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Detecting isomorphisms in the homotopy category

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We show that no generalization of Whitehead's theorem holds for unpointed spaces. More precisely, we show that the homotopy category of unpointed spaces admits no set of objects jointly reflecting isomorphisms. We give an explicit counterexample involving infinite symmetric groups. In contrast, we prove that the spheres do jointly reflect equivalences in the homotopy 2–category of spaces. We also show that homotopy colimits of transfinite sequential diagrams of spaces are not generally weak colimits in the homotopy category, and furthermore exhibit such a diagram with the property that none of its weak colimits is *privileged*, which means, roughly, that it sees the spheres as compact objects. The nonexistence of a set jointly reflecting isomorphisms in the homotopy category was originally claimed by Heller, but our results on weak colimits show that his argument had an inescapable gap, leading to the need for the new proof given here.

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1 Introduction

Let **Hot** denote the homotopy category of spaces, and let $\mathbf{Hot}_{*,c}$ denote the homotopy category of pointed, connected spaces. Whitehead's theorem says that, in $\mathbf{Hot}_{*,c}$, the set of spheres jointly reflects isomorphisms. One is naturally led to wonder whether there is a set of spaces in **Hot** which jointly reflects isomorphisms.

Brown [1] proved that a functor $\mathbf{Hot}_{*,c}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if it is half-exact, in the sense that it sends coproducts and weak pushouts in $\mathbf{Hot}_{*,c}$ to products and weak pullbacks in **Set**. Heller [4] proved an abstract representability theorem: if C is a category with coproducts and weak pushouts and C contains a "bounded" set $\mathcal{G}$

Kevin Arlin was formerly known as Kevin Carlson.

of objects that jointly reflects isomorphisms (see Definition 1.1 below), then a functor $C^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if it is half-exact. Heller also gave an example of a half-exact functor $\mathbf{Hot}^{\mathrm{op}} \to \mathbf{Set}$ which is not representable. He then claimed without proof [4, Proposition 1.2] that every set of spaces in $\mathbf{Hot}$ is bounded, and concluded [4, Corollary 2.3] that no set of spaces jointly reflects isomorphisms in $\mathbf{Hot}$.

We show that it is not true that every set of spaces is bounded, reopening the question of whether there is a set of spaces that jointly reflects isomorphisms in $\mathbf{Hot}$. We thus also give an independent proof that no set of spaces jointly reflects isomorphisms.

We now give the definitions needed in order to precisely state our results.

Definition 1.1 Let C be any category and let $\mathcal{G} \subseteq C$ be a set of objects.

(1) We say that $\mathcal{G}$ *jointly reflects isomorphisms* if a morphism $f : X \to Y$ in C is an isomorphism whenever $C(S, f) : C(S, X) \to C(S, Y)$ is a bijection for every $S \in \mathcal{G}$.[1]

(2) A *weak colimit* of a diagram $D : I \to C$ is a cocone through which every cocone factors, not necessarily uniquely.

(3) A cocone W of $D : I \to C$ is $\mathcal{G}$–*privileged* if the canonical map

$$\operatorname*{colim}_{\alpha \in I} C(S, D(\alpha)) \to C(S, W)$$

is a bijection for every $S \in \mathcal{G}$.

(4) For an ordinal β, we say that $\mathcal{G}$ is β–*bounded* if every diagram $D : \beta \to C$ has a $\mathcal{G}$–privileged weak colimit.

(5) We say that $\mathcal{G}$ is *left cardinally bounded*, or just *bounded*, if it is β–bounded for each sufficiently large regular cardinal β.

We use the word "set" to mean what is sometimes called a "small set", i.e. an object of the category $\mathbf{Set}$. All of our ordinals and cardinals are "small". We regard a cardinal as an ordinal which is least in its cardinality class. The *cofinality* of an ordinal α is the smallest ordinal that is the order type of a cofinal subset of α. A cardinal is *regular* if it is equal to its cofinality.

As mentioned above, $\mathbf{Hot}$ denotes the homotopy category of spaces, by which we mean the localization of the category of spaces at the weak homotopy equivalences, or, equivalently, the category whose objects are CW–complexes and whose morphisms are

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[1] Other terminology is in use, such as "$\mathcal{G}$ is a set of (weak) generators" or "the functors $C(S, -)$ are jointly conservative". Heller says that "$\mathcal{G}$ is left adequate".

homotopy classes of continuous maps. It is well known that every small diagram in **Hot** has a weak colimit, and that weak colimits are not unique.

We can now state our main results more precisely. First we give the result that shows that [4, Proposition 1.2] is false:

Theorem 3.1 *The set $\mathcal{G} = \{S^n \mid n \geq 0\}$ of spheres in* **Hot** *is not κ–bounded for any ordinal κ of uncountable cofinality. That is, for each such κ, there exists a diagram $D : \kappa \to$* **Hot** *that admits no $\mathcal{G}$–privileged weak colimit.*

Note that Theorem 3.1 applies to all uncountable regular cardinals, showing that the set of spheres is not left cardinally bounded. By adding one more space to the set, we can remove the uncountability assumption:

Corollary 3.2 *Let T denote a countably infinite, discrete space. Then the set $\{S^n \mid n \geq 0\} \cup \{T\}$ is not κ–bounded in* **Hot** *for any limit ordinal κ.*

The proof of Theorem 3.1 is somewhat involved and forms the bulk of the paper. We first show that it is sufficient to find a counterexample in the homotopy category **HoGpd** of groupoids. Then, given κ as in the statement, we consider the diagram $D : \kappa \to$ **HoGpd** sending α to the free group on $2 + \alpha$ generators. We make use of the theory of graphs of groups (see Serre [7]) and the associated fundamental groupoid (see Higgins [5]) in order to construct a sufficiently pathological cocone $D \to Z$, which we use to show that D admits no $\mathcal{G}'$–privileged weak colimit, where $\mathcal{G}' = \{B\mathbb{Z}\}$. This involves a detailed understanding of the morphisms in Z and how they are expressed as words in the given generators. It follows that the diagram $\kappa \to$ **Hot** sending α to the wedge of α circles has no $\mathcal{G}$–privileged weak colimit, where $\mathcal{G}$ is as in the statement of Theorem 3.1.

Heller's argument for his claim [4, Proposition 1.2] that any set $\mathcal{G}$ of objects in **Hot** is bounded was to take the cocone W to be the homotopy colimit, i.e. a generalized telescope. Since such homotopy colimits are $\mathcal{G}$–privileged, our result above implies that they are not, in general, even weak colimits in **Hot**. This is in contrast to the situation for telescopes of sequences indexed by ω, and for other homotopy colimits of diagrams indexed by freely generated categories. In the introduction to [2], Franke suggests using a Bousfield–Kan spectral sequence to show that Heller's claim is false, by comparing weak colimits to homotopy colimits, but we were unable to find an example in which we could prove that a certain differential was nonzero.

In the homotopy category of pointed, connected spaces, the set of spheres jointly reflects isomorphisms — this is the classical form of Whitehead's theorem. However,

we conjecture that the set of spheres is not bounded in $\mathbf{Hot}_{*,c}$. If this is true, it means that Heller's abstract representability theorem, as stated, does not imply Brown's representability theorem. That said, Heller's argument only requires a set of objects that jointly reflects isomorphisms and is β–bounded for *some* regular cardinal β. Thus, since the set of spheres is $\aleph_0$–bounded, the proof of Heller's theorem goes through in $\mathbf{Hot}_{*,c}$.

Next we state the result that shows that the statement of [4, Corollary 2.3] is nevertheless correct:

Theorem 2.1 *The category* $\mathbf{Hot}$ *contains no set* $\mathcal{G}$ *of spaces that jointly reflects isomorphisms. That is, there exists no set* $\mathcal{G}$ *of spaces such that, if* $f : X \to Y$ *is a map of spaces and* $f_* : \mathbf{Hot}(S, X) \to \mathbf{Hot}(S, Y)$ *is a bijection for every* $S \in \mathcal{G}$, *then* f *is an isomorphism in* $\mathbf{Hot}$.

This second result is easier to prove, and so we prove it first, in Section 2. Our method is a generalization of Proposition 4.1 of Matumoto, Minami and Sugawara [6], which gives a "phantom homotopy equivalence", that is, a map in $\mathbf{Hot}$ which, while not an isomorphism, is seen as one by all finite complexes. Our proof also shows that there is no set of connected spaces that jointly reflects isomorphisms in the homotopy category of connected spaces. Moreover, Theorem 2.1 implies similar results in other settings. For example, since $\mathbf{Hot}$ is a reflective subcategory of the homotopy category of $(\infty, 1)$–categories, it follows that there is no set of $(\infty, 1)$–categories that jointly reflects isomorphisms in that category.

Since the $(\infty, 1)$–category $\mathcal{S}$ of spaces certainly contains a set of objects jointly reflecting equivalences — namely the set whose only element is the one-point space — while its 1–categorical truncation $\mathbf{Hot}$ does not, one might ask which behaviours the n–categorical truncations of $\mathcal{S}$ exhibit for larger values of n. In fact, we show in Theorem 4.3 that, in the 2–category $\underline{\mathbf{Hot}}$ of spaces, morphisms and homotopy classes of homotopies between them, the set of spheres *does* jointly reflect equivalences, which is the natural generalization of joint reflection of isomorphisms to 2–category theory. Intuitively, the reason for the divergent behaviour of $\mathbf{Hot}$ and $\underline{\mathbf{Hot}}$ is that the 2–morphisms of $\underline{\mathbf{Hot}}$ retain the information about based homotopies that is lost in $\mathbf{Hot}$.

2 Hot admits no set that jointly reflects isomorphisms

We make the following definitions. For an ordinal α, write Σ_α for the group of all bijections of the set α, ignoring order. When $\beta < \alpha$, there is a natural inclusion $\Sigma_\beta \hookrightarrow \Sigma_\alpha$, and we define Σ_α^c to be the union of the images of Σ_β for all $\beta < \alpha$. We typically consider Σ_α^c when α is a cardinal, considered as the smallest ordinal with that cardinality, and we call the elements of Σ_α^c *essentially constant* permutations.

Theorem 2.1 *The category* **Hot** *contains no set* $\mathcal{G}$ *of spaces that jointly reflects isomorphisms.* (See Definition 1.1.)

Proof Let $\mathcal{G}$ be a set of spaces and let α be a regular cardinal larger than the cardinality of $\pi_1(S, s_0)$ for each $S \in \mathcal{G}$ and each $s_0 \in S$. We must construct a map $f : X \to Y$ which is not a homotopy equivalence but which induces bijections on homotopy classes of maps from spaces in $\mathcal{G}$.

Our example will be $Bs : B\Sigma_\alpha^c \to B\Sigma_\alpha^c$, where $s : \Sigma_\alpha^c \to \Sigma_\alpha^c$ is the shift homomorphism given by

$$(s\sigma)(\gamma) = \begin{cases} \sigma(\gamma') + 1 & \text{if } \gamma = \gamma' + 1, \\ \gamma & \text{if } \gamma \text{ is a limit ordinal,} \end{cases}$$

for $\sigma \in \Sigma_\alpha^c$. (Here and in what follows, if γ is a successor ordinal, we write γ' for its predecessor.) We must check that $s\sigma \in \Sigma_\alpha^c$. First, it is essentially constant: if $\beta < \alpha$ and σ fixes each $\gamma \geq \beta$, then, for $\gamma > \beta$, we have $(s\sigma)(\gamma) = \gamma$ if γ is a limit ordinal, and $(s\sigma)(\gamma) = \sigma(\gamma') + 1 = \gamma' + 1 = \gamma$ if γ is a successor. Next, we see that s is a homomorphism: $s(\sigma\tau)$ and $(s\sigma)(s\tau)$ both fix all limit ordinals, while for successors we have

$$(s\sigma)((s\tau)(\gamma)) = \sigma\big([\tau(\gamma') + 1]'\big) + 1 = \sigma\tau(\gamma') + 1 = s(\sigma\tau)(\gamma),$$

as desired. It follows that $s\sigma$ is a bijection, with inverse $s(\sigma^{-1})$.

Let H be a group with classifying space BH and let X be a connected space. If **Gp** denotes the category of groups, recall that $\mathbf{Hot}(X, BH)$ is isomorphic to $\mathbf{Gp}(\pi_1(X), H)$ modulo conjugation by elements of H. (See for example [8, Corollary V.4.4].) In particular, we have a natural isomorphism $\mathbf{Hot}(X, BH) \cong \mathbf{Hot}(B\pi_1(X), BH)$. It also follows that, for groups G and H, $\mathbf{Hot}(BG, BH)$ is isomorphic to $\mathbf{Gp}(G, H)$ modulo conjugation by elements of H, and that an element of $\mathbf{Hot}(BG, BH)$ is a homotopy equivalence if and only if it is represented by an isomorphism.

Note that s is not surjective, since $s\sigma$ always preserves limit ordinals. Therefore, $Bs : B\Sigma_\alpha^c \to B\Sigma_\alpha^c$ is not a homotopy equivalence. However, we will show that it

induces an isomorphism on $\mathcal{G}$. First observe that it suffices to prove this for connected components of spaces in $\mathcal{G}$. It follows that it is enough to prove this for spaces of the form BG, where G is a group of cardinality less than α.

Any map $BG \to B\Sigma_\alpha^c$ arises from a homomorphism $\varphi \colon G \to \Sigma_\alpha^c$, well defined up to conjugation. Since α is regular, there is a limit ordinal $\beta < \alpha$ such that $\varphi(g) \in \Sigma_\beta$ for every $g \in G$. We claim that $s \circ \varphi$ is conjugate to φ by an element $\tau \in \Sigma_\alpha^c$ defined by

$$\tau(\gamma) = \begin{cases} \gamma' & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \beta + \gamma & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \gamma + 1 & \text{if } \beta \leq \gamma < \beta + \beta, \\ \gamma & \text{otherwise.} \end{cases}$$

It is straightforward to check that τ is a permutation, and it clearly fixes ordinals greater than or equal to $\beta + \beta$, which is less than α. For $g \in G$, let $\sigma = \varphi(g)$. Then, noting that $\tau^{-1}(\gamma) = \gamma + 1$ for any $\gamma < \beta$, we have

$$(\tau^{-1}\sigma\tau)(\gamma) = \begin{cases} \tau^{-1}(\sigma(\gamma')) & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \tau^{-1}(\sigma(\beta + \gamma)) & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \tau^{-1}(\sigma(\gamma + 1)) & \text{if } \beta \leq \gamma < \beta + \beta, \\ \tau^{-1}(\sigma(\gamma)) & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \tau^{-1}(\sigma(\gamma')) & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \tau^{-1}(\beta + \gamma) & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \tau^{-1}(\gamma + 1) & \text{if } \beta \leq \gamma < \beta + \beta, \\ \tau^{-1}(\gamma) & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \sigma(\gamma') + 1 & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \gamma & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \gamma & \text{if } \beta \leq \gamma < \beta + \beta, \\ \gamma & \text{otherwise,} \end{cases}$$

$$= s(\sigma)(\gamma).$$

We have used that, if $\gamma \geq \beta$, then $\sigma(\gamma) = \gamma$, and the consequence that, if $\gamma < \beta$, then $\sigma(\gamma) < \beta$.

In summary, we have shown that Bs induces the identity on $\mathbf{Hot}(S, B\Sigma_\alpha^c)$ for every $S \in \mathcal{G}$, proving the claim. $\qquad\square$

Remark 2.2 Since the map $Bs \colon B\Sigma_\alpha^c \to B\Sigma_\alpha^c$ used in the proof has connected domain and codomain, it follows that there is no set of connected spaces that jointly reflects isomorphisms in the homotopy category of connected spaces.

We explain the origin of the maps s and τ. Morally, s is conjugation by the successor operation on ordinals, with limit ordinals handled specially. The map τ implements this by "making room" for the relevant limit ordinals in a range outside of the support of a particular permutation σ. In fact, if we denote the map τ above by τ_β, then s itself is conjugation by τ_α in Σ_γ^c for a regular cardinal $\gamma > \alpha$.

Remark 2.3 The referee pointed out an alternative proof of Theorem 2.1, which makes use of the techniques employed in [4, Lemma 2.2], namely the use of HNN extensions. It also involves a map between classifying spaces, but is less explicit. In addition, the referee and N Kuhn pointed out that the case when $\alpha = \omega$ was proved in [6, Proposition 4.1], using an approach very similar to ours.

3 The lack of privileged weak colimits

In this section, we give an example showing that Heller's privileged weak colimits do not generally exist.

Theorem 3.1 *The set $\mathcal{G} = \{S^n \mid n \geq 0\}$ of spheres in **Hot** is not κ–bounded for any ordinal κ of uncountable cofinality, e.g. for any uncountable regular cardinal. That is, for each such κ, there exists a diagram $D: \kappa \to$ **Hot** that admits no $\mathcal{G}$–privileged weak colimit.*

In particular, D admits no $\mathcal{G}$–privileged weak colimit for any set $\mathcal{G}$ containing the spheres. Note that the set of spheres is $\aleph_0$–bounded, so we learn that boundedness for one ordinal does not imply it for ordinals with larger cofinality.

Corollary 3.2 *Let T denote a countably infinite, discrete space. Then the set $\{S^n \mid n \geq 0\} \cup \{T\}$ is not κ–bounded in **Hot** for any limit ordinal κ.*

Proof Since κ is a limit ordinal, it has infinite cofinality. If κ has uncountable cofinality, then Theorem 3.1 applies. If κ has countable cofinality, then $\{T\}$ is not κ–bounded. □

In Section 3.1, we reduce the problem to finding a counterexample in the homotopy category of groupoids. In Section 3.2, we recall the theory of graphs of groups, and prove some general results about the word problem in the fundamental groupoid of a graph of groups. Finally, in Section 3.3, we give a counterexample in the homotopy category of groupoids and complete the proof of Theorem 3.1.

3.1 Reducing from spaces to groupoids

To prove Theorem 3.1 we will work primarily in the homotopy category **HoGpd** of groupoids, that is, the category of groupoids and isomorphism classes of functors. It is well known that the geometric realization of groupoids induces a reflective embedding B: **HoGpd** $\to$ **Hot** whose left adjoint is the fundamental groupoid functor Π_1 and whose image consists of the 1–types, i.e. those spaces X such that $\pi_n(X, x) = 0$ for every $n > 1$ and every $x \in X$. This follows from the adjunction between π_1 and the classifying space functor B that was used in the proof of Theorem 2.1.

Lemma 3.3 *Suppose given a diagram $D: J \to$ **HoGpd**, a set $\mathcal{G}'$ of groupoids and a set $\mathcal{G}$ of spaces containing $B\mathcal{G}'$ as well as S^n for all n. If D admits no $\mathcal{G}'$–privileged weak colimit in **HoGpd**, then $B \circ D: J \to$ **Hot** admits no $\mathcal{G}$–privileged weak colimit in **Hot**.*

Proof We prove the contrapositive. Let $\lambda: B \circ D \to X$ be a $\mathcal{G}$–privileged weak colimit, with $X \in$ **Hot**. Then, since left adjoints preserve weak colimits, $\Pi_1(\lambda): D \to \Pi_1 X$ is a weak colimit. We will show that it is $\mathcal{G}'$–privileged.

First, since λ is $\mathcal{G}$–privileged, every map $a: S^n \to X$ factors through a 1–type $BD(j)$ for some j. Thus, when $n > 1$, a is freely homotopic to a constant, which implies that $\pi_n(X, x)$ is trivial for all $x \in X$. We conclude that X is a 1–type itself, so that $X \simeq B(\Pi_1 X)$.

Since B is fully faithful, we see that $\Pi_1(\lambda): D \to \Pi_1 X$ is $\mathcal{G}'$–privileged. Indeed, if $G \in \mathcal{G}'$, then

$$\textbf{HoGpd}(G, \Pi_1 X) \cong \textbf{Hot}(BG, B(\Pi_1 X)) \cong \textbf{Hot}(BG, X)$$
$$\cong \operatorname*{colim}_{j} \textbf{Hot}(BG, BD(j)) \cong \operatorname*{colim}_{j} \textbf{HoGpd}(G, D(j)).$$

One can show that the composite isomorphism is induced by $\Pi_1(\lambda)$. $\qquad\square$

Thus, it suffices to exhibit appropriately pathological diagrams in **HoGpd**, and then to upgrade them to **Hot**. We aim to give a diagram in **HoGpd** admitting no weak colimit privileged with respect to the set $\mathcal{G}' = \{\boldsymbol{B}\mathbb{Z}\}$. Here $\boldsymbol{B}\mathbb{Z}$ denotes the groupoid freely generated by an automorphism, i.e. the groupoid with one object $*$ whose endomorphism group is the integers. Of course, $B(\boldsymbol{B}\mathbb{Z})$ is homotopy equivalent to S^1, so $\mathcal{G}$ in Lemma 3.3 can be taken to be the set of spheres.

Remark 3.4 For any groupoid G, a functor $f: B\mathbb{Z} \to G$ corresponds to an object $f(*)$ of G and an automorphism $f_*: f(*) \to f(*)$. Furthermore, two such functors $f, g: B\mathbb{Z} \to G$ are naturally isomorphic if and only if the automorphisms f_* and g_* are conjugate in G. In particular, a functor $f: B\mathbb{Z} \to G$ factors through $h: H \to G$ in **HoGpd** if and only if f_* is conjugate to an automorphism in the image of h.

3.2 Graphs of groups

To construct our example, we recall the notion of a graph of groups and prove Corollaries 3.7 and 3.8, and Lemma 3.9, which will be used in the next section.

Definition 3.5 A *graph of groups* Γ is given by

- a graph, i.e. a set X of vertices, a set Y of oriented edges, functions $s, t: Y \rightrightarrows X$, and an involution $\overline{(-)}: Y \to Y$ interchanging s and t;

- groups G_x and G_y for $x \in X$ and $y \in Y$ equipped with monomorphisms $\mu_y: G_y \to G_{s(y)}$ such that $G_y = G_{\bar{y}}$.

For simplicity, we assume that the groups G_x are disjoint. For more on graphs of groups, see [7, Section I.5; 3, Section 1.B].

Higgins [5] defined the fundamental groupoid $\Pi_1\Gamma$ of a graph of groups. The groupoid $\Pi_1\Gamma$ is the groupoid on objects X with generating morphisms the elements of the groups G_x, endowed with x as domain and codomain, together with the elements of Y viewed as morphisms $y: s(y) \to t(y)$. These generators are subject to the relations holding in the groups G_x, as well as new relations

$$\mu_{\bar{y}}(a) = y\mu_y(a)\bar{y}$$

for every y and every $a \in G_y$. Note in particular that $\bar{y} = y^{-1}$, and we shall use both notations. It may aid the intuition to consider $\Pi_1\Gamma$ as the fundamental groupoid of the space built from $\coprod_X BG_x$ with cylinders $BG_y \times I$ glued in for each set $\{y, \bar{y}\}$ of elements of Y related by the involution.

By definition, the groupoid $\Pi_1\Gamma$ is a quotient of the groupoid K with object set X and with morphisms freely generated by $\left(\coprod G_x\right) \amalg Y$, subject to the relations holding in the groups G_x. A morphism $x_0 \to x_n$ in K is given by a word $(a_n, y_n, \ldots, y_1, a_0)$, with $y_i \in Y$, $s(y_1) = x_0$, $t(y_n) = x_n$, and $s(y_{i+1}) = t(y_i) =: x_i$ for $1 \leq i < n$, while $a_i \in G_{x_i}$ for $0 \leq i \leq n$.

The natural realization functor $K \to \Pi_1 \Gamma$ will be denoted by $|(a_n, y_n, \ldots, y_1, a_0)| = a_n \circ y_n \circ \cdots \circ y_1 \circ a_0$. Higgins proved that every morphism of $\Pi_1 \Gamma$ is *uniquely* the image under $|\cdot|$ of a so-called "normal" word. We will not recall this concept, as we need only Higgins' corollary regarding the less rigid *irreducible* words.

A morphism $(a_n, y_n, \ldots, y_1, a_0)$ in K is called *reducible* if $n > 1$ and, for some i, $y_{i-1} = \bar{y}_i$ and $a_{i-1} \in \mu_{y_i}(G_{y_i})$. Otherwise, the morphism is said to be *irreducible*. Note that a reducible word can be shortened by the move

$$(\ldots, a_i, y_i, \mu_{y_i}(\hat{a}_{i-1}), \bar{y}_i, a_{i-2}, \ldots) \mapsto (\ldots, a_i \mu_{\bar{y}_i}(\hat{a}_{i-1}) a_{i-2}, \ldots)$$

to a word with the same realization. Therefore, every element of $\Pi_1 \Gamma$ is the realization of an irreducible word. We will use a key result of [5]:

Proposition 3.6 [5, Corollary 5] *Let w be an irreducible word in K. If $|w|$ is an identity morphism in $\Pi_1 \Gamma$, then $w = (e)$, where e is an identity element of some G_x.*

Define the *length* $\ell(w)$ of the word $w = (a_n, y_n, \ldots, y_1, a_0)$ to be n. We deduce the following:

Corollary 3.7 *Let Γ be a graph of groups and consider a word w in the groupoid K. If $\ell(w) > 0$ and $|w|$ is equal to the realization of a zero-length word, then w is reducible.*

Proof Suppose that $w = (a_n, y_n, \ldots, y_1, a_0)$ for $n > 0$ and that $|w| = |(a)|$ for some a in some G_x. Let $w' = (a_n, y_n, \ldots, y_1, a_0 a^{-1})$. Then $|w'|$ is an identity morphism in $\Pi_1 \Gamma$, so, by Proposition 3.6, w' is reducible. Since reduction occurs at interior points, w must be reducible as well.   □

Corollary 3.8 *Given a graph of groups Γ and a vertex x, the vertex group G_x embeds in the automorphism group of x in the fundamental groupoid $\Pi_1 \Gamma$.*

Because of this, we regard elements of the vertex groups as elements of the fundamental groupoid without explicitly naming the inclusion map.

Proof The map sends $a \in G_x$ to the realization of the word (a). Since the word (a) is irreducible, if the realization is an identity in $\Pi_1 \Gamma$, Proposition 3.6 tells us that a is the identity element of G_x. Therefore, this map is injective.   □

We next record some facts about free groups:

Lemma 3.9 *Let $A \subseteq B$ be nonabelian free groups, with A free on generators $\{a_i\}$ and B free on $\{a_i\} \cup \{b_j\}$.*

(1) *If $b \in B$ and, for all $a \in A$, we have $bab^{-1} = a$, then b is the identity.*

(2) *If $b \in B$ satisfies $bab^{-1} \in A$ for some $a \in A$, then either a is the identity or $b \in A$.*

Proof Fix $b \in B$. For part (1), if we take $a = a_i$ then the assumption that $ba_i b^{-1} = a_i$ shows that an irreducible word for b must have last letter a_i or a_i^{-1} for every i, which is absurd since there are at least two i's.

For part (2), we assume a is nontrivial and $b \notin A$. Factor b as $b'b''$, where $b'' \in A$ while b' is represented by an irreducible word with rightmost letter some b_j. Then $bab^{-1} = b'a'b'^{-1}$, where $a' := b''ab''^{-1}$ is a nontrivial element of A. The conclusion now follows from the observation that no reductions are possible in the concatenation of the irreducible words for b', a' and b'^{-1}, since concatenating those words gives no letter adjacent to its inverse. □

3.3 A counterexample in the homotopy category of groupoids

We now apply the generalities above to the problem of weak colimits in **HoGpd**.

We fix for the rest of the paper an ordinal κ of uncountable cofinality, and introduce the main characters in our counterexample. Note that Theorem 3.1 will follow if we replace $\kappa = [0, \kappa)$ by the interval $[2, \kappa)$, since the two categories are isomorphic. We use the latter because it allows us to use simple indexing while ensuring that all of the vertex groups below are nonabelian.

Definition 3.10 Define a graph of groups Γ with object set $[2, \kappa)$, vertex group G_α free on α generators, edge set $\{y_\alpha^\beta : \beta \to \alpha \mid \alpha \neq \beta \in [2, \kappa)\}$ and involution $y_\alpha^\beta \mapsto y_\beta^\alpha$. The edge group $G_{y_\alpha^\beta}$ is just $G_{\min(\beta,\alpha)}$. The edge morphism $\mu_{y_\alpha^\beta} : G_{\min(\beta,\alpha)} \to G_\beta$ is the natural inclusion. Let $Z = \Pi_1 \Gamma$.

Next, define a diagram $D : [2, \kappa) \to$ **HoGpd** by letting $D(\alpha) = G_\alpha$, with action on morphisms the natural inclusions, denoted by $D_\alpha^\beta : D(\beta) \to D(\alpha)$. We have a cocone $A : D \to Z$ with $A_\alpha : D(\alpha) \to Z$ the natural inclusion of the vertex group. To see that these maps do constitute a cocone, we note that y_α^β is the unique component of a natural isomorphism $A_\beta \cong A_\alpha \circ D_\alpha^\beta$.

We do not need this fact, but it may provide motivation to the reader to know that Z is the "standard" weak colimit of the diagram D, defined as the homotopy coequalizer of the natural diagram

$$\coprod_{\beta < \alpha} D(\beta) \rightrightarrows \coprod_{\beta} D(\beta).$$

Critically, we do *not* have the relations $y_\alpha^\beta y_\beta^\gamma = y_\alpha^\gamma$ in Z which would allow us to lift A into a cocone in the 2–category of groupoids. We now intend to show that D admits no privileged weak colimit by, roughly, showing that this failure is unavoidable: no choice of isomorphisms $A_\beta \cong A_\alpha \circ D_\alpha^\beta$ can give A such a lift.

Write Z_Y for the subgroupoid of Z generated by the edges of the graph. Any morphism of Z_Y can be uniquely written as a reduced word in the generators y_α^β. We say that such a morphism *passes through* a vertex α if this unique word involves a generator with source or target α. The identity id_α is said to *pass through* α and no other vertex.

Lemma 3.11 *Let $u \colon \beta \to \alpha$ in Z and let $2 \le \gamma \le \min(\alpha, \beta)$. Then u is in Z_Y and does not pass through any vertex less than γ if and only if u is the unique component of a natural isomorphism $A_\beta \circ D_\beta^\gamma \cong A_\alpha \circ D_\alpha^\gamma$ between functors $D(\gamma) \to Z$. Explicitly, for all $a \in D(\gamma)$, we must have $D_\alpha^\gamma(a) = u D_\beta^\gamma(a) u^{-1}$ in Z.*

Proof Suppose that u is in Z_Y and does not pass through any vertex less than γ. It suffices to show that y_α^β conjugates D_β^γ into D_α^γ when $\gamma \le \beta \le \alpha$. In this case, $\mu_{y_\alpha^\beta}$ is an identity map, and so the claim follows from the defining relations of Z:

$$y_\alpha^\beta D_\beta^\gamma(a) \bar{y}_\alpha^\beta = y_\alpha^\beta \mu_{y_\alpha^\beta}(D_\beta^\gamma(a)) \bar{y}_\alpha^\beta = \mu_{\bar{y}_\alpha^\beta}(D_\beta^\gamma(a)) = D_\alpha^\beta(D_\beta^\gamma(a)) = D_\alpha^\gamma(a).$$

For the converse, let u be the realization of an irreducible word $w = (a_n, y_n, \ldots, y_1, a_0)$. We proceed by induction on n. If $n = 0$, then $\alpha = \beta$ and $u = |(a_0)| \in G_\beta$. The assumption that $D_\beta^\gamma(a) = u D_\beta^\gamma(a) u^{-1}$ shows that u centralizes a nonabelian subgroup of a free group. By [Lemma 3.9](1), we see that u is trivial, as desired. And clearly u does not pass through a vertex less than γ; indeed, it passes through only β, and $\beta \ge \gamma$.

For the inductive step, assume $n > 0$. Then $s(y_1) = \beta$ and $t(y_n) = \alpha$. Let $t(y_1) = \delta$, and note that $\delta \ne \beta$. In terms of w, the assumption on u is that the word

$$w' = (a_n, y_n, \ldots, y_1, a_0 D_\beta^\gamma(a) a_0^{-1}, y_1^{-1}, a_1^{-1}, \ldots, y_n^{-1}, a_n^{-1})$$

has realization $D_\alpha^\gamma(a)$ for every $a \in G_\gamma$. Thus, by [Corollary 3.7](#), w' is reducible. Since, by assumption, w is irreducible, any reduction must occur at the central entry.

So, letting $\varepsilon := \min(\beta, \delta)$, we must have $a_0 D_\beta^\gamma(a) a_0^{-1} \in \mu_{y_1}(G_\varepsilon) = D_\beta^\varepsilon(G_\varepsilon)$. In particular, $a_0 D_\beta^\varepsilon(\hat{a}) a_0^{-1} \in D_\beta^\varepsilon(G_\varepsilon)$ for some nonidentity element $\hat{a}$ in $G_{\min(\gamma, \varepsilon)}$. So, by Lemma 3.9(2), we see that $a_0 \in D_\beta^\varepsilon(G_\varepsilon) \subseteq G_\beta$; that is, $a_0 = D_\beta^\varepsilon(\hat{a}_0)$ for some $\hat{a}_0 \in G_\varepsilon$. It then follows that $D_\beta^\gamma(a)$ is in the image of D_β^ε for every $a \in G_\gamma$, which means that $\gamma \leq \varepsilon$, since the inclusions of vertex groups are strict.

The reduction of w at its central entry is

$$(a_n, y_n, \ldots, y_2, a_1 D_\delta^\varepsilon(\hat{a}_0) D_\delta^\gamma(a) D_\delta^\varepsilon(\hat{a}_0)^{-1} a_1^{-1}, y_2^{-1}, a_2^{-1}, \ldots, a_n^{-1}).$$

Thus, if we define $u' : \delta \to \alpha$ to be $|w''|$, where $w'' = (a_n, y_n, \ldots, y_2, a_1 D_\delta^\varepsilon(\hat{a}_0))$, then $\ell(w'') < n$ and u' conjugates D_δ^γ to D_α^γ. By induction, $u' \in Z_Y$. Since

$$u' y_1 = a_n y_n \cdots y_2 a_1 D_\delta^\varepsilon(\hat{a}_0) y_1 = a_n y_n \cdots y_2 a_1 y_1 D_\beta^\varepsilon(\hat{a}_0) = u,$$

u is in Z_Y as well. Finally, recall that we observed that $\gamma \leq \varepsilon = \min(\beta, \delta)$. By induction, u' does not pass through any vertex less than γ. So the same is true of $u = u' y_1$. $\quad\square$

Let Z_X denote the subgroupoid of Z containing those morphisms in the image of G_x for some x. By Corollary 3.8, Z_X is isomorphic to the disjoint union of the groups G_x.

Lemma 3.12 *Consider a morphism $z : \alpha \to \alpha$ in Z. If there are morphisms $u : \alpha \to \beta$ and $v : \alpha \to \gamma$ in Z such that uzu^{-1} is in Z_X and vzv^{-1} is in Z_Y, then $z = \mathrm{id}_\alpha$.*

Proof Let $y = vzv^{-1}$. Note that the inclusion $Z_Y \to Z$ has a retraction $r : Z \to Z_Y$ defined by sending the generators of each vertex group to identity elements. Since $uv^{-1} y v u^{-1}$ is in Z_X, we have that $r(uv^{-1} y v u^{-1}) = r(uv^{-1}) y r(uv^{-1})^{-1}$ is an identity, and so y is an identity. Since $y = vzv^{-1}$ is an identity, we have that z is an identity as well. $\quad\square$

The following is the key technical result:

Lemma 3.13 *Suppose given a family $u_\alpha^\beta : \beta \to \alpha$ of morphisms of Z_Y for all $\beta < \alpha \in [2, \kappa)$ such that $u_\alpha^\gamma = u_\alpha^\beta u_\beta^\gamma$ for all triples $\gamma < \beta < \alpha$. Then there exists a pair $\beta < \alpha$ such that u_α^β passes through some γ with $\gamma < \beta$.*

Proof Assume that this is not the case. Let $\delta_0 = 2$ and $\delta_1 = 3$. Inductively, for each $n \in \omega$, let δ_n be an ordinal exceeding every vertex that $u_{\delta_{n-1}}^{\delta_{n-2}}$ passes through. This is possible because κ is a limit ordinal.

For each n, $u_{\delta_n}^{\delta_{n-1}}$ can be written uniquely as a reduced word in the free groupoid Z_Y. Let y_n be a letter in this word which is of the form y_α^β with $\beta < \delta_n \le \alpha$. Such a letter must exist since $u_{\delta_n}^{\delta_{n-1}}$ starts at a vertex less than δ_n and ends at δ_n. Note that y_n cannot occur in the reduced form of any $u_{\delta_k}^{\delta_{k-1}}$ with $k \neq n$. For $k < n$, this holds by definition of δ_n, and, for $k > n$, this holds by our assumption that each u_α^β only passes through γ with $\gamma \ge \beta$. In particular, the y_n are distinct.

Using that κ has uncountable cofinality, choose $\delta_\omega < \kappa$ to be an ordinal exceeding every δ_n. Consider the decompositions

$$u_{\delta_\omega}^{\delta_0} = u_{\delta_\omega}^{\delta_1} u_{\delta_1}^{\delta_0} = u_{\delta_\omega}^{\delta_2} u_{\delta_2}^{\delta_1} u_{\delta_1}^{\delta_0} = u_{\delta_\omega}^{\delta_3} u_{\delta_3}^{\delta_2} u_{\delta_2}^{\delta_1} u_{\delta_1}^{\delta_0} = \cdots .$$

In the expression $u_{\delta_\omega}^{\delta_1} u_{\delta_1}^{\delta_0}$, a y_1 occurs in the reduced form of the right-hand factor, and does not occur in the left-hand factor, so the reduced form of $u_{\delta_\omega}^{\delta_0}$ must contain a y_1. Similarly, the second decomposition involves a y_2, which can't be cancelled from either side, so the reduced form of $u_{\delta_\omega}^{\delta_0}$ must contain a y_2. Continuing, we see that the reduced form of $u_{\delta_\omega}^{\delta_0}$ must contain countably many distinct letters, a contradiction. □

Recall that κ is an arbitrary ordinal of uncountable cofinality.

Proposition 3.14 *There exists a diagram $C : [2, \kappa) \to \mathbf{HoGpd}$ valued in the homotopy category of groupoids such that, for any weak colimit with cocone $F : C \to W$, there exists an automorphism in W which is not conjugate to any morphism in the image of any leg $F_\alpha : C(\alpha) \to W$ of F.*

Proof We claim that the diagram D (see Definition 3.10) is an example of such a C.

Towards a contradiction, suppose $F : D \to W$ is a weakly colimiting cocone such that every automorphism in W is conjugate to one in the image of some component of F. Write F_α for functors representing the maps $D(\alpha) \to W$. Since F is a cocone in $\mathbf{HoGpd}$, for each $\beta < \alpha \in [2, \kappa)$ we may choose a natural isomorphism

$$h_\alpha^\beta : F_\beta \cong F_\alpha \circ D_\alpha^\beta$$

between functors $D(\beta) \to W$ in $\mathbf{Gpd}$. Denote by $\hat{h}_\alpha^\beta$ the unique component of h_α^β. As usual we shall denote $(h_a^\beta)^{-1}$ by h_β^α, and similarly for $\hat{h}$, as well as u below.

Recall the natural cocone $A : D \to Z$ from Definition 3.10 and suppose given a representative $f : W \to Z$ of a factorization of the cocone A through F. For each α, pick a natural isomorphism $k_\alpha : A_\alpha \cong f \circ F_\alpha$ with unique component $\hat{k}_\alpha$. For $\beta < \alpha$, let $u_\alpha^\beta = \hat{k}_\alpha^{-1} f(\hat{h}_\alpha^\beta) \hat{k}_\beta$, the unique component of the natural transformation $A_\beta \to A_\alpha \circ D_\alpha^\beta$

defined by $(k_\alpha^{-1} * D_\alpha^\beta) \circ (f * h_\alpha^\beta) \circ k_\beta$, where $*$ denotes whiskering.[2] By Lemma 3.11, we see that each u_α^β is in Z_Y, so the same holds for the morphism $u_{\alpha\beta\gamma} \colon \gamma \to \gamma$ defined as $u_\gamma^\alpha u_\alpha^\beta u_\beta^\gamma$ for $\gamma < \beta < \alpha$. Furthermore, the same lemma guarantees that no u_α^β passes through a vertex less than $\min(\beta, \alpha)$.

For each $\gamma < \beta < \alpha$, denote by $w_{\alpha\beta\gamma} \in W$ the unique component of the composite natural transformation

$$h_\gamma^\alpha \circ (h_\alpha^\beta * D_\beta^\gamma) \circ h_\beta^\gamma \colon F_\gamma \to F_\gamma.$$

We have $w_{\alpha\beta\gamma} = \hat{h}_\gamma^\alpha \hat{h}_\alpha^\beta \hat{h}_\beta^\gamma$, so

$$\hat{k}_\gamma^{-1} f(w_{\alpha\beta\gamma}) \hat{k}_\gamma = \hat{k}_\gamma^{-1} f(\hat{h}_\gamma^\alpha) \hat{k}_\alpha \hat{k}_\alpha^{-1} f(\hat{h}_\alpha^\beta) \hat{k}_\beta \hat{k}_\beta^{-1} f(\hat{h}_\beta^\gamma) \hat{k}_\gamma = u_{\alpha\beta\gamma}.$$

In particular, $u_{\alpha\beta\gamma}$ is conjugate to $f(w_{\alpha\beta\gamma})$.

On the other hand, by assumption on F, $w_{\alpha\beta\gamma}$ is conjugate to a morphism in the image of some $F_\theta \colon D(\theta) \to W$, say to $F_\theta(w'_{\alpha\beta\gamma})$. Composing with f, we see that $u_{\alpha\beta\gamma}$ is conjugate to $f(F_\theta(w'_{\alpha\beta\gamma}))$. Finally, using $\hat{k}_\gamma$, we see $u_{\alpha\beta\gamma}$ is conjugate to $A_\theta(w'_{\alpha\beta\gamma})$, so, in particular, to an element of Z_X. Since we saw above that $u_{\alpha\beta\gamma}$ is in Z_Y, Lemma 3.12 shows that $u_{\alpha\beta\gamma} = \mathrm{id}_\gamma$.

Finally, Lemma 3.13 implies that at least one u_α^β passes through a vertex less than β, contradicting what we saw above. □

Proof of Theorem 3.1 By Proposition 3.14 and Remark 3.4, the diagram D admits no weak colimit privileged with respect to the set $\mathcal{G}' = \{B\mathbb{Z}\}$. Thus, by Lemma 3.3, $B \circ D$ admits no weak colimit in **Hot** which is privileged with respect to the set of spheres. □

4 The spheres reflect equivalences in the 2–category of spaces

We saw in Theorem 2.1 that in the homotopy category of spaces there is no set of objects that jointly reflects isomorphisms. In this section, we show that in the homotopy 2–*category* of spaces, the spheres do jointly reflect equivalences. We first define the terms we are using.

Definition 4.1 By **Hot**, we mean the 2–category whose objects are spaces of the homotopy type of a CW–complex and whose hom categories are the fundamental groupoids of mapping spaces; that is, $\underline{\mathbf{Hot}}(X, Y) = \Pi_1(Y^X)$.

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[2]For instance, $f * h_\alpha^\beta \colon f \circ F_\beta \cong f \circ F_\alpha \circ D_\alpha^\beta$ has unique component $f(\hat{h}_\alpha^\beta)$.

Definition 4.2 A set $\mathcal{G}$ of objects in a 2–category $\mathcal{K}$ *jointly reflects equivalences* if, whenever $f: X \to Y$ is a morphism in $\mathcal{K}$ such that, for every $S \in \mathcal{G}$, the induced functor $\mathcal{K}(S, f): \mathcal{K}(S, X) \to \mathcal{K}(S, Y)$ is an equivalence of categories, f itself must be an equivalence in $\mathcal{K}$.

We shall show in Theorem 4.3 that the 2–category **Hot** admits a set $\mathcal{G}$ of objects that jointly reflects equivalences, namely $\mathcal{G} = \{S^n \mid n \geq 0\}$. Note that a map f in **Hot** is an equivalence if and only if it is a homotopy equivalence. This theorem is a corollary of [6, Theorem 1], which shows that, for a map $f: X \to Y$ of (arcwise connected) spaces which is surjective on all fundamental groups, bijectivity of f on higher homotopy groups is equivalent to that on free homotopy classes of maps from spheres.

With this, we are prepared to show that the spheres satisfy the analogue of Whitehead's theorem for **Hot**:

Theorem 4.3 *The set $\mathcal{G} = \{S^n\}$ of spheres jointly reflects equivalences in the 2–category* **Hot** *of spaces.*

Proof Let $f: X \to Y$ be such that $\underline{\mathbf{Hot}}(S^n, f): \underline{\mathbf{Hot}}(S^n, X) \to \underline{\mathbf{Hot}}(S^n, Y)$ is an equivalence of groupoids for every n. Consider an inclusion of $*$ into $S^0 \cong * \sqcup *$. Since this has a retraction, the functor $\underline{\mathbf{Hot}}(*, X) \to \underline{\mathbf{Hot}}(*, Y)$ is a retract of the equivalence $\underline{\mathbf{Hot}}(S^0, X) \to \underline{\mathbf{Hot}}(S^0, Y)$ and is therefore also an equivalence. That is, f induces an equivalence $\Pi_1(X) \to \Pi_1(Y)$ of fundamental groupoids. Thus, f induces an isomorphism on π_0 and on every π_1.

Therefore, we can apply [6, Theorem 1], so that f will be a homotopy equivalence as soon as it induces a bijection on free homotopy classes of maps from S^n. Now, the set of free homotopy classes of maps $S^n \to X$ is simply the set of connected components in the groupoid $\underline{\mathbf{Hot}}(S^n, X)$. Since f induces an equivalence $\underline{\mathbf{Hot}}(S^n, X) \to \underline{\mathbf{Hot}}(S^n, Y)$, a fortiori it induces an isomorphism on connected components, and the theorem is proven. □

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