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The upsilon invariant at 1 of 3–braid knots

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We provide explicit formulas for the integer-valued smooth concordance invariant $\nu(K) = \Upsilon_K(1)$ for every 3–braid knot K . We determine this invariant, which was defined by Ozsváth, Stipsicz and Szabó (2017), by constructing cobordisms between 3–braid knots and (connected sums of) torus knots. As an application, we show that for positive 3–braid knots K several alternating distances all equal the sum $g(K) + \nu(K)$, where $g(K)$ denotes the 3–genus of K . In particular, we compute the alternation number, the dealternating number and the Turaev genus for all positive 3–braid knots. We also provide upper and lower bounds on the alternation number and dealternating number of every 3–braid knot which differ by 1.

57K10; 20F36, 57K18

1 Introduction

We study *knots* in the 3–sphere S^3 , ie nonempty, connected, oriented, closed, smooth 1–dimensional submanifolds of S^3 , considered up to ambient isotopy. Two knots K and J are called *concordant* if there exists an annulus $A \cong S^1 \times [0, 1]$ smoothly and properly embedded in $S^3 \times [0, 1]$ such that $\partial A = K \times \{0\} \cup J \times \{1\}$ and such that the induced orientation on the boundary of the annulus agrees with the orientation of K , but is the opposite one on J . Knots up to concordance form a group, the *concordance group* \mathcal{C} , with the group operation induced by connected sum.

In [46], Ozsváth, Stipsicz and Szabó used the Heegaard Floer knot complex to define the invariant Υ_K of a knot K , which induces a homomorphism from the knot concordance group to the group of real-valued piecewise-linear functions on the interval $[0, 2]$. The function Υ_K evaluated at $t = 1$, $\nu(K) := \Upsilon_K(1)$, induces a homomorphism $\mathcal{C} \rightarrow \mathbb{Z}$. In this article, we will call $\nu(K)$ *upsilon of K* .

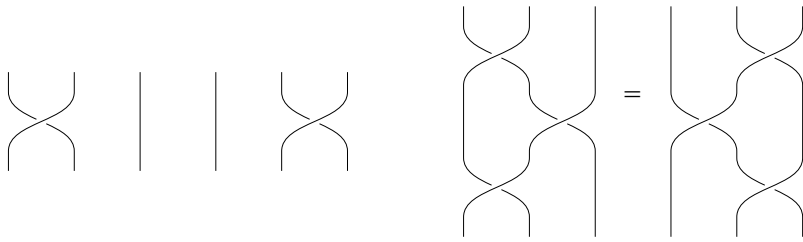


Figure 1: Generators and relation in the braid group B_3 . Left: the two generators a and b . Right: the braid relation $aba = bab$.

A 3–braid is an element of the *braid group on three strands*, denoted B_3 . The classical presentation of B_3 with generators a and b and relation $aba = bab$, the *braid relation*, was introduced by Artin [5]. A *braid word* γ — a word in the generators of B_3 and their inverses — defines a diagram for a (geometric) 3–braid; the generators a and b correspond to the geometric 3–braids given by braid diagrams as in Figure 1. In our figures, braid diagrams will always be oriented from bottom to top. We denote by Δ the braid $aba = bab$, and note that its square $\Delta^2 = (ab)^3$ (the positive full twist on three strands) generates the center of B_3 ; see Chow [14, Theorem 3]. A 3–braid knot is a knot that arises as the closure $\hat{\gamma}$ of a 3–braid γ .

As our main result, we determine the upsilon invariant for all 3–braid knots. More precisely, we show the following.

Theorem 1.1 *Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} a^{-p_2} b^{q_2} \dots a^{-p_r} b^{q_r}$ be a braid word in the generators a and b of B_3 for some integers $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$, where $\Delta^2 = (ab)^3$. Suppose that the closure $K = \hat{\gamma}$ of γ is a knot. Then its upsilon invariant is*

$$v(K) = \frac{1}{2} \left(\sum_{i=1}^r (p_i - q_i) \right) - 2\ell.$$

By Murasugi’s classification of the conjugacy classes of 3–braids [45, Proposition 2.1], indeed all 3–braid knots — except for the torus knots that are closures of 3–braids — are covered by Theorem 1.1. However, for torus knots the invariant v can be calculated explicitly by a combinatorial, inductive formula in terms of their Alexander polynomial [46, Theorem 1.15]; see (12) below. Hence, we have indeed determined $v(K)$ for all 3–braid knots K .

As an application of Theorem 1.1, we show that the following invariants coincide for positive 3–braid knots — knots that are the closure of positive 3–braids.

Corollary 1.2 *Let K be a knot that is the closure of a **positive 3–braid**, ie an element of B_3 that can be written as a word in the generators a and b only (no inverses). Then*

$$\text{alt}(K) = \text{dalt}(K) = g_T(K) = \mathcal{A}_s(K) = g(K) + v(K).$$

Here, the *alternation number* $\text{alt}(K)$, *dealternating number* $\text{dalt}(K)$ and *Turaev genus* $g_T(K)$ are different ways of measuring how far the knot K is from being alternating. The best known among them is certainly the first one: the alternation number $\text{alt}(K)$ of a knot K was first defined by Kawauchi [31] as the minimal *Gordian distance* of K to the set of alternating knots. In Section 5, we will review the precise definition and prove Corollary 1.2. The invariant $\mathcal{A}_s(K)$ introduced by Friedl, Livingston and Zentner [23] is defined as the minimal number of double point singularities in a generically immersed concordance from a knot K to an alternating knot. Lastly, $g(K)$ denotes the *3–genus* of K , the minimal genus of a compact, connected, oriented, smooth surface in S^3 with oriented boundary the knot K .

Two other corollaries of Theorem 1.1 for positive 3–braid knots are the following.

Corollary 1.3 *Let K be a positive 3–braid knot. Then the minimal r such that K is the closure of $a^{p_1}b^{q_1}a^{p_2}b^{q_2} \dots a^{p_r}b^{q_r}$ for positive integers p_i and q_i , for $i \in \{1, \dots, r\}$, is $r = g(K) + v(K) + 1$.*

Corollary 1.4 *If K and J are concordant knots that are both closures of positive 3–braids, then the minimal r from Corollary 1.3 is the same for both K and J .*

Proposition 3.2 provides a normal form for 3–braids, the *Garside normal form*, which is different from Murasugi’s normal form mentioned above (see Definition 4.15). The Garside normal form allows us to read off from a braid word whether it is conjugate to a positive braid word. In Section 6, we provide formulas for the fractional Dehn twist coefficient for all 3–braids in Garside normal form; see Corollary 6.1.

Proof strategy for Theorem 1.1 A crucial property of the invariant v is that it provides a lower bound on the *4–genus* $g_4(K)$ of a knot K , the minimal genus of a compact, connected, oriented surface smoothly embedded in the 4–ball B^4 with oriented boundary the knot K in $S^3 = \partial B^4$: we have

$$(1) \quad |v(K)| \leq g_4(K)$$

for any knot K [46, Theorem 1.11]. Our general strategy to find $\nu(K)$ for any 3–braid knot K will be to construct a cobordism between K and another knot J for which the value of ν is known. A *cobordism* between K and J is a smoothly and properly embedded oriented surface C in $S^3 \times [0, 1]$ with boundary $K \times \{0\} \cup J \times \{1\}$ such that the induced orientation on the boundary of C agrees with the orientation of K and disagrees with the orientation of J . We have

$$(2) \quad |\nu(K) - \nu(J)| \leq g(C)$$

for any cobordism C between K and J , where $g(C)$ denotes the genus of the cobordism; see inequality (15) in Section 4.1. This provides bounds on $\nu(K)$ in terms of $\nu(J)$ and $g(C)$.

We will find such cobordisms for example by algebraic modifications of a braid word representing K and by saddle moves corresponding to the addition or deletion of generators from such braid words. We will also repeatedly make use of the trick described in Example 4.1 in Section 4.1 of looking at cobordisms of genus 1 between $\hat{\gamma} \# T_{2,2n+1}$ and $\widehat{\gamma b^{2n}}$ for 3–braid words γ and $n \geq 1$.

To prove Theorem 1.1, we will first determine ν for all positive 3–braid knots and then generalize our computations to all 3–braid knots. This extension was somewhat unexpected for the author since, in contrast, the same method would not work to determine slice-torus invariants — see Lewark [33] — like the invariant τ defined by Ozsváth and Szabó [48] or Rasmussen’s invariant s [50] for all 3–braid knots. We will elaborate on this in Section 4.4.2.

Remark 1.5 As we will only use properties of the upsilon invariant (see Section 2.2) and not its definition, we can similarly determine any concordance homomorphism $C \rightarrow \mathbb{Z}$ whose absolute value bounds the 4–genus of a knot from below and which takes the same value as ν on torus knots of braid index 2 and 3. An example is $-\frac{1}{2}t$ for the concordance invariant t constructed by Ballinger [8] from the $E(-1)$ spectral sequence on Khovanov homology. The invariant t defines a concordance homomorphism valued in the even integers which satisfies $|\frac{1}{2}t(K)| \leq g_4(K)$ for any knot K [8, Theorem 1.1]. Moreover, it fulfills $t(T_{p,q}) = -2\nu(T_{p,q})$ for the torus knots $T_{p,q}$ for any coprime positive integers p and q [8, page 22]. The same method we use for the proof of Theorem 1.1 shows that $t(K) = -2\nu(K)$ for any 3–braid knot K .

Remark 1.6 Theorem 1.1 and a result of Erle [17] imply that $\sigma(K) = 2\nu(K)$ for all 3–braid knots K except when $K = \pm T_{3,3\ell+k}$ for odd $\ell > 0$ and $k \in \{1, 2\}$. Here $\sigma(K)$

denotes the classical signature of the knot K ; see Trotter [54].¹ In the exceptional cases, $\sigma(K) = 2\nu(K) - 2$. This observation improves a result by Feller and Kratovich who showed that $|\nu(K) - \frac{1}{2}\sigma(K)| \leq 2$ for all 3-braid knots K [20, Proposition 4.4]; see also Section 4.4.1.

Organization The remainder of this article is organized as follows. In Section 2, we will provide the necessary background on (positive) braids and the epsilon invariant before providing a normal form for 3-braids (Proposition 3.2) that we call the Garside normal form in Section 3. Then in Section 4, after a more detailed outline of our proof strategy (Section 4.1), we will prove Theorem 1.1 first for positive 3-braid knots (Section 4.2) and afterwards in the general 3-braid case (Section 4.3). We will prove Corollaries 1.3 and 1.4 in Section 4.2. Section 4.4 will provide further context on our results. Section 5 is concerned with the proof of Corollary 1.2 (Section 5.1) and the application of our result about the epsilon invariant to alternating distances of general 3-braid knots (Section 5.2). In particular, we determine the alternation number of any 3-braid knot up to an additive error of at most 1. Finally, in Section 6, we determine the fractional Dehn twist coefficient for all 3-braids in Garside normal form.

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2 Preliminaries

We recall important concepts about knots and braids, and also the necessary properties of the epsilon invariant and the knot invariant τ coming from Heegaard Floer homology.

2.1 Knots and braids

By a fundamental theorem of Alexander [4], every knot in S^3 can be represented as the closure of a geometric n -braid for some positive integer n . An n -braid is an element

¹We use the standard signature convention that the positive torus knots have negative signatures, eg $\sigma(T_{3,2}) = -2$.

of the *braid group on n strands*, denoted by B_n , which is presented by $n - 1$ generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1};$$

see [5]. We call a word in the generators of B_n and their inverses a *braid word*. A braid word defines a diagram for a (geometric) n -braid where the generators σ_i of the braid group correspond to the geometric n -braids given by the braid diagrams in which the i^{th} and $(i+1)^{\text{st}}$ strands cross once positively. In the following, we will always identify braid words with the corresponding geometric braids, and we suppress n if the context is clear.

By gluing the top ends of the (oriented) strands of a geometric braid $\gamma \in B_n$ to the corresponding bottom ends, we get a knot (or link) $\hat{\gamma}$, called the *closure* of γ . If γ induces a permutation with only one cycle on the ends of its n strands, then its closure $\hat{\gamma}$ is a knot and we call it an *n -braid knot*. Note that conjugate braids $\gamma_0, \gamma_1 \in B_n$, denoted by $\gamma_0 \sim \gamma_1$, have isotopic closures $\hat{\gamma}_0 = \hat{\gamma}_1$. For a more detailed account on braids, we refer the reader to [10].

A *positive braid* is an element of the braid group B_n for some n that can be written as a positive braid word $\sigma_{s_1} \sigma_{s_2} \cdots \sigma_{s_l}$ with $s_i \in \{1, \dots, n - 1\}$. A knot is called a *positive braid knot* if it can be represented as the closure of a positive braid. The set of positive braid knots contains the sets of (positive) torus knots and algebraic knots, while itself being a subset of the set of positive knots or, more generally, the frequently studied set of (strongly) quasipositive knots.

Let $\text{wr}(\gamma)$ denote the *writhe* of a braid word $\gamma \in B_n$, ie the exponent sum of the word γ . If γ is a positive n -braid such that $K = \hat{\gamma}$ is a knot, then, by work of Bennequin [9] and Rudolph [51] — the latter building on Kronheimer and Mrowka's proof of the local Thom conjecture [32] — we have

$$(3) \quad g_4(K) = g(K) = \frac{1}{2}(\text{wr}(\gamma) - n + 1).$$

2.2 The concordance invariants τ and Υ

In [48], Ozsváth and Szabó constructed the knot invariant τ via the knot filtration on the Heegaard Floer chain complex of S^3 ; the latter was also defined independently by Rasmussen [49]. The invariant τ induces a group homomorphism $\mathcal{C} \rightarrow \mathbb{Z}$ from the (smooth) knot concordance group \mathcal{C} to the group of integers \mathbb{Z} and gives a lower bound on the 4-ball genus $g_4(K)$: we have $|\tau(K)| \leq g_4(K)$ for any knot K . For the torus

knots $T_{p,q}$, where p and q are coprime positive integers, the invariant τ recovers the 3-genus [48, Corollary 1.7]; namely,

$$(4) \quad \tau(T_{p,q}) = g(T_{p,q}) = \frac{1}{2}(p-1)(q-1).$$

Moreover, it follows from [34, Theorem 4 and Corollary 7] together with (3) above that, for any knot K that is the closure of a positive n -braid γ ,

$$(5) \quad \tau(K) = \frac{1}{2}(\text{wr}(\gamma) - n + 1) = g_4(K) = g(K).$$

The invariant Υ was defined by Ozsváth, Stipsicz and Szabó in [46]. We will not recall the definition of Υ via the knot Floer complex $CFK^\infty(K)$ since the properties of Υ mentioned below will be enough for our later computations and we will not explicitly use the Heegaard Floer theory behind it. For an overview on the properties of Υ , see the original article [46] or Livingston’s notes on Υ [35]; see [28] for a survey on Heegaard Floer homology and knot concordance.

For every knot K , the knot invariant $\Upsilon_K : [0, 1] \rightarrow \mathbb{R}$ is a continuous, piecewise linear function with the following properties [46]:

$$(6) \quad \Upsilon_K(0) = 0,$$

$$(7) \quad \text{the slope of } \Upsilon_K(t) \text{ at } t = 0 \text{ is given by } -\tau(K),$$

$$(8) \quad \Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t) \text{ for all } 0 \leq t \leq 1 \text{ and all knots } K_1 \text{ and } K_2,$$

$$(9) \quad \Upsilon_{-K}(t) = -\Upsilon_K(t) \text{ for all } 0 \leq t \leq 1,$$

$$(10) \quad |\Upsilon_K(t)| \leq g_4(K)t \text{ for all } 0 \leq t \leq 1.$$

Here, $-K$ is the knot obtained by mirroring K and reversing its orientation. Its concordance class is the inverse of the class of K in the knot concordance group \mathcal{C} . It follows from (8)–(10) that Υ induces a homomorphism from the concordance group to the group of real-valued piecewise-linear functions on the interval $[0, 1]$.

For some classes of knots, the invariant Υ can be explicitly computed in terms of classical knot invariants like the signature and the Alexander polynomial.

Proposition 2.1 [46, Theorem 1.14] *We have $\Upsilon_K(t) = \frac{1}{2}\sigma(K)t$ for all alternating or quasialternating knots K and all $0 \leq t \leq 1$.*

For positive torus knots, $\Upsilon_K(t)$ is completely determined by a combinatorial formula in terms of their Alexander polynomial [46, Theorem 1.15]. For torus knots of braid index 2 or 3, the following holds; see eg [18]. For $\ell \geq 0$,

$$(11) \quad \Upsilon_{T_{2,2\ell+1}}(t) = -\tau(T_{2,2\ell+1}) \cdot t = -\ell \cdot t \quad \text{for all } 0 \leq t \leq 1.$$

For $\ell \geq 0$ and $k \in \{1, 2\}$,

$$\begin{aligned}
 & \Upsilon_{T_{3,3\ell+1}}(1) = \Upsilon_{T_{3,3\ell+2}}(1) + 1 = -2\ell, \\
 (12) \quad & \Upsilon_{T_{3,3\ell+k}}(t) = -\tau(T_{3,3\ell+k})t = -(3\ell + k - 1)t \quad \text{for all } 0 \leq t \leq \frac{2}{3} \\
 & \Upsilon_{T_{3,3\ell+k}}(t) \text{ is linear on } \left[\frac{2}{3}, 1\right].
 \end{aligned}$$

3 The Garside normal form for 3–braids

In this section, we provide a classification result on the conjugacy classes of 3–braids; see Proposition 3.2. This result is basically due to work of Garside [25] who gave the first solution to the conjugacy problem for all braid groups B_n with $n \geq 3$ in 1965. Proposition 3.2 might be known to the experts, but since the explicit formulas appear to be missing from the literature, we will provide them here.

Throughout, we denote the two generators of the braid group B_3 by $a := \sigma_1$ and $b := \sigma_2$ which are subject to the braid relation $aba = bab$. Recall that the braid $\Delta^2 = (aba)^2 = (ab)^3$ generates the center of B_3 .

Remark 3.1 Any 3–braid is conjugate to the same braid with generators a and b interchanged. More precisely, let $\gamma = a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for some $r \geq 1$ and integers p_i and q_i for $i \in \{1, \dots, r\}$ be a 3–braid. Then using $\Delta a = b\Delta$ and $\Delta b = a\Delta$, we have

$$\gamma = \Delta^{-1} \Delta a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r} = \Delta^{-1} b^{p_1} a^{q_1} \dots b^{p_r} a^{q_r} \Delta \sim b^{p_1} a^{q_1} \dots b^{p_r} a^{q_r}.$$

In Proposition 3.2, we will provide a certain standard form for the conjugacy classes of 3–braids.

Proposition 3.2 *Let γ be a 3–braid. Then γ is conjugate to one of the 3–braids*

- (A) $\Delta^{2\ell} a^p, \quad \ell \in \mathbb{Z}, p \geq 0,$
- (B) $\Delta^{2\ell} a^p b, \quad \ell \in \mathbb{Z}, p \in \{1, 2, 3\},$
- (C) $\Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}, \quad \ell \in \mathbb{Z}, r \geq 1, p_i, q_i \geq 2, i \in \{1, \dots, r\},$
- (D) $\Delta^{2\ell+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}, \quad \ell \in \mathbb{Z}, r \geq 1, p_r, p_i, q_i \geq 2, i \in \{1, \dots, r-1\}.$

If γ is a positive 3–braid, then $\ell \geq 0$. If $\hat{\gamma}$ is a knot, then only cases (B)–(D) can occur and p must be odd in case (B), at least one of the p_i and one of the q_i must be odd in case (C), and at least one of the p_i or q_i must be odd in case (D).

While we will never use it in this article, we note — without proof — the following uniqueness result related to Proposition 3.2.

Remark 3.3 Up to cyclic permutation of the powers $p_1, q_1, \dots, p_r, q_r$ in (C) and $p_1, q_1, \dots, p_{r-1}, q_{r-1}, p_r$ in (D), each 3-braid is conjugate to exactly one of the 3-braids listed in Proposition 3.2. This follows from Garside’s work [25]. In his notation, each of the 3-braids listed in (A)–(D) in Proposition 3.2 is the standard form of a certain element in the (so-called) summit set of γ . For 3-braids of the form (C) or (D), the summit set consists of those 3-braids obtained by cyclic permutation of the powers $p_1, q_1, \dots, p_r, q_r$ in (C) and $p_1, q_1, \dots, p_{r-1}, q_{r-1}, p_r$ in (D), respectively.

Definition 3.4 We call a braid word of the form in (A)–(D) a 3-braid in Garside normal form.

Remark 3.5 The advantage of the Garside normal form over Murasugi’s normal form for 3-braids used later in Section 4.3 (see Definition 4.15) is that positive 3-braids are easier to detect in this normal form: if γ is a positive 3-braid, then γ is conjugate to one of the braids in (A)–(D) with $\ell \geq 0$. Since Garside’s solution to the conjugacy problem works for any n -braid with $n \geq 3$, one might hope to generalize an explicit standard form as in Proposition 3.2 to n -braids for any $n \geq 3$.

Remark 3.6 For odd p , case (B) of Proposition 3.2 covers the torus knots of braid index 3. More precisely, if $\gamma \sim \Delta^{2\ell} ab = (ab)^{3\ell+1}$, then its closure is $\hat{\gamma} = T_{3,3\ell+1}$ for $\ell \geq 0$ and $\hat{\gamma} = -T_{3,3(-\ell-1)+2}$ for $\ell < 0$, and if $\gamma \sim \Delta^{2\ell} a^3 b \sim (ab)^{3\ell+2}$, then $\hat{\gamma} = T_{3,3\ell+2}$ for $\ell \geq 0$ and $\hat{\gamma} = -T_{3,3(-\ell-1)+1}$ for $\ell < 0$.

Proof of Proposition 3.2 The proof will follow from the following claim.

Claim 1 Let γ be a positive 3-braid. Then γ is conjugate to one of the 3-braids in (A)–(D) with $\ell \geq 0$.

We first deduce Proposition 3.2 from this claim. To that end, let γ be any 3-braid. If γ is a positive braid, we are done by Claim 1. If not, then γ can be written in the form $\gamma = \Delta^m \alpha$ where m is a negative integer and α a positive 3-braid [25, Theorem 5]. In fact, inserting $\Delta^{-1} \Delta$ if m is odd, we can assume γ to be of the form $\Delta^{-2n} \alpha$ for some $n \geq 1$ and a positive 3-braid α . The proposition then easily follows using the claim for α . It remains to prove Claim 1.

Proof of Claim 1 A positive 3-braid γ has the form $\gamma = a^{P_1} b^{Q_1} \dots a^{P_R} b^{Q_R}$ for integers $R \geq 1$ and $P_i, Q_i \geq 0$ for $i \in \{1, \dots, R\}$. If all the P_i or all the Q_i are 0, then

(possibly using Remark 3.1) γ is conjugate to a^p for some $p \geq 0$ and we are in case (A) for $\ell = 0$. Possibly after conjugation and reduction of R , we can thus assume that all of the integers P_i and Q_i are nonzero. If $P_1, Q_1 \geq 2$ applies for all $i \in \{1, \dots, R\}$, then γ is of the form in (C) for $\ell = 0$. If $R = 1$, ie $\gamma = a^{P_1}b^{Q_1}$ for integers $P_1, Q_1 \geq 1$, and $P_1 = 1$ or $Q_1 = 1$, then (possibly using Remark 3.1) γ is conjugate to a braid of the form in (B).

It remains to consider the case where $R \geq 2$ and at least one of the P_i or Q_i is 1. In that case — if necessary after conjugation — γ contains $\Delta = aba = bab$ as a subword and is thus conjugate to $\Delta\alpha$ for some positive 3–braid α . Now, let $n \geq 1$ be maximal with the property that γ is conjugate to $\Delta^n\alpha$ for some positive 3–braid α . Then, possibly after conjugation of γ , the braid word α must be one of the following:

$$\begin{aligned}
 & a^p, & p \geq 0, \\
 & a^p b, & p \geq 1, \\
 (13) \quad & a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}, & r \geq 1, p_i, q_i \geq 2, i \in \{1, \dots, r\}, \\
 & a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}, & r \geq 1, p_r \geq 2, p_i, q_i \geq 2, i \in \{1, \dots, r-1\}.
 \end{aligned}$$

Indeed, using Remark 3.1, up to conjugation these are the only possible words such that $\Delta^n\alpha$ does not contain any additional Δ as a subword. Note that α can be the empty word, which is covered by the first case in (13) for $p = 0$. Further, note that

$$\begin{aligned}
 & \Delta^{2\ell} a^p b \sim \Delta^{2\ell+1} a^{p-2}, \\
 & \Delta^{2\ell+1} a \sim \Delta^{2\ell} a^3 b, \\
 (14) \quad & \Delta^{2\ell+1} a^p b \sim \Delta^{2\ell+1} a^{p+1}, \\
 & \Delta^{2\ell+1} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r} \sim \Delta^{2\ell+1} a^{p_1+q_r} b^{q_1} a^{p_2} \dots b^{q_{r-1}} a^{p_r}, \\
 & \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \sim \Delta^{2\ell} a^{p_1+p_r} b^{q_1} a^{p_2} \dots a^{p_{r-1}} b^{p_{r-1}}
 \end{aligned}$$

for any $\ell \geq 0, p \geq 1$ and $p_i, q_i \geq 2$ for $i \in \{1, \dots, r\}$. It follows from a case by case analysis of the cases in (13), using (14) and taking the parity of n into account, that any positive 3–braid is conjugate to one of the 3–braids in (A)–(D) with $\ell \geq 0$. □

4 The upsilon invariant of 3–braid knots

In this section, we prove Theorem 1.1. Along the way, we compute the invariant v for positive 3–braid knots in Garside normal form (Proposition 4.2) and prove Corollaries 1.3 and 1.4.

4.1 Methodology

We first recall inequality (2) from the introduction — which will be repeatedly used in Section 4 — in more generality.

The cobordism distance $d(K, J)$ between two knots K and J is defined as the 4-genus $g_4(K \# -J)$ of the connected sum of K and the inverse of J . Equivalently, the cobordism distance $d(K, J)$ could be defined as the minimal genus of a smoothly and properly embedded oriented surface C in $S^3 \times [0, 1]$ with boundary $K \times \{0\} \cup J \times \{1\}$ such that the induced orientation on the boundary of C agrees with the orientation of K and disagrees with the orientation of J . Suppose the genus of a cobordism C between two knots K and J is $g(C)$. We then have $d(K, J) \leq g(C)$, so by the properties (8)–(10) of Υ from Section 2.2 we get

$$(15) \quad |\Upsilon_K(t) - \Upsilon_J(t)| = |\Upsilon_{K\#-J}(t)| \leq g_4(K \# -J)t = d(K, J)t \leq g(C)t$$

for all $0 \leq t \leq 1$. This provides bounds on $\Upsilon_K(t)$ in terms of $\Upsilon_J(t)$ and $g(C)$.

We now give an example for the cobordisms we will use later on.

Example 4.1 Among other things, we will frequently use the following trick the author first saw in [20, Example 4.5]. Let γ be a 3-braid such that $K = \hat{\gamma}$ is a knot. Consider the 3-braid $\alpha := \gamma b^{2n}$ for some $n \geq 1$. Then $\hat{\alpha}$ is also a knot and there is a cobordism between $\hat{\alpha}$ and the connected sum $K \# T_{2,2n+1}$ of genus 1. This cobordism can be realized by two saddle moves (1-handle attachments) of the form shown in Figure 2, right, performed in the two circled regions of Figure 2, left. One of them is used to add a generator b to the braid α to obtain the braid word γb^{2n+1} and the other is used to transform the closure of this new braid word into a connected sum of K and $T_{2,2n+1}$. Recall that our braid diagrams are oriented from bottom to top.

Using $v(T_{2,2n+1}) = -n$ by (11) and that the genus of the cobordism is 1, by (15) for $t = 1$ we have

$$(16) \quad |v(\hat{\alpha}) - v(K \# T_{2,2n+1})| \leq 1 \iff |v(\hat{\alpha}) - v(K) + n| \leq 1,$$

which provides the lower bound $v(K) \geq v(\hat{\alpha}) + n - 1$ on $v(K)$.

4.2 The epsilon invariant of positive 3-braid knots

In this section, we determine the invariant v for all positive 3-braid knots.

By Proposition 3.2 and Remark 3.6, positive 3-braid knots are either the torus knots $T_{3,3\ell+k}$ for $\ell \geq 0$ and $k \in \{1, 2\}$ which have braid representatives of Garside normal

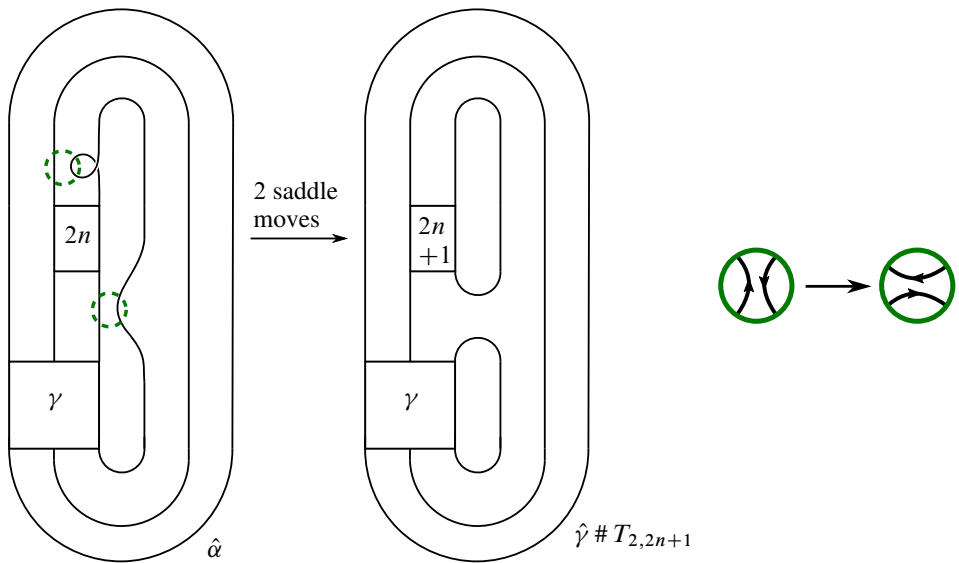


Figure 2: An example illustrating our proof strategy. Left: a schematic of a cobordism between the knots $\hat{\alpha}$ and $\hat{\gamma} \# T_{2,2n+1}$ realized by two saddle moves. Right: a saddle move.

form (B), or closures of positive 3–braids of Garside normal form (C) or (D) (see Definition 3.4). The following proposition thus proves Theorem 1.1 for all positive 3–braid knots.

Proposition 4.2 *Let γ be a positive 3–braid such that $K = \hat{\gamma}$ is a knot. Then*

$$\nu(K) = \begin{cases} -2\ell - \frac{1}{2}(p - 1) & \text{if } \gamma \text{ is conjugate to a braid in (B),} \\ -\frac{1}{2}(\sum_{i=1}^r (p_i + q_i)) + r - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in (C),} \\ -\frac{1}{2}(\sum_{i=1}^{r-1} (p_i + q_i) + p_r) + r - 2\ell - \frac{3}{2} & \text{if } \gamma \text{ is conjugate to a braid in (D).} \end{cases}$$

Remark 4.3 In fact, the formulas from Proposition 4.2 also give the correct upsilon invariant in terms of the Garside normal form of a 3–braid representative of a knot K if K is the closure of any 3–braid in Garside normal form (C) or (D), not necessarily a positive one. This follows from Theorem 1.1 (proved in the next section) and the observations of Section 4.4.3.

Recall that for the torus knots of braid index 3, we know the invariant ν by (12). In the following, we will determine the invariant ν for all knots that are closures of positive 3–braids of Garside normal form (C) or (D).

We first provide an upper bound on $\Upsilon_K(t)$ for positive 3-braid knots K and $0 \leq t \leq 1$. The following inequality (17) in Lemma 4.4 could also be shown using the dealternating number and a result of Abe and Kishimoto [2, Lemma 2.2], whereas the main work for the upper bound on ν for the knots in the second and third case in Proposition 4.2 will be to rewrite the braid words representing these knots. We use the approach below since it will also give bounds on the minimal cobordism distance between any positive 3-braid knot and an alternating knot; see Remark 4.14.

Lemma 4.4 *Let $\gamma = a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ be a positive 3-braid, where $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ are integers such that $K = \hat{\gamma}$ is a knot. Then*

$$(17) \quad \Upsilon_K(t) \leq (-g(K) + r - 1)t \quad \text{for all } 0 \leq t \leq 1.$$

Proof We claim that there is a cobordism C of genus

$$(18) \quad g(C) = \frac{1}{2}(r - 1 + \varepsilon)$$

between K and the connected sum

$$J_\varepsilon = T_{2, \sum_{i=1}^r p_i + \varepsilon_p} \# T_{2, q_1 + \varepsilon_1} \# T_{2, q_2 + \varepsilon_2} \# \dots \# T_{2, q_r + \varepsilon_r},$$

where $\varepsilon_1, \dots, \varepsilon_r, \varepsilon_p \in \{0, 1\}$ are chosen such that J_ε is a connected sum of torus knots (rather than links), ie such that $\sum_{i=1}^r p_i + \varepsilon_p, q_1 + \varepsilon_1, q_2 + \varepsilon_2, \dots, q_r + \varepsilon_r$ are all odd, and $\varepsilon := \varepsilon_p + \sum_{i=1}^r \varepsilon_i$. This cobordism C can be realized by $r - 1 + \varepsilon$ saddle moves as follows. Following the schematic in Figure 3, we add ε generators b by ε saddle moves and additionally perform $r - 1$ saddle moves of the form shown in Figure 2, right, in the circled regions of Figure 3. In Figure 3, a box on the left labeled p_i or q_i stands for the positive braid a^{p_i} or b^{q_i} , respectively. The Euler characteristic of the cobordism C is $\chi(C) = -r + 1 - \varepsilon$. Since C is connected and—as J_ε and K are knots—has two boundary components, the genus of C is $g(C) = -\frac{1}{2}\chi(C) = \frac{1}{2}(r - 1 + \varepsilon)$ as claimed. By (15), we get $|\Upsilon_K(t) - \Upsilon_{J_\varepsilon}(t)| \leq g(C)t$ for all $0 \leq t \leq 1$; hence

$$(19) \quad \Upsilon_K(t) \leq \Upsilon_{J_\varepsilon}(t) + g(C)t \quad \text{for all } 0 \leq t \leq 1.$$

By (8) and (11) from Section 2.2,

$$\begin{aligned} \Upsilon_{J_\varepsilon}(t) &= \left(-\frac{1}{2} \left(\sum_{i=1}^r p_i + \varepsilon_p - 1 \right) - \frac{1}{2}(q_1 + \varepsilon_1 - 1) - \frac{1}{2}(q_2 + \varepsilon_2 - 1) - \dots - \frac{1}{2}(q_r + \varepsilon_r - 1) \right) t \\ &= -\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) - (r + 1) + \varepsilon \right) t, \end{aligned}$$

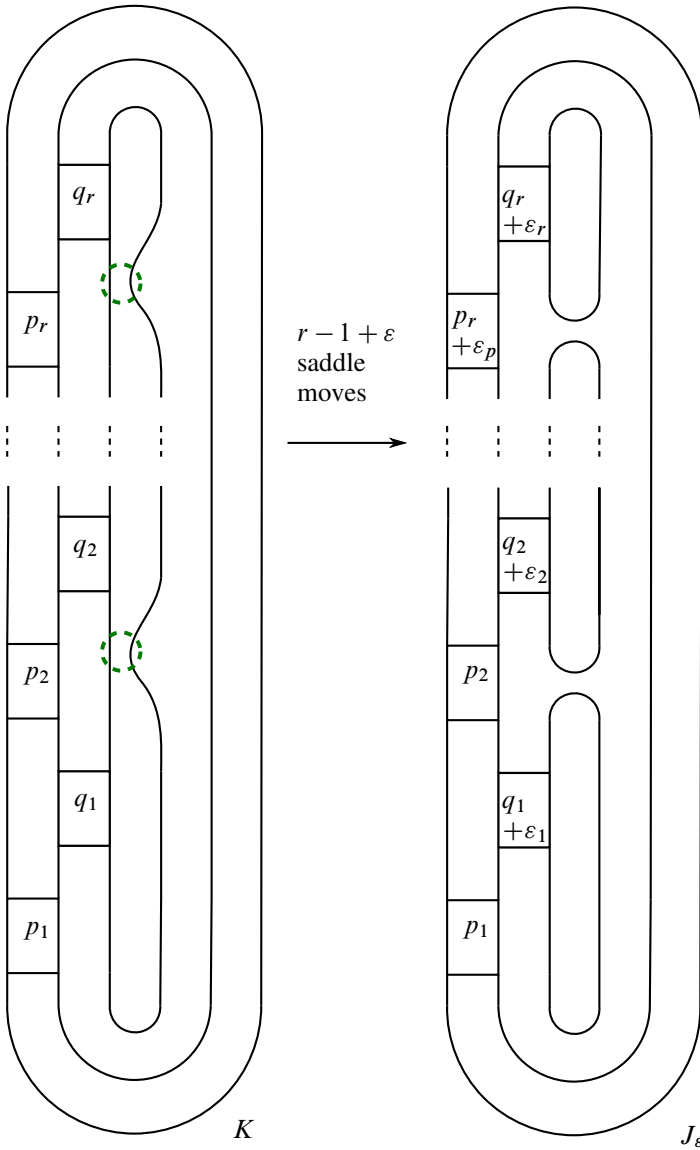


Figure 3: A schematic of a cobordism between $K = \hat{\gamma}$ and the connected sum of torus knots $J_\varepsilon = T_{2, \sum_{i=1}^r p_i + \varepsilon_p} \# T_{2, q_1 + \varepsilon_1} \# T_{2, q_2 + \varepsilon_2} \# \cdots \# T_{2, q_r + \varepsilon_r}$ realized by $r - 1 + \varepsilon$ saddle moves.

so (18) and (19) imply

$$\Upsilon_K(t) \leq \left(-\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) \right) + r \right) t \quad \text{for all } 0 \leq t \leq 1.$$

The claim follows, since by (3),

$$g(K) = \frac{\text{wr}(\gamma) - 2}{2} = \frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) - 2 \right). \quad \square$$

The following two lemmas improve the bound from Lemma 4.4 for knots that are closures of positive 3-braids of Garside normal form (C) or (D), respectively.

Lemma 4.5 *Let $\gamma = \Delta^{2\ell+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ for some $\ell \geq 0, r \geq 1, p_r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r-1\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$\Upsilon_K(t) \leq \left(-\frac{1}{2} \left(\sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + r - 2\ell - \frac{3}{2} \right) t \quad \text{for all } 0 \leq t \leq 1.$$

In the proof of Lemma 4.5, we will use that in B_3 ,

$$(20) \quad (ab)^{3n+1} = ab\Delta^{2n} = a^2ba^3(aba^3)^{n-1}ba^n \quad \text{for all } n \geq 1,$$

where $\Delta^2 = (aba)^2 = (ab)^3 = (ba)^3$; see [18, Proof of Proposition 22].

Proof of Lemma 4.5 Let $\Sigma_\gamma = \sum_{i=1}^{r-1} (p_i + q_i) + p_r$ and note that using (3),

$$(21) \quad g(K) = \frac{1}{2}(3(2\ell + 1) + \Sigma_\gamma - 2) = \frac{1}{2}\Sigma_\gamma + 3\ell + \frac{1}{2}.$$

If $\ell = 0$, then $\gamma = \Delta a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ is conjugate to

$$\gamma_1 = a^{p_1+1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r+1} b$$

and $\hat{\gamma}_1 = \hat{\gamma} = K$, so $g(\hat{\gamma}_1) = \frac{1}{2}\Sigma_\gamma + \frac{1}{2}$. By Lemma 4.4,

$$\Upsilon_K(t) \leq (-g(\hat{\gamma}_1) + r - 1)t = \left(-\frac{1}{2}\Sigma_\gamma + r - \frac{3}{2} \right) t \quad \text{for all } 0 \leq t \leq 1.$$

For $\ell \geq 1$, using $\Delta^{2\ell+1} = (ab)^{3\ell} aba = (ab)^{3\ell+1} a$,

$$\begin{aligned} \gamma &= \Delta^{2\ell+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \\ &= (ab)^{3\ell+1} a^{p_1+1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r} \\ &= a^2ba^3(aba^3)^{\ell-1}ba^{p_1+\ell+1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \quad \text{(by (20))} \\ &\sim a^{p_r+2}ba^3(aba^3)^{\ell-1}ba^{p_1+\ell+1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}} =: \gamma_1. \end{aligned}$$

We have $\hat{\gamma}_1 = \hat{\gamma} = K$ and $g(\hat{\gamma}_1) = \frac{1}{2}\Sigma_\gamma + 3\ell + \frac{1}{2}$ by (21). Again, Lemma 4.4 implies

$$\Upsilon_K(t) \leq (-g(\hat{\gamma}_1) + r + \ell - 1)t = \left(-\frac{1}{2}\Sigma_\gamma + r - 2\ell - \frac{3}{2} \right) t \quad \text{for all } 0 \leq t \leq 1,$$

which proves the claim of the lemma. □

Lemma 4.6 Let $\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for some $\ell \geq 0, r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then

$$\Upsilon_K(t) \leq \left(-\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) \right) + r - 2\ell \right) t \quad \text{for all } 0 \leq t \leq 1.$$

In the proof, we will need the following statement about positive 3–braids.

Lemma 4.7 In B_3 , we have $(ab)^{3n-1} = a^{2n} b (a^2 b^2)^{n-1} a$ for all $n \geq 1$.

Proof Starting with the left-hand side,

$$(ab)^{3n-1} = a(ba)^{3(n-1)}bab = a(ab)^{3(n-1)}aba,$$

which proves the lemma for $n = 1$. We now show by induction that

$$(22) \quad (ab)^{3(n-1)}a = a^{2n-1}b(a^2b^2)^{n-2}a^2b \quad \text{for all } n \geq 2,$$

which implies the lemma for all $n \geq 1$. For $n = 2$,

$$(ab)^3a = a(ba)^3 = a(ab)^3 = a^2babab = a^3ba^2b.$$

Assuming that (22) is true for some $n - 1 \geq 2$,

$$\begin{aligned} (ab)^{3(n-1)}a &= a(ba)^{3(n-1)} \\ &= a(ab)^{3(n-1)} \\ &= a^2(ba)^{3(n-2)}babab \\ &= a^2(ab)^{3(n-2)}aba^2b \\ &= a^2(a^{2n-3}b(a^2b^2)^{n-3}a^2b)ba^2b \\ &= a^{2n-1}b(a^2b^2)^{n-2}a^2b, \end{aligned}$$

using the induction hypothesis in the second to last equality. □

Proof of Lemma 4.6 Let $\Sigma_\gamma = \sum_{i=1}^r (p_i + q_i)$. If $\ell = 0$, then by (3) and Lemma 4.4,

$$\Upsilon_K(t) \leq (-g(K) + r - 1)t = \left(-\frac{1}{2}\Sigma_\gamma + r\right)t \quad \text{for all } 0 \leq t \leq 1.$$

For $\ell \geq 1$, using $\Delta^2 = (ba)^3$ and Lemma 4.7,

$$\begin{aligned} \gamma &= (ba)^{3\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r} \sim (ab)^{3\ell-1} a^{p_1+1} b^{q_1} \dots a^{p_r} b^{q_r+1} \\ &\sim a^{2\ell} b (a^2 b^2)^{\ell-1} a^{p_1+2} b^{q_1} \dots a^{p_r} b^{q_r+1} =: \gamma_1. \end{aligned}$$

Note that $\hat{\gamma}_1 = \hat{\gamma} = K$ and by (3),

$$g(\hat{\gamma}_1) = g(K) = \frac{1}{2}(6\ell + \Sigma_\gamma - 2) = \frac{1}{2}\Sigma_\gamma + 3\ell - 1.$$

Again by Lemma 4.4,

$$\Upsilon_K(t) \leq (-g(\hat{\gamma}_1) + r + \ell - 1)t = \left(-\frac{1}{2}\Sigma_\gamma + r - 2\ell\right)t \quad \text{for all } 0 \leq t \leq 1. \quad \square$$

We will now focus on $v(K) = \Upsilon_K(1)$ and prove Proposition 4.2 by showing that the upper bounds on $\Upsilon_K(t)$ from Lemmas 4.5 and 4.6 for $t = 1$ are also lower bounds. We will need the following observation used in [20, Example 4.5] about 3-braids, which we prove here for completeness.

Lemma 4.8 *In B_3 , $a^{2n+1}b(a^2b^2)^n = (ab)^{3n+1}$ and $b^{2n+1}a(b^2a^2)^n = (ba)^{3n+1}$ for all $n \geq 0$.*

Proof We prove the first statement by induction. For $n = 0$, the equality is clearly true. For $n = 1$, using $\Delta a = b\Delta$ and $\Delta b = a\Delta$, we have

$$a^3ba^2b^2 = a^2\Delta ab^2 = a^2ba\Delta b = a\Delta^2b = \Delta^2ab = (ab)^4.$$

We now assume the lemma is true for some $n - 1 \geq 0$. Using the induction hypothesis and the equality for $n = 1$,

$$\begin{aligned} a^{2n+1}b(a^2b^2)^n &= a^2(ab)^{3(n-1)+1}a^2b^2 = a^3b\Delta^{2(n-1)}a^2b^2 \\ &= \Delta^{2(n-1)}a^3ba^2b^2 = (ab)^{3(n-1)}(ab)^4 = (ab)^{3n+1}. \quad \square \end{aligned}$$

Lemma 4.9 *Let $\gamma = \Delta^{2\ell+1}a^{p_1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}}a^{p_r}$ for some $\ell \geq 0, r \geq 1, p_r \geq 3$ and $p_i, q_i \geq 2$ for $i \in \{1, \dots, r - 1\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$v(K) = -\frac{1}{2}\left(\sum_{i=1}^{r-1}(p_i + q_i) + p_r\right) + r - 2\ell - \frac{3}{2}.$$

Proof Let $\Sigma_\gamma = \sum_{i=1}^{r-1}(p_i + q_i) + p_r$. From Lemma 4.5, it follows directly that

$$v(K) = \Upsilon_K(1) \leq -\frac{1}{2}\Sigma_\gamma + r - 2\ell - \frac{3}{2},$$

so we are left to show that $v(K) \geq -\frac{1}{2}\Sigma_\gamma + r - 2\ell - \frac{3}{2}$. To that end, consider

$$\begin{aligned} \gamma &= \Delta^{2\ell+1}a^{p_1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}}a^{p_r} \\ &\sim \Delta^{2\ell}a\Delta a^{p_1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1} \\ &= \Delta^{2\ell}bab^2a^{p_1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1} =: \gamma_1, \end{aligned}$$

where we used $a\Delta = abab = bab^2$. Note that $\hat{\gamma}_1 = \hat{\gamma} = K$. Now, define

$$\alpha := b^{2r}\gamma_1 = \Delta^{2\ell}b^{2r+1}ab^2a^{p_1}b^{q_1} \dots a^{p_{r-1}}b^{q_{r-1}}a^{p_r-1}$$

and note that $\hat{\alpha}$ is a knot. By assumption, $p_r - 1 \geq 2$. There is a cobordism between $\hat{\alpha}$ and the connected sum $T_{2,2r+1} \# \hat{\gamma}_1 = T_{2,2r+1} \# K$ of genus 1 by using two saddle moves similar to the two saddle moves illustrated in [Figure 2](#). Similarly as in (16) from [Example 4.1](#), we have $\nu(K) \geq \nu(\hat{\alpha}) + r - 1$. In order to find a lower bound for $\nu(\hat{\alpha})$, note that there is a cobordism C between $\hat{\alpha}$ and the torus knot $T = T_{3,3(\ell+r)+1}$ of genus $g(C) = \frac{1}{2}\Sigma_\gamma - 2r + \frac{1}{2}$. Here we think of T as the closure of the braid word $\beta = \Delta^{2\ell} b^{2r+1} a(b^2 a^2)^r$, which is equal to $\Delta^{2\ell} (ba)^{3r+1} = (ba)^{3(\ell+r)+1}$ as 3-braids by [Lemma 4.8](#). The cobordism C between $\hat{\alpha}$ and $T = \hat{\beta}$ can thus be realized by

$$p_1 - 2 + q_1 - 2 + \dots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 3 = \Sigma_\gamma - 4r + 1$$

saddle moves corresponding to the deletion of the same number of generators a and b from the braid word α to obtain β . Hence the Euler characteristic of the cobordism C is $\chi(C) = -\Sigma_\gamma + 4r - 1$. Since C is connected and has two boundary components (as $\hat{\alpha}$ and $T = \hat{\beta}$ are knots), the genus of C is indeed $g(C) = \frac{1}{2}\Sigma_\gamma - 2r + \frac{1}{2}$. Now, by (15) and (12),

$$\nu(\hat{\alpha}) \geq \nu(T) - g(C) = -2(\ell + r) - \left(\frac{1}{2}\Sigma_\gamma - 2r + \frac{1}{2}\right) = -\frac{1}{2}\Sigma_\gamma - 2\ell - \frac{1}{2}.$$

It follows that

$$\nu(K) \geq \nu(\hat{\alpha}) + r - 1 \geq -\frac{1}{2}\Sigma_\gamma + r - 2\ell - \frac{3}{2}. \quad \square$$

Lemma 4.10 *Let $\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for some $\ell \geq 0, r \geq 1, p_r, q_r \geq 3$ and $p_i, q_i \geq 2$ for $i \in \{1, \dots, r-1\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$\nu(K) = -\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) \right) + r - 2\ell.$$

Proof The proof uses similar ideas as that of [Lemma 4.9](#). Let $\Sigma_\gamma = \sum_{i=1}^r (p_i + q_i)$. By [Lemma 4.6](#), $\nu(K) \leq -\frac{1}{2}\Sigma_\gamma + r - 2\ell$, so it remains to show that $\nu(K) \geq -\frac{1}{2}\Sigma_\gamma + r - 2\ell$. To that end, we consider

$$\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r} \sim \Delta^{2\ell} b a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r-1} =: \gamma_1.$$

Note that $\hat{\gamma}_1 = \hat{\gamma} = K$. We define

$$\alpha := a^{2r} \gamma_1 = a^{2r} \Delta^{2\ell} b a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r-1} \sim \Delta^{2\ell} b a^{2r} b a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r-2} =: \alpha_1.$$

Then $\hat{\alpha}_1 = \hat{\alpha}$ is a knot and by assumption we have $q_r - 2 \geq 1$. There is a cobordism between $\hat{\alpha}$ and $T_{2,2r+1} \# \hat{\gamma}_1 = T_{2,2r+1} \# K$ of genus 1 by using two saddle moves similar to the cobordism considered in [Example 4.1](#) and in the proof of [Lemma 4.9](#);

hence $\nu(K) \geq \nu(\hat{\alpha}_1) + r - 1$. To find a lower bound for $\nu(\hat{\alpha}_1)$, we observe that there is a cobordism C between the knot $\hat{\alpha}_1$ and the knot $\hat{\beta}$, where

$$\beta = \Delta^{2\ell} b a^{2r} b (a^2 b^2)^{r-1} a^3 b.$$

Using Lemma 4.8 for $n - 1$, in B_3 ,

$$b a^{2n} b (a^2 b^2)^{n-1} a^2 = b a (ab)^{3(n-1)+1} a^2 = b a \Delta^{2(n-1)} a b a^2 = \Delta^{2n} \quad \text{for all } n \geq 1.$$

We thus have $\beta = \Delta^{2\ell} \Delta^{2r} a b = (ab)^{3(\ell+r)+1}$, so the closure of β is the torus knot $T = T_{3,3(\ell+r)+1}$ with $\nu(T) = -2(\ell + r)$ by (12). The cobordism C between $\hat{\alpha}_1$ and $T = \hat{\beta}$ can be realized by

$$p_1 - 2 + q_1 - 2 + \dots + p_{r-1} - 2 + q_{r-1} - 2 + p_r - 3 + q_r - 3 = \Sigma_\gamma - 4r - 2$$

saddle moves corresponding to the deletion of the same number of generators a and b from the braid word α_1 to obtain β . By a similar Euler characteristic argument as in the proofs of Lemmas 4.4 and 4.9, the genus of this cobordism is $g(C) = \frac{1}{2} \Sigma_\gamma - 2r - 1$. Note that here we used $p_r \geq 3$ and $q_r \geq 3$. Now, by (15),

$$\nu(\hat{\alpha}_1) \geq \nu(T) - g(C) = -\frac{1}{2} \Sigma_\gamma - 2\ell + 1,$$

$$\nu(K) \geq \nu(\hat{\alpha}_1) + r - 1 \geq -\frac{1}{2} \Sigma_\gamma + r - 2\ell. \quad \square$$

Lemma 4.11 *Let $\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for some $\ell \geq 0, r \geq 2$ and $p_i, q_i \geq 2$ for $i \in \{1, \dots, r\}$. Suppose that $q_r \geq 3$ and $p_k \geq 3$ for some $1 \leq k < r$ and that $K = \hat{\gamma}$ is a knot. Then*

$$\nu(K) = -\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) \right) + r - 2\ell.$$

Proof We proceed as in the proof of Lemma 4.10, but here we will look at a different cobordism to obtain a lower bound for $\nu(\hat{\alpha}_1)$. The steps of the proof are exactly the same until then, so we consider

$$\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r} \sim \Delta^{2\ell} b a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r-1} =: \gamma_1$$

and define

$$\alpha := a^{2r} \gamma_1 \sim \Delta^{2\ell} b a^{2r} b a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r-2} =: \alpha_1.$$

Again, we have $\nu(K) \geq \nu(\hat{\alpha}_1) + r - 1$. Now, in order to find a lower bound for $\nu(\hat{\alpha}_1)$, we observe that there is a cobordism C between $\hat{\alpha}_1$ and the knot $\hat{\beta}$, where

$$\beta = \Delta^{2\ell} b a^{2r} b (a^2 b^2)^{k-1} a^3 b^2 (a^2 b^2)^{r-k-1} a^2 b.$$

We find the cobordism C by the deletion of generators from the braid word β to obtain α_1 , where we use the assumptions $q_r \geq 3$ and $p_k \geq 3$. In fact, the cobordism can be realized by

$$\begin{aligned}
 & p_1 - 2 + q_1 - 2 + \cdots + p_{k-1} - 2 + q_{k-1} - 2 + p_k - 3 + q_k - 2 \\
 & + p_{k+1} - 2 + q_{k+1} - 2 + \cdots + p_{r-1} - 2 + q_{r-1} - 2 \\
 & + p_r - 2 + q_r - 3 = \Sigma_\gamma - 4r - 2
 \end{aligned}$$

saddle moves, so its genus is $g(C) = \frac{1}{2}\Sigma_\gamma - 2r - 1$. Using

$$a^{2k-1}b(a^2b^2)^{k-1} = (ab)^{3k-2}$$

from [Lemma 4.8](#), we have

$$\begin{aligned}
 \beta &= \Delta^{2\ell}ba^{2r-2k+1}(ab)^{3k-2}a^3b^2(a^2b^2)^{r-k-1}a^2b \\
 &= \Delta^{2\ell}ba^{2r-2k+1}\Delta^{2(k-1)}aba^3b^2(a^2b^2)^{r-k-1}a^2b \\
 &\sim \Delta^{2(\ell+k-1)}\Delta a^2b^2(a^2b^2)^{r-k-1}a^2b^2a^{2r-2k+1} \\
 &= \Delta^{2(\ell+k-1)+1}(a^2b^2)^{r-k+1}a^{2r-2k+1} =: \beta_1.
 \end{aligned}$$

Note that by our assumptions on ℓ, r and k , we have $\ell + k - 1 \geq 0, r - k + 1 \geq 2$ and $2r - 2k + 1 \geq 3$, so β_1 has the form of the braid words considered in [Lemma 4.9](#). We thus have

$$\begin{aligned}
 v(\hat{\beta}) &= v(\hat{\beta}_1) = -\frac{1}{2}(4(r - k + 1) + 2r - 2k + 1) + (r - k + 2) - 2(\ell + k - 1) - \frac{3}{2} \\
 &= -2(\ell + r).
 \end{aligned}$$

By [\(15\)](#),

$$\begin{aligned}
 v(\hat{\alpha}_1) &\geq v(\hat{\beta}) - g(C) = -\frac{1}{2}\Sigma_\gamma - 2\ell + 1, \\
 v(K) &\geq v(\hat{\alpha}_1) + r - 1 \geq -\frac{1}{2}\Sigma_\gamma + r - 2\ell. \quad \square
 \end{aligned}$$

Proof of Proposition 4.2 The first case of [Proposition 4.2](#) follows from [Remark 3.6](#) and [\(12\)](#). Lemmas [4.10](#) and [4.11](#) together prove the second case, [Lemma 4.9](#) proves the third case. Note that up to conjugation, by [Remark 3.1](#) and the remarks in [Proposition 3.2](#), it is no restriction to assume that $p_r \geq 3$ in [Lemma 4.9](#) and that $q_r \geq 3$ and either $p_r \geq 3$ or $p_k \geq 3$ for some $1 \leq k < r$ in Lemmas [4.10](#) and [4.11](#), respectively. □

Before we proceed with the general case where the knot K is given as the closure of any 3-braid, let us prove the following corollaries of our results in this section.

Corollary 4.12 (Corollary 1.3) *Let K be a knot that is the closure of a positive 3–braid. Then*

$$r = g(K) + v(K) + 1$$

is minimal among all integers $r \geq 1$ such that K is the closure of a positive 3–braid $a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for integers $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$.

Proof By Lemma 4.4,

$$v(K) \leq -g(K) + r - 1 \iff g(K) + v(K) + 1 \leq r$$

whenever K is the closure of a positive 3–braid $a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for integers $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$. It remains to show that we can always find a positive braid representative for K of the form $a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ with $r = g(K) + v(K) + 1$. We will use Proposition 3.2. In fact, if K is the closure of a positive braid γ of the form in (C) with $\ell \geq 0$, then $g(K) + v(K) + 1 = r + \ell$ by (3) applied to γ and Lemmas 4.10 and 4.11. Moreover,

$$\begin{aligned} \gamma &= a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r} && \text{if } \ell = 0, \\ \gamma &\sim a^{2\ell} b (a^2 b^2)^{\ell-1} a^{p_1+2} b^{q_1} \dots a^{p_r} b^{q_r+1} && \text{if } \ell \geq 1 \end{aligned}$$

by the proof of Lemma 4.6; these give the desired braid representatives for K . Furthermore, if K is represented by a positive braid γ of the form in (D) with $\ell \geq 0$, then $g(K) + v(K) + 1 = r + \ell$ by (3) and Lemma 4.9, and we have

$$\begin{aligned} \gamma &\sim a^{p_1+1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r+1} b && \text{if } \ell = 0, \\ \gamma &\sim a^{p_r+2} b a^3 (aba^3)^{\ell-1} b a^{p_1+\ell+1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} && \text{if } \ell \geq 1 \end{aligned}$$

by the proof of Lemma 4.5. Finally, if $K = T_{3,3\ell+k}$ for $\ell \geq 0$ and $k \in \{1, 2\}$, then by (4) and (12), we have $g(K) + v(K) + 1 = \ell + 1$ and $T_{3,3\ell+1}$ and $T_{3,3\ell+2}$ are represented by the positive 3–braids $(ab)^{3\ell+1} = a^{2\ell+1} b (a^2 b^2)^\ell$ and $(ab)^{3\ell+2} \sim a^{2\ell+3} b (a^2 b^2)^\ell$, respectively, by Lemmas 4.8 and 4.7. \square

Corollary 4.13 (Corollary 1.4) *If K and J are concordant knots that are both closures of positive 3–braids, then the minimal r from Corollary 4.12 is the same for both K and J .*

Proof If K and J are concordant, then their 4–genus and their epsilon invariants are equal. So by (3) from Section 2.1 and by Corollary 4.12, positive 3–braids with closures K and J , respectively, will have the same minimal r . \square

Remark 4.14 Let $\mathcal{A}_g(K)$ denote the minimal genus of a cobordism between a knot K and an alternating knot, ie the cobordism distance $d(K, \{\text{alternating knots}\})$. By [23, Theorem 8], we have $\frac{1}{2}|\tau(K) + \nu(K)| \leq \mathcal{A}_g(K)$ for any knot K . It thus follows from our results in this section that

$$\frac{1}{2}(r + \ell - 1) \leq \mathcal{A}_g(K) \leq \frac{1}{2}(r + \ell - 1 + \varepsilon)$$

for any knot K that is the closure of a positive 3–braid in Garside normal form (C) or (D), where $\varepsilon \geq 0$ is an integer depending on K . The lower bound uses Proposition 4.2 and (5) from Section 2.2; see also the proof of Corollary 4.12. The upper bound follows from the proofs of Lemmas 4.5 and 4.6; see also the proof of Lemma 4.4. Note that for most positive 3–braid knots, we have $\varepsilon > 0$, so we do not get an equality.

A shorter proof of Lemma 4.4 without cobordisms follows from a result of Abe and Kishimoto on the dealternating number of positive 3–braid knots. Indeed, by (5), (24) and (27),

$$|\Upsilon_K(t) + g(K)t| = |\Upsilon_K(t) + \tau(K)t| \leq \text{alt}(K)t \leq \text{dalt}(K)t \leq (r - 1)t \quad \text{for all } 0 \leq t \leq 1.$$

The definitions of the dealternating number $\text{dalt}(K)$ and the alternation number $\text{alt}(K)$ of a knot K and more details on the inequalities used here will be provided in Section 5.

4.3 Proof of Theorem 1.1

It remains to show Theorem 1.1 when K is the closure of a not necessarily positive 3–braid. We first recall a result of Murasugi, which implies that indeed all 3–braid knots except for the torus knots of braid index 3 are covered by Theorem 1.1.

Let γ be a 3–braid. Then, by [45, Proposition 2.1], γ is conjugate to one and only one of the 3–braids

- (a) $\Delta^{2\ell} a^p$ or $\Delta^{2\ell+1}$ for $\ell \in \mathbb{Z}, p \in \mathbb{Z}$,
- (b) $\Delta^{2\ell} ab$ or $\Delta^{2\ell} (ab)^2$ for $\ell \in \mathbb{Z}$,
- (c) $\Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for $\ell \in \mathbb{Z}, r \geq 1, p_i, q_i \geq 1, i \in \{1, \dots, r\}$.

Definition 4.15 We call a braid word of the form in (a)–(c) a 3–braid in Murasugi normal form.

Remark 4.16 The closures of the 3–braids in Murasugi normal form (a) are links of two (if p is odd) or three components and the closures of the 3–braids in Murasugi normal form (b) are the torus knots of braid index 3 (see Remark 3.6).

If $\ell = 0$ in case (c), the braid word $\gamma = a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for integers $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ gives rise to an alternating braid diagram. If $K = \hat{\gamma}$ is a knot, by Proposition 2.1 we thus have $\nu(K) = \frac{1}{2}\sigma(K)$ in that case and the statement of Theorem 1.1 follows directly from a result by Erle on the signature of 3–braid knots.

Proposition 4.17 [17, Theorem 2.6] *Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for integers $\ell \in \mathbb{Z}, r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$\sigma(K) = \sum_{i=1}^r (p_i - q_i) - 4\ell.$$

We still need to show Theorem 1.1 when K is the closure of a 3–braid in Murasugi normal form (c) with $\ell \neq 0$. The proof will follow from the following two lemmas.

Lemma 4.18 *Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for some $\ell \geq 1, r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$\Upsilon_K(t) \leq \left(\frac{1}{2} \left(\sum_{i=1}^r (p_i - q_i) \right) - 2\ell \right) t \quad \text{for all } 0 \leq t \leq 1.$$

Lemma 4.19 *Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for some $\ell \geq 0, r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$\nu(K) \geq \frac{1}{2} \left(\sum_{i=1}^r (p_i - q_i) \right) - 2\ell.$$

Proof of Theorem 1.1 For $\ell \geq 1$, the statement of the theorem follows directly from Lemmas 4.18 and 4.19. If $\ell < 0$, the knot $-K$ is represented by the braid word $\Delta^{-2\ell} a^{-q_r} b^{p_r} \dots a^{-q_1} b^{p_1}$ with $-\ell \geq 1$ and accordingly we have

$$\nu(-K) = \frac{1}{2} \left(\sum_{i=1}^r (q_i - p_i) \right) + 2\ell.$$

Using that $\nu(-K) = -\nu(K)$ by (9) from Section 2.2, this implies the claim. □

The remainder of this section is devoted to the proofs of the above lemmas.

Proof of Lemma 4.18 We first consider the case where $p_1 \geq 2$ and $\ell \geq 2$. Using $\Delta a^{-1} = ab$ and

$$(ab)^{3n+2} = b^{n+1} a (b^3 ab)^{n-1} b^3 ab^3 \quad \text{for all } n \geq 1$$

from [18, Proof of Proposition 22], we have

$$\begin{aligned} \gamma &= \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r} \\ &= \Delta^{2(\ell-1)+1} a b a^{-p_1+1} b^{q_1} \dots a^{-p_r} b^{q_r} \\ &= (ba)^{3(\ell-1)+2} b a^{-p_1+1} b^{q_1} \dots a^{-p_r} b^{q_r} \\ &\sim (ab)^{3(\ell-1)+2} a^{-p_1+1} b^{q_1} \dots a^{-p_r} b^{q_r+1} \\ &\sim a(b^3 ab)^{\ell-2} b^3 a b^3 a^{-p_1+1} b^{q_1} \dots a^{-p_r} b^{q_r+\ell+1} =: \gamma_1. \end{aligned}$$

Now, we claim that there is a cobordism C of genus $g(C) = \frac{1}{2}(\ell + r - 1 + \varepsilon)$ between the closure K of γ_1 and the connected sum

$$J_\varepsilon = -T_{2,p_1-1-\varepsilon_1} \# -T_{2,p_2-\varepsilon_2} \# \dots \# -T_{2,p_r-\varepsilon_r} \# T_{2,\sum_{i=1}^r q_i+5\ell-1+\varepsilon_q},$$

where we choose $\varepsilon_1, \dots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that J_ε is a connected sum of torus knots, ie such that $\sum_{i=1}^r q_i + 5\ell - 1 + \varepsilon_q, p_1 - 1 - \varepsilon_1, p_2 - \varepsilon_2, \dots, p_r - \varepsilon_r$ are all odd; and $\varepsilon = \varepsilon_q + \sum_{i=1}^r \varepsilon_i$. This cobordism C can be realized using $\ell + r - 1 + \varepsilon$ saddle moves as follows. On the one hand, we add $\sum_{i=1}^r \varepsilon_i$ generators a and ε_q generators b to the braid word γ_1 ; on the other hand, we perform $\ell + r - 1$ saddle moves of the form as the $r - 1$ saddle moves used in the proof of Lemma 4.4 to get a connected sum of torus knots. The Euler characteristic of C is $\chi(C) = -\ell - r + 1 - \varepsilon$. Since C is connected and has two boundary components (as K and J_ε are knots), the genus of C is $g(C) = -\frac{1}{2}\chi(C) = \frac{1}{2}(\ell + r - 1 + \varepsilon)$ as claimed. By (8) and (11),

$$\Upsilon_{J_\varepsilon}(t) = \left(\frac{1}{2} \left(\sum_{i=1}^r (p_i - q_i) - \varepsilon - r - 5\ell + 1 \right) \right) t \quad \text{for all } 0 \leq t \leq 1,$$

and by (15),

$$\Upsilon_K(t) \leq \Upsilon_{J_\varepsilon}(t) + g(C)t = \left(\frac{1}{2} \left(\sum_{i=1}^r (p_i - q_i) \right) - 2\ell \right) t \quad \text{for all } 0 \leq t \leq 1.$$

If $p_1 \geq 2$ and $\ell = 1$, then

$$\gamma \sim (ab)^2 a^{-p_1+1} b^{q_1} \dots a^{-p_r} b^{q_r+1} \sim ab^2 a^{-p_1+1} b^{q_1} \dots a^{-p_r} b^{q_r+2} =: \gamma_1,$$

and similarly as above, there is a cobordism C of genus $g(C) = \frac{1}{2}(r + \varepsilon)$ between the closure K of γ_1 and the connected sum

$$J_\varepsilon = -T_{2,p_1-1-\varepsilon_1} \# -T_{2,p_2-\varepsilon_2} \# \dots \# -T_{2,p_r-\varepsilon_r} \# T_{2,\sum_{i=1}^r q_i+4+\varepsilon_q},$$

where we choose $\varepsilon_1, \dots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that J_ε is a connected sum of torus knots and $\varepsilon = \varepsilon_q + \sum_{i=1}^r \varepsilon_i$. The claim follows also in this case from equations (8) and (11), and the inequality in (15).

It remains to show the claim when $p_1 = 1$. In that case, using $\Delta a^{-1} = ab$,

$$\begin{aligned} \gamma &= \Delta^{2\ell} a^{-1} b^{q_1} \dots a^{-p_r} b^{q_r} \\ &= \Delta^{2\ell-1} ab^{q_1+1} \dots a^{-p_r} b^{q_r} \sim \Delta^{2\ell-1} b^{q_1+1} \dots a^{-p_r} b^{q_r+1}. \end{aligned}$$

If $\ell = 1$, then γ is conjugate to $\gamma_1 = ab^{q_1+2} a^{-p_2} b^{q_2} \dots a^{-p_r} b^{q_r+2}$ and if $\ell \geq 2$, then using (20) from Section 4.2,

$$\begin{aligned} \gamma &\sim \Delta^{2(\ell-1)+1} b^{q_1+1} a^{-p_2} b^{q_2} \dots a^{-p_r} b^{q_r+1} \\ &= (ba)^{3(\ell-1)+1} b^{q_1+2} a^{-p_2} b^{q_2} \dots a^{-p_r} b^{q_r+1} \\ &\sim ab^3 (bab^3)^{\ell-2} ab^{q_1+\ell+1} a^{-p_2} b^{q_2} \dots a^{-p_r} b^{q_r+3} =: \gamma_1. \end{aligned}$$

In both cases, there is a cobordism C of genus $g(C) = \frac{1}{2}(\ell + r - 2 + \varepsilon)$ between the closure K of γ_1 and the connected sum

$$J_\varepsilon = -T_{2,p_2-\varepsilon_2} \# \dots \# -T_{2,p_r-\varepsilon_r} \# T_{2,\sum_{i=1}^r q_i+5\ell-1+\varepsilon_q},$$

where we choose $\varepsilon_1, \dots, \varepsilon_r, \varepsilon_q \in \{0, 1\}$ such that J_ε is a connected sum of torus knots and $\varepsilon = \varepsilon_q + \sum_{i=1}^r \varepsilon_i$. Using (8), (11), and (15) again, the claim follows. \square

We will need the following two technical lemmas for the proof of Lemma 4.19.

Lemma 4.20 *Let $\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for some $\ell \geq 0, r \geq 1$ and integers p_i and q_i such that $p_i < 0$ or $p_i \geq 2$, and $q_i < 0$ or $q_i \geq 2$, for any $i \in \{1, \dots, r\}$. Moreover, assume that $K = \hat{\gamma}$ is a knot. Then*

$$v(K) \geq -\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) \right) + r - 2\ell - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\},$$

where $\#A$ denotes the cardinality of the set A .

Lemma 4.21 *Let $\gamma = \Delta^{2\ell+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ for some $\ell \geq 0, r \geq 1$ and integers p_i and q_i such that $p_i < 0$ or $p_i \geq 2$ for any $i \in \{1, \dots, r\}$ and $q_i < 0$ or $q_i \geq 2$ for any $i \in \{1, \dots, r-1\}$. Moreover, assume that $K = \hat{\gamma}$ is a knot. Then*

$$v(K) \geq -\frac{1}{2} \left(\sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + r - 2\ell - \frac{3}{2} - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\}.$$

For the proofs of Lemmas 4.20 and 4.21, we refer the reader to the very end of this section; we will first prove Lemma 4.19 using these lemmas.

Proof of Lemma 4.19 Let k be the number of exponents q_j of γ with $q_j = 1$ and let $\mathcal{J} = \{j_1, \dots, j_k\}$ for $0 \leq k \leq r$ be the set of indices such that $q_j = 1$ if and only if $j \in \mathcal{J}$. For all $j \in \mathcal{J}$, we rewrite the subword $a^{-p_j} b^{q_j}$ of γ using $\Delta^{-1} ab = a^{-1}$ as

$$a^{-p_j} b^{q_j} = a^{-p_j} b = a^{-p_j} a^{-1} \Delta \Delta^{-1} ab = a^{-p_j-1} \Delta a^{-1} = \Delta b^{-p_j-1} a^{-1}.$$

Note that if $j, j + 1 \in \mathcal{J}$, then $a^{-p_j} b^{q_j} a^{-p_{j+1}} b^{q_{j+1}} = \Delta^2 a^{-p_j-1} b^{-p_{j+1}-2} a^{-1}$. After rewriting $a^{-p_j} b^{q_j}$ for all $j \in \mathcal{J}$, the braid γ is conjugate to $\gamma_1 = \Delta^{2\ell+k} \alpha$ for some 3–braid α which is of the form

$$\alpha = \begin{cases} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_n} b^{\tilde{q}_n} & \text{for } n = r - \frac{1}{2}k & \text{if } k \text{ is even,} \\ b^{\tilde{p}_1} a^{\tilde{q}_1} \dots b^{\tilde{p}_{n-1}} a^{\tilde{q}_{n-1}} b^{\tilde{p}_n} & \text{for } n = r - \frac{1}{2}(k - 1) & \text{if } k \text{ is odd,} \end{cases}$$

where $\sum_{i=1}^n (\tilde{p}_i + \tilde{q}_i) = \sum_{i=1}^r (-p_i + q_i) - 3k$ and where the \tilde{p}_i and \tilde{q}_i fulfill the assumptions of Lemmas 4.20 and 4.21, respectively, ie where $\tilde{p}_i < 0$ or ≥ 2 and $\tilde{q}_i < 0$ or ≥ 2 for any i . The number of negative exponents in α equals the number of negative exponents $-p_i$ in γ , so

$$\#\{i \mid \tilde{p}_i < 0\} + \#\{i \mid \tilde{q}_i < 0\} = r.$$

If k is even, by Lemma 4.20,

$$\begin{aligned} v(\hat{\gamma}) &\geq -\frac{1}{2} \left(\sum_{i=1}^n (\tilde{p}_i + \tilde{q}_i) \right) + n - (2\ell + k) - \#\{i \mid \tilde{p}_i < 0\} + \#\{i \mid \tilde{q}_i < 0\} \\ &= -\frac{1}{2} \left(\sum_{i=1}^r (-p_i + q_i) - 3k \right) + r - \frac{k}{2} - (2\ell + k) - r \\ &= \frac{1}{2} \left(\sum_{i=1}^r (p_i - q_i) \right) - 2\ell. \end{aligned}$$

Similarly, if k is odd, the claim follows from Lemma 4.21. □

It remains to prove Lemmas 4.20 and 4.21.

Proof of Lemma 4.20 We will modify the braid word γ in $2r$ steps, where each step corresponds to one of the $2r$ exponents p_i or q_i , for $i \in \{1, \dots, r\}$, of γ . In every step, we will either just conjugate γ (if the corresponding exponent is positive) or perform a cobordism of genus 1 between the closure of $a^{2n}\gamma$ or $b^{2n}\gamma$ and the connected sum $T_{2,2n+1} \# \hat{\gamma}$ for some $n \geq 0$ — similar to the cobordism described in Example 4.1 and

used in the proofs of Lemmas 4.9, 4.10 and 4.11. We now describe these steps in more detail. First, let $\gamma'_{0,q} = \gamma$ and define

$$\begin{aligned} a^{-p_1+2+\varepsilon_{1,p}} \gamma'_{0,q} &= \Delta^{2\ell} a^{2+\varepsilon_{1,p}} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} \\ &\sim \Delta^{2\ell} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{2+\varepsilon_{1,p}} =: \gamma'_{1,p} \quad \text{if } p_1 < 0, \\ \gamma'_{0,q} &\sim \Delta^{2\ell} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{p_1} =: \gamma'_{1,p} \quad \text{if } p_1 > 0, \end{aligned}$$

so that $\gamma'_{1,p} = \Delta^{2\ell} b^{q_1} a^{p_2} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1}$ for some $\tilde{p}_1 \geq 2$ (note that we assumed $p_1 < 0$ or $p_1 \geq 2$). Here, if $p_1 < 0$, we choose $\varepsilon_{1,p} \in \{0, 1\}$ such that $-p_1 + 2 + \varepsilon_{1,p}$ is even and $\hat{\gamma}'_{1,p}$ is a knot. Second, let $\varepsilon_{1,q} \in \{0, 1\}$ be such that $-q_1 + 2 + \varepsilon_{1,q}$ is even if $q_1 < 0$, and define

$$\begin{aligned} \gamma_{1,q} &= b^{-q_1+2+\varepsilon_{1,q}} \gamma'_{1,p} = \Delta^{2\ell} b^{2+\varepsilon_{1,q}} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} \\ &\sim \Delta^{2\ell} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{2+\varepsilon_{1,q}} =: \gamma'_{1,q} \quad \text{if } q_1 < 0, \\ \gamma_{1,q} &= \gamma'_{1,p} \sim \Delta^{2\ell} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{q_1} =: \gamma'_{1,q} \quad \text{if } q_1 > 0, \end{aligned}$$

so that $\gamma'_{1,q} = \Delta^{2\ell} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{\tilde{q}_1}$ for some $\tilde{p}_1, \tilde{q}_1 \geq 2$. Inductively, for any $1 \leq i \leq r$, we let

$$\begin{aligned} a^{-p_i+2+\varepsilon_{i,p}} \gamma'_{i-1,q} &= \Delta^{2\ell} a^{2+\varepsilon_{i,p}} b^{q_i} a^{p_{i+1}} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_{i-1}} b^{\tilde{q}_{i-1}} \\ &\sim \Delta^{2\ell} b^{q_i} a^{p_{i+1}} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_{i-1}} b^{\tilde{q}_{i-1}} a^{2+\varepsilon_{i,p}} =: \gamma'_{i,p} \quad \text{if } p_i < 0, \\ \gamma'_{i-1,q} &\sim \Delta^{2\ell} b^{q_i} a^{p_{i+1}} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_{i-1}} b^{\tilde{q}_{i-1}} a^{p_i} =: \gamma'_{i,p} \quad \text{if } p_i > 0, \end{aligned}$$

so that

$$\gamma'_{i,p} = \Delta^{2\ell} b^{q_i} a^{p_{i+1}} \dots a^{p_r} b^{q_r} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_{i-1}} b^{\tilde{q}_{i-1}} a^{\tilde{p}_i}$$

for some integers $\tilde{p}_1, \tilde{q}_1, \dots, \tilde{p}_{i-1}, \tilde{q}_{i-1}, \tilde{p}_i \geq 2$. Here we choose $\varepsilon_{i,p} \in \{0, 1\}$ such that $-p_i + 2 + \varepsilon_{i,p}$ is even if $p_i < 0$. Moreover, for $1 \leq i \leq r$, we let $\varepsilon_{i,q} \in \{0, 1\}$ be such that $-q_i + 2 + \varepsilon_{i,q}$ is even, and define

$$\gamma_{i,q} = \begin{cases} b^{-q_i+2+\varepsilon_{i,q}} \gamma'_{i,p} & \text{if } q_i < 0, \\ \gamma'_{i,p} & \text{if } q_i > 0, \end{cases}$$

and we define $\gamma'_{i,q}$ similarly as $\gamma'_{1,q}$. Inductively, after $2r$ steps, we get the positive 3-braid

$$\gamma'_{r,q} = \Delta^{2\ell} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_r} b^{\tilde{q}_r}$$

with

$$\tilde{p}_i = \begin{cases} 2 + \varepsilon_{i,p} & \text{if } p_i < 0, \\ p_i & \text{if } p_i > 0, \end{cases} \quad \text{and} \quad \tilde{q}_i = \begin{cases} 2 + \varepsilon_{i,q} & \text{if } q_i < 0, \\ q_i & \text{if } q_i > 0, \end{cases}$$

for all $1 \leq i \leq r$; so $\tilde{p}_1, \tilde{q}_1, \dots, \tilde{p}_r, \tilde{q}_r \geq 2$. By Proposition 4.2,

$$v(\hat{\gamma}'_{r,q}) = -\frac{1}{2} \left(\sum_{\substack{i=1 \\ p_i > 0}}^r p_i + \sum_{\substack{i=1 \\ q_i > 0}}^r q_i + \sum_{\substack{i=1 \\ p_i < 0}}^r (2 + \varepsilon_{i,p}) + \sum_{\substack{i=1 \\ q_i < 0}}^r (2 + \varepsilon_{i,q}) \right) + r - 2\ell.$$

Now, note that if $p_i < 0$ for some $1 \leq i \leq r$, then there is a cobordism of genus 1 between $\hat{\gamma}'_{i,p}$ and $T_{2,2m+1} \# \hat{\gamma}'_{i-1,q}$ by using two saddle moves, where $m = \frac{1}{2}(-p_i + 2 + \varepsilon_{i,p})$, so similarly as in (16) from Example 4.1, we have

$$v(\hat{\gamma}'_{i-1,q}) \geq v(\hat{\gamma}'_{i,p}) + m - 1 = v(\hat{\gamma}'_{i,p}) + \frac{1}{2}(-p_i + \varepsilon_{i,p}).$$

Similarly, if $q_i < 0$ for some $1 \leq i \leq r$, then $v(\hat{\gamma}'_{i,p}) \geq v(\hat{\gamma}'_{i,q}) + \frac{1}{2}(-q_i + \varepsilon_{i,q})$. In addition, if $p_i > 0$, then $v(\hat{\gamma}'_{i,p}) = v(\hat{\gamma}'_{i-1,q})$, and if $q_i > 0$, then $v(\hat{\gamma}'_{i,q}) = v(\hat{\gamma}'_{i,p})$. We conclude that

$$\begin{aligned} v(\hat{\gamma}) &= v(\hat{\gamma}'_{0,q}) \geq v(\hat{\gamma}'_{r,q}) + \sum_{\substack{i=1 \\ p_i < 0}}^r \frac{-p_i + \varepsilon_{i,p}}{2} + \sum_{\substack{i=1 \\ q_i < 0}}^r \frac{-q_i + \varepsilon_{i,q}}{2} \\ &= -\frac{1}{2} \left(\sum_{\substack{i=1 \\ p_i > 0}}^r p_i + \sum_{\substack{i=1 \\ q_i > 0}}^r q_i + \sum_{\substack{i=1 \\ p_i < 0}}^r (p_i + 2) + \sum_{\substack{i=1 \\ q_i < 0}}^r (q_i + 2) \right) + r - 2\ell \\ &= -\frac{1}{2} \left(\sum_{i=1}^r (p_i + q_i) \right) + r - 2\ell - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\}. \quad \square \end{aligned}$$

Proof of Lemma 4.21 The strategy of the proof is the same as in the proof of Lemma 4.20. Here, we need $2r - 1$ steps corresponding to the $2r - 1$ exponents $p_1, q_1, \dots, p_{r-1}, q_{r-1}, p_r$ of γ . The steps are similar to the proof of Lemma 4.20, the only change is that we multiply $\gamma'_{i-1,q}$ by a power of b if $p_i < 0$, and $\gamma'_{i,p}$ by a power of a if $q_i < 0$ (since $a\Delta^{2\ell+1} = \Delta^{2\ell+1}b$ and $b\Delta^{2\ell+1} = \Delta^{2\ell+1}a$). Thus, starting with $\gamma'_{0,q} = \gamma$, after $2r - 1$ steps we obtain the positive 3-braid

$$\gamma'_{r,p} = \Delta^{2\ell+1} a^{\tilde{p}_1} b^{\tilde{q}_1} \dots a^{\tilde{p}_{r-1}} b^{\tilde{q}_{r-1}} a^{\tilde{p}_r}$$

with

$$\tilde{p}_i = \begin{cases} 2 + \varepsilon_{i,p} & \text{if } p_i < 0, \\ p_i & \text{if } p_i > 0, \end{cases} \quad \text{and} \quad \tilde{q}_i = \begin{cases} 2 + \varepsilon_{i,q} & \text{if } q_i < 0, \\ q_i & \text{if } q_i > 0. \end{cases}$$

By Lemma 4.9,

$$v(\gamma'_{r,p}) = -\frac{1}{2} \left(\sum_{\substack{i=1 \\ p_i > 0}}^r p_i + \sum_{\substack{i=1 \\ q_i > 0}}^{r-1} q_i + \sum_{\substack{i=1 \\ p_i < 0}}^r (2 + \varepsilon_{i,p}) + \sum_{\substack{i=1 \\ q_i < 0}}^{r-1} (2 + \varepsilon_{i,q}) \right) + r - 2\ell - \frac{3}{2}.$$

Since the steps we performed have similar effects on $\nu(\hat{\gamma})$ as the ones in the proof of Lemma 4.20, we get

$$\begin{aligned} \nu(\hat{\gamma}) &= \nu(\hat{\gamma}'_{0,q}) \geq \nu(\hat{\gamma}'_{r,p}) + \sum_{\substack{i=1 \\ p_i < 0}}^r \frac{-p_i + \varepsilon_{i,p}}{2} + \sum_{\substack{i=1 \\ q_i < 0}}^{r-1} \frac{-q_i + \varepsilon_{i,q}}{2} \\ &= -\frac{1}{2} \left(\sum_{\substack{i=1 \\ p_i > 0}}^r p_i + \sum_{\substack{i=1 \\ q_i > 0}}^{r-1} q_i + \sum_{\substack{i=1 \\ p_i < 0}}^r (p_i + 2) + \sum_{\substack{i=1 \\ q_i < 0}}^{r-1} (q_i + 2) \right) + r - 2\ell - \frac{3}{2} \\ &= -\frac{1}{2} \left(\sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + r - 2\ell - \frac{3}{2} - \#\{i \mid p_i < 0\} - \#\{i \mid q_i < 0\}. \quad \square \end{aligned}$$

4.4 Further discussion of Theorem 1.1

In this section, we provide some further context on our main result. In particular, in Section 4.4.2 we will discuss why it might be surprising that our proof strategy works for all 3–braid knots.

4.4.1 Comparison of epsilon and the classical signature

By Theorem 1.1 and Proposition 4.17,

$$(23) \quad \sigma(K) = 2\nu(K)$$

for any knot K that is the closure of a 3–braid $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for integers $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$. Computations of the signature for torus knots (and links) of braid index 3, first done by Hirzebruch, Murasugi and Shinora [45, Proposition 9.1, pages 34–35], together with (12) from Section 2.2 imply that the equality in (23) is in fact true for all 3–braid knots K except for the cases that $K = \pm T_{3,3\ell+1}$ for odd $\ell > 0$ or $K = \pm T_{3,3\ell+2}$ for odd $\ell > 0$. In the exceptional cases, we have $\sigma(K) = 2\nu(K) - 2$. As mentioned in the introduction, this improves the inequality $|\nu(K) - \frac{1}{2}\sigma(K)| \leq 2$ for all 3–braid knots K in [20, Proposition 4.4].

It was shown in [47, Theorem 1.2] that $|\nu(K) - \frac{1}{2}\sigma(K)|$ gives a lower bound on the nonorientable smooth 4–genus of a knot K , denoted by $\gamma_4(K)$, the minimal first Betti number of a nonorientable surface in B^4 that meets the boundary S^3 along K . The similarity of the invariant ν and the classical signature σ on 3–braid knots K described above clearly does not lead to a good lower bound on $\gamma_4(K)$.

However, the equality $\sigma(K) = 2\nu(K)$ for most 3–braid knots is actually no great surprise when noting that in fact $|\nu(K) - \frac{1}{2}\sigma(K)| \leq 1$ must be true for all 3–braid

knots K for the following reason. It is not hard to see that for every 3–braid knot K , there is a nonorientable band move to a 2–bridge knot J , which is alternating [26]. This implies that the *nonorientable cobordism distance* $d_\gamma(K, J) = \gamma_4(K \# -J)$ between K and J is bounded from above by 1. On the other hand, using that ν and σ induce homomorphisms $\mathcal{C} \rightarrow \mathbb{Z}$ (see Section 2.2 and [44]), the inequality $|\nu(K) - \frac{1}{2}\sigma(K)| \leq \gamma_4(K)$ implies that

$$|\nu(K) - \frac{1}{2}\sigma(K)| = |\nu(K \# -J) - \frac{1}{2}\sigma(K \# -J)| \leq d_\gamma(K, J) \leq 1,$$

where we used $\nu(J) = \frac{1}{2}\sigma(J)$ by Proposition 2.1.

Note that a similar argument shows that $|\nu(K) - \frac{1}{2}\sigma(K)| \leq 2$ for all 4–braid knots K , using two nonorientable band moves to transform K into a 2–bridge link, which is also alternating.

4.4.2 On the proof technique As mentioned in the introduction, it came as a surprise to the author that our proof strategy works not only for positive 3–braid knots, but for all 3–braid knots. Let us make this more precise.

The proofs in Sections 4.2 and 4.3 imply, for any 3–braid knot K , the existence of cobordisms C_1 and C_2 of genus $g(C_1)$ and $g(C_2)$ between K and (connected sums of) torus knots T_1 and T_2 , respectively, such that

$$g(C_1) + g(C_2) = |\nu(T_2) - \nu(T_1)|$$

and

$$\nu(K) = \nu(T_1) + g(C_1) = \nu(T_2) - g(C_2).$$

For example, for knots K that are closures of positive 3–braids of Garside normal form (D), the proof of Lemma 4.5 shows the existence of such a cobordism C_1 for $T_1 = J_\varepsilon$ as in the proof of Lemma 4.4; and the existence of such a cobordism C_2 between K and $T_2 = T_{3,3(\ell+r)+1} \# -T_{2,2r+1}$ follows from the proof of Lemma 4.9.

The same strategy would work to determine the concordance invariants s and τ for all positive 3–braid knots K . Indeed, every positive 3–braid knot can be realized as the slice of a cobordism C between the unknot U and a torus knot T of braid index 3 such that $g(C) = |\tau(U) - \tau(T)| = |s(U) - s(T)|$ [21, Proposition 4.1]. However, in contrast, there are 3–braid knots where this strategy provably fails to determine s and τ . A concrete example is the 3–braid knot 10_{125} — the closure of $a^{-5}ba^3b$ [36] — which is not squeezed [21, Example 3.1]. This means that every cobordism C between two connected sums of torus knots T_1 and T_2 that has 10_{125} as a slice satisfies $g(C) > |\tau(T_2) - \tau(T_1)| = |s(T_2) - s(T_1)|$.

4.4.3 Comparison of the normal forms for 3–braids An algorithm described in [11, Section 7] as Schreier’s solution to the conjugacy problem [52] can be used to convert 3–braids in Garside normal form (see Definition 3.4) to 3–braids in Murasugi normal form (see Definition 4.15): if γ is a 3–braid of Garside normal form (C), then

$$\gamma \sim \Delta^{2(\ell+r)} a^{-1} b^{p_1-2} a^{-1} b^{q_1-2} \dots a^{-1} b^{p_r-2} a^{-1} b^{q_r-2},$$

and if γ is of Garside normal form (D), then

$$\gamma \sim \Delta^{2(\ell+r)} a^{-1} b^{p_1-2} a^{-1} b^{q_1-2} \dots a^{-1} b^{p_{r-1}-2} a^{-1} b^{q_{r-1}-2} a^{-1} b^{p_r-2}.$$

In addition, it is easy to see how 3–braids of Garside normal form (A) or (B) are conjugate to braids of Murasugi normal form (a) or (b).

5 On alternating distances of 3–braid knots

In this section, we prove Corollary 1.2 from the introduction and provide lower and upper bounds on the alternation number and dealternating number of any 3–braid knot which differ by 1.

5.1 Alternating distances of positive 3–braid knots

We will prove the following proposition.

Proposition 5.1 *Let K be a knot that is the closure of a positive 3–braid. Then*

$$\begin{aligned} \text{alt}(K) &= \text{dalt}(K) = \tau(K) + \nu(K) \\ &= \begin{cases} \ell & \text{if } K \text{ is the torus knot } T_{3,3\ell+k} \text{ for } \ell \geq 0 \text{ and } k \in \{1, 2\}, \\ r + \ell - 1 & \text{if } K \text{ is the closure of a braid of the form in (C) or (D),} \end{cases} \end{aligned}$$

where (C) and (D) refer to the Garside normal forms from Proposition 3.2.

Remark 5.2 Some of the cases in Proposition 5.1 have already been proved by other authors. Indeed, Feller, Pohlmann and Zentner used the observation (25) below to show that $\text{alt}(T_{3,3\ell+k}) = \ell$ for all $\ell \geq 0$ and $k \in \{1, 2\}$ [22, Theorem 1.1]. The upper bound they used was provided by [30, Theorem 8]; in fact, the equality had already been shown by Kanenobu in half of the cases, namely when ℓ is even. Moreover, Abe and Kishimoto [2, Theorem 3.1] showed that $\text{alt}(K) = \text{dalt}(K) = r + \ell - 1$ if K is a knot that is the closure of a positive 3–braid of the form in (C). However, to the best of this author’s knowledge, it is new that $\text{alt}(K) = g(K) + \nu(K)$ for all positive 3–braid knots K . Recall that $\tau(K) = g(K)$ for all positive 3–braid knots K by (5) from Section 2.1.

Before we prove [Proposition 5.1](#), let us provide the necessary definitions and background. The *Gordian distance* $d_G(K, J)$ between two knots K and J is the minimal number of crossing changes needed to transform a diagram of K into a diagram of J , where the minimum is taken over all diagrams of K [43]. The *alternation number* $\text{alt}(K)$ of a knot K is defined as the minimal Gordian distance of the knot K to the set of alternating knots [31], ie

$$\text{alt}(K) = \min\{d_G(K, J) \mid J \text{ is an alternating knot}\}.$$

The *dealternating number* $\text{dalt}(K)$ of a knot K is defined via a more diagrammatic approach [3]: it is the minimal number n such that K has a diagram that can be turned into an alternating diagram by n crossing changes. It follows from the definitions that

$$(24) \quad \text{alt}(K) \leq \text{dalt}(K)$$

for any knot K and $\text{alt}(K) = \text{dalt}(K) = 0$ if and only if K is alternating. Note that there are families of knots for which the difference between the alternation number and the dealternating number becomes arbitrarily large [38, Theorem 1.1].

In the proof of [Proposition 5.1](#), we will use that

$$(25) \quad |\tau(K) + \nu(K)| \leq \text{alt}(K)$$

for any knot K . In fact, for all alternating knots K ,

$$(26) \quad \tau(K) = \frac{1}{2}s(K) = -\nu(K) = -\frac{1}{t}\Upsilon_K(t) = -\frac{1}{2}\sigma(K)$$

for any $t \in (0, 1]$ —see [46, Theorem 1.14; 48, Theorem 1.4; 50, Theorem 3]—where s denotes Rasmussen’s concordance invariant from Khovanov homology [50]. It follows from [1, Theorem 2.1]—which builds on ideas of Livingston [34, Corollary 3]—that the absolute value of the difference of any two of the invariants in (26) is a lower bound on $\text{alt}(K)$. It was first observed in [22] that the upsilon invariant fits very well in this context; see also [23, Lemma 8].

Another main ingredient of our proof of [Proposition 5.1](#) is the inequality

$$(27) \quad \text{dalt}(\hat{\gamma}) \leq r - 1$$

for any positive 3–braid $\gamma = a^{p_1}b^{q_1} \dots a^{p_r}b^{q_r}$ with integers $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ [2, Lemma 2.2].

Proof of Proposition 5.1 Let K be a knot that is the closure of a positive 3–braid γ of the form in (C) or (D) from [Proposition 3.2](#) with $\ell \geq 0$. We claim that

$$(28) \quad r + \ell - 1 = \tau(K) + \nu(K) = |\tau(K) + \nu(K)| \leq \text{alt}(K) \leq \text{dalt}(K) \leq r + \ell - 1,$$

which implies the statement of the proposition for these knots. The two equalities in (28) directly follow from our computations of $\nu(K)$ in Proposition 4.2 and (5) applied to γ . The first two inequalities are direct consequences of the inequalities (25) and (24). Finally, the last inequality follows from inequality (27) applied to the particular braid representatives of K considered in the proof of Corollary 4.12.

For torus knots of braid index 3, the statement follows analogously. More precisely, if $K = T_{3,3\ell+k}$ for $\ell \geq 0$ and $k \in \{1, 2\}$, then by (4) and (12), we have $|\tau(K) + \nu(K)| = \ell$. In addition, the inequality in (27) applied to the particular braid representatives of K considered in the proof of Corollary 4.12 implies that $\text{dalt}(T_{3,3\ell+k}) \leq \ell$. □

From Proposition 5.1, it is easy to deduce that the alternating positive 3–braid knots are precisely the unknot and the connected sums $T_{2,2p+1} \# T_{2,2q+1}$ of two torus knots of braid index 2 for $p, q \geq 0$. This was already known; in fact, the stronger statement is true that the only prime alternating positive braid knots are the torus knots of braid index 2 [6, Corollary 3]. Note that by [42]—see also [11, Corollary 7.2]—the only composite 3–braid knots are the connected sums $T_{2,2p+1} \# T_{2,2q+1}$ for $p, q \in \mathbb{Z}$.

By [1, Theorem 1.1], the only torus knots with alternation number 1 are the torus knots $T_{3,4}$ and $T_{3,5}$. A knot with dealternating number 1 is called *almost alternating*.

Corollary 5.3 *A positive 3–braid knot is almost alternating if and only if it is one of the torus knots $T_{3,4}$ or $T_{3,5}$, or it is represented by a braid of the form*

$$a^{p_1} b^{q_1} a^{p_2} b^{q_2}, \quad \Delta a^{p_1} b^{q_1} a^{p_2}, \quad \Delta^2 a^{p_1} b^{q_1} \quad \text{or} \quad \Delta^3 a^{p_1}$$

for some integers $p_1, p_2, q_1, q_2 \geq 2$.

Proof This follows directly from Proposition 5.1. □

Remark 5.4 In particular, the seven positive 3–braid knots with crossing number 12—see [36]—are all almost alternating.

Remark 5.5 Our results imply that the Turaev genus equals the alternation number for all positive 3–braid knots. Indeed, let K be a knot that is the closure of a positive braid of the form in (C) or (D) with $\ell \geq 0$. Then we have

$$(29) \quad g_T(K) = \text{alt}(K) = \text{dalt}(K) = r + \ell - 1,$$

where $g_T(K)$ denotes the Turaev genus of the knot K . The *Turaev genus* $g_T(K)$ of a knot K is another alternating distance [38], which was first defined in [15] as the minimal genus of a Turaev surface $F(D)$, where the minimum is taken over all diagrams D of K . The Turaev surface $F(D)$ is a closed orientable surface embedded in S^3 associated to the diagram D . It is formed by building the natural cobordism between the circles in the two extreme Kauffman states (the *all-A-state* and the *all-B-state*) of the diagram D via adding saddles for each crossing of D , and then capping off the boundary components with disks. More details on the definition can be found, for example, in a survey by Champanerker and Kofman [13].

The equality $g_T(K) = \text{dalt}(K)$ in (29) easily follows from Proposition 5.1, the inequalities $|\tau(K) + \frac{1}{2}\sigma(K)| \leq g_T(K)$ [16, Theorem 1.1] and $g_T(K) \leq \text{dalt}(K)$ [2, Corollary 5.4], and the fact that $\sigma(K) = 2\nu(K)$ for all knots that are closures of positive braids of Garside normal form (C) or (D) (see Section 4.4.1).

It is not known whether the alternation number and the Turaev genus of a knot are in general comparable; namely, it is not known whether $\text{alt}(K) \leq g_T(K)$ for all knots K —see [38, Question 3]. However, it was shown by Abe and Kishimoto that $g_T(T_{3,3\ell+k}) = \text{dalt}(T_{3,3\ell+k}) = \ell$ for all $\ell \geq 0$ and $k \in \{1, 2\}$ [2, Theorem 5.9], so $g_T(K) = \text{alt}(K) = \text{dalt}(K)$ is true for all positive 3–braid knots.

Remark 5.6 In [23], Friedl, Livingston and Zentner introduce the invariant $\mathcal{A}_s(K)$, the minimal number of double point singularities in a generically immersed concordance from a knot K to an alternating knot. In the case that the alternating knot is the unknot, this is the well studied invariant $c_4(K)$ called the *4–dimensional clasp number* [53]. A sequence of crossing changes in a diagram of a knot K leading to a diagram of an alternating knot J realizes an immersed concordance from K to J where any crossing change gives rise to a double point singularity in the concordance. We thus have $\mathcal{A}_s(K) \leq \text{alt}(K)$ for any knot K , which resembles the inequality $c_4(K) \leq u(K)$ between the 4–dimensional clasp number and the unknotting number $u(K)$ of K . Moreover, we have $|\nu(K) + \tau(K)| \leq \mathcal{A}_s(K)$ for any knot K [23, Theorem 18], so Proposition 5.1 implies $\mathcal{A}_s(K) = \text{alt}(K)$ for all positive 3–braid knots K .

We are now ready to prove Corollary 1.2 from the introduction.

Proof of Corollary 1.2 The corollary follows directly from Proposition 5.1 and Remarks 5.5 and 5.6. \square

5.2 Bounds on the alternation number of general 3–braid knots

In the following, we turn our attention to 3–braid knots in general, which are not necessarily the closure of positive 3–braids. We will use that

$$(30) \quad \left| \frac{1}{2}s(K) + v(K) \right| \leq \text{alt}(K)$$

for any knot K , which follows from [1, Theorem 2.1]; see also (26) from Section 5.1. Rasmussen’s invariant s was computed for all 3–braid knots in Murasugi normal form (see Definition 4.15) by Greene.²

Corollary 5.7 *Let $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for some $\ell \in \mathbb{Z}, r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$ such that $K = \hat{\gamma}$ is a knot. Then*

$$|\ell| - 1 \leq \text{alt}(K) \leq \text{dalt}(K) \leq |\ell| \quad \text{if } \ell \neq 0.$$

Proof The lower bound on the alternation number follows from (30), Theorem 1.1 and the values of the invariant s for $K = \hat{\gamma}$ [27, Proposition 2.4]; namely

$$s(K) = \begin{cases} -\sum_{i=1}^r (p_i - q_i) + 6\ell - 2 & \text{if } \ell > 0, \\ -\sum_{i=1}^r (p_i - q_i) + 6\ell + 2 & \text{if } \ell < 0. \end{cases}$$

Moreover, it follows from [2, Theorem 2.5] that $\text{dalt}(\hat{\gamma}) \leq |\ell|$. □

Remark 5.8 An alternative way to prove the upper bound on $\text{dalt}(K)$ in Corollary 5.7 for $\ell \geq 1$ follows from our observations in the proof of Lemma 4.18. In fact, the braid diagrams given by the braid representatives γ_1 of $K = \hat{\gamma}$ considered in that proof can easily be transformed into alternating diagrams by ℓ crossing changes: it is enough to change the positive crossings corresponding to the single generators a in γ_1 to negative crossings; we obtain generators a^{-1} in the corresponding braid words which then correspond to alternating braid diagrams.

Remark 5.9 If K is represented by a 3–braid of Garside normal form (C) or (D) (see Definition 3.4), then using the observations in Section 4.4.3, Corollary 5.7 implies

$$(31) \quad \begin{aligned} |r + \ell| - 1 \leq \text{alt}(K) \leq \text{dalt}(K) \leq |r + \ell| & \quad \text{if } |r + \ell| > 0, \\ \text{alt}(K) = \text{dalt}(K) = 0 & \quad \text{if } r + \ell = 0. \end{aligned}$$

By Proposition 5.1, the lower bound in (31) is sharp whenever K is the closure of a positive 3–braid of Garside normal form (C) or (D). However, there are examples

²These computations were generalized to all links that are closures of 3–braids in [41].

where the upper bound in (31) is sharp. The two easiest such examples in terms of crossing number are the nonalternating knots 8_{20} and 8_{21} , which are represented by the 3–braids

$$a^3 b^{-1} a^{-3} b^{-1} \sim \Delta^{-3} a^7, \quad a^3 b a^{-2} b^2 \sim \Delta^{-2} a^3 b^2 a^2 b^3,$$

respectively; see [36]. The lower bound on the alternation number from (31) is

$$|r + \ell| - 1 = 0$$

in both cases. Indeed, by [7, Theorem 8.6] both knots are quasialternating, so all the invariants from (26) are equal [7, Proposition 1.4; 40; 46].

Remark 5.10 In a similar fashion as Corollary 5.7, the Turaev genus of all 3–braid knots was determined up to an additive error of at most 1 by Lowrance using his computation of the Khovanov width for these knots [37, Proposition 4.15]. More precisely,

$$|\ell| - 1 \leq g_T(K) \leq |\ell| \quad \text{if } \ell \neq 0$$

for any knot K that is represented by $\gamma = \Delta^{2\ell} a^{-p_1} b^{q_1} \dots a^{-p_r} b^{q_r}$ for some $\ell \in \mathbb{Z}$, $r \geq 1$ and $p_i, q_i \geq 1$ for $i \in \{1, \dots, r\}$.

6 The fractional Dehn twist coefficient of 3–braids in Garside normal form

In this section, we compute the fractional Dehn twist coefficient of any 3–braid in Garside normal form (see Definition 3.4).

The *fractional Dehn twist coefficient* is a homogeneous quasimorphism on the braid group B_n that assigns to any n –braid γ a rational number $\omega(\gamma)$. Here, a *quasimorphism* on a group G is any map $\varphi: G \rightarrow \mathbb{R}$ such that

$$\sup_{(a,b) \in G \times G} |\varphi(ab) - \varphi(a) - \varphi(b)| =: D_\varphi < \infty,$$

where D_φ is called the *defect* of φ . A quasimorphism $\varphi: G \rightarrow \mathbb{R}$ is called *homogeneous* if $\varphi(a^k) = k\varphi(a)$ for all $k \in \mathbb{Z}$ and $a \in G$. Any homogeneous quasimorphism is invariant under conjugation, so $\omega(\gamma)$ is invariant under the conjugacy class of γ .

The fractional Dehn twist coefficient first appeared in [24] in a different language. It can be defined for mapping classes of general surfaces with boundary, where we here view braids as mapping classes of the n times punctured closed disk. Maljutin defined

the fractional Dehn twist coefficient $\omega: B_n \rightarrow \mathbb{R}$, with $n \geq 2$, for all braid groups and showed that its defect is 1 if $n \geq 3$ and 0 if $n = 2$ [39, Theorem 6.3]. We refer the reader to [39] for a more detailed account.

Corollary 6.1 *Let γ be a 3-braid. Then its fractional Dehn twist coefficient is*

$$\omega(\gamma) = \begin{cases} \ell & \text{if } \gamma \text{ is conjugate to a braid in (A),} \\ \frac{1}{6}(p + 1) + \ell & \text{if } \gamma \text{ is conjugate to a braid in (B),} \\ r + \ell & \text{if } \gamma \text{ is conjugate to a braid in (C) or (D),} \end{cases}$$

where (A)–(D) refer to the Garside normal forms from Proposition 3.2.

Remark 6.2 The fractional Dehn twist coefficient was computed for 3-braids in Murasugi normal form (see Definition 4.15) in [29, Proposition 6.6].

In the proof of Corollary 6.1, we will use that the fractional Dehn twist coefficient of any 3-braid γ is completely determined by the writhe $\text{wr}(\gamma)$ and the homogenized epsilon invariant \tilde{v} of γ : we have, by [19, Theorem 1.3],

$$(32) \quad \omega(\gamma) = \tilde{v}(\gamma) + \frac{1}{2} \text{wr}(\gamma)$$

for any 3-braid γ . The invariant \tilde{v} is another real-valued homogeneous quasimorphism on the braid group B_3 which can be defined as

$$\tilde{v}: B_3 \rightarrow \mathbb{R}, \quad \gamma \mapsto \tilde{v}(\gamma) = \lim_{k \rightarrow \infty} \frac{v(\widehat{\gamma^{6k} ab})}{6k}.$$

More generally, Brandenbursky [12, Theorem 2.6] showed that a homogeneous quasimorphism $B_n \rightarrow \mathbb{R}$ can be assigned to any concordance homomorphism $\mathcal{C} \rightarrow \mathbb{R}$ that is bounded above by a constant multiple of the 4-genus. We refer the reader to [12] or [19, Appendix A] for more details on homogenized concordance invariants.

Proposition 6.3 *Let γ be a 3-braid. Then*

$$\tilde{v}(\gamma) = \begin{cases} -\frac{1}{2}p - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in (A),} \\ -\frac{1}{3}(p + 1) - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in (B),} \\ -\frac{1}{2}(\sum_{i=1}^r (p_i + q_i)) + r - 2\ell & \text{if } \gamma \text{ is conjugate to a braid in (C),} \\ -\frac{1}{2}(\sum_{i=1}^{r-1} (p_i + q_i) + p_r) + r - 2\ell - \frac{3}{2} & \text{if } \gamma \text{ is conjugate to a braid in (D).} \end{cases}$$

Proof We will use that $\tilde{v}(\alpha\beta) = \tilde{v}(\alpha) + \tilde{v}(\beta)$ if α and β commute [19, Lemma A.1]. In particular, for any 3-braid γ and any $\ell \in \mathbb{Z}$,

$$(33) \quad \tilde{v}(\Delta^{2\ell} \gamma) = \tilde{v}(\Delta^{2\ell}) + \tilde{v}(\gamma).$$

Moreover, by the definition of \tilde{v} , equation (12) and the homogeneity of \tilde{v} ,

$$(34) \quad \tilde{v}(\Delta^{2\ell}) = -2\ell \quad \text{for all } \ell \in \mathbb{Z}.$$

We will now compute $\tilde{v}(\gamma)$ for the positive 3–braids γ of the form (A)–(D), ie assuming $\ell \geq 0$ in (A)–(D). The statement of Proposition 6.3 will then follow from (33) and (34).

First, let $\gamma = \Delta^{2\ell} a^p$ for some $\ell \geq 0$ and $p \geq 0$. If $p = 0$, we have $\tilde{v}(\gamma) = -2\ell$ by (34). If $p \geq 1$, we have

$$\gamma^{6k} ab = \Delta^{12\ell k} a^{6pk} ab \sim \Delta^{12\ell k + 1} a^{6pk - 1},$$

so by Lemma 4.9, for $k \geq 1$,

$$\begin{aligned} v(\widehat{\gamma^{6k} ab}) &= -\frac{1}{2}(6pk - 1) + 1 - 12\ell k - \frac{3}{2} = -3pk - 12\ell k, \\ \tilde{v}(\gamma) &= \lim_{k \rightarrow \infty} \frac{v(\widehat{\gamma^{6k} ab})}{6k} = \lim_{k \rightarrow \infty} \frac{-3pk - 12\ell k}{6k} = -\frac{p}{2} - 2\ell. \end{aligned}$$

Second, let $\gamma = \Delta^{2\ell} a^p b$ for some $\ell \geq 0$ and $p \in \{1, 2, 3\}$. We have

$$\begin{aligned} \gamma^{6k} ab &= \Delta^{12\ell k} (ab)^{6k} ab = \Delta^{12\ell k + 4k} ab && \text{if } p = 1, \\ \gamma^{6k} ab &= \Delta^{12\ell k} (a^2 b a^2 b)^{3k} ab = \Delta^{12\ell k} (ababab)^{3k} ab = \Delta^{12\ell k + 6k} ab && \text{if } p = 2, \\ \gamma^{6k} ab &= \Delta^{12\ell k} (a^3 b a^3 b a^3 b)^{2k} ab = \Delta^{12\ell k} (a^2 bababab a^2 b)^{2k} ab \\ &= \Delta^{12\ell k + 8k} ab && \text{if } p = 3. \end{aligned}$$

By (12),

$$\tilde{v}(\gamma) = \lim_{k \rightarrow \infty} \frac{-12\ell k - (2p + 2)k}{6k} = -2\ell - \frac{p + 1}{3}.$$

Third, let $\gamma = \Delta^{2\ell} a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r}$ for some $\ell \geq 0$, $r \geq 1$ and $p_i, q_i \geq 2$ for $i \in \{1, \dots, r\}$. Then

$$\begin{aligned} \gamma^{6k} ab &= \Delta^{12\ell k} (a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r})^{6k} ab \\ &\sim \Delta^{12\ell k + 1} a^{p_1 - 1} b^{q_1} \dots a^{p_r} b^{q_r} (a^{p_1} b^{q_1} \dots a^{p_r} b^{q_r})^{6k - 1} \\ &\sim \Delta^{12\ell k + 1} (b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{q_r} a^{p_1})^{6k - 1} b^{q_1} a^{p_2} b^{q_2} \dots a^{p_r} b^{p_1 + q_r - 1}, \end{aligned}$$

where $p_1 + q_r - 1 \geq 3$. By Lemma 4.9,

$$\begin{aligned} v(\widehat{\gamma^{6k} ab}) &= -3k \sum_{i=1}^r (p_i + q_i) + 6kr - 12\ell k - 1, \\ \tilde{v}(\gamma) &= -\frac{1}{2} \sum_{i=1}^r (p_i + q_i) + r - 2\ell. \end{aligned}$$

Finally, let $\gamma = \Delta^{2\ell+1} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r}$ for some $\ell \geq 0$, $r \geq 1$, $p_r \geq 2$, and $p_i, q_i \geq 2$ for $i \in \{1, \dots, r-1\}$. Then

$$\begin{aligned} \gamma^{6k} ab &= \Delta^{12\ell k} (\Delta a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r})^{6k} ab \\ &= \Delta^{12\ell k} (\Delta^2 b^{p_1} a^{q_1} \dots b^{p_{r-1}} a^{q_{r-1}} b^{p_r} a^{p_1} b^{q_1} \dots a^{p_{r-1}} b^{q_{r-1}} a^{p_r})^{3k} ab \\ &= \Delta^{12\ell k + 6k} (b^{p_1} \dots b^{p_r} a^{p_1} \dots a^{p_r})^{3k} ab \\ &\sim \Delta^{12\ell k + 6k} a^{q_1} b^{p_2} \dots b^{p_r} a^{p_1} \dots \\ &\quad \dots a^{p_r} (b^{p_1} \dots b^{p_r} a^{p_1} \dots a^{p_r})^{3k-2} b^{p_1} \dots b^{p_r} a^{p_1} \dots a^{p_r+1} b^{p_1+1}, \end{aligned}$$

where $p_r + 1, p_1 + 1 \geq 3$. By Lemma 4.10,

$$\begin{aligned} v(\widehat{\gamma^{6k} ab}) &= -3k \left(\sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + 6kr - 12\ell k - 9k - 1, \\ \tilde{v}(\gamma) &= -\frac{1}{2} \left(\sum_{i=1}^{r-1} (p_i + q_i) + p_r \right) + r - 2\ell - \frac{3}{2}. \quad \square \end{aligned}$$

Proof of Corollary 6.1 This follows directly from Proposition 6.3, (32), and a straightforward calculation of the writhe of the braids in (A)–(D). □

Remark 6.4 If γ is a 3–braid conjugate to a braid of the form in (C) or (D) such that $\hat{\gamma}$ is a knot, then Proposition 6.3 and Theorem 1.1 imply $\tilde{v}(\gamma) = v(\hat{\gamma})$. If γ additionally is a positive 3–braid, then $\omega(\gamma) = r + \ell = g(\hat{\gamma}) + v(\hat{\gamma}) + 1$ is the minimal number from Corollary 1.3 (ie Corollary 4.12).

Remark 6.5 Our computation of $\omega(\gamma)$ in Corollary 6.1 together with [19, Theorem 1.3] completely determines $\widetilde{\Upsilon}(t)(\gamma)$ for all $0 \leq t \leq 1$ for any 3–braid γ , where $\widetilde{\Upsilon}(t)(\gamma)$ is the homogenization of the invariant $\Upsilon(t): \mathcal{C} \rightarrow \mathbb{R}$, defined similarly as the homogenization \tilde{v} of v .

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