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We show that a closed orientable 3–dimensional manifold admits a round fold map into the plane, ie a fold map whose critical value set consists of disjoint simple closed curves isotopic to concentric circles, if and only if it is a graph manifold, generalizing the characterization for simple stable maps into the plane. Furthermore, we also give a characterization of closed orientable graph manifolds that admit directed round fold maps into the plane, ie round fold maps such that the number of regular fiber components of a regular value increases toward the central region in the plane.

57R45; 57K30, 58K30

Dedicated to Professor Kazuhiro Sakuma on the occasion of his 60th birthday

1 Introduction

Let M be a smooth closed manifold of dimension ≥ 2. It is known that if a smooth map $f\colon M \to \mathbb{R}^2$ is generic enough, then it has only fold and cusps as its singularities; see Levine [9; 10] and Whitney [16]. Furthermore, if M has even Euler characteristic (eg if $\dim M$ is odd), then the cusps can be eliminated by homotopy. In particular, every smooth closed orientable 3–dimensional manifold admits a smooth map into $\mathbb{R}^2$ with only fold singularities, ie a *fold map*.

In [13; 14], the second author considered the following smaller class of generic smooth maps. A fold map $f\colon M \to \mathbb{R}^2$ on a smooth closed orientable 3–dimensional manifold M is a *simple stable map* if for every $q \in \mathbb{R}^2$, each component of $f^{-1}(q)$ contains at most one singular point and $f|_{S(f)}$ is an immersion with normal crossings, where $S(f)(\subset M)$ denotes the set of singular points of f. Note that if f is a fold map, then $S(f)$ is a regular closed 1–dimensional submanifold of M. In particular, if $f|_{S(f)}$ is an embedding, then f is a simple stable map. In [14], it has been proved that for a smooth closed orientable 3–dimensional manifold M, the following three properties are equivalent to each other:

(1) M admits a fold map $f\colon M \to \mathbb{R}^2$ such that $f|_{S(f)}$ is an embedding.

(2) M admits a simple stable map into $\mathbb{R}^2$.

(3) M is a graph manifold, ie it is a finite union of S^1–bundles over compact surfaces attached along their torus boundaries.

Thus, for example, if M is hyperbolic, then M never admits such a fold map.

On the other hand, the first author introduced the notion of a round fold map [7; 6; 5]: a smooth map $f\colon M \to \mathbb{R}^2$ is a *round fold map* if it is a fold map and $f|_{S(f)}$ is an embedding onto the disjoint union of some concentric circles in $\mathbb{R}^2$; for details, see Section 2. As has been studied by the first author, round fold maps have various nice properties.

The first main result of this paper is Theorem 3.1, which states that every graph 3–manifold admits a round fold map into $\mathbb{R}^2$. This generalizes the characterization result obtained in [14] for simple stable maps mentioned above.

It is not difficult to observe that if $f\colon M \to \mathbb{R}^2$ is a round fold map of a closed orientable 3–dimensional manifold, then the number of components of the fiber over a regular value changes exactly by one when the regular value crosses the critical value set transversely once. We can thus put a normal orientation to each component of the critical value set in such a way that the orientation points in the direction that increases the number of components of a regular fiber. Then, a round fold map is said to be *directed* if all the circles in the critical value set are directed inward. The second main result of this paper (Theorem 3.2) characterizes those graph 3–manifolds which admit directed round fold maps. It will turn out that the class is strictly smaller than that of closed orientable graph 3–manifolds.

The paper is organized as follows. In Section 2, we prepare several definitions and a lemma concerning round fold maps and graph 3–manifolds necessary for our purposes. We also give an observation on fibered links or open book structures associated with round fold maps and give some examples. In Section 3, we state and prove the main theorems. Basically, we will follow the proof given in [14, Theorem 3.1]: however, in some steps we need to modify the strategy for the constructions of round fold maps. In Section 4, we give some corollaries and show that the class of 3–manifolds that admit directed round fold maps is strictly smaller than that of all graph 3–manifolds, using results obtained in [2; 12]. Finally, we give some open problems related to our results.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified. For a space X, id_X denotes the identity map of X. The symbol $\cong$ denotes a diffeomorphism between smooth manifolds.

2 Preliminaries

2.1 Round fold maps

In this subsection, we recall some notions related to round fold maps and give some examples.

Let M be a closed orientable 3–dimensional manifold and $f\colon M \to \mathbb{R}^2$ a smooth map. We denote by $S(f)$ $(\subset M)$ the set of all singular points of f.

Definition 2.1 A point $p \in S(f)$ is a *definite fold point* (resp. an *indefinite fold point*, or a *cusp point*) if f is represented by the map

$$(u, x, y) \mapsto (u, x^2 + y^2) \quad \text{(resp. } (u, x^2 - y^2)\text{, or } (u, y^2 + ux - x^3)\text{)}$$

around the origin with respect to certain local coordinates around p and $f(p)$. We call a point $p \in S(f)$ a *fold point* if it is a definite or an indefinite fold point. A smooth map $f\colon M \to \mathbb{R}^2$ is called a *fold map* if it has only fold points as its singular points. Note that then $S(f)$ is a closed 1–dimensional submanifold of M and that $f|_{S(f)}$ is an immersion.

Definition 2.2 Let $C^\infty(M, \mathbb{R}^2)$ denote the space of all smooth maps of M into $\mathbb{R}^2$, endowed with the Whitney C^∞–topology. A smooth map $f\colon M \to \mathbb{R}^2$ is a *stable map* if there exists a neighborhood $N(f)$ of f in $C^\infty(M, \mathbb{R}^2)$ such that for every $g \in N(f)$, there are diffeomorphisms $\Phi\colon M \to M$ and $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g = \varphi \circ f \circ \Phi^{-1}$.

It is known that a smooth map $f\colon M \to \mathbb{R}^2$ is a stable map if and only if the following three conditions hold; for example, see Levine [10].

(1) It has only fold and cusp points as its singularities.

(2) We have $f^{-1}(f(q)) \cap S(f) = \{q\}$ for every cusp point $q \in S(f)$.

(3) The restriction of f to the set of fold points is an immersion with normal crossings.

Definition 2.3 A stable map $f\colon M \to \mathbb{R}^2$ is *simple* if it has no cusp points and for every $q \in \mathbb{R}^2$, each component of $f^{-1}(q)$ contains at most one singular point; for details, see [13; 14].

In the following, for $r > 0$, C_r denotes the circle of radius r centered at the origin in $\mathbb{R}^2$.

Definition 2.4 A finite disjoint union of simple closed curves in $\mathbb{R}^2$ is said to be *concentric* if it is isotopic to a set of concentric circles

$$\bigcup_{i=1}^{m} C_i$$

for some positive integer m.

Definition 2.5 We say that a smooth map $f\colon M \to \mathbb{R}^2$ is a *round fold map* if it is a fold map and $f|_{S(f)}$ is an embedding onto a concentric family of simple closed curves. Note that a round fold map is a simple stable map. Note also that the outermost circle component of $f(S(f))$ consists of the images of definite fold points.

By composing a diffeomorphism of $\mathbb{R}^2$ if necessary, we always assume that a round fold map $f\colon M \to \mathbb{R}^2$ satisfies

$$f(S(f)) = \bigcup_{i=1}^{m} C_i \tag{2-1}$$

for some positive integer m.

In the following, A denotes the annulus $S^1 \times [-1,1]$, and P denotes the compact surface obtained from the 2–sphere by removing three open disks: in other words, P is a *pair of pants*.

Let $f\colon M \to \mathbb{R}^2$ be a round fold map of a closed *orientable* 3–dimensional manifold M. For a component C of $f(S(f))$, take a small arc $\alpha \cong [-1,1]$ in $\mathbb{R}^2$ that intersects $f(S(f))$ exactly at one point in C transversely. Then, $f^{-1}(\alpha)$ is a compact surface with boundary $f^{-1}(a) \cup f^{-1}(b)$, which is diffeomorphic to a finite disjoint union of circles, where a and b are the end points of α. Furthermore, $f|_{f^{-1}(\alpha)}\colon f^{-1}(\alpha) \to \alpha$ can be regarded as a Morse function with exactly one critical point. As M is orientable, we see that $f^{-1}(\alpha)$ is diffeomorphic to the union of D^2 (or P) and a finite number of copies of A; see [15], for example. Therefore, the number of components of $f^{-1}(a)$ differs from that of $f^{-1}(b)$ exactly by one. If $f^{-1}(a)$ has more components than $f^{-1}(b)$, then we normally orient C from b to a; otherwise, we orient C from a to b. It is easily shown that this normal orientation is independent of the choice of α. In this way, each component of $f(S(f))$ is normally oriented. If the normal orientation points inward, then the component is said to be *inward-directed*; otherwise, it is *outward-directed*.

Definition 2.6 Let $f\colon M \to \mathbb{R}^2$ be a round fold map. We say that f is *directed* if all the components of $f(S(f))$ are inward-directed. It is easy to see that a round fold map f is directed if and only if the number of components of a regular fiber over a point in the innermost component of $\mathbb{R}^2 \setminus f(S(f))$ coincides with the number of components of $S(f)$.

Let $f\colon M \to \mathbb{R}^2$ be a round fold map satisfying (2-1). Set $L = f^{-1}(0)$, which is an oriented link in M if it is not empty. Let D be the closed disk centered at the origin with radius $\frac{1}{2}$. Then, $f^{-1}(D)$ is diffeomorphic to $L \times D$, which can be identified with a tubular neighborhood $N(L)$ of L in M. Furthermore, the composition $\varphi = \pi \circ f\colon M \setminus \operatorname{Int} N(L) \to S^1$ is a submersion, where $\pi\colon \mathbb{R}^2 \setminus \operatorname{Int} D \to S^1$ is the standard radial projection and $\varphi|_{\partial N(L)}\colon \partial N(L) = L \times \partial D \to S^1$ corresponds to the projection to the second factor followed by a scalar multiplication. Hence, φ is a smooth fiber bundle and L is a fibered link. (In other words, M admits an open book structure with binding L.) The fiber (or the page) is identified with $F = f^{-1}(J)$, where

$$J = \left[\tfrac{1}{2}, m+1\right] \times \{0\} \subset \mathbb{R}^2,$$

and it is a compact oriented surface. Note that $g = f|_F\colon F \to J$ is a Morse function with exactly m critical points, and that a monodromy diffeomorphism of the fibration over S^1 can be chosen so that it preserves g.

Note that all these arguments work even when $L = \varnothing$. In this case, F is a closed orientable surface and M is the total space of an F–bundle over S^1.

Conversely, if we have a compact orientable surface F, a Morse function $g\colon F \to \left[\frac{1}{2}, m+1\right]$ such that $g(\partial F) = \frac{1}{2}$ and that g has no critical point near the boundary, and a diffeomorphism $h\colon F \to F$ which is the identity on the boundary and which satisfies $g \circ h = g$, then we can construct a round fold map $f\colon M \to \mathbb{R}^2$ in such a way that M is the union of $\partial F \times D^2$ and the total space of the F–bundle over S^1 with geometric monodromy h.

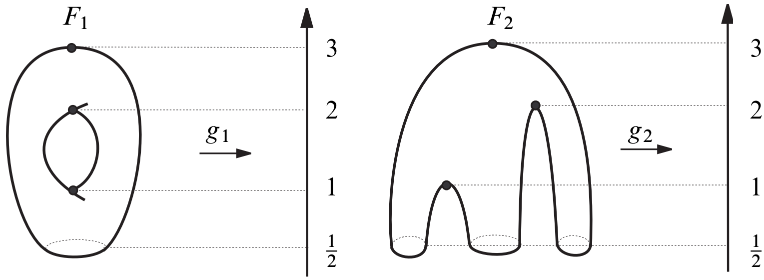

Figure 1: Morse functions on surfaces with Euler characteristic -1.

Example 2.7 Let F be a compact connected orientable surface with $\partial F \neq \varnothing$. Let us consider the identity diffeomorphism as the geometric monodromy in the above construction. Then, we see that the source 3–manifold M of the round fold map is diffeomorphic to $(\partial F \times D^2) \cup (F \times S^1) \cong \partial(F \times D^2)$. By using a handle decomposition argument, we can easily see that $F \times D^2$ is diffeomorphic to D^4 or a boundary connected sum of a finite number of copies of $S^1 \times D^3$. Therefore, M is diffeomorphic either to S^3 or to the connected sum of a finite number of copies of $S^1 \times S^2$.

For example, if we start with the Morse function $g_1 \colon F_1 \to \left[\frac{1}{2}, 4\right]$ as depicted in Figure 1, left, then the singular point set $S(f_1)$ of the resulting round fold map $f_1 \colon M_1 \to \mathbb{R}^2$ has three components and their images coincide with C_1, C_2 and C_3. The first one is outward directed, while the other two are inward directed. Therefore, the fold map f_1 is not directed. In this example, M_1 is diffeomorphic to $(S^1 \times S^2) \sharp (S^1 \times S^2)$.

On the other hand, if we start with the Morse function $g_2 \colon F_2 \to \left[\frac{1}{2}, 4\right]$ as depicted in Figure 1, right, then we get a round fold map $f_2 \colon M_2 \to \mathbb{R}^2$ with the same singular values: however, this round fold map is directed. We can also show that M_2 is again diffeomorphic to $(S^1 \times S^2) \sharp (S^1 \times S^2)$.

2.2 Graph 3–manifolds

In this subsection, we recall the notion of graph 3–manifolds and related results necessary to state our main theorems of the paper, and to get further results.

Definition 2.8 Let M be a compact orientable 3–dimensional manifold possibly with torus boundaries. It is called a *graph manifold* if it is diffeomorphic to a union of S^1–bundles over compact surfaces attached along their torus boundaries.

Let M be a graph manifold. For a boundary component of each S^1–bundle piece, we have a pair of distinguished simple closed curves: an S^1–fiber and a cross-section over the corresponding boundary component of the base surface with respect to a fixed trivialization. Note that such a pair of simple closed curves are unique up to isotopy once we fix a trivialization of the S^1–bundle over the boundary of the base surface. More precisely, the manifold M is oriented, and when the base surface is orientable, we orient the simple closed curves in a way consistent with the orientations of the base surface and the ambient 3–manifold. A decomposition of M as in Definition 2.8 is said to be of *plumbing type* if for each gluing of a pair of torus boundaries, the attaching diffeomorphism, which is orientation reversing, interchanges the S^1–fiber and the cross-section over the corresponding boundary component of the base surface. It is well-known that every graph manifold admits a decomposition of plumbing type; for example, see Hirzebruch, Neumann and Koh [4].

Each decomposition of plumbing type can be represented by a weighted graph: each vertex corresponds to an S^1–bundle piece over a connected surface and each edge corresponds to a gluing of the corresponding torus boundaries. A vertex is weighted with the genus of the base surface together with its orientability and the Euler number of the S^1–bundle. Furthermore, an edge is weighted by a sign $+$ or $-$ corresponding to the orientation preserving/reversing property of the gluing map on the pair of an S^1–fiber and a cross-section. Then, Neumann [12] listed up certain operations to weighted graphs in such a way that the two graph 3–manifolds corresponding to two weighted graphs are diffeomorphic if and only if the graphs are related by a finite iteration of the operations. Neumann also established the notion of normal form plumbing graphs as a special class of weighted graphs as above and showed that every graph 3–manifold has a unique normal form plumbing graph.

Now, in order to state one of our main theorems, we need the following.

Lemma 2.9 *Every closed orientable graph 3–manifold is diffeomorphic to a union of finite numbers of copies of $P \times S^1$ and the solid torus $D^2 \times S^1$ attached along their torus boundaries.*

Proof It is known that every such 3–manifold is diffeomorphic to a union of a finite number of S^1–bundles over compact connected orientable surfaces of genus zero attached along their torus boundaries; for example, see [14, Lemma 3.3]. In fact, if a base surface is nonorientable, then we can decompose the surface into the union of a compact orientable surface of genus zero and some copies of the Möbius band attached along their boundaries, and we see that the S^1–bundle over the Möbius band can be further decomposed into the union of S^1–bundles over compact orientable surfaces of genus zero. If a base surface is orientable of positive genus, then we can decompose it into the union of compact orientable surfaces of genus zero attached along their boundaries. Then, we can decompose the 3–manifold accordingly so that we obtain a desired decomposition.

Now, consider a base surface B, which is orientable of genus zero. If the number of boundary components is greater than or equal to 4, then we can decompose B into a union of a finite number of copies of P attached along their circle boundaries. If the number of boundary components is equal to two, then B is diffeomorphic to the union of P and a disk. If the surface B has no boundary, then we can decompose it into two disks. As orientable S^1–bundles over P or a disk are always trivial, the result follows. □

As a consequence, a graph manifold can be represented by a (multi-)graph, where each vertex corresponds to $P \times S^1$ or a solid torus and each edge corresponds to the gluing along a pair of boundary components. Note that each gluing corresponds to an element of the (orientation preserving) mapping class group of the torus, identified with $SL(2, \mathbb{Z})$.

3 Main theorems and proofs

In this section, we state our main results, Theorems 3.1 and 3.2, of this paper and give their proofs.

Theorem 3.1 *Let M be a closed orientable 3–dimensional manifold. Then, it admits a round fold map into $\mathbb{R}^2$ if and only if it is a graph manifold.*

Theorem 3.1 generalizes the characterization for simple stable maps obtained in [14].

In particular, every closed orientable graph 3–manifold admits a fibered link which is also a graph link. Compare this with Myers [11]. Here, a link in a graph 3–manifold is a *graph link* if its exterior is a graph manifold.

Theorem 3.2 *Let M be a closed connected orientable graph 3–manifold. Then, it admits a directed round fold map into $\mathbb{R}^2$ if and only if it can be decomposed into a union of finite numbers of copies of $P \times S^1$ and $D^2 \times S^1$ such that the corresponding graph is a tree.*

Proof of Theorem 3.1 As noted above, a round fold map is a simple stable map. Therefore, if a closed orientable 3–dimensional manifold admits such a map, then it is necessarily a graph manifold by [14].

Now, suppose M is a graph manifold. We will follow the proof of [14, Theorem 3.1] in order to construct a round fold map $f\colon M \to \mathbb{R}^2$, except for the first step, in which a nonsingular map is constructed for each S^1–bundle piece in [14] while we construct a fold map for each piece, as explained below.

By virtue of Lemma 2.9, we have disjointly embedded tori $T_1, T_2, \dots, T_\ell$ in M such that each of the components $X_1, X_2, \dots, X_k$ of $M \setminus \bigsqcup_{i=1}^{\ell} \operatorname{Int} N(T_i)$ is diffeomorphic either to $P \times S^1$ or to $D^2 \times S^1$, where $N(T_i)$ denotes a small tubular neighborhood of T_i in M, $1 \le i \le \ell$. By inserting pieces diffeomorphic to $A \times S^1 \cong T^2 \times [-1,1]$ if necessary, we may assume that the decomposition is of plumbing type (for details, see [4]), where $T^2 = S^1 \times S^1$ denotes the torus. Now, each X_i is diffeomorphic either to $P \times S^1$, $D^2 \times S^1$ or $A \times S^1$.

Take a component X_j, for some $1 \le j \le k$. Suppose it is diffeomorphic to $D^2 \times S^1$. Let $\delta\colon D^2 \to [-1,1]$ be the Morse function defined by $\delta(x,y) = -x^2 - y^2$, where D^2 is identified with the unit 2–disk in $\mathbb{R}^2$; see Figure 2, left. Then, define $f|_{X_j}$ to be the composition

$$\eta_j \circ (\delta \times \mathrm{id}_{S^1}) \circ \varphi_j : X_j \xrightarrow{\varphi_j} D^2 \times S^1 \xrightarrow{\delta \times \mathrm{id}_{S^1}} [-1,1] \times S^1 \xrightarrow{\eta_j} \mathbb{R}^2,$$

where φ_j is a diffeomorphism and η_j is an embedding whose image is a small tubular neighborhood of the circle of radius j centered at the origin. We also arrange η_j in such a way that $\eta_j(\{\pm 1\} \times S^1)$ coincides with the circle of radius $j \pm \frac{1}{3}$.

Suppose X_j is diffeomorphic to $P \times S^1$. We define $f|_{X_j}$ by the composition

$$\eta_j \circ (\iota \times \mathrm{id}_{S^1}) \circ \varphi_j : X_j \xrightarrow{\varphi_j} P \times S^1 \xrightarrow{\iota \times \mathrm{id}_{S^1}} [-1,1] \times S^1 \xrightarrow{\eta_j} \mathbb{R}^2,$$

where φ_j is a diffeomorphism, $\iota\colon P \to [-1,1]$ is the standard Morse function with exactly one saddle point (as depicted in Figure 2, right) and η_j is an embedding as described in the previous paragraph.

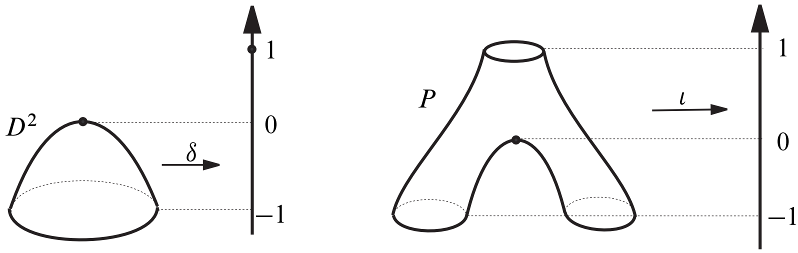

Figure 2: Morse functions δ and ι.

Finally, suppose X_j is diffeomorphic to $A \times S^1$. In this case, we define $f|_{X_j}$ by the composition

$$\eta_j \circ (\rho \times \mathrm{id}_{S^1}) \circ \varphi_j : X_j \xrightarrow{\varphi_j} A \times S^1 \xrightarrow{\rho \times \mathrm{id}_{S^1}} [-1, 1] \times S^1 \xrightarrow{\eta_j} \mathbb{R}^2,$$

where φ_j is a diffeomorphism, $\rho\colon A \cong S^1 \times [-1, 1] \to [-1, 1]$ is the projection to the second factor and η_j is an embedding as described above.

Now, the map $f|_{\bigsqcup_{j=1}^k X_j}$ has only fold singular points, and its restriction to the singular point set is an embedding onto a concentric family of circles in $\mathbb{R}^2$. Then, we can extend the map to get a round fold map $f : M \to \mathbb{R}^2$ as follows, by a method similar to that used in [14, Proof of Theorem 3.1].

In the following, we set $I = [0, 1]$. For the construction, we need two smooth maps h_1 and $h_2\colon T^2 \times I \to \mathbb{R}^2$ as follows. (For details, see [14, Section 3].) The map h_1 is constructed as the composition

$$T^2 \times I \xrightarrow{\cong} A \times S^1 \xrightarrow{\upsilon \times \mathrm{id}_{S^1}} [-1, 1] \times S^1 \xrightarrow{\eta} \mathbb{R}^2,$$

where $\upsilon\colon A \to [-1, 1]$ is a Morse function with exactly one saddle point and one maximum point such that $\upsilon^{-1}(-1) = \partial A$ (see [14, Figure 8]), and η is an embedding.

On the other hand, the smooth map $h_2\colon T^2 \times I \to \mathbb{R}^2$ enjoys the following properties.

(1) The image of h_2 is the disk of radius 3 centered at the origin.

(2) The singular point set $S(h_2)$ is a circle and consists of indefinite fold points.

(3) The map $h_2|_{S(h_2)}$ is an embedding onto the circle C_2.

(4) The inverse image $(h_2)^{-1}(C_3)$ coincides with a boundary component of $T^2 \times I$, and $(h_2)^{-1}(C_1)$ consists of two components one of which coincides with the other boundary component.

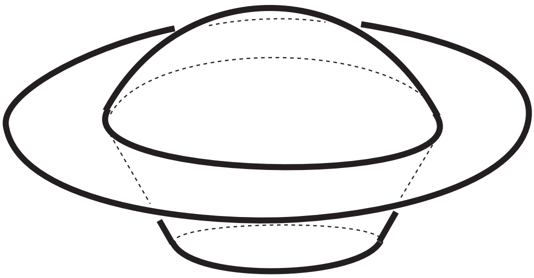

Figure 3: Quotient space of h_2.

(5) The quotient space in the Stein factorization of h_2 is a 2–dimensional polyhedron as depicted in Figure 3. (For the definition of the quotient space in the Stein factorization, see [14, Section 2], for example.)

Here, we omit the detailed construction of h_2, as it is fully explained in [14]. The idea is to use a Dehn surgery on the exterior of a 2–component trivial link in S^3.

Note that on the boundary components of $T^2 \times I$, the maps h_1 and h_2 are S^1–bundles over their images. Therefore, on the boundary components, we have distinguished pairs of simple closed curves: pairs of an S^1–fiber and a cross-section. Another important property of h_1 and h_2 is that for h_1 the canonical diffeomorphism between the components of $\partial(T^2 \times I)$ keeps the S^1–fiber and the cross-section, while for h_2 it interchanges them.

Now, let us proceed as in [14, Proof of Theorem 3.1]. Recall that in the proof there, a simple stable map into S^2 is first constructed: however, in our case, we can directly construct a map into $\mathbb{R}^2$ as the singular value set of $f|_{\bigsqcup_{j=1}^k X_j}$ is a concentric family of circles in $\mathbb{R}^2$. In order to extend the map, we need to arrange an appropriate map on each $N(T_i) \cong T^2 \times I$. Depending on the location of the images by the map $f|_{\bigsqcup_{j=1}^k X_j}$ of the small collar neighborhoods of the boundary tori for gluing, we have 4 cases.[1] Depending on the cases, we may need to decompose $N(T_i)$ into two or three parts, each of which is diffeomorphic to $T^2 \times I$, in order to glue the parts as prescribed by the weighted plumbing graph. The key ideas are to use h_2 in order to interchange the S^1–fiber and the cross-section for gluing, and to use h_1 in order to adjust the direction of the gluing. In order to use h_2, we need to use a disk region in the target: in such a case, we can choose the region that does not contain the point $\infty \in S^2 = \mathbb{R}^2 \cup \{\infty\}$.

[1] In [14, Proof of Theorem 3.1], we had only three cases, since S^2 was considered as the target. Here, as the target is $\mathbb{R}^2$, we have one more case for Case 2 there: however, the argument that we use is the same.

We can also arrange f in such a way that $f|_{S(f)}$ is an embedding by appropriately modifying f near $S(f)$ if necessary. This completes the proof of Theorem 3.1. $\square$

Let us go on to the proof of the second theorem. In the following, we put, for $0 < a < b$,

$$A_{[a,b]} = \{(x, y) \in \mathbb{R}^2 \mid a \leq \sqrt{x^2 + y^2} \leq b\}.$$

We can observe that $f^{-1}(C_{i-\frac{1}{2}})$ is a finite disjoint union of tori for each $i = 1, 2, \dots, m$, since M is orientable. Let K be the closure of a component of

$$M \setminus \Bigl(\bigsqcup_{i=1}^{m} f^{-1}(C_{i-1/2})\Bigr)$$

such that $f(K) \subset A_{[i-1/2,i+1/2]}$. Let $p_K : K \to S^1$ be the composition of $f|_K : K \to A_{[i-1/2,i+1/2]}$ and the radial projection $A_{[i-1/2,i+1/2]} \to S^1$. We can easily see that p_K and its restriction to the boundary are submersions and hence p_K is a locally trivial fibration. The fiber is a disjoint union of copies of D^2, A and P. Since $f|_{S(f)}$ is an embedding and K is connected, the fiber is diffeomorphic to D^2, P, or a finite disjoint union of copies of A. If the fiber is diffeomorphic to D^2, then K is diffeomorphic to $D^2 \times S^1$, since K is an orientable 3–dimensional manifold. If the fiber is diffeomorphic to P, then K is diffeomorphic either to $P \times S^1$ or a nontrivial P–bundle over S^1; see the proof of [14, Lemma 2.4].

Suppose that K is a nontrivial P–bundle over S^1 and that $C_i \subset f(S(f))$ is inward-directed. If $i = 1$, then this leads to a contradiction, since f is a trivial fiber bundle over the innermost region of $\mathbb{R}^2 \setminus f(S(f))$. If $i > 1$, then a component of $f^{-1}(A_{[i-3/2,i-1/2]})$ adjacent to K is either a nontrivial P–bundle over S^1, or a nontrivial $(A \sqcup A)$–bundle over S^1, where $A \sqcup A$ is the disjoint union of two copies of A and the monodromy for the latter bundle interchanges the two components of $A \sqcup A$. In the former case, $C_{i-1} \subset f(S(f))$ is outward-directed. In the latter case, we can repeat the argument toward inner components to find an outward-directed component.

Thus we have proved the following.

Lemma 3.3 *Let $f : M \to \mathbb{R}^2$ be a round fold map of a closed orientable 3–dimensional manifold such that $f(S(f)) = \bigsqcup_{i=1}^{m} C_i$. If f is directed, then the closure of a component of*

$$M \setminus \Bigl(\bigsqcup_{i=1}^{m} f^{-1}(C_{i-1/2})\Bigr)$$

is never diffeomorphic to the nontrivial P–bundle over S^1.

Proof of Theorem 3.2 First, suppose that there exists a directed round fold map $f: M \to \mathbb{R}^2$. We may assume that it satisfies (2-1). Then the disjoint union of tori $\bigsqcup_{i=1}^{m} f^{-1}(C_{i-1/2})$ decomposes M into a union of copies of $P \times S^1$, $A \times S^1$ and $D^2 \times S^1$ attached along their torus boundaries. Note that by Lemma 3.3, a nontrivial P–bundle over S^1 does not appear, since f is directed. Furthermore, we can ignore the components diffeomorphic to $A \times S^1 \cong T^2 \times [-1, 1]$ for obtaining a decomposition of M.

Since f is directed and M is connected, we see that the components diffeomorphic to $D^2 \times S^1$ are the outermost component $f^{-1}(A_{[m-1/2, m+1/2]})$ together with the components of the innermost part $f^{-1}(A_{[0,1/2]})$: no other components are diffeomorphic to $D^2 \times S^1$. Then, we can easily see that the corresponding graph describing this decomposition of M into copies of $D^2 \times S^1$ and $P \times S^1$ is a tree, as the number of components of regular fibers strictly increases toward the central region.

Conversely, suppose that the graph describing the decomposition of M into copies of $P \times S^1$ and $D^2 \times S^1$ is a tree. By inserting pieces diffeomorphic to $A \times S^1$ if necessary, we may assume that the decomposition is of plumbing type. Then, the graph Γ describing this new decomposition is also a tree. Note that Γ has at least one vertex of degree one. Let k denote the number of vertices of Γ. We label the vertices by $\{1, 2, \dots, k\}$ in such a way that

(1) the labeling gives a one-to-one correspondence between the set of vertices and the set $\{1, 2, \dots, k\}$,

(2) the degree of the vertex labeled k is equal to one,

(3) for each $j \in \{1, 2, \dots, k\}$, the vertices of labels $\geq j$ together with the edges connecting them constitute a connected subgraph of Γ.

This is possible, since Γ is a tree with only vertices of degrees one, two or three.

Then, we follow the procedure as in the proof of Theorem 3.1 for constructing a round fold map on M, except for the components corresponding to vertices of degree one whose label is different from k. Note that in the process described in the proof of [14, Theorem 3.1], we do not need to use $h_1: T^2 \times I \to \mathbb{R}^2$ in our situation. Furthermore, when we use h_2, we make sure that the corresponding image is contained in $A_{[0,k]}$. Finally, for the components corresponding to vertices of degree one with label $< k$, we just consider the projection $D^2 \times S^1 \to D^2$, where the target D^2 should be enlarged depending on the label. This matches with the construction for the adjacent components.

Now, it is not difficult to see that the resulting map $f: M \to \mathbb{R}^2$ is a directed round fold map. This completes the proof. □

4 Further results and open problems

4.1 Corollaries and examples

In this subsection, we give some corollaries of our main theorems. We also show that the class of 3–manifolds that admit directed round fold maps is strictly smaller than that of all graph 3–manifolds.

Corollary 4.1 *Suppose that M is a closed connected orientable graph 3–manifold. If $H_1(M;\mathbb{Q}) = 0$, then it admits a directed round fold map into $\mathbb{R}^2$.*

Proof Let G be the graph corresponding to a decomposition of M into $P \times S^1$ and $D^2 \times S^1$ as described in Lemma 2.9. Then, we can naturally construct a continuous map $\gamma: M \to G$ in such a way that for each piece, the complement of a small collar neighborhood of the boundary is mapped to the corresponding vertex. Then, we can show that γ induces a surjection $\gamma_*: \pi_1(M) \to \pi_1(G)$. Since $H_1(M;\mathbb{Q}) = 0$, we see that G is a tree. Then, the result follows from Theorem 3.2. □

Since every closed orientable Seifert 3–manifold over the 2–sphere admits a decomposition into a union of a finite number of copies of $P \times S^1$ and $D^2 \times S^1$ such that the corresponding graph is a tree, we have the following.

Corollary 4.2 *Every closed orientable Seifert 3–manifold over S^2 admits a directed round fold map into $\mathbb{R}^2$.*

By virtue of the realization result due to [1], as a corollary, we see that every linking form can be realized as that of a 3–manifold admitting a directed round fold map into $\mathbb{R}^2$. Thus, the linking form cannot detect the nonexistence of a directed round fold map.

On the other hand, as to the cohomology ring, we have the following.

Corollary 4.3 *If a closed orientable 3–manifold M admits a directed round fold map into $\mathbb{R}^2$, then for every pair $\xi, \eta \in H^1(M;\mathbb{Q})$, their cup product $\xi \smile \eta$ vanishes in $H^2(M;\mathbb{Q})$.*

The above corollary follows from [2, Theorem 5.2]. More precisely, let us consider the decomposition of M into the union of copies of $P \times S^1$ and $D^2 \times S^1$ attached along their torus boundaries such that the corresponding graph is a tree. As P and D^2 are of genus 0, and as the cohomology ring of $S^2 \times S^1$ satisfies the property described as in the corollary, we see that the cohomology ring of M also satisfies the same property.

Thus, for example, for every closed orientable surface Σ of genus ≥ 1, the 3–manifold $\Sigma \times S^1$ never admits a directed round fold map into $\mathbb{R}^2$, although it is a graph manifold.

Note that if we use coefficients other than $\mathbb{Q}$, the result might not hold. For example, $\mathbb{R}P^3$ admits a directed round fold map into $\mathbb{R}^2$, as it is the union of two copies of $D^2 \times S^1$ attached along their boundaries; however, for the generator of $H^1(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2$, its square does not vanish in $H^2(\mathbb{R}P^3; \mathbb{Z}_2)$. On the other hand, we do not know if the result in Corollary 4.3 holds for $\mathbb{Z}$–coefficients.

Now, let us consider the normal form plumbing graphs as explained in Section 2.2. The following lemma can be proved by following the proof of [12, Theorem 4.1].

Lemma 4.4 *If a closed connected orientable graph 3–manifold is decomposed into a union of finite numbers of copies of $P \times S^1$ and $D^2 \times S^1$ in such a way that the corresponding graph is a tree, then its normal form plumbing graph is a finite disjoint union of trees.*

Proof Let us consider the tree that represents a given decomposition into copies of $P \times S^1$ and $D^2 \times S^1$. This may not be of plumbing type; however, by inserting copies of $T^2 \times I$ if necessary, we may assume that the tree is of plumbing type.

On the other hand, as shown in the proof of [12, Theorem 4.1], there is an algorithm that turns a given plumbing graph into a normal form. It is not difficult to see that if we start with a tree, then each operation in the algorithm keeps the property that it is a finite disjoint union of trees. Then the result follows. □

As a corollary, we have the following.

Corollary 4.5 *Let M be a closed connected orientable graph 3–manifold whose normal-form plumbing graph contains a cycle. Then, M admits a round fold map into $\mathbb{R}^2$ but does not admit a directed round fold map into $\mathbb{R}^2$.*

For example, some torus bundles over S^1 as described in [12, Theorem 6.1] satisfy the assumption of the above corollary. (More precisely, those torus bundles over S^1 whose monodromy matrix has trace ≥ 3 or ≤ -3 give such examples.)

4.2 Open problems

Finally, we list some related open problems which may interest the reader.

Problem 4.6 (1) Generalize Theorems 3.1 and 3.2 for nonorientable 3–manifolds.

(2) The notion of round fold maps of 3–dimensional manifolds into $\mathbb{R}^2$ as in Definition 2.5 can naturally be generalized to that of round fold maps of n–dimensional manifolds into $\mathbb{R}^p$ for $n \geq p \geq 2$; for details, see [7; 6; 5]. For such a fixed pair (n, p) of dimensions, characterize those closed n–dimensional manifolds which admit round fold maps into $\mathbb{R}^p$.

For the dimension pair $(n, n-1)$, $n \geq 4$, such a generalization has been obtained in [8].

Problem 4.7 Classify the right–left equivalence classes of (directed) round fold maps on a given 3–manifold.

Refer to a certain classification result for simple stable maps given in [14]. For round fold maps of n–dimensional manifolds into $\mathbb{R}^{n-1}$, a classification result has been obtained in [8].

Recall that as explained in Section 2.1, a round fold map corresponds naturally to an open book structure. On the other hand, open book structures are closely related to contact structures on 3–manifolds; for example, see [3]. Therefore, the following problem seems to be reasonable.

Problem 4.8 Clarify the relationship between round fold maps and contact structures on 3–dimensional manifolds through open book decompositions.

One of the main motivations of Neumann's work [12] on plumbing graphs is to analyze the topology of the links of normal surface singularities. The following questions have been addressed by quite a few topologists to the authors.

Problem 4.9 Is there any relation between singularity links and round fold maps? Is it possible to construct explicit round fold maps on the singularity links in a natural way?

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