AGT

Algebraic & Geometric Topology

Volume 23 (2023)

Milnor–Witt motivic cohomology of complements of hyperplane arrangements

Keyao Peng

msp

Algebraic & Geometric Topology **23**:8 (2023) 3531–3552
DOI: 10.2140/agt.2023.23.3531
Published: 5 November 2023

Milnor–Witt motivic cohomology of complements of hyperplane arrangements

KEYAO PENG

We compute the (total) Milnor–Witt motivic cohomology of the complement of a hyperplane arrangement in an affine space as an algebra with given generators and relations. We also obtain some corollaries by realization to classical cohomology.

14C25, 14F42, 19E15

1 Introduction

Let K be a perfect field of characteristic different from 2, and let $U \subset \mathbb{A}_K^N$ be the complement of a finite union of hyperplanes. For $K = \mathbb{R}$, the cohomology ring $H^*_{\text{sing}}(U(\mathbb{R}), \mathbb{Z})$ is just the direct sum of $\mathbb{Z}$ corresponding to each regions (connected components), and those regions form a poset. In the special case when the hyperplanes arise from a root system, the resulting poset is the corresponding Weyl group with the weak Bruhat order. In general, the poset of regions is ranked by the number of separating hyperplanes and its Möbius function has been computed; see Edelman [8].

For any essentially smooth scheme X over K and any integers $p, q \in \mathbb{Z}$, one can define the Milnor–Witt (MW) motivic cohomology groups $H^{p,q}_{\text{MW}}(X, \mathbb{Z})$ introduced by Bachmann, Calmès, Déglise, Fasel and Østvær [1]. There are homomorphisms (functorial in X), for any $p, q \in \mathbb{Z}$,

$$H^{p,q}_{\text{MW}}(X, \mathbb{Z}) \to H^{p,q}_{M}(X, \mathbb{Z}),$$

where the right-hand side denotes the ordinary motivic cohomology of Voevodsky.

As illustrated by the list of properties in the following section, the Milnor–Witt motivic cohomology groups behave in a fashion similar to ordinary motivic cohomology groups, but there are crucial differences (for instance, there are no reasonable Chern classes).

In this paper, we compute the total Milnor–Witt cohomology ring of the complement of a hyperplane arrangement in affine spaces $H_{\mathrm{MW}}(U)$ using methods very similar to Chatzistamatiou [4], with some necessary modifications. To state our main result, we first recall a few facts.

Let R be a commutative ring. The Milnor–Witt K–theory of R is defined to be the graded algebra freely generated by elements of degree 1 of the form $[a]$ with $a \in R^\times$ and an element η in degree -1, subject to the relations

(1) $[a][1-a] = 0$ for any a such that $a, 1-a \in R^\times \setminus \{1\}$;

(2) $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in R^\times$;

(3) $\eta[a] = [a]\eta$ for any $a \in R^\times$;

(4) $\eta(2 + \eta[-1]) = 0$.

It defines a presheaf on the category of schemes over a perfect field K via $X \mapsto K_*^{\mathrm{MW}}(\mathcal{O}(X))$. On the other hand, one can also consider the Milnor–Witt motivic cohomology (bigraded) presheaf

$$X \mapsto H_{\mathrm{MW}}(X) := \bigoplus_{p,q} H_{\mathrm{MW}}^{p,q}(X, \mathbb{Z}).$$

By Déglise and Fasel [7, Theorem 4.2.2], there is a morphism of presheaves

$$s\colon \bigoplus_{n\in\mathbb{Z}} K_n^{\mathrm{MW}}(-) \to \bigoplus_{n\in\mathbb{Z}} H_{\mathrm{MW}}^{n,n}(-, \mathbb{Z}) \subset H_{\mathrm{MW}}(X),$$

which specializes to the above isomorphism if $X = \mathrm{Spec}(F)$, where F is a finitely generated field extension of K; see Calmès and Fasel [3].

Theorem 1.1 *Let K be a perfect field of characteristic different from 2 and let $U \subset \mathbb{A}_K^N$ be the complement of a finite union of hyperplanes. There is an isomorphism of $H_{\mathrm{MW}}(K)$–algebras*

$$H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \cong H_{\mathrm{MW}}(U)$$

defined by mapping $(f) \in \mathbb{G}_m(U)$ to the class $[f]$ in $H_{\mathrm{MW}}^{1,1}(U, \mathbb{Z})$ corresponding to f under s. Here, $H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$ is the free (associative) graded $H_{\mathrm{MW}}(K)$–algebra generated by $\mathbb{G}_m(U)$ in degree $(1,1)$ and J_U is the ideal generated by the elements

(1) $(f) - [f]$ *if* $f \in K^\times \subset \mathbb{G}_m(U)$;

(2) $(f) + (g) + \eta(f)(g) - (fg)$ *if* $f, g \in \mathbb{G}_m(U)$;

(3) $(f_1)(f_2)\cdots(f_t)$ *for any* $f_1, \dots, f_t \in \mathbb{G}_m(U)$ *such that* $\sum_{i=1}^t f_i = 1$;

(4) $(f)^2 - [-1](f)$ *if* $f \in \mathbb{G}_m(U)$.

As indicated above, this theorem and its proof are inspired by the computation of the (ordinary) motivic cohomology of U in [4]. We can recover the main theorem [4, Theorem 3.5] of the motivic cohomology case by taking $\eta = 0$. As a corollary, we obtain the following result:

Corollary 1.2 *Let $U \subset \mathbb{A}^N_K$ be the complement of a finite union of hyperplanes. The isomorphism of Theorem 1.1 induces an isomorphism*

$$\bigoplus_{n\in\mathbb{Z}} K_n^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \to \bigoplus_{n\in\mathbb{Z}} H^{n,n}_{\mathrm{MW}}(U,\mathbb{Z}).$$

We do not know if the left-hand side coincides with $K_*^{\mathrm{MW}}(U)$. To conclude, we spend a few lines on the real realization homomorphism

$$H_{\mathrm{MW}}(U,\mathbb{Z}) \to H^*_{\mathrm{sing}}(U(\mathbb{R}),\mathbb{Z})$$

when U is over $K = \mathbb{R}$. We prove in particular that both sides have essentially the same generators, and that the map is surjective.

Conventions The base field K is assumed to be perfect and of characteristic not 2. For a scheme X over K, we write $H_{\mathrm{MW}}(X)$ for the total MW motivic cohomology ring $\bigoplus_{p,q\in\mathbb{Z}} H^{p,q}_{\mathrm{MW}}(X,\mathbb{Z})$.

For each $f \in \mathbb{G}_m(U)$, we use (f) to indicate the corresponding generator in the corresponding free algebra (eg $K_n^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$) and $[f]$ to indicate the corresponding element in the cohomology group (eg $H^{1,1}_{\mathrm{MW}}(U,\mathbb{Z})$).

2 Milnor–Witt motivic cohomology

In this section, we define Milnor–Witt motivic cohomology and state some properties that will be used in the proof of Theorem 1.1. We start with the (big) category of motives $\widetilde{\mathrm{DM}}(K) := \widetilde{\mathrm{DM}}_{\mathrm{Nis}}(K,\mathbb{Z})$ defined in [7, Definition 3.3.2] and the functor

$$\widetilde{M}\colon \mathrm{Sm}/K \to \widetilde{\mathrm{DM}}(K).$$

The category $\widetilde{\mathrm{DM}}(K)$ is symmetric monoidal [7, Proposition 3.3.4] with unit $\mathbb{1} = \widetilde{M}(\mathrm{Spec}(K))$. For any integers $p, q \in \mathbb{Z}$, we obtain MW motivic cohomology groups

$$H^{p,q}_{\mathrm{MW}}(X,\mathbb{Z}) := \mathrm{Hom}_{\widetilde{\mathrm{DM}}(K)}(\widetilde{M}(X), \mathbb{1}(q)[p]).$$

By [7, Proposition 4.1.2], motivic cohomology groups can be computed as the Zariski hypercohomology groups of explicit complexes of sheaves.

We will make use of the following property of $\widetilde{\mathrm{DM}}(K)$. First, we note that $\widetilde{\mathrm{DM}}(K)$ is also a triangulated category.

Proposition 2.1 (Gysin triangle) *Let X be a smooth K–scheme, let $Z \subset X$ be a smooth closed subscheme of codimension c and let $U = X \setminus Z$. Suppose that the normal cone $N_X Z$ admits a trivialization $\phi\colon N_X Z \cong Z \times \mathbb{A}^c$. Then there is a Gysin triangle*

$$\widetilde{M}(U) \to \widetilde{M}(X) \to \widetilde{M}(Z)(c)[2c] \xrightarrow{+1},$$

where the last two arrows depend on the choice of ϕ.

Proof We have an adjunction of triangulated categories

$$\mathrm{SH}(K) \leftrightarrows \widetilde{\mathrm{DM}}(K)$$

obtained by combining the adjunction of [6, Section 4.1] and the classical Dold–Kan correspondence (eg [5, 5.3.35]). Here, $\mathrm{SH}(K)$ is the stable homotopy category of smooth schemes over K. The functor $\mathrm{SH}(K) \to \widetilde{\mathrm{DM}}(K)$ being exact, the statement follows for instance from [13, Chapter 3, Theorem 2.23]. $\square$

Furthermore, the Milnor–Witt motivic cohomology groups satisfy most of the formal properties of ordinary motivic cohomology and were computed in a few situations:

(1) If $q \leq 1$, there are canonical isomorphisms

$$H^{p,q}_{\mathrm{MW}}(X, \mathbb{Z}) \cong H^{p-q}_{\mathrm{Nis}}(X, \boldsymbol{K}^{\mathrm{MW}}_q) \cong H^{p-q}_{\mathrm{Zar}}(X, \boldsymbol{K}^{\mathrm{MW}}_q)$$

where $\boldsymbol{K}^{\mathrm{MW}}_q$ is the unramified Milnor–Witt K–theory sheaf (in weight q) introduced in [12].

(2) If L/K is a finitely generated field extension there are isomorphisms $H^{n,n}_{\mathrm{MW}}(L, \mathbb{Z}) \cong K^{\mathrm{MW}}_n(L)$ fitting in a commutative diagram, for any $n \in \mathbb{Z}$,

$$\begin{tikzcd} H^{n,n}_{\mathrm{MW}}(L, \mathbb{Z}) \arrow[r, "\sim"] \arrow[d] & K^{\mathrm{MW}}_n(L) \arrow[d] \\ H^{n,n}_{M}(L, \mathbb{Z}) \arrow[r, "\sim"] & K^{M}_n(L) \end{tikzcd}$$

where $K^M_n(L)$ is the (n^{th}) Milnor K–theory group of L, the bottom horizontal map is the isomorphism of Suslin, Nesterenko and Totaro, and the right-hand vertical map is the natural homomorphism from Milnor–Witt K–theory to Milnor K–theory. This result has the following consequence: the Milnor–Witt motivic cohomology groups are computed via an explicit complex of Nisnevich sheaves $\widetilde{\mathbb{Z}}(q)$ for any integer $q \in \mathbb{Z}$. The above result shows that there is a morphism of complexes of sheaves

$$\widetilde{\mathbb{Z}}(q) \to K_q^{\mathrm{MW}}[-q],$$

where the right-hand side is the complex whose only nontrivial sheaf is K_q^{MW} in degree $-q$. For any essentially smooth scheme X over K, this yields group homomorphisms

$$H_{\mathrm{MW}}^{p,q}(X,\mathbb{Z}) \to H^{p-q}(X,\boldsymbol{K}_q^{\mathrm{MW}}),$$

which are compatible with the ring structure on both sides. In the particular case $p=2n$ and $q=n$ for some $n \in \mathbb{Z}$, we obtain isomorphisms (functorial in X)

$$H_{\mathrm{MW}}^{2n,n}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{CH}}^n(X),$$

where the right-hand term is the n^{th} Chow–Witt group of X (defined in [2; 9]). Again, these isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H_{\mathrm{MW}}^{2n,n}(X,\mathbb{Z}) & \xrightarrow{\sim} & \widetilde{\mathrm{CH}}^n(X) \\ \downarrow & & \downarrow \\ H_M^{2n,n}(X,\mathbb{Z}) & \xrightarrow{\sim} & \mathrm{CH}^n(X) \end{array}$$

where the right-hand vertical homomorphism is the natural map from Chow–Witt groups to Chow groups.

(3) The total Milnor–Witt motivic cohomology has Borel classes for symplectic bundles [15] but in general the projective bundle theorem fails [14].

(4) If X is a smooth scheme over $\mathbb{R}$, there are two interesting realization maps. On the one hand, one may consider the composite

$$H_{\mathrm{MW}}^{p,q}(X,\mathbb{Z}) \to H_M^{p,q}(X,\mathbb{Z}) \to H_{\mathrm{sing}}^{p}(X(\mathbb{C}),\mathbb{Z}),$$

where the right-hand map is the complex realization map. On the other hand, one may also consider the composite

$$H_{\mathrm{MW}}^{p,q}(X,\mathbb{Z}) \to H^{p-q}(X,\boldsymbol{K}_q^{\mathrm{MW}}) \to H^{p-q}(X,\boldsymbol{I}^q) \to H_{\mathrm{sing}}^{p-q}(X(\mathbb{R}),\mathbb{Z}),$$

where $\boldsymbol{I}^q$ is the unramified sheaf associated to the q^{th} power of the fundamental ideal in the Witt ring, $\boldsymbol{K}_q^{\mathrm{MW}} \to \boldsymbol{I}^q$ is the canonical projection and $H^{p-q}(X,\boldsymbol{I}^q) \to H_{\mathrm{sing}}^{p-q}(X(\mathbb{R}),\mathbb{Z})$ is Jacobson's signature map [11].

We note here that these two realization maps show that Milnor–Witt motivic cohomology is in some sense the analogue of both the singular cohomology of the complex and the real points of X.

3 Basic structure of the cohomology ring

Let V be an affine space, ie $V \cong \mathbb{A}^N_K$ for some $N \in \mathbb{N}$. We consider finite families I of hyperplanes in V (which we suppose are distinct). We denote by $|I|$ the cardinality of I and set $U_I^V := V \setminus \left(\bigcup_{Y \in I} Y\right)$, and simply write U_I^N when $V = \mathbb{A}^N_K$. For any hyperplane Y, we put $I_Y := \{Y_i \cap Y \mid Y_i \in I,\ Y_i \neq Y\}$.

Proposition 3.1 *Let V and I be as above. We have*

$$\widetilde{M}(U_I^V) \cong \bigoplus_{j \in J} \mathbb{1}(n_j)[n_j]$$

for some set J and integers $n_j \geq 0$.

Proof We proceed by induction on the dimension N of V and $|I|$. If $|I| = 0$, then $\widetilde{M}(U_I^V) = \widetilde{M}(V) \cong \mathbb{1}$ and we are done. So let $|I| \geq 1$ and $Y \in I$. The Gysin triangle reads as

$$\widetilde{M}(U_I^V) \to \widetilde{M}(U^V_{I \setminus \{Y\}}) \xrightarrow{\phi} \widetilde{M}(U^Y_{I_Y})(1)[2] \xrightarrow{+1}. \tag{3-1}$$

If $\phi = 0$, then the triangle is split and consequently we obtain an isomorphism

$$\widetilde{M}(U_I^V) \cong \widetilde{M}(U^V_{I \setminus \{Y\}}) \oplus \widetilde{M}(U^Y_{I_Y})(1)[1]. \tag{3-2}$$

Since $|I \setminus \{Y\}| < |I|$ and $\dim(Y) = \dim(V) - 1$, we conclude by induction that the right-hand side has the correct form. We are then reduced to showing that $\phi = 0$.

By induction,

$$\begin{aligned}\phi \in \mathrm{Hom}_{\widetilde{\mathrm{DM}}(K)}&(\widetilde{M}(U^V_{I \setminus \{Y\}}), \widetilde{M}(U^Y_{I_Y})(1)[2]) \\ &\cong \bigoplus_{j,k} \mathrm{Hom}_{\widetilde{\mathrm{DM}}(K)}(\mathbb{1}(n_j)[n_j], \mathbb{1}(m_k)[m_k+1])\end{aligned}$$

for some integers $n_j, m_k \geq 0$, so it suffices to prove that $\mathrm{Hom}_{\widetilde{\mathrm{DM}}(K)}(\mathbb{1}, \mathbb{1}(m)[m+1]) = 0$ for any $m \in \mathbb{Z}$ to conclude. Now,

$$\mathrm{Hom}_{\widetilde{\mathrm{DM}}(K)}(\mathbb{1}, \mathbb{1}(m)[m+1]) = H^{m+1,m}_{\mathrm{MW}}(K, \mathbb{Z})$$

and the latter is trivial by [7, Proposition 4.1.2 and proof of Theorem 4.2.4]. □

As an immediate corollary, we obtain the following result:

Corollary 3.2 *The motivic cohomology $H_{\mathrm{MW}}(U_I^V)$ is a finitely generated, free $H_{\mathrm{MW}}(K)$–module.*

To obtain more precise results, we now study the Gysin (split) triangle (3-1) in more detail. We can rewrite it as

$$\widetilde{M}(U_{I_Y}^Y)(1)[1] \xrightarrow{\beta^Y} \widetilde{M}(U_I^V) \xrightarrow{\alpha^Y} \widetilde{M}(U_{I\setminus\{Y\}}^V) \xrightarrow{0}$$

and therefore we obtain the short (split) exact sequence, in which the morphisms are induced by the first two morphisms in the triangle,

$$(3\text{-}3)\quad 0 \to \bigoplus_{p,q} H_{\mathrm{MW}}^{p,q}(U_{I\setminus\{Y\}}^V, \mathbb{Z}) \xrightarrow{\alpha_*^Y} \bigoplus_{p,q} H_{\mathrm{MW}}^{p,q}(U_I^V, \mathbb{Z}) \xrightarrow{\beta_*^Y} \bigoplus_{p,q} H_{\mathrm{MW}}^{p-1,q-1}(U_{I_Y}^Y, \mathbb{Z}) \to 0.$$

The inclusion $Y \subset V$ yields a morphism $U_{I_Y}^Y \to U_{I\setminus\{Y\}}^V$ and therefore a morphism $\iota\colon \widetilde{M}(U_{I_Y}^Y) \to \widetilde{M}(U_{I\setminus\{Y\}}^V)$. The global section f of V corresponding to the equation of Y becomes invertible in U_I^V and therefore yields a morphism $[f]\colon \widetilde{M}(U_I^V) \to \mathbb{1}(1)[1]$ corresponding to the class $[f] \in H_{\mathrm{MW}}^{1,1}(U_I^V, \mathbb{Z})$ given by the morphism

$$s\colon \bigoplus_{n\in\mathbb{Z}} K_n^{\mathrm{MW}}(-) \to \bigoplus_{n\in\mathbb{Z}} H_{\mathrm{MW}}^{n,n}(-, \mathbb{Z}).$$

Lemma 3.3 *The following diagram commutes*:

$$\begin{tikzcd}
\widetilde{M}(U_{I_Y}^Y)(1)[1] \arrow[r, "\iota(1)[1]"] \arrow[d, "\beta^Y"'] & \widetilde{M}(U_{I\setminus\{Y\}}^V)(1)[1] \\
\widetilde{M}(U_I^V) \arrow[r, "\Delta"'] & \widetilde{M}(U_I^V) \otimes \widetilde{M}(U_I^V) \arrow[u, "\alpha^Y \otimes [f]"']
\end{tikzcd}$$

Proof The commutative diagram of schemes

$$\begin{tikzcd}
U_I^V \arrow[r] \arrow[d] & U_{I\setminus\{Y\}}^V \arrow[d, "(\mathrm{Id}, f)"] \\
U_{I\setminus\{Y\}}^V \times \mathbb{G}_m \arrow[r] & U_{I\setminus\{Y\}}^V \times \mathbb{A}_K^1
\end{tikzcd}$$

yields a morphism of Gysin triangles and thus a commutative diagram

$$\begin{tikzcd}
\widetilde{M}(U^Y_{I_Y})(1)[1] \arrow[r, "\beta^Y"] \arrow[d, "\iota(1)[1]"'] & \widetilde{M}(U^V_I) \arrow[r, "\alpha^Y"] \arrow[d] & \widetilde{M}(U^V_{I\setminus\{Y\}}) \arrow[r] \arrow[d] & \cdots \\
\widetilde{M}(U^V_{I\setminus\{Y\}})(1)[1] \arrow[r] \arrow[dr, equal] & \widetilde{M}(U^V_{I\setminus\{Y\}} \times \mathbb{G}_m) \arrow[r] \arrow[d] & \widetilde{M}(U^V_{I\setminus\{Y\}} \times \mathbb{A}^1_K) \arrow[r] & \cdots \\
 & \widetilde{M}(U^V_{I\setminus\{Y\}})(1)[1] & &
\end{tikzcd}$$

in which the map $\widetilde{M}(U^V_{I\setminus\{Y\}} \times \mathbb{G}_m) \to \widetilde{M}(U^V_{I\setminus\{Y\}})(1)[1]$ is just the projection. We conclude by observing that the middle vertical composite is just $(\alpha^Y \otimes [f]) \circ \Delta$. □

We may now prove the main result of this section.

Proposition 3.4 *The cohomology ring $H_{\mathrm{MW}}(U)$ is generated by the classes of units in U as an $H_{\mathrm{MW}}(K)$–algebra. In particular, the homomorphism*

$$s\colon \bigoplus_{n\in\mathbb{Z}} K^{\mathrm{MW}}_n(U) \to \bigoplus_{n\in\mathbb{Z}} H^{n,n}_{\mathrm{MW}}(U,\mathbb{Z})$$

is surjective.

Proof We again prove the result by induction on $|I|$ and the dimension of V, the case $|I| = 0$ being obvious. Suppose then that the result holds for $U^Y_{I_Y}$ and $U^V_{I\setminus\{Y\}}$ and consider the split sequence (3-3). For any $x \in H_{\mathrm{MW}}(U) = H_{\mathrm{MW}}(U^V_I)$, we have that $\beta^Y_*(x) \in H_{\mathrm{MW}}(U^Y_{I_Y})$ is in the subalgebra generated by $\{[f] \mid f \in \mathbb{G}_m(U^Y_{I_Y})\}$ and η. For any $f_1,\dots,f_n \in \mathbb{G}_m(U^V_{I\setminus\{Y\}})$, Lemma 3.3 yields

$$\beta^Y_*\big([(f_1)|_{U^V_I}]\cdots[(f_n)|_{U^V_I}]\cdot[t]\big) = [(f_1)|_{U^Y_{I_Y}}]\cdots[(f_n)|_{U^Y_{I_Y}}].$$

The map $\mathbb{G}_m(U^V_{I\setminus\{Y\}}) \to \mathbb{G}_m(U^Y_{I_Y})$ being surjective, it follows that there exists $x' \in H_{\mathrm{MW}}(U^V_I)$ in the subalgebra generated by units such that $\beta^Y_*(x - x') = 0$. Thus, $x - x' = \alpha_*(y)$ for some $y \in H_{\mathrm{MW}}(U^V_{I\setminus\{Y\}})$ and the result follows from the fact that α_* is just induced by the inclusion $U^V_I \subset U^V_{I\setminus\{Y\}}$. □

4 Relations in the cohomology ring

The purpose of this section is to prove that the relations of Theorem 1.1 hold in $H_{\mathrm{MW}}(U)$. The first two relations are obviously satisfied since the homomorphism is induced by the ring homomorphism

$$s\colon \bigoplus_{n\in\mathbb{Z}} K_n^{\mathrm{MW}}(U) \to \bigoplus_{n\in\mathbb{Z}} H_{\mathrm{MW}}^{n,n}(U,\mathbb{Z}).$$

Recall now that the last two relations are

(3) $[f_1][f_2]\cdots[f_t]$ if $f_i \in \mathbb{G}_m(U)$ for any i and $\sum_{i=1}^t f_i = 1$;

(4) $[f]^2 - [-1][f]$ if $f \in \mathbb{G}_m(U)$.

We will prove that they are equal to 0 in $H_{\mathrm{MW}}(U)$. Actually, it will be more convenient to work with the following relations, where $\epsilon := -\langle -1\rangle = -1 - \eta[-1]$:

(3′) $R(f_0,\dots,f_t)$, defined by

$$\sum_{i=0}^{t} \epsilon^{t+i}[f_0]\cdots\widehat{[f_i]}\cdots[f_t] + \sum_{0\le i_0<\cdots<i_k\le t} (-1)^k[-1]^k[f_0]\cdots\widehat{[f_{i_0}]}\cdots\widehat{[f_{i_k}]}\cdots[f_t]$$

for $f_i \in \mathbb{G}_m(U)$ such that $\sum_{i=0}^t f_i = 0$.

(4′) **Anticommutativity** $[f][g] - \epsilon[g][f]$.

Lemma 4.1 *The two groups of relations are equivalent in $H_{\mathrm{MW}}(U)$.*

Proof We first assume that (3) and (4) are satisfied. Since (1) and (2) are satisfied, we have $[-f] = [-1] + \langle -1\rangle[f]$. As (4) is satisfied and $[-1] = \epsilon[-1]$ in $K_*^{\mathrm{MW}}(K)$,

$$[-f][f] = [-1][f] + \langle -1\rangle[f]^2 = \epsilon([-1][f] - [f]^2) = 0$$

and then $[fg][-fg] = [f][g] + \epsilon[g][f]$ for any $g, f \in \mathbb{G}_m(U)$ by [12, proof of Lemma 3.7]. Suppose next that $\sum_{i=0}^t f_i = 0$, so that $\sum_{i=1}^t f_i/(-f_0) = 1$. Combining (3) and the anticommutativity law, we obtain

$$0 = [1] = [f_j^{-1}] + \langle f_j^{-1}\rangle[f_j] \qquad \text{(by (2))}, \tag{4-1}$$

$$\begin{aligned}\left[\frac{-f_i}{f_j}\right] &= \langle f_j^{-1}\rangle[-f_i] + [f_j^{-1}] \\ &= \langle f_j^{-1}\rangle([-f_i] - [f_j]) \qquad \text{(by (4-1))} \\ &= \langle f_j^{-1}\rangle(\langle -1\rangle[f_i] + [-1] - [f_j]),\end{aligned} \tag{4-2}$$

$$
\begin{aligned}
\text{(4-3)} \qquad ([f_0]-[-1])^k &= \sum_{i=0}^{k}\binom{k}{i}[-1]^{k-i}[f_0]^i \\
&= \Bigl(\sum_{i=0}^{k-1}\binom{k}{i}\Bigr)[-1]^{k-1}[f_0]+(-1)^k[-1]^k \qquad \text{(by (4))} \\
&= (-1)^{k-1}[-1]^{k-1}[f_0]+(-1)^k[-1]^k
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= (-\langle f_0\rangle)^t\Bigl[\frac{-f_1}{f_0}\Bigr]\Bigl[\frac{-f_2}{f_0}\Bigr]\cdots\Bigl[\frac{-f_t}{f_0}\Bigr] \qquad \text{(by (3))} \\
&= \bigl([f_0]-[-1]-\langle -1\rangle[f_1]\bigr)\cdots\bigl([f_0]-[-1]-\langle -1\rangle[f_t]\bigr) \qquad \text{(by (4-2))} \\
&= \epsilon^t\widehat{[f_0]}[f_1]\cdots[f_t]+\sum_{i=1}^{t}\epsilon^{t-1}[f_1]\cdots\widehat{[f_i]}([f_0]-[-1])\cdots[f_t] \\
&\qquad\qquad +\sum_{i<j}([f_0]-[-1])^2[f_1]\cdots\widehat{[f_i]}\cdots\widehat{[f_j]}\cdots[f_t]+\cdots \\
&= \sum_{i=0}^{t}\epsilon^{t+i}[f_0]\cdots\widehat{[f_i]}\cdots[f_t] \\
&\qquad +\sum_{0\le i_0<\cdots<i_k\le t}(-1)^k[-1]^k[f_0]\cdots\widehat{[f_{i_0}]}\cdots\widehat{[f_{i_k}]}\cdots[f_t] \qquad \text{(by (4-3))} \\
&= R(f_0,\ldots,f_t).
\end{aligned}
$$

Conversely, suppose that ($3'$) and ($4'$) hold. A direct calculation shows that

$$R(-1,f_1,\ldots,f_t)=(-\langle -1\rangle)^t[f_1]\cdots[f_t]=\epsilon^t[f_1]\cdots[f_t],$$

and consequently that (3) also holds. For every field $K\neq\mathbb{F}_2$, we have $1+a+b=0$ for some $a,b\neq 0$ and it follows from $[-a][-b]=0$ in $K_*^{\mathrm{MW}}(K)$ that

$$
\begin{aligned}
R(f,af,bf) &= R(f,af,bf)-[-a][-b]=R(f,af,bf)-\Bigl[-\frac{af}{f}\Bigr]\Bigl[-\frac{bf}{f}\Bigr] \\
&= R(f,af,bf)-(\langle -1\rangle[af]+[-1]-[f])(\langle -1\rangle[bf]+[-1]-[f]) \\
&= -[-1][f]+[-1]^2-([f]-[-1])^2=[-1][f]-[f]^2. \qquad \square
\end{aligned}
$$

Remark 4.2 The following properties of the relations R and anticommutativity hold:

(1) For any $a,b\in\mathbb{G}_m(U)$, we have $[a/b]=-\langle b^{-1}\rangle R(b,-a)$.

(2) For any $f_0,\dots,f_t \in \mathbb{G}_m(U)$, by direct computation, we have
$$R(f_0,\dots,f_t)-\epsilon^i[f_i]R(f_0,\dots,\hat{f_i},\dots,f_t)=P(f_0,\dots,\hat{f_i},\dots,f_t)$$
for some polynomial P. This uses the anticommutativity and the fact that $[-1]=\epsilon^j[-1]$ for any $j\ge 0$ in the computation.

(3) For any $f_0,\dots,f_t \in K^\times$ such that $\sum_{i=0}^t f_i=0$, we have $R(f_0,\dots,f_t)=0$ in $K_*^{\mathrm{MW}}(K)$.

The following lemma will prove useful in the proof of the main theorem:

Lemma 4.3 *Any morphism $\phi\colon \widetilde{M}(U_I^V)\to T$ in $\widetilde{\mathrm{DM}}(K)$ such that*
$$\widetilde{M}(U_{I_Y}^Y)(1)[1]\xrightarrow{\beta^Y}\widetilde{M}(U_I^V)\xrightarrow{\phi}T$$
is trivial for every $Y\in I$ factors through $\widetilde{M}(K)$, ie there is a morphism $\psi:\widetilde{M}(K)\to T$ such that the diagram
$$\begin{tikzcd} \widetilde{M}(U_I^V) \arrow[r, "\phi"] \arrow[d] & T \\ \widetilde{M}(K) \arrow[ur, "\psi"'] & \end{tikzcd}$$
is commutative.

Proof We prove as usual the result by induction on $|I|$, the result being trivial if $|I|=0$, ie if $U_I^V\cong\mathbb{A}_K^N$. By assumption, ϕ factors through $\widetilde{M}(U_{I\setminus\{Y\}}^V)$, ie we have a commutative diagram
$$\begin{tikzcd} \widetilde{M}(U_I^V) \arrow[r, "\alpha^Y"] \arrow[dr, "\phi"'] & \widetilde{M}(U_{I\setminus\{Y\}}^V) \arrow[d, "\phi_0"] \\ & T \end{tikzcd}$$

For $H\in I'=I\setminus\{Y\}$, we have an associated Gysin morphism $\beta^H:\widetilde{M}(U_{I_H}^H)(1)[1]\to\widetilde{M}(U_I^V)$ which induces a commutative diagram
$$\begin{tikzcd} \widetilde{M}(U_{I_H}^H)(1)[1] \arrow[r, "\beta^H"] \arrow[d, "\alpha^Y(1)[1]"'] & \widetilde{M}(U_I^V) \arrow[r, "\phi"] \arrow[d, "\alpha^Y"] & T \\ \widetilde{M}(U_{I'_H}^H)(1)[1] \arrow[r, "\beta^H"] & \widetilde{M}(U_{I'}^V) \arrow[ur, "\phi_0"'] & \end{tikzcd}$$

in which the morphism $\alpha^Y(1)[1]$ on the left is split surjective. It follows that

$$\phi_0 \circ \beta^H \circ \alpha^Y(1)[1] = \phi \circ \beta^H = 0$$

implies $\phi_0 \circ \beta^H = 0$. We conclude by induction. $\square$

Proposition 4.4 *Let S be an essentially smooth K–scheme and let $f_i \in \mathbb{G}_m(S)$ be such that $\sum_{i=0}^t f_i = 0$. Then*

$$R(f_0, \dots, f_t) = 0 \quad \textit{in } H_{\mathrm{MW}}(S).$$

Proof The global sections $f_0, \dots, f_t$ yield a morphism $j = (f_0, \dots, f_t)\colon S \to \mathbb{A}_K^{t+1}$ which restricts to a morphism $j : S \to U_I^H$, where $H \subset \mathbb{A}_K^{t+1}$ is given by $\sum_{i=0}^t x_i = 0$ and $I = \{\{x_1 = 0\}, \dots, \{x_t = 0\}\}$. Since $R(f_0, \dots, f_t) = j^*(R(x_0, \dots, x_t))$, we can reduce the proposition to the case $S = U_I^H$.

For any x_j, we set $Y_j := \{x_j = 0\} \subset H$ and we obtain a Gysin morphism

$$\beta_j : \widetilde{M}(U_{I_{Y_j}}^{Y_j})(1)[1] \to \widetilde{M}(U_I^H)$$

and a composite

$$\widetilde{M}(U_{I_{Y_j}}^{Y_j})(1)[1] \xrightarrow{\beta_j} \widetilde{M}(U_I^H) \xrightarrow{R(x_0,\dots,x_t)} \mathbb{1}(t)[t].$$

By Remark 4.2 and Lemma 3.3,

$$\begin{aligned}
&R(x_0, \dots, x_t) \circ \beta_j \\
&\qquad = (\epsilon^j [x_j] R(x_0, \dots, \hat{x}_j, \dots, x_t) + P(x_0, \dots, \hat{x}_j, \dots, x_t)) \circ \beta_j \\
&\qquad = \epsilon^j ([x_j] R(x_0, \dots, \hat{x}_j, \dots, x_t)) \circ \beta_j + P(x_0, \dots, \hat{x}_j, \dots, x_t) \circ \alpha_j \circ \beta_j \\
&\qquad = \epsilon^j R(x_0|_{U_{I_{Y_j}}^{Y_j}}, \dots, \hat{x}_j, \dots, x_t|_{U_{I_{Y_j}}^{Y_j}}).
\end{aligned}$$

As $R(f, -f) = 0$ for $f \in \mathbb{G}_m(S)$ by Remark 4.2, we obtain by induction that $R(x_0, \dots, x_t) \circ \beta_j = 0$ for any $j = 0, \dots, t$. Applying Lemma 4.3, we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{M}(U_I^H) \arrow[r, "R(x_0,\dots,x_t)"] \arrow[d] & \mathbb{1}(t)[t] \\
\widetilde{M}(K) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

As $\mathrm{char}(K) \neq 2$, U_I^H has a K–rational point $(\lambda_0, \dots, \lambda_t) \in \mathbb{A}_K^{t+1}$, and we obtain a diagram

$$\begin{tikzcd}
\widetilde{M}(K) \arrow[d, "u"'] \arrow[drr, "R(\lambda_0,\dots,\lambda_t)"] & & \\
\widetilde{M}(U_I^H) \arrow[rr, "R(x_0,\dots,x_t)"] \arrow[d] & & \mathbb{1}(t)[t] \\
\widetilde{M}(K) \arrow[urr, "\psi"'] & &
\end{tikzcd}$$

The vertical composite being the identity, $\psi = R(\lambda_0, \dots, \lambda_t)$, and the latter is trivial by the relations in Milnor–Witt K–theory. □

Applying Lemma 4.1, we obtain the following corollary:

Corollary 4.5 *Let S be an essentially smooth K–scheme.*

(1) *For any $f_1, \dots, f_t \in \mathbb{G}_m(S)$ such that $\sum_{i=1}^t f_i = 1$, we have*

$$[f_1][f_2]\cdots[f_t] = 0 \in H_{\mathrm{MW}}(S).$$

(2) *For any $f \in \mathbb{G}_m(S)$, we have $[f]^2 - [-1][f] = 0$ in $H_{\mathrm{MW}}(S)$.*

5 Proof of the main theorem

In this section, we prove Theorem 1.1. We denote by $J_U \subset H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$ the ideal generated by the relations

(1) $(f) - [f]$ for $f \in K^\times \subset \mathbb{G}_m(U)$;

(2) $(f) + (g) + \eta(f)(g) - (fg)$ for $f, g \in \mathbb{G}_m(U)$;

(3) $(f_1)(f_2)\cdots(f_t)$ for any $f_1, \dots, f_t \in \mathbb{G}_m(U)$ such that $\sum_{i=1}^t f_i = 1$;

(4) $(f)^2 - [-1](f)$ for $f \in \mathbb{G}_m(U)$.

By Lemma 4.1, $J_U \subset H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$ is in fact generated by

(1) $(f) - [f]$ for $f \in K^\times \subset \mathbb{G}_m(U)$;

(2) $(f) + (g) + \eta(f)(g) - (fg)$ for $f, g \in \mathbb{G}_m(U)$;

(3′) **Anticommutativity** $(f)(g) - \epsilon(g)(f)$ for any $f, g \in \mathbb{G}_m(U)$;

(4′) $R(f_0,\dots,f_t)$, given by

$$\sum_{i=0}^{t}\epsilon^{t+i}(f_0)\cdots\widehat{(f_i)}\cdots(f_t)+\sum_{0\le i_0<\cdots<i_k\le t}(-1)^k[-1]^k(f_0)\cdots\widehat{(f_{i_0})}\cdots\widehat{(f_{i_k})}\cdots(f_t)$$

for any $f_0,\dots,f_t\in\mathbb{G}_m(U)$ such that $\sum_{i=0}^{t}f_i=0$.

In view of Corollary 4.5, the morphism $H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}\to H_{\mathrm{MW}}(U)$ defined by $(f)\mapsto[f]$ induces a morphism of $H_{\mathrm{MW}}(K)$–algebras

$$\rho\colon H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U\to H_{\mathrm{MW}}(U).$$

Now, choose linear polynomials $\phi_1,\dots,\phi_s$ that define the hyperplanes $Y_i\in I$ and let $J'_U\subset H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$ be the ideal generated by the relations (1), (2), (3′) and (4′) for elements of the form $f_j=\lambda_j\phi_{i_j}$ or $f_j=\lambda_j$ for $\lambda_j\in K^\times$ and $\phi_{i_j}\in\{\phi_1,\dots,\phi_s\}$. We have a string of surjective morphisms of $H_{\mathrm{MW}}(K)$–algebras

$$H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J'_U\to H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U\xrightarrow{\rho}H_{\mathrm{MW}}(U),$$

whose composite we denote by ρ'.

Theorem 5.1 *The morphism of $H_{\mathrm{MW}}(K)$–algebras*

$$H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U\xrightarrow{\rho}H_{\mathrm{MW}}(U)$$

is an isomorphism.

Proof It suffices to prove that ρ' is an isomorphism. To see this, we work again by induction on $|I|$. If $|I|=0$, then $U\cong\mathbb{A}^N_K$ for some $N\in\mathbb{N}$. By homotopy invariance, we have to prove that the map

$$\rho'\colon H_{\mathrm{MW}}(K)\{\mathbb{G}_m(K)\}/J'_K\to H_{\mathrm{MW}}(K)$$

is an isomorphism. Now, the morphism of $H_{\mathrm{MW}}(K)$–algebras

$$H_{\mathrm{MW}}(K)\to H_{\mathrm{MW}}(K)\{\mathbb{G}_m(K)\}/J'_K$$

is surjective by relation (1). Its composite with ρ' is the identity and we conclude in that case.

Assume now that $Y\in I$ is defined by $\phi_1=0$ and that we have isomorphisms

$$H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U^V_{I'})\}/J'_{U^V_{I'}}\xrightarrow{\sim}H_{\mathrm{MW}}(U^V_{I'}),$$
$$H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U^Y_{I_Y})\}/J'_{U^Y_{I_Y}}\xrightarrow{\sim}H_{\mathrm{MW}}(U^Y_{I_Y}).$$

The morphism $U_I^V \to U_{I'}^V$ induces a morphism $\mathbb{G}_m(U_{I'}^V) \to \mathbb{G}_m(U_I^V)$ and then a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I'}^V)\}/J'_{U_{I'}^V} \arrow[r, "\cong"'] \arrow[d, "\widetilde{\alpha}"'] & H_{\mathrm{MW}}(U_{I'}^V) \arrow[d, "\alpha_*^Y"] \\
H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_I^V)\}/J'_{U_I^V} \arrow[r, "\rho'"] \arrow[d, "\widetilde{\beta}"'] & H_{\mathrm{MW}}(U_I^V) \arrow[d, "\beta_*^Y"] \\
H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I_Y}^Y)\}/J'_{U_{I_Y}^Y} \arrow[r, "\cong"'] & H_{\mathrm{MW}}(U_{I_Y}^Y) \arrow[d] \\
 & 0
\end{tikzcd}$$

in which $\widetilde{\beta}$ is the unique lift of $\beta_*^Y \circ \rho$ and the right column is exact. We are thus reduced to proving that the left vertical sequence is short exact to conclude. It is straightforward to check that $\widetilde{\alpha}$ is injective and $\widetilde{\beta}$ is surjective. Moreover, the commutativity of the diagram and the fact that $\beta_*^Y \circ \alpha_*^Y = 0$ imply that $\widetilde{\beta} \circ \widetilde{\alpha} = 0$, so we are left to prove exactness in the middle.

Let $x \in H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_I^V)\}/J'_{U_I^V}$. The group $\mathbb{G}_m(U_I^V)$ being generated by $\mathbb{G}_m(U_{I'}^V)$ and ϕ_1, we may use relations (2) and (4) to see that $x = (\phi_1)\widetilde{\alpha}(x_1) + \widetilde{\alpha}(x_0)$ in $H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_I^V)\}/J'_{U_I^V}$. By Lemma 3.3, we get $\widetilde{\beta}(x) = \tilde{\imath}(x_1)$, where $\tilde{\imath}$ is induced by the restriction $\mathbb{G}_m(U_{I'}^V) \to \mathbb{G}_m(U_{I_Y}^Y)$. Consequently, we need to prove that, if $\tilde{\imath}(x_1) = 0$, then $(\phi_1)\widetilde{\alpha}(x_1)$ is in the image of $\widetilde{\alpha}$. With this in mind, we now prove that the kernel of $\tilde{\imath}$ is generated by elements of the form

$$R(f_0, \dots, f_t),$$

where $f_j = \lambda\phi_{i_j}$ with $i_j > 1$ or $f_j = \lambda$ and $\sum_{i=0}^t f_i|_{U_{I_Y}^Y} = 0$. Denote by L' the ideal of $H_{\mathrm{MW}}(K)\{\mathbb{G}_m(U_{I'}^V)\}$ generated by such elements. By construction, the restriction induces a homomorphism

$$L' + J'_{U_{I'}^V} \to J'_{U_{I_Y}^Y},$$

which is surjective. Indeed, relations (1), (2) and (3′) can be lifted using the fact that the map $\mathbb{G}_m(U_{I'}^V) \to \mathbb{G}_m(U_{I_Y}^Y)$ is surjective, while an element satisfying relation (4) with every f_j of the form $f_j = \lambda_j\phi_{i_j}$ or $f_j = \lambda_j$ for $\lambda_j \in K^\times$ (with $i_j \neq 1$) lifts to an element in L'. As in [4, proof of Theorem 3.5], we see that the kernel of the group homomorphism $\mathbb{G}_m(U^V_{I'}) \to \mathbb{G}_m(U^Y_{I_Y})$ is generated by elements of the form

(1) $\lambda\phi_i/\phi_j$ with i and j such that $Y_1 \cap Y_i = Y_1 \cap Y_j$ and $\lambda = (\phi_j)|_{Y_1}/(\phi_i)|_{Y_1}$;

(2) $\lambda\phi_i$, where i is such that $Y_1 \cap Y_i = \varnothing$ and $\lambda = 1/(\phi_i)|_{Y_1}$.

Remark 4.2 yields

$$\left[\frac{\lambda\cdot\phi_i}{\phi_j}\right] = -\langle\phi_j^{-1}\rangle R(\phi_j, -\lambda\cdot\phi_i) \subset L',$$

while $[\lambda\phi_i] = \epsilon R(-1, \lambda\cdot\phi_i) \subset L'$ showing that $\ker(\mathbb{G}_m(U^V_{I'}) \to \mathbb{G}_m(U^Y_{I_Y})) \subset L' + J'_{U^V_{I'}}$. We deduce that $\ker(\tilde{\iota}) = L'$.

We now conclude. If $\tilde{\iota}(x_1) = 0$, then $x_1 \in L'$ and we may suppose that $x_1 = R(f_0, \dots, f_t)$ for $f_0, \dots, f_t$ such that $\sum_{i=0}^t f_i|_{U^Y_{I_Y}} = 0$. It follows that $\sum_{i=0}^t f_i = -\mu\phi_1$ for $\mu \in K$. If $\mu = 0$, there is nothing to do. Otherwise, use $R(\mu\phi_1, f_0, \dots, f_t) = 0$ and Remark 4.2 to get

$$\begin{aligned}(\phi_1)\widetilde{\alpha}(x_1) &= (\mu\phi_1)\widetilde{\alpha}(x_1) - \langle\phi_1\rangle(\mu)\widetilde{\alpha}(x_1)\\ &= (\mu\phi_1)\widetilde{\alpha}(x_1) + R(\mu\phi_1, f_0, \dots, f_t) - \langle\phi_1\rangle(\mu)\widetilde{\alpha}(x_1)\\ &= \widetilde{\alpha}(P(f_0, \dots, f_t)) - \widetilde{\alpha}(\langle\phi_1\rangle(\mu)x_1) \in \mathrm{image}(\widetilde{\alpha}).\end{aligned}$$ □

Corollary 5.2 *The graded ring isomorphism of Theorem 5.1 induces an isomorphism*

$$\bigoplus_{n\in\mathbb{Z}} K_n^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U \to \bigoplus_{n\in\mathbb{Z}} H^{n,n}_{\mathrm{MW}}(U, \mathbb{Z}).$$

Proof Notice that the ideal J_U of Theorem 5.1 is homogeneous, and it follows that $\bigoplus_{n\in\mathbb{Z}} H^{n,n}_{\mathrm{MW}}(U, \mathbb{Z})$ can be computed as $H^{*,*}_{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/J_U$, where $H^{*,*}_{\mathrm{MW}}(K)$ is the diagonal of $H_{\mathrm{MW}}(K)$. □

6 Combinatorial description

In this section, we fix an affine space $V = \mathbb{A}^N_K$, a family of hyperplanes I and we set $U := U^N_I$. We let $Q(U)$ be the cokernel of the group homomorphism $\mathbb{G}_m(K) \to \mathbb{G}_m(U)$, and we observe that the divisor map

$$\mathbb{G}_m(U) \xrightarrow{\mathrm{div}} \bigoplus_{Y_i\in I} \mathbb{Z}\cdot Y_i$$

in $\mathbb{A}^N_K$ induces an isomorphism $Q(U) \cong \bigoplus_{Y_i\in I} \mathbb{Z}\cdot Y_i$. We consider the exterior algebra $\Lambda_{\mathbb{Z}} Q(U)$ and write $\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U) := \mathbb{Z}[\eta]/2\eta \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}} Q(U)$. The abelian group $Q(U)$ being free, the $\mathbb{Z}[\eta]/2\eta$–module $\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)$ is also free, with usual basis. To provide a combinatorial description of $H_{\mathrm{MW}}(U)$, we will have to slightly modify the definition of the divisor map above, in order to incorporate the action of η. We then define a map

$$\mathbb{G}_m(U) \xrightarrow{\widetilde{\mathrm{div}}} \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)$$

as follows:

(1) If $f = \lambda\phi$ or $f = \lambda$, where $\lambda \in \mathbb{G}_m(K)$ and ϕ is a linear polynomial as above, then $\widetilde{\mathrm{div}}(f) = \mathrm{div}(f)$.

(2) If $f, g \in \mathbb{G}_m(U)$, then $\widetilde{\mathrm{div}}(fg) = \widetilde{\mathrm{div}}(f) + \widetilde{\mathrm{div}}(g) + \eta \cdot \widetilde{\mathrm{div}}(f) \wedge \widetilde{\mathrm{div}}(g)$.

Lemma 6.1 *The map* $\widetilde{\mathrm{div}}$ *is well defined.*

Proof We first notice that $\widetilde{\mathrm{div}}(fg) = \widetilde{\mathrm{div}}(gf)$, since

$$\widetilde{\mathrm{div}}(fg) - \widetilde{\mathrm{div}}(gf) = \eta \cdot \widetilde{\mathrm{div}}(f) \wedge \widetilde{\mathrm{div}}(g) - \eta \cdot \widetilde{\mathrm{div}}(g) \wedge \widetilde{\mathrm{div}}(f) = 2\eta \cdot \widetilde{\mathrm{div}}(f) \wedge \widetilde{\mathrm{div}}(g) = 0.$$

Let $f_1, f_2, g_1, g_2 \in \mathbb{G}_m(U)$ be such that $f_1 g_1 = f_2 g_2$. Let $Y \in I$ be such that $f_i = Y^{n_i} \cdot f_i'$ with $\mathrm{div}_Y(f_i') = 0$ and $g_i = Y^{m_i} \cdot g_i'$ with $\mathrm{div}_Y(g_i') = 0$ for $i = 1, 2$ and $m_i, n_i \in \mathbb{Z}$. We get

$$\widetilde{\mathrm{div}}(f_1 g_1) = (m_1 + n_1) \cdot Y + \widetilde{\mathrm{div}}(f_1' g_1') + (m_1 + n_1)\eta(Y \wedge \widetilde{\mathrm{div}}(f_1' g_1')),$$
$$\widetilde{\mathrm{div}}(f_2 g_2) = (m_2 + n_2) \cdot Y + \widetilde{\mathrm{div}}(f_2' g_2') + (m_2 + n_2)\eta(Y \wedge \widetilde{\mathrm{div}}(f_2' g_2')).$$

As $\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)$ is free with usual basis, we deduce that $\widetilde{\mathrm{div}}(f_2' g_2') = \widetilde{\mathrm{div}}(f_1' g_1')$, which allows us to conclude by induction on the number of nontrivial factors in the decomposition of $f_1 g_1$. □

Now let $L_U \subset \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)$ be the ideal generated by the elements

(1) $Y_1 \wedge \cdots \wedge Y_s$ for $Y_i \in I$ such that $Y_1 \cap \cdots \cap Y_s = \varnothing$;

(2) $\sum_{j=1}^{s} (-1)^k Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_s$ for $Y_i \in I$ such that $Y_1 \cap \cdots \cap Y_s \neq \varnothing$ and $\mathrm{codim}(Y_1 \cap \cdots \cap Y_s) < s$.

As a consequence of Lemma 6.1, the map $\widetilde{\mathrm{div}}$ induces a morphism of $\mathbb{Z}[\eta]/2\eta$–algebras

$$\psi \colon (\mathbb{Z}[\eta]/2\eta)\{\mathbb{G}_m(U)\} \to \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U.$$

It is now time to introduce the ring

$$A_0(U) := K_*^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}/\big(J_U + K^\times \cdot K_*^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}\big).$$

As $\epsilon = -1 - [-1]\eta \sim -1$ in $A_0(U)$, it follows that $A_0(U)$ is an exterior algebra. Moreover, the coefficient ring $K_*^{\mathrm{MW}}(K)$ can be reduced to $K_*^{\mathrm{MW}}(K)/(K^\times \cdot K_*^{\mathrm{MW}}(K)) \cong \mathbb{Z}[\eta]/2\eta$.

Proposition 6.2 *The morphism of $\mathbb{Z}[\eta]/2\eta$–algebras*

$$\psi\colon \mathbb{Z}[\eta]/2\eta\{\mathbb{G}_m(U)\} \to \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U$$

induces an isomorphism

$$\Psi\colon A_0(U) \to \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U.$$

Proof We first prove that Ψ is well defined, which amounts to showing that the image of J_U is contained in L_U. For $f \in K^\times$, we have $[f] \in K^\times \cdot K_*^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$ and $\widetilde{\mathrm{div}}(f) = 0$, showing that the first relation is satisfied. The second relation is satisfied by definition of $\widetilde{\mathrm{div}}$, while relation $(3')$ is satisfied as $\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U$ is an exterior algebra. As in the proof of Theorem 5.1, we are then left with elements of J'_U, ie elements of the form $R(f_0, \dots, f_t)$ for $\sum_{i=0}^t f_i = 0$, where $f_j = \lambda_j \phi_j$ or $f_j = \lambda_j$. Modulo $K^\times \cdot K_*^{\mathrm{MW}}(K)\{\mathbb{G}_m(U)\}$, we have $R(f_0, \dots, f_t) \sim \sum_{i=0}^t (-1)^{t+i} [f_0] \cdots \widehat{[f_i]} \cdots [f_t]$ and we just need to prove that

$$\alpha := (-1)^t \psi(R(f_0, \dots, f_t)) = \sum_{i=0}^t (-1)^i \widetilde{\mathrm{div}}(f_0) \wedge \cdots \wedge \widehat{\widetilde{\mathrm{div}}(f_i)} \wedge \cdots \wedge \widetilde{\mathrm{div}}(f_t)$$

is an element of L_U. Note that, if there are more than two constant functions among the f_j, α would be trivial. Suppose that $f_0 = \lambda_0$ is the only constant, and let $f_j = \lambda_j \phi_j$ with kernel $Y_j \in I$, so that $\alpha = Y_1 \wedge \cdots \wedge Y_t$. Since $\sum_{j=1}^t \lambda_j \phi_j = -\lambda_0 \neq 0$, we can easily get that $Y_1 \cap \cdots \cap Y_t = \varnothing$ and $\alpha = Y_1 \wedge \cdots \wedge Y_t \in L_U$. In the case where none of the f_j is constant, $\alpha = \sum_{i=0}^t (-1)^i Y_0 \wedge \cdots \wedge \widehat{Y_i} \wedge \cdots \wedge Y_t$. And, for every i, we have $\sum_{j=0, j\neq i}^t \lambda_j \phi_j = -\lambda_i \phi_i$, which means $Y_i \subseteq Y_0 \cap \cdots \cap \widehat{Y_i} \cap \cdots \cap Y_t = Y_0 \cap \cdots \cap Y_t$. If $Y_0 \cap \cdots \cap Y_t = \varnothing$, so is $Y_0 \cap \cdots \cap \widehat{Y_i} \cap \cdots \cap Y_t$, thus $Y_0 \wedge \cdots \wedge \widehat{Y_i} \wedge \cdots \wedge Y_t \in L_U$; otherwise, $\mathrm{codim}(Y_0 \cap \cdots \cap Y_t) = \mathrm{codim}(Y_0 \cap \cdots \cap \widehat{Y_i} \cap \cdots \cap Y_t) \leq t < t+1$, which just fits the condition (2) of L_U. This proves that Ψ is well defined.

To prove that Ψ is an isomorphism, we construct the inverse map by

$$\Phi\colon \Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U \to A_0(U),\ Y_i \mapsto (\phi_i)$$

and prove that it is well defined. As above, we just need to discuss elements of L_U. If $Y_1 \cap \cdots \cap Y_s = \varnothing$, then we can find $\lambda_i \in K^\times$ such that $\sum_i \lambda_i \phi_i = 1$, and thus $(\phi_1) \cdots (\phi_s) \sim (\lambda_1 \phi_1) \cdots (\lambda_s \phi_s) = 0$ in $A_0(U)$. In the case $\mathrm{codim}(Y_1 \cap \cdots \cap Y_s) < s$, we have $\sum_i \lambda_i \phi_i = 0$ for some $\lambda_i \in K^\times$. Then $\sum_{i=1}^s (-1)^i (\phi_1) \cdots \widehat{(\phi_i)} \cdots (\phi_s) \sim (-1)^{s-1} R(\lambda_1 \phi_1, \dots, \lambda_s \phi_s) = 0$ in $A_0(U)$. This shows that the inverse map is well defined. □

The following corollary shows that the rank of the free $H_{\mathrm{MW}}(K)$–module $H_{\mathrm{MW}}(U)$ is exactly the same as the rank of the free $H_M(K)$–module $H_M(U)$ [4, Proposition 3.11]:

Corollary 6.3 *The rank of the free $H_{\mathrm{MW}}(K)$–module $H_{\mathrm{MW}}(U)$ is equal to the rank of the free module $\Lambda_{\mathbb{Z}} Q(U)/L_U$.*

Proof It is clear that $\mathrm{rk}_{\mathbb{Z}[\eta]/2\eta}(\Lambda_{\mathbb{Z}[\eta]/2\eta} Q(U)/L_U) = \mathrm{rk}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}} Q(U)/L_U)$. As all generators in $H_{\mathrm{MW}}(U)$ are from $H^{p,p}_{\mathrm{MW}}(U,\mathbb{Z})$, we have

$$\mathrm{rk}_{H_{\mathrm{MW}}(K)}(H_{\mathrm{MW}}(U)) = \mathrm{rk}_{K^{\mathrm{MW}}_*(K)}\Bigl(\bigoplus_{n\in\mathbb{Z}} H^{n,n}_{\mathrm{MW}}(U,\mathbb{Z})\Bigr) = \mathrm{rk}_{\mathbb{Z}[\eta]/2\eta}(A_0(U)). \qquad \square$$

7 $\boldsymbol{I}$–cohomology and singular cohomology

In ordinary motivic cohomology theory, we have a realization functor to the topological cohomology of complex points. This yields the following comparative result:

Proposition 7.1 [4, Proposition 3.9] *In the case $K=\mathbb{C}$, there is an isomorphism of rings*

$$\bigoplus_n H^{n,n}_M(U,\mathbb{Q}) \otimes_{H_M(K)} K^M_*(K)/K^\times \cdot K^M_*(K) \xrightarrow{\cong} \bigoplus_n H^n_{\mathrm{sing}}(U(\mathbb{C}),\mathbb{Q}).$$

In this section, we provide an analogue for the singular cohomology of the real points of the complement of a hyperplane arrangement defined over $\mathbb{R}$. We start with some results about the $\boldsymbol{I}$–cohomology [9].

As recalled in Section 2, we have natural homomorphisms from Milnor–Witt motivic cohomology to $\boldsymbol{I}^*$–cohomology

$$H^{p,q}_{\mathrm{MW}}(X,\mathbb{Z}) \to H^{p-q}(X,\boldsymbol{K}^{\mathrm{MW}}_q) \to H^{p-q}(X,\boldsymbol{I}^q)$$

which induce a ring homomorphism $H_{\mathrm{MW}}(X) \to \bigoplus_{r,q} H^r(X,\boldsymbol{I}^q)$ (where $\boldsymbol{I}^q = \boldsymbol{K}^{\mathrm{MW}}_q = \boldsymbol{W}$ for $q<0$). In case $X=\mathrm{Spec}(K)$, we obtain in particular a ring homomorphism $H_{\mathrm{MW}}(K) \to \bigoplus_{r,q} H^r(K,\boldsymbol{I}^q) = \bigoplus_{q\in\mathbb{Z}} I^q(K)$.

Proposition 7.2 *The morphism of $\bigoplus_{q\in\mathbb{Z}} I^q(K)$–algebras*

$$j : H_{\mathrm{MW}}(U) \otimes_{H_{\mathrm{MW}}(K)} \Bigl(\bigoplus_{q\in\mathbb{Z}} I^q(K)\Bigr) \to \bigoplus_{r,q} H^r(U,\boldsymbol{I}^q)$$

is an isomorphism. Moreover, $H^r(U,\boldsymbol{I}^q)=0$ for $r\neq 0$.

Proof We write $H_{\mathrm{MW}}(U)\otimes \boldsymbol{I}$ for the graded ring $H_{\mathrm{MW}}(U)\otimes_{H_{\mathrm{MW}}(K)}\bigl(\bigoplus_q I^q(K)\bigr)$. We follow the same induction process as in the proof of the main theorem. When $|I|=0$, we only need to consider $\mathrm{Spec}(K)$ by homotopy invariance, and the result is trivial.

Assume now that $Y\in I$ and that we have isomorphisms for $U^V_{I'}$ and $U^Y_{I_Y}$. Notice that, for $\boldsymbol{I}$–cohomology, we still have a Gysin long exact sequence [9, remarque 9.3.5]. The proof of the main theorem yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & \bigoplus_q H^{-1}(U^Y_{I_Y},\boldsymbol{I}^{q-1}) \arrow[d] \\
H_{\mathrm{MW}}(U^V_{I'})\otimes \boldsymbol{I} \arrow[r, "\cong"] \arrow[d] & \bigoplus_q H^0(U^V_{I'},\boldsymbol{I}^q) \arrow[d] \\
H_{\mathrm{MW}}(U^V_{I})\otimes \boldsymbol{I} \arrow[r, "j"] \arrow[d] & \bigoplus_q H^0(U^V_{I},\boldsymbol{I}^q) \arrow[d] \\
H_{\mathrm{MW}}(U^Y_{I_Y})\otimes \boldsymbol{I} \arrow[r, "\cong"] \arrow[d] & \bigoplus_q H^0(U^Y_{I_Y},\boldsymbol{I}^{q-1}) \arrow[d] \\
0 & \bigoplus_q H^1(U^V_{I'},\boldsymbol{I}^q)
\end{tikzcd}$$

By our assumption, $H^{-1}(U^Y_{I_Y},\boldsymbol{I}^{q-1})$ and $H^1(U^V_{I'},\boldsymbol{I}^q)$ are both 0, so the right column is also short exact. We conclude that j is an isomorphism as well. The same argument implies that $H^r(U^V_I,\boldsymbol{I}^q)=0$ for $r\neq 0$. □

The analogue of Corollary 3.2 in this setting then reads as follows:

Corollary 7.3 *There is a finite set J and integers $n_j\geq 0$ for any $j\in J$ such that*

$$H^0(U^V_I,\boldsymbol{I}^q)\cong\bigoplus_{j\in J} I^{q-n_j}(K)b_j$$

as a free $\bigoplus_q I^q(K)$–module with basis elements $b_j\in H^0(U^V_I,\boldsymbol{I}^{n_j})$.

Proof Every step is the same as in Proposition 3.1, except the splitting, which comes from the identification with $H_{\mathrm{MW}}(U^V_I)\otimes\boldsymbol{I}$. □

As in [11; 10], we can compute the cohomology of the real spectrum using $\boldsymbol{I}$–cohomology.

Proposition 7.4 [10, Proposition 3.6] *The signature map induces an isomorphism*

$$H^r\big(X, \operatorname*{Colim}_{q\geq 0} \boldsymbol{I}^q\big) \xrightarrow{\mathrm{sign}_\infty} H^r_{\mathrm{sing}}(\mathrm{Sper}(X), \mathbb{Z}),$$

where $\mathrm{Sper}(X)$ *is the real spectrum. In particular*,

$$\operatorname*{Colim}_{q\geq 0} I^q(K) \cong H^0_{\mathrm{sing}}(\mathrm{Sper}(K), \mathbb{Z}).$$

In our case, since U is always noetherian and $\mathrm{Colim}_{q\geq 0}$ is filtered, we have a canonical isomorphism

$$H^r\big(U, \operatorname*{Colim}_{q\geq 0} \boldsymbol{I}^q\big) \cong \operatorname*{Colim}_{q\geq 0} H^r(U, \boldsymbol{I}^q). \tag{7-1}$$

Combining with Corollary 7.3, we obtain the following proposition:

Proposition 7.5 *There exists an integer* $N > 0$ *such that*

$$H^0(U_I^V, \boldsymbol{I}^N) \otimes_{\bigoplus_{q\geq 0} I^q(K)} H^0_{\mathrm{sing}}(\mathrm{Sper}(K), \mathbb{Z}) \xrightarrow{2^{-N}\mathrm{sign}} H^0_{\mathrm{sing}}(\mathrm{Sper}(U_I^V), \mathbb{Z})$$

is an isomorphism. Moreover, $H^r_{\mathrm{sing}}(\mathrm{Sper}(U_I^V), \mathbb{Z}) = 0$ *for* $r \neq 0$.

Proof By (7-1), we can rewrite the right-hand side as $\mathrm{Colim}_{q\geq 0} H^0(U_I^V, \boldsymbol{I}^q)$. Applying Corollary 7.3, we get

$$\operatorname*{Colim}_{q\geq 0}\bigg(\bigoplus_{j\in J} I^{q-n_j}(K)b_j\bigg) \cong \bigoplus_{j\in J}\big(\operatorname*{Colim}_{q\geq 0} I^{q-n_j}(K)b_j\big) \cong \bigoplus_{j\in J} H^0_{\mathrm{sing}}(\mathrm{Sper}(K), \mathbb{Z})b_j.$$

Let $N \in \mathbb{N}$ be such that $N \geq n_j$ for all $j \in J$. Using again Corollary 7.3,

$$H^0(U_I^V, \boldsymbol{I}^N) \cong \bigoplus_{j\in J} I^{N-n_j}(K)b_j,$$

which implies

$$\bigoplus_{j\in J} I^{N-n_j}(K)b_j \otimes_{\bigoplus_{q\geq 0} I^q(K)} H^0_{\mathrm{sing}}(\mathrm{Sper}(K), \mathbb{Z}) \cong \bigoplus_{j\in J} H^0_{\mathrm{sing}}(\mathrm{Sper}(K), \mathbb{Z})b_j$$

since, for every j, we have $N - n_j \geq 0$. That proves the first part, while the second part is trivial. □

Taking $K = \mathbb{R}$, we have $H^0_{\mathrm{sing}}(\mathbb{R}, \mathbb{Z}) = \mathbb{Z}$ and we recover the classical result for complements of hyperplane arrangements

$$H^0(U_I^V, \boldsymbol{I}^N) \xrightarrow[\cong]{2^{-N}\mathrm{sign}} H^0_{\mathrm{sing}}(U_I^V(\mathbb{R}), \mathbb{Z}) \cong \bigoplus_{R_i \in \text{connected components}} \mathbb{Z}\{R_i\}.$$

References

[1] **T Bachmann**, **B Calmès**, **F Déglise**, **J Fasel**, **P A Østvær**, *Milnor–Witt motives*, preprint (2020) arXiv 2004.06634

[2] **J Barge**, **F Morel**, *Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels*, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 287–290 MR Zbl

[3] **B Calmès**, **J Fasel**, *A comparison theorem for MW-motivic cohomology*, preprint (2017) arXiv 1708.06100

[4] **A Chatzistamatiou**, *Motivic cohomology of the complement of hyperplane arrangements*, Duke Math. J. 138 (2007) 375–389 MR Zbl

[5] **D-C Cisinski**, **F Déglise**, *Triangulated categories of mixed motives*, Springer (2019) MR Zbl

[6] **F Déglise**, **J Fasel**, *The Milnor–Witt motivic ring spectrum and its associated theories*, preprint (2017) arXiv 1708.06102

[7] **F Déglise**, **J Fasel**, *MW-motivic complexes*, preprint (2017) arXiv 1708.06095

[8] **P H Edelman**, *A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes*, Trans. Amer. Math. Soc. 283 (1984) 617–631 MR Zbl

[9] **J Fasel**, *Groupes de Chow–Witt*, Mém. Soc. Math. Fr. 113, Soc. Math. France, Paris (2008) MR Zbl

[10] **J Hornbostel**, **M Wendt**, **H Xie**, **M Zibrowius**, *The real cycle class map*, Ann. K–Theory 6 (2021) 239–317 MR Zbl

[11] **J A Jacobson**, *Real cohomology and the powers of the fundamental ideal in the Witt ring*, Ann. K–Theory 2 (2017) 357–385 MR Zbl

[12] **F Morel**, *$\mathbb{A}^1$–algebraic topology over a field*, Lecture Notes in Math. 2052, Springer (2012) MR Zbl

[13] **F Morel**, **V Voevodsky**, *$\mathbb{A}^1$–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR Zbl

[14] **N Yang**, *Projective bundle theorem in MW-motivic cohomology*, Doc. Math. 26 (2021) 1045–1083 MR Zbl

[15] **N Yang**, *Quaternionic projective bundle theorem and Gysin triangle in MW-motivic cohomology*, Manuscripta Math. 164 (2021) 39–65 MR Zbl

Institut Fourier, Université Grenoble Alpes
Grenoble, France

keyao.peng@univ-grenoble-alpes.fr

Received: 19 November 2020 Revised: 12 December 2021

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2023 is US $650/year for the electronic version, and $940/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 8 (pages 3417–3908) 2023