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The group of quasi-isometries of the real line cannot act effectively on the line

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We prove that the group $\mathrm{QI}^+(\mathbb{R})$ of orientation-preserving quasi-isometries of the real line is a left-orderable, nonsimple group, which cannot act effectively on the real line $\mathbb{R}$.

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1 Introduction

A function $f\colon X\to Y$ between metric spaces X and Y is a quasi-isometry if there exist real numbers $K\geq 1$ and $C\geq 0$ such that

$$\frac{1}{K}d(x_1,x_2)-C\leq d(f(x_1),f(x_2))\leq Kd(x_1,x_2)+C$$

for any $x_1,x_2\in X$, and $d(\mathrm{Im}\, f,y)\leq C$ for any $y\in Y$. Two quasi-isometries f and g are called equivalent if they are of bounded distance; ie $\sup_{x\in X} d(f(x),g(x))<\infty$. The quasi-isometry group $\mathrm{QI}(X)$ is the group of all equivalence classes $[f]$ of quasi-isometries $f\colon X\to X$ under composition. The notion of quasi-isometries is one of the fundamental concepts in geometric group theory. In this note, we consider the quasi-isometry group $\mathrm{QI}(\mathbb{R})$ of the real line. Gromov and Pansu [3, Section 3.3B] noted that the group of bi-Lipschitz homeomorphisms has a full image in $\mathrm{QI}(\mathbb{R})$. Sankaran [9] proved that the orientation-preserving subgroup $\mathrm{QI}^+(\mathbb{R})$ is torsion-free and many large groups, like Thompson groups and free groups of infinite rank, can be embedded into $\mathrm{QI}^+(\mathbb{R})$.

Recall that a group G is left-orderable if there is a total order $\leq$ on G such that $g\leq h$ implies $fg\leq fh$ for any $f\in G$. We will prove the following.

Theorem 1.1 *The quasi-isometry group* $\mathrm{QI}^+(\mathbb{R})$ *— or* $\mathrm{QI}([0,+\infty))$ *— is not simple.*

Theorem 1.2 *The quasi-isometry group* $\mathrm{QI}^+(\mathbb{R})$ — *or* $\mathrm{QI}([0,+\infty))$ — *is left-orderable.*

Theorem 1.3 *The quasi-isometry group* $\mathrm{QI}^+(\mathbb{R})$ *cannot act effectively on the real line* $\mathbb{R}$*.*

Other (uncountable) left-orderable groups that cannot act on the line are been known. For example, the germ group $\mathcal{G}_\infty(\mathbb{R})$, due to Mann [4] and Rivas; and the compact supported diffeomorphism group $\mathrm{Diff}_c(\mathbb{R}^n)$ for $n>1$, due to Chen and Mann [1].

2 The group structure of QI(ℝ)

Let $\mathrm{QI}(\mathbb{R}_+)$ (resp. $\mathrm{QI}(\mathbb{R}_-)$) be the quasi-isometry group of the ray $[0,+\infty)$ (resp. $(-\infty,0]$), viewed as subgroup of $\mathrm{QI}(\mathbb{R})$ fixing the negative (resp. positive) part.

Lemma 2.1 $\mathrm{QI}(\mathbb{R}) = (\mathrm{QI}(\mathbb{R}_+)\times\mathrm{QI}(\mathbb{R}_-))\rtimes\langle t\rangle$, *where* $t\in\mathrm{QI}(\mathbb{R})$ *is the reflection* $t(x)=-x$ *for any* $x\in\mathbb{R}$*.*

Proof Sankaran [9] proves that the group $\mathrm{PL}_\delta(\mathbb{R})$ consisting of piecewise linear homeomorphisms with bounded slopes has a full image in $\mathrm{QI}(\mathbb{R})$. Since every homeomorphism $f\in\mathrm{PL}_\delta(\mathbb{R})$ is of bounded distance to the map $f-f(0)\in\mathrm{PL}_\delta(\mathbb{R})$, we see that the subgroup

$$\mathrm{PL}_{\delta,0}(\mathbb{R}) = \{f\in\mathrm{PL}_\delta(\mathbb{R}) \mid f(0)=0\}$$

also has full image in $\mathrm{QI}(\mathbb{R})$. Let

$$\mathrm{PL}_{\delta,+}(\mathbb{R}) = \{f\in\mathrm{PL}_\delta(\mathbb{R}) \mid f(x)=x,\ x\le 0\},$$
$$\mathrm{PL}_{\delta,-}(\mathbb{R}) = \{f\in\mathrm{PL}_\delta(\mathbb{R}) \mid f(x)=x,\ x\ge 0\}.$$

Since $\mathrm{PL}_{\delta,+}(\mathbb{R})\cap\mathrm{PL}_{\delta,-}(\mathbb{R})=\{\mathrm{id}_\mathbb{R}\}$, we see that $\mathrm{PL}_{\delta,+}(\mathbb{R})\times\mathrm{PL}_{\delta,-}(\mathbb{R})$ has a full image in $\mathrm{QI}^+(\mathbb{R})$, the orientation-preserving subgroup of $\mathrm{QI}(\mathbb{R})$. It's obvious that $\mathrm{PL}_{\delta,+}(\mathbb{R})$ (resp. $\mathrm{PL}_{\delta,-}(\mathbb{R})$) has a full image in $\mathrm{QI}(\mathbb{R}_+)$ (resp. $\mathrm{QI}(\mathbb{R}_-)$). Therefore, $\mathrm{QI}(\mathbb{R}) = (\mathrm{QI}(\mathbb{R}_+)\times\mathrm{QI}(\mathbb{R}_-))\rtimes\langle t\rangle$. □

Let $\mathrm{Homeo}_+(\mathbb{R})$ be the group of orientation-preserving homeomorphisms of the real line. Two functions $f,g\in\mathrm{Homeo}_+(\mathbb{R})$ are of bounded distance if

$$\sup_{|x|\ge M} |f(x)-g(x)| < \infty$$

for a sufficiently large real number M. This means when we study elements $[f]$ in $\mathrm{QI}(\mathbb{R})$, we don't need to care too much about the function values $f(x)$ for x with small absolute values. We will implicitly use this fact in the following context. As $\mathrm{PL}_\delta(\mathbb{R})$ has a full image in $\mathrm{QI}(\mathbb{R})$ (by Sankaran [9]), we take representatives of quasi-isometries which are homeomorphisms in the rest of the article.

2.1 $\mathrm{QI}(\mathbb{R}_+)$ is not simple

Let $\mathrm{QI}(\mathbb{R}_+)$ be the quasi-isometry group of the half-line $[0,+\infty)$. Note that the quasi-isometry group $\mathrm{QI}^+(\mathbb{R}) = \mathrm{QI}(\mathbb{R}_+) \times \mathrm{QI}(\mathbb{R}_-)$ and $\mathrm{QI}(\mathbb{R}_+) \cong \mathrm{QI}(\mathbb{R}_-)$, by Lemma 2.1. Let $H = \{[f] \in \mathrm{QI}(\mathbb{R}_+) \mid \lim_{x\to\infty}(f(x)-x)/x = 0\}$. Theorem 1.1 follows from the following theorem.

Theorem 2.2 *H is a proper normal subgroup of $\mathrm{QI}(\mathbb{R}_+)$. In particular, $\mathrm{QI}(\mathbb{R}_+)$ is not simple.*

Proof For any $[f],[g] \in H$,

$$\frac{f(g(x))-x}{x} = \frac{f(g(x))-g(x)}{g(x)}\frac{g(x)}{x} + \frac{g(x)-x}{x}.$$

Since g is a quasi-isometry, we know that $(1/K)x - C \le g(x) - g(0) \le Kx + C$. Therefore, $1/K - 1 \le g(x)/x \le K+1$ for sufficiently large x. When $x \to \infty$, we have $g(x) \to \infty$. This means $(f(g(x))-g(x))/g(x) \to 0$. Therefore, $(f(g(x))-x)/x \to 0$ as $x \to \infty$. This proves that $[fg] \in H$.

Note that

$$\frac{|f^{-1}(x)-x|}{x} = \frac{|f^{-1}(x)-f^{-1}(f(x))|}{x} \le \frac{K|x-f(x)|+C}{x}.$$

Therefore,

$$\lim_{x\to\infty}\frac{|f^{-1}(x)-x|}{x} = 0.$$

This means $[f^{-1}] \in H$ and that H is a subgroup.

For any quasi-isometric homeomorphism $g \in \mathrm{Homeo}(\mathbb{R}_+)$ and any $[f] \in H$,

$$\begin{aligned}\frac{g^{-1}(f(g(x)))-x}{x} &= \frac{g^{-1}(f(g(x)))-g^{-1}(g(x))}{x} \\ &= \frac{g^{-1}(f(g(x)))-g^{-1}(g(x))}{g(x)}\frac{g(x)}{x}.\end{aligned}$$

Note that when $x\to\infty$, the function $g(x)/x$ is bounded. Let $y = g(x)$. We have

$$\frac{|g^{-1}(f(y))-g^{-1}(y)|}{y} \le \frac{K|f(y)-y|+C}{y} \to 0, \quad x \to \infty.$$

Therefore, $[g^{-1}fg] \in H$.

It's obvious that the function f defined by $f(x) = 2x$ is not an element in H. The function defined by $g(x) = x + \ln(x+1)$ gives a nontrivial element in H. Thus H is a proper normal subgroup of $\mathrm{QI}(\mathbb{R}_+)$. □

Lemma 2.3 *Let*

$$W(\mathbb{R}) = \Big\{ f \in \mathrm{Diff}(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} |f(x)-x| < \infty,\ \sup_{x\in\mathbb{R}} |f'(x)| < \infty \Big\}$$

be the group consisting of diffeomorphisms with bounded derivatives and of bounded distance from the identity. Define a homeomorphism $h\colon \mathbb{R}\to\mathbb{R}$ *by* $h(x) = e^x$ *when* $x \ge 1$, $h(x) = -h(-x)$ *when* $x \le -1$, *and* $h(x) = ex$ *when* $-1 \le x \le 1$. *Then* hfh^{-1} *is a quasi-isometry for any* $f \in W(\mathbb{R})$.

Proof For any $f \in W(\mathbb{R})$ and sufficiently large $x > 0$, its derivative satisfies that

$$\begin{aligned} |hfh^{-1}(x)'| &= |(e^{f(\ln x)})'| \\ &= |(xe^{f(\ln x)-\ln x})'| \\ &= |e^{f(\ln x)-\ln x}(1 + f'(\ln x) - 1)| \\ &= |e^{f(\ln x)-\ln x} f'(\ln x)| \\ &\le e^{\sup_{x\in\mathbb{R}} |f(x)-x|} \cdot \sup_{x\in\mathbb{R}} |f'(x)|. \end{aligned}$$

The case for negative $x < 0$ can be calculated similarly. This proves that hfh^{-1} is a quasi-isometry. □

The following result was proved by Sankaran [9].

Corollary 2.4 *The quasi-isometry group* $\mathrm{QI}(\mathbb{R})$ *contains* $\mathrm{Diff}_{\mathbb{Z}}(\mathbb{R})$ (*the lift of* $\mathrm{Diff}(S^1)$ *to* $\mathrm{Homeo}(\mathbb{R})$).

Proof For any $f \in \mathrm{Diff}_{\mathbb{Z}}(\mathbb{R})$, we have $f(x+1) = f(x) + 1$ for any $x \in \mathbb{R}$. This means $\sup_{x\in\mathbb{R}} |f(x) - x| < +\infty$. Since $f(x) - x$ is periodic, we know that $f'(x)$ is bounded. Suppose that $f(x) > x$ for some $x \in [0,1]$. Take $y_n = e^{x+n}$ for $n > 0$. Let h be the function defined in Lemma 2.3. We have

$$|hfh^{-1}(y_n) - y_n| = |e^{f(x+n)} - e^{x+n}| = |e^{f(x)} - e^x| e^n \to \infty,$$

which means $[hfh^{-1}] \neq [\mathrm{id}] \in \mathrm{QI}(\mathbb{R})$. □

Lemma 2.5 $\mathrm{QI}(\mathbb{R})$ *contains the semidirect product* $\mathrm{Diff}_{\mathbb{Z}}(\mathbb{R}) \ltimes H$.

Proof Since H is normal, it's enough to prove that $\mathrm{Diff}_{\mathbb{Z}}(\mathbb{R}) \cap H = \{e\}$, the trivial subgroup. Actually, for any $f \in \mathrm{Diff}_{\mathbb{Z}}(\mathbb{R})$, the conjugate hfh^{-1} is a quasi-isometry as in the proof of Corollary 2.4. If $hfh^{-1} \in H$, then

$$\lim_{x\to\infty} \frac{hfh^{-1}(x)}{x} = \lim_{x\to\infty} \frac{xe^{f(\ln x)-\ln x}}{x} = \lim_{x\to\infty} e^{f(\ln x)-\ln x} = 1.$$

Since $f(x)-x$ is periodic, we know that $f(\ln x) = \ln x$ for any sufficiently large x. But this means that $f(y) = y$ for any y, so f is the identity. □

2.2 Affine subgroups of QI(ℝ)

Lemma 2.6 *The quasi-isometry group* $\mathrm{QI}(\mathbb{R}_+)$ (*actually*, *the semidirect product* $\mathrm{Diff}_{\mathbb{Z}}(\mathbb{R}) \ltimes H$) *contains the semidirect product* $\mathbb{R}_{>0} \ltimes \big(\bigoplus_{i\in\mathbb{R}_{\geq 1}} \mathbb{R}\big)$, *generated by* A_t *and* $B_{i,s}$ *for* $t \in \mathbb{R}_{>0}$, $i \in \mathbb{R}_{\geq 1} = [1,\infty)$ *and* $s \in \mathbb{R}$ *satisfying*

$$A_t B_{i,s} A_t^{-1} = B_{i,st^{\frac{i}{i+1}}}, \quad B_{i,s_1} B_{i,s_2} = B_{i,s_1+s_2},$$
$$A_{t_1} A_{t_2} = A_{t_1 t_2}, \qquad B_{i,s_1} B_{j,s_2} = B_{j,s_2} B_{i,s_1},$$

for any $t_1, t_2 \in \mathbb{R}_{>0}$, $i, j \in \mathbb{R}_{\geq 1}$ *and* $s_1, s_2 \in \mathbb{R}$.

Proof Let

$$A_t(x) = tx, \qquad t \in \mathbb{R}_{>0},$$
$$B_{i,s}(x) = x + sx^{\frac{1}{i+1}}, \quad s \in \mathbb{R},$$

for $x \geq 0$. We define $A_t(x) = B_{i,s}(x) = x$ for $x \leq 0$. Since the derivatives

$$A_t'(x) = t, \quad B_{i,s}'(x) = 1 + \frac{s}{i+1} x^{\frac{-i}{i+1}}$$

are bounded for sufficiently large x, we know that A_t and $B_{i,s}$ are quasi-isometries. For any $x \geq 1$,

$$A_t B_{i,s} A_t^{-1}(x) = A_t B_{i,s}\Big(\frac{x}{t}\Big) = A_t\Big(\frac{x}{t} + s\Big(\frac{x}{t}\Big)^{\frac{1}{i+1}}\Big) = x + st^{\frac{i}{i+1}} x^{\frac{1}{i+1}} = B_{i,st^{\frac{i}{i+1}}}(x).$$

For any $x \geq 1$,

$$B_{i,s_1} B_{i,s_2}(x) = B_{i,s_1}(x + s_2 x^{\frac{1}{i+1}}) = x + s_2 x^{\frac{1}{i+1}} + s_1 (x + s_2 x^{\frac{1}{i+1}})^{\frac{1}{i+1}}$$

and

$$|B_{i,s_1} B_{s_2}(x) - B_{i,s_1+s_2}(x)| = |s_1((x + s_2 x^{\frac{1}{i+1}})^{\frac{1}{i+1}} - x^{\frac{1}{i+1}})| \leq \left| s_1 \frac{s_2 x^{\frac{1}{i+1}}}{x^{\frac{i}{i+1}}} \right| \leq |s_1 s_2|$$

by Newton's binomial theorem. This means that $B_{i,s_1} B_{i,s_2}$ and B_{i,s_1+s_2} are of bounded distance. It is obvious that $A_{t_1} A_{t_2} = A_{t_1 t_2}$.

When $i < j$ are distinct natural numbers,

$$\begin{aligned}
&|B_{i,s_1}B_{j,s_2}(x) - B_{j,s_2}B_{i,s_1}(x)| \\
&\qquad = |x + s_2x^{\frac{1}{j+1}} + s_1(x + s_2x^{\frac{1}{j+1}})^{\frac{1}{i+1}} - (x + s_1x^{\frac{1}{i+1}} + s_2(x + s_1x^{\frac{1}{i+1}})^{\frac{1}{j+1}})| \\
&\qquad = |s_1((x + s_2x^{\frac{1}{j+1}})^{\frac{1}{i+1}} - x^{\frac{1}{i+1}}) + s_2(x^{\frac{1}{j+1}} - (x + s_1x^{\frac{1}{i+1}})^{\frac{1}{j+1}})| \\
&\qquad \le \left|s_1\frac{s_2x^{\frac{1}{j+1}}}{x^{\frac{i}{i+1}}}\right| + \left|s_2\frac{s_1x^{\frac{1}{i+1}}}{x^{\frac{j}{j+1}}}\right| \\
&\qquad \le 2|s_1s_2|
\end{aligned}$$

for any $x \ge 1$. This proves that images $[A_t], [B_{i,s}] \in \mathrm{QI}(\mathbb{R}_{\ge 0})$ satisfy the relations. By abuse of notation, we still denote the classes by the same letters.

We prove that the subgroup generated by $\{B_{i,s} \mid i \in \mathbb{R}_{\ge 1}, s \in \mathbb{R}\}$ is the infinite direct sum $\bigoplus_{i\in\mathbb{R}_{\ge 1}} \mathbb{R}$. It's enough to prove that $B_{i_1,s_1}, B_{i_2,s_2}, \dots, B_{i_k,s_k}$ are $\mathbb{Z}$–linearly independent for distinct $i_1, i_2, \dots, i_k$ and nonzero $s_1, s_2, \dots, s_k \in \mathbb{R}$. This can directly checked. For integers $n_1, n_2, \dots, n_k$, suppose that $B_{i_1,s_1}^{n_1} \circ B_{i_2,s_2}^{n_2} \circ \dots \circ B_{i_k,s_k}^{n_k} = \mathrm{id} \in \mathrm{QI}(\mathbb{R}_{\ge 0})$. We have

$$\begin{aligned}
&\sup_{x\in\mathbb{R}_{>0}} |B_{i_1,s_1}^{n_1} \circ B_{i_2,s_2}^{n_2} \circ \dots \circ B_{i_k,s_k}^{n_k}(x) - x| \\
&\quad = \sup_{x\in\mathbb{R}_{>0}} |n_ks_kx^{\frac{1}{i_k+1}} + n_{k-1}s_{k-1}(x + n_ks_kx^{\frac{1}{i_k+1}})^{\frac{1}{i_{k-1}+1}} + \dots + n_1s_1(x + \cdots)^{\frac{1}{i_1+1}}| \\
&\quad < +\infty,
\end{aligned}$$

which implies $n_1 = n_2 = \dots = n_k = 0$, considering the exponents.

The subgroup $\mathbb{R}_{>0} \ltimes \left(\bigoplus_{i\in\mathbb{R}_{\ge 1}} \mathbb{R}\right)$ lies in $\mathrm{Diff}_{\mathbb{Z}}(\mathbb{R}) \ltimes H$ by the following construction. Let $a_t, b_{i,s} \colon \mathbb{R} \to \mathbb{R}$ be defined by $a_t(x) = x + \ln t$ and $b_{i,s}(x) = \ln(e^x + se^{\frac{x}{i+1}})$ for $t \in \mathbb{R}_{>0}$, $i \in \mathbb{R}_{\ge 1}$ and $s \in \mathbb{R}$. It can be directly checked that $a_t \in \mathrm{Diff}_{\mathbb{Z}}(\mathbb{R})$ and $b_{i,s} \in W(\mathbb{R})$ (defined in Lemma 2.3). Let $h(x) = e^x$. A direct calculation shows that $ha_th^{-1} = A_t$ and $hb_{i,s}h^{-1} = B_{i,s}$, as elements in $\mathrm{QI}(\mathbb{R}_+)$. □

3 Left-orderability

The following is well known; for a proof, see [7, Proposition 1.4]:

Lemma 3.1 *A group G is left-orderable if and only if, for every finite collection of nontrivial elements $g_1, \dots, g_k$, there exist choices $\varepsilon_i \in \{1, -1\}$ such that the identity is not an element of the semigroup generated by $\{g_i^{\varepsilon_i} \mid i = 1, 2, \dots, k\}$.*

The proof of Theorem 1.2 follows a similar strategy used by Navas to prove the left-orderability of the group $\mathcal{G}_\infty$ of germs at ∞ of homeomorphisms of $\mathbb{R}$; cf [2, Remark 1.1.13] or [4, Proposition 2.2].

Proof of Theorem 1.2 It's enough to prove that $\mathrm{QI}(\mathbb{R}_+)$ is left-orderable. Let $f_1, f_2, \dots, f_n \in \mathrm{QI}(\mathbb{R}_+)$ be any finitely many nontrivial elements. Note that any $1 \neq [f] \in \mathrm{QI}(\mathbb{R}_+)$ has $\sup_{x>0} |f(x) - x| = \infty$. This property doesn't depend on the choice of $f \in [f]$. Without confusion, we still denote $[f]$ by f. Choose a sequence $\{x_{1,k}\} \subset \mathbb{R}_+$ such that $\sup_{k\in\mathbb{N}} |f_1(x_{1,k}) - x_{1,k}| = \infty$. For each $i > 1$, we have either $\sup_{k\in\mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| = \infty$ or $\sup_{k\in\mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \le M$ for a real number M. After passing to subsequences, we assume for each $i = 1, 2, \dots, n$ that either $f_i(x_{1,k}) - x_{1,k} \to +\infty$, $f_i(x_{1,k}) - x_{1,k} \to -\infty$ or $\sup_{k\in\mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \le M$. We assign $\varepsilon_i = 1$ for the first case and $\varepsilon_i = -1$ for the second case. For the third case, let

$$S_1 = \{f_i \mid \sup_{k\in\mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \le M\}.$$

Note that $f_1 \notin S_1$. Choose $f_{i_0} \in S_1$ if S_1 is not empty. We choose another sequence $\{x_{2,k}\}$ such that $\sup_{k\in\mathbb{N}} |f_{i_0}(x_{2,k}) - x_{2,k}| = \infty$. Similarly, after passing to a subsequence, we have for each $f \in S_1$ that either $f(x_{2,k}) - x_{2,k} \to +\infty$, $f(x_{2,k}) - x_{2,k} \to -\infty$ or $\sup_{k\in\mathbb{N}} |f(x_{2,k}) - x_{2,k}| \le M'$ for another real number M'. Assign $\varepsilon_i = 1$ for the first case and $\varepsilon_i = -1$ for the second case. Continue this process to define $S_2, S_3, \dots$ and choose sequences $\{x_{i,k}\}, i = 3, 4, \dots$ to assign ε_i for each f_i. Note that the process will stop at n times, as the number of elements without assignment is strictly decreasing.

For an element $f \in \mathrm{QI}(\mathbb{R}_+)$ satisfying $f(x_i) - x_i \to \infty$ as $i \to \infty$ for some sequence $\{x_i\}$, we assume that $f(x_i) - x_i > 0$ for each i. Since f and f^{-1} are orientation-preserving,

$$\begin{aligned} f^{-1}(x_i) - x_i &= -(x_i - f^{-1}(x_i)) \\ &= -(f^{-1}(f(x_i)) - f^{-1}(x_i)) \le -\Big(\frac{1}{K}(f(x_i) - x_i) - C\Big) \to -\infty. \end{aligned}$$

Let $w = f_{i_1}^{\varepsilon_{i_1}} \cdots f_{i_m}^{\varepsilon_{i_m}} \in \langle f_1, f_2, \dots, f_n \rangle$ be a nontrivial word. If $\{i_1, \dots, i_m\} \not\subseteq S_1$, we have $w(x_{1,k}) - x_{1,k} \to \infty$. Otherwise, $\sup_{k\in\mathbb{N}} |w(x_{1,k}) - x_{1,k}| < \infty$. Suppose that $\{i_1, \dots, i_m\} \subset S_t$, but $\{i_1, \dots, i_m\} \not\subseteq S_{t+1}$ with the assumption that $S_0 = \{f_1, f_2, \dots, f_n\}$. We have $w(x_{t+1,k}) - x_{t+1,k} \to \infty$ as $k \to \infty$. This proves that $w \neq 1 \in \mathrm{QI}(\mathbb{R}_+)$. Therefore, $\mathrm{QI}(\mathbb{R}_+)$ is left-orderable by Lemma 3.1. □

Lemma 3.2 *The group* $\mathrm{QI}(\mathbb{R}_+)$ *is not locally indicable.*

Proof Note that $\mathrm{QI}(\mathbb{R}_+)$ contains the lift $\widetilde{\Gamma}$ of $\mathrm{PSL}(2,\mathbb{R}) < \mathrm{Diff}(S^1)$ to $\mathrm{Homeo}(\mathbb{R})$ (Corollary 2.4). But this lift $\widetilde{\Gamma}$ contains a subgroup $\Gamma = \langle f, g, h : f^2 = g^3 = h^7 = fgh \rangle$, the lift of the $(2,3,7)$–triangle group. There are no nontrivial maps from Γ to $(\mathbb{R},+)$; for more details see [2, page 94]. □

4 The quasi-isometric group cannot act effectively on the line

The following was proved by Mann [4, Proposition 6].

Lemma 4.1 *Consider the affine group $\mathbb{R}_{>0} \ltimes \mathbb{R}$, generated by A_t and B_s for $t \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}$ satisfying*

$$A_t B_s A_t^{-1} = B_{ts}, \quad B_{s_1} B_{s_2} = B_{s_1+s_2}, \quad A_{t_1} A_{t_2} = A_{t_1 t_2}.$$

The affine group $\mathbb{R}_{>0} \ltimes \mathbb{R}$ cannot act effectively on the real line $\mathbb{R}$ by homeomorphisms with A_t a translation for each t.

Proof Suppose that $\mathbb{R}_{>0} \ltimes \mathbb{R}$ acts effectively on the real line $\mathbb{R}$ with each A_t a translation. After passing to an index-2 subgroup, we assume that the group is orientation-preserving. If B_1 acts freely on $\mathbb{R}$, then it is conjugate to the translation $T : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x+1$. In such a case, we have $A_2 T A_2^{-1} = T^2$. Therefore, $A_2^{-1}(x+2) = A_2^{-1}(x) + 1$ for any x. Since A_2^{-1} maps intervals of length 2 to an interval of length 1, it is a contracting map, and thus has a fixed point.

If B_1 has a nonempty fixed point set $\mathrm{Fix}(B_1)$, choose I to be a connected component of $\mathbb{R} \setminus \mathrm{Fix}(B_1)$. Suppose that $A_2(x) = x + a$, a translation by some real number $a > 0$. Since $A_2 = A^n_{2^{1/n}}$, we have $A_{2^{1/n}}(x) = x + a/n$ for each positive integer n. For each n, let $F_n = A_{2^{1/n}} B_1 A^{-1}_{2^{1/n}}$. Since $A_{2^{1/n}} B_1 A^{-1}_{2^{1/n}}$ commutes with B_1, we see that $F_n \mathrm{Fix}(B_1) = \mathrm{Fix}(B_1)$. This means that either $F_n(I) = I$ or $F_n(I) \cap I = \varnothing$. Since $F_n(x) = B_1(x - a/n) + a/n$ for any $x \in \mathbb{R}$, we know that $F_n(I) = I$ for sufficiently large n. Without loss of generality, we assume that I is of the form (x, y) or $(-\infty, y)$. Choose a sufficiently large n such that $y - a/n \in I$. We have

$$A_{2^{1/n}} B_1 A^{-1}_{2^{1/n}}(y) = B_1\Big(y - \frac{a}{n}\Big) + \frac{a}{n} \neq y,$$

which is a contradiction to the fact that $F_n(I) = I$. □

Definition 4.2 A topologically diagonal embedding of a group $G < \mathrm{Homeo}(\mathbb{R})$ is a homomorphism $\phi \colon G \to \mathrm{Homeo}_+(\mathbb{R})$ defined as follows. Choose a collection of disjoint open intervals $I_n \subset \mathbb{R}$ and homeomorphisms $f_n \colon \mathbb{R} \to I_n$. Define ϕ by $\phi(g)(x) = f_n g f_n^{-1}(x)$ when $x \in I_n$ and $\phi(g)(x) = x$ when $x \notin I_n$.

The following is similar to a result proved by Militon [6].

Lemma 4.3 (Militon [6]) *Let* $\Gamma = \mathrm{PSL}_2(\mathbb{R})$ *and* $\widetilde{\Gamma} < \mathrm{Homeo}_+(\mathbb{R})$ *be the lift of* Γ *to the real line. Any effective action* $\phi \colon \widetilde{\Gamma} \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ *of* $\widetilde{\Gamma}$ *on the real line* $\mathbb{R}$ *is a topological diagonal embedding.*

Proof After passing to an index-2 subgroup, we assume the action is orientation-preserving. Let $\tau \colon \mathbb{R} \to \mathbb{R}$ be the translation $x \mapsto x+1$. Suppose that $\mathrm{Fix}(\phi(\tau)) \neq \varnothing$. Note that τ lies in the center of $\widetilde{\Gamma}$. The quotient group $\Gamma = \widetilde{\Gamma}/\langle\tau\rangle$ acts on the fixed point set $\mathrm{Fix}(\phi(\tau))$. For any $f \in \Gamma$ and $x \in \mathrm{Fix}(\phi(\tau))$, we denote the action by $f(x)$ without confusion. Choose any torsion-element $f \in \Gamma$ and any $x \in \mathrm{Fix}(\phi(\tau))$. We must have $x = f(x)$, for otherwise $x < f(x) < f^2(x) < \cdots < f^k(x)$ for any k. Since Γ is simple, we know that the action of $\widetilde{\Gamma}$ on $\mathrm{Fix}(\tau)$ is trivial. For each connected component $I_i \subset \mathbb{R} \setminus \mathrm{Fix}(\phi(\tau))$, we know that $\tau|_{I_i}$ is conjugate to a translation. The group $\Gamma = \widetilde{\Gamma}/\langle\tau\rangle$ acts on $I_i/\langle\phi(\tau)\rangle = S^1$. A result of Matsumoto [5, Theorem 5.2] says that the group Γ is conjugate to the natural inclusion $\mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{Homeo}_+(S^1)$ by a homeomorphism $g \in \mathrm{Homeo}_+(S^1)$. Therefore, the group $\phi(\widetilde{\Gamma})|_{I_i}$ is conjugate to the image of the natural inclusion $\widetilde{\Gamma} \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$. □

For a real number $a \in \mathbb{R}$, let

$$t_a \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + a$$

be the translation. Denote by $A = \langle t_a : a \in \mathbb{R}\rangle$, the subgroup of translations in the lift $\widetilde{\Gamma}$ of $\mathrm{PSL}_2(\mathbb{R})$.

Corollary 4.4 *For any injective group homomorphism* $\phi \colon \widetilde{\Gamma} \to \mathrm{Homeo}(\mathbb{R})$, *the image* $\phi(A)$ *is a continuous one-parameter subgroup; ie* $\lim_{a \to a_0} \phi(t_a) = \phi(t_{a_0})$ *for any* $a_0 \in \mathbb{R}$.

Proof If ϕ is injective, the previous lemma says that ϕ is a topological diagonal embedding. Therefore, $\phi(A)$ is continuous. □

We will need the following elementary fact.

Lemma 4.5 *Let* $\phi \colon (\mathbb{R}, +) \to (\mathbb{R}, +)$ *be a group homomorphism. If* ϕ *is continuous at any* $x \neq 0$, *then* ϕ *is* $\mathbb{R}$*–linear.*

Proof For any nonzero integer n, we have $\phi(n) = n\phi(1)$ and $\phi(1) = \phi\big(\frac{1}{n}n\big) = n\phi\big(\frac{1}{n}\big)$. Since ϕ is additive, we have $\phi\big(\frac{m}{n}\big) = m\phi\big(\frac{1}{n}\big) = \frac{m}{n}\phi(1)$ for any integers $m, n \neq 0$.

For any nonzero real number $a \in \mathbb{R}$, choose a rational sequence $r_i \to a$. When ϕ is continuous, we have that $\phi(r_i) \to \phi(a)$ and $\phi(r_i) = r_i\phi(1) \to a\phi(1) = \phi(a)$. □

The following is the classical theorem of Hölder: a group acting freely on $\mathbb{R}$ is semiconjugate to a group of translations; see Navas [8, Section 2.2.4].

Lemma 4.6 *Let Γ be a group acting freely on the real line $\mathbb{R}$. There is an injective group homomorphism $\phi\colon \Gamma \to (\mathbb{R}, +)$ and a continuous nondecreasing map $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi(h(x)) = \varphi(x) + \phi(h)$$

for any $x \in \mathbb{R}$ and $h \in \Gamma$.

Corollary 4.7 *Suppose that the affine group $\mathbb{R}_{>0} \ltimes \mathbb{R} = \langle a_t : t \in \mathbb{R}_{>0}\rangle \ltimes \langle b_s : s \in \mathbb{R}\rangle$ acts on the real line $\mathbb{R}$ by homeomorphisms satisfying*

(1) *the action of the subgroup $\mathbb{R} = \langle b_s : s \in \mathbb{R}\rangle$ is free;*
(2) *for any fixed $x \in \mathbb{R}$, $a_t(x)$ is continuous with respect to $t \in \mathbb{R}_{>0}$.*

Let $\phi\colon \langle b_s : s \in \mathbb{R}\rangle \to (\mathbb{R}, +)$ be the additive map in Lemma 4.6 for $\Gamma = \langle b_s : s \in \mathbb{R}\rangle$. Then ϕ is an $\mathbb{R}$–linear map.

Proof Note that $a_t b_s a_t^{-1} = b_{ts}$. We have

$$\varphi(b_{ts}(x)) = \varphi(x) + \phi(b_{ts}).$$

Since $b_{ts}(x) = a_t b_s a_t^{-1}(x) \to b_s(x)$ when $t \to 1$, we have that

$$\varphi(x) + \phi(b_{ts}) \to \varphi(b_s(x)) = \varphi(x) + \phi(b_s).$$

This implies that $\phi(b_{ts}) \to \phi(b_s)$ as $t \to 1$. For any nonzero $x \in \mathbb{R}$ and sequence $x_n \to x$,

$$\phi(b_{x_n}) = \phi(b_{\frac{x_n}{x}x}) \to \phi(b_x).$$

The map ϕ is $\mathbb{R}$–linear by Lemma 4.5. □

Theorem 4.8 *Consider $G = \mathbb{R}_{>0} \ltimes \left(\bigoplus_{i\in\mathbb{R}_{\geq 1}} \mathbb{R}\right)$, generated by A_t and $B_{i,s}$ for $t \in \mathbb{R}_{>0}$, $i \in \mathbb{R}_{\geq 1} = [1, \infty)$ and $s \in \mathbb{R}$ satisfying*

$$A_t B_{i,s} A_t^{-1} = B_{i,st^{\frac{i}{i+1}}}, \quad B_{i,s_1} B_{i,s_2} = B_{i,s_1+s_2},$$
$$A_{t_1} A_{t_2} = A_{t_1 t_2}, \qquad B_{i,s_1} B_{j,s_2} = B_{j,s_2} B_{i,s_1}$$

for any $t_1, t_2 \in \mathbb{R}_{>0}$, $i, j \in \mathbb{R}_{\geq 1}$ and $s_1, s_2 \in \mathbb{R}$. Then G cannot act effectively on the real line $\mathbb{R}$ by homeomorphisms when the induced action of $\langle A_t : t \in \mathbb{R}_{>0}\rangle$ is a topologically diagonal embedding of the translation subgroup $(\mathbb{R}, +) \hookrightarrow \mathrm{Homeo}(\mathbb{R})$.

Proof Suppose that G acts effectively on $\mathbb{R}$ with the induced action of $\langle A_t : t \in \mathbb{R}_{>0}\rangle$, a topologically diagonal embedding of the translation subgroup $(\mathbb{R},+) \hookrightarrow \mathrm{Homeo}(\mathbb{R})$. Let I be a connected component of $\mathbb{R} \setminus \mathrm{Fix}(\langle A_t, B_{i,s} : t \in \mathbb{R}_{>0}, i = 1, s \in \mathbb{R}\rangle)$.

Suppose that there is an element $B_{1,s}$ having a fixed point $x \in I$ for some $s > 0$. Since $A_4 B_{1,s} A_4^{-1} = B_{1,s}^2$, we know that $A_4 x \in \mathrm{Fix}(B_{1,s}) = \mathrm{Fix}(B_{1,s}^2)$. Since there are no fixed points in I for $\langle A_t, B_{1,s} : t \in \mathbb{R}_{>0}, s \in \mathbb{R}\rangle$, we know that $\lim_{n\to\infty} A_4^n x \notin I$.[1] This implies that A_4 has no fixed point in I. Since the group homomorphism

$$\langle A_t : t \in \mathbb{R}_{>0}\rangle \to \mathrm{Homeo}(\mathbb{R})$$

is a diagonal embedding, we see that each A_t has no fixed point in I and the action of $\langle A_t : t \in \mathbb{R}_{>0}\rangle$ on I is conjugate to a group of translations. By Lemma 4.1, the affine group $\langle A_t, B_{1,s} : t \in \mathbb{R}_{>0}, s \in \mathbb{R}\rangle$ cannot act effectively on I. Suppose that $A_t B_{1,s'}$ acts trivially on I for some $t > 0$ and $s' > 0$. We have that $A_t B_{1,s} = A_{s^2 s'^{-2}} (A_t B_{1,s'}) A_{s^2 s'^{-2}}^{-1}$ acts trivially on I. But $A_t B_{1,s}(x) = A_t(x) = x$ implies that $t = 1$. Therefore, the element $B_{1,s}$ (and any $B_{1,t} = A_{t^2 s^{-2}} B_{1,s} A_{t^2 s^{-2}}^{-1}$ for $t \in \mathbb{R}_{>0}$) acts trivially on I. This means that the action of $\langle B_{1,s} : s \in \mathbb{R}\rangle$ on the connected component I is either trivial or free. Since the action of G is effective, there is a connected component I_1 on which $B_{1,s}$ acts freely. A similar argument shows that $B_{i,s'}$ acts freely on a component I_i for each $i \in \mathbb{R}_{\geq 1}$ and any $s' \in \mathbb{R} \setminus \{0\}$.

Since $B_{i,s'}$ commutes with $B_{j,s}$, we have $B_{i,s'}(I_1) \subset \mathbb{R} \setminus \mathrm{Fix}(\langle B_{j,s} : s \in \mathbb{R}\rangle)$. Moreover, $B_{i,s'}(I_j) \cap I_j$ is either I_j or the empty set. Suppose that $I_i \cap I_j \neq \varnothing$ and the right end b_i of I_i lies in I_j. Choose $x \in I_i \cap I_j$. Note that $B_{j,s}([x, b_i)) \cap [x, b_i) = \varnothing$ for any $s > 0$. This is impossible as $B_{j,s/n}(x) \to x$ as $n \to \infty$. Therefore, $I_i \cap I_j = I_i$ or is empty for distinct $i, j \in \mathbb{R}_{\geq 1}$. Since we have uncountably many $i \in \mathbb{R}_{>0}$, there must be some distinct $i, j \in \mathbb{R}_{\geq 1}$ such that $I_i = I_j$. This means that the subgroup $\mathbb{R} \oplus \mathbb{R}$ spanned by the i, j–components acts freely on I_i. Hölder's theorem (Lemma 4.6) gives an injective group homomorphism $\phi\colon \mathbb{R} \oplus \mathbb{R} \to (\mathbb{R},+)$ and a continuous nondecreasing map $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(h(x)) = \varphi(x) + \phi(h)$$

for any $x \in \mathbb{R}$. Since $\langle A_t : t \in \mathbb{R}_{>0}\rangle \to \mathrm{Homeo}(\mathbb{R})$ is a topological embedding, we have that for any fixed $x \in \mathbb{R}$, $A_t(x)$ is continuous with respect to $t \in \mathbb{R}_{>0}$. By Corollary 4.7,

[1] Otherwise, $\lim_{n\to\infty} A_4^n x \in I$. But $A_t(\lim_{n\to\infty} A_4^n x) = \lim_{n\to\infty} A_4^n x$ for any $t > 0$ by the topologically diagonal embedding. For any s', we have $B_{1,s'} = A_{s'^2 s^{-2}} B_{1,s} A_{s'^2 s^{-2}}^{-1}$ and $B_{1,s'}(\lim_{n\to\infty} A_4^n x) = \lim_{n\to\infty} A_4^n x$. This would imply that $\lim_{n\to\infty} A_4^n x$ is a global fixed point of $\langle A_t, B_{1,s} : t \in \mathbb{R}_{>0}, s \in \mathbb{R}\rangle$.

the restriction map $\phi|_{\mathbb{R}}$ is $\mathbb{R}$–linear for each direct summand $\mathbb{R}$. This is a contradiction to the fact that ϕ is injective. Therefore, the group G cannot act effectively. □

Proof of Theorem 1.3 Suppose that $\mathrm{QI}^+(\mathbb{R})$ acts on the real line by an injective group homomorphism $\phi\colon \mathrm{QI}^+(\mathbb{R}) \to \mathrm{Homeo}(\mathbb{R})$. The group $\mathrm{QI}^+(\mathbb{R})$ contains the semidirect product $\mathbb{R}_{>0} \ltimes \left(\bigoplus_{i\in\mathbb{R}_{\geq 1}} \mathbb{R}\right)$, by Lemma 2.6. The subgroup $\mathbb{R}_{>0}$ (as the image of the exponential map) is a homomorphic image of the subgroup $\mathbb{R} < \widetilde{\Gamma}$, which is the lift of $\mathrm{SO}(2)/\{\pm I_2\} < \mathrm{PSL}_2(\mathbb{R})$ to $\mathrm{Homeo}(\mathbb{R})$. Note that $\widetilde{\Gamma}$ is embedded into $\mathrm{QI}^+(\mathbb{R})$ (see Corollary 2.4 and its proof). By Lemma 4.3, any effective action of $\widetilde{\Gamma}$ on the real line $\mathbb{R}$ is a topological diagonal embedding. This means that the action of $\mathbb{R}_{>0}$ is a topological diagonal embedding (Corollary 4.4). Theorem 4.8 shows that the action of $\mathbb{R}_{>0} \ltimes \left(\bigoplus_{i\in\mathbb{R}_{\geq 1}} \mathbb{R}\right)$ is not effective. □

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