The group of quasi-isometries of the real line cannot act effectively on the line

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We prove that the group $\text{QI}^+(\mathbb{R})$ of orientation-preserving quasi-isometries of the real line is a left-orderable, nonsimple group, which cannot act effectively on the real line $\mathbb{R}$.

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1 Introduction

A function $f : X \to Y$ between metric spaces $X$ and $Y$ is a quasi-isometry if there exist real numbers $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K}d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + C$$

for any $x_1, x_2 \in X$, and $d(\text{Im} f, y) \leq C$ for any $y \in Y$. Two quasi-isometries $f$ and $g$ are called equivalent if they are of bounded distance; ie $\sup_{x \in X} d(f(x), g(x)) < \infty$. The quasi-isometry group $\text{QI}(X)$ is the group of all equivalence classes $[f]$ of quasi-isometries $f : X \to X$ under composition. The notion of quasi-isometries is one of the fundamental concepts in geometric group theory. In this note, we consider the quasi-isometry group $\text{QI}(\mathbb{R})$ of the real line. Gromov and Pansu [3, Section 3.3B] noted that the group of bi-Lipschitz homeomorphisms has a full image in $\text{QI}(\mathbb{R})$. Sankaran [9] proved that the orientation-preserving subgroup $\text{QI}^+(\mathbb{R})$ is torsion-free and many large groups, like Thompson groups and free groups of infinite rank, can be embedded into $\text{QI}^+(\mathbb{R})$.

Recall that a group $G$ is left-orderable if there is a total order $\leq$ on $G$ such that $g \leq h$ implies $fg \leq fh$ for any $f, g \in G$. We will prove the following.

**Theorem 1.1** The quasi-isometry group $\text{QI}^+(\mathbb{R})$ — or $\text{QI}([0, +\infty))$ — is not simple.

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Theorem 1.2  The quasi-isometry group \( \text{QI}^+ (\mathbb{R}) \) — or \( \text{QI}([0, +\infty)) \) — is left-orderable.

Theorem 1.3  The quasi-isometry group \( \text{QI}^+ (\mathbb{R}) \) cannot act effectively on the real line \( \mathbb{R} \).

Other (uncountable) left-orderable groups that cannot act on the line are been known. For example, the germ group \( G_\infty (\mathbb{R}) \), due to Mann \[4\] and Rivas; and the compact supported diffeomorphism group \( \text{Diff}_c (\mathbb{R}^n) \) for \( n > 1 \), due to Chen and Mann \[1\].

2  The group structure of \( \text{QI}(\mathbb{R}) \)

Let \( \text{QI}(\mathbb{R}_+) \) (resp. \( \text{QI}(\mathbb{R}_-) \)) be the quasi-isometry group of the ray \([0, +\infty)\) (resp. \((-\infty, 0]\)), viewed as subgroup of \( \text{QI}(\mathbb{R}) \) fixing the negative (resp. positive) part.

Lemma 2.1  \( \text{QI}(\mathbb{R}) = (\text{QI}(\mathbb{R}_+) \times \text{QI}(\mathbb{R}_-)) \rtimes \langle t \rangle \), where \( t \in \text{QI}(\mathbb{R}) \) is the reflection \( t(x) = -x \) for any \( x \in \mathbb{R} \).

Proof  Sankaran \[9\] proves that the group \( \text{PL}_\delta (\mathbb{R}) \) consisting of piecewise linear homeomorphisms with bounded slopes has a full image in \( \text{QI}(\mathbb{R}) \). Since every homeomorphism \( f \in \text{PL}_\delta (\mathbb{R}) \) is of bounded distance to the map \( f - f(0) \in \text{PL}_\delta (\mathbb{R}) \), we see that the subgroup

\[
\text{PL}_{\delta,0} (\mathbb{R}) = \{ f \in \text{PL}_\delta (\mathbb{R}) \mid f(0) = 0 \}
\]

also has full image in \( \text{QI}(\mathbb{R}) \). Let

\[
\text{PL}_{\delta, +} (\mathbb{R}) = \{ f \in \text{PL}_\delta (\mathbb{R}) \mid f(x) = x, x \leq 0 \},
\]

\[
\text{PL}_{\delta, -} (\mathbb{R}) = \{ f \in \text{PL}_\delta (\mathbb{R}) \mid f(x) = x, x \geq 0 \}.
\]

Since \( \text{PL}_{\delta, +} (\mathbb{R}) \cap \text{PL}_{\delta, -} (\mathbb{R}) = \{ \text{id}_\mathbb{R} \} \), we see that \( \text{PL}_{\delta, +} (\mathbb{R}) \times \text{PL}_{\delta, -} (\mathbb{R}) \) has a full image in \( \text{QI}^+ (\mathbb{R}) \), the orientation-preserving subgroup of \( \text{QI}(\mathbb{R}) \). It’s obvious that \( \text{PL}_{\delta, +} (\mathbb{R}) \) (resp. \( \text{PL}_{\delta, -} (\mathbb{R}) \)) has a full image in \( \text{QI}(\mathbb{R}_+) \) (resp. \( \text{QI}(\mathbb{R}_-) \)). Therefore, \( \text{QI}(\mathbb{R}) = (\text{QI}(\mathbb{R}_+) \times \text{QI}(\mathbb{R}_-)) \rtimes \langle t \rangle \). \( \square \)

Let \( \text{Homeo}_+(\mathbb{R}) \) be the group of orientation-preserving homeomorphisms of the real line. Two functions \( f, g \in \text{Homeo}_+(\mathbb{R}) \) are of bounded distance if

\[
\sup_{|x| \geq M} |f(x) - g(x)| < \infty
\]

for a sufficiently large real number \( M \). This means when we study elements \([f]\) in \( \text{QI}(\mathbb{R}) \), we don’t need to care too much about the function values \( f(x) \) for \( x \) with small
The group of quasi-isometries of the real line cannot act effectively on the line absolute values. We will implicitly use this fact in the following context. As \( \text{PL}_g(\mathbb{R}) \) has a full image in \( \text{QI}(\mathbb{R}) \) (by Sankaran [9]), we take representatives of quasi-isometries which are homeomorphisms in the rest of the article.

### 2.1 QI(\( \mathbb{R}_+ \)) is not simple

Let \( \text{QI}(\mathbb{R}_+) \) be the quasi-isometry group of the half-line \([0, +\infty)\). Note that the quasi-isometry group \( \text{QI}^+(\mathbb{R}) = \text{QI}(\mathbb{R}_+) \times \text{QI}(\mathbb{R}_-) \) and \( \text{QI}(\mathbb{R}_+) \cong \text{QI}(\mathbb{R}_-) \), by Lemma 2.1. Let \( H = \{ [f] \in \text{QI}(\mathbb{R}_+) \mid \lim_{x \to \infty} (f(x) - x)/x = 0 \} \). Theorem 1.1 follows from the following theorem.

**Theorem 2.2**  \( H \) is a proper normal subgroup of \( \text{QI}(\mathbb{R}_+) \). In particular, \( \text{QI}(\mathbb{R}_+) \) is not simple.

**Proof** For any \([f], [g] \in H\),

\[
\frac{f(g(x)) - x}{x} = \frac{f(g(x)) - g(x)}{g(x)} \frac{g(x)}{x} + \frac{g(x) - x}{x}.
\]

Since \( g \) is a quasi-isometry, we know that \((1/K)x - C \leq g(x) - g(0) \leq Kx + C\). Therefore, \(1/K - 1 \leq g(x)/x \leq K + 1\) for sufficiently large \( x \). When \( x \to \infty \), we have \( g(x) \to \infty \). This means \((f(g(x)) - g(x))/g(x) \to 0\). Therefore, \((f(g(x)) - x)/x \to 0\) as \( x \to \infty \). This proves that \([fg] \in H\).

Note that

\[
\left| \frac{f^{-1}(x) - x}{x} \right| = \left| \frac{f^{-1}(x) - f^{-1}(f(x))}{x} \right| \leq K \left| x - f(x) \right| + C.
\]

Therefore,

\[
\lim_{x \to \infty} \frac{\left| f^{-1}(x) - x \right|}{x} = 0.
\]

This means \([f^{-1}] \in H\) and that \( H \) is a subgroup.

For any quasi-isometric homeomorphism \( g \in \text{Homeo}(\mathbb{R}_+) \) and any \([f] \in H\),

\[
\frac{g^{-1}(f(g(x))) - x}{x} = \frac{g^{-1}(f(g(x))) - g^{-1}(g(x))}{x} = \frac{g^{-1}(f(g(x))) - g^{-1}(g(x))}{g(x)} \frac{g(x)}{x}.
\]

Note that when \( x \to \infty \), the function \( g(x)/x \) is bounded. Let \( y = g(x) \). We have

\[
\left| \frac{g^{-1}(f(y)) - g^{-1}(y)}{y} \right| \leq K \left| f(y) - y \right| + C \to 0, \quad x \to \infty.
\]

Therefore, \([g^{-1} fg] \in H\).
It’s obvious that the function \( f \) defined by \( f(x) = 2x \) is not an element in \( H \). The function defined by \( g(x) = x + \ln(x + 1) \) gives a nontrivial element in \( H \). Thus \( H \) is a proper normal subgroup of \( \text{QI}(\mathbb{R}_+) \).

**Lemma 2.3** Let

\[
W(\mathbb{R}) = \{ f \in \text{Diff}(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |f(x) - x| < \infty, \sup_{x \in \mathbb{R}} |f'(x)| < \infty \}
\]

be the group consisting of diffeomorphisms with bounded derivatives and of bounded distance from the identity. Define a homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) by \( h(x) = e^x \) when \( x \geq 1 \), \( h(x) = -h(-x) \) when \( x \leq -1 \), and \( h(x) = e^x \) when \(-1 \leq x \leq 1 \). Then \( hf h^{-1} \) is a quasi-isometry for any \( f \in W(\mathbb{R}) \).

**Proof** For any \( f \in W(\mathbb{R}) \) and sufficiently large \( x > 0 \), its derivative satisfies that

\[
|hfh^{-1}(x)'| = |(e^{f(x)} - f(x))'|
= |(xe^{f(x)} - f(x))'|
= |e^{f(x)}(1 + f'(x)) - f'(x)|
\leq e^{\sup_{x \in \mathbb{R}} |f(x) - x|} \cdot \sup_{x \in \mathbb{R}} |f'(x)|.
\]

The case for negative \( x < 0 \) can be calculated similarly. This proves that \( hf h^{-1} \) is a quasi-isometry.

The following result was proved by Sankaran [9].

**Corollary 2.4** The quasi-isometry group \( \text{QI}(\mathbb{R}) \) contains \( \text{Diff}_Z(\mathbb{R}) \) (the lift of \( \text{Diff}(S^1) \) to \( \text{Homeo}(\mathbb{R}) \)).

**Proof** For any \( f \in \text{Diff}_Z(\mathbb{R}) \), we have \( f(x + 1) = f(x) + 1 \) for any \( x \in \mathbb{R} \). This means \( \sup_{x \in \mathbb{R}} |f(x) - x| < +\infty \). Since \( f(x) - x \) is periodic, we know that \( f'(x) \) is bounded. Suppose that \( f(x) > x \) for some \( x \in [0, 1] \). Take \( y_n = e^{x+n} \) for \( n > 0 \). Let \( h \) be the function defined in **Lemma 2.3**. We have

\[
|hfh^{-1}(y_n) - y_n| = |e^{f(x+n)} - e^{x+n}| = |e^{f(x)} - e^x|e^n \to \infty,
\]

which means \( [hf h^{-1}] \neq [\text{id}] \in \text{QI}(\mathbb{R}) \).

**Lemma 2.5** \( \text{QI}(\mathbb{R}) \) contains the semidirect product \( \text{Diff}_Z(\mathbb{R}) \ltimes H \).
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**Proof** Since \( H \) is normal, it’s enough to prove that \( \text{Diff}_Z(\mathbb{R}) \cap H = \{e\} \), the trivial subgroup. Actually, for any \( f \in \text{Diff}_Z(\mathbb{R}) \), the conjugate \( hf h^{-1} \) is a quasi-isometry as in the proof of Corollary 2.4. If \( hf h^{-1} \in H \), then

\[
\lim_{x \to \infty} \frac{h f h^{-1}(x)}{x} = \lim_{x \to \infty} \frac{e f (\ln x) - \ln x}{x} = \lim_{x \to \infty} e f (\ln x) - \ln x = 1.
\]

Since \( f(x) - x \) is periodic, we know that \( f(\ln x) = \ln x \) for any sufficiently large \( x \). But this means that \( f(y) = y \) for any \( y \), so \( f \) is the identity. \( \square \)

### 2.2 Affine subgroups of \( \text{QI}(\mathbb{R}) \)

**Lemma 2.6** The quasi-isometry group \( \text{QI}(\mathbb{R}_+) \) (actually, the semidirect product \( \text{Diff}_Z(\mathbb{R}) \ltimes H \)) contains the semidirect product \( \mathbb{R}_{>0} \ltimes (\bigoplus_{i \in \mathbb{R}_{\geq 1}} \mathbb{R}) \), generated by \( A_t \) and \( B_{i,s} \) for \( t \in \mathbb{R}_{>0}, i \in \mathbb{R}_{\geq 1} = [1, \infty) \) and \( s \in \mathbb{R} \) satisfying

\[
A_t B_{i,s} A_t^{-1} = B_{i,st \frac{i}{t+1}}, \quad B_{i,s_1} B_{i,s_2} = B_{i,s_1+s_2},
\]

\[
A_{t_1} A_{t_2} = A_{t_1 t_2}, \quad B_{i,s_1} B_{j,s_2} = B_{j,s_2 B_{i,s_1}},
\]

for any \( t_1, t_2 \in \mathbb{R}_{>0}, i, j \in \mathbb{R}_{\geq 1} \) and \( s_1, s_2 \in \mathbb{R} \).

**Proof** Let

\[
A_t(x) = tx, \quad t \in \mathbb{R}_{>0},
\]

\[
B_{i,s}(x) = x + sx \frac{i}{i+1}, \quad s \in \mathbb{R},
\]

for \( x \geq 0 \). We define \( A_t(x) = B_{i,s}(x) = x \) for \( x \leq 0 \). Since the derivatives

\[
A'_t(x) = t, \quad B'_{i,s}(x) = 1 + \frac{s}{i + 1} x \frac{i}{i+1}
\]

are bounded for sufficiently large \( x \), we know that \( A_t \) and \( B_{i,s} \) are quasi-isometries. For any \( x \geq 1 \),

\[
A_t B_{i,s} A_t^{-1}(x) = A_t B_{i,s} \left( \frac{x}{t} \right) = A_t \left( \frac{x}{t} + s \left( \frac{x}{t} \right) \frac{i}{i+1} \right) = x + st \frac{i}{i+1} x \frac{i}{i+1} = B_{i,st \frac{i}{i+1}}(x).
\]

For any \( x \geq 1 \),

\[
B_{i,s_1} B_{i,s_2}(x) = B_{i,s_1} (x + s_2 x \frac{i}{i+1}) = x + s_2 x \frac{i}{i+1} + s_1 (x + s_2 x \frac{i}{i+1}) \frac{i}{i+1}
\]

and

\[
|B_{i,s_1} B_{s_2}(x) - B_{i,s_1+s_2}(x)| = |s_1 ((x + s_2 x \frac{i}{i+1}) \frac{i}{i+1} - x \frac{i}{i+1})| \leq \left| s_1 \frac{s_2 x \frac{i}{i+1}}{x \frac{i}{i+1}} \right| \leq |s_1 s_2|
\]

by Newton’s binomial theorem. This means that \( B_{i,s_1} B_{i,s_2} \) and \( B_{i,s_1+s_2} \) are of bounded distance. It is obvious that \( A_{t_1} A_{t_2} = A_{t_1 t_2} \).
When $i < j$ are distinct natural numbers,

$$|B_{i,s} B_{j,s_2}(x) - B_{j,s_2} B_{i,s_1}(x)|$$

$$= |x + s_2 x^\frac{1}{i+1} + s_1 (x + s_2 x^\frac{1}{i+1})^\frac{1}{i+1} - (x + s_1 x^\frac{1}{i+1} + s_2 (x + s_1 x^\frac{1}{i+1})^\frac{1}{i+1})|$$

$$= |s_1 ((x + s_2 x^\frac{1}{i+1})^\frac{1}{i+1} - x^\frac{1}{i+1}) + s_2 (x^\frac{1}{i+1} - (x + s_1 x^\frac{1}{i+1})^\frac{1}{i+1})|$$

$$\leq \left| \frac{s_2 x^\frac{1}{i+1}}{x^\frac{1}{i+1}} \right| + \left| \frac{s_1 x^\frac{1}{i+1}}{x^\frac{1}{i+1}} \right|$$

$$\leq 2|s_1 s_2|$$

for any $x \geq 1$. This proves that images $[A_t], [B_{i,s}] \in \mathbb{QI}(\mathbb{R}\geq 0)$ satisfy the relations. By abuse of notation, we still denote the classes by the same letters.

We prove that the subgroup generated by $\{B_{i,s} \mid i \in \mathbb{R}\geq 1, s \in \mathbb{R}\}$ is the infinite direct sum $\bigoplus_{i \in \mathbb{R}\geq 1} \mathbb{R}$. It’s enough to prove that $B_{i_1,s_1}, B_{i_2,s_2}, \ldots, B_{i_k,s_k}$ are $\mathbb{Z}$–linearly independent for distinct $i_1, i_2, \ldots, i_k$ and nonzero $s_1, s_2, \ldots, s_k \in \mathbb{R}$. This can directly checked. For integers $n_1, n_2, \ldots, n_k$, suppose that $B_{i_1,s_1}^{n_1} \circ B_{i_2,s_2}^{n_2} \circ \cdots \circ B_{i_k,s_k}^{n_k} = \text{id} \in \mathbb{QI}(\mathbb{R}\geq 0)$. We have

$$\sup_{x \in \mathbb{R}\geq 0} |B_{i_1,s_1}^{n_1} \circ B_{i_2,s_2}^{n_2} \circ \cdots \circ B_{i_k,s_k}^{n_k}(x) - x|$$

$$= \sup_{x \in \mathbb{R}\geq 0} |n_1 s_1 x \frac{1}{i_1+1} + n_2 s_2 x \frac{1}{i_2+1} + \cdots + n_k s_k x \frac{1}{i_k+1} + n_{k-1} s_{k-1} (x + n_k s_k x \frac{1}{i_k+1})^\frac{1}{i_k+1} + \cdots + n_1 s_1 (x + \cdots)^\frac{1}{i_1+1}|$$

$$< +\infty,$$

which implies $n_1 = n_2 = \cdots = n_k = 0$, considering the exponents.

The subgroup $\mathbb{R}\geq 0 \ltimes (\bigoplus_{i \in \mathbb{R}\geq 1} \mathbb{R})$ lies in $\text{Diff}_\mathbb{Z}(\mathbb{R}) \ltimes H$ by the following construction. Let $a_t, b_{i,s} : \mathbb{R} \to \mathbb{R}$ be defined by $a_t(x) = x + \ln t$ and $b_{i,s}(x) = \ln(e^x + se^{\frac{x}{i+1}})$ for $t \in \mathbb{R}\geq 0, i \in \mathbb{R}\geq 1$ and $s \in \mathbb{R}$. It can be directly checked that $a_t \in \text{Diff}_\mathbb{Z}(\mathbb{R})$ and $b_{i,s} \in W(\mathbb{R})$ (defined in Lemma 2.3). Let $h(x) = e^x$. A direct calculation shows that $ha_t h^{-1} = A_t$ and $hb_{i,s} h^{-1} = B_{i,s}$, as elements in $\mathbb{QI}(\mathbb{R}_+)$. \[\square\]

### 3 Left-orderability

The following is well known; for a proof, see [7, Proposition 1.4]:

**Lemma 3.1** A group $G$ is left-orderable if and only if, for every finite collection of nontrivial elements $g_1, \ldots, g_k$, there exist choices $\varepsilon_i \in \{1, -1\}$ such that the identity is not an element of the semigroup generated by $\{g_i^{\varepsilon_i} \mid i = 1, 2, \ldots, k\}$. 

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The proof of Theorem 1.2 follows a similar strategy used by Navas to prove the left-orderability of the group $G_\infty$ of germs at $\infty$ of homeomorphisms of $\mathbb{R}$; cf [2, Remark 1.1.13] or [4, Proposition 2.2].

**Proof of Theorem 1.2** It’s enough to prove that $\text{QI}(\mathbb{R}_+)$ is left-orderable. Let $f_1, f_2, \ldots, f_n \in \text{QI}(\mathbb{R}_+)$ be any finitely many nontrivial elements. Note that any $1 \neq [f] \in \text{QI}(\mathbb{R}_+)$ has $\sup_{x>0} |f(x) - x| = \infty$. This property doesn’t depend on the choice of $f \in [f]$. Without confusion, we still denote $[f]$ by $f$. Choose a sequence $\{x_{1,k}\} \subset \mathbb{R}_+$ such that $\sup_{k \in \mathbb{N}} |f_1(x_{1,k}) - x_{1,k}| = \infty$. For each $i > 1$, we have either $\sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| = \infty$ or $\sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \leq M$ for a real number $M$. After passing to subsequences, we assume for each $i = 1, 2, \ldots, n$ that either $f_i(x_{1,k}) - x_{1,k} \to +\infty$, $f_i(x_{1,k}) - x_{1,k} \to -\infty$ or $\sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \leq M$.

We assign $\varepsilon_i = 1$ for the first case and $\varepsilon_i = -1$ for the second case. For the third case, let

$$S_1 = \{f_i \mid \sup_{k \in \mathbb{N}} |f_i(x_{1,k}) - x_{1,k}| \leq M\}.$$  

Note that $f_1 \notin S_1$. Choose $f_{i_0} \in S_1$ if $S_1$ is not empty. We choose another sequence $\{x_{2,k}\}$ such that $\sup_{k \in \mathbb{N}} |f_{i_0}(x_{2,k}) - x_{2,k}| = \infty$. Similarly, after passing to a subsequence, we have for each $f \in S_1$ that either $f(x_{2,k}) - x_{2,k} \to +\infty$, $f(x_{2,k}) - x_{2,k} \to -\infty$ or $\sup_{k \in \mathbb{N}} |f(x_{2,k}) - x_{2,k}| \leq M'$ for another real number $M'$. Assign $\varepsilon_i = 1$ for the first case and $\varepsilon_i = -1$ for the second case. Continue this process to define $S_2, S_3, \ldots$ and choose sequences $\{x_{i,k}\}, i = 3, 4, \ldots$ to assign $\varepsilon_i$ for each $f_i$.

Note that the process will stop at $n$ times, as the number of elements without assignment is strictly decreasing.

For an element $f \in \text{QI}(\mathbb{R}_+)$ satisfying $f(x_i) - x_i \to \infty$ as $i \to \infty$ for some sequence $\{x_i\}$, we assume that $f(x_i) - x_i > 0$ for each $i$. Since $f$ and $f^{-1}$ are orientation-preserving,

$$f^{-1}(x_i) - x_i = -(x_i - f^{-1}(x_i))$$

$$= -(f^{-1}(f(x_i)) - f^{-1}(x_i)) \leq -\left(\frac{1}{K}(f(x_i) - x_i) - C\right) \to -\infty.$$

Let $w = f_{i_1}^{e_{i_1}} \cdots f_{i_m}^{e_{i_m}} \in \langle f_1, f_2, \ldots, f_n \rangle$ be a nontrivial word. If $\{i_1, \ldots, i_m\} \not\subseteq S_1$, we have $w(x_{1,k}) - x_{1,k} \to \infty$. Otherwise, $\sup_{k \in \mathbb{N}} |w(x_{1,k}) - x_{1,k}| < \infty$. Suppose that $\{i_1, \ldots, i_m\} \subset S_t$, but $\{i_1, \ldots, i_m\} \not\subseteq S_{t+1}$ with the assumption that $S_0 = \{f_1, f_2, \ldots, f_n\}$. We have $w(x_{t+1,k}) - x_{t+1,k} \to \infty$ as $k \to \infty$. This proves that $w \neq 1 \in \text{QI}(\mathbb{R}_+)$. Therefore, $\text{QI}(\mathbb{R}_+)$ is left-orderable by Lemma 3.1.

□

**Lemma 3.2** The group $\text{QI}(\mathbb{R}_+)$ is not locally indicable.
The affine group $\mathbb{R}$ which is a contradiction to the fact that $F$ (Corollary 2.4). But this lift $\Gamma$ contains a subgroup $\Gamma = (f, g, h : f^2 = g^3 = h^7 = fg)$, the lift of the $(2, 3, 7)$–triangle group. There are no nontrivial maps from $\Gamma$ to $(\mathbb{R}, +)$; for more details see [2, page 94].

4 The quasi-isometric group cannot act effectively on the line

The following was proved by Mann [4, Proposition 6].

**Lemma 4.1** Consider the affine group $\mathbb{R}_> \ltimes \mathbb{R}$, generated by $A_t$ and $B_s$ for $t \in \mathbb{R}_>$ and $s \in \mathbb{R}$ satisfying

$$A_t B_s A_t^{-1} = B_{ts}, \quad B_{s_1} B_{s_2} = B_{s_1 + s_2}, \quad A_{t_1} A_{t_2} = A_{t_1 t_2}. $$

The affine group $\mathbb{R}_> \ltimes \mathbb{R}$ cannot act effectively on the real line $\mathbb{R}$ by homeomorphisms with $A_t$ a translation for each $t$.

**Proof** Suppose that $\mathbb{R}_> \ltimes \mathbb{R}$ acts effectively on the real line $\mathbb{R}$ with each $A_t$ a translation. After passing to an index-2 subgroup, we assume that the group is orientation-preserving. If $B_1$ acts freely on $\mathbb{R}$, then it is conjugate to the translation $T : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x + 1$. In such a case, we have $A_2 TA_2^{-1} = T^2$. Therefore, $A_2^{-1}(x + 2) = A_2^{-1}(x) + 1$ for any $x$. Since $A_2^{-1}$ maps intervals of length 2 to an interval of length 1, it is a contracting map, and thus has a fixed point.

If $B_1$ has a nonempty fixed point set $\text{Fix}(B_1)$, choose $I$ to be a connected component of $\mathbb{R} \setminus \text{Fix}(B_1)$. Suppose that $A_2(x) = x + a$, a translation by some real number $a > 0$. Since $A_2 = A_2^n 1/n$, we have $A_2^{1/n}(x) = x + a/n$ for each positive integer $n$. For each $n$, let $F_n = A_2^{1/n} B_1 A_2^{-1/n}$. Since $A_2^{1/n} B_1 A_2^{-1/n}$ commutes with $B_1$, we see that $F_n \text{Fix}(B_1) = \text{Fix}(B_1)$. This means that either $F_n(I) = I$ or $F_n(I) \cap I = \emptyset$. Since $F_n(x) = B_1(x - a/n) + a/n$ for any $x \in \mathbb{R}$, we know that $F_n(I) = I$ for sufficiently large $n$. Without loss of generality, we assume that $I$ is of the form $(x, y)$ or $(-\infty, y)$. Choose a sufficiently large $n$ such that $y - a/n \in I$. We have

$$A_2^{1/n} B_1 A_2^{-1/n}(y) = B_1(y - \frac{a}{n}) + \frac{a}{n} \neq y,$$

which is a contradiction to the fact that $F_n(I) = I$.

**Definition 4.2** A topologically diagonal embedding of a group $G < \text{Homeo}(\mathbb{R})$ is a homomorphism $\phi : G \to \text{Homeo}_+ (\mathbb{R})$ defined as follows. Choose a collection of disjoint open intervals $I_n \subset \mathbb{R}$ and homeomorphisms $f_n : \mathbb{R} \to I_n$. Define $\phi$ by $\phi(g)(x) = f_n g f_n^{-1}(x)$ when $x \in I_n$ and $\phi(g)(x) = x$ when $x \notin I_n$. 

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The following is similar to a result proved by Militon [6].

**Lemma 4.3** (Militon [6]) Let $\Gamma = \text{PSL}_2(\mathbb{R})$ and $\tilde{\Gamma} < \text{Homeo}_+(\mathbb{R})$ be the lift of $\Gamma$ to the real line. Any effective action $\phi: \tilde{\Gamma} \hookrightarrow \text{Homeo}_+(\mathbb{R})$ of $\tilde{\Gamma}$ on the real line $\mathbb{R}$ is a topological diagonal embedding.

**Proof** After passing to an index-2 subgroup, we assume the action is orientation-preserving. Let $\tau: \mathbb{R} \to \mathbb{R}$ be the translation $x \mapsto x + 1$. Suppose that $\text{Fix}(\phi(\tau)) \neq \emptyset$. Note that $\tau$ lies in the center of $\tilde{\Gamma}$. The quotient group $\Gamma = \tilde{\Gamma}/\langle \tau \rangle$ acts on the fixed point set $\text{Fix}(\phi(\tau))$. For any $f \in \Gamma$ and $x \in \text{Fix}(\phi(\tau))$, we denote the action by $f(x)$ without confusion. Choose any torsion-element $f \in \Gamma$ and any $x \in \text{Fix}(\phi(\tau))$. We must have $x = f(x)$, for otherwise $x < f(x) < f^2(x) < \cdots < f^k(x)$ for any $k$. Since $\Gamma$ is simple, we know that the action of $\tilde{\Gamma}$ on $\text{Fix}(\tau)$ is trivial. For each connected component $I_i \subset \mathbb{R} \setminus \text{Fix}(\phi(\tau))$, we know that $\tau|_{I_i}$ is conjugate to a translation. The group $\Gamma = \tilde{\Gamma}/\langle \tau \rangle$ acts on $I_i/\langle \phi(\tau) \rangle = S^1$. A result of Matsumoto [5, Theorem 5.2] says that the group $\tilde{\Gamma}$ is conjugate to the natural inclusion $\text{PSL}_2(\mathbb{R}) \hookrightarrow \text{Homeo}_+(S^1)$ by a homeomorphism $g \in \text{Homeo}_+(S^1)$. Therefore, the group $\phi(\tilde{\Gamma})|_{I_i}$ is conjugate to the image of the natural inclusion $\tilde{\Gamma} \hookrightarrow \text{Homeo}_+(\mathbb{R})$. □

For a real number $a \in \mathbb{R}$, let
\[
\tau_a: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + a
\]
be the translation. Denote by $A = \langle \tau_a : a \in \mathbb{R} \rangle$, the subgroup of translations in the lift $\tilde{\Gamma}$ of $\text{PSL}_2(\mathbb{R})$.

**Corollary 4.4** For any injective group homomorphism $\phi: \tilde{\Gamma} \to \text{Homeo}(\mathbb{R})$, the image $\phi(A)$ is a continuous one-parameter subgroup; i.e. $\lim_{a \to a_0} \phi(t_a) = \phi(t_{a_0})$ for any $a_0 \in \mathbb{R}$.

**Proof** If $\phi$ is injective, the previous lemma says that $\phi$ is a topological diagonal embedding. Therefore, $\phi(A)$ is continuous. □

We will need the following elementary fact.

**Lemma 4.5** Let $\phi: (\mathbb{R}, +) \to (\mathbb{R}, +)$ be a group homomorphism. If $\phi$ is continuous at any $x \neq 0$, then $\phi$ is $\mathbb{R}$–linear.

**Proof** For any nonzero integer $n$, we have $\phi(n) = n\phi(1)$ and $\phi(1) = \phi(\frac{1}{n}n) = n\phi(\frac{1}{n})$. Since $\phi$ is additive, we have $\phi\left(\frac{m}{n}\right) = m\phi\left(\frac{1}{n}\right) = \frac{m}{n}\phi(1)$ for any integers $m, n \neq 0$. □
For any nonzero real number $a \in \mathbb{R}$, choose a rational sequence $r_i \to a$. When $\phi$ is continuous, we have that $\phi(r_i) \to \phi(a)$ and $\phi(r_i) = r_i\phi(1) \to a\phi(1) = \phi(a)$. □

The following is the classical theorem of Hölder: a group acting freely on $\mathbb{R}$ is semi-conjugate to a group of translations; see Navas [8, Section 2.2.4].

**Lemma 4.6** Let $\Gamma$ be a group acting freely on the real line $\mathbb{R}$. There is an injective group homomorphism $\phi: \Gamma \to (\mathbb{R}, +)$ and a continuous nondecreasing map $\varphi: \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(h(x)) = \varphi(x) + \phi(h)$$

for any $x \in \mathbb{R}$ and $h \in \Gamma$.

**Corollary 4.7** Suppose that the affine group $\mathbb{R}_{>0} \ltimes \mathbb{R} = \langle a_t : t \in \mathbb{R}_{>0} \rangle \ltimes \langle b_s : s \in \mathbb{R} \rangle$ acts on the real line $\mathbb{R}$ by homeomorphisms satisfying

1. the action of the subgroup $\mathbb{R} = \langle b_s : s \in \mathbb{R} \rangle$ is free;
2. for any fixed $x \in \mathbb{R}$, $a_t(x)$ is continuous with respect to $t \in \mathbb{R}_{>0}$.

Let $\phi: \langle b_s : s \in \mathbb{R} \rangle \to (\mathbb{R}, +)$ be the additive map in Lemma 4.6 for $\Gamma = \langle b_s : s \in \mathbb{R} \rangle$. Then $\phi$ is an $\mathbb{R}$–linear map.

**Proof** Note that $a_t b_s a_t^{-1} = b_{ts}$. We have

$$\varphi(b_{ts}(x)) = \varphi(x) + \phi(b_{ts}).$$

Since $b_{ts}(x) = a_t b_s a_t^{-1}(x) \to b_s(x)$ when $t \to 1$, we have that

$$\varphi(x) + \phi(b_{ts}) \to \varphi(b_s(x)) = \varphi(x) + \phi(b_s).$$

This implies that $\phi(b_{ts}) \to \phi(b_s)$ as $t \to 1$. For any nonzero $x \in \mathbb{R}$ and sequence $x_n \to x$,

$$\phi(b_{x_n}) = \phi(b_{\frac{x_n}{x}x}) \to \phi(b_x).$$

The map $\phi$ is $\mathbb{R}$–linear by Lemma 4.5. □

**Theorem 4.8** Consider $G = \mathbb{R}_{>0} \ltimes \bigoplus_{i \in \mathbb{R}_{\geq 1}} \mathbb{R}$, generated by $A_t$ and $B_{i,s}$ for $t \in \mathbb{R}_{>0}$, $i \in \mathbb{R}_{\geq 1} = [1, \infty)$ and $s \in \mathbb{R}$ satisfying

$$A_t B_{i,s} A_t^{-1} = B_{i,s+(i+1)t}, \quad B_{i,s_1} B_{i,s_2} = B_{i,s_1+s_2},$$

$$A_{t_1} A_{t_2} = A_{t_1+t_2}, \quad B_{i,s_1} B_{j,s_2} = B_{i+s_1+s_2} B_{j,s_2}$$

for any $t_1, t_2 \in \mathbb{R}_{>0}$, $i, j \in \mathbb{R}_{\geq 1}$ and $s_1, s_2 \in \mathbb{R}$. Then $G$ cannot act effectively on the real line $\mathbb{R}$ by homeomorphisms when the induced action of $\langle A_t : t \in \mathbb{R}_{>0} \rangle$ is a topologically diagonal embedding of the translation subgroup $(\mathbb{R}, +) \hookrightarrow \text{Homeo}(\mathbb{R})$. 

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This implies that \( A \) is a diagonal embedding. For any \( 1 \leq i,j \leq n \), spanned by the \( i,j \)-components acts freely on \( A \). Since \( h \) means that the action of \( I \) is a diagonal embedding, we see that each \( I \) has no fixed point in \( I \). Since the group homomorphism

\[
\langle A_t : t \in \mathbb{R} > 0 \rangle \rightarrow \text{Homeo}(\mathbb{R})
\]

is a diagonal embedding, we see that each \( A_t \) has no fixed point in \( I \) and the action of \( \langle A_t : t \in \mathbb{R} > 0 \rangle \) on \( I \) is conjugate to a group of translations. By Lemma 4.1, the affine group \( \langle A_t, B_{1,s} : t \in \mathbb{R} > 0, s \in \mathbb{R} \rangle \) cannot act effectively on \( I \). Suppose that \( A_t B_{1,s} \) acts trivially on \( I \) for some \( t > 0 \) and \( s' > 0 \). We have that \( A_t B_{1,s} = A_{s^2s' - 2}(A_t B_{1,s'})A_{s^2s' - 2}^{-1} \) acts trivially on \( I \). But \( A_t B_{1,s}(x) = A_t(x) = x \) implies that \( t = 1 \). Therefore, the element \( B_{1,s} \) and any \( B_{1,t} = A_{s^2s' - 2}B_{1,s}A_{t^2s' - 2}^{-1} \) for \( t \in \mathbb{R} > 0 \) acts trivially on \( I \). This means that the action of \( \langle B_{1,s} : s \in \mathbb{R} \rangle \) on the connected component \( I \) is either trivial or free. Since the action of \( G \) is effective, there is a connected component \( I_1 \) on which \( B_{1,s} \) acts freely. A similar argument shows that \( B_{i,s'} \) acts freely on a component \( I_i \) for each \( i \in \mathbb{R} > 1 \) and any \( s' \in \mathbb{R} \setminus \{0\} \).

Since \( B_{i,s'} \) commutes with \( B_{j,s} \), we have \( B_{i,s'}(I_1) \subset \mathbb{R} \setminus \text{Fix}(\langle B_{j,s} : s \in \mathbb{R} \rangle) \). Moreover, \( B_{i,s'}(I_j) \cap I_j \) is either \( I_j \) or the empty set. Suppose that \( I_i \cap I_j \neq \emptyset \) and the right end \( b_i \) of \( I_i \) lies in \( I_j \). Choose \( x \in I_i \cap I_j \). Note that \( B_{j,s'}([x,b_j]) \cup [x,b_i] = \emptyset \) for any \( s > 0 \). This is impossible as \( B_{j,s/2}(x) \rightarrow x \) as \( n \rightarrow \infty \). Therefore, \( I_i \cap I_j = I_i \) or is empty for distinct \( i, j \in \mathbb{R} > 1 \). Since we have uncountably many \( i \in \mathbb{R} > 0 \), there must be some distinct \( i, j \in \mathbb{R} > 1 \) such that \( I_i = I_j \). This means that the subgroup \( \mathbb{R} \oplus \mathbb{R} \) spanned by the \( i,j \)-components acts freely on \( I_i \). Hölder’s theorem (Lemma 4.6) gives an injective group homomorphism \( \phi : \mathbb{R} \oplus \mathbb{R} \rightarrow (\mathbb{R},+) \) and a continuous nondecreasing map \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\varphi(h(x)) = \varphi(x) + \phi(h)
\]

for any \( x \in \mathbb{R} \). Since \( \langle A_t : t \in \mathbb{R} > 0 \rangle \rightarrow \text{Homeo}(\mathbb{R}) \) is a topological embedding, we have that for any fixed \( x \in \mathbb{R} \), \( A_t(x) \) is continuous with respect to \( t \in \mathbb{R} > 0 \). By Corollary 4.7,

\[\text{lim}_{n \rightarrow \infty} A^n x \in I.\]

But \( A_t \) (\( \text{lim}_{n \rightarrow \infty} A^n x \)) = \( \text{lim}_{n \rightarrow \infty} A^n x \) for any \( t > 0 \) by the topologically diagonal embedding. For any \( s' \), we have \( B_{1,s'} = A_{s^2s' - 2}B_{1,s}A_{s^2s' - 2}^{-1} \) and \( B_{1,s'} (\text{lim}_{n \rightarrow \infty} A^n x) = \text{lim}_{n \rightarrow \infty} A^n x \). This would imply that \( \text{lim}_{n \rightarrow \infty} A^n x \) is a global fixed point of \( \langle A_t, B_{1,s} : t \in \mathbb{R} > 0, s \in \mathbb{R} \rangle \).
the restriction map $\phi|_{\mathbb{R}}$ is $\mathbb{R}$–linear for each direct summand $\mathbb{R}$. This is a contradiction to the fact that $\phi$ is injective. Therefore, the group $G$ cannot act effectively. \qed

**Proof of Theorem 1.3** Suppose that $\text{QI}^+(\mathbb{R})$ acts on the real line by an injective group homomorphism $\phi: \text{QI}^+(\mathbb{R}) \to \text{Homeo}(\mathbb{R})$. The group $\text{QI}^+(\mathbb{R})$ contains the semidirect product $\mathbb{R}_0^+ \ltimes \bigoplus_{i \in \mathbb{R}_1^+} \mathbb{R}_i$, by Lemma 2.6. The subgroup $\mathbb{R}_0^+$ (as the image of the exponential map) is a homomorphic image of the subgroup $\mathbb{R} < \tilde{\Gamma}$, which is the lift of $\text{SO}(2)/\{\pm I_2\} < \text{PSL}_2(\mathbb{R})$ to $\text{Homeo}(\mathbb{R})$. Note that $\tilde{\Gamma}$ is embedded into $\text{QI}^+(\mathbb{R})$ (see Corollary 2.4 and its proof). By Lemma 4.3, any effective action of $\tilde{\Gamma}$ on the real line $\mathbb{R}$ is a topological diagonal embedding. This means that the action of $\mathbb{R}_0^+$ is a topological diagonal embedding (Corollary 4.4). Theorem 4.8 shows that the action of $\mathbb{R}_0^+ \ltimes \bigoplus_{i \in \mathbb{R}_1^+} \mathbb{R}_i$ is not effective. \qed

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