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On a problem of Hopf for circle bundles over aspherical manifolds with hyperbolic fundamental groups

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We prove that a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group admits a self-map of absolute degree greater than one if and only if it is virtually trivial. This generalizes in every dimension the case of circle bundles over hyperbolic surfaces, for which the result was known by the work of Brooks and Goldman on the Seifert volume. As a consequence, we verify the following strong version of a problem of Hopf for the above class of manifolds: every self-map of nonzero degree of a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group is either homotopic to a homeomorphism or homotopic to a nontrivial covering and the bundle is virtually trivial. As another application, we derive the first examples of nonvanishing numerical invariants that are monotone with respect to the mapping degree on nontrivial circle bundles over aspherical manifolds with hyperbolic fundamental groups in any dimension.

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1 Introduction

A long-standing question of Hopf (see Problem 5.26 in Kirby's list [14]) asks:

Problem 1.1 (Hopf) *For a closed oriented manifold M, is every self-map $f: M \to M$ of degree ± 1 a homotopy equivalence?*

A complete solution to Hopf's problem seems to be currently out of reach. Nevertheless, some affirmative answers are known for certain classes of manifolds and dimensions, most notably for simply connected manifolds (by Whitehead's theorem), for manifolds of dimension at most four with Hopfian fundamental groups (see Hausmann [13], and recall that a group is called Hopfian if every surjective endomorphism is an isomorphism), and for aspherical manifolds with hyperbolic fundamental groups (eg negatively curved manifolds). These last groups are Hopfian (see Maltsev [19] and Sela [27]), thus the asphericity assumption, together with the simple fact that any map of degree ± 1 is π_1–surjective, affirmatively answer Problem 1.1 for closed aspherical manifolds with hyperbolic fundamental groups.

In fact, the assumption about degree ± 1 is unnecessary in affirmatively answering Problem 1.1 for aspherical manifolds with nonelementary hyperbolic fundamental groups, because those manifolds cannot admit self-maps of degree other than ± 1 or zero; see Bridson, Hinkkanen and Martin [5], Mineyev [20; 21] and Sela [26; 27], and Section 3.1. Hence, every self-map of nonzero degree of a closed oriented aspherical manifold with nonelementary hyperbolic fundamental group is a homotopy equivalence. Of course, this statement does not hold for all (aspherical) manifolds because, for example, the circle admits self-maps of any degree. Nevertheless, every self-map of the circle of degree greater than one is homotopic to a (nontrivial) covering. The same is true for every self-map of nilpotent manifolds (see Belegradek [3]) and for certain solvable mapping tori of homeomorphisms of the n–dimensional torus; see Neofytidis [23] and Wang [29]. In addition, every nonzero degree self-map of a 3–manifold M is either a homotopy equivalence or homotopic to a covering map, unless the fundamental group of each prime summand of M is finite or cyclic; see Wang [30]. The above results suggest the following question for aspherical manifolds:

Problem 1.2 (strong version of Hopf's problem for aspherical manifolds) *Is every nonzero degree self-map of a closed oriented aspherical manifold either a homotopy equivalence or homotopic to a nontrivial covering*?

In dimension three, hyperbolic manifolds and manifolds containing a hyperbolic piece in their JSJ decomposition do not admit any self-map of degree greater than one (equivalently, of absolute degree greater than one, by taking f^2 whenever $\deg(f) < -1$) due to the positivity of the simplicial volume; see Gromov [11]. (Recall that the simplicial volume $\|\cdot\|$ satisfies $\|M'\| \geq |\deg(f)| \cdot \|M\|$ for every map $f : M' \to M$.) The other classes of aspherical 3–manifolds which do not admit self-maps of degree greater than one are $\widetilde{\mathrm{SL}}_2$–manifolds (see Brooks and Goldman [6]) and graph manifolds (see Derbez and Wang [10]), since those manifolds have another (virtually) positive invariant that is monotone with respect to mapping degrees, namely the Seifert volume (introduced in [6] by Brooks and Goldman). In particular, nontrivial circle bundles over closed hyperbolic surfaces (which are modeled on the $\widetilde{\mathrm{SL}}_2$ geometry) do not admit self-maps of degree greater than one. On the other hand, it is clear that trivial circle bundles over (hyperbolic) surfaces, ie products $S^1 \times \Sigma$, admit self-maps of any degree (and those maps are either homotopy equivalences or homotopic to nontrivial coverings [30]).

Recall that a circle bundle $M \xrightarrow{\pi} N$ is classified by its Euler class $e \in H^2(N;\mathbb{Z})$. In particular, M is virtually trivial if and only if e is torsion. For a circle bundle M over a closed oriented surface Σ, its Euler class $e \in H^2(\Sigma) = \mathbb{Z}$ is either zero and the bundle is trivial (ie $M = S^1 \times \Sigma$), or e is not zero and nontorsion and the bundle is not virtually trivial. Our main result is that the nonexistence of self-maps of degree greater than one on nontrivial circle bundles over closed oriented hyperbolic surfaces (ie over closed oriented aspherical 2–manifolds with hyperbolic fundamental groups) can be extended to any dimension. In fact, we prove a stronger statement:

Theorem 1.3 *An oriented circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group admits a self-map of absolute degree greater than one if and only if it is virtually trivial.*

The "if" direction holds more generally without any assumption on the hyperbolicity of the fundamental group of the base:

Example 1.4 Let M be a virtually trivial oriented circle bundle over a closed oriented manifold N. Then its Euler class $e \in H^2(N)$ is k–torsion for some k. Since M is fiberwise oriented, M is a principal U(1)–bundle and thus M can be viewed as the (associated) complex line bundle whose first Chern class is the Euler class e. Consider the tensor product $M \otimes \cdots \otimes M$ of $k+1$ copies of M. Then the first Chern class of $M \otimes \cdots \otimes M$ is

$$c_1(M \otimes \cdots \otimes M) = (k+1)c_1(M) = c_1(M),$$

and so $M \otimes \cdots \otimes M$ is isomorphic to M. Taking the $k+1$ power of a section of M gives us a fiberwise map

$$f : M \to M \otimes \cdots \otimes M,$$

which has degree $k+1$ on the fibers and degree one on the base N. Thus $\deg(f) = k+1$.

Outline of the proof of the main theorem

In view of Example 1.4, the proof of Theorem 1.3 amounts to showing that if an oriented circle bundle M over a closed oriented aspherical manifold N with $\pi_1(N)$ hyperbolic admits a self-map f of degree greater than one, then M must be virtually trivial. We will show that such an f is in fact homotopic to a fiberwise nontrivial self-covering of M, and thus the powers of f induce a purely decreasing sequence

$$\pi_1(M) \supsetneq f_*(\pi_1(M)) \supsetneq \cdots \supsetneq f_*^m(\pi_1(M)) \supsetneq f_*^{m+1}(\pi_1(M)) \supsetneq \cdots. \tag{1}$$

Using this sequence, we will obtain an infinite-index subgroup of $\pi_1(M)$ given by

$$G := \bigcap_m f_*^m(\pi_1(M)).$$

The last part of the proof uses the concept of groups infinite-index presentable by products (IIPP) and characterizations of groups fulfilling this condition [22]. More precisely, we will see that the multiplication map

$$\varphi\colon C(\pi_1(M)) \times G \to \pi_1(M)$$

defines a presentation by products for $\pi_1(M)$, where both G and the center $C(\pi_1(M))$ have infinite index in $\pi_1(M)$. This will lead us to the conclusion that $\pi_1(M)$ has a finite-index subgroup isomorphic to a product and M is virtually trivial.

Remark 1.5 In the proof of Theorem 1.3 we will use the fact that the base is an aspherical manifold which does not admit self-maps of degree greater than one, and its fundamental group is Hopfian with trivial center. Thus we can extend Theorem 1.3 (and its consequences; see Section 2) to any circle bundle over a closed oriented manifold N that fulfills the aforementioned properties. For instance, if N is an irreducible locally symmetric space of noncompact type, then it is aspherical, it has positive simplicial volume by Bucher-Karlsson [7] and Lafont and Schmidt [16] (and thus does not admit self-maps of degree greater than one), and $\pi_1(N)$ is Hopfian (see Maltsev [19]) without center; see Raghunathan [25].

Remark 1.6 A decreasing sequence (1) exists whenever an aspherical manifold M admits a self-map f of degree greater than one and $\pi_1(\overline{M})$ is Hopfian for every finite cover $\overline{M}$ of M (which is conjectured to be true; see Section 2). This gives further evidence towards an affirmative answer to Problem 1.2, since the existence of such a sequence is a necessary condition for f to be homotopic to a nontrivial covering. Now, every finite-index subgroup of the fundamental group of a circle bundle over a closed aspherical manifold with hyperbolic fundamental group is indeed Hopfian, and therefore this gives us an alternative way of obtaining sequence (1). We will discuss the Hopf property for those circle bundles and Problem 1.2 more generally in Section 5.

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2 Applications of the main result

Before proceeding to the proof of Theorem 1.3, let us mention a few consequences of Theorem 1.3, or of parts of its proof.

It is a long-standing question (motivated by Problem 1.1) whether the fundamental group of every closed aspherical manifold is Hopfian (see [24] for a discussion). If this is true, then every self-map of an aspherical manifold of degree ± 1 is a homotopy equivalence. In the course of the proof of Theorem 1.3, we will see that every self-map of a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group is homotopic to a fiberwise covering map, and this alone shows that Problem 1.1 and, in most cases, Problem 1.2 indeed have affirmative answers for self-maps of those manifolds. More interestingly, Theorem 1.3 implies the following complete characterization with respect to Problem 1.2:

Corollary 2.1 *A self-map of nonzero degree of an oriented circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group either is a homotopy equivalence or is homotopic to a fiberwise nontrivial covering* (*and to a nontrivial covering in dimensions other than four and five*) *and the bundle is virtually trivial.*

Remark 2.2 (the Borel conjecture: from homotopy equivalences to homeomorphisms) In most cases, an even stronger conclusion holds for the homotopy equivalences of Corollary 2.1. Recall that the Borel conjecture asserts that any homotopy equivalence between two closed aspherical manifolds is homotopic to a homeomorphism. (Note that the Borel conjecture does not hold in the smooth category or for nonaspherical manifolds; see for example the survey paper [18] and the discussion in [28].) A complete affirmative answer to the Borel conjecture is known in dimensions less than four (see again [18] for a survey). Moreover, by [1; 2], the fundamental group of a circle bundle M over a closed aspherical manifold N with $\pi_1(N)$ hyperbolic and $\dim(N) \neq 3, 4$ satisfies the Farrell–Jones conjecture, and therefore the Borel conjecture, and so every homotopy equivalence of such a circle bundle is in fact homotopic to a homeomorphism. (See also [5] for self-maps of the base N.)

Beyond the Seifert volume for nontrivial circle bundles over hyperbolic surfaces [6], no other nonvanishing monotone invariant respecting the degree seems to have been known on higher-dimensional circle bundles over aspherical manifolds with hyperbolic fundamental groups (note that the simplicial volume vanishes as well [11]). A consequence of Theorem 1.3 is that such a monotone invariant exists and is given by the domination seminorm. Recall that the domination seminorm is defined by

$$\nu_M(M') := \sup\{|\deg(f)| \mid f : M' \to M\},$$

and it was introduced in [9]. Theorem 1.3 implies:

Corollary 2.3 *If M is a not virtually trivial circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group, then $\nu_M(M) = 1$.*

However, the domination seminorm is not finite in general, because M might admit maps of infinitely many different degrees from another manifold M'. Nevertheless, Theorem 1.3 and the nonvanishing of the Seifert volume for nontrivial circle bundles over hyperbolic surfaces suggest:

Conjecture 2.4 *In every dimension n, there is a homotopy n–manifold numerical invariant I_n satisfying $I_n(M) \geq |\deg(f)| I_n(N)$ for each map $f : M \to N$ which is positive and finite on every not virtually trivial circle bundle over a closed aspherical manifold with hyperbolic fundamental group.*

3 Infinite sequences of coverings

In this section we reduce our discussion to self-coverings of a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group, and thus obtain a purely decreasing sequence of finite-index subgroups of the fundamental group of this bundle.

3.1 Self-maps of aspherical manifolds with hyperbolic fundamental groups

First, we observe that the hyperbolicity of the fundamental group of the base implies strong restrictions on the possible degrees of its self-maps:

Proposition 3.1 [5] *Every self-map of nonzero degree of a closed aspherical manifold with nonelementary hyperbolic fundamental group is a homotopy equivalence.*

Proof There are two ways to see this. The first (given in [5]) is purely algebraic, using the co-Hopf property of torsion-free nonelementary hyperbolic groups [26; 27]. The other way uses bounded cohomology and the simplicial volume; see [11; 20; 21].

Suppose N is a closed oriented aspherical manifold whose fundamental group is nonelementary hyperbolic and $f: N \to N$ is a map of nonzero degree. By [26; 27] (see also [5, Lemma 4.2]), $\pi_1(N)$ is co-Hopfian (ie every injective endomorphism is an isomorphism), and so by the asphericity of N it suffices to show that f_* is injective. Suppose the contrary, and let x be a nontrivial element in $\ker(f_*)$. Since $f_*(\pi_1(N))$ has finite index in $\pi_1(N)$, there is some $n \in \mathbb{N}$ such that $x^n \in f_*(\pi_1(N))$, ie there is some $y \in \pi_1(N)$ such that $f_*(y) = x^n$. Clearly $x^n \neq 1$, because $\pi_1(N)$ is torsion-free, and so $y \notin \ker(f_*)$. Now, $f_*^2(y) = f_*(x^n) = 1$, which means that $y \in \ker(f_*^2)$. By iterating this process, we obtain a purely increasing sequence

$$\ker(f_*) \subsetneq \ker(f_*^2) \subsetneq \cdots \subsetneq \ker(f_*^m) \subsetneq \ker(f_*^{m+1}) \subsetneq \cdots.$$

But this sequence contradicts Sela's result [26; 27] that, for every endomorphism ψ of a torsion-free hyperbolic group, there exists $m_0 \in \mathbb{N}$ such that $\ker(\psi^k) = \ker(\psi^{m_0})$ for all $k \geq m_0$. We deduce that f_* is injective, and therefore an isomorphism as required.

Alternatively to the above argument, since $\pi_1(M)$ is nonelementary hyperbolic, the comparison map from bounded cohomology to ordinary cohomology

$$\psi_{\pi_1(M)}: H_b^n(\pi_1(M); \mathbb{R}) \to H^n(\pi_1(M); \mathbb{R})$$

is surjective; see [20; 21; 12]. Thus, by the duality of the simplicial ℓ^1–seminorm and the bounded cohomology ℓ^∞–seminorm (see [11]), we deduce that M has positive simplicial volume. This implies that every nonzero degree map $f: M \to M$ has degree ± 1. In particular, f is π_1–surjective, and thus an isomorphism, because $\pi_1(M)$ is Hopfian [19; 27]. □

3.2 Fundamental group and finite covers

Let $M \xrightarrow{\pi} N$ be an oriented circle bundle, where N is a closed oriented aspherical manifold with $\pi_1(N)$ hyperbolic. We may assume that $\dim(N) \geq 2$, otherwise we deal with the well-known case of T^2. The fundamental group of M fits into the central extension (see [4; 8])

$$1 \to C(\pi_1(M)) \to \pi_1(M) \xrightarrow{\pi_*} \pi_1(N) \to 1,$$

where $C(\pi_1(M)) = \mathbb{Z}$. (Note that $C(\pi_1(N)) = 1$ because $\pi_1(N)$ is torsion-free nonelementary hyperbolic.)

It is easy to observe that every finite covering of M is of the same type. More precisely:

Lemma 3.2 [22, Lemma 4.6] *Every finite cover $\overline{M} \xrightarrow{p} M$ is a circle bundle $\overline{M} \xrightarrow{\bar{\pi}} \overline{N}$, where $\overline{N} \xrightarrow{\bar{p}} N$ is a finite covering.*

In particular, p is a generalized bundle map covering $\bar{p}$ and the (infinite cyclic) center of $\pi_1(\overline{M})$ is mapped under p_* into the center of $\pi_1(M)$.

3.3 Reduction to fiberwise covering maps

Now, let $f: M \to M$ be a map of nonzero degree. We observe that f is homotopic to a fiberwise covering map:

Proposition 3.3 *The map f is homotopic to a fiberwise covering where the induced map $f_{S^1}: S^1 \to S^1$ has degree $\pm\deg(f)$.*

Proof Consider the composite map $\pi \circ f: M \to N$ and the induced homomorphism

$$(\pi \circ f)_*: \pi_1(M) \to \pi_1(N).$$

Since the center of $\pi_1(N)$ is trivial, we derive, after lifting f to a π_1–surjective map $\bar{f}: M \to \overline{M}$ (where $\overline{M} \xrightarrow{p} M$ corresponds to $f_*(\pi_1(M))$), that the center of $\pi_1(M)$ is mapped under $(\pi \circ f)_*$ to the trivial element of $\pi_1(N)$; see Lemma 3.2 and the lines above that. Thus f factors, up to homotopy, through a self-map $g: N \to N$, ie $\pi \circ f = g \circ \pi$ (up to homotopy).

Clearly $\deg(g) \neq 0$, otherwise f would factor through the degree-zero map from the pullback bundle of g along π to M, which is impossible because $\deg(f) \neq 0$. Now, Proposition 3.1 implies that g is a homotopy equivalence of N (in particular $\deg(g) = \pm 1$). Hence, the induced map f_{S^1} on the S^1 fiber is homotopic to a self-covering of degree

$$\deg(f_{S^1}) = \pm\deg(f). \qquad \square$$

Since every map of degree ± 1 is π_1–surjective, every self-map of M of degree ± 1 is a homotopy equivalence, thus answering Problem 1.1 in the affirmative. More interestingly, the above proposition gives the following strong affirmative answer to Problem 1.2 (see Remark 2.2):

Corollary 3.4 *Let M be an oriented circle bundle over a closed oriented aspherical manifold N with hyperbolic fundamental group and $\dim(N) \neq 3, 4$. Every self-map of M of nonzero degree is either homotopic to a homeomorphism or homotopic to a nontrivial covering.*

Consider now the iterates

$$f^m : M \to M, \quad m \geq 1.$$

By Proposition 3.3, each f^m is homotopic to a fiberwise covering of degree

$$(\deg(f))^m = [\pi_1(M) : f_*^m(\pi_1(M))].$$

That is, for each m, the homomorphism

$$f_*^m : \pi_1(M) \to \pi_1(M)$$

maps every element $x \in C(\pi_1(M)) = \mathbb{Z} = \langle z \rangle$ to $x^{\pm\deg(f^m)} \in C(\pi_1(M))$ and induces an isomorphism on $\pi_1(N) = \pi_*(\pi_1(M))$. In particular, when $\deg(f) > 1$, we obtain:

Corollary 3.5 *If $f : M \to M$ has degree greater than one, then there is a purely decreasing infinite sequence of subgroups of $\pi_1(M)$ given by*

$$\pi_1(M) \supsetneq f_*(\pi_1(M)) \supsetneq \cdots \supsetneq f_*^m(\pi_1(M)) \supsetneq f_*^{m+1}(\pi_1(M)) \supsetneq \cdots. \tag{2}$$

4 Distinguishing between trivial and nontrivial bundles

Now we show that the existence of sequence (2) implies that $\pi_1(M)$ has a finite-index subgroup which is isomorphic to a direct product, and thus M is virtually trivial. To this end, we construct a presentation of $\pi_1(M)$ by a product of two infinite-index subgroups.

4.1 Groups infinite-index presentable by products

An infinite group Γ is said to be *infinite-index presentable by products* or *IIPP* if there exist two infinite subgroups $\Gamma_1, \Gamma_2 \subset \Gamma$ that commute elementwise, such that $[\Gamma : \Gamma_i] = \infty$ for both Γ_i and the multiplication homomorphism

$$\Gamma_1 \times \Gamma_2 \to \Gamma$$

surjects onto a finite-index subgroup of Γ.

The notion of IIPP groups was introduced in [22] in the study of maps of nonzero degree from direct products to aspherical manifolds with nontrivial center. The concept of groups presentable by products (ie without the constraint on the index) was introduced in [15]. It is clear that when Γ is a *reducible* group, that is, virtually a product of two infinite groups, then Γ is IIPP. Thus, a natural problem is to determine when these two properties are equivalent. In general they are not equivalent, as shown in [22, Section 8], however, equivalence is achieved under certain assumptions:

Theorem 4.1 [22, Theorem D] *Suppose Γ fits into a central extension*

$$1 \to C(\Gamma) \to \Gamma \to \Gamma/C(\Gamma) \to 1,$$

where $\Gamma/C(\Gamma)$ is not presentable by products. Then Γ is IIPP if and only if it is reducible.

The following theorem characterizes aspherical circle bundles when the fundamental group of the base is not presentable by products:

Theorem 4.2 [22, Theorem C] *Let $M \xrightarrow{\pi} N$ be a circle bundle over a closed aspherical manifold N whose fundamental group $\pi_1(N)$ is not presentable by products. Then the following are equivalent*:

(i) *M admits a map of nonzero degree from a direct product.*

(ii) *M is finitely covered by a product $S^1 \times \overline{N}$ for some finite cover $\overline{N} \to N$.*

(iii) *$\pi_1(M)$ is reducible.*

(iv) *$\pi_1(M)$ is IIPP.*

Since nonelementary hyperbolic groups are not presentable by products [15], each circle bundle M over a closed aspherical manifold N with $\pi_1(N)$ hyperbolic fulfills the assumptions of Theorems 4.1 and 4.2. Using this, we will be able to deduce that M is virtually trivial. Furthermore, our presentation by products for $\pi_1(M)$ will have trivial kernel; see Remark 4.3.

4.2 An infinite-index presentation by products

Under the assumption of the existence of $f^m : M \to M$ with $\deg(f^m) = (\deg(f))^m > 1$ for all $m \geq 1$, and thus of sequence (2), we consider the subgroup of $\pi_1(M)$ defined by

$$G := \bigcap_m f_*^m(\pi_1(M)).$$

First, we observe the general fact (ie without using the specific forms of the $f_*^m(\pi_1(M))$) that G has infinite index in $\pi_1(M)$. Suppose, to the contrary, that $[\pi_1(M) : G] < \infty$. Then since

$$[\pi_1(M) : f_*^m(\pi_1(M))] \leq [\pi_1(M) : G]$$

for all m, and $\pi_1(M)$ contains only finitely many subgroups of a fixed index, we deduce that there exists n such that $f_*^n(\pi_1(M)) = f_*^k(\pi_1(M))$ for all $k \geq n$. This is, however, impossible by Corollary 3.5. Now we will show that $\pi_1(M)$ admits a presentation by the product $C(\pi_1(M)) \times G$. Let

$$\varphi\colon C(\pi_1(M)) \times G \to \pi_1(M) \tag{3}$$

be the multiplication map. Since each element of $C(\pi_1(M))$ commutes with every element of G, we deduce that φ is in fact a well-defined homomorphism.

We claim that φ surjects onto a finite-index subgroup of $\pi_1(M)$, ie that $C(\pi_1(M))G$ has finite index in $\pi_1(M)$. To this end, we will use the specific description of f^m and $f_*^m(\pi_1(M))$. In Section 3.3 we saw that, for every m, the composite f^m is a fiberwise covering of degree $\deg(f^m)$ on the fibers that induces an isomorphism on $\pi_1(N)$, and even more it induces a homotopy equivalence of N. In particular, for every $m \geq 1$, we obtain a short exact sequence

$$1 \to \langle z^{\deg(f)^m} \rangle \to f_*^m(\pi_1(M)) \to \pi_1(N_m) \to 1,$$

where $\pi_1(N_m) \cong \pi_1(N)$. Hence, $\pi_1(M)/f_*^m(\pi_1(M)) \cong \mathbb{Z}/\deg(f)^m\mathbb{Z}$ for all $m \geq 1$, and so $\pi_1(M)/G \cong \mathbb{Z}$. Thus, we obtain a short exact sequence (induced by π_*),

$$1 \to (C(\pi_1(M))G)/G \to \pi_1(M)/G \xrightarrow{\bar{\pi}_*} \pi_1(N)/\pi_*(G) \to 1.$$

Since $(C(\pi_1(M))G)/G \cong \mathbb{Z}$, we conclude that $\pi_*(G)$ is a finite-index subgroup of $\pi_1(N)$.

Now let $x \in \pi_1(M)$. If $x = z^s \in C(\pi_1(M)) = \langle z \rangle$, then $\varphi(x, 1) = x$. If $x \notin C(\pi_1(M))$, then $\pi_*(x)$ is not trivial in $\pi_1(N)$ and so there exists $t \geq 0$ such that $\pi_*(x^t) \in \pi_*(G)$, that is $\pi_*(x^t) = \pi_*(g)$ for some $g \in G$. Thus $x^t = z^a g$ for some $a \in \mathbb{Z}$, and so $\varphi(z^a, g) = x^t$. We conclude that $\varphi(C(\pi_1(M)) \times G) = C(\pi_1(M))G$ has finite index in $\pi_1(M)$.

Since moreover $C(\pi_1(M))$ and G have infinite index in $\pi_1(M)$, we conclude that the presentation given in (3) is an infinite-index presentation by products. Theorem 4.2 implies that $\pi_1(M)$ is reducible and M is virtually a trivial circle bundle.

This finishes the proof of Theorem 1.3.

Remark 4.3 The kernel of φ must be trivial because it is isomorphic to $C(\pi_1(M)) \cap G$, which is trivial. Thus $C(\pi_1(M))G$ is isomorphic to the fundamental group of a trivial circle bundle that covers M. In particular, the property of $\pi_1(N)$ that is not presentable by products was not necessary for our proof.

An alternative way to see that $C(\pi_1(M))\cap G$ is trivial is to observe that

$$[C(\pi_1(M)):C(\pi_1(M))\cap G]=[\pi_1(M):G]=\infty.$$

Together with the fact that $C(\pi_1(M))=\mathbb{Z}$, we conclude that $C(\pi_1(M))\cap G$ is trivial.

The proof of Corollary 2.1 is now straightforward:

Proof of Corollary 2.1 Let M be a circle bundle over a closed oriented aspherical manifold N with $\pi_1(N)$ hyperbolic and $f\colon M\to M$ be a map of nonzero degree. As we have seen in Section 3.3, if $\deg(f)=\pm1$ then f is a homotopy equivalence, and if $\deg(f)\neq\pm1$ then f is homotopic to a nontrivial fiberwise covering (and to a nontrivial covering when $\dim(N)\neq 3,4$; see Remark 2.2). In the latter case, Theorem 1.3 implies moreover that M is virtually $S^1\times\overline{N}$ for some finite covering $\overline{N}\to N$. □

Remark 4.4 When M has torsion Euler class $e\in H^2(N)$, we have seen in Example 1.4 that M admits a self-map f of degree greater than one. Recall that a product finite covering $S^1\times\overline{N}\to M$ is obtained by pulling back $M\xrightarrow{\pi}N$ along the finite covering $\overline{N}\to N$ that corresponds to the finite-index subgroup

$$H:=\ker(\pi_1(N)\xrightarrow{h}H_1(N)\xrightarrow{\pi_T}\operatorname{Tor}H_1(N))\subseteq\pi_1(N),$$

where h denotes the Hurewicz map and π_T is the projection to the torsion of $H_1(N)$. (Note that e lies in $\operatorname{Tor}H_1(M)$ by the universal coefficient theorem.) The groups H and $\pi_*(G)$ are commensurable in $\pi_1(N)$ because

$$[\pi_*(G):\pi_*(G)\cap H]\leq[\pi_1(N):H]<\infty$$

and

$$[H:\pi_*(G)\cap H]\leq[\pi_1(N):\pi_*(G)]<\infty.$$

5 The Hopf property and strong version of Hopf's problem

In this section we discuss the Hopf property for circle bundles over aspherical manifolds with hyperbolic fundamental groups and Problem 1.2 more generally.

5.1 The Hopf property

First, we show that the fundamental groups of circle bundles over aspherical manifolds with hyperbolic fundamental groups are Hopfian:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C(\pi_1(M)) & \longrightarrow & \pi_1(M) & \xrightarrow{\pi_*} & \pi_1(N) & \longrightarrow & 1 \\ & & \Big\downarrow \phi|_{C(\pi_1(M))} & & \Big\downarrow \phi & & \Big\downarrow \bar{\phi} & & \\ 1 & \longrightarrow & C(\pi_1(M)) & \longrightarrow & \pi_1(M) & \xrightarrow{\pi_*} & \pi_1(N) & \longrightarrow & 1 \end{array}$$

Figure 1: The Hopf property for $\pi_1(M)$.

Proposition 5.1 *If M is a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group, then every finite-index subgroup of $\pi_1(M)$ is Hopfian.*

Proof Let $M \xrightarrow{\pi} N$ be a circle bundle where N is a closed oriented aspherical manifold with $\pi_1(N)$ hyperbolic. (As before, we can assume that $\pi_1(N)$ is not cyclic.) Since every finite covering of M is of the same type (see Lemma 3.2), it suffices to show that $\pi_1(M)$ is Hopfian.

Let $\phi\colon \pi_1(M) \to \pi_1(M)$ be a surjective homomorphism. Then $\phi(C(\pi_1(M))) \subseteq C(\pi_1(M))$, and so the composite homomorphism $\pi_* \circ \phi\colon \pi_1(M) \to \pi_1(N)$ maps $C(\pi_1(M))$ to the trivial element of $\pi_1(N)$. In particular, there exists a surjective homomorphism $\bar{\phi}\colon \pi_1(N) \to \pi_1(N)$ such that $\bar{\phi} \circ \pi_* = \pi_* \circ \phi$. Now, $\bar{\phi}$ is injective as well (and so an isomorphism), because $\pi_1(N)$ is Hopfian, being hyperbolic and torsion-free [19; 27]. Then, again using the surjectivity of ϕ, we deduce that

$$\phi|_{C(\pi_1(M))}\colon C(\pi_1(M)) \to C(\pi_1(M))$$

is also surjective. Since $C(\pi_1(M)) = \mathbb{Z}$ is Hopfian, we conclude that $\phi|_{C(\pi_1(M))}$ is in fact an isomorphism. Then the five lemma for the commutative diagram in Figure 1 implies that ϕ is an isomorphism as well. □

In this way, we obtain also an alternative proof of the fact that every self-map of M of degree ± 1 is a homotopy equivalence. Of course, the above group-theoretic argument uses the same line of argument as the proof of Proposition 3.3, with the difference that it starts with a stronger assumption, namely that ϕ is surjective.

5.2 Infinite decreasing sequences and Problem 1.2

The fact that every finite-index subgroup of the fundamental group of a circle bundle over an aspherical manifold N with hyperbolic $\pi_1(N)$ has the Hopf property is actually conjectured to be true for all aspherical manifolds. Not only would this immediately verify Problem 1.1 for every aspherical manifold, it also gives evidence for an affirmative answer to Problem 1.2. Namely, let $f\colon M \to M$ be a map of degree $\deg(f) > 1$ and suppose that every finite-index subgroup of $\pi_1(M)$ is Hopfian. Then, as in the case of nontrivial coverings, there is a purely decreasing infinite sequence

$$\pi_1(M) \supsetneq f_*(\pi_1(M)) \supsetneq \cdots \supsetneq f_*^m(\pi_1(M)) \supsetneq f_*^{m+1}(\pi_1(M)) \supsetneq \cdots.$$

The proof of this claim can be found along the lines of the proof of Theorem 14.40 of [17], but let us give the details for completeness: Suppose to the contrary that there is some n such that $f_*^n(\pi_1(M)) = f_*^k(\pi_1(M))$ for all $k \geq n$. Let $M_n \xrightarrow{p_n} M$ be the finite covering of M corresponding to $f_*^n(\pi_1(M))$ and denote by $\overline{f^n}\colon M \to M_n$ the lift of f^n, which induces a surjection on π_1. Since $f_*^n(\pi_1(M)) = f_*^{2n}(\pi_1(M))$, we deduce that the composite map $\overline{f^n} \circ p_n\colon M_n \to M_n$ induces a surjection

$$(\overline{f^n} \circ p_n)_*\colon \pi_1(M_n) \to \pi_1(M_n).$$

Since $\pi_1(M_n)$ is Hopfian, we deduce that $(\overline{f^n} \circ p_n)_*$ is an isomorphism, and so a homotopy equivalence, because M_n is aspherical. In particular,

$$\deg(\overline{f^n}), \deg(p_n) \in \{\pm 1\},$$

which leads to the absurd conclusion that $\deg(f) = \pm 1$.

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