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Anchored foams and annular homology

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We describe equivariant SL(2) and SL(3) homology for links in the thickened annulus via foam evaluation. The thickened annulus is replaced by 3–space with a distinguished line in it. Generators of state spaces for annular webs are represented by foams with boundary that may intersect the distinguished line; intersection points, called anchor points, contribute additional terms, reminiscent of square roots of the Hessian, to the foam evaluation. Both oriented and unoriented SL(3) foams are treated.

57K18; 18N25, 57K16

1 Introduction

Asaeda–Przytycki–Sikora [2] homology of links in the thickened annulus has led to a number of interesting developments — see the first author [1], Baldwin, Beliakova, Grigsby, Licata, Putyra and Wehrli [3; 5; 11; 12; 13] and Roberts [35] — and extensions of their work to SL(N) and GL(N) link homology in the thickened annulus — see Queffelec, Rose, Sartori and Wedrich [30; 31; 32].

GL(N) and SL(N) link homology theories are closely related to foam evaluation. This connection was made the most transparent by the work of Robert and Wagner [34], who wrote down a combinatorial formula for GL(N) closed foam evaluation that allows to build GL(N) link homology from the ground up, bypassing categorical approaches to the latter. A variation of their formula was used by Robert and the second author [18] to evaluate unoriented SL(3) foams, giving a combinatorial approach to some of the structures discovered by Kronheimer and Mrowka [23].

In this paper we extend foam evaluation framework to build equivariant SL(2) and SL(3) state spaces for annular webs and, consequently, equivariant SL(2) and SL(3) homology for links in the thickened annulus. Our construction complements earlier work [30; 32] on the subject. The same approach allows us to define state spaces for unoriented SL(3) annular webs, extending the construction in [18]. As in [18], the unoriented SL(3) theory yields state spaces and skein relations for planar webs but does not extend to a link homology theory.

In the APS (Asaeda–Przytycki–Sikora) annular homology and its equivariant and SL(N) generalizations, one first defines state spaces for annular SL(2) and SL(N) webs, where annular SL(2) webs are just collections of embedded circles in an annulus. See also Boerner [7; 8], where the APS theory is reformulated using embedded surfaces.

Our idea is to think of an open thickened annulus as the complement to a line L in $\mathbb{R}^3$, chosen for convenience to be the z–axis. An annular SL(N) web Γ is then placed into the xy–plane with $(0,0)$ removed. To define its state space $\langle\Gamma\rangle$, we consider SL(N) foams F in the half-space $\mathbb{R}^3_-$ bounded by the xy–plane such that Γ is the boundary of F. These foams may intersect the z–axis, and we refer to the intersection points as *anchor points* and to such foams as *anchored foams*. Anchor points additionally carry a label from 1 to N, and we modify foam evaluation by adding a new type of factors associated to anchor points.

We treat $N=2$ and $N=3$ cases, with modified evaluations given by formulas (2) and (77), respectively; also see (35) for the unoriented SL(3) anchored foam evaluation.

Anchored foam evaluation take values in the ring of polynomials rather than the ring of symmetric polynomials. One starts with an admissible coloring c of facets of a foam F, as usual. An anchor point labeled i lying on a facet of color j contributes $\delta_{i,j}\sqrt{\pm f'(x_i)}$ to the evaluation $\langle F,c\rangle$, where, in the SL(3) case as an example,

$$f(x)=(x-x_1)(x-x_2)(x-x_3)$$

is the polynomial of degree three with roots x_1, x_2 and x_3. The full evaluation $\langle F\rangle$ is given by summing over $\langle F,c\rangle$ for all admissible colorings c. We check integrality of these evaluations, with $\langle F\rangle$ a polynomial in x_1, x_2 and x_3, in the SL(3) case.

Given evaluations of anchored closed foams, one can form state spaces for annular webs. We show that this modified evaluation, with anchor points contributing $\delta_{i,j}\sqrt{\pm f'(x_i)}$, perfectly matches the structure of state spaces of annular homology, in SL(2) and SL(3) cases. The construction also allows us to define unoriented SL(3) homology for annular trivalent graphs, extending [18] to the annular framework.

With state spaces at hand, it is straightforward to define annular SL(2) and SL(3) link homology, by analogy with [1; 2; 4; 14] in the SL(2) setting, with [18] in the unoriented

SL(3) setting, and with [15; 28; 34] in the oriented SL(3) setting. State spaces and link homology carry additional gradings coming from intersection points of foams with the z–axis. We show that the result matches equivariant SL(2) homology [1] of the first author. A simple modification of the construction (truncating the ground ring by sending the x_i to 0 upon evaluation) gives a foam approach to the original APS homology. We expect that the nonequivariant variant of our SL(3) construction recovers the $N = 3$ case of the homology in [30]. It seems that the equivariant annular SL(3) homology, as described in the present paper, is new.

Section 2 describes SL(2) homology via anchored foams. The evaluation is defined in Section 2.1, which also contains the skein relations for anchored SL(2) foams. The state spaces are studied in Section 2.2. The state space of n circles in the annulus is a free module of rank 2^n over the ground ring R_α of polynomials in two variables; see Theorem 2.11. The numbers of contractible and essential circles control the bigraded rank. This section also discusses categories of *anchored* and *annular* cobordisms. Annular cobordisms between annular SL(2) webs are disjoint from the z–axis, while anchored cobordism may intersect it.

Theorem 2.20 identifies the annular cobordism functor with that constructed in [1]. Consequently, equivariant annular SL(2) link homology [1] can be rederived via anchored foams. To obtain the original APS homology, one can use anchored foam evaluation, combined with the homomorphism $R_\alpha \to \mathbb{Z}$ taking α_1 and α_2 to 0 to get state spaces and cobordism maps in the APS theory.

Section 3 constructs the state spaces for the annular unoriented SL(3) foam theory, extending the construction of [18]. We start with the evaluation (Section 3.1), followed by skein relations on annular foams (Section 3.2) and properties of state spaces (Section 3.3). Section 3.4 describes similarities between anchor points contributions and Lee's theory, given by inverting the discriminant in the ground ring. Similar to the planar case [18], we don't know a way to describe the state space of an annular web when regions of valency at most four, allowing an inductive simplification, are absent.

In Section 4 we describe annular equivariant SL(3) link homology, based on anchored (annular) oriented SL(3) foams. This homology extends Mackaay–Vaz [28] equivariant SL(3) homology of links in $\mathbb{R}^3$; also see Clark [10], the second author [15], Morrison and Nieh [29], and Robert [33] for the nonequivariant homology in $\mathbb{R}^3$. We start with a review of oriented SL(3) foams in Section 4.1 and then follow a similar route to that of the earlier sections.

Our constructions of annular equivariant link homology via foam evaluation requires working with $U(1)^{\times N}$–equivariant homology rather than $U(N)$ or $\mathrm{GL}(N)$–equivariant homology. In these G–equivariant theories homology of the empty link is $\mathrm{H}_G(p,\mathbb{Z})$, the G–equivariant cohomology of a point. For $U(1)^{\times N}$ that cohomology consists of polynomials in N variables (denoted here by α_1 and α_2 for $N=2$, and x_1 x_2 and x_3 for $N=3$), which is a larger ring than its subring of symmetric polynomials, which is the corresponding equivariant cohomology of a point for $U(N)$ and $\mathrm{GL}(N)$. Having a larger background ring gives additional freedom and allows a "symmetry breaking" between these polynomial variables, necessary in our case as clear from the evaluation (also see Remark 2.1 below).

Working with that larger ring and $U(1)^{\times N}$–equivariant cohomology is a recent phenomenon. It was used by T Sano [37] in resolving the minus sign ambiguity in the functorial extension of Khovanov homology to link cobordisms, bypassing earlier constructions that required additional decorations of links and cobordisms (see [19] for more references and a short discussion). We expect this symmetry breaking of the ground ring generators to find more applications to link homology in the future.

A recent paper of R Lipshitz and S Sarkar [25] contains an application of annular equivariant link homology. The authors use maps associated to moving a strand across the puncture. These maps come for free from the anchored foam perspective of the present paper; see [25, Remark 3.2].

Unoriented SL(3) homology for planar graphs (webs) is closely related to the 4–color theorem and Kronheimer–Mrowka instanton homology for 3–orbifolds [18; 23]. This homology of webs remains a mysterious structure which has only been computed for reducible webs (see Boozer [9] for a computational approach to homology of the dodecahedron and other nonreducible webs). In the annular case, nonreducible webs have fewer vertices, with the smallest such web shown in Figure 10, and annular homology may shed light on and aid in understanding unoriented SL(3) homology of nonreducible webs and related structures.

We expect that our construction admits a generalization to $\mathrm{SL}(N)$ homology for all N via an extension of the Robert–Wagner formula [34] to the anchored case.

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2 SL(2) anchored homology

2.1 Anchored surfaces and their evaluations

Consider the integral polynomial ring $R_\alpha = \mathbb{Z}[\alpha_1, \alpha_2]$ in two variables α_1, α_2. Define a grading on R_α by setting

$$\deg(\alpha_1) = \deg(\alpha_2) = 2. \tag{1}$$

Denote by τ the nontrivial involution of $\{1,2\}$. It is given by $\tau(i) = 3-i$ for $i \in \{1,2\}$. Also denote by τ the induced involution of R_α which permutes α_1 and α_2, so that $\tau(\alpha_i) = \alpha_{\tau(i)} = \alpha_{3-i}$. Let R be the τ–invariant subring of R_α, which consists of symmetric polynomials in α_1 and α_2. The subring R is itself a polynomial ring, $R = \mathbb{Z}[E_1, E_2]$, where E_1 and E_2 are elementary symmetric polynomials in α_1 and α_2,

$$E_1 = \alpha_1 + \alpha_2, \quad E_2 = \alpha_1\alpha_2.$$

Degrees of E_1 and E_2 are 2 and 4, respectively.

Let $L \subset \mathbb{R}^3$ denote the z–axis, $L = (0,0) \times \mathbb{R}$. Let $S \subset \mathbb{R}^3$ be a closed, smoothly embedded surface which intersects L transversely. The surface S may be decorated by dots, disjoint from L, that can otherwise float freely on components of S. The intersection points $S \cap L$ are called *anchor points*. Fix a labeling ℓ, which is a map from the set of anchor points to $\{1,2\}$,

$$\ell\colon S \cap L \to \{1,2\}.$$

Order the anchor points by $1, \dots, 2k$, read from bottom to top, so that the labeling ℓ consists of a choice $\ell(j) \in \{1,2\}$ for each $1 \le j \le 2k$. We will define an evaluation

$$\langle S \rangle \in R_\alpha$$

for S with the fixed labeling ℓ, which is omitted from the notation.

Let $\mathrm{Comp}(S)$ denote the set of connected components of S. A *coloring* of S is a function $c\colon \mathrm{Comp}(S) \to \{1,2\}$, and we denote by $\mathrm{adm}(S)$ the set of colorings of S. The surface S has $2^{|\mathrm{Comp}(S)|}$ colorings. Fix a coloring c. For $i = 1,2$, let $d_i(c)$ denote the number of dots on components colored i. Let S_2 denote the union of the 2–colored components. For $1 \le j \le 2k$, let $c(j)$ denote the color of the j^{th} anchor point, induced by c, which may in general be different from the fixed label $\ell(j)$. Define

$$\langle S, c\rangle = (-1)^{\chi(S_2)/2} \frac{\alpha_1^{d_1(c)}\alpha_2^{d_2(c)}\left(\prod_{j=1}^{2k}(\alpha_{c(j)} - \alpha_{\ell(j)})\right)^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}. \tag{2}$$

Note that $\chi(S_2)$ is even since S_2 is a closed surface in $\mathbb{R}^3$. Let us explain the square root in the above equation.

Each component S' of S can be made disjoint from L via a homotopy. Since the mod 2 intersection number is preserved under homotopy, it follows that S' intersects L at an even number of points $p_1, \dots, p_{2m}$, which can be ordered as encountered along L, from bottom to top. Suppose S' is colored by $c(S') = j$, and moreover S' contains an anchor point labeled j. Then the product $\prod_{j=1}^{2m}(\alpha_{c(j)} - \alpha_{\ell(j)}) = 0$, since it contains a term $\alpha_j - \alpha_j = 0$, and the entire evaluation $\langle S, c\rangle = 0$. Thus, the evaluation (2) is only nonzero when the anchor points on a component S' colored j are all labeled by the complementary color $\tau(j)$. In this case, each component contributes an even number of factors of either $\alpha_1 - \alpha_2$ or $\alpha_2 - \alpha_1$ to the product $\prod_{j=1}^{2m}(\alpha_{c(j)} - \alpha_{\ell(j)})$, and we define the square root to be $(\alpha_1 - \alpha_2)^m$ or $(\alpha_2 - \alpha_1)^m$, respectively. If S' has no anchor points, this term is 1 and can be removed from the product.

Note that the evaluation is the product of evaluations of individual components,

(3) $$\langle S, c\rangle = \prod_{S' \in \mathrm{Comp}\,(S)} \langle S', c(S')\rangle.$$

Thus, if S' is colored 1 by $c' = c(S')$, has $2k$ anchor points all labeled 2 and carries d dots, then

(4) $$\langle S', c'\rangle = \alpha_1^d (\alpha_1 - \alpha_2)^{k - \chi(S')/2}.$$

If S' is colored 2 by $c' = c(S')$, has $2k$ anchor points all labeled 1 and carries d dots, then

(5) $$\langle S', c'\rangle = (-1)^{\chi(S')/2 + k} \alpha_2^d (\alpha_1 - \alpha_2)^{k - \chi(S')/2} = \alpha_2^d (\alpha_2 - \alpha_1)^{k - \chi(S')/2}.$$

Otherwise, if one of the anchor points has the same label as the color of S', the evaluation $\langle S', c'\rangle = 0$ and $\langle S, c\rangle = 0$.

Define the evaluation of S by

(6) $$\langle S\rangle = \sum_c \langle S, c\rangle,$$

where the sum is over all colorings of S. Note that if $S \cap L = \varnothing$, then $\langle S\rangle$ agrees with the evaluation in [19; 34]. Also note that $\langle S\rangle = 0$ if a component of S has two anchor points with different labels 1, 2.

We have

$$\langle S\rangle = \prod_{S'\in \operatorname{Comp} S} \langle S'\rangle, \tag{7}$$

that is, evaluation of S is the product of evaluations over connected components of S.

We can rewrite $\langle S\rangle$ as follows. First, suppose S is connected, carrying d dots, with $2k \geq 0$ anchor points. For $i = 1, 2$, let c_i denote the coloring of S by i. Define

$$\langle S, c_1\rangle = \frac{\alpha_1^d((\alpha_1-\alpha_{\ell(1)})\cdots(\alpha_1-\alpha_{\ell(2k)}))^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}, \tag{8}$$

$$\langle S, c_2\rangle = (-1)^{\chi(S)/2}\frac{\alpha_2^d((\alpha_2-\alpha_{\ell(1)})\cdots(\alpha_2-\alpha_{\ell(2k)}))^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}. \tag{9}$$

Again, square roots in the above equations are taken in the natural way. If S has oppositely labeled anchor points then both (8) and (9) are zero. If all anchor points are labeled 1, then (8) is zero, whereas (9) is equal to

$$\langle S, c_2\rangle = (-1)^{\chi(S)/2}\frac{\alpha_2^d(\alpha_2-\alpha_1)^k}{(\alpha_1-\alpha_2)^{\chi(S)/2}}.$$

On the other hand, if all anchor points are labeled by 2 then (9) is zero and (8) equals

$$\frac{\alpha_1^d(\alpha_1-\alpha_2)^k}{(\alpha_1-\alpha_2)^{\chi(S)/2}}.$$

Then for connected S with anchor points,

$$\langle S\rangle = \langle S, c_1\rangle + \langle S, c_2\rangle,$$

where at most one of the summands $\langle S, c_i\rangle$ is nonzero.

Clearly the evaluation is multiplicative under disjoint union. That is, if $S = S_1 \sqcup \cdots \sqcup S_n$, then

$$\langle S\rangle = \langle S_1\rangle \cdots \langle S_n\rangle.$$

Remark 2.1 Unlike closed foam evaluations appearing elsewhere [16; 18; 19; 34; 36], our evaluation does not in general produce a symmetric function. The following examples illustrate this.

Example 2.2 Let S be a sphere intersecting L in two points with labels i and j and carrying d dots. If $i \neq j$, then each coloring c yields $\langle S, c\rangle = 0$. If both anchor points are labeled 1, then only coloring S by 2 contributes to the sum, and we have

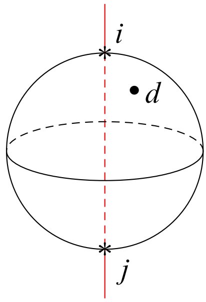
$$\langle S\rangle = \langle S, c_2\rangle = -\frac{\alpha_2^d(\alpha_2-\alpha_1)}{\alpha_1-\alpha_2} = \alpha_2^d.$$

On the other hand, if both anchor points are labeled 2, then

$$\langle S\rangle = \langle S, c_1\rangle = \alpha_1^d.$$

This is summarized pictorially by

(10)



$$= \delta_{ij}\tau(\alpha_i)^d.$$

Both signs are positive since $k + \chi(\mathbb{S}^2)/2 = 1 + 1 = 2$ is even.

Note that these evaluations are not symmetric in α_1 and α_2.

Example 2.3 More generally, let S be a genus g surface with d dots and $2k$ anchor points. If $k = 0$ (that is, if S is disjoint from L) then the evaluation is

$$\langle S\rangle = \frac{\alpha_1^d + (-1)^{g-1}\alpha_2^d}{(\alpha_1 - \alpha_2)^{1-g}}.$$

On the other hand, if $k > 0$, then

$$\langle S\rangle = \begin{cases} \alpha_2^d(\alpha_2-\alpha_1)^{k+g-1} & \text{if } \ell(1) = \cdots = \ell(2k) = 1, \\ \alpha_1^d(\alpha_1-\alpha_2)^{k+g-1} & \text{if } \ell(1) = \cdots = \ell(2k) = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

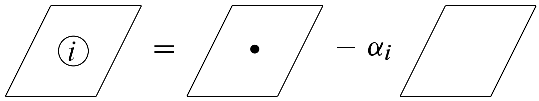
Proposition 2.4 *For any anchored surface $S \subset \mathbb{R}^3$ with d dots and $2k$ anchor points, its evaluation $\langle S\rangle$ is a homogeneous polynomial in α_1 and α_2 of degree*

$$-\chi(S) + 2d + 2k.$$

Proof If S does not intersect L, then this follows from Example 2.3. Suppose that S intersects L. It suffices to verify the statement for connected surfaces. If S intersects L, then the statement follows from (11), since $k > 0$. $\square$

We recall the following notation from [19]. For $i = 1, 2$, we allow surfaces to carry decorations (i) consisting of i inscribed in a small circle. They must be disjoint from

L and are allowed to float freely along the connected component on which they appear. We call these *shifted* dots. Diagrammatically, a shifted dot ⓘ is the difference between a dot and α_i:

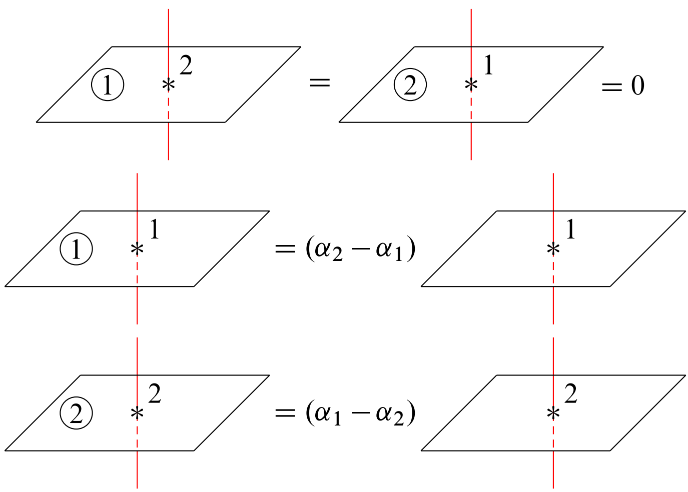
(12)


Lemma 2.5 *Let S be an anchored foam and let $S \cup$ ⓘ denote the anchored foam obtained by placing a shifted dot* ⓘ *on some component S' of S. Then*

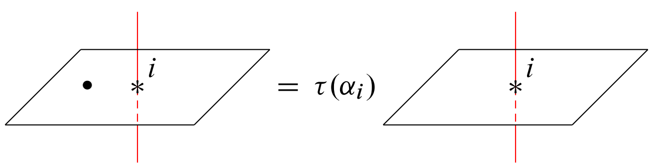
$$\langle S \cup \textcircled{i} \rangle = \begin{cases} 0 & \text{if } S' \text{ has an anchor point labeled } \tau(i), \\ (-1)^i(\alpha_1 - \alpha_2)\langle S \rangle & \text{if all anchor points on } S' \text{ are labeled } i. \end{cases}$$

Proof This is clear from the definitions. □

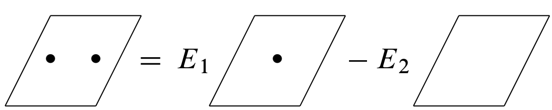
Lemma 2.5 is summarized diagrammatically by

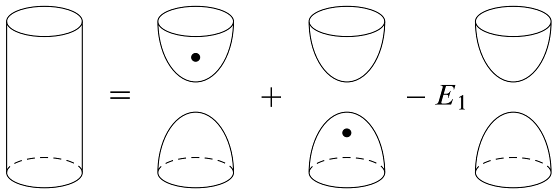
(13)


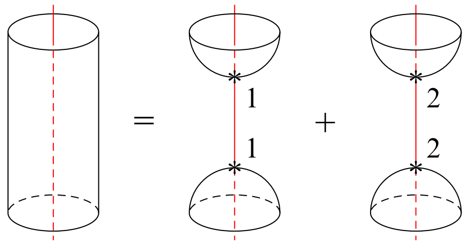
Alternatively, the skein relations (13) may written compactly as

(14)


Lemma 2.6 *The following local relations hold*:

(15)


(16) 

(17) 

Proof The relation (15) is straightforward. Let us now verify (16), which is proved in the same way as for nonanchored foams, see [19, Lemma 3.5]. Let S denote the surface on the left, and let F denote the surface obtained by surgering S as shown on the right. Denote by F^t (resp. F^b) the surface obtained from F by placing an additional dot on the top (resp. bottom) depicted disk. Note that anchor points, as well as their labels, are the same for F^t, F^b, and F. Colorings of F, F^t, and F^b are in a canonical bijection. There are four local models for a coloring of F, illustrated in Figure 1.

Let c be a coloring of F of the type shown in Figure 1(c), with the corresponding coloring of F^t and F^b still denoted by c. We have

$$\langle F^t, c\rangle = \alpha_1 \langle F, c\rangle, \quad \langle F^b, c\rangle = \alpha_2 \langle F, c\rangle,$$

hence $\langle F^t, c\rangle + \langle F^b, c\rangle - E_1 \langle F, c\rangle = 0$. A similar calculation holds for a coloring c of Figure 1(d) type.

There is a natural bijection between colorings of S and colorings of F of Figures 1(a) and 1(b) types. Let c be a coloring of F of Figure 1(a) type, and continue to denote by

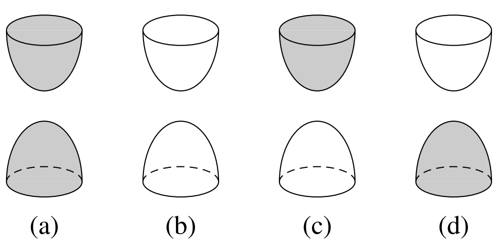


Figure 1: Local models for colorings of F. Shaded indicates color 1 and solid white indicates color 2.

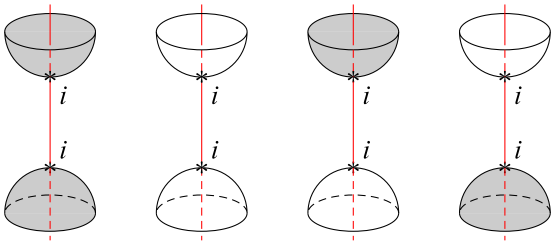


Figure 2: Local models for colorings of F^i. Shaded indicates color 1 and solid white indicates color 2.

c the corresponding coloring of S. Then

$$\chi(F)=\chi(S)+2, \quad \chi(F_2(c))=\chi(S_2(c)),$$
$$\langle F^t,c\rangle=\alpha_1\langle F,c\rangle, \qquad \langle F^b,c\rangle=\alpha_1\langle F,c\rangle,$$

so we have

$$\langle F^t,c\rangle+\langle F^b,c\rangle-E_1\langle F,c\rangle=(\alpha_1-\alpha_2)\langle F,c\rangle=\langle S,c\rangle.$$

Finally, if c is a coloring of F of the Figure 1(b) type, then

$$\chi(F)=\chi(S)+2, \qquad \langle F^t,c\rangle=\alpha_2\langle F,c\rangle,$$
$$\chi(F_2(c))=\chi(S_2(c))+2, \quad \langle F^b,c\rangle=\alpha_2\langle F,c\rangle,$$

which yields

$$\langle F^t,c\rangle+\langle F^b,c\rangle-E_1\langle F,c\rangle=(\alpha_2-\alpha_1)\langle F,c\rangle=-(\alpha_2-\alpha_1)\frac{\langle S,c\rangle}{\alpha_1-\alpha_2}=\langle S,c\rangle.$$

We now address (17), where anchor points are present. Let S denote the surface on the left-hand side of the equality. Let F^1 and F^2 denote the two anchored foams obtained by surgery on S in which the new anchor points are both labeled 1 or 2, respectively, so that (17) reads $\langle S\rangle=\langle F^1\rangle+\langle F^2\rangle$. For each $i=1,2$ there are four local models for a coloring of F^i, shown in Figure 2. Colorings c in Figures 2(c) and 2(d) evaluate to zero for both $i=1,2$,

$$\langle F^1,c\rangle=\langle F^2,c\rangle=0,$$

and they don't correspond to any colorings of S. There is a natural bijection between colorings of S and colorings of F^i of the types in Figures 2(a) and 2(b).

Let c be a coloring of S in which the depicted region of S in (17) is colored 1, with the corresponding colorings of F^1 and F^2 still denoted by c. We have immediately that $\langle F^1, c\rangle = 0$. On the other hand,

$$\chi(F^2) = \chi(S) + 2, \quad \chi(F_2^2(c)) = \chi(S_2(c)),$$

and F^2 has two additional anchor points compared to S, both labeled 2 and their regions colored 1. Therefore,

$$\langle F^1, c\rangle + \langle F^2, c\rangle = \langle F^2, c\rangle = (\alpha_1 - \alpha_2)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle.$$

Now let c be a coloring of S in which the depicted region of (17) is colored 2, and continue to denote by c the corresponding colorings of F^1 and F^2. Then $\langle F^2, c\rangle = 0$. Since

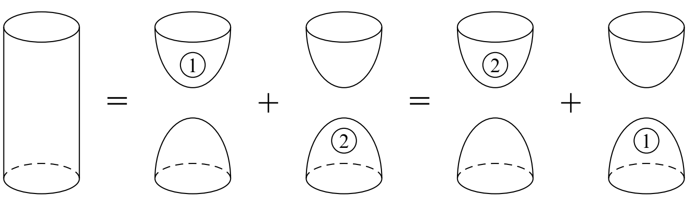
$$\chi(F^1) = \chi(S) + 2, \quad \chi(F_2^1(c)) = \chi(S_2(c)) + 2,$$

and F^1 contains two more anchor points labeled 1 and colored 2 than S does, we obtain

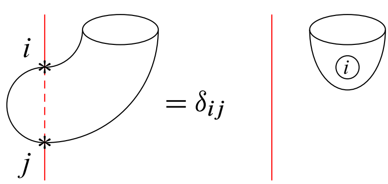
$$\langle F^1, c\rangle + \langle F^2, c\rangle = \langle F^1, c\rangle = -(\alpha_2 - \alpha_1)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle.$$

Relation $\langle S\rangle = \langle F^1\rangle + \langle F^2\rangle$ in (17) follows. □

Equation (16) can also be written using shifted dots:

(18) 

Corollary 2.7 *The following local relation holds*:

(19) 

Proof This can be seen by applying the neck-cutting relation (16) near the depicted contractible circle and evaluating the resulting anchored sphere according to (10). □

2.2 State spaces

Following [6; 19], we can apply the universal construction to the evaluation described above. Let $\mathcal{P} = \mathbb{R}^2 \setminus (0,0)$ denote the punctured plane. Given a collection C of disjoint simple closed curves in $\mathcal{P}$, let $\mathrm{Fr}(C)$ denote the free R_α–module with a basis consisting of properly embedded compact surfaces $S \subset \mathbb{R}^2 \times (-\infty, 0]$ with $\partial S = C$ and which are transverse to the ray $L_- := (0,0) \times (-\infty, 0]$. The intersection $S \cap L_-$ is a 0–submanifold of L_- and consists of finitely many points. Moreover, each such surface S must carry a labeling, a map

$$\ell = \ell_S : S \cap L_- \to \{1, 2\}$$

from the set of its intersection points with the ray L_- (its *anchor points*) to $\{1,2\}$. For a basis element $S \in \mathrm{Fr}(C)$, let $\overline{S} \subset \mathbb{R}^2 \times [0, \infty)$ denote its reflection through the plane $\mathbb{R}^2$. Labels of anchor points do not change upon reflection. For two basis elements $S, S' \in \mathrm{Fr}(C)$, denote by $\overline{S}S'$ the closed anchored surface obtained by gluing $\overline{S}$ to S' along their common boundary C.

Define a bilinear form

$$(-,-) : \mathrm{Fr}(C) \times \mathrm{Fr}(C) \to R_\alpha \tag{20}$$

by setting $(S, S') = \langle \overline{S}S' \rangle$. A direct computation shows that the form is symmetric, since for a closed surface T the evaluation satisfies $\langle \overline{T} \rangle = \langle T \rangle$.

Define the *state space of* C, denoted by $\langle C \rangle$, to be the quotient of $\mathrm{Fr}(C)$ by the kernel

$$\{x \in \mathrm{Fr}(C) \mid (x, y) = 0 \text{ for all } y \in \mathrm{Fr}(C)\}$$

of this bilinear form. For a basis element $S \in \mathrm{Fr}(C)$, we will write $[S]$ to denote its equivalence class in $\langle C \rangle$.

Equip the ground ring R_α with a bigrading by placing α_1 and α_2 in bidegree $(2,0)$. We extend this bigrading $(\mathrm{qdeg}, \mathrm{adeg})$ to $\mathrm{Fr}(C)$ as follows. For a basis element $S \in \mathrm{Fr}(C)$ with d dots and m anchor points, set the quantum grading $\mathrm{qdeg}(S) \in \mathbb{Z}$ to be

$$\mathrm{qdeg}(S) = -\chi(S) + 2d + m. \tag{21}$$

Note that if S is a closed surface, then $\langle S \rangle \in R_\alpha$ is a homogeneous polynomial of degree $\mathrm{qdeg}(S)$, following the degree convention (1).

	label 1	label 2
i odd	1	-1
i even	-1	1

Table 1: The contribution of the i^{th} anchor point on S to $\operatorname{adeg}(S)$.

Next, let $\ell(1), \dots, \ell(m)$ denote the labels of the anchor points of S, ordered from bottom to top, and define the annular grading $\operatorname{adeg}(S) \in \mathbb{Z}$ by setting

$$\operatorname{adeg}(S) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}. \tag{22}$$

In other words, if the i^{th} anchor point p_i is labeled 1, then it contributes 1 to adeg if i is odd and -1 if i is even. Likewise, if p_i has label 2 then it contributes -1 if i is odd and 1 if i is even; see also Table 1. Multiplication by α_1 or α_2 increases the $(\operatorname{qdeg}, \operatorname{adeg})$–bidegree by $(2, 0)$.

Example 2.8 Let C consist of two noncontractible circles. The bidegree $(\operatorname{qdeg}, \operatorname{adeg})$ of the four anchored surfaces in $\operatorname{Fr}(C)$ whose underlying surface consists of two disks each intersecting L_- once are recorded in Figure 3.

Lemma 2.9 *Let S be an anchored surface. Then $\langle S \rangle = 0$ or $\operatorname{adeg}(S) = 0$.*

Proof If some component of S has anchor points with different labels then $\langle S \rangle = 0$. Assume that all anchor points on any component of S are labeled identically. We also assume that S intersects L, otherwise $\operatorname{adeg}(S) = 0$ is immediate. As usual, order the anchor points $p_1, \dots, p_m$ from bottom to top.

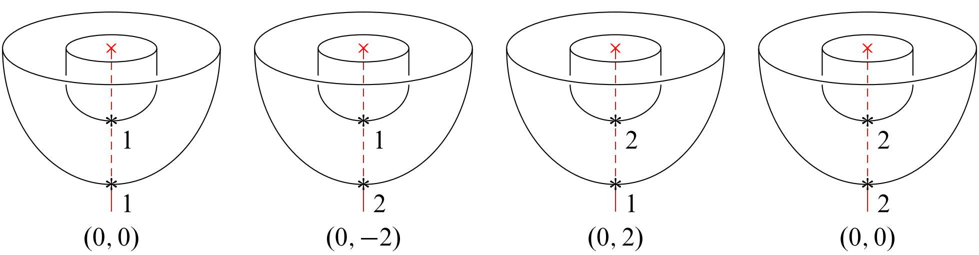


Figure 3: The $(\operatorname{qdeg}, \operatorname{adeg})$–bidegrees of some anchored surfaces whose boundary consists of two noncontractible circles.

Take a generic half-plane P in $\mathbb{R}^3$ containing the anchor line L, so that $P \cap S$ consists of finitely many arcs (with boundary on L) and circles (disjoint from L). For any arc a in $P \cap S$ with boundary $\partial a = \{p_i, p_j\}$, necessarily i and j have opposite parities. To see this, any anchor point between p_i and p_j is one boundary point of an arc in $P \cap S$, and the other boundary point of this arc must also be between p_i and p_j, which shows that the number of anchor points between p_i and p_j is even. Moreover, $\ell(p_i) = \ell(p_j)$ by assumption. Therefore the total contribution of the anchor points p_i and p_j to $\operatorname{adeg}(S)$ is zero. Summing over all arcs in $P \cap S$ yields the statement of the lemma. □

The subspace $\ker((-,-)) \subset \operatorname{Fr}(C)$ respects this bigrading on $\operatorname{Fr}(C)$. Consequently, the bigrading descends to the state space $\langle C \rangle$.

Note that the relations (16) and (17) are bihomogeneous. Let $S \in \operatorname{Fr}(C)$ be a basis element of the form $S = S_1 \sqcup S_2$ where $S_1, S_2 \in \operatorname{Fr}(C)$ are anchored surfaces with S_2 closed. Then in $\langle C \rangle$,

$$[S] = \langle S_2 \rangle [S_1], \quad \langle S_2 \rangle \in R_\alpha. \tag{23}$$

Moreover, the relation (23) is bihomogeneous. That it is homogeneous with respect to qdeg follows from the fact that $\langle S_2 \rangle \in R_\alpha$ is a polynomial of degree $\operatorname{qdeg}(S_2)$. Lemma 2.9 ensures that $\operatorname{adeg}(S_2) = \operatorname{adeg}(\langle S_2 \rangle) = 0$, so $\operatorname{adeg}(S) = \operatorname{adeg}(S_1)$.

Given a bigraded module $M = \bigoplus_{(i,j)\in\mathbb{Z}^2} M_{i,j}$ over a commutative domain such that each $M_{i,j}$ has finite rank, define its *graded rank* to be

$$\operatorname{grank}(M) = \sum_{i,j} \operatorname{rank}(M_{i,j}) q^i a^j .$$

Lemma 2.10 *Let $C \subset \mathcal{P}$ be a single circle. Then the state space $\langle C \rangle$ is a free R_α–module of rank 2. Moreover,*

$$\operatorname{grank}(\langle C \rangle) = \begin{cases} q + q^{-1} & \text{if } C \text{ is contractible,} \\ a + a^{-1} & \text{if } C \text{ is noncontractible.} \end{cases}$$

Proof We consider two cases. If C is contractible, then by applying the neck-cutting relation (16) near C and evaluating closed anchored surfaces as in (23), we see that $\langle C \rangle$ is spanned by the two elements S and $S_\bullet$ shown in Figure 4. Bidegrees of S and $S_\bullet$ are $(-1,0)$ and $(1,0)$, respectively. Computing the matrix of the bilinear form (20) for these elements yields

$$\begin{pmatrix} \bar{S}S & \bar{S}S_\bullet \\ \bar{S}_\bullet S & \bar{S}_\bullet S_\bullet \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & E_1 \end{pmatrix},$$

which is invertible; thus S and $S_\bullet$ constitute a basis for $\langle C \rangle$.

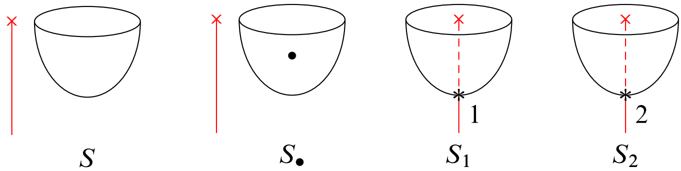


Figure 4: Basis elements for the state space of a single circle C. The first two surfaces form a basis if C is contractible, and the last two form a basis if C is noncontractible.

Now suppose C is noncontractible. Applying the neck-cutting relation (17) near C and evaluating closed anchored surfaces shows that the two elements S_1 and S_2 depicted in Figure 4 span $\langle C\rangle$. Bidegrees of S_1 and S_2 are $(0,1)$ and $(0,-1)$, respectively. The matrix of the bilinear form is

$$\begin{pmatrix} \overline{S}_1 S_1 & \overline{S}_1 S_2 \\ \overline{S}_2 S_1 & \overline{S}_2 S_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence S_1 and S_2 are linearly independent and constitute a basis of $\langle C\rangle$. □

Theorem 2.11 *Let $C \subset \mathcal{P}$ consist of n contractible circles and m noncontractible circles. Then the state space $\langle C\rangle$ is a free R_α–module of rank 2^{n+m}. Moreover,*

$$\operatorname{grank}(\langle C\rangle) = (q+q^{-1})^n (a+a^{-1})^m.$$

Proof Consider a 2^{n+m}–element set $B(C)$ of basis vectors of $\mathrm{Fr}(C)$ consisting of surfaces S satisfying:

- Each component of S is a disk.
- Each disk in S with contractible boundary is disjoint from L_- and carries either zero or one dot.
- Each disk in S with noncontractible boundary intersects L_- exactly once, and its intersection point may be labeled by either 1 or 2.

That $B(C)$ spans $\langle S\rangle$ follows from applying the two neck-cutting relations (16) and (17) near the circles in C and evaluating closed anchored surfaces. Linear independence of $B(C)$ and the statement regarding graded rank follow from the computations in Lemma 2.10. □

Elements of the basis $B(C)$ constructed above are *standard generators*. For such a $\Sigma \in B(C)$ with d dots and anchor points labeled $\ell_1, \dots, \ell_m$, we have

$$\mathrm{qdeg}(\Sigma) = -n + 2d, \quad \mathrm{adeg}(\Sigma) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}. \tag{24}$$

Let $C_0, C_1 \subset \mathcal{P}$ be two collections of disjoint circles in the punctured plane. An *anchored cobordism from C_0 to C_1* is a smoothly and properly embedded compact surface $S \subset \mathbb{R}^2 \times [0,1]$ with boundary $\partial S = C_0 \sqcup C_1$, such that $C_i \subset \mathbb{R}^2 \times \{i\}$ for $i = 0, 1$. Moreover, S is required to intersect the arc $L_{[0,1]} := (0,0) \times [0,1]$ transversely and come equipped with a labeling of these intersection points (called *anchor points*), which is a map

$$\ell = \ell_S : S \cap L_{[0,1]} \to \{1, 2\}$$

from the set of its anchor points to $\{1,2\}$. Anchored cobordisms are allowed to carry dots which can float on components but cannot jump to a different component.

We compose anchored cobordisms in the usual manner. For anchored cobordisms $S_1 : C_0 \to C_1$ and $S_2 : C_1 \to C_2$, let $S_2 S_1 : C_0 \to C_2$ denote the anchored cobordism obtained by gluing along the common boundary C_1 and rescaling. Labels of anchor points of $S_2 S_1$ are inherited from labels of S_1 and S_2.

As above, if an anchored cobordism S from C_0 to C_1 has m anchor points and carries d dots, define

$$\mathrm{qdeg}(S) = -\chi(S) + 2d + m.$$

Let $\ell(1), \dots, \ell(m)$ denote the labels of anchor points of S, ordered from bottom to top, and let n be the number of noncontractible circles in C_0. Set

$$\mathrm{adeg}(S) = (-1)^n \sum_{i=1}^{m} (-1)^{i+\ell(i)}.$$

Remark 2.12 If $C_0 = \varnothing$, then S is a basis element of $\mathrm{Fr}(C_1)$, and moreover the two degrees $\mathrm{qdeg}(S)$, $\mathrm{adeg}(S)$ defined above for anchored cobordisms agree with the definitions in (21) and (22) for elements of $\mathrm{Fr}(C_1)$.

An anchored cobordism S from C_0 to C_1 induces an R_α–linear map

$$S : \mathrm{Fr}(C_0) \to \mathrm{Fr}(C_1)$$

defined on the basis by gluing along the common boundary C_0. The definition of state spaces via universal construction immediately implies that we have an induced map

$$\langle S \rangle : \langle C_0 \rangle \to \langle C_1 \rangle. \tag{25}$$

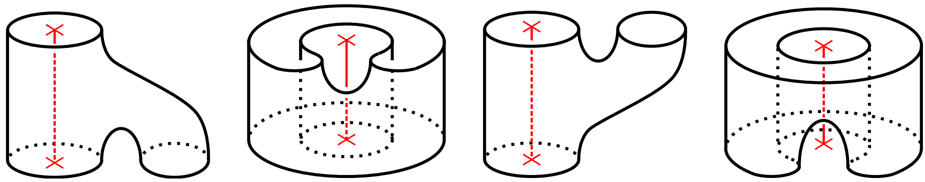


Figure 5: Elementary saddles involving noncontractible circles.

Lemma 2.13 *Let $S_1\colon C_0 \to C_1$ and $S_2\colon C_1 \to C_2$ be anchored cobordisms. Then*

$$\operatorname{qdeg}(S_2S_1) = \operatorname{qdeg}(S_2) + \operatorname{qdeg}(S_1), \quad \operatorname{adeg}(S_2S_1) = \operatorname{adeg}(S_2) + \operatorname{adeg}(S_1).$$

In particular, $\langle S_1\rangle\colon \langle C_0\rangle \to \langle C_1\rangle$ is a map of bidegree $(\operatorname{qdeg}(S_1), \operatorname{adeg}(S_1))$.

Proof The first equality involving qdeg is straightforward. Let n and m denote the number of noncontractible circles in C_0 and C_1 respectively, and let k denote the number of anchor points of S_1. We have

$$\operatorname{adeg}(S_2S_1) = \operatorname{adeg}(S_1) + (-1)^{n+m+k} \operatorname{adeg}(S_2).$$

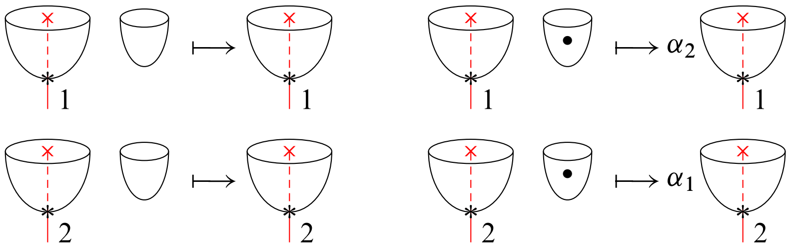
Note $n+m+k$ is even, since it is equal to the number of anchor points of the closed surface obtained by gluing disks to all boundary circles of S_1.

The final statement concerning the bidegree of $\langle S_1\rangle$ follows from interpreting generators of $\langle C_0\rangle$ as anchored cobordisms $\varnothing \to C_0$, as in Remark 2.12. □

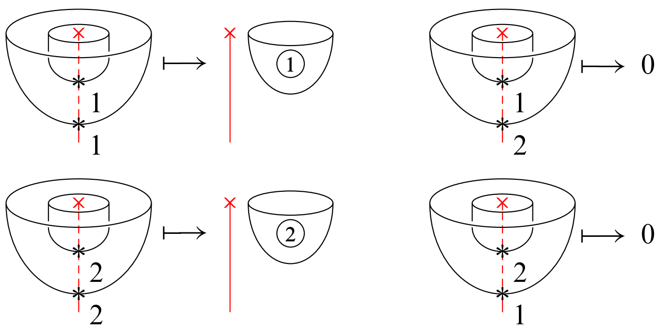
Definition 2.14 An *annular cobordism* is an anchored cobordism $S \subset \mathbb{R}^2 \times [0,1]$ which is disjoint from the arc $L_{[0,1]} = (0,0) \times [0,1]$. An *elementary* annular cobordism is one with a single nondegenerate critical point with respect to the height function $\mathbb{R}^2 \times [0,1] \to [0,1]$.

Elementary annular cobordisms consist of a union of a product cobordism with a cup, cap, or saddle. Every annular cobordism may be obtained by composing finitely many elementary ones. Cup and cap annular cobordisms always have contractible boundary. There are four types of elementary annular saddles involving at least one noncontractible circle, illustrated in Figure 5. In the next four examples we write down the maps assigned to these four cobordisms in the standard bases of state spaces, as defined in the proof of Theorem 2.11. We also use the notation of shifted dots from (12).

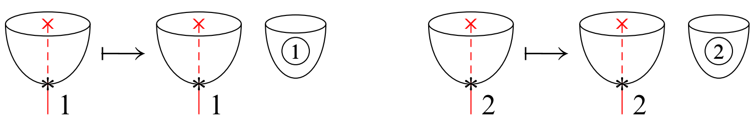
Example 2.15 (Figure 5, type A map) The calculation for this map follows at once from the skein relation (14):



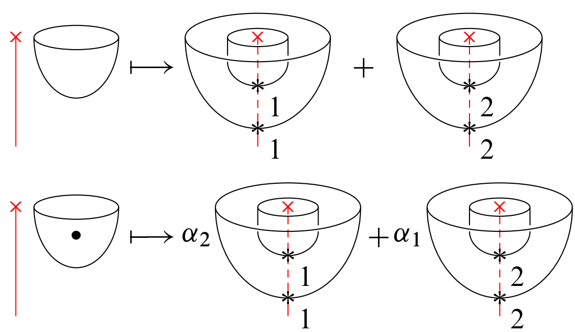
Example 2.16 (Figure 5, type B map) This calculation follows easily from the skein relation (19):



Example 2.17 (Figure 5, type C map) A convenient way to perform this calculation is to use neck-cutting with shifted dots (18) near the contractible circle and then simplify using the relations (13):



Example 2.18 (Figure 5, type D map) The neck-cutting relation (17) is helpful here. For the dotted cup we also use (14) to simplify further:



Recall the involution τ of R_α that transposes α_1 and α_2, and extend it to an antilinear involution, also denoted τ, of the free R_α–module $\mathrm{Fr}(C)$ as follows. Involution τ on $\mathrm{Fr}(C)$ sends a surface S to the same surface with the labeling ℓ of anchor points reversed and acts on linear combinations by

$$\tau\Bigl(\sum_i \lambda_i S_i\Bigr) = \sum_i \tau(\lambda_i)\tau(S_i).$$

For a closed surface S we have, by direct computation, $\langle \tau(S)\rangle = \tau(\langle S\rangle)$, showing compatibility of the two involutions. If S, in addition, carries shifted dots, involution τ reverses their labels, so that $\tau(①) = ②$ and $\tau(②) = ①$. Involution τ descends to an involution, also denoted τ, on $\langle C\rangle$. Annular degree is negated under τ: $\mathrm{adeg}(\tau(S)) = -\mathrm{adeg}(S)$ for an anchored cobordism S.

2.3 Annular link homology

Let ACob denote the category whose objects consist of collections of finitely many disjoint simple closed curves in the punctured plane $\mathcal{P}$. A morphism from C_0 to C_1 in ACob is an anchored cobordism from C_0 to C_1, up to ambient isotopy fixing the boundary pointwise and mapping $L_{[0,1]}$ to itself. Let ACob$'$ denote the subcategory of ACob with the same objects as ACob but whose morphisms are isotopy classes of annular cobordisms, disjoint from the anchor line L. The composition of annular cobordisms is again annular.

Let R_α–ggmod denote the category of bigraded R_α–modules and homogeneous maps (of any bidegree) between them. We have a functor

$$\langle - \rangle\colon \mathrm{ACob} \to R_\alpha\text{–ggmod},$$

which sends a collection of circles $C \subset \mathcal{P}$ to the state space $\langle C\rangle$ and sends an anchored cobordism S from C_0 to C_1 to the map $\langle S\rangle\colon \langle C_0\rangle \to \langle C_1\rangle$ as in (25). By Lemma 2.13, $\langle S\rangle$ is a map of bidegree $(\mathrm{qdeg}(S), \mathrm{adeg}(S))$. We can restrict to the category of annular cobordisms to get a functor

$$\langle - \rangle'\colon \mathrm{ACob}' \to R_\alpha\text{–ggmod},$$

which assigns to an annular cobordism S a map $\langle S\rangle' = \langle S\rangle$ of bidegree $(\mathrm{qdeg}(S), 0)$. The restriction $\langle - \rangle'$ does not change the state space assigned to a collection of circles $C \subset \mathcal{P}$.

On the other hand, a functor

$$\mathcal{G}_\alpha\colon \mathrm{ACob}' \to R_\alpha\text{–ggmod}$$

was introduced in [1]. We briefly recall $\mathcal{G}_\alpha$ below.

Consider the algebra

$$A_\alpha = R_\alpha[X]/((X-\alpha_1)(X-\alpha_2)).$$

It is a free R_α–module with basis $\{1, X\}$. The trace $\epsilon_\alpha\colon A_\alpha \to R_\alpha$ given by $1 \mapsto 0$ and $X \mapsto 1$ makes A_α into a Frobenius algebra, which defines a (1+1)–dimensional TQFT, a functor $\mathcal{F}_\alpha$ from the category of dotted cobordisms to the category of R_α–modules. A dot on a cobordism is interpreted as multiplication by $X \in A_\alpha$. Define a grading on A_α by setting

$$\mathrm{qdeg}(1) = -1, \quad \mathrm{qdeg}(X) = 1. \tag{26}$$

With this grading, a cobordism S with d dots is assigned by $\mathcal{F}_\alpha$ a map of degree $-\chi(S) + 2d$. Alternatively, the TQFT $\mathcal{F}_\alpha$ is the result of applying the universal construction to the closed surface evaluation (6) when restricted to surfaces disjoint from L and collections of contractible circles in $\mathcal{P}$. See [19] for further details about the Frobenius pair (R_α, A_α).

Let $C \subset \mathcal{P}$ be a collection of n contractible and m noncontractible circles. Define the bigraded R_α–module $\mathcal{G}_\alpha(C)$ as follows. As an R_α–module, we set

$$\mathcal{G}_\alpha(C) = \mathcal{F}_\alpha(C) = A_\alpha^{\otimes(n+m)}.$$

Define the *annular grading*, denoted adeg, on $\mathcal{F}_\alpha(C)$ as follows.

Every tensor factor A_α corresponding to a contractible circle is concentrated in annular degree zero. Order the noncontractible circles in C from outermost (furthest from the puncture) to innermost. Introduce the notation

$$v_0 = 1, \quad v_1 = X - \alpha_1, \quad v_0' = 1, \quad v_1' = X - \alpha_2. \tag{27}$$

Both $\{v_0, v_1\} = \{1, X-\alpha_1\}$ and $\{v_0', v_1'\} = \{1, X-\alpha_2\}$ constitute an R_α–basis for A_α. Set

$$\mathrm{adeg}(v_0) = \mathrm{adeg}(v_0') = -1, \quad \mathrm{adeg}(v_1) = \mathrm{adeg}(v_1') = 1. \tag{28}$$

The annular grading on noncontractible circle is defined by assigning the homogeneous basis $\{v_0, v_1\}$ or $\{v_0', v_1'\}$ to the corresponding tensor factor of A_α in an alternating

	1	X	v_0	v_1	v'_0	v'_1
qdeg$'$	-1	1	0	0	0	0
adeg	0	0	-1	1	-1	1

Table 2: The $(\mathrm{qdeg}', \mathrm{adeg})$–bidegrees of relevant elements, where $\{1, X\}$ is a basis for a contractible circle and $\{v_0, v_1\}$ and $\{v'_0, v'_1\}$ are bases for noncontractible circles.

manner with respect to nesting in $\mathcal{P}$, with the convention that the outermost circle is assigned $\{v_0, v_1\}$.

It is convenient to distinguish between the modules assigned to different types of circles in $\mathcal{P}$. Let V_α and V'_α denote the R_α–modules A_α with bases $\{v_0, v_1\}$ and $\{v'_0, v'_1\}$, respectively. The notation A_α will be reserved for the module assigned to a contractible circle, with basis $\{1, X\}$.

The R_α–module $\mathcal{G}_\alpha(C)$ also carries a quantum grading qdeg inherited from (26). Define a modified quantum grading qdeg$'$ on $\mathcal{G}_\alpha(C)$ by

$$\mathrm{qdeg}' = \mathrm{qdeg} - \mathrm{adeg}. \tag{29}$$

We will consider $\mathcal{G}_\alpha(C)$ as a bigraded R_α–module with bigrading $(\mathrm{qdeg}', \mathrm{adeg})$. Bidegrees are recorded in Table 2.

Remark 2.19 The modified quantum grading qdeg$'$ appears elsewhere in the literature and is more natural in the context of annular link homology. In [12] this grading was denoted j'. Similarly, the annular link homology defined in [5] carries the modified quantum grading.

We now define $\mathcal{G}_\alpha$ on annular cobordisms. For an annular cobordism $S \subset \mathbb{R}^2 \times [0, 1]$, if the boundary of S is contractible in $\mathcal{P}$ then $\mathcal{G}_\alpha(S) = \mathcal{F}_\alpha(S)$, where $\mathcal{F}_\alpha$ is the TQFT corresponding to the Frobenius algebra A_α as above. Formulas for the maps assigned by $\mathcal{G}_\alpha$ to the four elementary cobordisms in Figure 5 are recorded below. If other essential circles are present, then due to parity the formulas may be slightly different from those below. To obtain the full set of formulas, one interchanges $v_0 \leftrightarrow v'_0$, $v_1 \leftrightarrow v'_1$, and $\alpha_1 \leftrightarrow \alpha_2$:

$$\begin{gathered} V_\alpha \otimes A_\alpha \xrightarrow{\text{(A)}} V_\alpha, \\ v_0 \otimes 1 \mapsto v_0, \quad v_1 \otimes 1 \mapsto v_1, \quad v_0 \otimes X \mapsto \alpha_1 v_0, \quad v_1 \otimes X \mapsto \alpha_2 v_1, \end{gathered} \tag{30}$$

$$V_\alpha \otimes V'_\alpha \xrightarrow{\text{(B)}} A_\alpha, \tag{31}$$
$$v_0 \otimes v'_0 \mapsto 0, \quad v_1 \otimes v'_0 \mapsto X-\alpha_1, \quad v_0 \otimes v'_1 \mapsto X-\alpha_2, \quad v_1 \otimes v'_1 \mapsto 0,$$

$$V_\alpha \xrightarrow{\text{(C)}} V_\alpha \otimes A_\alpha, \tag{32}$$
$$v_0 \mapsto v_0 \otimes (X-\alpha_2), \quad v_1 \mapsto v_1 \otimes (X-\alpha_1),$$

$$A_\alpha \xrightarrow{\text{(D)}} V_\alpha \otimes V'_\alpha, \tag{33}$$
$$1 \mapsto v_0 \otimes v'_1 + v_1 \otimes v'_0, \quad X \mapsto \alpha_1 v_0 \otimes v'_1 + \alpha_2 v_1 \otimes v'_0.$$

Theorem 2.20 *The functors* $\langle - \rangle' : \mathrm{ACob}' \to R_\alpha$–ggmod *and* $\mathcal{G}_\alpha : \mathrm{ACob}' \to R_\alpha$–ggmod *are naturally isomorphic via bidegree-preserving maps.*

Proof Let $C \subset \mathcal{P}$ be a collection of circles. We will define an R_α–linear, bidegree preserving isomorphism $\Phi_C : \langle C \rangle \to \mathcal{G}_\alpha(C)$ and show that it is natural with respect to annular cobordisms.

Let n and m denote the number of contractible and noncontractible circles in C, respectively. Fix an ordering $1, \dots, n$ of the contractible circles in C. The R_α–module $\mathcal{G}_\alpha(C)$ is free with basis given by elements of the form

$$y_1 \otimes \cdots \otimes y_n \otimes z_1 \otimes \cdots \otimes z_m,$$

where each y_i is in $\{1, X\}$, specifying a basis element of the i^{th} contractible circle, and each z_j is in either $\{v_0, v_1\}$ or $\{v'_0, v'_1\}$, depending on nesting, specifying basis elements of the noncontractible circles. The ordering of factors $z_1 \otimes \cdots \otimes z_m$ corresponding to noncontractible circles is from outermost to innermost as usual, so that the first factor z_1 labels the outermost noncontractible circle.

We now define the isomorphism $\Phi_C : \langle C \rangle \to \mathcal{G}_\alpha(C)$. Recall the standard basis $B = B(C)$ for $\langle C \rangle$ defined in the proof of Theorem 2.11. For $\Sigma \in B$ with anchor points labeled $\ell_1, \dots, \ell_m$, read from bottom to top, set

$$\Phi_C(\Sigma) = y_1 \otimes \cdots \otimes y_n \otimes z_1 \otimes \cdots \otimes z_m,$$

where $y_i = 1$ if the corresponding cup in Σ is undotted and $y_i = X$ if the corresponding cup in Σ is dotted. The generators z_j of noncontractible circles are determined using the rule

$$z_j = \begin{cases} v_1 & \text{if } j \text{ is odd and } \ell_j = 1, \\ v_0 & \text{if } j \text{ is odd and } \ell_j = 2, \\ v'_0 & \text{if } j \text{ is even and } \ell_j = 1, \\ v'_1 & \text{if } j \text{ is even and } \ell_j = 2. \end{cases}$$

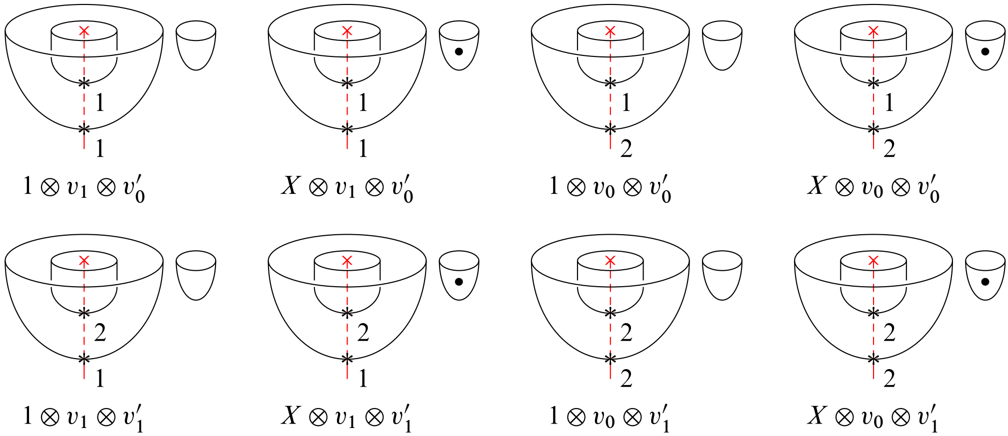


Figure 6: An example of the isomorphism Φ_C when C consists of one contractible circle and two noncontractible circles. Basis elements Σ of $\langle C \rangle$ are drawn with the corresponding basis element $\Phi_C(\Sigma) \in \mathcal{G}_\alpha(C)$ written underneath.

See Figure 6 for an example of the assignment Φ_C when $n = 1$ and $m = 2$. By comparing the bidegree formula (24) for Σ with the bidegree of $\Phi_C(\Sigma)$ (see Table 2), we see that Φ_C is a bidegree-preserving isomorphism. Recall that we use the modified quantum grading (29) for $\mathcal{G}_\alpha(C)$.

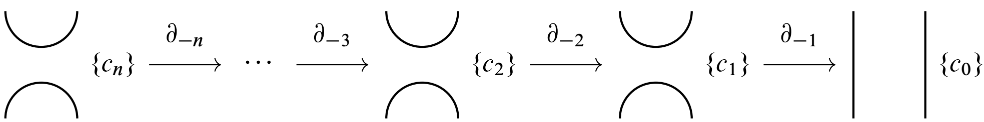
Now let $S \colon C_1 \to C_2$ be an annular cobordism. To complete the proof, we check that the square

$$\begin{tikzcd}
\langle C_1 \rangle \arrow[r, "\Phi_{C_1}"] \arrow[d, "\langle S \rangle"'] & \mathcal{G}_\alpha(C_1) \arrow[d, "\mathcal{G}_\alpha(S)"] \\
\langle C_2 \rangle \arrow[r, "\Phi_{C_2}"] & \mathcal{G}_\alpha(C_2)
\end{tikzcd}$$

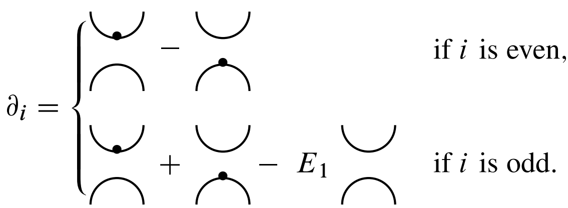
commutes. If all the boundary circles of S are contractible, then commutativity of the square is straightforward. Otherwise, if S has at least one noncontractible boundary circle, it suffices to consider the case where S is one of the elementary annular cobordisms depicted in Figure 5. Formulas for these maps were recorded in Examples 2.15–2.18. Comparing with the formulas (30)–(33) completes the proof. $\square$

Let $\mathbb{A} := S^1 \times [0,1]$ denote the annulus. For an oriented link $L \subset \mathbb{A} \times [0,1]$ in the thickened annulus, a generic projection of L onto $\mathbb{A} \times \{0\}$ yields a link diagram D in the interior of $\mathbb{A}$. Identifying the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P}$, we may form the cube of resolutions of D in the usual way, for instance as described in [4, Section 2], with all smoothings drawn in $\mathcal{P}$. The result is a commutative cube in the category ACob'. Introducing signs to make the cube anticommutative, taking direct sums along diagonals, adding homological and quantum grading shifts, and applying the functor $\langle - \rangle' \colon \mathrm{ACob}' \to R_\alpha$–ggmod, one obtains a chain complex $C(D)$ of bigraded R_α–modules. Diagrams representing isotopic annular links are related by Reidemeister moves away from the puncture. By standard arguments [4; 14], the chain homotopy class of $C(D)$ is an invariant of the annular link L. We write $H(L)$ to denote the homology of $C(D)$, for any diagram D of L. Theorem 2.20 implies that the resulting annular homology is isomorphic to that of [1].

Example 2.21 As an explicit example, let σ denote the positive crossing generator of the 2–strand braid group, and let L_n denote the annular link obtained as the annular closure of σ^{-n}. Consider the complex $C(n)$:



The right-most term is in homological degree zero and the quantum grading shifts c_i are given by $c_0 = n$ and $c_i = n + 2i - 1$ for $1 \le i \le n$. The right-most differential ∂_{-1} is the saddle cobordism, and for $-n \le i \le -2$ the differentials are



The above schematic depiction of ∂_i is interpreted as follows: each ∂_i is an R_α–linear combination of surfaces, each of which is given by the product cobordism on the depicted planar tangle, with a dot on a component of the surface if the corresponding tangle component is dotted. One can show that the chain complex $C(L_n)$ is chain homotopy equivalent to the annular closure of $C(n)$.

Note that the annular closure of chain groups of $C(n)$ in negative homological degree are each a contractible circle, contributing a free module with basis 1 and X (represented by the surfaces S and $S_\bullet$ in Figure 4). In homological degree zero the result is two essential circles. We also see that, upon taking the annular closure, that $\partial_i = 0$ for i even, and that ∂_i for $i \le -3$ odd is given by $\partial_i(1) = 2X - E_1$ and $\partial_i(X) = E_1 X - 2E_2$,

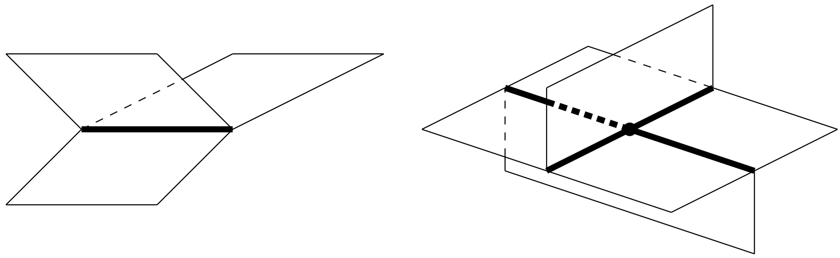


Figure 7: Local model of a prefoam near singular points. Left, seam points where three facets meet. Right, a seam vertex where six facets meet. The singular graph $s(F)$ is drawn bold.

which is injective. The differential ∂_{-1} is the map in Example 2.18, which is also injective. Therefore, in homological degree $i \leq 0$,

$$H^i(L_n) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \dfrac{R_\alpha\{n-2i-2,0\} \oplus R_\alpha\{n-2i,0\}}{\langle(-E_1,2),(-2E_2,E_1)\rangle} & \text{if } i < 0 \text{ and } i \text{ is even,} \\ R_\alpha\{n,-2\} \oplus R_\alpha\{n,2\} \oplus (R_\alpha\{n,0\}/\langle\alpha_2-\alpha_1\rangle) & \text{if } i = 0, \end{cases}$$

where the curly brackets $\{j,k\}$ denote an upwards (qdeg, adeg) shift of (j,k), and the angled brackets denote the R_α–submodule generated by the enclosed elements.

3 Unoriented SL(3) anchored homology of planar annular webs

We recall definitions and notations from [18], including that of (unoriented) SL(3) foams and refer the reader to [18, Section 2.1] for more details.

Definition 3.1 A (closed) SL(3) *prefoam* is a compact 2–dimensional CW complex equipped with a PL–structure such that each point has an open neighborhood that is either an open disk, the product of a tripod and an open interval (Figure 7, left), or the cone over the 1–skeleton of a tetrahedron (Figure 7, right). Points of the first type are called *regular*, those of the second are called *seam points*, and those of the third are called *seam vertices*. A (closed) SL(3) *foam* is a closed SL(3) prefoam together with a PL embedding into $\mathbb{R}^3$.

We will simply write *prefoam* and *foam* in place of *closed* SL(3) (*pre*)*foam*. For a prefoam F, denote by $v(F)$ the set of seam vertices and by $s(F)$ the set of seam points

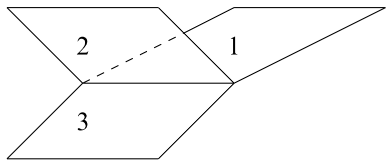


Figure 8: The local model for a preadmissible coloring near a seam point.

and seam vertices. The subspace $s(F)$ is a 4–valent graph which may contain closed loops. Connected components of $s(F) \setminus v(F)$ are called *seams*.

The subspace $F \setminus s(F)$ is a (not necessarily compact) surface, and a connected component of $F \setminus s(F)$ will be called a *facet* of F. The (finite) set of facets of F is denoted by $f(F)$. Facets of prefoams may be decorated by a finite number of dots, which are allowed to float freely on their facets but may not cross seams or enter seam vertices.

A *coloring* of a prefoam F is a map

$$c\colon f(F) \to \{1,2,3\}.$$

That is, a coloring assigns 1, 2 or 3 to each facet of F. A coloring is called *preadmissible* if the three facets meeting at each seam of F have distinct colors; see Figure 8. For a preadmissible coloring c and $1 \le i, j \le 3$ with $i \ne j$, let $F_{ij}(c)$ denote the union of facets colored i or j. The preadmissibility condition guarantees that each $F_{ij}(c)$ is a closed surface; see [18, Proposition 2.2].

A coloring c is called *admissible* if each $F_{ij}(c)$ is orientable. For a foam F (that is, a prefoam embedded in $\mathbb{R}^3$), every preadmissible coloring is admissible, since $F_{ij}(c)$ is a closed surface in $\mathbb{R}^3$.

3.1 Unoriented anchored SL(3) foams and their evaluations

Fix a field $\Bbbk$ of characteristic 2. In this section the following commutative rings will be used:

- $R'_x = \Bbbk[x_1, x_2, x_3]$ is the ring of polynomials in three variables.
- $R_x = \Bbbk[E_1, E_2, E_3]$ the subring of R'_x that consists of symmetric polynomials in x_1, x_2 and x_3, with generators E_i being elementary symmetric polynomials:

$$E_1 = x_1 + x_2 + x_3, \quad E_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad E_3 = x_1x_2x_3.$$

- $R''_x = R'_x[(x_1+x_2)^{-1}, (x_2+x_3)^{-1}, (x_1+x_3)^{-1}]$ is a localization of R'_x given by inverting $x_i + x_j$, for $1 \le i < j \le 3$.

- $\widetilde{R}'_x = \Bbbk[\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}]$ is the extension of R'_x obtained by introducing square roots of x_1, x_2 and x_3.
- $\widetilde{R}''_x = \Bbbk[\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, (x_1+x_2)^{-1}, (x_2+x_3)^{-1}, (x_1+x_3)^{-1}]$ is a localization of $\widetilde{R}'_x$ given by inverting $x_i + x_j$, for $1 \le i < j \le 3$.

All five of these rings are graded by setting $\deg(x_i) = 2$ for $i = 1, 2, 3$. Inclusions of the above rings are summarized in the following diagram:

$$\begin{array}{ccccc} & & \widetilde{R}'_x & \subset & \widetilde{R}''_x \\ & & \cup & & \cup \\ R_x & \subset & R'_x & \subset & R''_x \end{array} \tag{34}$$

We follow the notation established in [18] for these rings with the additional subscript x to distinguish from the notation in Section 2.

Definition 3.2 An *anchored* SL(3) *foam* F is an SL(3) foam $F' \subset \mathbb{R}^3$ that may intersect the line L at finitely many points away from the singular graph $s(F')$ of F'. Thus each intersection point belongs to some facet f of F', and intersection of facets with L are required to be transverse. Denote by $p(F) = F \cap L$ the set of intersection points (anchor points) of F. Intersection points carry labels in $\{1,2,3\}$; that is, F comes equipped with a fixed map

$$\ell \colon p(F) \to \{1,2,3\}.$$

It is convenient to order anchor points $p_1, \dots, p_m$ from bottom to top, with labels $\ell_i = \ell(p_i)$, $i = 1, \dots, m$.

We now refine the notion of admissible coloring of a foam to that of admissible coloring of an anchored foam F. Consider an anchored foam F with the underlying foam F'. A coloring $c \in \mathrm{adm}(F')$ induces a coloring of anchor points in F', by assigning to each point the color of its facet. We say that c is admissible if that's exactly the labeling of anchor points of F, that is, $\ell(p) = c(f)$ for each anchor point p in a facet f, and then set $c(p) = \ell(p)$.

In this way, the set of admissible colorings of F' is in a bijection with the set of admissible colorings of anchored foams F that become F' upon forgetting the labeling of anchor points:

$$\mathrm{adm}(F') \cong \coprod_F \mathrm{adm}(F).$$

Various constructions with SL(3) foams in [18] extend directly to anchored foams. In particular, bicolored surfaces $F_{ij}(c)$ are well defined, associated to an admissible coloring c. We will also call an admissible coloring simply a coloring. We will use i, j and k to denote the three elements of $\{1,2,3\}$, not necessarily in that order.

We refine [18, Definition 2.9] for anchored foams.

Definition 3.3 Let F be an anchored foam, $c \in \mathrm{adm}(F)$ be an admissible coloring, and Σ a connected component of $F_{ij}(c)$ which is disjoint from L. Define a coloring c' of F which swaps the colors i and j on facets of Σ, and leaves all other facets colored according to c. We say that c and c' are related by an *ij–Kempe move along* Σ. Note that since Σ has no anchor points, c' is still an admissible coloring of F.

Kempe moves can be done on components Σ of $F_{ij}(c)$ that intersect L as well, but the resulting anchored foam F_0 is different from F due to carrying different labels on anchor points on Σ.

For $k \in \{1,2,3\}$, denote by k' and k'' its two complementary elements, so that $\{k,k',k''\} = \{1,2,3\}$. Let F be an anchored foam with labeling ℓ. Let $c \in \mathrm{adm}(F)$ be an admissible coloring. For an anchor point $p \in p(F)$ lying on a facet $f \in f(F)$, we set $c(p) = c(f) = \ell(p)$; that is, $c(p)$ is the color of the facet, according to c, on which p lies, which equals $\ell(p)$ since c is admissible. For $1 \le i \le 3$, let $d_i(c)$ denote the number of dots on facets colored i. For $1 \le i \neq j \le 3$, let $F_{ij}(c)$ be the union of facets of F colored i or j. The space $F_{ij}(c)$ is a closed surface in $\mathbb{R}^3$ and hence has even Euler characteristic. Set

$$\langle F, c\rangle = \frac{P(F,c)}{Q(F,c)}, \tag{35}$$

where

$$P(F,c) = \prod_{i=1}^{3} x_i^{d_i(c)} \cdot \Bigl(\prod_{p\in p(F)} (x_{c(p)} + x_{\ell(p)'})(x_{c(p)} + x_{\ell(p)''})\Bigr)^{1/2}, \tag{36}$$

$$Q(F,c) = \prod_{1\le i<j\le 3} (x_i + x_j)^{\chi(F_{ij}(c))/2}. \tag{37}$$

The product of the two terms under the square root, for a given anchor point p, is equal to

$$\begin{aligned} &(x_1+x_2)(x_1+x_3) \quad \text{if } c(p) = 1,\\ &(x_2+x_1)(x_2+x_3) \quad \text{if } c(p) = 2,\\ &(x_3+x_1)(x_3+x_2) \quad \text{if } c(p) = 3.\end{aligned}$$

Remark 3.4 This product is the inverse of the square decoration $\square$ in [18, Section 4.1]. The square decoration was used to study a separable version of the unoriented SL(3) theory, with the discriminant $\mathcal{D} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$ inverted, which is a version of the Lee theory. Here, we use the defect line L rather than freely floating square dots in [18, Section 4.1] in the opposite way, to add factors to the evaluation rather than divide by terms in the discriminant.

Remark 3.5 If c is an admissible coloring of the underlying foam F' of F but not of the anchored foam F, then the evaluation (35) is still defined and equal to zero;

$$\langle F, c\rangle = 0, \quad c \in \operatorname{adm} F' \setminus \operatorname{adm} F. \tag{38}$$

This holds since, for some $p \in p(F)$, its color $c(p)$ differs from its label $\ell(p)$, so that $x_{c(p)} + x_{c(p)} = 0$ appears under the square root in (36) and $P(F, c) = 0$. Thus,

$$(x_{c(p)} + x_{\ell(p)'})(x_{c(p)} + x_{\ell(p)''}) = \begin{cases} (x_{\ell(p)} + x_{\ell(p)'})(x_{\ell(p)} + x_{\ell(p)''}) & \text{if } c(p) = \ell(p), \\ 0 & \text{otherwise.} \end{cases}$$

Define the *evaluation of* F to be

$$\langle F\rangle = \sum_{c \in \operatorname{adm}(F)} \langle F, c\rangle. \tag{39}$$

Alternatively, we can sum over the larger set of $c \in \operatorname{adm}(F')$, due to (38).

Let us explain the square root in (36). The equality $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ holds in a commutative ring of characteristic 2, so $\langle F, c\rangle$ is in the ring $\widetilde{R}''_x$; see (34). We will show in Proposition 3.11 that, in fact, no square roots appear, so that $\langle F, c\rangle \in R''_x$. Likewise, in Proposition 3.12 we show that $\langle F\rangle \in R'_x$.

The evaluation (39) is multiplicative with respect to disjoint union and does not depend on a particular embedding of F into $M = (\mathbb{R}^3, L)$ as long as anchor points on F and their labels are specified.

If an anchored foam F is a disjoint union of anchored foams $F_1 \sqcup \cdots \sqcup F_k$, then

$$\langle F\rangle = \langle F_1\rangle \cdots \langle F_k\rangle.$$

If F is disjoint from L, then $\langle F\rangle$ is equal to the evaluation in [18, Section 2.3].

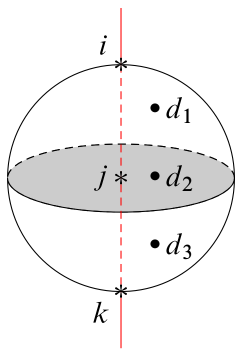
Example 3.6 Let F be a 2–sphere $\mathbb{S}^2$ with two anchor points and d dots. Its evaluation is zero unless both points have the same label $i \in \{1, 2, 3\}$, in which case there is only admissible coloring c which colors F by i. Let $j, k \in \{1, 2, 3\}$ denote the complementary elements to i. The surfaces $F_{ij}(c)$ and $F_{ik}(c)$ are 2–spheres, while $F_{jk}(c) = \varnothing$. Then the evaluation is

$$\langle F \rangle = \frac{x_i^d((x_i + x_j)^2(x_i + x_k)^2)^{1/2}}{(x_i + x_j)(x_i + x_k)} = x_i^d.$$

Example 3.7 More generally, let F be a genus g surface carrying d dots and $2n > 0$ anchor points. It evaluates to zero unless all points are labeled by the same $i \in \{1, 2, 3\}$. In this case, letting $j, k \in \{1, 2, 3\}$ be the complementary elements to i, the evaluation is

$$\langle F \rangle = \frac{x_i^d((x_i + x_j)(x_i + x_k))^n}{((x_i + x_j)(x_i + x_k))^{1-g}} = x_i^d((x_i + x_j)(x_i + x_k))^{n+g-1}.$$

Example 3.8 Consider the theta foam F whose facets each intersect L once, with anchor points labeled $i, j, k \in \{1, 2, 3\}$ and facets carrying d_1, d_2 and d_3 dots,



In an admissible coloring of the underlying foam, the three facets must have distinct colors, so $\langle F \rangle = 0$ if i, j and k are not distinct. If i, j and k are distinct, then there is one admissible coloring c which colors the top, middle, and bottom facets, respectively, by i, j and k. The surfaces $F_{ij}(c)$, $F_{ik}(c)$, $F_{jk}(c)$ are 2–spheres, and the evaluation is

$$\langle F \rangle = x_i^{d_1} x_j^{d_2} x_k^{d_3}.$$

Remark 3.9 Note that the evaluation of an anchored foam is in general not a symmetric function in x_1, x_2 and x_3, whereas in [18] the evaluation is always an element of R_x.

Let us call a sequence $\ell \in \{1, 2, 3\}^m$ *preadmissible* if the following holds. Let u_1, u_2 and u_3 be three nonzero elements of the abelian group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Sequence ℓ is *preadmissible* if and only if

$$\sum_{i=1}^{m} u_{\ell(i)} = 0 \in \mathbb{Z}/2 \times \mathbb{Z}/2. \tag{40}$$

Proposition 3.10 *If an anchored foam F has an admissible coloring, the sequence ℓ of its anchor points is preadmissible.*

Proof Consider a generic intersection of F with a half-plane in $\mathbb{R}^3$ bounding L. This intersection is a trivalent graph Γ in the half-plane. Coloring c of F induces a coloring c' of edges of Γ such that around each trivalent vertex of Γ the colors of the three edges are distinct (Tait coloring). On the boundary points (one-valent vertices) of Γ the coloring is given by labeling ℓ. The sum on the left hand side of (40) is zero since it can alternatively be written as the sum of triples of vectors $u_1+u_2+u_3=0$ over all trivalent vertices of Γ. Each inner edge of Γ, that bounds two trivalent vertices, contributes $u_i+u_i=0$ to the sum, where i is the color of the edge. An edge with one or both endpoints on the boundary contributes the sum of the u_i over its boundary points. □

For an anchored foam F and $1\le i\le 3$, let $\mathrm{an}(i)$ denote the number of anchor points of F with label i (the dependence on F is omitted).

Proposition 3.11 *For an anchored foam F and an admissible coloring c, we have $\langle F,c\rangle\in R''_x$.*

Proof Recall the rings R''_x and $\widetilde{R}''_x$ defined in (34). It's clear that $\langle F,c\rangle$ belongs to the larger ring $\widetilde{R}''_x$.

The expression in (35) under the square root is equal to

$$(x_1+x_2)^{\mathrm{an}(1)+\mathrm{an}(2)}(x_2+x_3)^{\mathrm{an}(2)+\mathrm{an}(3)}(x_1+x_3)^{\mathrm{an}(1)+\mathrm{an}(3)}.$$

For $1\le i<j\le 3$, the integer $\mathrm{an}(i)+\mathrm{an}(j)$ is even since it is equal to the number of intersection points of the closed surface $F_{ij}(c)$ with L; see also Proposition 3.10. Consequently, taking the square root produces integral exponent of x_i+x_j, implying that $\langle F,c\rangle$ is in R''_x. □

Using the above notation, the square root term in (36) is equal to

$$\widetilde{Q}(F,c):=\prod_{1\le i<j\le 3}(x_i+x_j)^{(\mathrm{an}(i)+\mathrm{an}(j))/2}, \tag{41}$$

so formula (35) can be rewritten as

$$\langle F,c\rangle=\prod_{i=1}^{3}x_i^{d_i(c)}\prod_{1\le i<j\le 3}(x_i+x_j)^{(\mathrm{an}(i)+\mathrm{an}(j)-\chi(F_{ij}(c)))/2}. \tag{42}$$

Proposition 3.12 *For an anchored foam F, we have $\langle F\rangle \in R'_x = \Bbbk[x_1,x_2,x_3]$.*

Proof The proof of Theorem 2.17 in [18] extends with minor changes to this case. Note that the evaluation is no longer a symmetric function. We must show that positive powers of $x_i + x_j$ for $1 \le i < j \le 3$, do not appear in the denominator of $\langle F\rangle$. Let us specialize to $i = 1$ and $j = 2$. Denominators $x_1 + x_2$ in the evaluations $\langle F, c\rangle$ may appear only from the components of $F_{12}(c)$ that are 2–spheres. If a 2–sphere does not intersect L, the proof in [18] works in this case as well. Suppose a 2–sphere component Σ of $F_{12}(c)$ intersects L in an(1) points colored 1 and an(2) points colored 2 (necessarily in the corresponding facets of F carrying those colors under c). These points contribute

$$(x_1+x_2)^{\mathrm{an}(1)+\mathrm{an}(2)}(x_1+x_3)^{\mathrm{an}(1)}(x_2+x_3)^{\mathrm{an}(2)}$$

to the expression under the square root, and $\mathrm{an}(1)+\mathrm{an}(2) \ge 2$, allowing to cancel the denominator term x_1+x_2 that Σ contributes. Summing over all admissible colorings and otherwise following the arguments in [18, Theorem 2.17] implies the result. □

Remark 3.13 Contributions of anchor points to the evaluation $\langle F, c\rangle$ can be interpreted as follows. Consider polynomial $f(x) = (x-x_1)(x-x_2)(x-x_3) \in R'_x[x]$. Then

$$f'(x) = (x-x_2)(x-x_3) + (x-x_1)(x-x_3) + (x-x_1)(x-x_2)$$

and

$$\begin{aligned} f'(x_1) &= (x_1-x_2)(x_1-x_3), \\ f'(x_2) &= (x_2-x_1)(x_2-x_3), \\ f'(x_3) &= (x_3-x_1)(x_3-x_2). \end{aligned}$$

Contribution of an anchor point p with a label $i = \ell(p)$ to the evaluations $\langle F, c\rangle$ and $\langle F\rangle$ is then $\sqrt{f'(x_i)}$, the square root of the derivative of f at the root x_i of the polynomial f. In characteristic two, signs do not matter, but this observation hints how to extend the evaluation to characteristic 0.

Since the labels $i_1, \dots, i_m$ of anchor points are fixed in a given F, these marked points contribute the same term,

$$\sqrt{f'|_{L\cap F}} := \left(\prod_{r=1}^{m} f'(x_{i_r})\right)^{1/2},$$

and we have

$$\langle F, c\rangle = \sqrt{f'|_{L\cap F}} \cdot \langle F', c\rangle, \quad \langle F\rangle = \sqrt{f'|_{L\cap F}} \cdot \langle F'\rangle, \tag{43}$$

where F' is the foam F viewed as a regular foam with anchored points and their labels ignored. When coloring c of F is not compatible with labels of anchor points, though, we should define $\sqrt{f'|_{L\cap F}} = 0$ to match the formula $\langle F, c\rangle = 0$.

Also notice that, switching to characteristic 0 and from the matrix factorization viewpoint [20], $f(x) = w'(x)$ is the derivative of the potential

$$w(x) = \tfrac{1}{4}x^4 - \tfrac{1}{3}E_1x^3 + \tfrac{1}{2}E_2x^2 - E_3x,$$

so the contributions of anchor points are given by square roots of the second derivative $\sqrt{w''(x_i)}$ at critical points of w, analogous to the square root of the Hessian factor that appears, for example, in the steepest descent method formulas.

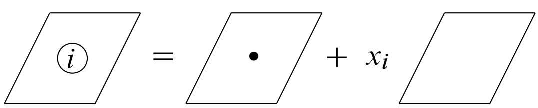
3.2 Skein relations

In this subsection we record several local relations satisfied by the evaluation of anchored SL(3) foams. We start with the following proposition concerning the relations in [18, Section 2.5], which should be understood as occurring away from the anchor line L.

Proposition 3.14 *The twelve local relations in* [18, Propositions 2.22–2.33] *hold.*

Proof The arguments in [18] apply without modification. □

We will use shifted dots in this section, as in (12). For $1 \le i \le 3$, we allow anchored foams to carry decorations of the form $i = \bullet + x_i$ on a facet. They are required to be disjoint from L, float freely on their facets, but cannot move past seams or seam vertices:



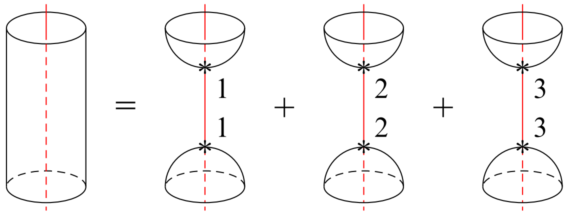
For an anchored foam F carrying i on some facet $f \in f(F)$, any coloring $c \in \mathrm{adm}(F)$ which colors f by i evaluates to zero, $\langle F, c\rangle = 0$. An anchor point labeled i has the same effect as placing

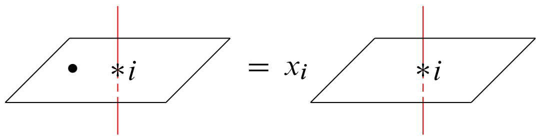
$$\sqrt{j\;k} = \sqrt{i'\;i''}$$

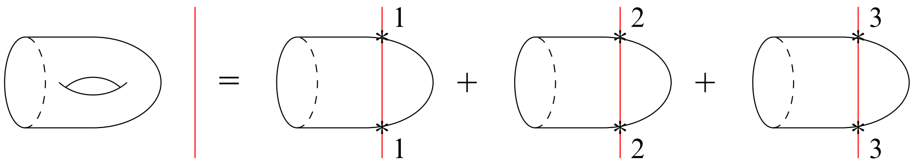
on the facet on which it lies (recall our conventions that $\{1,2,3\} = \{i,j,k\} = \{i,i',i''\}$). See also (47) and the discussion in Section 3.4.

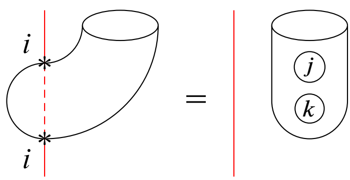
We also have relations involving the anchor line.

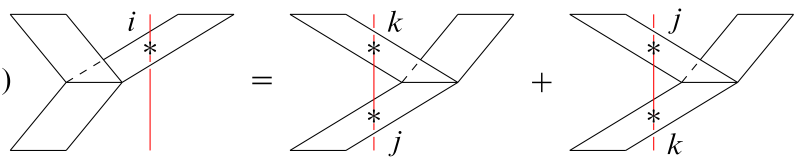
Lemma 3.15 *The following local relations hold*:

(44)


(45)


(46)


(47)


(48) $(x_j + x_k)$


In the last two equations, $\{i, j, k\} = \{1, 2, 3\}$.

Proof Let us verify (44); the other four relations are easier to check and the proof is left to the reader. Denote by F the anchored foam on the left-hand side, and by G^1, G^2 and G^3 the three foams on the right-hand side, with the superscript corresponding to the labels of the depicted anchor points. For $1 \le i \le 3$, let $\mathrm{adm}_i(F)$ be the set of admissible colorings of F in which the depicted tube is colored by i. Admissible colorings of G^i must color the two disks by i, so there is a natural bijection $\mathrm{adm}_i(F) \cong \mathrm{adm}(G^i)$.

For $c \in \mathrm{adm}_i(F)$, let $c' \in \mathrm{adm}(G^i)$ denote the corresponding coloring. We will show that

$$\langle F, c\rangle = \langle G^i, c'\rangle,$$

which completes the proof.

The anchored foam G^i carries two more anchor points, both labeled i, than F does, while the dot placement for G^i and F is the same, so

$$P(G, c') = (x_i + x_j)(x_i + x_k)P(F, c),$$

where $\{i, j, k\} = \{1, 2, 3\}$. On the other hand,

$$\chi(G^i_{ij}(c')) = \chi(F_{ij}(c)) + 2, \quad \chi(G^i_{ik}(c')) = \chi(F_{ik}(c)) + 2, \quad \chi(G^i_{jk}(c')) = \chi(F_{jk}(c)),$$

which yields

$$Q(G, c') = (x_i + x_j)(x_i + x_k)Q(F, c).$$

Thus $\langle F, c\rangle = \langle Q, c'\rangle$ as desired. Summing over all admissible colorings of F we get

$$\langle F\rangle = \langle G^1\rangle + \langle G^2\rangle + \langle G^3\rangle,$$

completing the proof. □

3.3 State spaces

We generalize the notion of webs and cobordisms between them from [18, Section 3.1] in the presence of the anchor line L.

Definition 3.16 A *web* is a trivalent graph Γ which is PL–embedded into the punctured plane $\mathcal{P} = \mathbb{R}^2 \setminus \{(0, 0)\}$. We allow webs to have closed loops with no vertices. A *anchored foam with boundary* V is obtained by intersecting a closed anchored foam $F \subset \mathbb{R}^3$ carrying no dots with a thickened plane $\mathbb{R}^2 \times [0, 1]$ such that $F \cap (\mathcal{P} \times \{i\})$ for $i = 0, 1$ is a web (in particular, F is disjoint from the two points $(0, 0, 0)$ and $(0, 0, 1)$). A connected component of the complement of singular points in $F \cap (\mathbb{R}^2 \times [0, 1])$ is called a *facet*. Each facet may be decorated by finitely many dots which can float freely along the facet but cannot intersect the anchor line or cross singular points.

Foams with boundary are considered equivalent if there is an orientation-preserving homeomorphism of $\mathbb{R}^2 \times [0, 1]$ taking one to the other which fixes the boundary of $\mathbb{R}^2 \times [0, 1]$ pointwise and maps the line segment $L_{[0,1]} := \{(0, 0)\} \times [0, 1]$ to itself.

For a foam with boundary V, let

$$p(V) = V \cap L_{[0,1]}$$

denote its intersection points with the anchor line, called *anchor points*. Each anchor point is required to carry a label in $\{1, 2, 3\}$.

We view V as a cobordism from the web $\partial_0 V := V \cap (\mathbb{R}^2 \times \{0\})$ to the web $\partial_1 V := V \cap (\mathbb{R}^2 \times \{1\})$. A closed foam is then a cobordism from the empty web to itself. We will often refer to foams with boundary simply as *foams* when the meaning is clear from context. Composition WV of foams V and W with $\partial_1 V = \partial_0 W$ is defined in the natural way. We obtain a category AFoam of webs and anchored foams.

The category AFoam has a contravariant involution ω which is the identity on webs and which sends a foam to its reflection about $\mathbb{R}^2 \times \{\frac{1}{2}\}$, preserving the labels of anchor points. As for closed foams, denote by $s(V)$ and $v(V)$ the singular graph and singular vertices, respectively, of a foam with boundary V. Define the *degree* of V to be

$$\deg(V) = 2\big(|d(V)| + |p(V)| - \chi(V)\big) - \chi(s(V)), \tag{49}$$

where $d(V)$ is the set of dots on V.

The definition of admissible colorings extends naturally to anchored foams with boundary. An admissible coloring induces a Tait coloring on the boundary webs. If a foam with boundary V has an admissible coloring c, then by [18, Remark 2.8],

$$\deg(V) = 2|d(V)| + 2|p(V)| - \big(\chi(V_{12}(c)) + \chi(V_{13}(c)) + \chi(V_{23}(c))\big). \tag{50}$$

It follows that for a closed foam F, its evaluation $\langle F \rangle \in R'_x$ is a homogeneous polynomial of degree $\deg(F)$.

Lemma 3.17 *For composable foams V and W,*

$$\deg(WV) = \deg(W) + \deg(V).$$

Proof This follows from [18, Proposition 3.1] and $|p(WV)| = |p(W)| + |p(V)|$. □

We now define state spaces for webs via universal construction and the evaluation formula (39). For a web Γ, let

$$\mathrm{Fr}(\Gamma)$$

denote the free R'_x–module generated by all anchored foams V from the empty web to Γ. Define a bilinear form

$$(-,-)\colon \mathrm{Fr}(\Gamma) \times \mathrm{Fr}(\Gamma) \to R'_x$$

by $(V, W) = \langle \omega(V) W \rangle$. This bilinear form is symmetric since $\langle F \rangle = \langle \omega(F) \rangle$ for any closed anchored foam F. Define the state space $\langle \Gamma \rangle := \mathrm{Fr}(\Gamma)/\ker((-,-))$ to be the

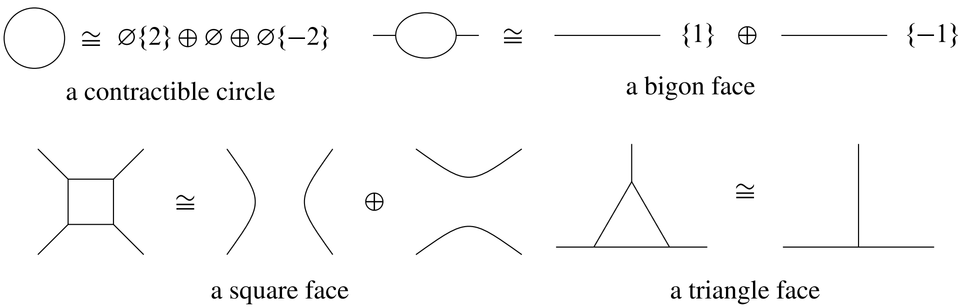


Figure 9: Direct sum decompositions from [18, Section 3.3], where the depicted regions do not contain the puncture.

quotient of $\mathrm{Fr}(\Gamma)$ by the kernel

$$\ker((-,-)) = \{x \in \mathrm{Fr}(\Gamma) \mid (x,y) = 0 \text{ for all } y \in \mathrm{Fr}(\Gamma)\}$$

of the bilinear form. Note that $(-,-)$ is degree-preserving, so its kernel and the state space $\langle \Gamma \rangle$ are graded R'_x–modules.

An anchored foam $V : \Gamma_0 \to \Gamma_1$ naturally induces a map

$$\langle V \rangle : \langle \Gamma_0 \rangle \to \langle \Gamma_1 \rangle$$

of degree $\deg(V)$, defined by sending the equivalence class of a basis element $U \in \mathrm{Fr}(\Gamma_0)$ to the class of the composition VU. This is functorial with respect to composition of anchored foams, $\langle WV \rangle = \langle W \rangle \langle V \rangle$ for composable anchored foams with boundary V and W.

Remark 3.18 For a web Γ and basis elements $V_1, V_2 \in \mathrm{Fr}(\Gamma)$, an admissible coloring of the closed foam $\omega(V_2)V_1$ induces a Tait coloring of Γ. Thus $\langle \Gamma \rangle = 0$ if Γ has no Tait colorings; see also [18, Proposition 3.16].

Proposition 3.19 *The local*[1] *isomorphisms in* [18, Propositions 3.12–3.15], *also shown in Figure 9, hold.*

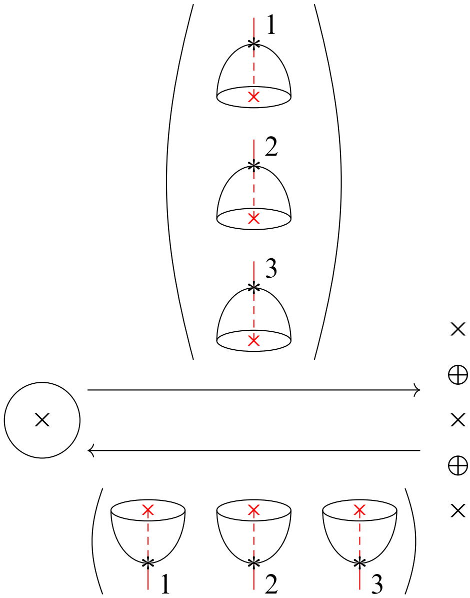
Proof Proposition 3.14 guarantees that the explicit isomorphisms defined in [18] hold in the anchored setting as well. □

[1] Here local means that the webs involved in the isomorphisms are identical outside of a disk which is disjoint from the puncture, and in this disk they are related as in the figures accompanying the statements of the propositions.

Proposition 3.20 *Let $\Gamma \subset \mathcal{P}$ be a web with a noncontractible circle C which bounds a disk in $\mathbb{R}^2 \setminus \Gamma$, and let $\Gamma' = \Gamma \setminus C$ be the web obtained by removing C. Then there is an isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle$$

given by the maps



Proof This follows from Example 3.6 and the relation (44). Note that there are no grading shifts in the three copies of $\langle \Gamma' \rangle$. □

It is an interesting and nontrivial problem to identify the state spaces $\langle \Gamma \rangle$. In the construction in [18] without the anchor line, state spaces can be simplified using the relations in [18, Section 3.3]; see Figure 9. In particular, bipartite webs always contain a contractible circle, bigon, or square, so the state space in the bipartite case is a free module of graded rank equal to the Kuperberg bracket [24], normalized as in [15]; see also [18, Propositions 3.17 and 4.15]. The simplest web which cannot be simplified using the relations in Figure 9 and for which the state space is unknown is the dodecahedral graph, as explored in [9; 17], and, on the gauge theory side, in [21; 22; 23].

One may also ask to identify state spaces in the presence of the anchor line and the modified evaluation considered in this paper. Propositions 3.19 and 3.20 give some ways to simplify state spaces. In general, we are not able to decompose the bigon, square, and triangle regions in Figure 9 if they contain the puncture. An extended evaluation, obtained by introducing additional types of intersection points of L with a foam, is discussed in Section 3.5. The following lemma addresses reducibility of smallest webs.

Lemma 3.21 *Let $\Gamma \subset \mathbb{R}^2$ be a connected, planar, trivalent graph with no edges connecting a vertex to itself.* [2]

(1) *If Γ is bipartite, then Γ has at least two bounded faces with at most four edges each.*

(2) *If at most one of the bounded faces of Γ has fewer than five edges, then Γ has at least eight vertices.*

Proof Let v, e, and f denote the number of vertices, edges, and faces (including the unbounded face) of Γ, respectively. Label the faces $1, \dots, f$, and for $1 \le i \le f$, let r_i denote the number of edges that form the boundary of the i^{th} face. We have

$$\sum_{i=1}^{f} r_i = 2e = 3v, \tag{51}$$

where the second equality holds since Γ is trivalent.

We first prove statement (1). Since Γ is bipartite, each r_i is even. Suppose for the sake of contradiction that at most one bounded face of Γ has four or fewer edges. Then (51) implies

$$\sum_{i=1}^{f} r_i > 6(f-2),$$

so $12 > 6f - 3v$. On the other hand, an Euler characteristic computation gives

$$12 = 6(f - e + v) = 6f - 3v,$$

which is a contradiction.

Let us now address statement (2). From (51) we obtain

$$3v \ge 5(f-2) + 4 = 5f - 6$$

since, by assumption, there are $f-2$ faces with at least five edges each, and the remaining two faces each have at least two edges. This together with an Euler characteristic computation gives $f \ge 6$, and it follows that $v \ge 8$. □

[2] A graph with such an edge has trivial state space; see Remark 3.18.

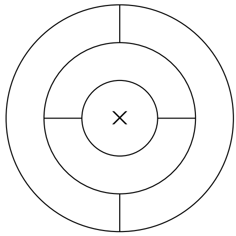


Figure 10: The simplest nonreducible web in the punctured plane.

Corollary 3.22 *Let $\Gamma \subset \mathcal{P}$ be a bipartite web. Then $\langle \Gamma \rangle$ is a free R'_x–module of rank equal to the number of Tait colorings of Γ.*

Proof By statement (1) of Lemma 3.21, any such web has either an innermost noncontractible circle or a region, not containing the puncture, which either bounds a closed loop, or is a bigon or square face. Thus state space can be reduced using Propositions 3.19 and 3.20. Since the resulting web remains bipartite we can continue the procedure until the state space is reduced to a direct sum of empty webs, each of which is free of rank 1. On the other hand, the number of Tait colorings can be computed using the same relations. □

It is natural to ask what is the simplest web for which the state space cannot be reduced using Propositions 3.19 and 3.20. By statement (2) of Lemma 3.21, such a web has at least eight vertices. The web shown in Figure 10 has precisely eight vertices and cannot be simplified using our local relations. We have not identified the state space of this web, but it can be approached via the 4–periodic (and, in general, nonexact) complex described in [18, Section 4.3]. It can be applied along any of the four edges of Figure 10 web near either the marked or the infinite point. One of the other three webs in the complex contains a loop and has trivial homology, but additional computations are needed to identify the state space due to nonexactness of the complex.

An annular graph $\Gamma \subset \mathcal{P}$ is called *reducible* if its state space can be reduced to a sum of those for the empty annular graph by recursively applying the relations in Figure 9 and relation in Proposition 3.20. It may make sense to also allow reductions to annular graphs without Tait colorings (including graphs with loops), since such graphs have trivial state spaces.

A reducible annular graph allows an identification of its state space with a suitable free graded R_x–module by recursively applying the above state sum decompositions. As a special case, we have the following decomposition formula for collections of simple closed curves in an annulus.

Proposition 3.23 *Let $\Gamma \subset \mathcal{P}$ consist of n contractible circles and m noncontractible circles. Then the state space $\langle \Gamma \rangle$ is a free R'_x–module of graded rank $3^m(q^2+1+q^{-2})^n$.*

In particular, for a reducible Γ, the graded rank of the free R'_x–module $\langle \Gamma \rangle$ can be computed recursively.

Anchored foams and state spaces carry an additional $(\mathbb{Z}/2\times\mathbb{Z}/2)$–grading as follows. Recall that u_1, u_2 and u_3 denote the nonzero elements of $\mathbb{Z}/2\times\mathbb{Z}/2$. For a foam V with (possibly empty) boundary, define

$$\operatorname{adeg}(V) = \sum_{p \in p(V)} u_{\ell(p)}.$$

We call adeg the *annular degree*. Clearly adeg is additive under disjoint union and composition.

The annular degree extends to a $(\mathbb{Z}/2\times\mathbb{Z}/2)$–grading on $\mathrm{Fr}(\Gamma)$, for a web $\Gamma \subset \mathcal{P}$, by setting the ground ring R'_x to be concentrated in annular degree zero. Proposition 3.10 implies that $\langle F \rangle = 0$ or $\operatorname{adeg}(\langle F \rangle) = 0$ for any closed foam F. It follows that $(-,-)$ preserves annular degree, so adeg descends to a $(\mathbb{Z}/2\times\mathbb{Z}/2)$–grading on the state space $\langle \Gamma \rangle$. The annular grading is the unoriented version of the grading on state spaces of annular oriented webs by the integral weight lattice of $\mathfrak{sl}_3$ — see Section 4.4 — even though the action of the latter is lacking on the equivariant annular state spaces.

In [18, Section 4] the authors consider localization of the unoriented SL(3) theory given by inverting the discriminant $\mathcal{D} = (x_1+x_2)(x_1+x_3)(x_2+x_3)$. This localization results in a significant simplification of the theory, making it separable, so to speak. In particular, a suitable 4–term sequence of web state spaces in [18, Section 4.3] is exact.

This localization easily extends to the annular case. The corresponding 4–term sequence is exact in the annular case as well. The ground ring for that theory is $R'_{\mathcal{D}} := \Bbbk[x_1, x_2, x_3, \mathcal{D}^{-1}]$, with $\Bbbk$ a characteristic two field. The analogue of [18, Proposition 4.13] holds: the localized state space of an annular web Γ is a projective $R'_{\mathcal{D}}$–module of rank equal to the number of Tait colorings of Γ. The latter is the number of edge colorings of Γ into three colors such that at each vertex the colors are distinct. Proof of this result in [18] easily adapts to the annular case, with the modification that the region around the marked point can be inductively simplified, if necessary, by reducing to the other three terms in the exact sequence, until it has a single edge (a loop around the marked point).

3.4 Remark on Lee's theory

Recall the function

$$f(x) = (x + x_1)(x + x_2)(x + x_3) = x^3 + E_1 x^2 + E_2 x + E_3 \tag{52}$$

(in characteristic 2 signs do not matter) with coefficients in the ring R_x and roots in $R'_x \supset R_x$. One can form the quotient ring $A := R'_x[x]/(f(x))$, naturally isomorphic to the homology of a contractible circle in our theory. Let

$$\mathcal{D} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = E_1 E_2 + E_3 \tag{53}$$

be the discriminant. Consider the localization

$$R'_{\mathcal{D}} := R'_x[\mathcal{D}^{-1}], \quad A_{\mathcal{D}} := R'_{\mathcal{D}} \otimes_{R'_x} A. \tag{54}$$

Introduce idempotents $e_1, e_2, e_3 \in A_{\mathcal{D}}$,

$$e_i := \frac{(x + x_j)(x + x_k)}{(x_i + x_j)(x_i + x_k)}, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{55}$$

We have

$$1 = e_1 + e_2 + e_3, \quad e_i e_j = \delta_{i,j} e_i. \tag{56}$$

These idempotents decompose the ring $A_{\mathcal{D}}$ into the direct product

$$A_{\mathcal{D}} \cong R'_{\mathcal{D}} e_1 \times R'_{\mathcal{D}} e_2 \times R'_{\mathcal{D}} e_3 \cong R'_{\mathcal{D}} \times R'_{\mathcal{D}} \times R'_{\mathcal{D}}. \tag{57}$$

An idempotent e_i can be visualized as floating on a facet of a foam F, in the localized theory. These idempotents allow us to decompose an evaluation of a foam F with n facets into 3^n terms by summing over all ways to place each of these three idempotents onto facets of F. Each term is straightforward to compute and equals zero unless the idempotents define a Tait coloring (an admissible coloring) of F.

Idempotent e_i bears a close relation to an anchor point labeled i. The anchor point p on a facet f contributes the term $\sqrt{f'(x_{c(f)})} = \sqrt{(x_{c(f)} + x_j)(x_{c(f)} + x_k)}$ to the evaluation $\langle F, c \rangle$. The square of this term is either 0 (if $i \neq c(f)$) or the denominator of e_i, if $i = c(f)$, for any coloring c of F.

Comparing e_i and an anchor point p labeled i, when coloring c associates color $c(f) \neq i$ to the facet f carrying e_i or p, both evaluations are zero. When $c(f) = i$, the idempotented dot e_i contributes 1 to the evaluation, while the anchor point contributes $\sqrt{f'(x_i)}$. The denominator of e_i is $f'(x_i)$.

One can try to unify e_i and anchor points p by considering anchor lines and circles L in $\mathbb{R}^3$ possibly intersecting a foam F. Intersection points (anchor points) carry labels $i \in \{1,2,3\}$ and a circle anchor points labeled i is the idempotent e_i. Then a "small" circle intersecting a facet f at two points, both labeled i, can also be converted into e_i. Notice that once e_i are allowed, integrality is lost and an evaluation of such a foam may contain denominators which are products of $x_i + x_j$.

For a different generalization, instead of a single line $L \subset \mathbb{R}^3$ consider a 1–manifold L properly embedded in $\mathbb{R}^3$, say a finite union of lines and circles, possibly knotted. All anchor points (intersection points with L) on a foam F carry labels, with the usual contribution to the evaluation, as in formula (36). The integrality Theorem 4.15 still holds for such generalized evaluation. In particular, given k points on a plane, one can define various state spaces for webs Γ embedded in the plane and disjoint from these marked points. Also note that for $k \geq 2$ punctures, bipartite graphs are in general not reducible, which makes it harder to understand corresponding state spaces in the oriented SL(3) case.

Remark 3.24 A handle next to but disjoint from an anchor line can be written as a sum of three lower genus terms intersecting the line — see (46) — which follows from the formula

$$\begin{aligned} m \circ \Delta(1) &= (x_1+x_2)(x_1+x_3) + (x_1+x_2)(x_2+x_3) + (x_1+x_3)(x_2+x_3) \\ &= f'(x_1) + f'(x_2) + f'(x_3). \end{aligned}$$

3.5 Unlabeled anchor points and bigon decomposition

Direct sum decompositions for webs Γ containing a bigon, triangle, or square face which do not contain the puncture are given in Proposition 3.19. On the other hand, Proposition 3.20 describes how to simplify a web containing an innermost noncontractible circle. In order to have direct sum decompositions for more general regions containing the puncture, we introduce additional types of intersections of the anchor line L with a foam and modify the evaluation $\langle - \rangle$.

In addition to anchor points, which carry labels in $\{1,2,3\}$ as in Definition 3.2, we allow finitely many transverse intersections of L with a foam F away from the singular

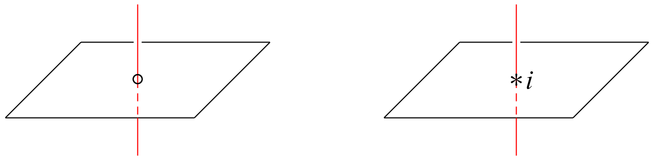


Figure 11: Left, a type 1 anchor point marked $\circ$ and carrying no label. Right, a type 2 anchor point marked $*$ with label $i \in \{1,2,3\}$.

graph $s(F)$, and we do not require labels. We will call the usual (labeled) anchor points *type* 2, and the new (unlabeled) anchor points *type* 1. In the figures, we denote type 2 anchor points by an asterisk $*$ as usual, along with a label in $\{1,2,3\}$, and type 1 anchor points will be indicated by a small unshaded circle $\circ$. Figure 11 illustrates the convention. Let $p_1(F)$ and $p_2(F)$ denote the set of type 1 and type 2 anchor points, respectively (using the notation in Section 3.1, $p(F) = p_2(F)$). The definition of admissible coloring remains the same.

We modify the evaluation in the presence of type 1 points as follows. Let $c \in \mathrm{adm}(F)$. For $p \in p_1(F)$ lying on some facet $f \in f(F)$, let $c(p) := c(f)$ denote the coloring of the facet on which p lies. Also recall that for $i \in \{1,2,3\}$, we write i', i'' and j, k to denote the two complementary elements, so $\{1,2,3\} = \{i,j,k\} = \{i,i',i''\}$.

Define

$$\widetilde{Q}_\circ(F,c) = \prod_{p \in p_1(F)} \sqrt{x_{c(p)'} + x_{c(p)''}}, \tag{58}$$

$$P_\circ(F,c) = P(F,c) \cdot \widetilde{Q}_\circ(F,c), \tag{59}$$

$$\langle F,c\rangle_\circ = \frac{P_\circ(F,c)}{Q(F,c)}, \tag{60}$$

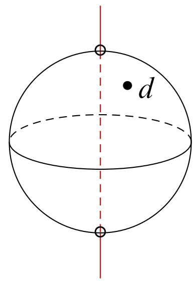
$$\langle F\rangle_\circ = \sum_{c \in \mathrm{adm}(F)} \langle F,c\rangle_\circ, \tag{61}$$

where $P(F,c)$ and $Q(F,c)$ are as defined in (36) and (37). In other words, a type 1 point p on an i–colored facet contributes a factor of $\sqrt{x_j + x_k}$ to the evaluation $\langle F,c\rangle_\circ$.

Remark 3.25 Type 1 intersection points are related to the triangle decoration from [18, Section 4.1]. Precisely, the contribution of a type 1 point p to the square root in (58) equals the inverse of placing a triangle decoration on the facet where p lies. See relation (62), as well as Remark 3.4 for a related discussion.

Note that a type 1 intersection point contributes half the degree of a type 2 point to the degree of the evaluation and, thus, to the degree of a cobordism represented by a foam with boundary.

Example 3.26 Consider a 2–sphere F carrying d dots and intersecting L in two type 1 anchor points,

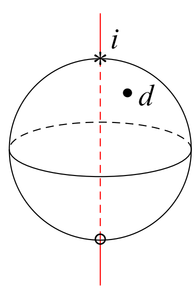


For $1 \le i \le 3$, let $c_i \in \mathrm{adm}(F)$ color F by i. Then

$$\begin{aligned}\langle F, c_i\rangle_\circ &= \frac{x_i^d(x_j+x_k)}{(x_i+x_j)(x_i+x_k)},\\ \langle F\rangle_\circ &= \langle F,c_1\rangle_\circ + \langle F,c_2\rangle_\circ + \langle F,c_3\rangle_\circ\\ &= \frac{x_1^d(x_2+x_3)^2 + x_2^d(x_1+x_3)^2 + x_3^d(x_1+x_2)^2}{(x_1+x_2)(x_1+x_3)(x_2+x_3)}\\ &= \frac{x_1^d(x_2^2+x_3^2) + x_2^d(x_1^2+x_3^2) + x_3^d(x_1^2+x_2^2)}{(x_1+x_2)(x_1+x_3)(x_2+x_3)}.\end{aligned}$$

Thus, $\langle F\rangle_\circ = 0$ if $d = 0, 2$, and $\langle F\rangle_\circ = 1$ if $d = 1$. For $d \ge 3$, the last expression above equals the ratio of the antisymmetrizer with exponent $(d, 2, 0)$ and antisymmetrizer with exponent $(2, 1, 0)$ (up to adding signs, which does not matter in characteristic 2). Thus $\langle F\rangle_\circ$ equals the Schur function $s_\lambda(x_1, x_2, x_3)$ for the partition $\lambda = (d-2, 1, 0)$ when $d \ge 3$.

Example 3.27 Consider a 2–sphere F carrying d dots and intersecting L in one type 1 anchor point and one type 2 anchor point,



Then F has one admissible coloring, and

$$\langle F\rangle_\circ = \frac{x_i^d\sqrt{(x_i+x_j)(x_i+x_k)(x_j+x_k)}}{(x_i+x_j)(x_i+x_k)} = \frac{x_i^d\sqrt{x_j+x_k}}{\sqrt{(x_i+x_j)(x_i+x_k)}}.$$

From Example 3.27 we see that the evaluation $\langle F\rangle_\circ$ in general has denominators and square roots, so we can only conclude that

$$\langle F\rangle_\circ \in \widetilde{R}_\circ := \Bbbk[x_1,x_2,x_3,(x_1+x_2)^{-1/2},(x_2+x_3)^{-1/2},(x_1+x_3)^{-1/2}].$$

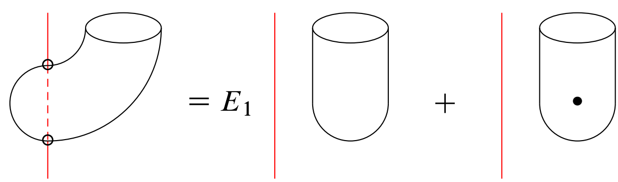
Note that $\widetilde{R}_\circ$ is a subring of $\widetilde{R}''_x$; see Section 3.1 and diagram (34).

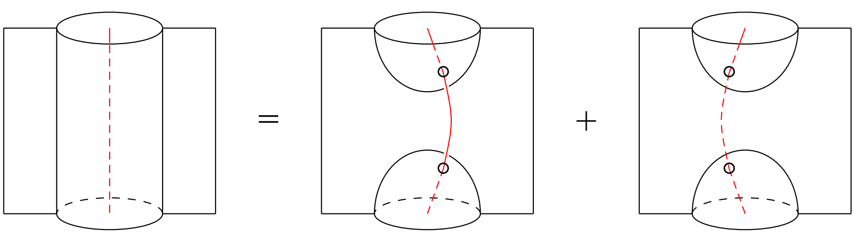
We use $\widetilde{R}_\circ$ as the ground ring of the theory. Evaluations of closed anchored foams F with two types of anchor points belong to this ring. We define the state space $\langle\Gamma\rangle_\circ$ of a trivalent graph $\Gamma\subset\mathcal{P}$ using this evaluation and following the general recipe of Section 3.3. The state space is a graded $\widetilde{R}_\circ$–module, but, due to the presence of invertible elements $(x_i+x_j)^{1/2}$ of degree 1, grading carries little information, and for many purposes one can downsize and consider the degree zero part $\langle\Gamma\rangle_\circ^0$ of the state space, which is a module over the degree 0 subring $\widetilde{R}_\circ^0$ of $\widetilde{R}_\circ$.

This theory is functorial and foams with top and bottom boundary and anchor points of those two different types induce maps between the corresponding state spaces. Various direct sum decompositions that hold for the unoriented SL(3) theory $\langle -\rangle$ hold for this theory as well.

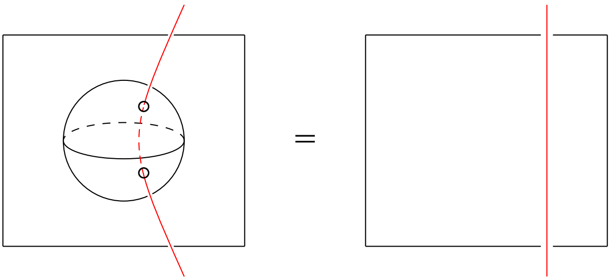
We also have local relations involving type 1 intersection points.

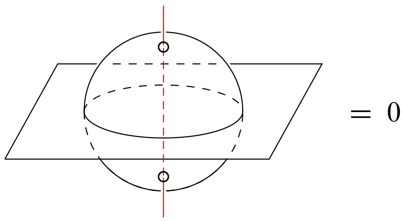
Lemma 3.28 *The following local relations*[3] *hold for the theory $\langle -\rangle_\circ$:*

(62) 

(63) 

[3]To clarify relation (63): the first term on the right-hand side of the equality has a type 1 anchor point on each of two front-facing half-bubbles, while the second term has a type 1 anchor point on each of the two back-facing half-bubbles.

(64) 

(65) 

Proof Relation (62) is straightforward and left to the reader. Let us verify relation (63). Denote by F the foam on the left-hand side of the equality, and denote by F^1 and F^2 the two foams on the right-hand side. There is a natural identification $\mathrm{adm}(F^1) = \mathrm{adm}(F^2)$.

Let $c \in \mathrm{adm}(F^1)$ be a coloring in which the front two half-bubble facets are differently colored, say the top front half-bubble is colored j, the bottom front half-bubble is colored k, and the remaining "big" facet is colored i. Continue to denote by $c \in \mathrm{adm}(F^2)$ the corresponding coloring of F^2. The top type 1 intersection point of F^1 contributes $\sqrt{x_i + x_k}$ to $\langle F^1, c\rangle$ and the bottom type 1 intersection point of F^1 contributes $\sqrt{x_i + x_j}$, while the contributions of these points to $\langle F^2, c\rangle$ are reversed. Thus in characteristic two we have

$$\langle F^1, c\rangle + \langle F^2, c\rangle = 0.$$

Next, the admissible colorings of F are in natural bijection with the admissible colorings of F^1 (and of F^2) in which the front half-bubbles of F^1 are colored the same. Let $c \in \mathrm{adm}(F)$, and let $c' \in \mathrm{adm}(F^1) \cong \mathrm{adm}(F^2)$ denote the corresponding colorings. Suppose that c' colors the front half-bubbles of F^1 by j, the "big" facet by i, and the back half-bubbles by k. Then

$$\langle F^1, c'\rangle = \frac{x_i + x_k}{x_j + x_k}\langle F, c\rangle \quad \text{and} \quad \langle F^2, c'\rangle = \frac{x_i + x_j}{x_j + x_k}\langle F, c\rangle,$$

from which we obtain

$$\langle F, c\rangle = \langle F^1, c'\rangle + \langle F^2, c'\rangle,$$

which completes the proof of relation (63).

We now address the relation (64). Let G denote the foam on the left-hand side of the equation, and let G' denote the foam on the right-hand side. Let $c \in \mathrm{adm}(G)$, and assume c colors the "big" facet of G by i, the front bubble by j, and the back bubble by k. Let $c' \in \mathrm{adm}(G)$ denote the coloring which is identical to c except the front and back bubbles are colored by k and j, respectively. Let $c'' \in \mathrm{adm}(G')$ denote the coloring of G' in which the depicted facet is colored i, and the remaining facets are colored according to c (equivalently, c'). We claim that

$$\langle G, c\rangle + \langle G, c'\rangle = \langle G', c''\rangle,$$

which completes the proof. To verify the above equality, observe that

$$\langle G, c\rangle = \frac{x_i + x_k}{x_j + x_k}\langle G', c''\rangle \quad \text{and} \quad \langle G, c'\rangle = \frac{x_i + x_j}{x_j + x_k}\langle G', c''\rangle.$$

The proof of relation (65) is similar and left to the reader. □

The previous lemma allows us to simplify the state space $\langle \Gamma \rangle_\circ$ assigned to a web $\Gamma \subset \mathcal{P}$ with a bigon region containing the puncture.

Proposition 3.29 *The two maps shown in Figure 12 are mutually inverse isomorphisms between state spaces of graphs in the theory* $\langle - \rangle_\circ$.

Proof This follows from the relations in Lemma 3.28. □

4 Oriented SL(3) anchored homology

In this section we recall oriented SL(3) foams, which were introduced in [15] in the context of $\mathfrak{sl}(3)$ link homology. An equivariant analogue was defined in [28]; see also [10; 26; 27; 29; 33] for various aspects of SL(3) foams and link homology. In Section 4.1 we define an evaluation of oriented SL(3) foams via colorings in the style of Robert and Wagner [34] and show in Theorem 4.26 that our evaluation agrees with that of [28]. In Section 4.2 we deform the evaluation in the presence of the anchor line L. In Theorem 4.15 we show that our evaluation is always a polynomial.

To avoid introducing new notation, in this section we will reuse the notation for various rings from Section 3:

- $R'_x = \mathbb{Z}[x_1, x_2, x_3]$ is the ring of polynomials in three variables.
- $R_x = \mathbb{Z}[E_1, E_2, E_3]$ is the subring of R'_x that consists of symmetric polynomials in x_1, x_2 and x_3, with generators E_i being the elementary symmetric polynomials

$$E_1 = x_1 + x_2 + x_3, \quad E_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad E_3 = x_1x_2x_3.$$

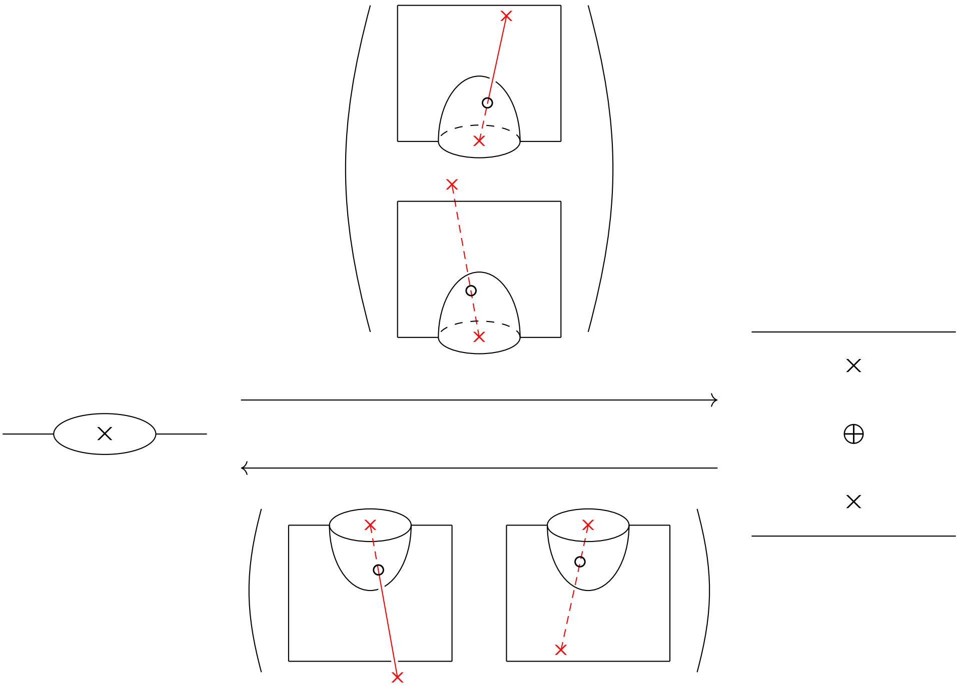


Figure 12: Isomorphisms which simplify a bigon region containing the puncture, for the theory $\langle - \rangle_\circ$. In the top map, the top foam has a type 1 point on the front half-bubble, and the bottom foam has a type 1 point on the back half-bubble. In the bottom map, the first foam has a type 1 point on the front half-bubble, and the second foam has a type 1 point on the back half-bubble.

- $R''_x = R'_x[(x_1 - x_2)^{-1}, (x_2 - x_3)^{-1}, (x_1 - x_3)^{-1}]$ is a localization of R'_x given by inverting $x_i - x_j$, for $1 \le i < j \le 3$.
- $\widetilde{R}'_x = R'_x[\sqrt{x_1 - x_3}, \sqrt{x_2 - x_3}, \sqrt{x_1 - x_3}]$ is the extension of R'_x obtained by introducing square roots of $\sqrt{x_i - x_j}$, for $1 \le i < j \le 3$.
- $\widetilde{R}''_x = \widetilde{R}'_x[(x_1 - x_2)^{-1}, (x_2 - x_3)^{-1}, (x_1 - x_3)^{-1}]$ is a suitable localization of the ring $\widetilde{R}'_x$.

All five of these rings are graded by setting $\deg(x_1) = \deg(x_2) = \deg(x_3) = 2$. Inclusions of the above rings are summarized in the following diagram:

(66)
$$\begin{array}{ccccc} & & \widetilde{R}'_x & \subset & \widetilde{R}''_x \\ & & \cup & & \cup \\ R_x & \subset & R'_x & \subset & R''_x \end{array}$$

4.1 Oriented SL(3) foams and their evaluations

We begin by recalling the definition of oriented SL(3) foams from [15, Section 3.2].

Definition 4.1 A (closed) *oriented* SL(3) *prefoam* F consists of the following data:

- An orientable surface F' with connected components $F_1, \dots, F_k$ and a partition of the boundary components of F' into triples. The underlying CW structure of F is obtained by identifying the three circles in each triple. The image of the three circles in each triple becomes a single circle in F, called a *singular circle*. The image of the surfaces F_i are called *facets*. Three facets meet at each singular circle.
- For each singular circle Z, we fix a cyclic ordering of the three facets meeting at Z. There are two possible choices of cyclic ordering for each Z.
- Each facet may carry some number of dots, which are allowed to float freely along the facet but cannot cross singular circles.

A *oriented* SL(3) *foam* is a prefoam as above equipped with an embedding into $\mathbb{R}^3$, along with an orientation on each facet such that any two of the three facets meeting at each singular circle are incompatibly oriented, as shown in Figure 13, left. Each singular circle Z acquires an induced orientation; see Figure 13, middle. This induced orientation on Z specifies a cyclic ordering of the three facets meeting at Z by following the left-hand rule — Figure 13, right — and we require this to match the cyclic ordering specified by the prefoam F.

Note that unlike unoriented foams considered in Section 3, the oriented SL(3) prefoams in the present section do not contain singular vertices. When there is no risk of confusion

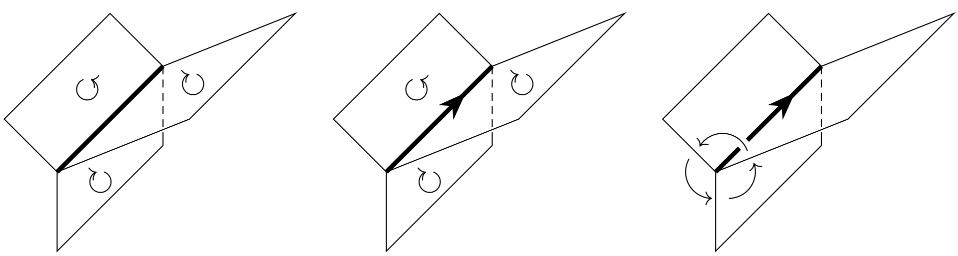


Figure 13: Left: orientations of three facets meeting at a singular circle. Middle: the induced orientation of a singular circle. Right: the induced cyclic ordering.

between the foams introduced in the Definition 4.1 and those of Section 3, in this section we will simply write (pre)foam rather than oriented SL(3) (pre)foam.

For a prefoam F, let $\Theta(F)$ denote the set of its singular circles and $\theta(F) = |\Theta(F)|$ the number of singular circles. Each $Z \in \Theta(F)$ has a neighborhood homeomorphic to the product of a circle $\mathbb{S}^1$ and a tripod. Let $f(F)$ denote the set of facets of F. We use the definitions of preadmissible and admissible colorings of prefoams and foams from Section 3 in the present situation. For a prefoam F, $\mathrm{adm}(F)$ denotes the set of admissible colorings of F. Note that if F is a foam, every preadmissible coloring is also admissible.

Fix a prefoam F and an admissible coloring $c \in \mathrm{adm}(F)$. For $1 \le i \neq j \le 3$, bicolored surfaces $F_{ij}(c)$ consist of all facets colored i or j; each $F_{ij}(c)$ is a closed, orientable surface. For $1 \le i \le 3$, let $F_i(c)$ be the surface consisting of all facets of F which are colored i by c; the surface $F_i(c)$ is orientable and has $\theta(F)$ boundary components. Denote by $\overline{F}_i(c)$ the closed surface obtained by gluing disks along boundary components of $F_i(c)$. We have

$$\begin{aligned} \chi(\overline{F}_i(c)) &= \chi(F_i(c)) + \theta(F), && 1 \le i \le 3, \\ \chi(F_{ij}(c)) &= \chi(F_i(c)) + \chi(F_j(c)), && 1 \le i < j \le 3. \end{aligned} \tag{67}$$

The three facets meeting at each singular circle are colored by i, j and k, whereas before we used i, j and k to denote the three elements of $\{1,2,3\}$. We now define quantities $\theta^{\pm}(c)$ and $\theta^{\pm}_{ij}(c)$ associated with the set of singular circles $\Theta(F)$ and the admissible coloring c.

Definition 4.2 Let F be a prefoam with admissible coloring c, and let $1 \le i < j \le 3$. A singular circle $Z \in \Theta(F)$ is *positive* with respect to (i, j) if the cyclic ordering of the colors of the three facets meeting at Z is $(i\ k\ j)$. If F is a foam, then an equivalent formulation is as follows: when looking along the orientation of Z with the facet colored k, as in Figure 14, the i–colored facet is to the left of the j–colored facet. Otherwise, we say Z is negative with respect to (i, j). See Figure 14, left, for a pictorial definition. Let $\theta^+_{ij}(c)$ (resp. $\theta^-_{ij}(c)$) denote the number of positive (resp. negative) circles with respect to (i, j). We have

$$\theta^+_{ij}(F,c) + \theta^-_{ij}(F,c) = \theta(F).$$

We say that a singular circle Z is *positive* with respect to c if the colors of the three facets meeting at Z are $(1\ 2\ 3)$ in the cyclic ordering, and otherwise Z is negative;

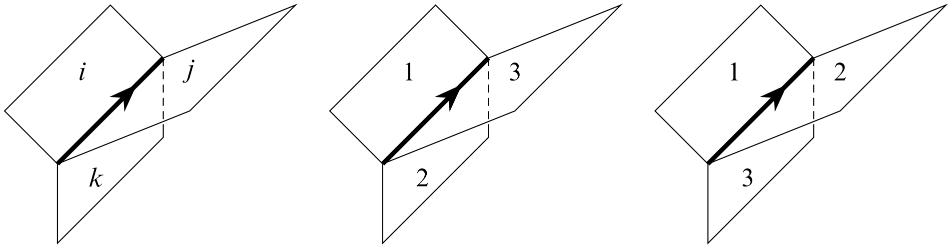


Figure 14: Left: a positive (i,j)–circle, where $i<j$. Middle: a positive singular circle. Right: a negative singular circle.

see Figure 14, middle and right. Let $\theta^+(F,c)$ (resp. $\theta^-(F,c)$) denote the number of positive (resp. negative) circles in F with respect to c. We have

$$\theta^+(F,c)+\theta^-(F,c)=\theta(F). \tag{68}$$

We will often omit F from the notation and simply write θ, $\theta^{\pm}_{ij}(c)$, and $\theta^{\pm}(c)$.

We now define the evaluations $\langle F,c\rangle$ and $\langle F\rangle$. For a prefoam F, $c\in\mathrm{adm}(F)$, and $1\le i\le 3$, let $d_i(c)$ denote the number of dots on facets colored i. Define

$$P(F,c)=\prod_{i=1}^{3}x_i^{d_i(c)}, \tag{69}$$

$$Q(F,c)=\prod_{1\le i<j\le 3}(x_i-x_j)^{\chi(F_{ij}(c))/2}, \tag{70}$$

$$s(F,c)=\sum_{i=1}^{3}i\chi(\bar F_i(c))/2+\sum_{1\le i<j\le 3}\theta^+_{ij}(c). \tag{71}$$

Set

$$\langle F,c\rangle=(-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c)}, \tag{72}$$

$$\langle F\rangle=\sum_{c\in\mathrm{adm}(F)}\langle F,c\rangle. \tag{73}$$

A priori, the evaluations $\langle F,c\rangle$ and $\langle F\rangle$ lie in the ring R''_x; see diagram (66).

In what follows, we use the symbol $\equiv$ to mean equality modulo 2. Note that

$$\sum_{i=1}^{3}i\chi(\bar F_i(c))/2\equiv\frac{\chi(\bar F_1(c))+\chi(\bar F_3(c))}{2}, \tag{74}$$

since $\chi(\bar{F}_2(c))$ is even. Moreover, from (67) we obtain

$$\sum_{i=1}^{3} i\chi(\bar{F}_i(c))/2 \equiv \theta + \sum_{i=1}^{3} i\chi(F_i(c))/2. \tag{75}$$

Lemma 4.3 *For a prefoam F and $c \in \mathrm{adm}(F)$,*

$$\sum_{1\le i<j\le 3} \theta_{ij}^{+}(c) \equiv \theta^{+}(c).$$

It follows that

$$s(F,c) \equiv \sum_{i=1}^{3} i\chi(F_i(c))/2 + \theta^{-}(c). \tag{76}$$

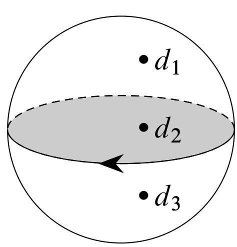
Proof Let $Z \in \Theta(F)$. Observe that if Z is positive with respect to c, then it contributes only to $\theta_{13}^{+}(c)$. Likewise, if Z is negative then it contributes to both $\theta_{12}^{+}(c)$ and $\theta_{23}^{+}(c)$ but not to $\theta_{13}^{+}(c)$, which verifies the first equality. The second equality follows from (75) and (68). □

Example 4.4 Let F be a 2–sphere $\mathbb{S}^2$ with d dots. For $1 \le i \le 3$, let $c_i \in \mathrm{adm}(F)$ color F by i. We have

$$\begin{aligned}
\langle F\rangle &= \langle F,c_1\rangle + \langle F,c_2\rangle + \langle F,c_3\rangle \\
&= -\frac{x_1^d}{(x_1-x_2)(x_1-x_3)} + \frac{x_2^d}{(x_1-x_2)(x_2-x_3)} - \frac{x_3^d}{(x_1-x_3)(x_2-x_3)} \\
&= \frac{-x_1^d(x_2-x_3)+x_2^d(x_1-x_3)-x_3^d(x_1-x_2)}{(x_1-x_2)(x_2-x_3)(x_1-x_3)} \\
&= -s_{(d-2,0,0)}(x_1,x_2,x_3) = -h_{d-2}(x_1,x_2,x_3) = -\sum_{i+j+k=d-2} x_1^i x_2^j x_3^k,
\end{aligned}$$

where $s_{(d-2,0,0)}(x_1,x_2,x_3)$ is the Schur function of the partition $(d-2,0,0)$, and $h_{d-2}(x_1,x_2,x_3)$ is the complete symmetric function of degree $d-2$. In particular $\langle F\rangle = 0$ if $d=0$ or $d=1$, and $\langle F\rangle = -1$ if $d=2$.

Example 4.5 Let F be the theta foam



Given any $c \in \operatorname{adm}(F)$, each capped-off surface $\bar{F}_i(c)$ and each bicolored surface $F_{ij}(c)$ is a 2–sphere. In particular,

$$s(F,c) \equiv \theta^+(c).$$

For $\sigma \in S_3$, let $c(\sigma) \in \operatorname{adm}(F)$ denote the coloring which colors the top facet by $\sigma(1)$, the middle facet by $\sigma(2)$, and the bottom facet by $\sigma(3)$. We have

$$\langle F \rangle = \sum_{\sigma \in S_3} \langle F, c(\sigma) \rangle = \frac{\sum_{\sigma \in S_3} (-1)^{\theta^+(c(\sigma))} x_{\sigma(1)}^{d_1} x_{\sigma(2)}^{d_2} x_{\sigma(3)}^{d_3}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)},$$

and moreover

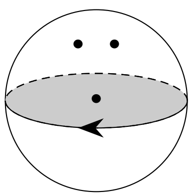
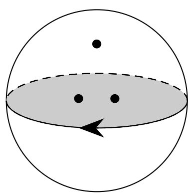
$$\theta^+(c(\sigma)) \equiv |\sigma|,$$

where $|\sigma|$ is the length of σ.

Therefore if $d_1 \geq d_2 \geq d_3$,

$$\langle F \rangle = s_{(d_1-2, d_2-1, d_3)}(x_1, x_2, x_3),$$

the Schur function with partition $(d_1 - 2, d_2 - 1, d_3)$. In particular, $\langle F \rangle = 0$ if d_1, d_2 and d_3 are not distinct. If d_1, d_2 and d_3 are distinct and $d_1 + d_2 + d_3 \leq 3$, then up to cyclic permutation there are two choices:

 $= 1,$  $= -1.$

The symmetric group S_3 naturally acts on $\operatorname{adm}(F)$ and on the five rings in the diagram (66). The following lemma is analogous to [34, Lemma 2.16].

Lemma 4.6 *Let F be a prefoam, $c \in \operatorname{adm}(F)$, and $\sigma \in S_3$. Then*

$$\sigma(\langle F, c \rangle) = \langle F, \sigma(c) \rangle.$$

Proof We may assume that σ is a transposition $(i\ i+1)$ for $i = 1, 2$. We have

$$\sigma(P(F,c)) = P(F, \sigma(c)), \quad \sigma(Q(F,c)) = (-1)^{\chi(F_{i(i+1)}(c))/2} Q(F, \sigma(c)).$$

Let $k \in \{1,2,3\} \setminus \{i, i+1\}$. Note that a singular circle Z is positive with respect to c if and only if Z is negative with respect to $\sigma(c)$, so

$$\theta^+(c) + \theta^+(\sigma(c)) = \theta = \theta^-(c) + \theta^-(\sigma(c)).$$

Moreover,

$$F_i(c) = F_{i+1}(\sigma(c)), \quad F_{i+1}(c) = F_i(\sigma(c)), \quad F_k(c) = F_k(\sigma(c)).$$

Therefore

$$\begin{aligned} s(F,c) - s(F,\sigma(c)) &= \frac{\chi(F_{i+1}(c)) - \chi(F_i(c))}{2} + \theta^-(c) - \theta^-(\sigma(c)) \\ &\equiv \frac{\chi(F_{i+1}(c)) - \chi(F_i(c))}{2} + \theta \\ &\equiv \frac{\chi(F_{i+1}(c)) + \chi(F_i(c))}{2} \\ &\equiv \frac{\chi(F_{i(i+1)}(c))}{2}. \end{aligned}$$

□

Corollary 4.7 *The evaluation $\langle F \rangle$ is a symmetric rational function.*

Later we will prove that $\langle F \rangle$ is in fact a polynomial; see Corollary 4.16.

Lemma 4.8 *Let $i \in \{1,2\}$, let F be a prefoam, and let $c \in \mathrm{adm}(F)$ be an admissible coloring. Suppose $c' \in \mathrm{adm}(F)$ is obtained from c by a $(1,2)$–Kempe move along a surface $\Sigma \subset F_{12}(c)$. Then*

$$s(F,c) \equiv s(F,c') + \tfrac{1}{2}\chi(\Sigma).$$

Proof Note that this is analogous to [34, Lemma 2.19]. Letting $\theta(\Sigma)$ denote the number of seam circles on Σ, we have

$$\theta^-(c) + \theta^-(c') \equiv \theta(\Sigma) \equiv \chi(F_1(c) \cap \Sigma).$$

Note also that

$$\begin{aligned} \chi(F_1(c)) - \chi(F_1(c')) &= \chi(F_1(c) \cap \Sigma) - \chi(F_2(c) \cap \Sigma), \\ \chi(F_2(c)) - \chi(F_2(c')) &= \chi(F_2(c) \cap \Sigma) - \chi(F_1(c) \cap \Sigma). \end{aligned}$$

We compute

$$\begin{aligned} s(F,c) - s(F,c') &\equiv \frac{\chi(F_1(c)) - \chi(F_1(c'))}{2} + \frac{2\big(\chi(F_2(c)) - \chi(F_2(c'))\big)}{2} + \theta(\Sigma) \\ &\equiv \frac{\chi(F_2(c) \cap \Sigma) - \chi(F_1(c) \cap \Sigma)}{2} + \chi(F_1(c) \cap \Sigma) \\ &\equiv \tfrac{1}{2}\chi(\Sigma). \end{aligned}$$

□

4.2 Oriented anchored SL(3) foams and their evaluations

Definition 4.9 An *oriented anchored* SL(3) *foam* F is an oriented foam $F' \subset \mathbb{R}^3$ that may intersect the anchor line L at finitely many points away from the singular circles of F', so that each intersection point belongs to some facet of F', and moreover these intersections are required to be transverse. Denote by $p(F) = F' \cap L$ the set of intersection points (anchor points) of F. The anchor points carry labels in $\{1,2,3\}$; that is, F comes equipped with a fixed map

$$\ell\colon p(F) \to \{1,2,3\}.$$

Fix an anchored foam F and an admissible coloring c of the underlying foam F'. Each anchor point $p \in p(F)$ lying on a facet f inherits a color $c(p) := c(f)$. As in Section 3, we say that c is an admissible coloring of the anchored foam F if for each $p \in p(F)$, the color of p equals the label of p, that is, $c(p) = \ell(p)$. Denote by $\operatorname{adm}(F)$ the set of admissible colorings of F.

For $i \in \{1,2,3\}$, let i' and i'' denote the complementary elements, so that $\{i,i',i''\} = \{1,2,3\}$. Define the evaluations

$$\langle F,c\rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)} \Biggl(\prod_{p\in p(F)} (-1)^{c(p)-1}(x_{c(p)} - x_{\ell(p)'})(x_{c(p)} - x_{\ell(p)''})\Biggr)^{1/2}, \tag{77}$$

$$\langle F\rangle = \sum_{c\in\operatorname{adm}(F)} \langle F,c\rangle, \tag{78}$$

where $P(F,c)$, $Q(F,c)$ and $s(F,c)$ are as defined in (69), (70) and (71), respectively.

Let us explain the square root in (77). We have $c(p) = \ell(p)$ for every anchor point $p \in p(F)$. If p is labeled i, then it contributes

$$(-1)^{i-1}(x_i - x_j)(x_i - x_k)$$

to the product under the square root. More concretely, the product of the two terms under the square root, for a fixed anchor point p, is equal to

$$\begin{aligned} &(x_1 - x_2)(x_1 - x_3) \quad \text{if } c(p) = 1,\\ &(x_1 - x_2)(x_2 - x_3) \quad \text{if } c(p) = 2,\\ &(x_1 - x_3)(x_2 - x_3) \quad \text{if } c(p) = 3. \end{aligned}$$

Let $\operatorname{an}(i)$ be the number of anchor points p with $c(p) = i$. Then for $1 \le i < j \le 3$ the sum $\operatorname{an}(i) + \operatorname{an}(j)$ is even, which follows from Proposition 3.10.

We define the square root as the product

$$\widetilde{Q}(F,c) := \prod_{1\le i<j\le 3} (x_i - x_j)^{(\operatorname{an}(i)+\operatorname{an}(j))/2} \tag{79}$$

and rewrite formula (77) as

$$\begin{aligned}\langle F, c\rangle &:= (-1)^{s(F,c)} \frac{P(F,c)\widetilde{Q}(F,c)}{Q(F,c)} \\ &= (-1)^{s(F,c)} P(F,c) \prod_{1\le i<j\le 3} (x_i - x_j)^{(\mathrm{an}(i)+\mathrm{an}(j)-\chi(F_{ij}(c)))/2}.\end{aligned}$$

Note that $\widetilde{Q}(F,c)$ depends only on the labels of anchor points and not on the coloring c, as long as c respects labels of anchor points (otherwise, the evaluation is 0). Consequently, it can also be denoted by $\widetilde{Q}(F)$. Alternatively, it may be useful to allow more general colorings c, with $\widetilde{Q}(F,c) = 0$ for c not compatible with the labels of anchor points.

Recall diagram (66) and the surrounding discussion for notations of various rings. The above formula implies the following proposition.

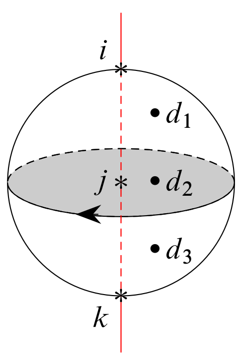
Proposition 4.10 *The evaluation $\langle F, c\rangle$ is an element of R''_x.*

Remark 4.11 As discussed in Remark 3.5, if c is an admissible coloring of the underlying foam F' but not of the anchored foam F, then the evaluation (77) is still well-defined and equal to zero. Even if we don't restrict the notion of admissible colorings of an anchored foam to those which color anchor points according to their labels, additional terms in the evaluation will each be 0, not contributing anything.

Example 4.12 Let F be a 2–sphere $\mathbb{S}^2$ carrying d dots and intersecting L twice. Then $\langle F\rangle = 0$ unless both anchor points are labeled by $i \in \{1,2,3\}$. In this case, there is one admissible coloring c which colors F by i. We see that $s(F,c) \equiv i$, and the evaluation is

$$\langle F\rangle = (-1)^i x_i^d.$$

Example 4.13 Consider the theta foam F whose facets each intersect L exactly once,



There is one admissible coloring c, and we have

$$\langle F\rangle = \langle F,c\rangle = \begin{cases} x_i^{d_1}x_j^{d_2}x_k^{d_3} & \text{if } (i,j,k)=(1,3,2) \text{ or a cyclic permutation,} \\ -x_i^{d_1}x_j^{d_2}x_k^{d_3} & \text{if } (i,j,k)=(1,2,3) \text{ or a cyclic permutation.} \end{cases}$$

The symmetric group S_3 acts on all five of the rings in diagram (66). Recall also that S_3 acts on the set of admissible colorings of an unanchored foam (ie those considered in Section 4.1). However, for an anchored foam F, $c \in \mathrm{adm}(F)$, and $\sigma \in S_3$, the coloring $\sigma(c)$ is in general not admissible for F.

Consider instead the anchored foam $\sigma(F)$ defined as follows. The underlying foam of $\sigma(F)$ agrees with the underlying foam of F. If anchor points of F are labeled by $\ell\colon p(F)\to\{1,2,3\}$, then the anchor points of $\sigma(F)$ are labeled by $\sigma(l)\colon p \mapsto \sigma(\ell(p))$. Note that σ provides a bijection $\mathrm{adm}(F)\cong\mathrm{adm}(\sigma(F))$ via $c\mapsto\sigma(c)$. The following lemma says that the evaluations $\langle F\rangle$ and $\langle\sigma(F)\rangle$ differ by a sign, and moreover the sign depends only on σ and on labels of anchor points of F.

Lemma 4.14 *For an anchored foam F, $c \in \mathrm{adm}(F)$, and $\sigma \in S_3$, we have*

$$\sigma(\langle F,c\rangle) = (-1)^{\varepsilon(F,\sigma)}\langle\sigma(F),\sigma(c)\rangle,$$

where

$$\varepsilon(F,\sigma) = \sum_{\substack{1\le i<j\le 3\\ \sigma(i)>\sigma(j)}} \frac{\mathrm{an}(i)+\mathrm{an}(j)}{2}. \tag{80}$$

It follows that

$$\sigma(\langle F\rangle) = (-1)^{\varepsilon(F,\sigma)}\langle\sigma(F)\rangle.$$

Proof By Lemma 4.6,

$$\sigma\left((-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c)}\right) = (-1)^{s(\sigma(F),\sigma(c))}\frac{P(\sigma(F),\sigma(c))}{Q(\sigma(F),\sigma(c))}.$$

It is clear that

$$\sigma(\widetilde{Q}(F)) = (-1)^{\varepsilon(F,\sigma)}\widetilde{Q}(\sigma(F)),$$

and the first equality follows. For the second equality, we have

$$\begin{aligned}\sigma(\langle F\rangle) &= \sum_{c\in\mathrm{adm}(F)}\sigma(\langle F,c\rangle)\\ &= (-1)^{\varepsilon(F,\sigma)}\sum_{c\in\mathrm{adm}(F)}\langle\sigma(F),\sigma(c)\rangle\\ &= (-1)^{\varepsilon(F,\sigma)}\langle\sigma(F)\rangle.\end{aligned}$$

□

For $1 \leq i \neq j \leq 3$, consider the ring

$$R''_{ij} := R'_x[(x_i - x_k)^{-1}, (x_j - x_k)^{-1}].$$

Each R''_{ij} is a subring of R''_x. A permutation $\sigma \in S_3$ sends R''_{ij} isomorphically onto $R''_{\sigma(i)\sigma(j)}$.

We are now ready for the main result of this section.

Theorem 4.15 *The evaluation $\langle F \rangle$ of an anchored foam is an element of R'_x, the polynomial ring in variables x_1, x_2 and x_3.*

Proof The proof is similar to that of [18, Theorem 2.17] and [34, Proposition 2.18]. By Lemma 4.14, it suffices to show that $\langle F \rangle \in R''_{12}$ for any anchored foam F. This is because we may take a permutation $\sigma \in S_3$ sending 1 to i and 2 to j, and consider the anchored foam $\sigma^{-1}(F)$. Then $\langle \sigma^{-1}(F) \rangle \in R''_{12}$ implies that

$$\pm\langle F \rangle = \pm\langle \sigma(\sigma^{-1}(F)) \rangle = \pm\sigma(\langle \sigma^{-1}(F) \rangle) \in R''_{ij},$$

where the first equality comes from Lemma 4.14. It follows that

$$\langle F \rangle \in R''_{12} \cap R''_{23} \cap R''_{13} = R'_x.$$

Let us show that $\langle F \rangle \in R''_{12}$. Partition $\mathrm{adm}(F)$ into equivalence classes as follows. For $c \in \mathrm{adm}(F)$, the class C_c containing c consists of colorings obtained from c by performing a sequence of $(1,2)$ Kempe moves along surfaces in $F_{12}(c)$ which are disjoint from L. If $F_{12}(c)$ has n connected components, $k \geq 0$ of which are disjoint from L, then C_c consists of 2^k elements. We will show that

$$\sum_{c' \in C_c} \langle F, c' \rangle \in R''_{12},$$

which will conclude the proof.

Write $\Sigma := F_{12}(c)$ as a disjoint union

$$\Sigma = \Sigma' \cup \Sigma_1 \cup \cdots \cup \Sigma_k,$$

where each Σ_a, for $a = 1, \ldots, k$, is connected and disjoint from L, and where each component of Σ' intersects L. For $i = 1, 2$ and $a = 1, \ldots, k$, let $t_i(a)$ denote the number of dots on i–colored facets (according to c) of Σ_a, and let t_3 denote the number of dots on 3–colored facets (according to c) of F. We claim that

$$\sum_{c' \in C_c} \langle F, c' \rangle = \frac{x_3^{t_3} \cdot \prod_{a=1}^{k} \left(x_1^{t_1(a)} x_2^{t_2(a)} + (-1)^{\chi(\Sigma_a)/2} x_2^{t_1(a)} x_1^{t_2(a)} ((x_1-x_3)/(x_2-x_3))^{\ell_{\Sigma_a}(c)/2} \right) \cdot \widetilde{Q}(F)}{(x_1-x_2)^{\chi(\Sigma)/2} (x_1-x_3)^{\chi(F_{13}(c))/2} (x_2-x_3)^{\chi(F_{23}(c))/2}}, \tag{81}$$

where

- $\ell_{\Sigma_a}(c) \in 2\mathbb{Z}$ is an even integer such that
$$\chi(F_{13}(c')) = \chi(F_{13}(c)) - \ell_{\Sigma_a}(c), \quad \chi(F_{23}(c')) = \chi(F_{23}(c)) + \ell_{\Sigma_a}(c)$$
for the coloring $c' \in C_c$ which is obtained from c by a $(1,2)$ Kempe move along Σ_a. See [18, Lemma 2.12(3)] for details regarding this integer.
- $\widetilde{Q}(F)$ is the contribution from the anchor points of F; see (79).

To verify the claimed equality, expand the product to obtain 2^k terms, each of which corresponds to one of the 2^k colorings in C_c. That the sign is correct follows from Lemma 4.8.

Finally, we argue that $(x_1 - x_2)^{\chi(\Sigma)/2}$ divides the numerator of (81). Positive contributions to $\chi(\Sigma)$ come from 2–sphere components of Σ. Each Σ_a which is a 2–sphere contributes one to the exponent $\chi(\Sigma)/2$. On the other hand, the corresponding factor in the product in the numerator of (81) is divisible by $x_1 - x_2$. The remaining positive contributions to $\chi(\Sigma)/2$ come from 2–sphere components of Σ'. Such a component Σ_0 contains at least two anchor points, each labeled 1 or 2, so the contribution from Σ_0 can be canceled with terms in $\widetilde{Q}(F)$. □

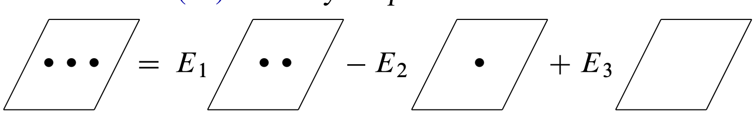
Corollary 4.16 *If F is a prefoam or a foam which is disjoint from L, then $\langle F \rangle \in R_x$, the ring of symmetric polynomials in x_1, x_2 and x_3.*

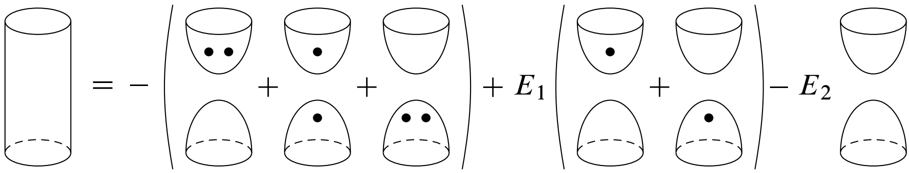
Proof This follows from Lemma 4.14 and Theorem 4.15. □

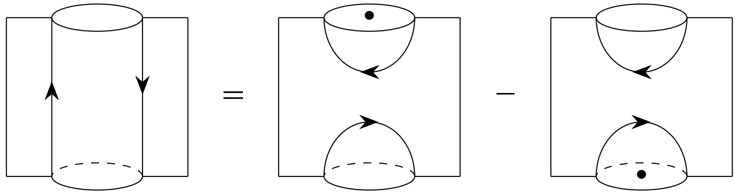
4.3 Skein relations

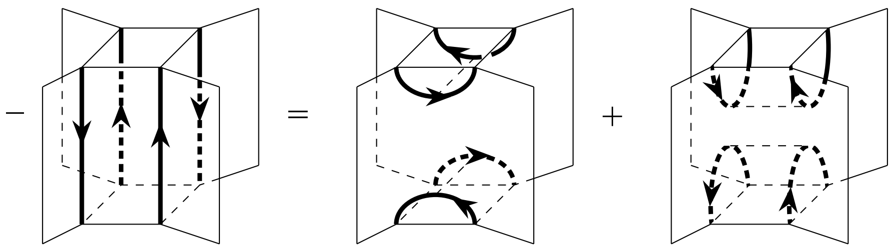
In this section we record several local relations involving oriented anchored SL(3) foams.

Lemma 4.17 *The following local relations hold for anchored foams. Seam lines are drawn in bold in relation (85) to clarify the picture*:

(82)


(83) 

(84) 

(85) 

Proof Proofs of these four relations are similar to Propositions 2.33, 2.22, 2.23 and 2.24 in [18], respectively, with the caveat that we must keep track of the sign (71). Moreover, S_3 symmetry is used in [18] to simplify the calculations. Anchor points and their labels are the same for the foams depicted in each of these four relations, so Lemma 4.14 implies that we may use S_3 symmetry in a similar manner.

We verify relation (83) and leave the remaining three relations to the reader. Let F denote the foam appearing on the left-hand side of the equality. The six foams on the right-hand side are identical except for placement of dots. We denote them by $G^1, \dots, G^6$, so that the relation reads

$$\langle F\rangle = -(\langle G^1\rangle + \langle G^2\rangle + \langle G^3\rangle) + E_1(\langle G^4\rangle + \langle G^5\rangle) - E_2\langle G^6\rangle.$$

Admissible colorings of $G^1, \dots, G^6$ are in canonical bijection. For $c \in \mathrm{adm}(G^1)$, let

$$\langle G, c\rangle := -(\langle G^1, c\rangle + \langle G^2, c\rangle + \langle G^3, c\rangle) + E_1(\langle G^4, c\rangle + \langle G^5, c\rangle) - E_2\langle G^6, c\rangle.$$

There are two types of colorings of G^1: those which color the two depicted disks the same, and those which color them differently. Those of the first type are in canonical bijection with colorings of F.

Suppose $c \in \mathrm{adm}(G^1)$ colors both disks the same color, say i, and denote by $c \in \mathrm{adm}(G^2) \cong \dots \cong \mathrm{adm}(G^6)$ and $c' \in \mathrm{adm}(F)$ the corresponding colorings. We will show that $\langle F, c' \rangle = \langle G, c \rangle$. We may assume $i = 1$. Then

$$\langle G^1, c \rangle = \langle G^2, c \rangle = \langle G^3, c \rangle = x_1^2 \langle G^6, c \rangle, \quad \langle G^4, c \rangle = \langle G^5, c \rangle = x_1 \langle G^6, c \rangle,$$

which yields

$$\langle G, c \rangle = -3x_1^2 \langle G^6, c \rangle + 2E_1 x_1 \langle G^6, c \rangle - E_2 \langle G^6, c \rangle = -(x_1 - x_2)(x_1 - x_3) \langle G^6, c \rangle.$$

To compare this with $\langle F, c' \rangle$, observe that

$$\chi(F_1(c')) + 2 = \chi(G_1^6(c)), \quad \chi(F_2(c')) = \chi(G_2^6(c)), \quad \chi(F_3(c')) = \chi(G_3^6(c)),$$

which implies $s(F, c') \equiv s(G, c) + 1$. Moreover,

$$\chi(F_{12}(c')) + 2 = \chi(G_{12}^6(c)), \quad \chi(F_{13}(c')) + 2 = \chi(G_{13}^6(c)), \quad \chi(F_{23}(c')) = \chi(G_{23}^6(c)).$$

Therefore,

$$\langle G^6, c \rangle = -\frac{\langle F, c' \rangle}{(x_1 - x_2)(x_1 - x_3)},$$

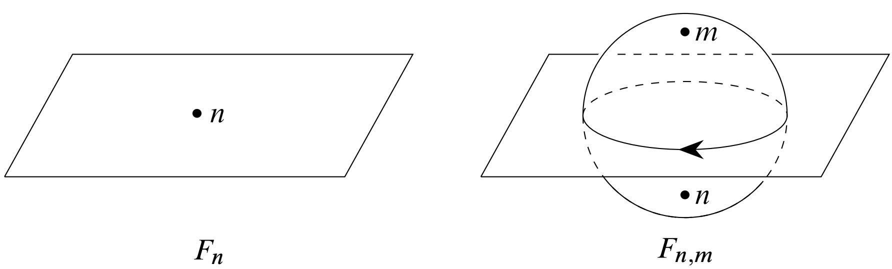
which verifies $\langle F, c' \rangle = \langle G, c \rangle$.

To complete the proof, suppose that c colors the top depicted disk by i and the bottom disk by j, with $i \neq j$. We have

$$\langle G^1, c \rangle = x_i^2 \langle G^6, c \rangle, \quad \langle G^2, c \rangle = x_i x_j \langle G^6, c \rangle, \quad \langle G^3, c \rangle = x_j^2 \langle G^3, c \rangle,$$
$$\langle G^4, c \rangle = x_i \langle G^6, c \rangle, \quad \langle G^5, c \rangle = x_j \langle G^6, c \rangle.$$

Therefore $\langle G, c \rangle = 0$, which concludes the proof. □

Lemma 4.18 *Let F be an anchored foam. Denote by $F_{n,m}$ the anchored foam obtained from F by adding a bubble (disjoint from L) to some facet in F, with the two new facets carrying n and m dots respectively, such that the facet with n dots directly precedes the facet with m dots in the cyclic ordering. Let F_n denote the foam obtained from F by adding n dots to the same facet,*

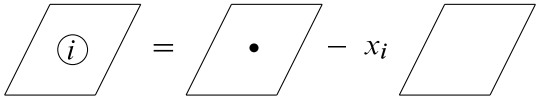


Then

$$\langle F_{n,n} \rangle = 0, \quad \langle F_{1,0} \rangle = -\langle F_{0,1} \rangle = \langle F \rangle, \quad \langle F_{2,0} \rangle = -\langle F_{0,2} \rangle = E_1 \langle F \rangle - \langle F_1 \rangle.$$

Remark 4.19 The relations in Lemmas 4.17 and 4.18 also hold for prefoams.

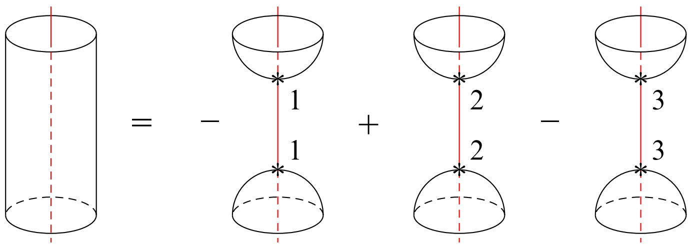
Similar to the SL(2) and unoriented SL(3) setting, for oriented SL(3) foams we allow shifted dots $\textcircled{i} = \bullet - x_i$ $(1 \le i \le 3)$ on a facet:



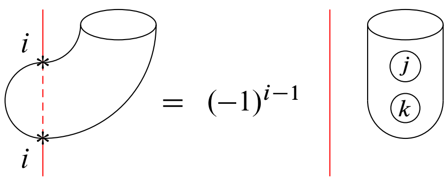
They must be disjoint from L and are allowed to float freely on their facets but cannot cross seam lines.

Lemma 4.20 *The following local relations hold*:

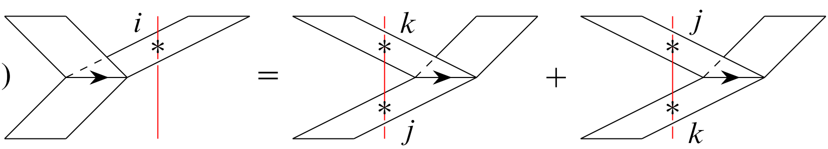
(86)



(87)



(88) $(x_j - x_k)$



In the last equation we assume $j < k$.

Proof We verify (86) and leave the remaining relations to the reader. The argument is similar to that of relation (44) in Lemma 3.15, so we will be brief. Let F denote the foam on the left-hand side, and let G^1, G^2 and G^3 denote the three foams on the right-hand side, with superscript corresponding to labels of the anchor points. For $1 \le i \le 3$, let $\mathrm{adm}_i(F)$ consist of all admissible colorings of F which color the depicted tube by i. There is a natural bijection $\mathrm{adm}_i(F) \cong \mathrm{adm}(G^i)$.

Given $c \in \mathrm{adm}_i(F)$, let $c' \in \mathrm{adm}(G^i)$ denote the corresponding coloring. Arguing as in the proof of Lemma 3.15, we obtain

$$\langle F, c\rangle = \pm \langle G^i, c'\rangle.$$

Figure 15: The orientations at each trivalent vertex of an oriented SL(3) web must be either all outgoing or all incoming.

It remains to show that the above sign is equal to $(-1)^i$. We have

$$\chi(F_j(c)) = \chi(G^i_j(c')), \quad \chi(F_k(c)) = \chi(G^j_k(c')), \quad \chi(F_i(c)) = \chi(G^i_i(c')) - 2,$$
$$\theta^{\pm}(F,c) = \theta^{\pm}(G^i, c'),$$

so $s(F,c) \equiv s(G^i, c') + i$ as needed. □

4.4 State spaces

In this section we define state spaces associated to oriented SL(3) webs. Much of this is analogous to notions in Section 3.3.

Definition 4.21 An *oriented* SL(3) *web* is a planar trivalent graph $\Gamma \subset \mathcal{P}$ in the punctured plane, which may have closed loops with no vertices. Moreover, edges and loops of Γ carry orientations such that each vertex is either a source or a sink, as shown in Figure 15. In this section we will simply write *web* rather than oriented SL(3) web.

The definition of an anchored foam with boundary in the oriented setting is analogous to that of Definition 3.16. The singular graph of a foam with boundary V is a union of finitely many arcs (with boundary in $\mathbb{R}^2 \times \{0,1\}$) and circles (disjoint from $\mathbb{R}^2 \times \{0,1\}$). Intersection points of V with $L_{[0,1]}$ (anchor points) must be disjoint from the singular graph and carry labels in $\{1,2,3\}$. Facets of V are required to carry orientations satisfying the convention in Figure 13, left, near singular points. As usual, we will use the left-hand rule to specify these orientations and cyclic orderings by orienting each singular circle and arc, as shown in Figure 13, middle and right.

As in Section 3.3, let $\partial_i V := V \cap (\mathbb{R}^2 \times \{0\})$ for $i = 0, 1$. The orientation of facets of V induces an orientation on $\partial_0 V$ and $\partial_1 V$ via the convention in Figure 16. We view V as a cobordism from the oriented web $\partial_0 V$ to the oriented web $\partial_1 V$. Composition WV of foams V and W with $\partial_1 V = \partial_0 W$ is defined in the natural way.

Denote by $p(V) = V \cap L_{[0,1]}$ the set of anchor points of V and by $|d(V)|$ the number of dots. The degree of V is defined to be

$$\deg(V) = 2(|d(V)| + |p(V)| - \chi(V)) + \chi(\partial V). \tag{89}$$

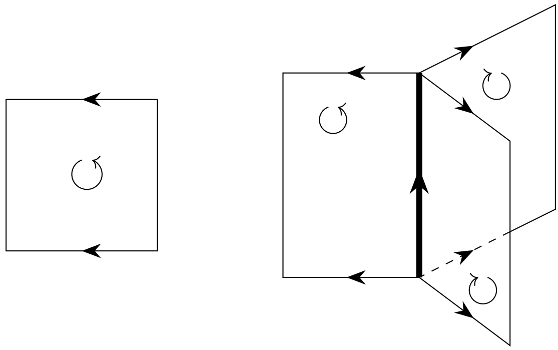


Figure 16: Our convention for the induced orientation on the webs $\partial_0 V$ (bottom) and $\partial_1 V$ (top).

Degree is clearly additive under composition and is compatible with the grading on R'_x, in the sense that if V is a closed foam, then $\deg(V) = \deg(\langle V \rangle)$.

As in Definition 2.14, by an *annular foam* we mean a foam (with boundary) which is disjoint from L. The composition of two annular foams is again annular.

There is an involution ω defined by reflecting a foam with boundary through $\mathbb{R}^2 \times \{1/2\}$. We have $\partial_1 V = \partial_0(\omega(V))$ and $\partial_0 V = \partial_1(\omega(V))$ for any foam with boundary V. Given a web $\Gamma \subset \mathcal{P}$, let $\mathrm{Fr}(\Gamma)$ denote the free R'_x–module generated by foams with boundary V from the empty web to Γ (that is, $\partial_0 V = \varnothing$, $\partial_1 V = \Gamma$). Define a bilinear form

$$(-,-)\colon \mathrm{Fr}(\Gamma) \times \mathrm{Fr}(\Gamma) \to R'_x$$

by $(V, W) = \omega(V)W$. This bilinear form is symmetric since $\langle F \rangle = \langle \omega(F) \rangle$ for any closed foam F. The state space $\langle \Gamma \rangle$ is the quotient of $\mathrm{Fr}(\Gamma)$ by the kernel

$$\ker((-,-)) = \{x \in \mathrm{Fr}(\Gamma) \mid (x, y) = 0 \text{ for all } y \in \mathrm{Fr}(\Gamma)\}$$

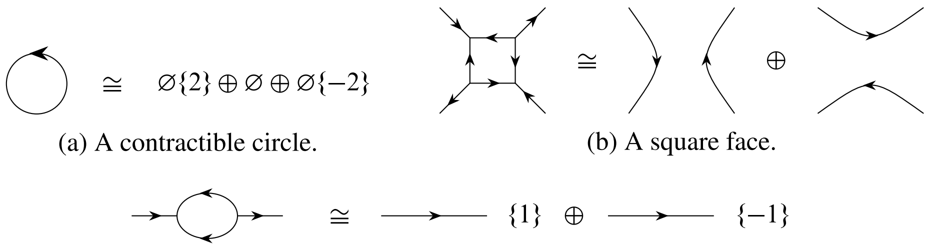
of the bilinear form,

$$\langle \Gamma \rangle := \mathrm{Fr}(\Gamma)/\ker((-,-)).$$

The state space $\langle \Gamma \rangle$ inherits the grading from $\mathrm{Fr}(\Gamma)$ since $(-,-)$ is degree-preserving. A foam with boundary V from Γ_0 to Γ_1 naturally induces a map

$$\langle V \rangle\colon \langle \Gamma_0 \rangle \to \langle \Gamma_1 \rangle$$

of degree $\deg(V)$, defined by sending the equivalence class of a basis element $U \in \mathrm{Fr}(\Gamma_0)$ to the equivalence class of $VU \in \mathrm{Fr}(\Gamma_1)$. This assignment is functorial with respect to composition of foams, $\langle WV \rangle = \langle W \rangle \langle V \rangle$ for composable V and W.



(a) A contractible circle.

(b) A square face.

(c) A bigon face.

Figure 17: Local relations for state spaces of oriented SL(3) webs, where the depicted regions do not contain the puncture.

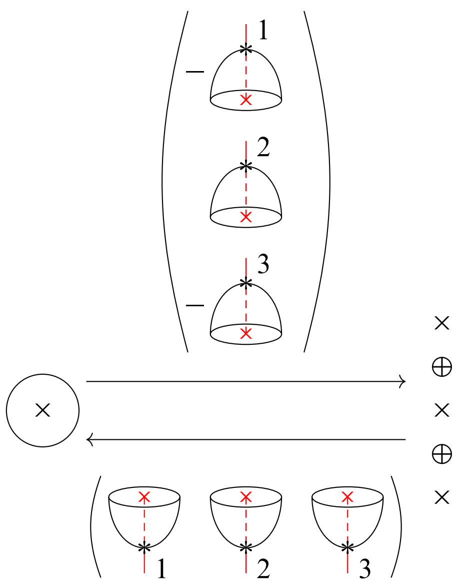
Lemma 4.22 *The three local isomorphisms shown in Figure 17 hold.*

Proof The arguments for relations (a), (b), and (c) of the figure are analogous to Propositions 7, 9, and 8, respectively, of [15]. The relevant relations are Lemmas 4.17 and 4.18. □

Proposition 4.23 *Let $\Gamma \subset \mathcal{P}$ be a web with a noncontractible circle C which bounds a disk in $\mathbb{R}^2 \setminus \Gamma$, and let $\Gamma' = \Gamma \setminus C$ be the web obtained by removing C. Then there is an isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle$$

given by the following maps (orientation of the circle is omitted):



Proof This follows from Example 4.12 and the neck-cutting relation (86). □

Theorem 4.24 *For any web $\Gamma \subset \mathcal{P}$, the state space $\langle \Gamma \rangle$ is a free graded R'_x–module of rank equal to the number of Tait colorings of Γ. Moreover, if Γ is contractible, then the graded rank of $\langle \Gamma \rangle$ equals the Kuperberg polynomial [24] of Γ, normalized as in* [15, Section 2].

Proof Lemma 3.21(1) guarantees that we can reduce $\langle \Gamma \rangle$ to a direct sum of empty webs by recursively applying the local isomorphisms in Lemma 4.22 and Proposition 4.23. It is then clear that the rank equals the number of Tait colorings.

If Γ is contractible, $\langle \Gamma \rangle$ can be simplified using only the isomorphisms in Lemma 4.22. Upon taking graded ranks, these isomorphisms recover the recursive relations for computing the Kuperberg polynomial. □

Theorem 4.24 does not address the graded rank of state spaces of noncontractible webs. These may be computed recursively. As a special case, if Γ consists of n contractible and m noncontractible circles, then $\langle \Gamma \rangle$ is free of graded rank $3^m(q^2+1+q^{-2})^n$.

Given a web $\Gamma \subset \mathcal{P}$, we can forget the puncture and the anchor line L and apply the universal construction to the evaluation (73). Precisely, let $\mathrm{Fr}(\Gamma)_{\mathrm{forget}}$ denote the free R_x–module generated by all foams with boundary Γ (forgetting the anchor line). By Corollary 4.16, we can define the bilinear form $(-,-)\colon \mathrm{Fr}(\Gamma)_{\mathrm{forget}} \times \mathrm{Fr}(\Gamma)_{\mathrm{forget}} \to R_x$ and the corresponding state space $\langle \Gamma \rangle_{\mathrm{forget}}$ in the usual way. Thus we obtain state spaces for webs in $\mathbb{R}^2$, functorial with respect to foams in $\mathbb{R}^2 \times [0,1]$. These state spaces and maps induced by foams are graded via (89), where $|p(V)| = 0$.

Proposition 4.25 *For a contractible web $\Gamma \subset \mathcal{P}$, there is a degree-preserving isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma \rangle_{\mathrm{forget}},$$

natural with respect to foams with contractible boundary and which are disjoint from L.

Proof This follows from Theorem 4.24. □

On the other hand, Mackaay and Vaz [28] define an evaluation $\langle - \rangle_{\mathrm{MV}}$ for oriented $\mathrm{SL}(3)$ prefoams and use it to define an equivariant (also called *universal*) version of the $\mathfrak{sl}(3)$ link homology introduced in [15]. They work over the ground ring $\mathbb{Z}[a,b,c]$ and associate a state space $\langle \Gamma \rangle_{\mathrm{MV}}$ to each web $\Gamma \subset \mathbb{R}^2$ via the universal construction applied to their prefoam evaluation $\langle - \rangle_{\mathrm{MV}}$. To compare with our situation, identify $\mathbb{Z}[a,b,c]$ with the ring $R_x = \mathbb{Z}[E_1, E_2, E_3]$ of symmetric functions in x_1, x_2 and x_3 via a ring isomorphism φ defined by $\varphi(a) = E_1$, $\varphi(b) = -E_2$ and $\varphi(c) = E_3$.

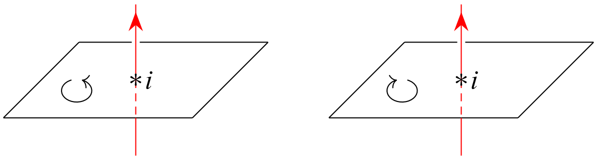


Figure 18: The oriented intersection number between a facet and L.

Theorem 4.26 *For any closed prefoam F,*

$$\langle F \rangle = \varphi(\langle F \rangle_{\mathrm{MV}}).$$

It follows that there are isomorphisms $\langle \Gamma \rangle_{\mathrm{forget}} \cong \langle \Gamma \rangle_{\mathrm{MV}} \otimes_{\mathbb{Z}[a,b,c]} R_x$ for any web $\Gamma \subset \mathbb{R}^2$, natural with respect to maps induced by foams with boundary.

Proof The evaluation $\langle - \rangle_{\mathrm{MV}}$ is defined by applying the local relations (3D), (CN), (S), and (Θ) in [28, Section 2.1] to reduce any foam to an element of $\mathbb{Z}[a,b,c]$. Under the change of variables $a \mapsto E_1$, $b \mapsto -E_2$ and $c \mapsto E_3$, these four relations hold for our evaluation $\langle - \rangle$ by relation (82), relation (83), Example 4.4, and Example 4.5. The statement follows. □

As in the SL(2) and unoriented SL(3) setting considered earlier in the paper, we can define an additional grading on oriented SL(3) foams and state spaces. Define the abelian group

$$\Lambda = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \oplus \mathbb{Z}w_3/(w_1 + w_2 + w_3), \tag{90}$$

on three generators and one relation. Λ is a free abelian group of rank two.

Orient the anchor line L from bottom to top. For an anchored foam V with boundary and $p \in p(V)$ an anchor point lying on some facet f, let $s(p) \in \{\pm 1\}$ denote the oriented intersection number between f and L ($s(p)$ does not depend on the label of p); see Figure 18 for the convention. Define the *annular degree of V* to be

$$\mathrm{adeg}(V) = \sum_{p \in p(V)} s(p) w_{\ell(p)} \in \Lambda. \tag{91}$$

Proposition 4.27 *If F is a closed anchored foam with an admissible coloring c, then* $\mathrm{adeg}(F) = 0$.

Proof The proof is similar to that of Proposition 3.10. The intersection of F with a generic half-plane that bounds L is an oriented web Γ with boundary points on L.

An admissible coloring c of F induces a Tait coloring of Γ. The boundary points (one-valent vertices) of Γ are colored according to their label. The sum in (91) may be rewritten as the sum of terms $\pm(w_1+w_2+w_3)=0$ over all trivalent vertices of Γ, where the sign is $+1$ if all edges are incoming and -1 if all edges are outgoing. Each i–colored inner edge e of Γ bounds two trivalent vertices and contributes $\pm(w_i-w_i)=0$ since e is oriented towards one of its boundary vertices and away from the other. The remaining edges, with one or both endpoints on L, contribute precisely $\operatorname{adeg}(F)$. $\square$

Let $\Gamma\subset\mathcal{P}$ be an (annular oriented) SL(3) web. An anchored foam $F\subset\mathbb{R}^3_-$ with $\partial F=\Gamma$ has a well-defined degree $\operatorname{adeg}(F)\in\Lambda$ via (91). Furthermore, we equip the coefficient ring R'_x with a Λ–grading, with all elements of degree 0. This makes free R'_x–module $\operatorname{Fr}(\Gamma)$ into a Λ–graded module, and Proposition 4.27 implies that the kernel of the bilinear form on $\operatorname{Fr}(\Gamma)$ is Λ–graded as well. Consequently, the grading descends to a Λ–grading on the state space $\langle\Gamma\rangle$. A foam V with boundary induces a map $\langle V\rangle\colon\langle-\partial_0\Gamma\rangle\to\langle\partial_1\Gamma\rangle$ which changes adeg by $\operatorname{adeg}(V)$. If V has no anchor points, it induces an annular degree 0 map between the state spaces of its boundaries. The state space of a contractible web is concentrated in annular degree zero.

Λ–grading on $\langle\Gamma\rangle$ is the analogue of grading on finite-dimensional SL(3) representations by the weight lattice. In fact, in the nonequivariant version of our construction, where all the x_i are set to 0 upon closed foam evaluation (and state spaces are defined accordingly, over a ground field rather than the ring R'_x), the state space $\langle\Gamma\rangle$ is naturally an $\mathfrak{sl}_3$–representation. We also refer the reader to Queffelec and Rose [30] for the construction of sutured annular $\mathfrak{sl}_n$–homology, with state spaces of annular webs carrying an $\mathfrak{sl}_n$–action. In the equivariant case, it is not clear how to define an $\mathfrak{sl}_3$–action or what's the substitute for it.

Denote by $\mathrm{AFoam}_{\mathrm{or}}$ the category whose objects consist of oriented SL(3) webs in $\mathcal{P}$ and whose morphisms are R'_x–linear combinations of anchored cobordisms between webs. Morphism spaces in this category are triply graded via (qdeg, adeg). The state space construction assembles into a functor

$$\langle-\rangle\colon \mathrm{AFoam}_{\mathrm{or}}\to R'_x\text{–}\mathrm{g}_3\mathrm{mod}$$

landing in the category of triply graded R'_x–modules.

This functor respects the trigradings on the hom spaces in the two categories. Restricting to the subcategory of annular cobordisms and their linear combinations, the induced maps have annular degree 0.

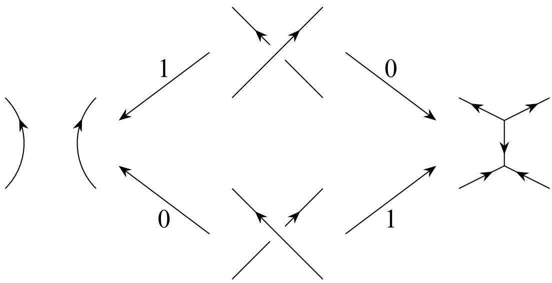


Figure 19: The 0– and 1–smoothings used to define the SL(3) chain complex.

4.5 Annular SL(3)–link homology

Let $L \subset \mathbb{A} \times [0,1]$ be a link in the thickened annulus. Projecting onto $\mathbb{A} \times \{0\} = \mathbb{A}$ and identifying the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P}$, we obtain a link diagram $D \subset \mathcal{P}$. Following [15, Section 4; 28], form the *cube of resolutions* of D. Order the crossings of D by $1, \dots, n$ and use the rule in Figure 19 to decorate each vertex $u \in \{0,1\}^n$ by the corresponding web $D_u \subset \mathcal{P}$.

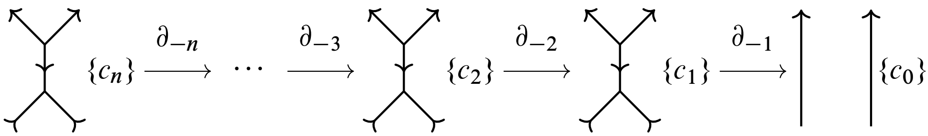
Introducing signs to make the cube anticommute, collapsing the cube to a chain complex, adding internal and homological degree shifts, and applying the functor $\langle - \rangle \colon \mathrm{AFoam}_{\mathrm{or}} \to R'_x\text{–}\mathrm{g}_3\mathrm{mod}$ yields a chain complex $C(D)$ of $\mathbb{Z} \oplus \Lambda$–graded R'_x–modules. In homological degree i, the complex is given by

$$C^i(D) = \bigoplus_{|u| = i + n_+} \langle D_u \rangle \{2(n_+ - n_-) - i\},$$

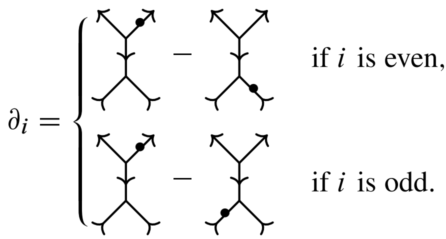
where n_+ and n_- are the number of positive and negative crossings of D. The $\mathbb{Z}$–grading is given by deg — see (89) — and the Λ–grading given by adeg — see (91). Degree shifts in the cube of resolutions are applied only to the $\mathbb{Z}$–degree deg. Diagrams in $\mathcal{P}$ representing isotopic annular links are related by Reidemeister moves away from the puncture. Proofs of Reidemeister invariance in [28] are local, and all local relations (away from the anchor line) on foams in [28] also hold for our evaluation $\langle - \rangle$ by (83), Example 4.4, and Example 4.5. It follows that the chain homotopy class of $C(D)$ is an invariant of the annular link L. We define *equivariant annular* SL(3) *homology* as cohomology groups $H(C(D))$.

Moreover, foams between webs appearing in the cube of resolutions are disjoint from L. Thus the differential preserves annular degree throughout the complex. Consequently, equivariant annular SL(3) link homology carries a homological grading as well as an internal $\mathbb{Z} \oplus \Lambda$–grading $(\deg, \operatorname{adeg})$. Cohomology groups $H(C(D))$ are trigraded R'_x–modules.

Example 4.28 We conclude with an explicit calculation. Let σ denote the positive crossing generator of the 2–strand braid group, let L_n denote the annular link obtained as the annular closure of σ^n, and let $C(L_n)$ denote the corresponding chain complex. Consider the complex $C(n)$,



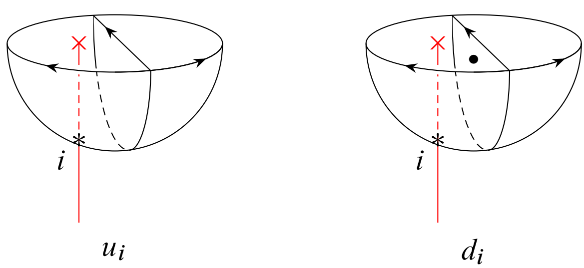
The right-most term is in homological degree zero and the quantum grading shifts c_i are $c_0 = 2n$ and $c_i = 2n + 2i - 1$ for $1 \leq i \leq n$. The right-most differential ∂_{-1} is the unzip cobordism, and for $-n \leq i \leq -2$ the differentials are



One can show that the chain complex $C(L_n)$ is chain homotopy equivalent to the annular closure of $C(n)$.

Upon taking annular closures, the differential ∂_i for even i is zero. Consider the annular closure Γ of the web appearing in negative homological degree. The state space $\langle \Gamma \rangle$ is a free R'_x–module of rank six, and we choose a basis $\{u_1, d_i, u_2, d_2, u_3, d_3\}$ shown in (92). Bidegrees of u_i and d_i are $(-1, -w_i)$ and $(1, -w_i)$, respectively (not accounting for grading shifts):

(92)



After taking the annular closure, the differential ∂_i, for $i \leq -3$ odd, is given as the difference of foams $F - G$, where F puts a dot on the right-most facet and G puts a dot on the middle facet of the depicted generators. We have

$$\begin{aligned} F(u_i) &= (x_j + x_k)u_i - d_i, & F(d_i) &= x_j x_k u_i, \\ G(u_i) &= d_i, & G(d_i) &= (x_j + x_k)d_i - x_j x_k u_i. \end{aligned}$$

In particular, ∂_i for $i \leq -3$ and i odd is injective.

Let us now compute the right-most differential, which is the annular closure of the unzip cobordism. Let Γ_0 denote the web consisting of two essential counterclockwise oriented circles, which is the annular closure of the term in homological degree zero in $C(n)$. For $1 \leq i, j \leq 3$, let $g_{ij} \colon \varnothing \to \Gamma_0$ be the foam consisting of two cups, each intersecting the anchor line once, with the anchor point of the inner cup labeled i and the anchor point of the outer cup labeled j. By Proposition 4.23, $\{g_{ij}\}_{1 \leq i, j \leq 3}$ is a basis for $\langle \Gamma_0 \rangle$. After introducing the grading shift, the generator g_{ij} is in quantum degree $2n$ and in annular degree $w_i + w_j = -w_k$. Let $Z \colon \Gamma \to \Gamma_0$ denote the unzip cobordism. By applying the neck-cutting relation (86) near the two circles that constitute Γ_0, we write $\partial_{-1}(u_i)$ as a sum

$$\partial_{-1}(u_i) = \sum_{1 \leq s, t \leq 3} (-1)^{s+t} g_{st} \sqcup \tau_{st},$$

where τ_{st} is a theta foam as in Example 4.13, with no dots, and anchor points labeled i, s and t read from bottom to top. These theta foams evaluate to zero unless $\{i, s, t\} = \{1, 2, 3\}$, and otherwise they evaluate to ± 1. Moreover, $\langle \tau_{st} \rangle = -\langle \tau_{ts} \rangle$. Therefore,

$$\partial_{-1}(u_i) = \pm (g_{jk} - g_{kj}).$$

A similar procedure yields $\partial_{-1}(d_i) = \pm (x_j g_{jk} - x_k g_{kj})$.

Thus, in homological degree $s \leq 0$ and annular degree $-w_i$, the homology of L_n is given by

$$H^{s,-w_i}(L_n) = \begin{cases} 0 & \text{if } s \text{ is odd}, \\ \dfrac{R'_x\{2n-2s-2\} \oplus R'_x\{2n-2s\}}{\langle (x_j + x_k, -2), (2x_j x_k, -(x_j + x_k)) \rangle} & \text{if } s < 0 \text{ and } s \text{ is even}, \\ (R'_x/(x_j - x_k))\{2n\} & \text{if } s = 0. \end{cases}$$

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