathcal{S}_{1b} \cup \mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e}$.*

(3) *$Y_{\boldsymbol{a}}^{0}$ bounds a $\mathbb{Q}B^4$ if and only if $\boldsymbol{a} \in \mathcal{S}_2$ or $\boldsymbol{d} \in \mathcal{S}_2$.*

Proof The sufficient conditions of Theorem 1.7 follow from the calculations in Section 4. The necessary conditions of Theorem 1.7 follow from Theorem 5.8 and the fact that $Y_{\boldsymbol{a}}^{t}$ bounds a $\mathbb{Q}B^4$ if and only if $Y_{\boldsymbol{d}}^{-t}$ bounds a $\mathbb{Q}B^4$. □

The proof of Theorem 5.8 will span the next three sections. The proof will begin in earnest in Section 6. The proof applies two strategies. The first will be to reduce certain cyclic subsets to good subsets and standard subsets and appeal to Lisca's work [8; 9]. The second will be to reduce certain cyclic subsets (via *contractions*) to a small list of base cases. In the upcoming subsection, we will recall Lisca's classification of standard subsets. In the subsequent subsection, we will describe how to perform contractions and list the relevant base cases. In the final subsection, we will prove a few preliminary lemmas that will be useful going forward.

5.1 Lisca's standard and good subsets

In Section 7, we will construct good subsets and standard subsets satisfying $I < 0$ from cyclic subsets, thus reducing the problem of classifying certain cyclic subsets to Lisca's work [8; 9]. In this section, we collect relevant results proved by Lisca. The first two propositions can be found in [8, Sections 3–7]. In particular, the "moreover" statements in Proposition 5.10 are obtained by examining the proofs of [8, Lemmas 7.1–7.3].

Proposition 5.9 *Let $T = \{v_1, \dots, v_n\}$ be a standard subset with $I(T) < 0$. Then:*

(1) $I(T) \in \{-1, -2, -3\}$.

(2) $|v_i \cdot e_j| \le 1$ *for all* i *and* j.

(3) $p_1(T) = 1$ *if and only if* $I(T) = -3$ *and, if* $p_1(T) = 0$, *then* $p_2(T) > 0$.

(4) *If* $I(T) = -3$, *then* $p_1(T) = p_2(T) = 1$ *and* $p_3(T) = n - 2$.

(5) *If* $I(T) = -2$, *then* $p_2(T) = 3$, $p_4(T) = 1$, *and* $p_3(T) = n - 4$.

(6) *If* $I(T) = -1$, *then* $p_2(T) = 2$, $p_4(T) = 1$ *and* $p_3(T) = n - 3$.

Proposition 5.10 *Let T be standard with $I(T) < 0$. Let $x, y \ge 0$.*

(1) *If $I(T) = -3$, then, if $E_i = \{s\}$, then v_s is internal (ie $1 < s < n$) and $v_s \cdot v_s = -2$; if $|E_j| = 2$, then $E_j = \{1, n\}$; either $v_1 \cdot v_1 = -2$ or $v_n \cdot v_n = -2$; and $v_1 \cdot e_j = -v_n \cdot e_j$. Moreover, T has associated string of the form $(b_1, \dots, b_k, 2, c_l, \dots, c_1)$, where $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear-dual strings.*

(2) *If $I(T) = -2$, then (up to reversal) T has associated string of the form*

(a) $(2^{[x]}, 3, 2 + y, 2 + x, 3, 2^{[y]})$,

(b) $(2^{[x]}, 3 + y, 2, 2 + x, 3, 2^{[y]})$, *or*

(c) $(b_1, \dots, b_{k-1}, b_k + 1, 2, 2, c_l + 1, c_{l-1}, \dots, c_1)$, *where the strings $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear-dual.*

Moreover, up to the action of $\mathrm{Aut}(\mathbb{Z}^n)$, the corresponding embeddings are of the form

(a) $$\Big\{e_{x+4} - e_{x+3}, e_{x+3} - e_{x+2}, \dots, e_5 - e_4, e_4 - e_2 - e_3, e_2 + e_1 + \sum_{\alpha=x+5}^{x+y+4} e_i, -e_2 - e_4 - \sum_{\alpha=5}^{x+4} e_i, e_2 - e_1 - e_3, e_1 - e_{x+5}, e_{x+5} - e_{x+6}, \dots, e_{x+y+3} - e_{x+y+4}\Big\},$$

(b) $$\Big\{e_{x+4} - e_{x+3}, e_{x+3} - e_{x+2}, \dots, e_5 - e_4, e_4 - e_2 - e_3 - \sum_{\alpha=x+5}^{x+y+4} e_i, e_2 + e_1, -e_2 - e_4 - \sum_{\alpha=5}^{x+4} e_i, e_2 - e_1 - e_3, e_3 - e_{x+5}, e_{x+5} - e_{x+6}, \dots, e_{x+y+3} - e_{x+y+4}\Big\},$$

(c) $\{u_1, \dots, u_{k-1}, u_k + e_4 - e_2 - e_3, e_2 + e_1, -e_2 - e_4, e_2 - e_1 - e_3 + w_1, w_2, \dots, w_l\}$, *where $k+l \geq 3$, $u_k = 0$ or $w_1 = 0$, $|E_1| = |E_4| = 2$. Furthermore (up to reversal), in* (c) *we may assume that $u_1^2 = -2$; consequently, there exist integers j_1 and j_2 such that $|E_{j_1}| = 2$, $|E_{j_2}| = 3$, $u_1 \cdot e_{j_2} = -u_2 \cdot e_{j_2} = -w_l \cdot e_{j_2} = 1$, and $|u_1 \cdot e_{j_2}| = |w_l \cdot e_{j_2}| = 1$.*

(3) *If $I(T) = -1$, then (up to reversal) T has associated string of the form*

- (a) $(2+x, 2+y, 3, 2^{[x]}, 4, 2^{[y]})$,
- (b) $(2+x, 2, 3+y, 2^{[x]}, 4, 2^{[y]})$, *or*
- (c) $(3+x, 2, 3+y, 3, 2^{[x]}, 3, 2^{[y]})$.

Moreover, up to the action of $\mathrm{Aut}(\mathbb{Z}^n)$, *the corresponding embeddings are of the form*

(a) $$\Big\{e_2 + e_4 + \sum_{\alpha=5}^{x+4} e_\alpha, e_1 - e_2 + \sum_{\alpha=x+5}^{x+y+4} e_\alpha, e_2 - e_3 - e_4, e_4 - e_5, e_5 - e_6, \dots, e_{x+3} - e_{x+4}, e_{x+4} - e_1 - e_2 - e_3, e_1 - e_{x+5}, e_{x+5} - e_{x+6}, \dots, e_{x+y+3} - e_{x+y+4}\Big\},$$

(b) $$\Big\{e_2 + e_4 + \sum_{\alpha=5}^{x+4} e_\alpha, e_1 - e_2, e_2 - e_3 - e_4 - \sum_{\alpha=x+5}^{x+y+4} e_\alpha, e_4 - e_5, \dots, e_{x+3} - e_{x+4}, e_{x+4} - e_1 - e_2 - e_3, e_3 - e_{x+5}, e_{x+5} - e_{x+6}, \dots, e_{x+y+3} - e_{x+y+4}\Big\},$$

(c) $$\Big\{e_1 - e_2 - e_5 - \sum_{\alpha=6}^{x+5} e_\alpha, e_2 + e_3, -e_2 - e_1 - e_4 - \sum_{\alpha=x+6}^{x+y+5} e_\alpha, -e_5 + e_2 - e_3, e_5 - e_6, e_6 - e_7, \dots, e_{x+4} - e_{x+5}, e_{x+5} + e_1 - e_4, e_4 - e_{x+6}, e_{x+6} - e_{x+7}, \dots, e_{x+y+4} - e_{x+y+5}\Big\}.$$

The next proposition follows from the first case (S irreducible) of the proof of the main theorem in [9, page 2160ff] and [8, Lemma 6.2] (see also [1, Lemma 6.6]). See [8, Definition 4.1] for the definition of *bad component*.

Proposition 5.11 [9] *Let $G \subset \mathbb{Z}^n$ be a good subset with two components and $I(G) \leq -2$. If G has no bad components, then $I(G) = -2$ and G has associated string of the form $(b_1, \dots, b_k) \cup (c_1, \dots, c_l)$, where $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear-dual strings. Moreover, if $G = \{v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}\}$, where $-v_i^2 = b_i$ for $1 \leq i \leq k$ and $-v_{k+j}^2 = c_j$ for all $1 \leq j \leq l$, then there exist integers α and β such that $E_\alpha = \{1, k+1\}$ and $E_\beta = \{k, k+l\}$.*

5.2 Contractions, expansions and base cases

In this section, we discuss how to reduce the length of certain cyclic subsets via contractions.

Definition 5.12 Suppose $S=\{v_1,\dots,v_n\}$ with $n\geq 3$ is a cyclic subset and suppose there exist integers i, s and t such that $E_i=\{s,\tilde{s},t\}$, where $\tilde{s}\in\{s\pm 1\}$, $V_{\tilde{s}}\cap V_s=\{i\}$, $|v_u\cdot e_i|=1$ for all $u\in E_i$, and $a_t\geq 3$. After possibly cyclically reordering and reindexing S, we may assume that $s\notin\{1,n\}$. Let $S'\subset\mathbb{Z}^{n-1}=\langle e_1,\dots,e_{i-1},e_{i+1},\dots,e_n\rangle$ be the subset defined by

$$S'=(S\setminus\{v_s,v_{\tilde{s}},v_t\})\cup\{v_s+v_{\tilde{s}},\pi_{e_i}(v_t)\},$$

where $\pi_{e_i}(v_t)=v_t+(v_t\cdot e_i)e_i$. We say that S' is obtained from S by a *contraction* and S is obtained from S' by an *expansion*.

Since $s\notin\{1,n\}$ and $|v_{\tilde{s}}\cdot e_i|=|v_s\cdot e_i|=1$, we have $v_{s-1}\cdot e_i=-v_s\cdot e_i$. Thus

$$(v_s+v_{\tilde{s}})\cdot v_u=\begin{cases}1 & \text{if } \tilde{s}=s+1 \text{ and } u\in\{s-1,s+2\},\\ 1 & \text{if } \tilde{s}=s-1 \text{ and } u\in\{s-2,s+1\},\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, $(\pi_{e_i}(v_t))^2=v_t^2+1\leq -2$ and

$$\pi_{e_i}(v_t)\cdot v_u=\begin{cases}1 & \text{if } u=t\pm 1,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, S' is a positive/negative cyclic subset if and only if S is positive/negative cyclic. Moreover, $I(S')=I(S)$, $p_j(S')=p_j(S)$ for all $j\neq 3$, and $p_3(S')=p_3(S)-1$.

Definition 5.13 Using the notation above, if $v_t\cdot v_s=1$ (so that $t=s\pm 1$ if $\tilde{s}=s\mp 1$) and $a_{\tilde{s}}=2$, then we say

- v_s is the *center of S relative to e_i*,
- S' is obtained by a *contraction of S centered at v_s*, and
- S is obtained by a -2*–expansion of S*.

Note that a subset obtained by a contraction of S centered at v_s is unique. Indeed, if $E_i=\{s-1,s,s+1\}$, $a_{s-1}=2$ and $a_{s+1}\geq 3$, then $V_{s-1}\cap V_s=\{i\}$ and the only contraction centered at v_s is $S\setminus\{v_s,v_{s-1},v_{s+1}\}\cup\{v_{s-1}+v_s,\pi_{e_i}(v_{s+1})\}$. Similarly, if $E_i=\{s-1,s,s+1\}$, $a_{s-1}=2$ and $a_{s+1}\geq 3$, then $V_{s-1}\cap V_s=\{i\}$ and the only contraction centered at v_s is $S \setminus \{v_s, v_{s-1}, v_{s+1}\} \cup \{v_s + v_{s+1}, \pi_{e_i}(v_{s-1})\}$. Now let S have associated string $(a_1, \dots, a_n)$. Then, under the contraction centered at v_s, the associated string changes via

$$(a_1, \dots, a_{s-2}, 2, \boldsymbol{a}_s, a_{s+1}, a_{s+2}, \dots, a_n) \to (a_1, \dots, a_{s-2}, \boldsymbol{a}_s, a_{s+1} - 1, a_{s+2}, \dots, a_n)$$

or

$$(a_1, \dots, a_{s-2}, a_{s-1}, \boldsymbol{a}_s, 2, a_{s+2}, \dots, a_n) \to (a_1, \dots, a_{s-2}, a_{s-1} - 1, \boldsymbol{a}_s, a_{s+1}, \dots, a_n).$$

Notice that two strings $(b_1, \dots, b_k)$ and $(c_l, \dots, c_1)$ are reverse linear-dual if and only if $(b_1, \dots, b_{k-1})$ and $(c_l - 1, \dots, c_1)$ or $(b_1, \dots, b_k - 1)$ and $(c_{l-1}, \dots, c_1)$ are reverse linear-dual. Thus the substrings on either side of a_s in the associated string of S are reverse linear-dual if and only if the substrings on either side of a_s in the associated string of the contraction of S centered at v_s are reverse linear-dual.

More generally, let $S = \{v_1, \dots, v_n\}$ and consider a sequence of contractions $S^0 = S$, S^1, $S^2, \dots, S^m$ such that S^k is obtained from S^{k-1} by performing a contraction centered at $v_s^{(k-1)} \in S^{k-1}$, where $v_s^{(0)} = v_s$. We call such a sequence of contractions *the sequence of contractions centered at* v_s and call the reverse sequence of expansions *a sequence of* -2*–expansions centered at* $v_s^{(m)}$. Notice that, for all $1 \le k \le m$, $v_s^{(k)} = v_s^{(k-1)} + v_{\tilde{s}}^{(k-1)}$, where $v_{\tilde{s}}^{(k-1)}$ is the unique vertex of S^{k-1} adjacent to $v_s^{(k-1)}$ with square -2. We have proven the following:

Lemma 5.14 *Let S' be obtained from S by a sequence of contractions centered at v and let $v^2 = -a$. Then S has associated string of the form $(b_1, \dots, b_k, a, c_l, \dots, c_1)$, where $(b_1, \dots, b_k)$ and $(c_l, \dots, c_1)$ are reverse linear-dual, if and only if S' has associated string of the form $(b'_1, \dots, b'_{k'}, a, c'_{l'}, \dots, c'_1)$, where $(b'_1, \dots, b'_{k'})$ and $(c'_{l'}, \dots, c'_1)$ are reverse linear-dual.*

When $I(S) \le 0$ and either $p_1(S) > 0$ or $p_1(S) = p_2(S) = 0$, we will be able to sequentially perform contractions until we arrive at certain base cases. In light of Example 5.7, we will restrict our attention to cyclic subsets containing at least one vector with square at most -3. We will now list all such cyclic subsets of length 2 and 3 with $I(S) \le 0$. It can be concretely checked case by case that the only such cyclic subsets are positive and (up to the action of $\mathrm{Aut}(\mathbb{Z}^2)$) are of the form

- $\{2e_1, -e_1 + e_2\}$, which has associated string $(4, 2) \in \mathcal{S}_{2a}$;
- $\{2e_1 - e_3, e_3 + e_2, -e_1 - e_3\}$, which has associated string $(5, 2, 2) \in \mathcal{S}_{2a}$; and
- $\{e_1 - e_2 - e_3, e_3 - e_1 - e_2, e_2 - e_3 - e_1\}$, which has associated string $(3, 3, 3) \in \mathcal{S}_{2c}$.

Notice that the second and third vertices of the subset with associated string $(5,2,2)$ are both centers relative to e_3. If we perform a contraction centered at either vertex relative to e_3, we obtain the subset with associated string $(4,2)$. Note that, when $n=3$, centers are not unique, but when $n \geq 4$, centers are necessarily unique.

Remark 5.15 We will usually denote cyclic subsets by S, standard subsets by T, and good subsets by G. Moreover, S' will be reserved for contractions of S.

5.3 Preliminary lemmas

The following lemmas will be important in future sections. The first follows from the proof of [8, Lemma 2.5].

Lemma 5.16 [8, Lemma 2.5] *If $S=\{v_1,\dots,v_n\} \subset \mathbb{Z}^n = \langle e_1,\dots,e_n\rangle$ is any subset, then*

$$2p_1(S)+p_2(S)+I(S) \geq \sum_{j=4}^{n}(j-3)p_j(S),$$

with equality if and only if $|v_\alpha \cdot e_\beta| \leq 1$ for all $1 \leq \alpha, \beta \leq n$.

Lemma 5.17 *Let S be cyclic and such that $p_2(S)>0$ and $|v_\alpha \cdot e_\beta| \leq 1$ for all $1 \leq \alpha,\beta \leq n$. Then $\sum_i p_{2i}(S) \equiv -I(S) \bmod 4$.*

Proof First notice that, since $I(S)=\sum_{i=1}^n (a_i-3)$, we have $\sum_{i=1}^n a_i = 3n+I(S)$. Now

$$-\Big(\sum_{i=1}^n v_i\Big)^2 = \sum_{i=1}^n a_i - \sum_{i=1}^{n-1} 2v_i \cdot v_{i+1} - 2v_1 \cdot v_n = \begin{cases} n+I(S) & \text{if } S \text{ is positive,} \\ n+4+I(S) & \text{if } S \text{ is negative.} \end{cases}$$

On the other hand, set $\sum_{i=1}^n v_i = \sum_{i=1}^n \lambda_i e_i$ and let $k_\alpha = \big|\{i : |\lambda_i| = 2\alpha+1\}\big|$ and $x_\beta = \big|\{i : |\lambda_i| = 2\beta\}\big|$. Finally, let $m \in \mathbb{Z}$ be the largest integer such that $k_m \neq 0$ and $k_t = 0$ for all $t>m$, and let $y \in \mathbb{Z}$ be the largest integer such that $x_y \neq 0$ and $x_t=0$ for all $t>y$. Since $|v_\alpha \cdot e_\beta| \leq 1$ for all α and β, we have $\sum_i p_{2i}(S) = x_0+\cdots+x_y$. Hence,

$$\begin{aligned} -\Big(\sum_{i=1}^n v_i\Big)^2 \\ = -\sum_{i=1}^n \lambda_i^2 = \Big(n - \Big(\sum_{\alpha=1}^m k_\alpha\Big) - \Big(\sum_{\beta=0}^y x_\beta\Big)\Big) + \sum_{\alpha=1}^m (2\alpha+1)^2 k_\alpha + \sum_{\beta=0}^y (2\beta)^2 x_\beta \end{aligned}$$

$$= n + \sum_{\alpha=1}^{m}(4\alpha^2+4\alpha)k_\alpha + \sum_{\beta=0}^{y}(4\beta^2-1)x_\beta$$
$$= n + \sum_{\alpha=1}^{m}(4\alpha^2+4\alpha)k_\alpha + \sum_{\beta=0}^{y}(4\beta^2)x_\beta - \Bigl(\sum_i p_{2i}(S)\Bigr).$$

Thus,

$$\sum_{\alpha=1}^{m}(4\alpha^2+4\alpha)k_\alpha + \sum_{\beta=1}^{y}(4\beta^2)x_\beta = \begin{cases} \sum_i p_{2i}(S) + I(S) & \text{if } S \text{ is positive,} \\ \sum_i p_{2i}(S) + 4 + I(S) & \text{if } S \text{ is negative.} \end{cases}$$

It follows that $\sum_i p_{2i}(S) \equiv -I(S) \bmod 4$. □

Lemma 5.18 *If $G = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ is a good subset with $I(G) = 0$, $p_3(G) = n$, and n components, then, up to the action of* Aut $\mathbb{Z}^n$, *negating vertices, and permuting vertices,*

- $G = \{e_1 - e_2 + e_3 - e_4, e_1 + e_2, -e_1 + e_2 + e_3 - e_4, e_3 + e_4\}$ *with associated string* $(4, 2, 4, 2)$, *or*
- $G = \{e_1 - e_2 - e_3, e_1 + e_2 - e_4, e_2 - e_3 + e_4, e_1 + e_3 + e_4\}$ *with associated string* $(3, 3, 3, 3)$.

Proof First notice that, by Lemma 5.16, $|v_\alpha \cdot e_\beta| \le 1$ for all α and β. Let i, s, t and u be integers such that $E_i = \{s, t, u\}$. Since every vertex of G is isolated, up to negating vertices we may assume that $v_s \cdot e_i = v_t \cdot e_i = v_u \cdot e_i = -1$.

First suppose $a_s = 2$ and let $v_s = e_i + e_j$. Then, since $v_s \cdot v_t = v_s \cdot v_u = 0$, we have $v_t = e_i - e_j + a$ and $v_u = e_i - e_j + b$. Since $v_t \cdot v_u = 0$, there are integers $k, l \in V_t \cap V_u$ such that $v_t = e_i - e_j + e_k - e_l + a'$ and $v_u = e_i - e_j - e_k + e_l + b'$. If $(a')^2 \ne 0$, then let $R = \{v'_1, \dots, v'_{s-1}, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-2} = \langle e_1, \dots, e_n\rangle / \langle e_i, e_j\rangle$, where $v'_t = \pi_{e_j}(\pi_{e_i}(v_t))$, $v'_u = \pi_{e_j}(\pi_{e_i}(v_u))$, and $v'_x := v_x$ for all $x \notin \{t, u\}$. Then $(v'_t)^2 \le -3$, $v'_t \cdot v'_u = 2$, and $v'_t \cdot v_x = v'_u \cdot v'_x = 0$ for all $x \notin \{t, u\}$. Consequently, R is the union of a positive cyclic subset $\{v'_t, v'_u\}$ and a good subset $R \setminus \{v'_t, v'_u\}$. Thus, by Remark 5.3, R is a linearly independent set of $n-1$ vectors in $\mathbb{Z}^{n-2}$, which is impossible. Thus $(a')^2 = 0$ and, similarly, $(b')^2 = 0$; hence, $v_t = e_i - e_j + e_k - e_l$ and $v_u = e_i - e_j - e_k + e_l$. Now, since $|E_k| = |E_l| = 3$, there exists an integer z such that $k, l \in V_z$ and, since $v_z \cdot v_t = 0$, we may assume that $v_z = e_k + e_l + c$. By a similar argument as above, $c^2 = 0$ and so $v_z = e_k + e_l$. Since G is irreducible, it follows that

$n = 4$ and so G has associated string of the form $(4, 2, 4, 2)$. Setting $i = 3$, $j = 4$, $k = 1$ and $l = 2$, we have the subset listed in the statement of the lemma.

Next suppose $a_s, a_t, a_u \geq 3$. Assume $a_s > 3$. Let $R = \{v'_1, \dots, v'_{s-1}, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n \rangle / \langle e_i \rangle$, where $v'_s = \pi_{e_i}(v_s)$, $v'_t = \pi_{e_i}(v_t)$, $v'_u = \pi_{e_i}(v_u)$, and $v'_x := v_x$ for all $x \notin \{s, t, u\}$. Then $(v'_s)^2 < -2$ and $v'_s \cdot v'_t = v'_s \cdot v'_u = v'_t \cdot v'_u = 1$; hence, $\{v'_s, v'_t, v'_u\}$ is a positive cyclic subset. Moreover, $v'_s \cdot v'_x = v'_t \cdot v'_x = v'_u \cdot v'_x = 0$ for all $x \notin \{s, t, u\}$. Thus R is the union of a positive cyclic subset and a good subset and so, by Remark 5.3, R is a linearly independent set of $n-1$ vectors in $\mathbb{Z}^{n-2}$, which is impossible. Thus $a_s = 3$; similarly, $a_t = a_u = 3$. Without loss of generality, $v_s = e_i - e_j - e_k$, $v_t = e_i + e_j - e_l$ and $v_u = e_i + e_k + e_l$ for some integers j, k and l. Since $|E_j| = 3$, there exists an integer z such that $j \in V_z$. Since $v_z \cdot v_s = v_z \cdot v_t = v_z \cdot v_u = 0$, we have $v_z = e_j - e_k + e_l + a$. If $a^2 \neq 0$, then we can define a subset R as above and arrive at a similar contradiction. Thus $v_z = e_j - e_k - e_l$. Since G is irreducible, it follows that $n = 4$ and so G has associated string of the form $(3, 3, 3, 3)$. Setting $i = 1$, $j = 2$, $k = 3$ and $l = 4$, we have the subset listed in the statement of the lemma. □

6 Lattice analysis, case I: $p_1(S) > 0$

Throughout this section, we will assume that $S = \{v_1, \dots, v_n\}$ is a cyclic subset with $I(S) \leq 0$ and $p_1(S) > 0$. Thus there exist integers i and s such that $E_i = \{s\}$. Lemmas 6.1–6.3 will ensure that we can contract such subsets.

Lemma 6.1 *Let S be a cyclic subset of length 4 such that $I(S) \leq 0$ and $E_i = \{s\}$ for some integers i and s. If $a_{s+1} \geq 3$ or $a_{s-1} \geq 3$, then S is positive and has associated string of the form $(6, 2, 2, 2)$ or $(5, 2, 2, 3)$. If $a_{s\pm1} = 2$, then S is either negative and has associated string of the form $(2, 2, 2, 2)$ or $(2, 2, 2, 5)$, or positive and has associated string of the form $(2, 2, 2, 3)$ or $(2, 2, 2, 6)$.*

Proof If $|V_s| = 1$, then, since $E_i = \{s\}$, we obtain $v_s \cdot v_{s+1} = 0$, which is a contradiction. Thus $|V_s| \geq 2$.

Suppose $a_{s-1} \geq 3$. If $|V_s| \geq 3$, then let $R \subset \mathbb{Z}^3$ be the subset obtained by replacing v_s by $v_s + (v_s \cdot e_i)e_i$. Then R is a cyclic subset and, by Remark 5.3, R is made of four linearly independent vectors in $\mathbb{Z}^3$, which is not possible. Thus $|V_s| = 2$. Let $V_s = \{i, j\}$. Then $E_j = \{s-1, s, s+1\}$, since otherwise we would necessarily have that $|E_i| > 1$. Moreover, since $V_{s-1} \cap V_s = V_{s+1} \cap V_s = \{j\}$, we necessarily have that $|v_{s-1} \cdot e_j| = |v_s \cdot e_j| = |v_{s+1} \cdot e_j| = 1$. If S is positive cyclic, then it is clear that $v_{s-1} \cdot e_j = v_{s+1} \cdot e_j = -v_s \cdot e_j$. If S is negative cyclic, then, by possibly moving the negative intersection (see Remark 5.4), we may assume that $v_{s-1} \cdot e_j = v_{s+1} \cdot e_j = -v_s \cdot e_j$. Thus we may perform a contraction of S centered at v_s relative to e_j to obtain a length 3 cyclic subset S' with $I(S') = I(S) \leq 0$ and $p_1(S') > 0$. By considering the base cases in Section 5.2, it is clear that $S' = \{2e_1 - e_3, e_3 + e_2, -e_1 - e_3\}$ (up to the action of $\mathrm{Aut}(\mathbb{Z}^3)$), which has associated string $(5,2,2)$. Thus $i = 2$, $j = 4$, and either $S = \{2e_1 - e_3 - e_4, e_2 + e_4, -e_4 + e_3, -e_1 - e_3\}$ or $S = \{2e_1 - e_3, e_3 - e_4, e_4 + e_2, -e_4 - e_1 - e_3\}$. Therefore, S is positive and has associated string $(6,2,2,2)$ or $(5,2,2,3)$.

Now suppose $a_{s-1} = a_{s+1} = 2$. Without loss of generality, assume $s = j = 4$. Let $T = \{v_1, v_2, v_3\} \subset \mathbb{Z}^3 = \langle e_1, e_2, e_3 \rangle$ be the length 3 standard subset obtained by removing v_s from S. Then T has associated string of the form $(2, a_2, 2)$. Since $I(S) \leq 0$, we must have $a_2 \leq 6$. It is easy to see that $a_2 \neq 6$, since otherwise $v_2 = 2e_1 - e_2 - e_3$ (up to the action of $\mathrm{Aut}(\mathbb{Z}^3)$), implying that $v_1 \cdot v_2 \neq \pm 1$, which is a contradiction. If $a_2 = 5$, then T is of the form $\{e_1 - e_2, e_2 + 2e_3, -e_2 - e_1\}$ and therefore S must be of the form $\{e_1 - e_2, e_2 + 2e_3, -e_2 - e_1, e_1 + e_4\}$ (up to the action of $\mathrm{Aut}(\mathbb{Z}^3)$). Thus S is negative with associated string $(2,5,2,2)$ (equivalently $(2,2,2,5)$). If $a_2 \leq 4$, then $I(T) < 0$. By Proposition 5.10, the only such length 3 standard subset has associated string $(2,2,2)$. Moreover, T is of the form $T = \{e_1 - e_2, e_2 - e_3, -e_2 + e_1\}$ (see [8, Lemma 2.4]). Since $v_3 \cdot v_4 = \pm 1$, either $1 \in V_4^S$, $2 \in V_4^S$, or both. If $1, 2 \in V_4^S$, then since $v_2 \cdot v_4 = 0$, we must have $3 \in V_4^S$; thus $|V_4^S| = 4$. Moreover, since $v_1 \cdot v_4 = \pm 1$, we must have that $v_4 \cdot e_1 = v_4 \cdot e_2 \pm 1$, implying that $a_4 \geq 7$, which is not possible. Thus either $1 \in V_4^S$ or $2 \in V_4^S$, but not both. If $1 \in V_4^S$, then S is negative and of the form $\{e_1 - e_2, e_2 - e_3, -e_2 - e_1, e_1 + e_4\}$ or $\{e_1 - e_2, e_2 - e_3, -e_2 - e_1, e_1 + 2e_4\}$, which have associated strings $(2,2,2,2)$ and $(2,2,2,5)$ (note that we found the latter subset above). If $2 \in V_4^S$, then $3 \in V_4^S$ and S is positive and of the form $\{e_1 - e_2, e_2 - e_3, -e_2 - e_1, e_2 + e_3 + e_4\}$ or $\{e_1 - e_2, e_2 - e_3, -e_2 - e_1, e_2 + e_3 + 2e_4\}$, which have associated strings $(2,2,2,3)$ and $(2,2,2,6)$. □

Lemma 6.2 *Let S be a cyclic subset of length at least 5 such that $E_i = \{s\}$ for some i and s. Then $|V_s| = 2$. Moreover, if $V_s = \{i, j\}$, then $E_j = \{s-1, s, s+1\}$ and $v_{s-1} \cdot e_j = v_{s+1} \cdot e_j = -v_s \cdot e_j = \pm 1$.*

Proof First note that, if $|V_s| = 1$, then, since $E_i = \{s\}$, we obtain $v_s \cdot v_{s+1} = 0$, which is a contradiction. Now suppose $|V_s| \geq 3$. Then, by replacing v_s with $v_s' = v_s + (v_s \cdot e_i) e_i$

and relabeling $v'_u = v_u$ for all $u \neq s$, we obtain a subset

$$R = \{v'_1, \dots, v'_{s-1}, v'_s, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n\rangle.$$

Let $(a'_1, \dots, a'_n)$ be the string associated to R, where $-a'_s := v'_s \cdot v'_s \leq -2$ and $a'_j = a_j$ for all $j \neq i$. If S is negative cyclic, then so is R and thus, by Remark 5.3, R is made of n linearly independent vectors in $\mathbb{Z}^{n-1}$, which is not possible. If S is positive cyclic and either $a'_s \geq 3$ or $a_i \geq 3$ for some $i \neq s$, then R is also positive cyclic, and we obtain a similar contradiction. Now suppose S is positive cyclic, $a'_s = 2$ and $a'_t = a_t = 2$ for all $t \neq s$. Let T be the subset obtained by removing v_s from S. Then T has associated string $(2^{[n-1]})$ and so $I(T) = -(n-1) \leq -4$. If $|E_k^S| \geq 2$ for all $k \in V_s^S$, where $k \neq i$, then T is a standard subset of $\mathbb{Z}^{n-1}$ with $I(T) \leq -4$, which contradicts Proposition 5.9. If $|E_k^S| = 1$ for some $k \in V_s^S$ such that $k \neq i$, then, by Remark 5.3, T consists of $n-1$ linearly independent vectors in $\mathbb{Z}^m$, where $m < n-1$, which is not possible. Thus $|V_s| = 2$. Let $V_s^S = \{i, j\}$. Then, as in the proof of Lemma 6.1, $E_j = \{s-1, s, s+1\}$ and $v_{s-1} \cdot e_j = v_{s+1} \cdot e_j = -v_s \cdot e_j = \pm 1$. □

Lemma 6.3 *Let S be a cyclic subset of length at least 5 such that $I(S) \leq 0$ and $E_i = \{s\}$ for some i and s. Then either $a_{s-1} \geq 3$ or $a_{s+1} \geq 3$. Moreover, if $a_{s\pm 1} \geq 3$, then S is positive with associated string $(2,3,2,3,2)$ or $(2,3,5,3,2)$.*

Proof By Lemma 6.2, $V_s = \{i, j\}$ and $E_j = \{s-1, s, s+1\}$. Assume that $a_{s-1} = a_{s+1} = 2$. Then $V_{s-1} = \{j, k\}$ for some k, $V_{s+1} = \{j, k'\}$ for some k', and $|v_{s\pm 1} \cdot e_j| = |v_{s-1} \cdot e_k| = |v_{s+1} \cdot e_{k'}| = 1$. Since $v_{s-1} \cdot v_{s+1} = 0$, we must have $k = k'$. Since $|v_{s-2} \cdot v_{s-1}| = 1$ and $j \notin V_{s-2}$, we must have $k \in V_{s-2}$. But then $v_{s-2} \cdot v_{s+1} \neq 0$, which is a contradiction.

Now suppose $a_{s-1}, a_{s+1} \geq 3$ and let R be the subset obtained by removing v_s and replacing $v_{s\pm 1}$ with $v'_{s\pm 1} = v_{s\pm 1} + (v_{s\pm 1} \cdot e_j) \cdot e_j$. Note that $v'_{s-1} \cdot v'_{s+1} = \pm 1$. As in the proof of Lemma 6.2, either R is cyclic or S is positive cyclic and R has associated string of the form $(2^{[n-1]})$. In the former case, by Remark 5.3, $R \subset \mathbb{Z}^{n-2}$ contains $n-1$ linearly independent vectors, which is not possible. In the latter case, let $T \subset \mathbb{Z}^{n-1}$ be the standard subset obtained from S by only removing v_s. Then T has associated string $(3, 2, \dots, 2, 3)$. By Proposition 5.10, the only such standard subset is $\{e_4 + e_3 - e_2, e_2 + e_1, -e_2 - e_4, e_2 + e_3 - e_1\}$ (up to the action of $\mathrm{Aut}(\mathbb{Z}^4)$), which has associated string $(3,2,2,3)$. Thus $j = 3$, $|v_s \cdot e_3| = 1$. Since $I(S) \leq 0$, S is of the form $\{-e_2 - e_4, e_2 + e_3 - e_1, e_5 - e_3, e_4 + e_3 - e_2, e_2 + e_1\}$ or $\{-e_2 - e_4, e_2 + e_3 - e_1, 2e_5 - e_3, e_4 + e_3 - e_2, e_2 + e_1\}$, which are positive and have associated strings $(2,3,2,3,2)$ and $(2,3,5,3,2)$, respectively. □

Let $S=\{v_1,\dots,v_n\}$ be a cyclic subset such that $n\geq 6$, $I(S)\leq 0$ and $E_i^S=\{s\}$ for some integers i and s. By Lemma 6.2, we may assume that $V_s^S=\{i,j\}$ and $E_j^S=\{s-1,s,s+1\}$ for some integer j. Thus v_s is the center vertex of S relative to e_j. By Lemma 6.3, we may further assume that $a_{s+1}\geq 3$ and $a_{s-1}=2$ and so $V_{s-1}^S=\{j,j_1\}$ for some integer j_1. Let $S'=\{v_1',\dots,v_{s-2}',v_s',v_{s+1}',\dots,v_n'\}$ be the contraction of S centered at v_s, where $v_x'=v_x$ for all $x\notin\{s-1,s,s+1\}$, $v_s'=v_{s-1}+v_s$, and $v_{s+1}'=\pi_{e_j}(v_t)$. Since $V_s^{S'}=\{i,j_1\}$ and $E_{j_1}^{S'}=\{s-2,s,s+1\}$, v_s' is the center vertex of S' relative to e_{j_1} and, by Lemma 6.3, either $(v_{s-2}')^2\leq -3$ or $(v_{s+1}')^2\leq -3$. If $(v_{s-2}')^2\leq -3$ and $(v_{s+1}')^2\leq -3$, then, by Lemma 6.3, S' is positive and has associated string of the form $(2,3,2,3,2)$ or $(2,3,5,3,2)$. If $(v_{s-2}')^2=-2$ or $(v_{s+1}')^2=-2$, then we can perform the contraction centered at v_s' relative to e_{j_1}, as above. Continuing in this way, we have a sequence of contractions centered at v_s, which ends in a subset $\widehat{S}$ either of length 4 or of length 5 with associated string $(2,3,2,3,2)$ or $(2,3,5,3,2)$. Let $\hat{v}_s$ denote the resulting center vertex of $\widehat{S}$. Then $V_s^{\widehat{S}}=\{i,k\}$ for some integer k and $|E_k^{\widehat{S}}|=3$.

Suppose that $\widehat{S}$ has length 4. By considering the length 4 cyclic subsets in the proof of Lemma 6.1, it is clear that $\widehat{S}$ is either negative and of the form

- $\{e_1-e_2,e_2-e_3,-e_2-e_1,e_1+e_4\}$ with associated string $(2,\mathbf{2},2,\mathbf{2})$, or
- $\{e_1-e_2,e_2-e_3,-e_2-e_1,e_1+2e_4\}$ with associated string $(2,\mathbf{2},2,\mathbf{5})$;

or positive and of the form

- $S=\{2e_1-e_3-e_4,e_2+e_4,-e_4+e_3,-e_1-e_3\}$ with associated string $(6,\mathbf{2},2,2)$, or
- $S=\{2e_1-e_3,e_3-e_4,e_4+e_2,-e_4-e_1-e_3\}$ with associated string $(5,2,\mathbf{2},3)$.

Each bold number in the above strings corresponds to a vertex $\hat{v}_m$ satisfying $E_\alpha^{\widehat{S}}=\{m\}$ for some integers α and m. In particular, one of the bold numbers in each of the above strings corresponds to $\hat{v}_s$. In the first two cases, notice that the substrings between the bold numbers (ie (2) and (2)) are reverse linear-dual. Thus, by Lemma 5.14, S has associated string of the form $(b_1,\dots,b_k,2,c_l,\dots,c_1,2)$ or $(b_1,\dots,b_k,2,c_l,\dots,c_1,5)$, where $(b_1,\dots,b_k)$ and $(c_l,\dots,c_1)$ are reverse linear-dual. Similarly, the third and fourth strings are of the form $(b_1+3,b_2,\dots,b_k,2,c_l,\dots,c_1)$, where $(b_1,\dots,b_k)$ and $(c_l,\dots,c_1)$ are reverse linear-dual, and so S has associated string of the same form. Note that the strings $(5,2,2)$ and $(4,2)$ also fall under this family (recall that the linear-dual of (1) is the empty string).

Now suppose $\widehat{S}$ has length 5. Then, by the proof of Lemma 6.3, $\widehat{S}$ is positive and of the form

- $\{-e_2-e_4, e_2+e_3-e_1, e_5-e_3, e_4+e_3-e_2, e_2+e_1\}$ with associated string $(2,3,\mathbf{2},3,2)$, or
- $\{-e_2-e_4, e_2+e_3-e_1, 2e_5-e_3, e_4+e_3-e_2, e_2+e_1\}$ with associated string $(2,3,\mathbf{5},3,2)$.

As above, the bold numbers in these two strings correspond to the vertex $\hat{v}_s$. Notice that, after performing a -2–expansion centered at $\hat{v}_s$, the first and last entries in each string remain unchanged. Moreover, the substrings adjacent to the bold numbers are (3) and (3); notice $(3-1)=(2)$ and $(3-1)=(2)$ are reverse linear-dual strings. Thus, as above, S has associated string of the form $(2, b_1+1, b_2, \dots, b_k, 2, c_l, \dots, c_2, c_1+1, 2)$ or $(2, b_1+1, b_2, \dots, b_k, 5, c_l, \dots, c_2, c_1+1, 2)$, where $(b_1, \dots, b_k)$ and $(c_l, \dots, c_1)$ are reverse linear-dual strings.

Remark 6.4 Consider the length 5 subsets above. We can perform contractions to obtain the cyclic subsets of Lemma 6.1 with associated strings $(2,2,2,3)$ and $(2,2,2,6)$. However, these do not fall under the general formulas listed above. Moreover, the string $(2,2,2,6)$ is also the associated string of a different subset, as seen in Lemma 6.1. This string already appeared in first set of cases we considered and so we will not count this string again.

Combining all of these cases, we have proven the following:

Proposition 6.5 *Let S be a cyclic subset with $I(S) \le 0$ and $p_1(S) > 0$. Then S is either negative with associated string in $\mathcal{S}_{1a} \cup \mathcal{S}_{1b}$, or positive with associated string in $\mathcal{S}_{2a} \cup \mathcal{S}_{2b} \cup \mathcal{S}_{2e}$.*

7 Lattice analysis, case II: $p_1(S) = 0$

In this section, we will assume that $S = \{v_1, \dots, v_n\}$ is cyclic with $I(S) \le 0$ and $p_1(S) = 0$. By Lemma 5.16, $p_2(S) \ge \sum_{j=4}^{n} (j-3) p_j(S)$. If $p_2(S) = 0$, then the inequality is necessarily an equality and so $p_j(S) = 0$ for all $4 \le j \le n$. Thus, in this case, $I(S) = 0$ and $p_3(S) = n$. Therefore, we have two cases to consider: $p_2(S) = 0$ and $p_2(S) > 0$.

7.1 Case IIa

Let S be cyclic and $p_1(S) = p_2(S) = 0$. Then, as shown above, $I(S) = 0$ and $p_3(S) = n$. The next two lemmas provide some general properties of S.

Lemma 7.1 *If S is cyclic and $p_1(S) = p_2(S) = 0$, then $|v_\alpha \cdot e_\beta| \leq 1$ for all $1 \leq \alpha, \beta \leq n$.*

Proof Let $v_i = \sum_{j=1}^n m_{ij} e_j$ for each i, where $m_{ij} = v_i \cdot e_j$. Then, since $I(S) = 0$, we have $3n = -\sum_{i=1}^n v_i^2 = \sum_{i,j} m_{ij}^2 \geq \sum_{i,j} |m_{ij}| \geq 3n$. Thus $m_{ij}^2 = |m_{ij}|$ for all i and j and so $|v_i \cdot e_j| = |m_{ij}| \leq 1$ for all i and j. □

Lemma 7.2 *If S is cyclic and $p_1(S) = p_2(S) = 0$, then S is positive cyclic.*

Proof Again, let $v_i = \sum_{j=1}^n m_{ij} e_j$. By Lemma 7.1, $|m_{ij}| \leq 1$ for all i and j. Let $\sum_{i=1}^n v_i = \sum_{i=1}^n \lambda_i e_i$. Then, since $p_3(S) = n$, $\lambda_i \in \{\pm 1, \pm 3\}$ for all i. Now, if S is negative, then $-3n = \sum_{i=1}^n v_i^2 = \left(\sum_{i=1}^n v_i\right)^2 - 2\sum_{i<j} v_i \cdot v_j = \left(-\sum_{i=1}^n \lambda_i^2\right) - 2(n-2)$ or $\sum_{i=1}^n \lambda_i^2 = n+4$. Thus there must exist j such that $\lambda_j = \pm 3$. But then $n - 1 \leq \sum_{i \neq j} \lambda_i^2 = n-5$, which is impossible. Thus S must be positive. □

If $p_3(S) = n$, then it is clear that $n \geq 3$. If $n = 3$, then S is the subset with associated string $(3,3,3) \in \mathcal{S}_{2b} \cap \mathcal{S}_{2c}$ found in Section 5.2. From now on, we will assume that $n \geq 4$.

Lemma 7.3 *Let S be cyclic with $p_1(S) = p_2(S) = I(S) = 0$. Suppose there exist integers i and s such that $E_i = \{s-1, s, s+1\}$. Then S is positive and has associated string in $\mathcal{S}_{2b}$.*

Proof By Lemma 7.2, we know that S is necessarily positive. Now, since $E_i = \{s-1, s, s+1\}$, we necessarily have that $a_s \geq 3$; otherwise, if $a_s = 2$ and $V_s = \{i, i'\}$, then $|E_{i'}| = 1$, which is a contradiction. We further claim that $a_{s-1} \geq 3$ or $a_{s+1} \geq 3$. Suppose otherwise: $a_{s-1} = a_{s+1} = 2$. Then $V_{s-1} = V_{s+1} = \{i, j\}$ for some integer j and, since $|E_i| = 3$, we necessarily have that $j \in V_{s-2} \cap V_{s+2}$. Since $|E_j| = 3$, we necessarily have that $n = 4$. But then there exists an integer $k \in V_s$ such that either $|E_k| = 1$ or $|E_k| = 2$, which is a contradiction. Without loss of generality, assume that $a_{s-1} \geq 3$.

First assume that $v_{s-1} \cdot e_i = v_s \cdot e_i$ (or similarly $v_{s+1} \cdot e_i = v_s \cdot e_i$). Let $x \geq 0$ be the smallest integer such that $a_{s+x+1} \geq 3$. Since $a_{s+1} = \cdots = a_{s+x} = 2$, we have $V_{s+\alpha} = \{i_{\alpha-1}, i_\alpha\}$ for all $1 \leq \alpha \leq x$, where $i_0 := i$ and $\{i_0, \ldots, i_x\}$ contains $x+1$ distinct integers.

Moreover, $E_{i_\alpha}=\{s-1,s+\alpha,s+\alpha+1\}$ for all $1\le\alpha\le x$. Since $v_{s-1}\cdot e_i=v_s\cdot e_i$, by Lemmas 7.1 and 7.2, there exist integers $m,k\in V_{s-1}\cap V_s$ such that $v_{s-1}\cdot e_m=-v_s\cdot e_m$ and $v_{s-1}\cdot e_k=-v_s\cdot e_k$. Thus $a_{s-1}\ge x+3$. Let $R=\{v_1',\dots,v_{s-1}',v_{s+x+1}',\dots,v_n\}\subset\mathbb{Z}^{n-x-1}=\langle e_1,\dots,e_n\rangle/\langle e_{i_0},\dots,e_{i_x}\rangle$, where $v_{s-1}'=\pi_{e_{i_0}}\bigl(\pi_{e_{i_1}}(\cdots(\pi_{e_{i_x}}(v_{s-1}))\cdots)\bigr)$, $v_{s+x+1}'=\pi_{e_{i_x}}(v_{s+x+1})$, and $v_y'=v_y$ for all $y\notin\{s-1,\dots,s+x+1\}$. Then R is negative cyclic with $I(R)=1-a_s\le-2$. By Proposition 7.14 in Section 7.2, R must have associated string in $\mathcal{S}_{1c}\cup\mathcal{S}_{1d}\cup\mathcal{S}_{1e}\cup\mathcal{O}\cup\{(2^{[n]})\mid n\ge2\}$ and hence either $I(R)=-(n-x-1)$ or $I(R)=-2$. In the former case, we necessarily have that $a_{s-1}=3+x$, $a_s=n+x$, and $a_{s+x+1}=3$; hence S has associated string of the form $(3+x,n+x,2^{[x]},3,2^{[n-x-3]})\in\mathcal{S}_{2b}$. In the latter case, $a_s=3$ and so $V_s^S\cap V_{s-1}^S=\{i,m,k\}$. Since $v_s^2=-3$, it follows that $V_m^S=V_k^S=\{s-1,s,z\}$ for some integer $z\notin\{s-1,s\}$. It is easy to see that $v_{s-1}^2\le-(4+x)$ and $\tilde v_z^2\le-3$. Let $T=(S\setminus\{v_z,v_s,v_{s-1}\})\cup\{\pi_{e_k}(v_s),\pi_{e_m}(\pi_{e_k}(v_{s-1}))\}$. Then T is standard with $I(T)\le-3$ and $E_m^T=\{s\}$. By Proposition 5.9, $I(T)=-3$ and so $v_z^2=-3$; by Proposition 5.10(1), T has associated string of the form $(b_1,\dots,b_k,2,c_l,\dots,c_1)$, where $(b_1,\dots,b_k)$ and $(c_1,\dots,c_l)$ are linear-dual strings and the middle vertex with square -2 is $\pi_{e_k}(v_s)$. Thus S has associated string $(3,b_1,\dots,b_k+2,3,c_l,\dots,c_1)$. Since $(\beta_1,\dots,\beta_\kappa)=(b_1,\dots,b_k+1)$ has linear-dual $(\gamma_1,\dots,\gamma_\lambda)=(2,c_1,\dots,c_l)$ (see Lemma 3.6), we have

$$(3,b_1,\dots,b_k+2,3,c_l,\dots,c_1)=(3,\beta_1,\dots,\beta_{\kappa-1},\beta_\kappa+1,\gamma_\lambda+1,\gamma_{l-1},\dots,\gamma_1)\in\mathcal{S}_{2b}.$$

Now assume that $v_{s-1}\cdot e_i=-v_s\cdot e_i=v_{s+1}\cdot e_i$. Suppose $a_{s+1}=2$ and set $V_{s+1}=\{i,j\}$. Note that $E_j=\{s-1,s+1,s+2\}$ and $V_s\cap V_{s+1}=\{i\}$. Thus v_s is the center of S relative to e_i. Perform the contraction of S centered at v_s to obtain the positive cyclic subset $S'=\{v_1',\dots,v_s',v_{s+2}',\dots,v_n'\}$, where $v_x'=v_x$ for all $x\notin\{s-1,s,s+1\}$, $v_s'=v_s+v_{s+1}$, and $v_{s-1}'=\pi_{e_i}(v_{s-1})$. Then $I(S')=0$ and $p_3(S')=n-1$. Now the vertices v_{s-1}', v_s', and v_{s+2}' are adjacent in S', $E_j^{S'}=\{s-1,s,s+2\}$, and $(v_s')^2=v_s^2\le-3$. Thus v_s' is the center of S' relative to e_j. Moreover, $v_{s-2}'\cdot e_j=-v_s'\cdot e_j=v_{s+2}'\cdot e_j$. If $(v_{s-2}')^2=-2$ or $(v_{s+1}')^2=-2$, then we can contract S' centered at v_s'. Continuing in this way, we have a sequence of contractions centered at v_s which terminates in a positive subset $\widetilde S$ such that the resulting center vertex $\tilde v_s$ has adjacent vertices whose squares are both at most -3. Reindex $\widetilde S$ chronologically and let $u=s$ under the new indexing. Then $\tilde v_u^2=v_s^2\le-3$, $\tilde v_{u\pm1}^2\le-3$, and there is an integer l such that $E_l^{\widetilde S}=\{u-1,u,u+1\}$ and $\tilde v_{u-1}\cdot e_l=-\tilde v_u\cdot e_l=\tilde v_{u+1}\cdot e_l$. Note that, if $a_{s+1}\ge3$, then $\widetilde S=S$. Let C be the subset obtained from $\widetilde S$ by removing $\tilde v_u$, replacing $\tilde v_{u\pm1}$ with $\tilde v_{u\pm1}'=\pi_{e_l}(\tilde v_{u\pm1})$, and setting $\tilde v_x'=\tilde v_x$ for all $x\notin\{u-1,u,u+1\}$. Then $I(C)\le-2$, $p_1(C)=0$, $p_2(C)>0$, and

$\tilde{v}_{u-1}\cdot\tilde{v}_{u+1}=1$. If there exists a vertex of C with square at most -3, then C is a positive cyclic subset. However, by Proposition 7.14 in Section 7.2, positive cyclic subsets with $p_1=0$ and $p_2>0$ have associated strings in $\mathcal{S}_{2c}\cup\mathcal{S}_{2d}$ and thus have $I\in\{-1,0\}$. Since $I(C)\le -2$, every vertex of C must have square equal to -2 and so $\widetilde{S}$ has associated string of the form $(3+x,3,2^{[x]},3)$, where $-(\tilde{v}_u)^2=3+x$. Notice that $(3-1)=(2)$ and $(3-1)=(2)$ are reverse linear-dual strings. Thus, by Lemma 5.14, S has associated string of the form $(3+x,b_1,\dots,b_{k-1},b_k+1,2^{[x]},c_l+1,c_{l-1},\dots,c_1)\in\mathcal{S}_{2b}$, where $(b_1,\dots,b_k)$ and $(c_1,\dots,c_l)$ are linear-dual strings. □

Lemma 7.4 *Let S be a cyclic subset with $p_1(S)=p_2(S)=I(S)=0$. Suppose that, for all $1\le i\le n$, $E_i\ne\{s-1,s,s+1\}$ for some integer s. Then S is positive with associated string in $\mathcal{S}_{2c}$.*

Proof Let s be an integer such that $a_s\ge 3$. Let i be an integer such that $v_s\cdot e_i=-v_{s+1}\cdot e_i$, which exists by Lemmas 7.1 and 7.2. Finally, let $E_i=\{s-1,s,t\}$. By assumption, $t\notin\{s-2,s+1\}$. Let $x\ge 0$ be the smallest integer such that $a_{s+x+1}\ge 3$. Since $a_{s+1}=\cdots=a_{s+x}=2$, we have $V_{s+\alpha}=\{i_{\alpha-1},i_\alpha\}$ for all $1\le\alpha\le x$, where $i_0:=i$ and $\{i_0,\dots,i_x\}$ contains $x+1$ distinct integers. Since $i\in V_t$ and $v_t\cdot v_{s+\alpha}=0$ for all $1\le\alpha\le x-1$, we have $i_0,\dots,i_{x-1}\in V_t$. If $t=s+x+1$, then it is clear that $i_x\notin V_t$ and so $|E_{i_x}|=1$, which is a contradiction. Thus $v_t\cdot v_{s+x}=0$ and so $i_x\in V_t\cap V_{s+x+1}$, and $a_t\ge x+1$. Moreover, since $E_{i_x}=\{s+x,s+x+1,t\}$, by assumption, $t\ne s+x+2$. Now, since $v_t\cdot v_{s-1}=v_t\cdot v_{s+x+1}=0$, there exist integers $m_1\in(V_t\setminus\{i_0,\dots,i_x\})\cap V_{s-1}$ and $m_2\in(V_t\setminus\{i_0,\dots,i_x\})\cap V_{s+x+1}$, implying that $a_t\ge 2+x$. If $a_t=2+x$, then $m_1=m_2$; set $m:=m_1=m_2$. But then $m\in V_{t\pm1}$, implying that $|E_m|\ge 5$, which is a contradiction. Thus $a_t\ge 3+x$. Let $G=\{v'_1,\dots,v'_{s-1},v'_{s+x+1},\dots,v'_{t-1},v'_{t+1},\dots,v_n\}\subset\mathbb{Z}^{n-x-1}=\langle e_1,\dots,e_n\rangle/\langle e_{i_0},\dots,e_{i_x}\rangle$, where $v'_{s-1}=\pi_{e_i}(v_{s-1})$, $v'_{s+x+1}=\pi_{e_{i_x}}(v_{s+x})$, and $v'_\alpha=v_\alpha$ for all $\alpha\notin\{s-1,\dots,s+x+1,t\}$. Then G has two components and $p_1(G)=p_4(G)=0$ and $I(G)\le -2$.

We first claim that G is irreducible and thus a good subset. Suppose otherwise. Then G is the union of two standard subsets G_1 and G_2. By Proposition 5.9, $I(G_1),I(G_2)\ge -3$. Since $p_1(G)=p_4(G)=0$, Proposition 5.9 tells us that $I(G_1),I(G_2)\ge 0$. Consequently, $-2=I(G)=I(G_1)+I(G_2)\ge 0$, a contradiction. Thus G is a good subset. Moreover, by the hypothesis, there do not integers l and z such that $E_l^G=\{z-1,z,z+1\}$, implying that neither component of G is bad (see [8, Definition 4.1]). By Proposition 5.11, $I(G)=-2$ (so $a_t=3+x$) and G_1 and G_2 have associated strings of the form $(b_1,\dots,b_k)$ and $(c_1,\dots,c_l)$, where $(b_1,\dots,b_k)$ and $(c_1,\dots,c_l)$

are linear-dual strings. Thus G has associated string of the form $(b_1,\dots,b_k,c_1,\dots,c_l)$ or $(b_1,\dots,b_k,c_l,\dots,c_1)$.

To determine which string is correct, we first claim that $m_1 \neq m_2$. Assume otherwise, and set $m := m_1 = m_2$. Since $a_t = 3+x$, we have $V_t^S = \{i_0,\dots,i_x,m,z\}$ for some integer z. Since $E_m^S = \{s-1,s+x+1,t\}$, we necessarily have that $E_z^S = \{t-1,t,t+1\}$, contradicting the hypothesis of the lemma. Thus $m_1 \neq m_2$ and $V_t^S = \{i_0,\dots,i_x,m_1,m_2\}$. Once again by the hypothesis, we may assume that $m_1 \in V_{t-1}^S$ and $m_2 \in V_{t+1}^S$. Thus $E_{m_1}^G = \{s-1,t-1\}$ and $E_{m_2}^G = \{s+x+1,t+1\}$. By Proposition 5.11, G must have associated string $(b_1,\dots,b_k,c_1,\dots,c_l)$. Consequently, S has associated string of the form $(3+x,b_1,\dots,b_{k-1},b_k+1,2^{[x]},c_1+1,c_2,\dots,c_l)$. Note that, by Lemma 3.6, $(\beta_1,\dots,\beta_\kappa) = (2+x,b_1,\dots,b_k)$ has linear-dual $(\gamma_1,\dots,\gamma_\lambda) = (2^{[x]},c_1+1,c_2,\dots,c_1)$; hence S has associated string

$$(\beta_1+1,\beta_2,\dots,\beta_{\kappa-1},\beta_\kappa+1,\gamma_1,\dots,\gamma_\lambda) \in \mathcal{S}_{2c}. \qquad \square$$

Combining the above two lemmas, we have proven the following:

Proposition 7.5 *Let S be a cyclic subset with $I(S) \leq 0$ and $p_1(S) = p_2(S) = 0$. Then S is positive with associated string in $\mathcal{S}_{2b} \cup \mathcal{S}_{2c}$.*

7.2 Case IIb: $p_2(S) > 0$

Throughout this section, we will consider cyclic subsets satisfying $p_1(S) = 0$ and $p_2(S) > 0$. In light of Example 5.7, we will further restrict ourselves to cyclic subsets containing at least one vertex with square at most -3. By the discussion in Section 5.2, there are no such cyclic subsets of length 2 or 3. Thus we assume that $n \geq 4$. We start with some useful notation and some preliminary lemmas.

Definition 7.6 Let $S = \{v_1,\dots,v_n\} \subset \mathbb{Z}^n$ be any subset. We define the sets

$$\mathcal{I}^S = \{i \mid E_i = \{s,t\} \text{ and } a_s = 2 \text{ or } a_t = 2\}, \quad \mathcal{J}^S = \{i \mid E_i = \{s,t\} \text{ and } a_s, a_t \geq 3\}.$$

In some cases, we will drop the superscript S from the notation if the subset being considered is understood. Notice that $p_2(S) = |\mathcal{I}^S| \cup |\mathcal{J}^S|$. For each $i \in \mathcal{I}^S \cup \mathcal{J}^S$, let $E_i = \{s(i),t(i)\}$. For each $i \in \mathcal{I}^S$, assume $a_{s(i)} = 2$.

Lemma 7.7 *Let S be cyclic, $I(S) \leq 0$, $p_1(S) = 0$, $p_2(S) > 0$, and $n \geq 4$. If $i \in \mathcal{I}$, then $a_{t(i)} \geq 3$.*

Proof Set $s := s(i)$ and $t := t(i)$. Assume $a_t = 2$. Suppose $v_s \cdot v_t = 0$. Then $V_s = V_t = \{i, j\}$ for some j, and $E_j \supseteq \{s-1, s, s+1, t-1, t, t+1\}$. If $n \geq 5$, then either $v_{s-1} \cdot v_t = 0$ or $v_{s+1} \cdot v_t = 0$, and so $i \in V_{s-1}$ or $i \in V_{s+1}$, which is a contradiction. If $n = 4$, then $t \pm 1 = s \mp 1$. Since $v_{t-1} \cdot v_{t+1} = 0$, there exists an integer k such that $k \in V_{t\pm 1}$. Moreover, there exists a fourth integer m such that $m \in V_{t+1}$ or V_{t-1}, but not both, since $v_{t-1} \cdot v_{t+1} = 0$. Thus $p_1(S) > 0$, contradicting the hypothesis.

Now suppose $|v_s \cdot v_t| = 1$ and, without loss of generality, let $t = s+1$. Since $a_s = a_{s+1} = 2$, we have $V_s = \{i, j\}$ and $V_{s+1} = \{i, i_1\}$, where $i_1 \neq j$. Let $l \geq 2$ be the smallest integer such that $a_{s+l} \geq 3$. Then it is easy to see that $V_{s+\alpha} = \{i_{\alpha-1}, i_\alpha\}$ for all $1 \leq \alpha \leq l-1$, where $i_0 := i$, $i_\alpha \notin \{i, j\}$ for all $1 \leq \alpha \leq l-1$ and the i_α are all distinct. Similarly, let $m \geq 1$ be the smallest integer such that $a_{s-m} \geq 3$. Then, as above, $V_{s-\beta} = \{j_{\beta-1}, j_\beta\}$ for all $1 \leq \beta \leq m-1$, where $j_0 := j$ and the set $\{j_\beta, i, i_\alpha\}$ has $m+l$ distinct elements. Now, since $|v_{s+l-1} \cdot v_{s+l}| = 1$, we must have that $V_{s+l-1} \cap V_{s+l} = \{i_{l-1}\}$ and $|v_{s+l} \cdot e_{i_{l-1}}| = 1$. Similarly, $V_{s-m+1} \cap V_{s-m} = \{j_{m-1}\}$ and $|v_{s-m} \cdot e_{j_{m-1}}| = 1$. Moreover, $E_{i_\alpha} = \{s+\alpha, s+\alpha+1\}$ and $E_{j_\beta} = \{s-\beta, s-\beta-1\}$ for all α and β.

If $v_{s-m} = v_{s+l} = v_u$, then $\{i_{l-1}, j_{m-1}\} \subset V_u$. Since $|v_u \cdot e_{i_{l-1}}| = |v_u \cdot e_{j_{m-1}}| = 1$ and $a_u \geq 3$, we have $|V_u| \geq 3$. Thus there is an integer p such that $E_p = \{u\}$, which contradicts $p_1(S) = 0$. Now suppose $v_{s-m} \neq v_{s+l}$. Let $T = \{v'_1, \dots, v'_{s-1}, v'_{s+l}, \dots, v'_n\} \subset \mathbb{Z}^{n-(m+l)} = \langle e_1, \dots, e_n \rangle / \langle e_{i_0}, \dots, e_{i_{l-1}}, e_{j_0}, \dots, e_{j_{m-1}} \rangle$, where $v'_{s-m} = \pi_{e_{j_{m-1}}}(v_{s-m})$ and $v'_{s+l} = \pi_{e_{i_{l-1}}}(v_{s+l})$. Since $|v_{s+l} \cdot e_{i_{l-1}}| = |v_{s-m} \cdot e_{j_{m-1}}| = 1$ and $a_{s-m}, a_{s+l} \geq 3$, we have $(v'_{s-m})^2, (v'_{s+l})^2 \leq -2$. Thus T is a standard subset made of $n-(l+m-1)$ vectors. However, by Remark 5.3, these vectors are linearly independent in $\mathbb{Z}^{n-(l+m)}$, which is not possible. □

Lemma 7.8 *Let S be cyclic, $I(S) \leq 0$, $p_1(S) = 0$, $p_2(S) > 0$, and $n \geq 4$. If $i \in \mathcal{I}$, then $v_{s(i)} \cdot v_{t(i)} = 0$.*

Proof Set $s := s(i)$ and $t := t(i)$. Let $V_s = \{i, j\}$. Then, by Lemma 7.7, $a_t \geq 3$. Assume $|v_s \cdot v_t| = 1$ and, without loss of generality, assume $t = s+1$. Then $\{s-1, s\} \subseteq E_j$. If there exists an integer $u \notin \{s-1, s, s+1\}$ such that $u \in E_j$, then we necessarily have that $i \in V_u$, implying that $|E_i| \geq 3$, which is not possible. Thus either $E_j = \{s-1, s\}$ or $E_j = \{s-1, s, s+1\}$.

If $E_j = \{s-1, s\}$, then, by Lemma 7.7, $a_{s-1} \geq 3$. Moreover, since $|v_s \cdot e_i| = |v_s \cdot e_j| = 1$, $V_s \cap V_{s-1} = \{j\}$ and $V_s \cap V_{s+1} = \{i\}$, we have $|v_{s+1} \cdot e_i| = |v_{s-1} \cdot e_j| = 1$. Let $T = \{v'_1, \dots, v'_{s-1}, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-2} = \langle e_1, \dots, e_n \rangle / \langle e_i, e_j \rangle$, where $v'_{s+1} = \pi_{e_j}(v_{s+1})$,

$v'_{s-1} = \pi_{e_j}(v_{s-1})$, and $v'_x = v_x$ for all $x \notin \{s-1, s, s+1\}$. Then $(v'_{s\pm 1})^2 \le -2$ and $v'_{s-1} \cdot v'_{s+1} = 0$. Thus T is standard with final vertices v'_{s-1} and v'_{s+1}. By Remark 5.3, $T \subset \mathbb{Z}^{n-2}$ contains $n-1$ linearly independent vectors, which is impossible.

If $E_j = \{s-1, s, s+1\}$, then, since $v_{s-1} \cdot v_{s+1} = 0$, there exists an integer $k \notin \{i, j\}$ such that $k \in V_{s-1} \cap V_{s+1}$. Moreover, $|v_{s-1} \cdot e_j| = 1$ and, since $V_{s+1} \cap V_s = \{i, j\}$ and $|v_{s+1} \cdot v_s| = 1$, we have $|v_{s+1} \cdot e_i| = x$ and $|v_{s+1} \cdot e_j| = x \pm 1$, where $x, x \pm 1 \neq 0$. Thus $a_{s+1} \ge x^2 + (x \pm 1)^2 + 1 \ge 6$. If $|v_{s+1} \cdot e_i| = x \ge 2$, let $T = \{v'_1, \dots, v'_{s-1}, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n\rangle / \langle e_i\rangle$, where $v'_{s+1} = \pi_{e_i}(v_{s+1})$ and $v'_x = v_x$ for all $x \notin \{s, s+1\}$. Then T is standard and $0 \ge I(S) = I(T) + x^2 + (a_s - 3) = I(T) + x^2 - 1$. Thus $I(T) \le 1 - x^2 < 0$ and so, by Proposition 5.9, we necessarily have that $I(T) = -3$ and $p_1(T) = 1$. But then $p_1(S) = p_1(T) = 1$, which contradicts our assumption that $p_1(S) = 0$. Now suppose $|v_{s+1} \cdot e_i| = 1$, so that $|v_{s+1} \cdot e_j| = 2$. Since $|v_{s-1} \cdot e_j| = 1$ and $|v_{s-1} \cdot v_{s+1}| = 0$, either $a_{s-1} \ge 3$ or $a_{s-1} = 2$ and $|v_{s+1} \cdot e_k| = 2$. In the latter case, note that $E_k = \{s-2, s-1, s+1\}$ and $E_j = \{s-1, s, s+1\}$. In this case, let $T' = \{v'_1, \dots, v'_{s-2}, v'_{s+1}, \dots, v'_n\} =\subset \mathbb{Z}^{n-2} = \langle e_1, \dots, e_n\rangle / \langle e_i, e_j\rangle$, where $v'_{s+1} = \pi_{e_i}(\pi_{e_j}(v_{s+1}))$ and $v'_x = v_x$ for all $x \notin \{s-1, s, s+1\}$. Then T' is standard with $p_1(T') = 0$ and $0 \ge I(S) = I(T') + 5 + (a_{s-1} - 3) + (a_s - 3) = I(T') + 3$, implying that $I(T') \le -3$. But, by Proposition 5.9, no such subset exists. In the former case ($a_{s-1} \ge 3$), let $T'' = \{v'_1, \dots, v'_{s-1}, v'_{s+2}, \dots, v'_n\} \subset \mathbb{Z}^{n-2} = \langle e_1, \dots, e_n\rangle / \langle e_i, e_j\rangle$, where $v'_{s-1} = \pi_{e_j}(v_{s-1})$ and $v'_x = v_x$ for all $x \notin \{s-1, s, s+1\}$. Then T'' is a standard subset such that $0 \ge I(S) = I(T'') + 1 + (a_s - 3) + (a_{s+1} - 3) \ge I(T'') + 3$. By Proposition 5.9, we necessarily have that $I(T'') = -3$ and $p_1(T'') = 1$. Thus $a_{s+1} = 6$ and $V^S_{s+1} = \{i, j, k\}$. This implies that $|E^{T''}_k| = 1$. But $k \in V^{T''}_{s-1}$ and v_{s-1} is a final vertex of T''. By Proposition 5.10(a), no such standard subset exists. □

Lemma 7.9 *Let S be cyclic, $I(S) \le 0$, $p_1(S) = 0$, $|\mathcal{I}| > 0$, and $n \ge 4$.*

(a) *If there exist integers $i, i' \in \mathcal{I}$ such that $|v_{s(i)} \cdot v_{s(i')}| = 1$, then S is negative and has associated string in $\mathcal{S}_{1d}$, $|\mathcal{J}| = 0$, and $|v_\alpha \cdot v_\beta| \le 1$ for all $1 \le \alpha, \beta \le n$.*

(b) *If $v_{s(i)} \cdot v_{s(i')} = 0$ for all $i, i' \in \mathcal{I}$, then $p_4(S) \ge |\mathcal{I}|$.*

Proof Suppose $|v_{s(i)} \cdot v_{s(i')}| = 1$ and, without loss of generality, assume $s(i') = s(i) + 1$. Then $t(i) = s(i) + 2$, $t(i') = s(i) - 1$, and there exists an integer j such that $E_j = \{s(i)-1, s(i), s(i)+1, s(i)+2\}$. Set $s := s(i)$. By Lemma 7.7, $a_{s-1}, a_{s+2} \ge 3$; consequently, $n \ge 5$. Without loss of generality, assume $v_{s-1} \cdot v_s = v_s \cdot v_{s+1} = 1$, so that $v_{s-1} \cdot e_j = -v_s \cdot e_j = v_{s+1} \cdot e_j \in \{\pm 1\}$. Let $S' = \{v'_1, \dots, v'_{s-1}, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n\rangle / \langle e_i\rangle$, where $v'_{s+2} = \pi_{e_i}(v_{s+2})$, $v'_{s-1} = \pi_{e_{i'}}(v_{s-1})$, and $v'_x := v_x$

for all $x \notin \{s-1, s, s+2\}$. Then S' is cyclic with $I(S') = I(S) - 1 < 0$ and $p_1(S') = 1$ (since $E_{i'}^{S'} = \{s+1\}$). Moreover, $v'_{s-1} \cdot e_j = v'_{s+1} \cdot e_j$ and so S' is positive if and only if S is negative. By the proof of Proposition 6.5, the only cyclic subset with $p_1 = 1$ and $I < 0$ is positive and has associated string of the form $(2, b_1+1, b_2, \dots, b_k, 2, c_l, \dots, c_2, c_1+1, 2) \in \mathcal{S}_{2e}$. Moreover, the vertex with square 2 in the middle of the string is v'_{s+1}. Thus S is negative and has associated string of the form $(2, b_1+1, b_2, \dots, b_k+1, 2, 2, c_l+1, \dots, c_2, c_1+1, 2) \in \mathcal{S}_{1d}$. Furthermore, by the proof of Proposition 6.5, it is easy to see that $|v'_\alpha \cdot v'_\beta| \le 1$ for all α and β and $|\mathcal{J}^{S'}| = 0$; hence $|v_\alpha \cdot v_\beta| \le 1$ for all $1 \le \alpha, \beta \le n$ and $|\mathcal{J}^S| = 0$.

By Lemma 7.8, for all $i \in \mathcal{I}^S$, there exists an integer $j(i)$ such that

$$E_{j(i)} = \{s(i)-1, s(i), s(i)+1, t(i)\}.$$

If $v_{s(i)} \cdot v_{s(i')} = 0$ for some $i, i' \in \mathcal{I}^S$, it follows that $j(i) \neq j(i')$; hence, if $v_{s(i)} \cdot v_{s(i')} = 0$ for all $i, i' \in \mathcal{I}^S$, then $p_4(S) \ge |\mathcal{I}^S|$. □

Lemma 7.10 *Let S be cyclic, $I(S) \le 0$, $p_1(S) = 0$, $p_2(S) > 0$, and $n \ge 4$. Then $|v_\alpha \cdot e_\beta| \le 1$ for all integers α and β.*

Proof By Lemma 7.9, we may assume that $v_{s(i)} \cdot v_{s(i')} = 0$ for all $i, i' \in \mathcal{I}$, so that $p_4(S) \ge |\mathcal{I}|$. First suppose that $|\mathcal{J}| \neq 0$. Let $i \in \mathcal{J}$ and set $s := s(i)$ and $t := t(i)$. Notice that we cannot have $|V_s| = |V_t| = 2$. Without loss of generality, assume that $|V_s| \ge 3$. Let $T = \{v'_1, \dots, v'_{t-1}, v'_{t+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n \rangle / \langle e_i \rangle$, where $v'_s = \pi_{e_i}(v_s)$ and $v'_x = v_x$ for all $x \notin \{s, t\}$. Then $(v'_s)^2 \le -2$ and $v'_{t-1} \cdot v'_{t+1} = 0$, and so T is standard. Let $|v_s \cdot e_i| = x \ge 1$. Then

$$0 \ge I(S) = I(T) + x^2 + (a_t - 3) \ge I(T) + x^2 \ge I(T) + 1.$$

Thus $I(T) \le -1$ and so, by Proposition 5.9, $I(T) \in \{-1, -2, -3\}$. Thus $a_t \le 5$ and $|v_s \cdot e_i| = x = 1$. Moreover, by Proposition 5.9, $|v'_\alpha \cdot e_\beta| \le 1$ for all α and β. Thus $|v_\alpha \cdot e_\beta| \le 1$ for all $\alpha \neq t$ and all β. If $|v_t \cdot e_j| \ge 2$ for some j, then, since $a_t \le 5$, we necessarily have $V_t = \{i, j\}$ and $a_t = 5$; consequently, $I(T) = -3$ and, by Proposition 5.9, $p_1(T) = 1$. In particular, $|E_j^T| = 1$ and $E_j^S = \{s, t\}$. If $v_s \cdot v_t = 0$, then $v_t \cdot v_{t\pm1} = 0$, which is a contradiction. If $|v_s \cdot v_t| = 1$ and, say, $t = s+1$, then $v_{s+1} \cdot v_{s+2} = 0$, which is a contradiction. Thus $|v_\alpha \cdot e_\beta| \le 1$ for all α and β.

Now suppose $|\mathcal{J}| = 0$. Then $p_4(S) \ge p_2(S)$ and so, by Lemma 5.16, $I(S) = 0$, $p_2(S) = p_4(S)$ and $p_j(S) = 0$ for all $j = 5, \dots, n$. Thus $p_3(S) = n - 2p_2(S)$. Let

$m_{ij} := v_i \cdot e_j$. Then

$$3n = \sum a_i = \sum_{i,j} m_{ij}^2 \geq \sum_{i,j} |m_{ij}| \geq \sum i p_i(S) = 2p_2(S) + 4p_2(S) + 3(n - 2p_2(S)) = 3n.$$

Thus $|v_i \cdot e_j| = |m_{ij}| \leq 1$ for all i and j. □

In light of Lemma 7.10, it will now be a standing assumption that $|v_\alpha \cdot e_\beta| \leq 1$ for all integers α and β.

Lemma 7.11 *Suppose S is cyclic with $n \geq 4$ and $|\mathcal{J}| \neq 0$. If there exists $i \in \mathcal{J}$ with $a_{s(i)}, a_{t(i)} \geq 4$, then S is positive with associated string $(4,4,2,2,2) \in \mathcal{S}_{2d}$.*

Proof By cyclically reordering and negating vertices, we may assume $s(i) = 1$ and $t(i) = k$ for some integer k. Let $R = \{v_1', \dots, v_n'\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n \rangle / \langle e_i \rangle$, where $v_1' = \pi_{e_i}(v_1)$, $v_k' = \pi_{e_i}(v_k)$, and $v_i' := v_i$ for all $i \neq 1, k$.

Case 1 ($v_1 \cdot v_k = 0$, so $k \notin \{2, n\}$) By Lemma 7.10, $-(v_1')^2 = a_1 - 1$, $-(v_k')^2 = a_k - 1$, and $v_1' \cdot v_k' = \pm 1$. Let A be the intersection matrix $A = (v_i' \cdot v_j')$. Assume $a_1, a_k \geq 4$. By Lemma A.4, if S is negative cyclic or S is positive cyclic with $v_1 \cdot e_i = -v_k \cdot e_i$, then A is negative definite; in these cases R is a linearly independent set of n vectors in $\mathbb{Z}^{n-1}$, which is not possible. Thus we may assume that S is positive and $v_1 \cdot e_i = v_k \cdot e_i$. Again by Lemma A.4, we arrive at another linear independence contradiction unless $a_1 = a_k = 4$ and $a_x = 2$ for all $x \notin \{1, k\}$. Thus $I(S) = -(n-4)$. Let $T = \{v_2', \dots, v_n'\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n \rangle / \langle e_i \rangle$, where $v_k' = \pi_{e_i}(v_k)$ and $v_x' = v_x$ for all $x \notin \{1, k\}$. Then T is a standard subset and $I(T) = I(S) - 2 = -(n-2)$. Since $I(T) \geq -3$ by Proposition 5.9, it follows that $n \leq 5$. If $n = 5$, then $I(S) = -1$, $I(T) = -3$, and T has length 4. By Proposition 5.10(1), up to reversal, T has associated string of the form $(3,2,2,2)$. Since $a_t = 4$, this implies that $k = 2$, a contradiction. If $n = 4$, then $I(S) = 0$, $I(T) = -2$, and T has length 3. But, by Proposition 5.10(2), no such standard subset exists.

Case 2 ($|v_1 \cdot v_k| = 1$) Without loss of generality, assume $k = 2$. If $v_1 \cdot e_i = -v_2 \cdot e_i$, then $v_1' \cdot v_2' = 0$; hence R is standard and so, by Remark 5.3, R is a linearly independent set of n vectors in $\mathbb{Z}^{n-1}$, a contradiction. If $v_1 \cdot e_i = v_2 \cdot e_i$, then $v_1' \cdot v_2' = 2$; by applying Lemma A.5 as in Case 1, we obtain a contradiction unless S is positive, $a_1 = a_2 = 4$, and $a_3 = \dots = a_n = 2$. In this case, $I(S) = -(n-4)$. As in Case 1, we necessarily have that $n \le 5$. If $n = 4$, then $I(T) = -2$ and T has length 3; by Proposition 5.10(2), no such subset exists. Suppose $n = 5$, so that $I(T) = -3$ and T has length 4. By Proposition 5.10(1), up to reversal, T has associated string of the form $(3, 2, 2, 2)$. Hence S is positive and has associated string of the form $(4, 4, 2, 2, 2) \in \mathcal{S}_{2d}$. □

We are now ready to finish the classification of cyclic subsets with $I(S) \le 0$, $p_1(S) = 0$, and $p_2(S) > 0$. We will consider two cases: $|\mathcal{J}| \ne 0$ and $|\mathcal{J}| = 0$. These cases are handled respectively in the next two propositions.

Proposition 7.12 *Let S be cyclic, $I(S) \le 0$, $p_1(S) = 0$, $p_2(S) > 0$, and $n \ge 4$. If $|\mathcal{J}| \ne 0$, then S is positive with associated string in $\mathcal{S}_{2c} \cup \mathcal{S}_{2d}$ or negative with associated string in $\mathcal{S}_{1c} \cup \mathcal{S}_{1e} \cup \mathcal{O}$.*

Proof Let $i \in \mathcal{J}$ and set $s := s(i)$ and $t := t(i)$. If $a_s, a_t \ge 4$, then, by Lemma 7.11, S is positive with associated string in $\mathcal{S}_{2d}$. Without loss of generality, we may now assume that $a_s = 3$. Moreover, by Lemma 7.9, $v_{s(i_1)} \cdot v_{s(i_2)} = 0$ for all $i_1, i_2 \in \mathcal{I}$, implying that $p_4(S) \ge |\mathcal{I}|$. Let $T = \{v'_1, \dots, v'_{s-1}, v'_{s+1}, \dots, v'_n\} \subset \mathbb{Z}^{n-1} = \langle e_1, \dots, e_n\rangle / \langle e_i \rangle$, where $v'_t = \pi_{e_i}(v_t)$ and $v'_x = v_x$ for all $x \notin \{s, t\}$. By Lemma 7.10, $(v'_t)^2 = v_t^2 + 1$ and so T is a standard subset and $I(T) = I(S) - 1 \le -1$. By Proposition 5.9, $I(T) \in \{-3, -2, -1\}$. We will work case by case, considering each of the standard subsets listed in Proposition 5.10.

Case 1 ($I(T) = -1$) By Proposition 5.9, $p_1(T) = 0$, $p_2(T) = 2$, $p_4(T) = 1$, and $p_j(T) = 0$ for all $j \ge 5$. Thus $p_2(S) \le 3$, $p_4(S) \le 3$, $p_5(S) \le 1$, and $p_j(S) = 0$ for all $j \ge 6$. Note that, since $a_s = 3$, if $p_5(S) = 1$, then $p_4(S) = p_2(S) - 2$, and if $p_5(S) = 0$, then $p_2(S) = p_4(S)$. By Lemma 5.17, $p_2(S) + p_4(S) \equiv 0 \bmod 4$, implying that either $p_5(S) = 1$, $p_2(S) = 3$ and $p_4(S) = 1$, or $p_5(S) = 0$ and $p_2(S) = p_4(S) = 2$. By Proposition 5.10(3), T is of one of the forms (a)–(c) given there.

Case 1(a) Without loss of generality, we may assume that the listed vertices are $v'_{s+1}, \dots, v'_n, v'_1, \dots, v'_{s-1}$. First assume $p_5(S) = 1$, $p_2(S) = 3$, and $p_4(S) = 1$. Then $2 \in V_s^S$ and $3, x + y + 4 \notin V_s^S$ (where $x + y + 4 = 1$ if $y = 0$). If $y = 0$, then, since $v_{s+2} \cdot v_s = 0$ and $1 \notin V_s^S$, we have $i \in V_{s+2}^S$. Since $v_{s+3} \cdot v_s = 0$ and $2 \in V_s^S$, we have $4 \in V_s^S$ and $v_s \cdot e_2 = v_s \cdot e_4$. Since $V_s^S = \{i, 2, 4\}$, if $x \ge 1$, then $v_{s+4} \cdot v_s \ne 0$, which is a contradiction, and if $x = 0$, then $v_{s-1} \cdot v_s = 0$, which is a contradiction. Thus we may assume $y \ge 1$. Since $v_s \cdot v_{s+2} = v_s \cdot v_{s+3} = 0$ and $a_s = 3$, either $i \in V_{s+2}^S$ and $4 \in V_s^S$, or $i \in V_{s+3}^S$ and $|\{1, x+5, \dots, x+y+3\} \cap V_s^S| = 1$. In the former case, $V_s^S = \{i, 2, 4\}$ and so $|v_s \cdot v_{s+1}| \neq 1$, which is a contradiction. In the latter case, if $1 \in V_s^S$, then $V_s^S = \{i, 1, 2\}$ and $v_s \cdot e_1 = v_s \cdot e_2$ (since $v_s \cdot v_{s+2} = 0$); but then $|v_{s+x+4} \cdot v_s| = 2$, which is a contradiction. On the other hand, if $|\{x+5, \dots, x+y+3\} \cap V_s^S| = 1$, then, since $v_s \cdot v_{s-\alpha} = 0$ for all $2 \le \alpha \le y$, $\{x+5, \dots, x+y+3\} \subset V_s^S$, implying that $y = 1$ and $1 \in V_s^S$, which is again a contradiction.

Now assume $p_5(S) = 0$ and $p_2(S) = p_4(S) = 2$. Then $2 \notin V_s^S$ and either $x+y+4 \in V_s^S$ or $3 \in V_s^S$, but not both (where $x+y+4 = 1$ if $y = 0$). First assume $x+y+4 \in V_s^S$. Since $x+y+4 \in V_{s+2}^S$ and $v_{s+2} \cdot v_s = 0$, either $|\{1, x+5, \dots, x+y+3\} \cap V_s^S| = 1$ or $i \in V_{s+2}^S$. In the former case, $y \ge 1$ and, since $v_{s-\alpha} \cdot v_s = 0$ for all $2 \le \alpha \le y$, it follows that $\{1, x+5, \dots, x+y+3\} \subset V_s^S$, implying that $|v_s \cdot v_{s-1}| \neq 1$, which is a contradiction. In the latter case, since $|v_s \cdot v_{s+1}| = 1$, we have $|\{4, 5, \dots, x+4\} \cap V_s^S| = 1$. Since $v_{s+\alpha} \cdot v_s = 0$ for all $4 \le \alpha \le x+4$, we have $\{4, 5, \dots, x+4\} \subset V_s^S$, which implies that $x = 0$ and $V_s^S = \{i, 4, x+y+4\}$; but then $|v_{s+3} \cdot v_s| = 1$, which is a contradiction.

Now suppose $3 \in V_s^S$. Since $v_s \cdot v_{s+3} = 0$ and $3 \in V_{s+3}^S$, either $i \in V_{s+3}^S$ or $4 \in V_s^S$. In the former case, since $|v_s \cdot v_{s+1}| = 1$, we have $|\{4, 5, \dots, x+4\} \cap V_s^S| = 1$. As in the previous case, we see that $x = 0$ and $V_s^S = \{i, 3, 4\}$ and so $v_{s+3} \cdot v_s \neq 0$, which is a contradiction. In the latter case, since $4 \in V_{s+4}^S$, we have $i \in V_{s+4}^S$ and, since $|v_s \cdot v_{s-1}| = 1$, we necessarily have that $y = 0$. Consequently, S is of the form

$$\Big\{e_i - e_4 + e_3, e_2 + e_4 + \sum_{\alpha=5}^{x+4} e_\alpha, e_1 - e_2, e_2 - e_3 - e_4, e_i + e_4 - e_5, \\ e_5 - e_6, \dots, e_{x+3} - e_{x+4}, e_{x+4} - e_1 - e_2 - e_3\Big\},$$

which is positive and has associated string $(3, 2+x, 2, 3, 3, 2^{[x-1]}, 4) \in \mathcal{S}_{2c}$.

Case 1(b) As in the previous case, we may label the vertices $v'_{s+1}, \dots, v'_n, v'_1, \dots, v'_{s-1}$. Note that, if $y = 0$, then S is also of the form in Case 1(a), which we already covered. Thus we may assume $y \ge 1$. Consequently, $|\mathcal{I}^T| = 2$. If $p_5(S) = 1$, then $p_2(S) = 3$ and so $|\mathcal{I}^S| = 2$; but we also have that $p_4(S) = 1 \ge |\mathcal{I}^S|$, which is a contradiction. Thus $p_5(S) = 0$ and $p_2(S) = p_4(S) = 2$; hence $2 \notin V_s^S$ and either $1 \in V_s^S$ or $x+y+4 \in V_s^S$, but not both. Assume $x+y+4 \in V_s^S$. Since $x+y+4 \in V_{s+3}^S$, either $i \in V_{s+3}^S$ or $|\{3, 4, x+5, \dots, x+y+3\} \cap V_s^S| = 1$. In the former case, since $|v_s \cdot v_{s+1}| = 1$, following as in Case 1(a) we see that $x = 0$ and $V_s^S = \{i, x+y+4, 4\}$, which implies that $|v_s \cdot v_{s+3}| = 1$, which is a contradiction. In the latter case, since $v_{s-\alpha} \cdot v_s = 0$ for all $2 \le \alpha \le y$, it is clear that $3, x+5, \dots, x+y+3 \notin V_s^S$ and so $4 \in V_s^S$. Since

$4, x+y+4 \in V^S_{s+3}$ and $4 \in V^S_{s+4}$, we have $i \in V^S_{s+4}$. Hence, if $x \geq 1$, S is of the form

$$\Big\{e_i - e_4 + e_{x+y+4}, e_2 + e_4 + \sum_{\alpha=5}^{x+4} e_\alpha, e_1 - e_2, e_2 - e_3 - e_4 - \sum_{\alpha=x+5}^{x+y+4} e_\alpha, \\ e_i + e_4 - e_5, \dots, e_{x+3} - e_{x+4}, e_{x+4} - e_1 - e_2 - e_3, e_3 - e_{x+5}, \\ e_{x+5} - e_{x+6}, \dots, e_{x+y+3} - e_{x+y+4}\Big\},$$

which is positive and has associated string $(3, 2+x, 2, 3+y, 3, 2^{[x-1]}, 4, 2^{[y]}) \in \mathcal{S}_{2c}$, and if $x = 0$, then S is of the form

$$\Big\{e_i - e_4 + e_{y+4}, e_2 + e_4, e_1 - e_2, e_2 - e_3 - e_4 - \sum_{\alpha=5}^{y+4} e_\alpha, e_i + e_4 - e_1 - e_2 - e_3, \\ e_3 - e_5, e_6 - e_7, \dots, e_{y+3} - e_{y+4}\Big\},$$

which is positive and has associated string $(3, 2, 2, 3+y, 5, 2^{[y]}) \in \mathcal{S}_{2c}$.

Next assume $3 \in V^S_s$. Since $v_s \cdot v_{s+3} = v_s \cdot v_{s+x+4} = 0$ and $3 \in V^S_{s+3} \cap V^S_{s+x+4}$, either $i \in V^S_{s+3}$ or $i \in V^S_{s+x+4}$. Since $y \geq 1$ and $|v_{s-1} \cdot v_s| = 1$, it follows that $x+y+3 \in V^S_s$ (where $x+y+3 = 3$ if $y = 1$). But then $v_s \cdot v_{s+1} = 0$, which is a contradiction.

Case 1(c) Label the vertices $v'_{s+1}, \dots, v'_n, v'_1, \dots, v'_{s-1}$. Notice that $|\mathcal{I}^T| = 2$ if $y \geq 1$. By the same argument as in Case 1(b), if $y \geq 1$, then $p_5(S) \neq 0$. Suppose $y = 0$, $p_5(S) = 1$, and $p_2(S) = 3$. Then $2 \in V^S_s$ and $3, 4 \notin V^S_s$. Since $2, 3 \in V^S_{s+2}$ and $v_s \cdot v_{s+2} = 0$, we necessarily have that $i \in V^S_{s+2}$. Now, since $V^S_{s+3} \cap V^S_{s+4} = \{2\}$, it follows that either $v_s \cdot v_{s+3} \neq 0$ or $v_s \cdot v_{s+4} \neq 0$, which is a contradiction. Thus we may assume that $p_5(S) = 0$ and $p_2(S) = p_4(S) = 2$. Thus $2 \notin V^S_s$ and either $3 \in V^S_s$ or $x+y+5 \in V^S_s$, but not both (where $x+y+5 = 4$ if $y = 0$). If $x+y+5 \in V^S_{s+3}$, then either $i \in V^S_{s+3}$ or $|\{1, 4, x+6, \dots, x+y+3\} \cap V^S_s| = 1$. In the former case, we obtain a contradiction as in Cases 1(a) and 1(b). In the latter case, we obtain similar contradictions unless $1 \in V^S_s$. In this case, since $1, x+y+5 \in V^S_{s+3}$ and $1 \in V^S_{s+x+5}$, we have $i \in V^S_{s+x+4}$. Thus S is of the form

$$\Big\{e_i - e_1 + e_{x+y+5}, e_1 - e_2 - e_5 - \sum_{\alpha=6}^{x+5} e_\alpha, e_2 + e_3, -e_2 - e_1 - e_4 - \sum_{\alpha=x+6}^{x+y+5} e_\alpha, \\ -e_5 + e_2 - e_3, e_5 - e_6, \dots, e_{x+4} - e_{x+5}, -e_i + e_{x+5} + e_1 - e_4, \\ e_4 - e_{x+6}, e_{x+6} - e_{x+7} \dots, e_{x+y+4} - e_{x+y+5}\Big\},$$

which is positive and has associated string $(3, 3+x, 2, 3+y, 3, 2^{[x]}, 4, 2^{[y]}) \in \mathcal{S}_{2c}$.

Next suppose $3 \in V_s^S$. Since $2 \notin V_{s+2}^S$ and $v_s \cdot v_{s+2} = 0$, we necessarily have that $i \in V_{s+2}^S$. Since $v_s \cdot v_{s+4} = 0$, we have $5 \in V_s^S$ and so $V_s^S = \{i, 3, 5\}$. Moreover, since $5 \in V_{s+5}^S$, $v_s \cdot v_{s+5} = 0$, and $|v_s \cdot v_{s-1}| = 1$, we must have that $x = y = 0$. Hence S is of the form

$$\{e_i - e_3 + e_5, e_1 - e_2 - e_5, e_2 + e_3 + e_i, -e_2 - e_1 - e_4, -e_5 + e_2 - e_3, e_5 + e_1 - e_4\},$$

which is negative cyclic with associated string $(3,3,3,3,3,3) \in \mathcal{O}$.

Case 2 $(I(T) = -2$, so that $I(S) = -1)$ By Proposition 5.10(2), $p_1(T) = 0$, $p_2(T) = 3$, $p_4(T) = 1$, $p_j(T) = 0$ for all $j \geq 5$, and $|\mathcal{I}^T| = 2$. Then, since $a_s = 3$, $p_2(S) \leq 4$, $p_4(S) \leq 3$, and $p_5(S) \leq 1$. By Lemma 5.17, $p_2(S) + p_4(S) = 1 \bmod 4$. By a similar argument as in Case 1(b), $p_5(S) = 0$ and so $p_2(S) = 3$ and $p_4(S) = 2$. By Proposition 5.10(2), T is of one of the forms (a)–(c) given there.

Case 2(a) Label the vertices $v'_{s+1}, \dots, v'_n, v'_1, \dots, v'_{s-1}$. Notice that, if $y = 0$, then T is also of the form given in Case 2(b). Moreover, if $x = 0$, then the reverse of T is of the form given in Case 2(b). We will assume that $x, y \geq 1$ and handle the cases $x = 0$ and $y = 0$ in Case 2(b). Since $p_5(S) = 0$ and $p_2(S) = 3$, we have $2 \notin V_s^S$ and $|\{x+4, x+y+4, 3\} \cap V_s^S| = 1$. If $x + 4 \in V_s^S$ or $x + y + 4 \in V_s^S$, then, arguing as in Case 1, we arrive at contradictions. Assume $3 \in V_s^S$. Since $3 \in V_{s+x+4}$ and $v_s \cdot v_{s+x+4} = 0$, either $i \in V_{s+x+4}^S$ or $1 \in V_s^S$, but not both. In the former case, since $|v_s \cdot v_{s\pm1}| = 1$, we have $x+3, x+y+3 \in V_s^S$, implying that $a_s \geq 4$, which is a contradiction. In the latter case, $V_s^S = \{i, 1, 3\}$, implying that $v_s \cdot v_{s+1} = 0$, which is a contradiction.

Case 2(b) Label the vertices $v'_{s+1}, \dots, v'_n, v'_1, \dots, v'_{s-1}$. Notice that, if $x = 0$, then T is of the form in Case 2(c). We will assume that $x \geq 1$ and handle $x = 0$ in Case 2(c). Since $p_5(S) = 0$ and $p_2(S) = 3$, we have $2 \notin V_s^S$ and $|\{x+4, x+y+4, 1\} \cap V_s^S| = 1$ (where $x + y + 4 = 3$ if $y = 0$). If $1 \in V_s^S$, then, since $v_{s+x+2} \cdot v_s = 0$, we necessarily have that $i \in V_{s+x+2}^S$. Now, since $|v_{s+1} \cdot v_s| = 1$, we have $x + 3 \in V_s^S$ and so $V_s^S = \{i, 1, x+3\}$; but then $|v_s \cdot v_{s+2}| = 1$, which is a contradiction. If $x + 4 \in V_s^S$, then, since $v_s \cdot v_{s+\alpha} = 0$ for all $2 \leq \alpha \leq x$, it follows that $4, \dots, x+3 \notin V_s^S$. Since $x + 4 \in V_{s+x+3}^S$, we must have that $i \in V_{s+x+3}^S$; consequently, since $|v_{s-1} \cdot v_s| = 1$, we necessarily have that $y \geq 1$ and $x + y + 3 \in V_s^S$. But then $v_{s-2} \cdot v_s \neq 0$, which is a contradiction. Thus $x + y + 4 \in V_s^S$. As above, it is easy to see that $3, x+5, \dots, x+y+3 \notin V_s^S$. Since $x + y + 4 \in V_{s+x+1}^S$, it follows that either $i \in V_{s+x+1}^S$ or $4 \in V_s^S$. In the former case, since $|v_s \cdot v_{s+1}| = 1$, we have $x + 3 \in V_s^S$, which leads to a contradiction. In the latter case, since $4 \in V^S_{s+x+3}$, we see that $i \in V^S_{s+x+4}$. Since $|v_s \cdot v_{s-1}| = 1$, it follows that $x = 1$. Thus S is of the form

$$\Big\{e_i + e_4 + e_{x+y+4}, e_5 - e_4, e_4 - e_2 - e_3 - \sum_{i=x+5}^{x+y+4} e_i, e_2 + e_1, e_i - e_2 - e_4 - e_5, \\ e_2 - e_1 - e_3, e_3 - e_{x+5}, e_{x+5} - e_{x+6}, \dots, e_{x+y+3} - e_{x+y+4}\Big\},$$

which is positive cyclic with associated string $(3, 2, 3 + y, 2, 4, 3, 2^{[y]}) \in \mathcal{S}_{2d}$.

Case 2(c) Label the vertices $v'_{s+1}, \dots, v'_n, v'_1, \dots, v'_{s-1}$. As usual, since $p_5(S) = 0$, $2 \notin V^S_s$. Notice $2 \in V^S_{s+k+1} \cap V^S_{s+k+2}$. By our standing assumption that $v_{s(i)} \cdot v_{s(i')} = 0$ for all $i, i' \in \mathcal{I}^S$, we necessarily have that either $1 \in V^S_s$ or $4 \in V^S_s$, but not both. Consequently, since $v_s \cdot v_{s+k+1} = v_s \cdot v_{s+k+2} = 0$, either $i \in V^S_{s+k+1}$ or $i \in V^S_{s+k+2}$. Moreover, since $p_2(S) = 3$, $j_1 \notin V^S_s$ and so $j_2 \in V^S_s$. Now, since $j_2 \in V^S_{s+2}$ and $v_s \cdot v_{s+2} = 0$, we necessarily have that $k = 2$ and $4 \in V^S_s$. Hence $V^S_s = \{4, i, j_2\}$, $i \in V^S_{s+k+2}$, and T has associated string of the form $(2, 3 + x, 2, 2, 3, 2^{[x-1]}, 3)$. Moreover, $v_s \cdot e_{j_2} = \pm v_{s-1} \cdot e_{j_2} = \mp v_{s+1} \cdot e_{j_2}$. Thus S is negative and has associated string of the form $(3, 2, 3 + x, 2, 3, 3, 2^{[x-1]}, 3) \in \mathcal{S}_{1e}$.

Case 3 $(I(T) = -3$, so that $I(S) = -2)$ By Proposition 5.9, $p_1(T) = 1$, $p_2(T) = 1$, and $p_j(T) = 0$ for all $j \geq 4$. Thus $p_j(S) = 0$ for all $j \geq 5$. Let l be the unique integer such that $|E^T_l| = 1$ and let u be the integer such that $E^T_l = \{u\}$, where $u \neq s \pm 1$. Then, since $p_1(S) = 0$, $l \in V^S_s$. Since $a_s = 3$, we have $p_2(S) \in \{2, 3\}$ and $p_4(S) = p_2(S) - 2$. By Lemma 5.17, $p_2(S) + p_4(S) = 2p_2(S) - 2 \equiv 2 \bmod 4$, implying that $p_2(S) = 2$ and $p_4(S) = 0$. By Proposition 5.10(1), there is an integer k such that $E^T_k = \{s - 1, s + 1\}$ and $v_{s-1} \cdot e_k = -v_{s+1} \cdot e_k$. Since $p_2(S) = 2$, $k \in V^S_s$, and so $V^S_s = \{i, l, k\}$. Since $k \notin V^S_u$, we must have that $i \in V^S_u$. Thus $a_u = 3$. Now, by Proposition 5.10(1), T has associated string $(b_1, \dots, b_k, 2, c_l, \dots, c_1)$, where the middle entry "2" corresponds to the square of v'_u. Now, since $v_{s-1} \cdot e_k = -v_{s+1} \cdot e_k$, we have $v_s \cdot e_k = \pm v_{s-1} \cdot e_k = \mp v_{s+1} \cdot e_k$ and so S is negative and has associated string of the form $(3, b_1, \dots, b_k, 3, c_l, \dots, c_1) \in \mathcal{S}_{1c}$. □

Proposition 7.13 *Let S be cyclic, $I(S) \leq 0$, $p_1(S) = 0$, $p_2(S) > 0$, and $n \geq 4$. If $|\mathcal{J}| = 0$, then S is negative and has associated string in $\mathcal{S}_{1d} \cup \mathcal{O}$.*

Proof Note that $|\mathcal{I}| = p_2(S)$. By Lemma 7.7, $a_{t(i)} \geq 3$ for all $i \in \mathcal{I}$. If there exist $i_1, i_2 \in \mathcal{I}$ such that $v_{s(i_1)} \cdot v_{s(i_2)} \neq 0$, then, by Lemma 7.9, S is negative with associated string in $\mathcal{S}_{1d}$. Now assume that $v_{s(i_1)} \cdot v_{s(i_2)} = 0$ for all $i_1, i_2 \in \mathcal{I}$. Then, by Lemmas 5.16 and 7.9, $p_4(S) = p_2(S)$, $I(S) = 0$, and $p_j(S) = 0$ for all $j \notin \{2, 3, 4\}$. Let $G = (S \setminus \{v_{s(i)}, v_{t(i)} \mid i \in \mathcal{I}\}) \cup \{\pi_{e_i}(v_{t(i)}) \mid i \in \mathcal{I}\}$ and set $v'_{t(i)} = \pi_{e_i}(v_{t(i)})$ for all $i \in \mathcal{I}$, $v'_x := v_x$ for all $x \notin \{s(i), t(i) \mid i \in \mathcal{I}\}$, and $a'_x = -(v'_x)^2$ for all x. Then $p_2(G) = p_4(S) = 0$, $I(G) = 0$, $p_3(G) = n - p_2(G)$, and, by Lemma 7.9, G has $|\mathcal{I}|$ components. Finally, since, for each $i \in \mathcal{I}$, there exists an integer $j(i)$ such that $E^S_{j(i)} = \{s(i)-1, s(i), s(i)+1, t(i)\}$, G is irreducible and hence a good subset.

Assume C is a component of G of length at least 2. After possibly relabeling, let $C = \{v'_1, \dots, v'_m\}$. Since $v'_1 \cdot v'_2 = 1$, by Lemma 7.10, there is an integer $k \in V^G_1 \cap V^G_2$ such that $v'_1 \cdot e_k = -v'_2 \cdot e_k$. Since $|E^G_k| = 3$, there exists an integer z such that $k \in V^G_z$. Since v'_1 is a final vertex, $v'_z \cdot v'_1 = 0$ and so there exists an integer $l \in V^G_1 \cap V^G_z$. Moreover, since $|E^G_l| = 3$, we necessarily have that $a'_1 \geq 3$. We claim that, if $a'_z = 2$, then $v'_z = v'_3$. If $v'_z \neq v'_3$, then it is clear that v'_z must be isolated. In this case, since $v'_z \cdot v'_2 = 0$, we have $l \in V^G_2$ and $v'_1 \cdot e_l = -v'_2 \cdot e_l$. Since $v'_1 \cdot v'_2 = 1$, there exists another integer $m \in V^G_1 \cap V^G_2$ and so $a'_1, a'_2 \geq 3$. Let $L = (G \setminus \{v'_1, v'_2\}) \cup \{\pi_{e_k}(v'_1), \pi_{e_k}(v'_2)\}$; then L is good and $p_1(L) = 1$. By [8, Corollary 3.5], $I(L) = -3$; but it is clear that $I(L) = I(G) - 2 = -2$, which is a contradiction.

Thus, if $a'_z = 2$, then $v'_z = v'_3$ and we can perform a contraction yielding the subset $G' = G \setminus \{v'_1, v'_2, v'_3\} \cup \{\pi_{e_k}(v'_1), v'_2 + v'_3\}$. Notice that G' is a good subset with $I(G') = 0$ and $p_j(G') = 0$ for all $j \neq 3$; moreover, the component $C' = \{\pi_{e_k}(v'_1), v'_2 + v_3, v'_4, \dots, v'_m\}$ has length one less than the length of C. On the other hand, if $a'_z \geq 3$, then we can perform a contraction yielding the subset $G'' = G \setminus \{v'_1, v'_2, v'_z\} \cup \{v'_1 + v'_2, \pi_{e_k}(v'_z)\}$. As above, G'' is a good subset with $I(G'') = 0$ and $p_j(G'') = 0$ for all $j \neq 3$, and the component C'' resulting from C has length one less than the length of C. We may continue performing contractions in this way until the component C is reduced to an isolated vertex. We can similarly perform contractions on all remaining components until they are all isolated vertices. We obtain a good subset K that contains only isolated vertices. By Lemma 5.18, K is of the form

- $\{e_1 - e_2 + e_3 - e_4, e_1 + e_2, -e_1 + e_2 + e_3 - e_4, e_3 + e_4\}$, or
- $\{e_1 - e_2 - e_3, e_1 + e_2 - e_4, e_2 - e_3 + e_4, e_1 + e_3 + e_4\}$.

It is easy to see that no expansion of either subset exists. Thus $K = G$. Moreover, by construction, $|\mathcal{I}| = 4$ and we may assume that $1 = j(i_1)$, $2 = j(i_2)$, $3 = j(i_3)$, and $4 = j(i_4)$, where $\mathcal{I} = \{i_1, i_2, i_3, i_4\}$. Thus (up to the action of $\operatorname{Aut}\mathbb{Z}^8$), S is of the form either

$$\{e_1 - e_2 + e_3 - e_4 - e_{i_2} + e_{i_3}, e_{i_1} - e_1, e_1 + e_2, \\ e_{i_2} - e_2, -e_1 + e_2 + e_3 - e_4 - e_{i_1} - e_{i_4}, e_{i_3} - e_3, e_3 + e_4, e_{i_4} - e_4\}$$

or

$$\{e_1 - e_2 - e_3 - e_{i_2}, e_{i_1} - e_1, e_1 + e_2 - e_4 - e_{i_4}, e_{i_2} - e_2, e_2 + e_3 + e_4 + e_{i_3}, \\ e_{i_4} - e_4, e_1 + e_3 + e_4 + e_{i_1}, e_{i_3} - e_3\},$$

So S is negative cyclic with associated string $(6,2,2,2,6,2,2,2)$ or $(4,2,4,2,4,2,4,2)$, both of which are in $\mathcal{O}$. □

To summarize, we have proven the following:

Proposition 7.14 *Let S be a cyclic subset with $p_1(S) = 0$, $p_2(S) > 0$ and $I(S) \leq 0$. Then S is positive with associated string in $\mathcal{S}_{2c} \cup \mathcal{S}_{2d}$ or negative with associated string in $\mathcal{S}_{1c} \cup \mathcal{S}_{1d} \cup \mathcal{S}_{1e} \cup \mathcal{O} \cup \{(2^{[n]} \mid n \geq 2)\}$.*

Appendix

Given a sequence of integers $(a_1, \dots, a_n)$ the (Hirzebruch–Jung) continued fraction expansion is given by

$$[a_1, \dots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}.$$

If $a_i \geq 2$ for all $1 \leq i \leq n$, then this fraction is well defined and the numerator is greater than the denominator. In fact, for coprime $p > q > 0 \in \mathbb{Z}$, there exists a unique continued fraction expansion $[a_1, \dots, a_n] = p/q$, where $a_i \geq 2$ for all $1 \leq i \leq n$. Moreover, by reversing the order of the continued fraction, $[a_n, \dots, a_1] = p/q'$, where q' is the unique integer such that $1 \leq q' < p$ and $qq' \equiv 1 \bmod p$.

Lemma A.1 *Let $p/q = [a_1, \dots, a_n]$, $s/r = [a_1, \dots, a_{n-1}]$, and $\boldsymbol{a} = (a_1, \dots, a_n)$. Then $|\mathrm{Tor}(H_1(\boldsymbol{T}_{\pm A(\boldsymbol{a})}))| = p - (r \pm 2)$.*

Proof Let $\boldsymbol{a} = (a_1, \dots, a_n)$. By [11, Theorem 6.1], hyperbolic torus bundles are of the form $\boldsymbol{T}_{\pm A(\boldsymbol{a})} = T^2 \times [0,1]/(\boldsymbol{x}, 1) \sim (\pm A\boldsymbol{x}, 0)$, where

$$A = A(\boldsymbol{a}) = \begin{pmatrix} p & q \\ -s & -r \end{pmatrix}, \quad \frac{p}{q} = [a_1, \dots, a_n] \quad \text{and} \quad \frac{s}{r} = [a_1, \dots, a_{n-1}].$$

Note that, since $A \in \mathrm{SL}_2(\mathbb{Z})$, we have $qs - pr = 1$. Moreover, since $\boldsymbol{T}_{\pm A(\boldsymbol{a})}$ is hyperbolic, $\operatorname{tr} A(\boldsymbol{a}) = p - r > 2$. Now, by [12, Lemma 10], $|\mathrm{Tor}(H_1(\boldsymbol{T}_{\pm A(\boldsymbol{a})}))| = |\mathrm{tr}(\pm A(\boldsymbol{a})) - 2| = |\pm(p-r) - 2| = |\pm(p-(r\pm 2))| = p - (r \pm 2)$. □

Lemma A.2 *Let $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ be linear-dual strings, where $l + k \geq 2$, $x \geq 1$ be an integer, and $[b_1, \dots, b_k] = p/q$. Then $[b_1, \dots, b_k, x+1, c_l, \dots, c_1] = xp^2/(xpq+1)$ and $[c_1, \dots, c_l, x+1, b_k, \dots, b_1] = xp^2/(xp^2 - xpq + 1)$.*

Proof Given the first conclusion, the second follows since $(xpq+1)(xp^2 - xpq + 1) = xp^2(xpq - q^2 + 1) + 1$. We will now prove that $[b_1, \dots, b_k, x+1, c_l, \dots, c_1] = xp^2/(xpq+1)$.

Let $n = k + l + 1$ be the length of $(b_1, \dots, b_k, x+1, c_l, \dots, c_1)$. We proceed by induction on n. If $n = 3$, then $k = 1$, $l = 1$, $(b_1) = \frac{2}{1}$, and $[2, x+1, 2] = 4x/(2x+1) = x2^2/(x(2)(1)+1)$. Suppose the lemma is true for all length $n-1$ continued fractions and consider $[b_1, \dots, b_k, x+1, c_l, \dots, c_1]$. By definition of linear-dual strings, either $b_1 = 2$ and $c_1 \geq 3$ or vice versa.

Assume that $b_1 = 2$. Then the strings $(b_2, \dots, b_k)$ and $(c_1 - 1, \dots, c_l)$ are linear-dual and, by the inductive hypothesis,

$$[b_2, \dots, b_k, x+1, c_l, \dots, c_1 - 1] = \frac{xm^2}{xmn+1},$$
$$[c_1 - 1, c_2, \dots, c_l, x+1, b_k, \dots, b_2] = \frac{xm^2}{xm^2 - xmn + 1},$$

where $[b_2, \dots, b_k] = m/n$. Thus,

$$[c_1, c_2, \dots, c_l, x+1, b_k, \dots, b_2] = 1 + \frac{xm^2}{xm^2 - xmn + 1} = \frac{2xm^2 - xmn + 1}{xm^2 - xmn + 1}.$$

Since $(2xmn - xn^2 + 2)(xm^2 - xmn + 1) = (2xm^2 - xmn + 1)(xmn - xn^2 + 1) + 1$,

$$[b_2, \dots, b_k, x+1, c_l, \dots, c_1] = \frac{2xm^2 - xmn + 1}{2xmn - xn^2 + 2}.$$

Thus,

$$[b_1, \dots, b_k, x+1, c_l, \dots, c_1] = 2 - \frac{2xmn - xn^2 + 2}{2xm^2 - xmn + 1} = \frac{x(2m-n)^2}{x(2m-n)m + 1},$$
$$[b_1, \dots, b_k] = 2 - \frac{n}{m} = \frac{2m-n}{m}.$$

Setting $p = 2m - n$ and $q = m$ yields the result.

Now suppose $c_1 = 2$. Then $(b_1 - 1, \dots, b_k)$ and $(c_2, \dots, c_l)$ are linear-dual and

$$[b_1 - 1, \dots, b_k, x+1, c_l, \dots, c_2] = \frac{xm^2}{xmn+1},$$

$$[c_2, \dots, c_l, x+1, b_k, \dots, b_1 - 1] = \frac{xm^2}{xm^2 - xmn + 1},$$

where $[b_1 - 1, \dots, b_k] = m/n$. Thus,

$$[c_1, \dots, c_l, x+1, b_k, \dots, b_1 - 1] = 2 - \frac{xm^2 - xmn + 1}{xm^2} = \frac{xm^2 + xmn - 1}{xm^2}.$$

Since $(xmn + xn^2 + 1)xm^2 = (xm^2 + xmn - 1)(xmn + 1) + 1$,

$$[b_1 - 1, \dots, b_k, x+1, c_l, \dots, c_2, c_1] = \frac{xm^2 + xmn - 1}{xmn + xn^2 + 1}.$$

Thus,

$$[b_1, \dots, b_k, x+1, c_l, \dots, c_2, c_1] = 1 + \frac{xm^2 + xmn - 1}{xmn + xn^2 + 1} = \frac{x(m+n)^2}{x(m+n)n + 1},$$

$$[b_1, \dots, b_k] = 1 + \frac{m}{n} = \frac{m+n}{n}.$$

Setting $p = m + n$ and $q = n$ yields the result. □

Proposition A.3 *Let $[b_1, \dots, b_k] = p/q$ and let $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathcal{S}_{1a}$. Then $|\mathrm{Tor}(H_1(\boldsymbol{T}_{-A(\boldsymbol{a})}))| = p^2$.*

Proof Let $\boldsymbol{a} = (2, b_1, \dots, b_k, 2, c_l, \dots, c_1)$, where $(b_1, \dots, b_k)$ and $(c_1, \dots, c_l)$ are linear-dual (up to cyclic reordering). By Lemma A.2, $[b_1, \dots, b_k, 2, c_l, \dots, c_1] = p^2/(pq+1)$ and so

$$[2, b_1, \dots, b_k, 2, c_l, \dots, c_1] = 2 - \frac{pq+1}{p^2} = \frac{2p^2 - pq - 1}{p^2}.$$

By Lemma A.1, $|\mathrm{Tor}(H_1(\boldsymbol{T}_{-A(\boldsymbol{a})}))| = |2p^2 - pq - 1 - (\alpha - 2)|$, where α is the denominator of $[2, b_1, \dots, b_k, 2, c_l, \dots, c_2]$. By Lemma A.2,

$$[c_1, \dots, c_l, 2, b_k, \dots, b_1] = \frac{p^2}{p^2 - pq + 1}$$

and so

$$[c_2, \dots, c_l, 2, b_k, \dots, b_1] = \frac{p^2 - pq + 1}{(c_1 - 1)p^2 - c_1 pq + c_1}.$$

Thus,

$$[b_1, \dots, b_k, 2, c_l, \dots, c_2] = \frac{p^2 - pq + 1}{s} \quad \text{for some } s.$$

Now it is clear that $\alpha = p^2 - pq + 1$ and so

$$|\mathrm{Tor}(H_1(\boldsymbol{T}_{-A(\boldsymbol{a})}))| = |2p^2 - pq - 1 - (p^2 - pq + 1 - 2)| = p^2.$$ □

Lemma A.4 *Let*

$$A=(a_{ij})=\begin{bmatrix} -a_1 & 1 & & (-1)^t & & & & (-1)^r \\ 1 & -a_2 & & & & & & \\ & & \ddots & 1 & & & & \\ (-1)^t & & 1 & -a_k & 1 & & & \\ & & & 1 & \ddots & & & \\ & & & & & -a_{n-1} & 1 & \\ (-1)^r & & & & & 1 & -a_n & \end{bmatrix}.$$

Suppose $a_i \ge 2$ for all $1 \le i \le n$, $a_1 \ge 3$, $a_k \ge 3$, and $r,t \in \{0,1\}$.

(1) *If $r=1$ or $t=1$, then A is negative definite.*

(2) *If $r=t=0$ and either $a_1 \ge 4$, $a_k \ge 4$, or there exists an integer $i \notin \{1,k\}$ such that $a_i \ge 3$, then A is negative definite.*

Proof Let $s_i=\sum_{j=1}^n a_{ij}$ be the i^{th} row sum of A. Then $s_i \le 0$ for all i. Moreover, since either $a_1 \ge 4$, $a_k \ge 4$, or there exists an integer $i \notin \{1,s\}$ such that $a_i \ge 3$, there exists a row sum that is strictly less than 0. Let $w \in \mathbb{Z}^n$. Then

$$\begin{aligned} w^TAw &= \sum_{i,j} a_{ij}w_iw_j = \frac{1}{2}\sum_{i,j} a_{ij}(w_i^2+w_j^2-(w_i-w_j)^2) \\ &= \sum_{i,j} a_{ij}w_i^2 - \sum_{i<j} a_{ij}(w_i-w_j)^2 = \sum_i s_iw_i^2 - \sum_{i<j} a_{ij}(w_i-w_j)^2. \end{aligned}$$

First suppose $r=t=0$. Then every term in the above expression is at most zero and so $w^TAw \le 0$. Moreover, if either $a_1 \ge 4$, $a_k \ge 4$ or there exists an integer $i \notin \{1,k\}$ such that $a_i \ge 3$, then one of the row sums s_i is strictly less than 0. In this case, $w^TAw=0$ if and only if $w=0$. Thus A is negative definite. Next suppose $r=1$ and $t=0$. Then $s_1, s_n \le -2$ and so

$$\begin{aligned} w^TAw &= s_1w_1^2+s_nw_n^2+(w_1-w_n)^2+\sum_{i\ne 1,n} s_iw_i^2 - \sum_{\substack{i<j \\ (i,j)\ne(1,n)}} (w_i-w_j)^2 \\ &\le -2w_1^2-2w_n^2+(w_1-w_n)^2+\sum_{i\ne 1,n} s_iw_i^2 - \sum_{\substack{i<j \\ (i,j)\ne(1,n)}} (w_i-w_j)^2 \\ &= -(w_1+w_n)^2+\sum_{i\ne 1,n} s_iw_i^2 - \sum_{\substack{i<j \\ (i,j)\ne(1,n)}} (w_i-w_j)^2. \end{aligned}$$

Each term in this expression is clearly negative. If $w^TAw=0$, then, from the first term, $w_1=-w_n$. From the terms in the last summand, $w_1=w_2=\cdots=w_n$. Hence

$w_n = -w_n$, implying that $w_1 = \cdots = w_n = 0$. Therefore, A is negative definite. We obtain a similar result if $r = 0$ and $t = 1$. Finally assume $r = t = 1$. Then $s_1 \leq -4$ and $s_k, s_n \leq -2$. Arguing as above,

$$\begin{aligned} w^T A w &= s_1 w_1^2 + s_k w_k^2 + s_n w_n^2 + (w_1 - w_n)^2 + (w_1 - w_k)^2 \\ &\qquad + \sum_{i \neq 1,k,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j) \neq (1,n),(1,k)}} (w_i - w_j)^2 \\ &\leq -(w_1 + w_n)^2 - (w_1 + w_k)^2 + \sum_{i \neq 1,n} s_i w_i^2 - \sum_{\substack{i<j \\ (i,j) \neq (1,n),(1,k)}} (w_i - w_j)^2. \end{aligned}$$

Once again, we can see that A is necessarily negative definite. $\square$

Lemma A.5 *Let*

$$A = \begin{bmatrix} -a_1 & 2 & & & & (-1)^r \\ 2 & -a_2 & 1 & & & \\ & 1 & -a_3 & & & \\ & & & \ddots & & \\ & & & & -a_{n-1} & 1 \\ (-1)^r & & & & 1 & -a_n \end{bmatrix}.$$

Suppose $a_i \geq 2$ *for all* $1 \leq i \leq n$, $a_1 \geq 3$, $a_2 \geq 3$, *and* $r \in \{0, 1\}$.

(a) *If* $r = 1$, *then* A *is negative definite.*

(b) *If* $r = 0$ *and either* $a_1 \geq 4$, $a_2 \geq 4$ *or there exists an integer* $i \notin \{1, k\}$ *such that* $a_i \geq 3$, *then* A *is negative definite.*

Proof The proof is similar to the proof of Lemma A.4. $\square$

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Received: 10 July 2020 Revised: 1 March 2022

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 6 (pages 2415–2924) 2023