Algebraic & Geometric
Topology

Volume 23 (2023)

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Algebraic & Geometric Topology **23**:6 (2023) 2561–2591
DOI: 10.2140/agt.2023.23.2561
Published: 7 September 2023

Legendrian large cables and new phenomenon for nonuniformly thick knots

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We define the notion of a knot type having Legendrian large cables and show that having this property implies that the knot type is not uniformly thick. We then show that there exists an infinite family of ribbon knots that have Legendrian large cables. These knots fail to be uniformly thick in several ways not previously seen. We also give a general construction of ribbon knots, and show when they give similar such examples.

57K10, 57K33, 57R65

1 Introduction

The *contact width* $w(K)$ of a knot $K \subset (S^3, \xi_{\mathrm{std}})$ was defined by Etnyre and Honda in [4] as follows.[1] An embedding $\phi\colon S^1 \times D^2 \hookrightarrow S^3$ is said to represent K if the core curve of $\phi(S^1 \times D^2)$ is isotopic to K. (To simplify notation, we will not distinguish between $S^1 \times D^2$ and its image under ϕ.) Define the *slope* of homotopically nontrivial curves on $\partial(S^1 \times D^2)$ by identifying $\partial(S^1 \times D^2) \simeq \mathbb{R}^2/\mathbb{Z}^2$, where the meridian has slope ∞ and the longitude (which is well defined since K is inside S^3) has slope 0. Now define the *contact width* $w(K)$ as

$$w(K) = \sup \operatorname{slope}(\Gamma_{\partial(S^1 \times D^2)})$$

where the supremum is taken over $S^1 \times D^2 \hookrightarrow S^3$ representing K with $\partial(S^1 \times D^2)$ convex.

Etnyre and Honda [4] also defined K to have the *uniform thickness property* if

(1) any solid torus representing the knot type $\mathcal{K}$ can be thickened to a *standard neighborhood* of a Legendrian representative of K, and

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[1] These definitions are slightly different than those originally made in [4] since we are using a different slope convention in this paper; see Remark 1.1.

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(2) $w(K)$ is equal to the maximal Thurston–Bennequin invariant $\overline{\mathrm{tb}}(K)$ of Legendrian representatives of K.

Here a *standard neighborhood* $N(L)$ of a Legendrian knot L is an embedded solid torus $N(L)$ representing L with convex boundary ∂N such that $\#\Gamma_{\partial N} = 2$ and $\mathrm{tb}(L) = \mathrm{slope}(\Gamma_{\partial N})$. Standard neighborhoods are contact isotopic to any sufficiently small tubular neighborhood N of L with convex boundary and $\#\Gamma_{\partial N} = 2$.

The usefulness of this property became evident when Etnyre and Honda showed in the same work that if $L \subset S^3$ is Legendrian simple and uniformly thick, then cables of L are Legendrian simple as well. Recall that a knot type is Legendrian simple if Legendrian knots in this knot type are completely determined (up to Legendrian isotopy) by their Thurston–Bennequin invariant and rotation number. They also showed that if the cables are sufficiently negative, then they too satisfy the uniform thickness property. This allows that certain iterated cables of Legendrian simple knots are Legendrian simple, for example.

Uniform thickness has become a key hypothesis in work since then. For example, generalizing the above work on cables, Etnyre and Vértesi [6] showed that given a companion knot $L \subset S^3$ which is both Legendrian simple and uniformly thick, and a pattern $P \subset S^1 \times D^2$ satisfying certain symmetry hypothesis, the knots in the satellite knot type P_K may be understood.

Broadly, if one wants to classify Legendrian knots in a satellite knot type with companion knot $K \subset S^3$, and a pattern $P \subset S^1 \times D^2$, then as a first step one needs to understand

(1) contact structures on the complement of a neighborhood N of K,

(2) contact structures on a neighborhood N of K, and

(3) a classification of Legendrian knots in the knot type of the pattern P in the possible contact structures on N.

If K is uniformly thick, then N can always be taken to be a standard neighborhood of K with dividing curves on the boundary of slope $\overline{\mathrm{tb}}(K)$ (ie maximal Thurston–Bennequin invariant of K), which reduces the problem to items (1) and (3) above. Moreover, if K is Legendrian simple and uniformly thick, then (1) is more or less known as well [4]. If K is not uniformly thick, then understanding satellites is much more complicated.

Similarly, uniform thickness can be useful in understanding contact surgery constructions. A typical way to obtain a new contact 3–manifold is by removing a solid torus in the knot type K, and gluing in some new contact solid torus. To understand the new manifold, one needs to understand items (1) and (2) above, and the gluing map defining the surgery. If K is uniformly thick, then N can always be taken to be a standard neighborhood of K with dividing curves on the boundary of slope $\overline{\mathrm{tb}}(K)$, which simplifies (1) and (2) considerably.

On the other hand, there are knot types that are not uniformly thick. For such knot types, it is important to understand in what ways they can fail to be uniformly thick.

1.1 New phenomenon for nonuniformly thick knots

Given a knot type $\mathcal{K} \subset S^3$, the *contact width* of $\mathcal{K}$ is

$$w(\mathcal{K}) = \sup\{\mathrm{slope}(\Gamma_{\partial N}) \mid N \text{ is a solid torus representing } \mathcal{K} \text{ with convex boundary}\}.$$

We say a solid torus represents $\mathcal{K}$ if its core is in the knot type of $\mathcal{K}$. The contact width satisfies the inequality $\overline{\mathrm{tb}}(\mathcal{K}) \leq w(\mathcal{K}) \leq \overline{\mathrm{tb}}(\mathcal{K}) + 1$; see [4].

Remark 1.1 A word about slope conventions: If μ and λ are the meridional and longitudinal curves, respectively, on a torus T then $[\lambda]$ and $[\mu]$ form a basis for $H_1(T)$. A (p,q) curve, or a curve of slope q/p, will refer to any simple closed curve in T that is in the homology class of $p[\lambda] + q[\mu]$, where $p, q \in \mathbb{Z}$ are relatively prime. This is the opposite convention to the one used in several of the main references in this paper, which were some of the first works in convex surface theory. However, it is the convention that is standard in low-dimensional topology. We caution however that, when the phrase "integer slope" is used, it would correspond to the phrase "one over integer slope" in Etnyre and Honda [3; 4; 10] among others.

We are now in position to define uniform thickness. We say that a knot type $\mathcal{K}$ has the *uniform thickness property* or is *uniformly thick* if

(1) $\overline{\mathrm{tb}}(\mathcal{K}) = w(\mathcal{K})$, and
(2) every solid torus representing $\mathcal{K}$ can be thickened to a standard neighborhood of a maximal tb representative of $\mathcal{K}$.

By a standard neighborhood of a Legendrian knot L, we mean a solid torus neighborhood N of L with convex boundary, and dividing set $\Gamma_{\partial N}$ consisting of two curves with slope $\mathrm{tb}(L)$.

In past work, it is shown that a knot type $\mathcal{K}$ can fail to have the uniform thickness property in two ways. It can have neighborhoods whose slopes are larger than $\overline{\mathrm{tb}}$, as is the case with the unknot U which has $\overline{\mathrm{tb}}(U) = -1$ and $w(U) = 0$. It can also happen that there are neighborhoods with slope strictly less than $\overline{\mathrm{tb}}$, but that do not thicken.

The first and only such examples are due to [4] and Etnyre, LaFountain and Tosun [5] where it is shown that all positive torus knots $T_{p,q}$ have tori N with slopes satisfying $\mathrm{slope}(\Gamma_{\partial N}) < \overline{\mathrm{tb}}(T_{p,q})$ but that do not thicken. Moreover, the contact structure on all of these N is universally tight.

In what follows we will denote the set of Legendrian knots, up to isotopy, in the same topological knot type as K by $\mathcal{L}(K)$. We also use the convention that for a pair of relatively prime integers p and q, the (p,q) cable of K, that is, the knot type of a curve of slope q/p on the boundary of a torus neighborhood of K, is denoted by $K_{p,q}$. Notice that if $p = \pm 1$, then $K_{p,q}$ is a trivial cable in the sense that it is isotopic to the underlying knot K. The following theorem of Etnyre and Honda motivates us to define some new terminology.

Theorem 1.2 (Etnyre and Honda [4]) *If $K \subset S^3$ satisfies the uniform thickness property, then for $|p| > 1$ and any $L \in \mathcal{L}(K_{p,q})$ we have $\mathrm{tb}(L) \leq pq$.*

We generalize this result in Lemma 3.3 below. Notice that if we have a uniformly thick knot K and we fix a Legendrian representative $L \in \mathcal{L}(K)$ with $\mathrm{tb}(L) = k$, then there is an isotopy of K which arranges that L is a trivial cable $L = K_{1,k-1}$. But then we have that $\mathrm{tb}(K_{1,k-1}) = \mathrm{tb}(L) = k \nleq k-1$, so the inequality in Theorem 1.2 is not satisfied.

Definition 1.3 Given $|p| > 1$, we will say that a Legendrian cable $L \in \mathcal{L}(K_{p,q})$ is *large* if $\mathrm{tb}(L) > pq$, and call $K_{p,q}$ *Legendrian large* if there exists large $L \in \mathcal{L}(K_{p,q})$. We will then say that K has *Legendrian large cables*, or has the *Legendrian large cable* (*LLC*) *property*, if any of its nontrivial cables are Legendrian large.

Notice the example above indicates that if we allowed trivial cables, the LLC property would be vacuous. Our main theorem relates the LLC property to uniform thickness.

Theorem 1.4 *If K has Legendrian large cables, then there exist solid tori $V = S^1 \times D^2$ representing K such that $\xi|_V$ is virtually overtwisted. Moreover, V cannot be thickened to a standard neighborhood of a Legendrian knot, and so K is not uniformly thick.*

Recall that the term *universally tight* refers to a contact structure that is tight, and that, when lifted to the universal cover, remains tight. If the lift becomes overtwisted, then we will refer to the contact structure as *virtually overtwisted.*

Theorem 1.5 *Given K, if there exists a slope $q/p > \overline{\mathrm{tb}}(K)$ with $|p| > 1$ such that $K_{p,q}$ is Legendrian large, then $w(K) > \overline{\mathrm{tb}}(K)$.*

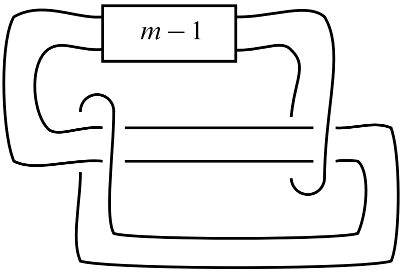

Figure 1: The ribbon knots K^m. There are $m-1$ right-handed full twists.

Question 1 Are there knots K and slopes $q/p < \overline{\mathrm{tb}}(K)$ such that $K_{p,q}$ is Legendrian large?

Question 2 If ξ is a virtually overtwisted contact structure on $S = S^1 \times D^2$, for which p and q is there a Legendrian (p,q) knot L in S with $\mathrm{tw}(L) > pq$?

In [15], Yasui gave some interesting examples of ribbon knots, which we will denote by K^m, shown in Figure 1. Yasui [15] shows that these knots have $\overline{\mathrm{tb}}(K^m) = -1$. In what follows, we will be concerned with integers $m < 0$. Building on his work, we observe that $K^m_{(-n,1)}$ is Legendrian large whenever $m \le -5$ and $1 < n \le \lfloor \frac{1}{4}(3-m) \rfloor$. This leads to the following theorem.

Theorem 1.6 *The knots K^m in Figure 1, with $m \le -5$, are not uniformly thick in $(S^3, \xi_{\mathrm{std}})$. In particular, there are solid tori T representing K^m such that*

$$\mathrm{slope}(\Gamma_{\partial T}) > \overline{\mathrm{tb}}(K^m)$$

and $\xi|_T$ is tight, but virtually overtwisted.

Remark 1.7 Previously, there were no known examples of $\mathcal{K}$ in $(S^3, \xi_{\mathrm{std}})$ with $w(\mathcal{K}) > \overline{\mathrm{tb}}(\mathcal{K})$ except for the unknot. These are also the first examples of solid tori in $(S^3, \xi_{\mathrm{std}})$ with virtually overtwisted contact structures.

It would be interesting to know what $w(K^m)$ is, and what the possible nonthickenable tori in the knot type of K^m are. We have the following partial result, following from Theorem 1.6 and its proof.

Proposition 1.8 *For $m \le -5$, the knots K^m in Figure 1 have*

$$w(K^m) \ge \frac{-1}{\lfloor \frac{1}{4}(3-m) \rfloor}.$$

The origin of the examples in Theorem 1.6 come from an interesting connection between contact structures and the famous cabling conjecture first observed by Lidman and Sivek in [12] where it is shown that for a knot K with $\overline{\mathrm{tb}}(K) > 0$, Legendrian surgery on K — ie $(\mathrm{tb}(K)-1)$–surgery — never yields a reducible manifold. They conjectured that this might be true with no condition on $\overline{\mathrm{tb}}(K)$. This is equivalent to the following conjecture for any K in S^3.

Conjecture 1.9 *For a Legendrian representative in the knot type* $L \in \mathcal{L}(K_{p,q})$, *we have* $\mathrm{tb}(L) \leq pq$.

If $\mathrm{tb}(L) > pq$ for such an L, then there exists L' with $\mathrm{tb}(L') = pq+1$ (we can always stabilize to achieve this). Legendrian surgery on this L' would then yield a reducible manifold; see Moser [14].

Theorem 1.10 (Yasui [15]) *There exist infinitely many Legendrian knots in* $(S^3, \xi_{\mathrm{std}})$ (see Figure 1), *each of which yields a reducible* 3*–manifold by a Legendrian surgery in the standard tight contact structure. Furthermore*, K *can be chosen so that the surgery coefficient is arbitrarily less than* $\overline{\mathrm{tb}}(K)$.

Yasui shows that for infinitely many pairs of integers $m, n \in \mathbb{Z}$ with $m \leq -5$, Legendrian surgery on the cables $K^m_{n,-1}$ yields a reducible manifold. This shows Lidman and Sivek's conjecture to be false, and stands in contrast with Theorem 1.2 of Etnyre and Honda.

We can now easily see that K^m — see Figure 1 — does not have the uniform thickness property. The interesting features of how K^m fails to be uniformly thick, given in Theorem 1.6, require much more work.

In [15], Yasui shows that for integers $n \leq \frac{1}{4}(3-m)$, the cables $K^m_{n,-1}$ have the property that $\overline{\mathrm{tb}}(K^m_{n,-1}) = -1$. But by Theorem 1.2, if K^m is uniformly thick, then we must have that $\overline{\mathrm{tb}}(K^m_{n,-1}) \leq -n$. So, for any $m \leq -5$ and any $1 < n \leq \lfloor \frac{1}{4}(3-m) \rfloor$, we arrive at a contradiction. This addresses the first assertion of Theorem 1.6.

Theorem 1.6 can be used to address the following conjecture.

Conjecture 1.11 *If* $K \subset S^3$ *is fibered, then* K *is uniformly thick if and only if* $\xi_K \neq \xi_{\mathrm{std}}$, *where* ξ_K *is the contact structure induced by an open book decomposition of* K.

Building on our above work, Hyunki Min [13] recognized that the K^m are counterexamples. Min showed that the K^m are all fibered. We also know that K^m are slice (since they are ribbon knots) and not strongly quasipositive (since they are obtained

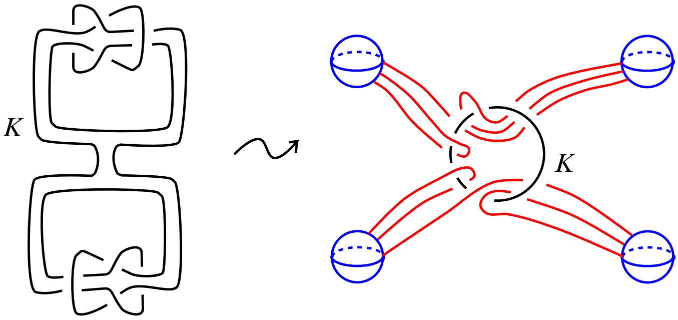

Figure 2: An example ribbon knot before running the algorithm in Theorem 1.12 (left), and after running the algorithm (right).

by attaching negative bands to two parallel disks), which implies that $\xi_K \neq \xi_{\mathrm{std}}$ by a result of Matthew Hedden [9, Proposition 2.1]. Theorem 1.6 tells us that the K^m are not uniformly thick however, so at least one direction of this conjecture is false. The other direction remains an interesting open question.

1.2 Ribbon knots and Legendrian large cable examples

Yasui's examples are all ribbon knots with Legendrian large cables, and can be generalized to other families of ribbon knots. We first observe a folk result that any ribbon knot can be described in a simple way.

Theorem 1.12 *Suppose $K \subset S^3$ is an arbitrary ribbon knot with $n \in \mathbb{N}$ ribbon singularities. Then there is an algorithm to construct a 2–handlebody for D^4 having $n-1$ or fewer 1–2–handle-canceling pairs such that there is an unknot U in the boundary of the 1–subhandlebody which, after attaching the 2–handles, is isotopic to K.*

A representation of a ribbon knot K as in Theorem 1.12 will be called a *handlebody picture for K*. The proof of Theorem 1.12 will be given in Section 3. Figure 2 gives an example ribbon knot and its image after running the algorithm.

Theorem 1.13 *Given an arbitrary ribbon knot K, we can associate to it a handlebody picture. If it is possible to Legendrian realize the attaching circles of the 2–handles so that the handle attachments are Stein (ie framings are all $\mathrm{tb}-1$), and also Legendrian realize K so that $\mathrm{tb}(K) = -1$, then K is a Legendrian ribbon knot that bounds a Lagrangian disk in $(B^4, \omega_{\mathrm{std}})$.*

Proof Given a handlebody picture for K, there is an unknot U in the boundary of the 1–subhandlebody which, by hypothesis, can be realized with $\mathrm{tb}(U) = -1$. Such

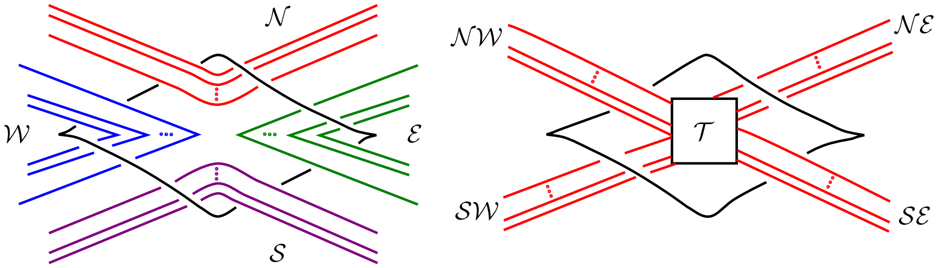

Figure 3: Possible examples of knots with Legendrian large cables. The ellipses are meant to indicate a finite number of strands bundled as shown, while $\mathcal{T}$ is an arbitrary Legendrian tangle.

an unknot bounds a Lagrangian disk in the 1–subhandlebody. Since the 2–handles are attached disjointly from this disk, K bounds a Lagrangian disk after they are attached, that is, K bounds a Lagrangian disk in $(B^4, \omega_{\mathrm{std}})$. □

Conway, Etnyre and Tosun [1] make use of this fact to describe when contact surgery on a knot in $(S^3, \xi_{\mathrm{std}})$ preserves symplectic fillability.

Corollary 1.14 *Given an arbitrary ribbon knot K, we can associate to it a handlebody picture. If it is possible to Legendrian realize the attaching circles of the 2–handles so that the handle attachments are Stein, Legendrian realize K so that $\mathrm{tb}(K) = -1$, and also arrange the local picture of K to be as in Figure 3, left, then K has Legendrian large cables.*

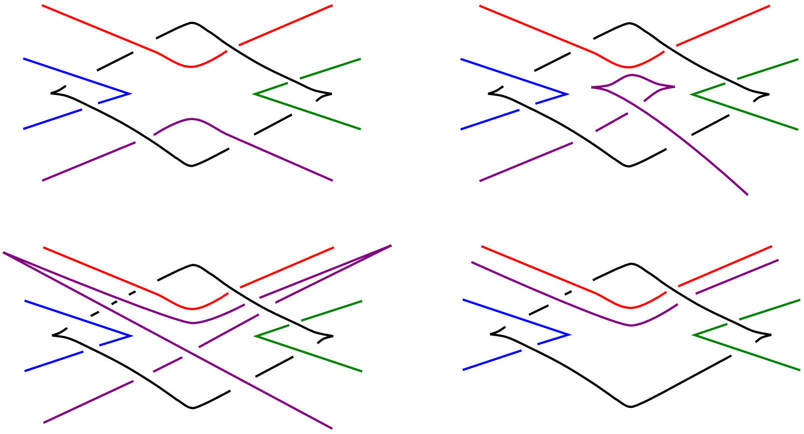

Figure 4: The steps in a Legendrian isotopy to change strands of type $\mathcal{S}$ into strands of type $\mathcal{N}$.

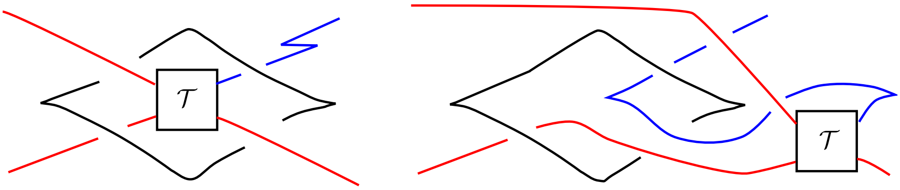

Figure 5: A Legendrian isotopy of the tangle $\mathcal{T}$. In this example, strands of type $\mathcal{NE}$ are assumed to have stabilizations.

Proof The proof is exactly the same as that of Yasui's Theorem 1.3 [15, pages 7–13], when there are only strands of type $\mathcal{N}$, since everything in the arguments can be done locally. The rest of the cases follow by Legendrian isotopy of Figure 3, left. For example, we can change all strands of type $\mathcal{S}$ into strands of type $\mathcal{N}$ by the Legendrian isotopy shown in Figure 4. We can also change all strands of types $\mathcal{E}$ and $\mathcal{W}$ into strands of type $\mathcal{N}$ by even easier isotopies. □

Remark 1.15 If the framings of the 2–handles allow stabilizations, then there are more examples. Given an arbitrary ribbon knot K, we can associate to it a handlebody picture. If it is possible to Legendrian realize the attaching circles of the 2–handles so that the handle attachments are Stein, Legendrian realize K so that $\mathrm{tb}(K) = -1$, arrange the local picture of K to be as in Figure 3, left, and arrange that there is a stabilization on each of the strands of at least one group of strands $\mathcal{NE}$, $\mathcal{NW}$, $\mathcal{SE}$, or $\mathcal{SW}$, then K has Legendrian large cables. This is true since we can isotope the stabilizations to have the form of Figure 5, left, Legendrian isotope the tangle $\mathcal{T}$ off to the side as shown in Figure 5, right, and then apply Corollary 1.14.

Acknowledgements The author would like to express profound gratitude to his advisor John B Etnyre for his patience, encouragement, and many helpful comments and suggestions, without which this paper would not have been possible. He would also like to extend thanks to James Conway for making him aware of Conjecture 1.11, and to Sudipta Kolay, Hyunki Min, Surena Hozoori and Peter Lambert-Cole for many useful and productive conversations.

2 Background

We will assume that the reader is familiar with Legendrian knots and basic convex surface theory. Some excellent sources for this material are [3; 7; 10; 11]. We will need to understand the twisting of a contact structure along a Legendrian curve with respect to two different framings. Suppose we are given a solid torus $S \subset (S^3, \xi)$ with convex boundary which represents the knot K. This just means that $S = D^2 \times S^1$ and $K = \{\mathrm{pt}\} \times S^1$ for some point in $\mathrm{int}(D^2)$. Further suppose that we are given a Legendrian (p, q) curve L in S. Since L is null homologous in S^3, there is a well defined framing on L given by any Seifert surface Σ, and measuring the twisting of ξ along L with respect to this framing gives us $\mathrm{tw}(L; \Sigma) = \mathrm{tb}(L)$, that is, the Thurston–Bennequin invariant of L. We can also find a boundary parallel torus $T^2 \subset S$ containing L, and measure the twisting of ξ along L with respect to the framing coming from T^2. We will denote this twisting by $\mathrm{tw}(L; \partial S)$. The relationship between these twistings is given by the expression

$$\mathrm{tw}(L; \partial S) + pq = \mathrm{tb}(L);$$

see [4].

Consider a contact structure ξ on $T^2 \times I$ with convex boundary, let T_1 and T_2 be its two torus boundary components, and assume without loss of generality that

$$s_1 = \mathrm{slope}(\Gamma_{T_1}) \le \mathrm{slope}(\Gamma_{T_2}) = s_2,$$

where Γ_S denotes the dividing curves on a convex surface S. Then we will say that ξ is *minimally twisting* if every convex, boundary parallel torus $S \subset T^2 \times I$ has $s_1 \le \mathrm{slope}(\Gamma_S) \le s_2$. This is the same notion of minimal twisting that Honda defined in [10]. We will also need to make use of his basic slices to decompose $T^2 \times I$ into layers. Using the same notation as above, we will call $(T^2 \times I, \xi)$ a *basic slice* if

(1) ξ is tight, and minimally twisting;
(2) T_i are convex and $\#\Gamma_{T_i} = 2$;
(3) s_i form an integral basis for $\mathbb{Z}^2$.

Honda showed that, up to isotopy fixing the boundary, there are exactly two tight contact structures on a basic slice, distinguished by their relative Euler classes in $H^2(T^2 \times I, \partial(T^2 \times I); \mathbb{Z})$.

The Farey tessellation, Figure 6, gives a convenient way to describe curves on T^2.

To construct the eastern half of the Farey tessellation, first label the north pole by $0 = \frac{0}{1}$, the south pole by $\infty = \frac{1}{0}$, and connect them by an edge (by edge, we mean a hyperbolic geodesic). Next, label the eastern most point that is midway between 0 and ∞ by $1 = \frac{1}{1}$, as shown in Figure 6. Connect 1 by edges to 0 and ∞. For rational numbers on the tessellation with the same sign, we can define an addition on the Farey tessellation by $a/b + c/d = (a + c)/(b + d)$, locate $(a + c)/(b + d)$ midway between a/b and

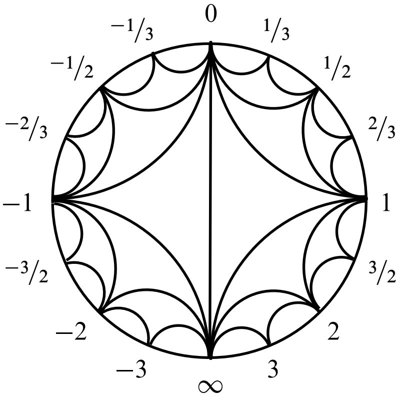

Figure 6: Farey tessellation.

c/d, and connect $(a+c)/(b+d)$ by edges with a/b and c/d respectively. Thus we can fill in the rest of the positive side of the Farey tessellation by iterating this addition. Notice that, if a/b and c/d are assumed to be an integral basis for $\mathbb{Z}^2$, then both

$$\begin{vmatrix} a & a+c \\ b & b+d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$$

and, similarly,

$$\begin{vmatrix} a+c & c \\ b+d & d \end{vmatrix} = \pm 1,$$

so any two points connected by an edge are an integral basis for $\mathbb{Z}^2$. Also notice that, given two positive rational numbers $a/b > c/d$, there are exactly two other points with edges to both a/b and c/d, namely $(a+c)/(b+d)$ and $(a-c)/(b-d)$.

To construct the western (negative) half of the Farey tessellation, first relabel the north pole by $0 = \frac{0}{-1}$. Next, label the western most point that is midway between 0 and ∞ by $-1 = \frac{1}{-1}$, as shown in Figure 6. Connect -1 by edges to 0 and ∞. Now using the same addition we defined above, we can iteratively build up the negative side of our Farey tessellation. Notice that the only point which was labeled twice was the north pole, which is now given by $\frac{0}{\pm 1}$.

For any two points p_1 and p_2 on the Farey tessellation, we define the interval $[p_1, p_2]$ to be the set of all points encountered starting from p_1 and moving clockwise around the tessellation until reaching p_2. Given a clockwise sequence of three points connected by edges p_1, p_2 and p_3 on the Farey tessellation, we say that a jump from p_2 to p_3 is *half maximal* if p_3 is the half way point of the maximum possible clockwise jump one could make in the interval (p_2, p_1). We will consider only clockwise paths in the

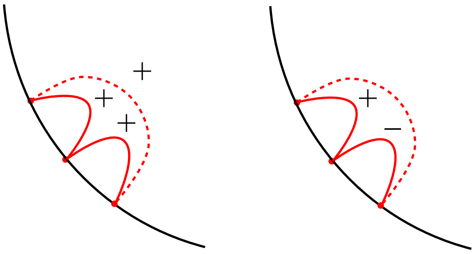

Figure 7: Left: a consistent shortening. Right: a shortening which is not consistent.

Farey tessellation, where a path is a sequence of jumps along edges. We call a path between two points $s_1, s_2 \in \mathbb{Q}$ a *continued fraction block* if, after the first jump, every jump is half maximal. Notice that, by construction, a path that is a continued fraction block cannot be shortened. We will also need to consider decorated paths (ie paths for which each jump gets a "+" or "−"). We can define an equivalence relation "$\sim$" on decorated paths in the Farey tessellation which says that any two paths with the same endpoints and which differ only by shuffling of signs within continued fraction blocks are in the same class. The following result, due to Honda [10], and in a different terminology Giroux [7], describes a relationship between contact structures on $T^2 \times I$ and minimal decorated paths in the Farey tessellation. Given a manifold M and a multicurve Γ in ∂M, let $\mathrm{Tight}(M, \Gamma)$ denote the set of isotopy classes of tight contact structures on M with convex boundary such that Γ is a set of dividing curves for ∂M. Similarly, given $T^2 \times I$ with boundary $T_1 \sqcup T_2$, and two multicurves Γ_i on T_i, let $\mathrm{Tight}(T^2 \times I, T_1 \cup T_2)$ denote the set of tight, minimally twisting contact structures on $T^2 \times I$ with convex boundary such that Γ_i is a set of dividing curves for T_i.

Theorem 2.1 (Honda [10]) *Given $T^2 \times I$ with boundary $T_1 \sqcup T_2$, and two multicurves Γ_i on T_i with $\#\Gamma_i = 2$ such that $s_1 = \mathrm{slope}(\Gamma_1) \leq \mathrm{slope}(\Gamma_2) = s_2$, there is a one-to-one correspondence*

$$\mathrm{Tight}(T^2 \times I, \Gamma_1 \cup \Gamma_2) \leftrightarrow \{\textit{minimal decorated paths from } s_1 \textit{ to } s_2\}/\sim.$$

Given $T^2 \times I$ with a two-component multicurve on each of its two torus boundary components, and with boundary slopes $s_1, s_2 \in \mathbb{Q}$, then any decorated path starting from s_1 and ending at s_2 describes a contact structure on $T^2 \times I$. Each jump in the path describes a basic slice, and therefore has two possible contact structures distinguished by the relative Euler class. We then get $T^2 \times I$ by concatenating these basic slices. For more details, see [10]. It follows from Theorem 2.1 that within any continued fraction block, shuffling the signs of the jumps results in isotopic contact structures.

Suppose we have a decorated path which can be shortened; see Figure 7. It follows from Honda's gluing theorem that if the two jumps which are being combined into a single jump have different signs, then the contact structure on $T^2 \times I$ described by this path is overtwisted. If the signs agree, then the contact structure will be tight. For this reason, we say that a shortening is *consistent* if the signs of the smaller jumps agree, and make the following theorem owing to Honda.

Theorem 2.2 *Given a decorated path in the Farey tessellation from* s_1 *to* s_2*, the contact structure on* $T^2 \times I$ *with convex boundary* $T_1 \sqcup T_2$, $\#\Gamma_{T_i} = 2$, $s_1 = \mathrm{slope}(\Gamma_{T_1})$ *and* $\mathrm{slope}(\Gamma_{T_2}) = s_2$ *described by this path is tight if and only if every shortening is consistent.*

To classify the tight contact structures on solid tori, we will consider a slightly different type of path. Let a *truncated path* be a decorated path, as defined above, with the sign of the first jump omitted from consideration. In other words, the first jump is not decorated. Suppose we have $S^1 \times D^2$ with a two-component multicurve on its torus boundary, and with boundary slope $s_2 \in \mathbb{Q}$. If the meridian of $\partial(S^1 \times D^2)$ has slope $s_1 \in \mathbb{Q}$, then we have the following classification. Given $S^1 \times D^2$ with boundary T, and a multicurve Γ on T, let $\mathrm{Tight}(S^1 \times D^2, \Gamma)$ denote the set of isotopy classes of tight, minimally twisting contact structures on $S^1 \times D^2$ with convex boundary, such that Γ is a set of dividing curves for T.

Theorem 2.3 (Honda [10]) *Given* $S^1 \times D^2$ *with boundary* T*, and a multicurve* Γ *on* T *with* $\#\Gamma = 2$ *such that* $s_2 = \mathrm{slope}(\Gamma)$ *and* $s_1 = \mathrm{slope}(\mu)$*, where* μ *is a meridional curve for* T*,*

$$\mathrm{Tight}(S^1 \times D^2, \Gamma) \leftrightarrow \{\textit{minimal truncated paths from } s_1 \textit{ to } s_2\}/\sim.$$

Theorem 2.4 (Honda [10]) (1) *Given* $T^2 \times I$ *with boundary* $T_1 \sqcup T_2$*, and two multicurves* Γ_i *on* T_i *with* $\#\Gamma_i = 2$ *such that* $s_1 = \mathrm{slope}(\Gamma_1) \leq \mathrm{slope}(\Gamma_2) = s_2$*, there are exactly two tight contact structures on* $T^2 \times I$*, and these contact structures are universally tight. The paths describing these two structures are the same, one decorated entirely by "+", and the other decorated entirely by "−".*

(2) *Given* $S^1 \times D^2$ *with boundary* T*, and a multicurve* Γ *on* T *with* $\#\Gamma = 2$ *such that* $s_2 = \mathrm{slope}(\Gamma)$*, and* $s_1 = \mathrm{slope}(\mu)$*, where* μ *is a meridional curve for* T*, then, if* $s_1 \cdot s_2 \neq \pm 1$*, there are exactly two tight contact structures on* $S^1 \times D^2$*, and these contact structures are universally tight. The paths describing these two structures are the same, one decorated entirely by "+", and the other decorated entirely by "−". If*

$s_1 \cdot s_2 = \pm 1$, *then there exists a unique tight contact structure on* $S^1 \times D^2$, *and this contact structure is universally tight.*

It follows from Theorem 2.4 that if we have a path with a mixture of signs, then the contact structure described by this path on either $T^2 \times I$, or on $S^1 \times D^2$, must be virtually overtwisted.

3 Cables in solid tori

In this section, we will give the proof of Theorems 1.4, 1.5 and 1.6 and Proposition 1.8. We would like to record and make use of the following result.

Theorem 3.1 (Etnyre and Honda [4]) *Any cable in a standard neighborhood of a Legendrian knot can be put on a convex torus.*

Proposition 3.2 *If* ξ *is a universally tight contact structure on a solid torus* S *with convex boundary, then any Legendrian* (p,q) *knot* $L \subset S$ *has* $\operatorname{tw}(L; \partial S) \leq 0$.

We delay the proof of Proposition 3.2 to the end of this section, but use it here to give proofs of our main theorems stated in the introduction.

Proof of Theorem 1.4 If K has Legendrian large cables, then there exists $L \in \mathcal{L}(K_{p,q})$ such that $\operatorname{tb}(L) > pq$. Take a solid torus S representing K and containing L as a (p,q) curve. Perturb S to have convex boundary. By hypothesis, $\operatorname{tw}(L; \partial S) > 0$, so by Proposition 3.2, $\xi|_S$ must be virtually overtwisted. Suppose that it were possible to thicken S to a standard neighborhood $\widetilde{S}$ of K. Then $\operatorname{slope}(\Gamma_{\partial \widetilde{S}}) \in \mathbb{Z}$, which implies, by a result of Kanda [11], that $\xi|_{\widetilde{S}}$ is the unique tight contact structure on $\widetilde{S}$, and moreover that $\xi|_{\widetilde{S}}$ is universally tight. But this is a contradiction since $S \subset \widetilde{S}$ and $\xi|_S$ is virtually overtwisted, so no such thickening exists. If K were uniformly thick, then any neighborhood of K would be thickenable to a $\operatorname{slope}(\overline{\operatorname{tb}}(K))$ standard neighborhood of K, which we have just seen is not possible. □

Proof of Theorem 1.5 By assumption, there exists $L \in \mathcal{L}(K_{p,q})$ such that $\operatorname{tb}(L) > pq$. Stabilize L to obtain $\widetilde{L}$ such that $\operatorname{tb}(\widetilde{L}) = pq$. There is a solid torus S representing K for which $\widetilde{L} \subset \partial S$, and as discussed at beginning of Section 2, we see that $\operatorname{tw}(\widetilde{L}; \partial S) = 0$. We can therefore C^0 perturb a collar neighborhood N of $\widetilde{L}$ in ∂S to be convex, and then C^∞ perturb $\partial S \setminus N$ to obtain a solid torus $\widetilde{S}$ representing K with convex boundary. Since $\operatorname{tw}(\widetilde{L}; \partial \widetilde{S}) = 0$, and since $\operatorname{slope}(\widetilde{L}) = q/p$, we must have that $\operatorname{slope}(\Gamma_{\partial \widetilde{S}}) = q/p$, owing to the fact that $\mathrm{tw}(\widetilde{L};\widetilde{S}) = -\frac{1}{2}|\widetilde{L}\bullet\Gamma_{\partial\widetilde{S}}|$ where $C_1\bullet C_2$ denotes the geometric intersection number of two curves on a torus. But $q/p > \overline{\mathrm{tb}}(K)$ by assumption, so $w(K) > \overline{\mathrm{tb}}(K)$. □

Proof of Theorem 1.6 In [15], Yasui shows that for integers $n \leq \frac{1}{4}(3-m)$, the cables $K^m_{n,-1}$ have the property that $\overline{\mathrm{tb}}(K^m_{n,-1}) = -1$. So, for any $m \leq -5$ and any $1 < n \leq \lfloor \frac{1}{4}(3-m) \rfloor$, we see that K^m has Legendrian large cables $L \in \mathcal{L}(K^m_{n,-1})$. Then, by Theorem 1.4, K^m is not uniformly thick and has virtually overtwisted neighborhoods, and by Theorem 1.5 we have that $w(K^m) > \overline{\mathrm{tb}}(K^m)$ □

Proof of Proposition 1.8 The slope of the cable $K^m_{n,-1}$ is $\mathrm{slope}(K^m_{n,-1}) = -1/n$. Whenever $n \leq \frac{1}{4}(3-m)$, we know there exist $L \in \mathcal{L}(K^m_{n,-1})$ which are Legendrian large. Stabilize L to obtain $\widetilde{L}$ such that $\mathrm{tb}(\widetilde{L}) = -n$. There is a solid torus S representing K^m for which $\widetilde{L} \subset \partial S$, and we have seen that $\mathrm{tw}(\widetilde{L};\partial S) = 0$. Using the strategy of the proof of Theorem 1.5, we can C^0 perturb a collar neighborhood N of $\widetilde{L}$ in ∂S to be convex, and then C^∞ perturb $\partial S \setminus N$ to obtain a solid torus $\widetilde{S}$ representing K^m with convex boundary. Since $\mathrm{tw}(\widetilde{L};\partial\widetilde{S}) = 0$, and since $\mathrm{slope}(\widetilde{L}) = -1/n$, we must have $\mathrm{slope}(\Gamma_{\partial\widetilde{S}}) = -1/n$, and therefore $w(K^m) \geq -1/n$. □

Now we will give a series of results leading to the proof of Proposition 3.2.

Lemma 3.3 *If S is a solid torus with convex boundary, $\#\Gamma_{\partial S} = 2$, and $\mathrm{slope}(\Gamma_{\partial S}) \in \mathbb{Z}$ with its unique tight contact structure ξ, then any Legendrian (p,q) knot $L \subset S$ has $\mathrm{tw}(L;\partial S) \leq 0$.*

Proof Notice that this follows immediately from Theorem 3.1, since S is a standard neighborhood, and any Legendrian curve L on a convex torus T must have $\mathrm{tw}(L;T) = \mathrm{tw}(L;\partial S) \leq 0$. Alternatively, we can reason in the following way. Recall that Kanda [11] showed that any solid torus with integer slope and two dividing curves has a unique tight contact structure. Suppose that S is a solid torus with convex boundary, $\#\Gamma_{\partial S} = 2$, and $\mathrm{slope}(\Gamma_{\partial S}) = k \in \mathbb{Z}$ with its unique tight contact structure ξ, and that $L \subset S$ is a Legendrian (p,q) knot. Then S is a standard neighborhood of a Legendrian core curve K. Any two standard neighborhoods are contactomorphic, so we can find a neighborhood $N \subset (S^3,\xi_{\mathrm{std}})$ of a Legendrian unknot $U \subset S^3$ with $\mathrm{tb}(U) = -1$, and a contactomorphism $\varphi\colon S \to N$ which sends $\varphi(K) = U$. This contactomorphism sends torus knots to torus knots, so our (p,q) knot L is mapped to a $(p, q-p(k+1))$ knot $\varphi(L)$, as one can easily check. But now $\varphi(L)$ is a torus knot in (S^3,ξ_{std}), and Etnyre and Honda [3] have shown that $\mathrm{tb}(\varphi(L)) \leq p(q-p(k+1))$. But

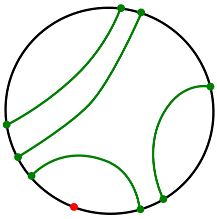

Figure 8: An arbitrary disk with arcs.

we understand how to switch between the Seifert framing and the framing coming from the torus ∂N, that is, $\operatorname{tw}(\varphi(L); \partial N) = \operatorname{tb}(\varphi(L)) - p(q - (k+1)) \le 0$. This implies that $\operatorname{tw}(L; \partial S) \le 0$, since N and S are contactomorphic. □

We can strengthen Lemma 3.3 slightly by dropping the assumption that $\#\Gamma = 2$.

Lemma 3.4 *If S is a solid torus with convex boundary, and* $\operatorname{slope}(\Gamma_{\partial S}) \in \mathbb{Z}$ *with any tight contact structure ξ, then any Legendrian (p,q) knot $L \subset S$ has* $\operatorname{tw}(L; \partial S) \le 0$.

Proof We will show that (S, ξ) will embed in a tight contact structure $(\widetilde{S}, \tilde{\xi})$ that satisfies the hypothesis of Lemma 3.3, and therefore show that $\operatorname{tw}(L; \partial S) \le 0$. To this end, we note that we can assume $\operatorname{slope}(\Gamma_{\partial S}) = 0$ by applying a diffeomorphism to S. Recall from [10], that ξ is completely determined by the dividing set Γ_D on a meridional disk D of S. We will build a model situation for S in which we can construct $(\widetilde{S}, \tilde{\xi})$. Since $\#\Gamma_{\partial S} > 2$ we see that $\#\Gamma_D > 1$. Suppose that we have a convex disk D with an arbitrary collection of dividing curves Γ, as in Figure 8.

Let v be a vector field on D that guides the characteristic foliation. We can label the regions in $D \setminus \Gamma$ as either Σ^+ or Σ^- so that no adjacent pair share the same label. There exists an area form ω on D which satisfies that $\pm\operatorname{div}_\omega v > 0$ on $\Sigma^\pm$. Assign a 1–form $\lambda = \iota_v \omega$; then we know from Giroux [7] that there exists a function $u \colon D \to \mathbb{R}$

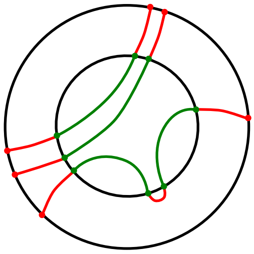

Figure 9: An annulus has been attached, and the number of curves has been reduced by one.

such that $u\,dt+\lambda$ gives rise to a contact structure ξ on $D\times\mathbb{R}$ that is invariant in the $\mathbb{R}$ direction. Moreover, we know from a theorem of Giroux that ξ is tight, since there are no homotopically trivial dividing curves. This invariance means that we can mod out by $\mathbb{Z}$ to obtain a tight contact structure on a solid torus $(D\times\mathbb{R})/\mathbb{Z}=D\times S^1$. The solid torus and contact structure we obtain in this way are contactomorphic to our original (S,ξ), that is, there exist v, ω and $u\colon D\to\mathbb{R}$ for which this construction exactly reproduces (S,ξ).

Now suppose that the number of properly embedded arcs is greater than 1. We would now like to reduce the number of dividing curves by taking a larger disk containing our original D. So we attach an annulus to D to obtain $D_{\mathrm{ext}}=D\cup_\varphi(S^1\times[0,1])$, where $\varphi\colon S^1\times\{0\}\to\partial D$ is the gluing map. Denote the endpoints of the properly embedded arcs by $\{x_1,\dots,x_{2k}\}$. Notice that if we fix a point on $p\in\partial D$ and move counterclockwise from p along ∂D, then it must happen that we encounter an x_i followed by an x_{i+1} which are not endpoints of the same curve. If this were not so, then there could only be one curve, which we have supposed not to be the case. Without loss of generality, assume that these two points are x_1 and x_2. Now connect these points by an arc in $S^1\times[0,1]$. Form arcs from the remaining points $\{x_3,\dots,x_{2k}\}$ to $\partial D_{\mathrm{ext}}$ by using $\{x_i\}\times[0,1]$, as in Figure 9. Notice that D_{ext} has one fewer embedded arc than D. So we can iterate this procedure to obtain a disk $\widetilde{D}\supset D$ which has only 1 properly embedded arc. Call this arc $\widetilde{\Gamma}$. Notice that we can arrange the gluing map φ to be smooth and such that the extension of Γ to $\widetilde{\Gamma}$ is smooth. We can also smoothly extend ω and v to $\widetilde{D}$ so that the singular foliation on $\widetilde{D}$ guided by v has $\widetilde{\Gamma}$ as a dividing curve. We can now build, just as we did above, a contact structure $\tilde{\xi}$ on $\widetilde{D}\times S^1=\widetilde{S}$ having $\widetilde{D}$ as a convex meridional disk, with convex boundary. Since $\#\Gamma_{\widetilde{D}}=1$, we have $\operatorname{tb}(\partial\widetilde{D})=-1$, which in turn implies that $\#\Gamma_{\partial\widetilde{S}}=2$. Notice that $\tilde{\xi}|_S=\xi$. Also notice that, by construction, the method of reducing the number of dividing curves on ∂S yields $\operatorname{slope}(\Gamma)=\operatorname{slope}(\widetilde{\Gamma})$. Now, by Lemma 3.3, any Legendrian (p,q) knot $L\subset S$ has $\operatorname{tw}(L;\partial S)\le 0$. □

Lemma 3.5 *If ξ is a universally tight contact structure on a solid torus S with convex boundary and $\#\Gamma_{\partial S}=2$, then any Legendrian (p,q) knot $L\subset S$ has $\operatorname{tw}(L;\partial S)\le 0$.*

Proof By a diffeomorphism of S, we can assume that $\operatorname{slope}(\Gamma_{\partial S})=-r/s$ where $-\infty\le -r/s\le -1$, and that the meridional slope is $-\infty$. Let $n=\lceil r/s\rceil$. Then since ξ is universally tight, we know that any path in the Farey tessellation describing our contact structure has the property that each jump must be decorated with the same

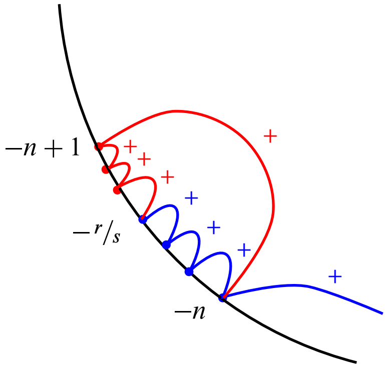

Figure 10: Farey tessellation picture describing the contact structure on our solid torus. The original solid torus, S, is shown in blue, while the red indicates the $T^2 \times I$ which is glued on to obtain the larger solid torus $\widetilde{S}$.

sign by Theorem 2.1. A portion of the Farey tessellation shows this in Figure 10. We can obtain a larger solid torus $\widetilde{S} \supset S$, which is convex, has two dividing curves, and with $\operatorname{slope}(\Gamma_{\partial \widetilde{S}}) = -n+1$ in the following way. Take a shortest path in the Farey tessellation from $-r/s$ to $-n+1$, and decorate each jump with the sign which appears in the description of the contact structure on S. This describes a contact structure on $T^2 \times I$ which extends S to $\widetilde{S}$, and since the signs are all the same we know that $\widetilde{S}$ is tight by Theorem 2.2. Moreover, we see that $\widetilde{S}$ has integer slope giving it a unique tight contact structure. Now we have that $\operatorname{tw}(K;\partial S) \leq 0$ by Lemma 3.3. □

Remark 3.6 In the above proof, we are able to thicken S to a larger solid torus $\widetilde{S} \supseteq S$ with $\operatorname{slope}(\Gamma_{\partial \widetilde{S}}) = -n+1$ because we are thinking of $S = S^1 \times D^2$ abstractly as a contact 3–manifold with convex boundary, and not embedded in any particular contact manifold. There is a shortest path in the standard Farey tessellation picture from any negative rational $-r/s$ to $-n+1$ which describes our contact structure. We

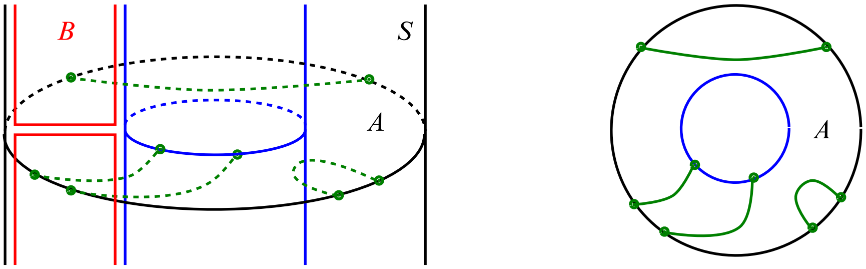

Figure 11: Left: $X = T^2 \times I$. Right: the annulus A and its dividing curves.

are not claiming that if S is a solid torus representing a knot $K \subseteq S^3$ it must always be thickenable in S^3, for example, Etnyre, LaFountain, and Tosun have given examples of nonthickenable tori in [5].

Proposition 3.2 strengthens Lemma 3.5 slightly by dropping the assumption that $\#\Gamma = 2$.

Proof of Proposition 3.2 Suppose we are given a solid torus S with convex boundary, a universally tight contact structure ξ, and we have a Legendrian (p,q) knot L in S. Again, by a diffeomorphism of S, we can assume that $\operatorname{slope}(\Gamma_{\partial S}) = -r/s$, where $-\infty \le -r/s \le -1$, and that the meridional slope is $-\infty$. Let $n = \lceil r/s \rceil$. If

$$\#\Gamma_{\partial S} = 2k > 2,$$

then we can attach a bypass to ∂S along a Legendrian ruling curve to obtain a smaller solid torus $S' \subset S$ which has $\operatorname{slope}(\Gamma_{\partial S'}) = -r/s$ and $\#\Gamma_{\partial S'} = 2k-2$. We can repeat this procedure until we have a solid torus $\widetilde{S} \subset S$ which has $\operatorname{slope}(\Gamma_{\partial \widetilde{S}}) = -r/s$ and $\#\Gamma_{\partial \widetilde{S}} = 2$. Notice that the contact structure on $\widetilde{S}$ is just $\xi|_{\widetilde{S}}$. If we look at a meridional disk $D \subset S$, we know that along ∂D there are $2sk$ intersection points with $\Gamma_{\partial S}$; however, there exists a slope γ for which curves on ∂S of slope γ have exactly $2k$ intersection points with $\Gamma_{\partial S}$. For convenience, change coordinates on S so that $\operatorname{slope}(\gamma) \mapsto -\infty$ and $\operatorname{slope}(\Gamma_{\partial S}) \mapsto 0$. Notice that we have a $T^2 \times I$ layer $X = S \setminus \widetilde{S}$, and we can find a convex annulus A in X with Legendrian boundary of slope γ. We would like to show that the contact structure on X is completely determined by the dividing curves on A. Since $\#\Gamma_{\partial \widetilde{S}} = 2$, $\#\Gamma_{\partial S} = 2k$, and $\operatorname{slope}(\Gamma_{\partial \widetilde{S}}) = \operatorname{slope}(\Gamma_{\partial S}) = 0$, we know that the dividing curves on A must have the form shown in Figure 11, right, by the green arcs.

We know from Giroux [7] that the contact structure on a neighborhood of A is determined by its dividing curves. If we cut X along A, and round corners, we obtain a solid torus Y with convex boundary. Using the edge rounding lemma [10], it is easy to see that $\#\Gamma_{\partial Y} = 2$ and $\operatorname{slope}(\Gamma_{\partial Y}) = -1$. Notice in Figure 11, left, that we have a meridional disk B of Y which we have just seen has $\operatorname{tw}(\partial B) = -1$, and which we can perturb to be convex. There is a unique choice of dividing curves on such a disk. Finally, if we cut Y along B and round corners, we obtain a B^3 with convex boundary, which has a unique tight contact structure from work of Eliashberg [2]. So we have seen that the contact structure of X is determined solely by the dividing curves on A.

Let v be a vector field on A that guides the characteristic foliation. We can label the regions in $A \setminus \Gamma$ as either Σ^+ or Σ^- so that no adjacent pair share the same label. There exists an area form ω on A which satisfies that $\pm\operatorname{div}_\omega v > 0$ on $\Sigma^\pm$. Assign a

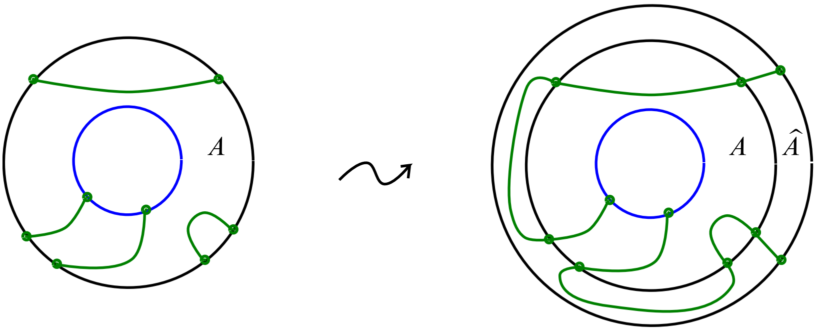

Figure 12: Reducing the number of dividing curves on A by extending with an annulus $\widehat{A}$.

1–form $\lambda = \iota_v \omega$; then we know from Giroux [7] that there exists a function $u\colon A \to \mathbb{R}$ such that $u\,dt + \lambda$ gives rise to a contact structure ξ on $A \times \mathbb{R}$ that is invariant in the $\mathbb{R}$ direction. Moreover, we know from a theorem of Giroux that ξ is tight since there are no homotopically trivial dividing curves. This invariance means that we can mod out by $\mathbb{Z}$ to obtain a tight contact structure on $(A \times \mathbb{R})/\mathbb{Z} = T^2 \times I$. The $T^2 \times I$ layer and contact structure we obtain in this way are contactomorphic to our original (X, ξ); that is, there exist v, ω and $u\colon A \to \mathbb{R}$ for which this construction exactly reproduces (X, ξ).

Now observe that we can smoothly extend A, abstractly, by an annulus $\widehat{A}$ causing the number of dividing curves to be reduced to 2, just as we did with the disk in the proof of Lemma 3.4; see Figure 12.

We can arrange that the extension of Γ_A to $\Gamma_{A \cup \widehat{A}}$ is smooth, and we can also smoothly extend ω and v to a neighborhood of $\widehat{A}$ so that the singular foliation on $\widehat{A}$ guided by v has $\Gamma_{A \cup \widehat{A}}$ as a set of dividing curves. We can now build, just as we did above, a contact structure $\hat{\xi}$ on $(A \cup \widehat{A}) \times S^1 = \widehat{X}$ with convex boundary. Since $\#\Gamma_{A \cup \widehat{A}} = 2$, we see that $\mathrm{tb}(\partial \widehat{A} \cap \partial \widehat{X}) = -1$, which implies that $\#\Gamma_{\partial \widehat{X}} = 2$. Notice that $\hat{\xi}|_X = \xi$. Also notice that, by construction, the method of reducing the number of dividing curves on ∂X yields $\mathrm{slope}(\Gamma_{\partial X}) = \mathrm{slope}(\Gamma_{\partial \widehat{X}})$. But now we have a minimally twisting $T^2 \times I$ layer $\widehat{X}$ whose boundary tori each have two dividing curves with slope 0. Honda [10] showed that there are an integers worth of tight contact structures satisfying these boundary conditions, and that each one is I–invariant. Adding the I–invariant thickened torus $X \cup \widehat{X}$ to $\widetilde{S}$, we get a new solid torus with contact structure contactomorphic to $\xi|_{\widetilde{S}}$, thus universally tight. Clearly S is contained in this solid torus. Now, by Lemma 3.5, L has $\mathrm{tw}(L; \partial S) = \mathrm{tw}(L; \partial S') \leq 0$. □

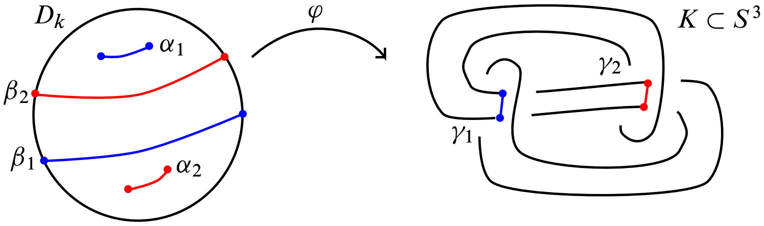

Figure 13: The immersion of a ribbon disk D_K.

4 Building ribbon knots from canceling handles

We are concerned here with ribbon knots, which we take to be the following: A knot $K \subset S^3$ is a *ribbon knot* if there is an immersed disk $\varphi\colon D_K \to S^3$ such that:

(1) $\partial\varphi(D_K) = K$.

(2) All of the double points of $\varphi(D_K) = \widetilde{D}_K$ (we will use the symbol $\sim$ to denote image under φ) occur transversely along arcs $\gamma_i \subset S^3$ whose preimage $\varphi^{-1}(\gamma_i) \subset D_K$ consists of exactly two arcs. One of these, α_i, must be contained entirely in the interior, $\alpha_i \subset \operatorname{int}(D_K)$, and the other, β_i (meant to suggest boundary), must be a properly embedded arc in D_K (ie $\partial\beta_i \subset \partial D_K$ and $\operatorname{int}(\beta_i) \cap \partial D_K = \varnothing$).

An example ribbon disk and its image under φ are shown in Figure 13.

Note that by transversality, the preimages of the γ_i's are 1–dimensional submanifolds of the compact manifold D_K, so there are only finitely many ribbon singularities γ_i.

We want to give a construction of an arbitrary ribbon knot using 1–2–handle-canceling pairs. Given any ribbon knot $K \subset S^3$, it has a ribbon disk D_K by definition. Notice, every ribbon singularity γ_i must appear exactly twice on the ribbon disk: once as a properly embedded arc, and once as an arc contained entirely in the interior of D_K. We will use a common color when picturing these pairs. So a general ribbon disk might look something like the one seen in Figure 14.

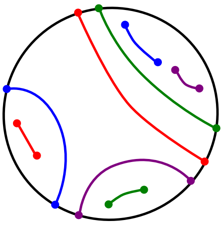

Figure 14: A general ribbon disk example.

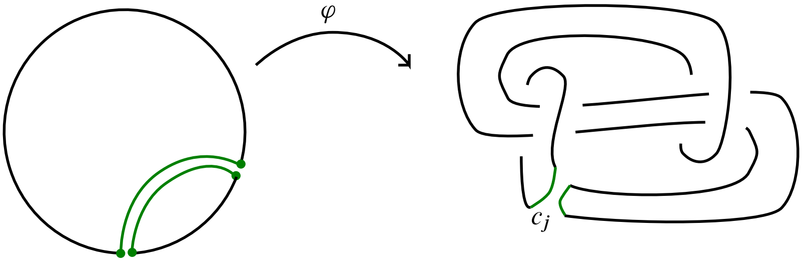

Figure 15: Cutting a ribbon disk along an arc c_j.

We will want to make cuts c_j, by pushing off two parallel copies of an arc in D_K and removing a small ϵ–strip. The result of this cut is shown in Figure 15.

We also need to set up a tool for manipulating ribbon disks and their images. Suppose we have an arc $b \subset \partial D_K$ whose endpoints are the endpoints of one of our β's. Further suppose that the subdisk D they bound contains no other singular points, as in Figure 16.

Let $N = I \times [0, \epsilon]$ be a collar neighborhood of β in D_K such that $(t, 0) = \beta$. We can form a new disk $D_\epsilon = D \cup N$ with boundary

$$\partial D_\epsilon = b \cup (0, s) \cup (1, s) \cup (t, \epsilon)$$

and notice that $\operatorname{int}(\beta) \subset \operatorname{int}(D_\epsilon)$. By choosing $\epsilon > 0$ sufficiently small, we can assume that $\widetilde{D}_\epsilon$ is embedded. Then we can see that D_ϵ guides an isotopy, supported in a small neighborhood of D_K, taking b to (t, ϵ) so that the disk $\overline{D_K - D_\epsilon} = D'_K$ does not contain β. We will refer to such a move as a *disk slide*; Figure 17 shows a typical one.

Theorem 4.1 *Given an arbitrary ribbon knot $K \subset S^3$ with $n \in \mathbb{N}$ ribbon singularities γ_i, we can make $n-1$ or fewer cuts c_j, so that what remains of K is an unlink, and what remains of $\widetilde{D}_K$ is, after n or fewer disk slides, embedded. That is, it is a collection of disjoint disks.*

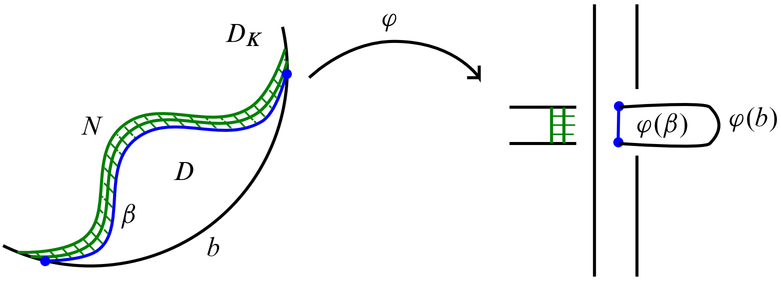

Figure 16: A subdisk and collar neighborhood, and its immersed image under φ.

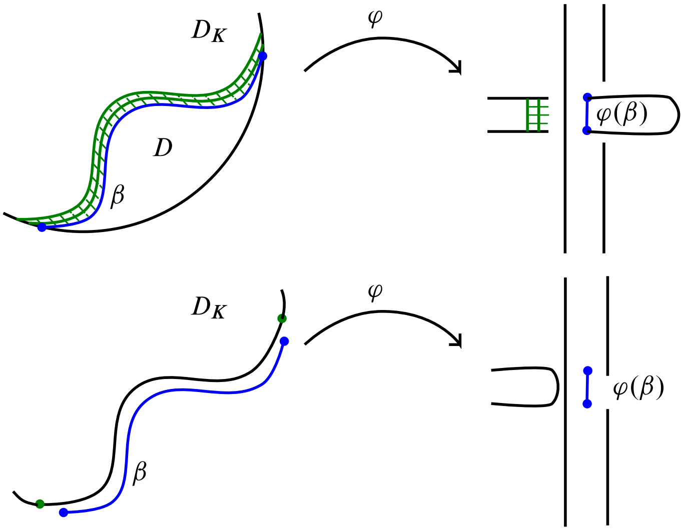

Figure 17: An illustration of a disk slide.

To prove this we will need the following.

Lemma 4.2 *Given a ribbon knot K with n ribbon singularities, if we can find a subdisk $D \subset D_K$ such that*

$$\partial D = (\text{an arc in } \partial D_K) \cup \beta_i$$

for one of our properly embedded arcs β_i, and $\mathrm{int}(D)$ is disjoint from all α's and β's, then a disk slide gives an isotopy of K supported in a small neighborhood of D so that the new slicing disk $\widetilde{D}'_K$ has $n-1$ ribbon singularities.

Proof For reference, let $b_i = \partial D \cap \partial D_K$ so that $\partial D = b_i \cup \beta_i$. Also, let $N = I \times [0, \epsilon]$ be a collar neighborhood of β_i in D_K such that $(t, 0) = \beta_i$, similar to the one shown in Figure 16.

Then we can define a new subdisk $D_\epsilon = D \cup N$ with boundary

$$\partial D_\epsilon = b_i \cup (0, s) \cup (1, s) \cup (t, \epsilon)$$

and notice that $\mathrm{int}(\beta_i) \subset \mathrm{int}(D_\epsilon)$. By choosing $\epsilon > 0$ sufficiently small, we can assume that $\widetilde{D}_\epsilon$ is embedded. Then there is a disk slide taking b_i to (t, ϵ) so that the disk $\overline{D_K - D_\epsilon} = D'_K$ does not contain β_i. But then it also cannot contain α_i, since the preimages of singularities occur in pairs, and hence the singularity γ_i has been eliminated. We also have that the resulting knot $\partial \widetilde{D}'_K$ is isotopic to K. □

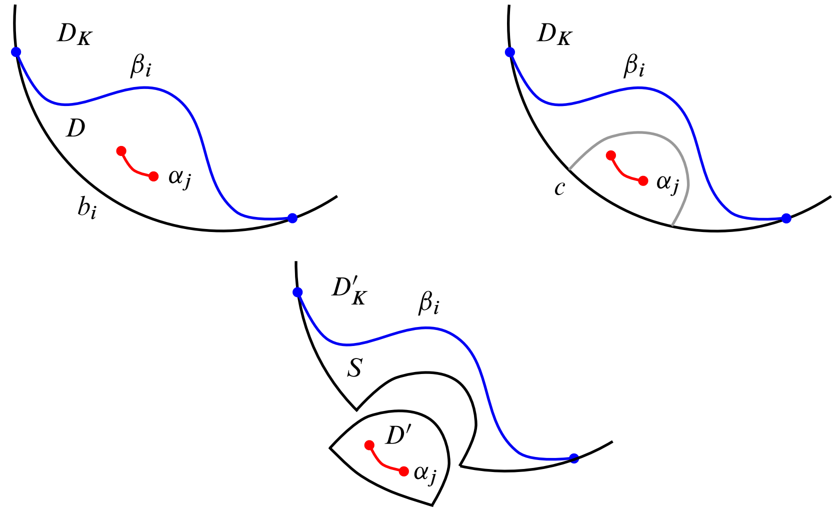

Figure 18: Cutting a ribbon disk.

Notice that Lemma 4.2 says that if we see a boundary parallel arc in D_K with no other singular points between that arc and some portion of ∂D_K, then we can eliminate that arc and its interior partner from the picture by an isotopy of K. Now back to our general picture and the proof of Theorem 4.1.

Proof of Theorem 4.1 We will assume that our ribbon disk is reduced in the sense that, if it were possible to simplify with a disk slide, then we have done so already. We will consider Figure 14 as our prototypical ribbon disk, and recall the convention that for each singularity γ_i, $\varphi^{-1}(\gamma_i)$ consists of $\alpha_i \cup \beta_i$ with β_i properly embedded. Given an arbitrary ribbon knot $K \subset S^3$ with $n \in \mathbb{N}$ ribbon singularities γ_i, and ribbon disk $\varphi\colon D_K \to S^3$, there will always be an "outermost" properly embedded arc β_i. This means that in some subdisk D, whose boundary is β_i together with an arc $b_i \subset \partial D_K$, there are only interior singular points α_j, and no other properly embedded arcs. Figure 18, top left, shows one such case.

Let c be a properly embedded arc in $D \subset D_K$ such that c cuts D into $D' \cup S$ with $\beta_i \subset S$ and D' containing all arcs $\alpha_j \subset D$. We may cut D_K along c so that φ is defined on $D'_K = \overline{D_K - D'}$ and D', and after a small isotopy of $\varphi\,|_{D'}$ we have that $\varphi(D')$ and $\varphi(D'_K)$ are disjoint, as pictured in Figure 18, bottom. Then a disk slide eliminates β_i by Lemma 4.2. Notice that when we eliminate a particular β_i using a disk slide, that automatically eliminates the corresponding α_i since they occur in pairs. Also notice, each cut eliminates at least one β_i, but could allow for the removal of more than one.

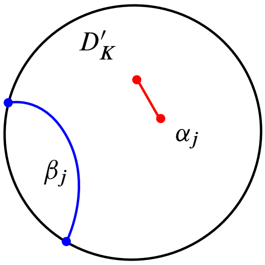

Figure 19: Final iteration.

But after at most $n-1$ cuts we have at most one β_j and its corresponding α_j. Since β_j cuts the disk it sits on into two components, one of them contains no α curves — see Figure 19 — and so β_j can be removed with no further cuts. Thus we never need to make the n^{th} cut since this last β curve may be eliminated by a disk slide without making a cut. Then the image under φ is now n embedded disks whose boundary is an unlink. □

We remark that this gives an upper bound on the number of cuts needed, but there are certainly cases where this number is not optimal as the following example shows.

Example 4.3 Consider the ribbon knot in Figure 20. This knot has $n+2$ ribbon singularities for any $n \in \mathbb{N}$, and yet only one cut (shown in green) will reduce the picture to two disjoint disks.

Now we will introduce handles and obtain a Kirby picture in which our knot K takes a particularly simple form. We assume that the reader is familiar with basic handlebody

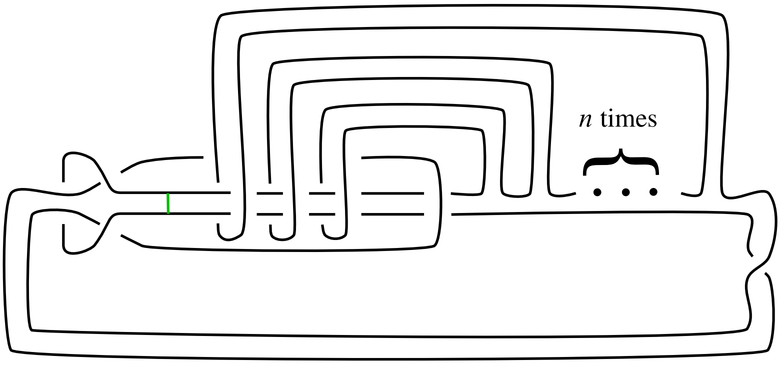

Figure 20: An example ribbon knot with $n+2$ singularities for which a single cut suffices.

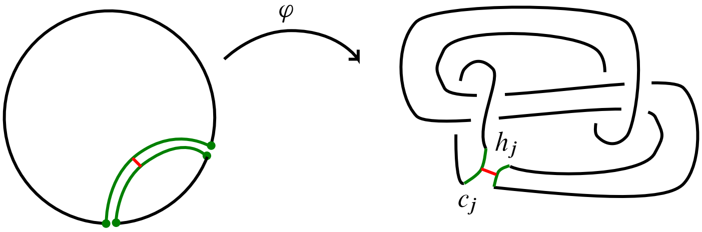

Figure 21: A 2–handle h_j associated to a cut c_j.

theory; an excellent reference for this material is [8]. For every cut c_j, we will attach an arc h_j seen in Figure 21. We will think of h_j as a thin ribbon, which would recover K if glued along. For this reason we will give the arc h_j a framing, by which we mean a parallel arc, and keep track of this framing through any isotopies of K. By a 1–*subhandlebody*, we will mean the subhandlebody consisting of the 0–handles and the 1–handles.

Proof of Theorem 1.12 Using Theorem 4.1, we can make $k < n$ cuts to the ribbon disk to obtain the unlink. So we have a diagram in which there are k disjoint disks, and $k-1$ framed arcs h_j. We know that by taking a band sum along these arcs (paying attention to framings) we can recover our diagram for K. Let K_{cut} be the union of the boundaries of these disks. Now in a small neighborhood of the end points of each h_j we insert the attaching spheres of a 1–handle, letting h_j be the attaching circle of a 2–handle as seen in Figure 22.

This pair cancels by construction, and also has the effect of doing the band sum that recovers K for the cut c_j as seen in Figure 23. Notice that we make two handle slides that free K_{cut} from the 1–handle, and then cancel the pair. Also notice that this has exactly the same effect that a band sum of K_{cut} along h_j would have had.

There is no obstruction to this handle slide and cancellation caused by the possible presence of other handle pairs since the double band sum shown on the left can be

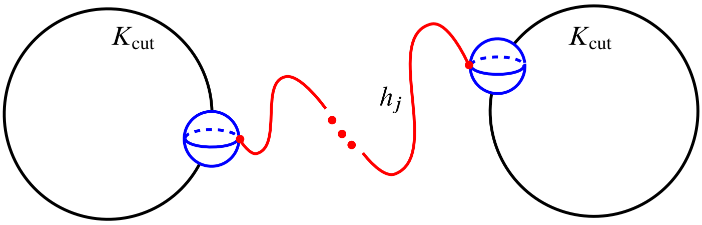

Figure 22: A 1–2–handle-canceling pair.

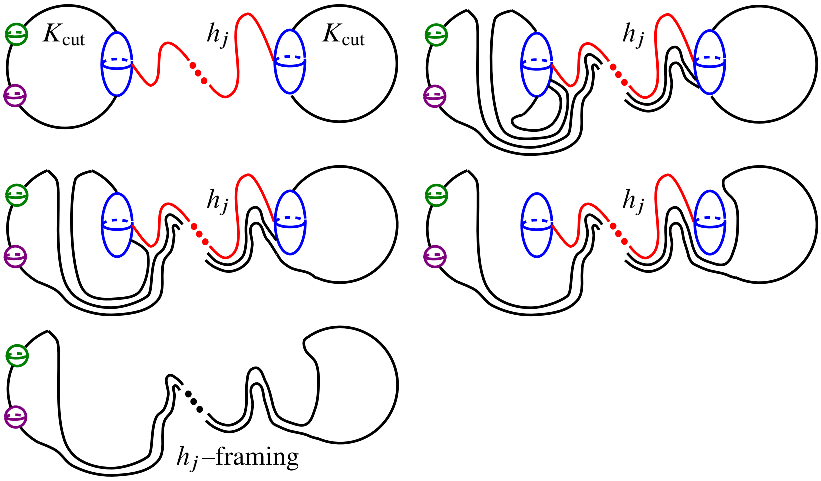

Figure 23: An example of handle cancellation to recover K.

carried out in a small neighborhood of the attaching sphere on the left. So, after $n-1$ or fewer iterations, we have recovered our diagram for K. It is worth noting that framings on 2–handles denote an even number of half twists; therefore the framings on the h_j must be even. If our diagram for K requires an odd number of half twists then we can accommodate this by inserting any number of half twists in one of the disks spanning K_{cut}, shown in Figure 24 for the case of a single half twist.

We would like to think of our diagram in which there are k disjoint disks connected by $k-1$ arcs h_j abstractly as a graph in order to show that K_{cut} can be pulled free of the 1–handles. To do this, we first work in the boundary of the 1–subhandlebody. We think of each of our disjoint disks as a vertex, and put an edge between vertices

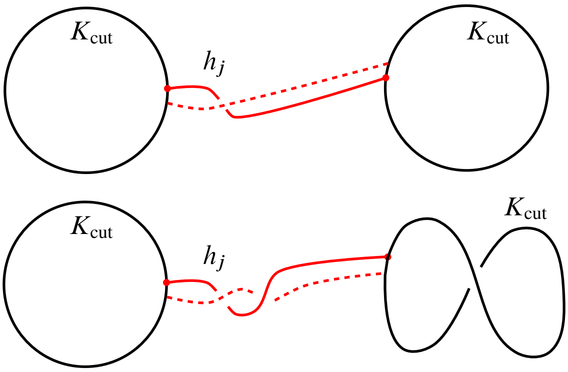

Figure 24: Framing adjustment.

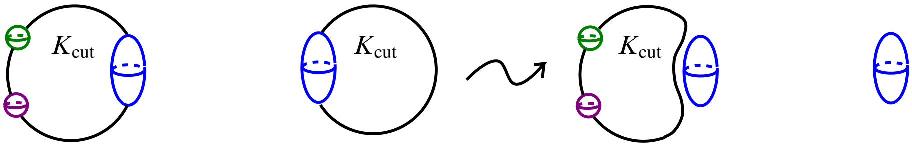

Figure 25: Handle picture corresponding to a univalent vertex of G.

if the corresponding disks are joined by a 1–handle. Notice G embeds in D_K as the "dual" graph to D_K cut along $\varphi^{-1}(c_j)$; that is, there is a vertex in the center of each component of $D_K - \bigcup_{j=1}^{k-1} \varphi^{-1}(c_j)$ and an edge for each $\varphi^{-1}(c_j)$. Then G is homotopy equivalent to D_K, and so we see that $\chi(G) = \chi(D_K) = 1$. It is well known that the Euler characteristic of a connected graph is one if and only if that graph is a tree, so G is a tree. Each univalent vertex of G is now associated to a portion of our picture consisting of two disks connected by a 1–handle, where one disk might have many 1–handle attaching spheres, but the other must have exactly one 1–handle attaching sphere as shown in Figure 25. In the 1–subhandlebody it is clear that K_{cut} may be isotoped off this 1–handle. Notice that the effect of this isotopy on G is to remove the corresponding edge and univalent vertex from the graph. Since G is a tree, we can iterate this procedure revealing that K_{cut} can be pulled completely free of the 1–handles. This may be seen in Figure 26 by simply ignoring the attaching circles of the 2–handles h_j.

The above iteration gives an isotopy of K_{cut} which extends to an ambient isotopy of the boundary of the 1–subhandlebody. This, in turn, induces an isotopy on the attaching circles of the 2–handles h_j, resulting in a 2–handlebody as claimed in Theorem 1.12. See Figure 26. By construction, handle slides and cancellations give us a knot isotopic to $K \subset S^3$. □

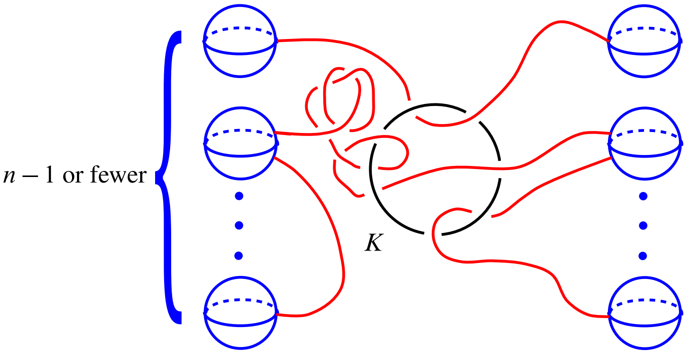

Figure 26: A 2–handlebody picture where K appears as the unknot in the boundary of the 1–subhandlebody.

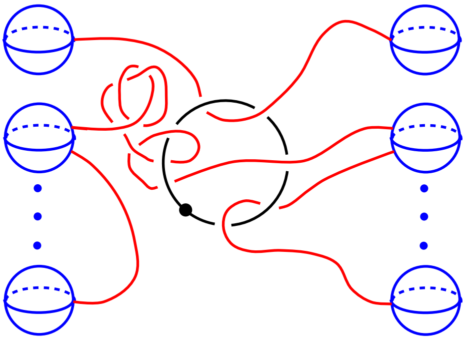

Figure 27: A 2–handlebody picture of the complement of the slice disk for K.

So we have shown that any ribbon knot with n ribbon singularities may be constructed by starting with the unknot in $\#_k S^1 \times S^2$, where $k \le n-1$, and attaching 2–handles to cancel each of the 1–handles in an appropriate manner.

Example 4.4 It is an exercise in Kirby calculus to show that images in Figure 2 are two pictures of the same ribbon knot in S^3.

Corollary 4.5 *In Figure 26, if we replace the unknot in the* 1*–subhandlebody with a dotted circle, then we obtain a picture of the* 4*–manifold which is the complement of the slicing disk in* D^4*, shown in Figure 27.*

Proof The slicing disk can be seen in the picture as the disk filling the unknot that we have in the 1–subhandlebody. This is because canceling the 1–2–handle pairs not only recovers K, but also recovers the ribbon disk $\widetilde{D}_K$. The definition of the dotted circle notation is that we remove a small neighborhood of the dotted unknot along with a small neighborhood of the disk after pushing it into D^4. And so this is exactly the complement of the slicing disk, $D^4 - \widetilde{D}_K$. □

One nice fact is that, since disk slides, isotopies and handle cancellations can be done locally, and since ribbon knots always bound an immersed ribbon disk, this construction actually works in any 3–manifold. We did not rely on any special properties of S^3 during the process. One can create examples by combining a 2–handlebody picture for a ribbon knot $K \subset S^3$ as in the above construction with a Kirby picture of a 4–manifold W whose boundary is the intended 3–manifold $M^3 = \partial W$. When combining the two pictures, K may be allowed to run across noncanceling 1–handles to form nontrivial examples as shown in Figure 28, where we have a Kirby picture of a 4–manifold whose

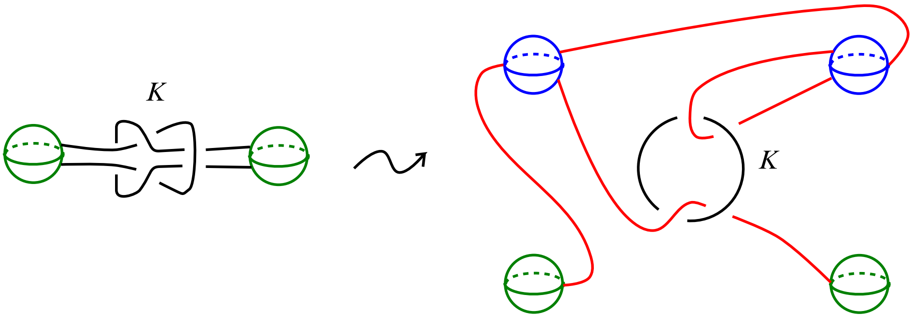

Figure 28: An example ribbon knot in $S^1 \times S^2$ and its decomposition.

boundary is $S^1 \times S^2$. We can see the ribbon disk for K in the image on the left. The image on the right shows the result using the technique developed above.

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Received: 7 October 2020 Revised: 30 August 2021

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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