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Two-dimensional extended homotopy field theories

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We give another definition of 2–dimensional extended homotopy field theories (EHFTs) with aspherical targets and classify them. When the target of EHFT is chosen to be a $K(G,1)$–space, we classify EHFTs taking values in the symmetric monoidal bicategory of algebras, bimodules, and bimodule maps by certain Frobenius G–algebras called quasibiangular G–algebras. As an application, for any discrete group G, we verify a special case of the $(G \times \mathrm{SO}(2))$–structured cobordism hypothesis due to Lurie.

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1 Introduction

Extended topological field theories (ETFTs) are generalizations of topological field theories (usually called TQFTs or TFTs) to manifolds with corners and higher categories; see Freed [7], Lawrence [14] and J Baez and J Dolan [2]. A different generalization of TFTs is obtained by considering manifolds equipped with principal G–bundles. When G is a discrete group, such a generalization was introduced by Turaev [27], who called them homotopy (quantum) field theories (HFTs). These theories are defined by applying axioms of TFTs to manifolds and cobordisms endowed with maps to a fixed target space. In this paper, we combine ETFTs and HFTs in dimension 2. More precisely, we define 2–dimensional extended homotopy field theories (EHFTs) with aspherical targets and classify them.

1.1 Main results

To define a 2–dimensional EHFT with target $X \simeq K(G,1)$, we introduce an X–cobordism bicategory $X\mathrm{Bord}_2$. The objects of $X\mathrm{Bord}_2$ are compact oriented 0–dimensional manifolds and the 1–morphisms are oriented cobordisms between such manifolds equipped with homotopy classes of maps to X. The 2–morphisms of $X\mathrm{Bord}_2$ are equivalence classes of pairs (S, P), where S is a certain type of oriented surface with corners and P is a homotopy class of a map from S to X. The equivalence relation is given by diffeomorphisms respecting P and restricting to the identity on the boundary. The disjoint union operation turns $X\mathrm{Bord}_2$ into a symmetric monoidal bicategory and *a 2–dimensional EHFT with target X (extended X–HFT)* is defined as a symmetric monoidal 2–functor from $X\mathrm{Bord}_2$ to any other symmetric monoidal bicategory.

For a given symmetric monoidal bicategory $\mathcal{C}$, our classification of $\mathcal{C}$–valued 2–dimensional extended X–HFTs comprises two steps. Firstly, we define certain combinatorial diagrams in $I = [0,1]$, I^2, and I^3, called G–linear, G–planar, and G–spatial diagrams, respectively. These diagrams generalize the ones of Schommer-Pries [22] and they possess the same information as the morphisms of $X\mathrm{Bord}_2$. As the second step, we define a symmetric monoidal bicategory $\mathrm{XB}^{\mathrm{PD}}$ whose 1– and 2–morphisms are defined using these diagrams. This bicategory is equivalent to $X\mathrm{Bord}_2$ and has a convenient description in terms of generators and relations. Figures 25 and 26 show the corresponding list of generators and relations for $X\mathrm{Bord}_2$. Then, the classification of 2–dimensional extended X–HFTs reduces to an application of the cofibrancy theorem [22], which is a coherence theorem for symmetric monoidal 2–functors explained below.

For a computadic symmetric monoidal bicategory $\mathsf{F}(\mathbb{P})$ constructed from a list of generators and relations $\mathbb{P}$, let $\mathrm{SymMon}(\mathsf{F}(\mathbb{P}), \mathcal{C})$ denote the bicategory of symmetric monoidal 2–functors, transformations, and modifications. The cofibrancy theorem gives an equivalence $\mathrm{SymMon}(\mathsf{F}(\mathbb{P}), \mathcal{C}) \simeq \mathbb{P}(\mathcal{C})$ of bicategories, where $\mathbb{P}(\mathcal{C})$ is the bicategory, called *$\mathbb{P}$–data in $\mathcal{C}$*, whose objects are assignments of generators in $\mathbb{P}$ to the objects, 1–morphisms, and 2–morphisms of $\mathcal{C}$ subject to relations (see Section 4.2). Applying this theorem to the list of generators and relations $\mathbb{XP}$ of $\mathrm{XB}^{\mathrm{PD}}$, along with the equivalence $X\mathrm{Bord}_2 \simeq \mathrm{XB}^{\mathrm{PD}}$, gives the following classification theorem:

Theorem 4.10 *For any symmetric monoidal bicategory $\mathcal{C}$, there is an equivalence of bicategories*

$$\mathrm{SymMon}(X\mathrm{Bord}_2, \mathcal{C}) \simeq \mathbb{XP}(\mathcal{C}).$$

Next, we consider a specific target bicategory $\mathrm{Alg}^2_{\Bbbk}$ of $\Bbbk$–algebras, bimodules, and bimodule maps for a commutative ring $\Bbbk$ with unit. The following notions are the main ingredients of our result on $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended X–HFTs. For a discrete group G with identity element e, a strongly graded G–algebra is a G–graded associative $\Bbbk$–algebra $A = \bigoplus_{g \in G} A_g$ with unity such that $A_g A_{g'} = A_{gg'}$ for all $g, g' \in G$. The opposite G–algebra of A is $A^{\mathrm{op}} = \bigoplus_{g \in G} A_{g^{-1}}$, where the order of multiplication is reversed.

A Frobenius G–algebra is a pair (A, η), where $A = \bigoplus_{g \in G} A_g$ is a G–algebra such that each A_g is a finitely generated projective $\Bbbk$–module, and $\eta\colon A \otimes A \to \Bbbk$ is a nondegenerate bilinear form satisfying $\eta(ab, c) = \eta(a, bc)$ for any $a, b, c \in A$. A quasibiangular G–algebra is a strongly graded Frobenius G–algebra (A, η) in which the identity component A_e is separable and η satisfies certain conditions (see Section 4.3). We also need G–graded Morita contexts between G–algebras, which were introduced by Boisen [3]. We recall their definition and introduce a notion of compatibility with Frobenius structures in Section 4.3.

Theorem 4.19 *Let $\Bbbk$ be a commutative ring and X be a CW–complex which is a $K(G,1)$–space for a discrete group G. Then any $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended X–HFT $Z\colon X\mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}$ whose precomposition $\mathrm{XB}^{\mathrm{PD}} \xrightarrow{\simeq} X\mathrm{Bord}_2 \xrightarrow{Z} \mathrm{Alg}^2_{\Bbbk}$ gives a strict symmetric monoidal 2–functor determines a triple (A, B, ζ), where A and B are quasibiangular G–algebras and ζ is a compatible G–graded Morita context between A and B^{op}. Conversely, for any such triple (A, B, ζ) there exists an $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended X–HFT.*

This generalizes Schommer-Pries' classification of $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended TFTs [22] in terms of separable symmetric Frobenius algebras. Theorem 4.10 suggests studying the bicategory $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ to understand $\mathrm{SymMon}(X\mathrm{Bord}_2, \mathrm{Alg}^2_{\Bbbk})$, more specifically to answer the question of which triples yield equivalent extended X–HFTs. This study leads us to define a bicategory, Frob^G, which has quasibiangular G–algebras as objects, compatible G–graded Morita contexts as 1–morphisms, and equivalences of such Morita contexts as 2–morphisms (see Section 4.3).

Theorem 4.24 *Under the assumptions of Theorem 4.19, there is an equivalence of bicategories*

$$\mathrm{SymMon}(X\mathrm{Bord}_2, \mathrm{Alg}^2_{\Bbbk}) \simeq \mathrm{Frob}^G.$$

On the level of objects, this equivalence maps a triple (A, B, ζ) to A. Consequently, Theorem 4.24 implies the following corollary:

Corollary 4.25 *Under the assumptions of Theorem 4.19, two triples (A_1, B_1, ζ_1) and (A_2, B_2, ζ_2) produce equivalent 2–dimensional extended X–HFTs if and only if there exists a compatible G–graded Morita context between A_1 and A_2.*

A different approach to the classification of 2–dimensional EHFTs with $K(G,1)$ targets is given by the $(G\times\mathrm{SO}(2))$–structured cobordism hypothesis due to J Lurie [15]. This hypothesis states a classification of such EHFTs in terms of homotopy $(G\times\mathrm{SO}(2))$–fixed points (see Section 4.5). Davidovich [6] computed these fixed points in $\mathrm{Alg}^2_{\Bbbk}$ when $\Bbbk$ is an algebraically closed field of characteristic zero. By comparing Theorem 4.24 with Davidovich's results, we verify a special case of the $(G\times\mathrm{SO}(2))$–structured cobordism hypothesis as follows:

Corollary 4.27 *For any discrete group G and any algebraically closed field $\Bbbk$ of characteristic zero, the $(G\times\mathrm{SO}(2))$–structured cobordism hypothesis for $\mathrm{Alg}^2_{\Bbbk}$–valued oriented EHFTs with target $X \simeq K(G,1)$ holds true.*

In the definition of $X\mathrm{Bord}_2$, we use oriented manifolds. By using unoriented manifolds, we define the unoriented X–cobordism bicategory $X\mathrm{Bord}_2^{\mathrm{un}}$ and provide a list of generators and relations. Then, parallel to the oriented case, we classify 2–dimensional extended unoriented HFTs and verify a special case of the $(G\times O(2))$–structured cobordism hypothesis.

1.2 Related works

Our main reference is Schommer-Pries' thesis [22] on the classification of 2–dimensional extended TFTs. In addition to detailed classification of oriented and unoriented extended TFTs, Schommer-Pries sketched the classification of 2–dimensional structured extended TFTs (see [22, Section 3.5]). In this approach the structured cobordism bicategory is defined using topological stacks. In particular, a stack corresponding to principal G–bundles and orientation structures provides an alternative formulation for extended $(G\times\mathrm{SO}(2))$–structured TFTs, or equivalently 2–dimensional extended HFTs with $K(G,1)$ targets.

When defining 2–dimensional extended TFTs, Schommer-Pries [22] defined the cobordism bicategory in all dimensions, not only in dimension 2. Schweigert and Woike [23] extended this cobordism bicategory to the setting of homotopy field theories and defined extended HFTs in all dimensions. Similarly, Müller and Szabo [18] constructed a geometric cobordism bicategory $\mathrm{Cob}^{\mathcal{F}}_{n,n-1,n-2}$, where $\mathcal{F}$ is a general stack which encodes the arbitrary background fields in the corresponding quantum field theory (see also Müller [17]). In their work [19], when the background fields $\mathcal{F}$ are chosen for 2–dimensional Dijkgraaf–Witten theories, namely principal bundles with finite structure group G and orientations, the symmetric monoidal bicategory $\mathrm{Cob}^{\mathcal{F}}_{2,1,0}$ is equivalent to $X\mathrm{Bord}_2$, described above. Using the n–dimensional version $\mathrm{Cob}^{\mathcal{F}}_{n,n-1,n-2}$ of this bicategory, they defined an n–dimensional extended homotopy field theory for all $n \geq 2$. Moreover, Müller and Woike [20] constructed an n–dimensional extended HFT from a flat $(n-1)$–gerbe on a target space represented by a $U(1)$–valued singular cocycle (see also [17] and Bunke, Turner and Willerton [4]). This construction was generalized to unoriented extended HFTs by Young [29].

A state sum approach to both 2–dimensional extended TFTs and HFTs was taken by Davidovich [6]. As mentioned above, Davidovich [6] also classified $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended $(G\times\mathrm{SO}(2))$–structured TFTs following the cobordism hypothesis.

Conventions Throughout the paper, G is a discrete group with identity element e and the target space is a pointed aspherical CW–complex (X, x) with $\pi_1(X, x) = G$. All manifolds are assumed to be smooth and all algebras are unital. By a closed manifold we mean a compact manifold without boundary. For smooth manifolds M and N, the space of smooth maps $C^\infty(M, N)$ is provided with the Whitney C^∞–topology. For subsets $K \subset M$ and $L \subset N$, the notation $[(M, K), (N, L)]$ stands for the set of relative homotopy classes of maps between pairs.

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2 The 2–dimensional X–cobordism bicategory

In his study of HFTs, Turaev [27] introduced notions of X–manifold and X–cobordism using pointed manifolds, where X is a connected CW–complex with a specified point $x \in X$. In this paper, X is always a $K(G,1)$–space. In this case, a *pointed manifold* is a manifold with a basepoint on each connected component. We denote the set of basepoints of a pointed manifold M by $\mathtt{bp}_M$. An *n–dimensional X–manifold* is a pair $(M, \mathtt{g})$ consisting of a closed pointed n–manifold M and a homotopy class $\mathtt{g} \in [(M, \mathtt{bp}_M), (X, x)]$, called the *characteristic map*. An *X–cobordism* between X–manifolds $(M, \mathtt{g})$ and $(M', \mathtt{g}')$ is a pair $(W, \mathtt{P})$ consisting of a cobordism W between M and M' and a homotopy class $\mathtt{P} \in [(W, \mathtt{bp}_M \cup \mathtt{bp}_{M'}), (X, x)]$ restricting to $\mathtt{g}$ and $\mathtt{g}'$ on the corresponding boundary components.

Given an aspherical space X, a 2–dimensional extended X–HFT is a symmetric monoidal 2–functor from the 2–dimensional X–cobordism bicategory $X\mathrm{Bord}_2$ to another symmetric monoidal bicategory. Therefore, this bicategory plays a key role in the definition of 2–dimensional extended X–HFTs.

There are existing definitions of structured or equivariant cobordism bicategories related to $X\mathrm{Bord}_2$. These include the homotopy bicategory of the $(\infty,2)$–category of cobordisms with $(G\times\mathrm{SO}(2))$–structures (see [15; 5]), the structured cobordism bicategory $\mathrm{Bord}_2^{\mathcal{F}}$ introduced in [22] with a topological stack $\mathcal{F}$ corresponding to principal G–bundles and orientation structures, the G–equivariant cobordism bicategory G–$\mathrm{Cob}(2,1,0)$ introduced in [23], and the structured cobordism bicategory $\mathrm{Cob}_{2,1,0}^{\mathcal{F}}$ introduced in [18] with an appropriate choice of a stack $\mathcal{F}$ (see [19; 17]). It can be shown that these symmetric monoidal bicategories are equivalent to $X\mathrm{Bord}_2$ and hence the corresponding extended HFTs with aspherical targets are equivalent.

We start this section with a descriptive definition of $X\mathrm{Bord}_2$ to motivate the types of X–manifolds and structures on them, which form the 1– and 2–morphisms of this bicategory. Then we provide the complete definition of $X\mathrm{Bord}_2$.

2.1 Surfaces with corners

Roughly, the objects of $X\mathrm{Bord}_2$ are compact oriented 0–manifolds, 1–morphisms are 1–dimensional X–cobordisms, and 2–morphisms are X–homeomorphism classes of 2–dimensional X–cobordisms between those X–cobordisms. This hints that the underlying manifold of a 2–morphism must be a surface with corners. Recall that a surface with corners M is a 2–dimensional topological manifold whose coordinate charts are of the form $\varphi\colon U \to \mathbb{R}^2_+$, where $U \subset M$ is open and $\mathbb{R}^2_+ = [0,\infty) \times [0,\infty)$. Compatibility of charts is given by diffeomorphisms; that is, two charts (U,φ) and (U',φ') with $U \cap U' \neq \varnothing$ are compatible if the composition $\varphi' \circ \varphi^{-1}\colon \varphi(U \cap U') \to \varphi'(U \cap U')$ is a diffeomorphism. Here, when $\varphi(U \cap U') \cap \partial\mathbb{R}^2_+ \neq \varnothing$, the map $\varphi' \circ \varphi^{-1}$ is a diffeomorphism if it is a restriction of a diffeomorphism defined on an open set containing $\varphi(U \cap U')$.

The composition of 2–morphisms in the X–cobordism bicategory is given by gluing surfaces with corners along their common boundaries. Recall that the first step of gluing construction is to choose collar neighborhoods. Nevertheless, not every surface with corners admits collar neighborhoods. Following [22], we use $\langle 2\rangle$–surfaces, which are special cases of $\langle n\rangle$–manifolds defined in [13]. These surfaces admit collar neighborhoods (see [13]) and they are a certain type of surfaces with faces.

A surface with faces M is a surface with corners such that any point $m \in M$ belongs to $\mathrm{index}(m)$ different connected faces. Here, the index of a point m is the number of zeros in $\varphi(m) \in \mathbb{R}^2_+$, where (U,φ) is a chart with $m \in U$ and a connected face of a surface with corners M is the closure of a component of $\{m \in M \mid \mathrm{index}(m) = 1\}$. A face is a disjoint union of connected faces.

Definition 2.1 A $\langle 2\rangle$*–surface* is a 2–dimensional compact manifold with faces S equipped with two submanifolds with faces $\partial_h S$ and $\partial_v S$, called the *horizontal* and *vertical faces*, respectively, such that $\partial S = \partial_h S \cup \partial_v S$ and $\partial_h S \cap \partial_v S$ is either empty or a face of both. A $\langle 2\rangle$–surface S is *pointed* if it is equipped with a finite set $R \subset \partial S$ such that $\partial_h S \cap \partial_v S \subset R$, $\partial_v S \cap R = \partial_h S \cap \partial_v S$, and every connected component of $\partial_h S$ contains at least two elements of R.

Definition 2.2 A $\langle 2\rangle$*–X–surface* is a triple $(S, R, \mathtt{P})$, where (S, R) is a pointed oriented $\langle 2\rangle$–surface and $\mathtt{P} \in [(S,R),(X,x)]$ is a homotopy class. A $\langle 2\rangle$–X–surface $(S,R,\mathtt{P})$ is said to be of *cobordism type* if $\partial_v S$ is diffeomorphic to a product X–manifold with a constant characteristic map, ie $(\partial_v S, \mathtt{P}|_{\partial_v S}) \cong (M \times I, \mathtt{P}|_{M\times I})$, where $I = [0,1]$, $(M, \mathtt{P}|_M)$ is a 0–dimensional X–manifold, and the restriction of $\mathtt{P}|_{M\times I} \in [(M\times I, \partial(M\times I)),(X,x)]$ to each connected component is the constant homotopy class.

Figure 1 shows an example of a cobordism type $\langle 2\rangle$–X–surface $(S,R,\mathtt{P})$, where we encode the data of relative homotopy class $\mathtt{P}$ by arrows and G–labels are determined uniquely by $\mathtt{P}$ and arrows. Observe that the horizontal boundary of a $\langle 2\rangle$–X–surface $(S',R',\mathtt{P}')$ is not a 1–dimensional X–cobordism if $R' \neq \partial(\partial_h S')$. Since we regard

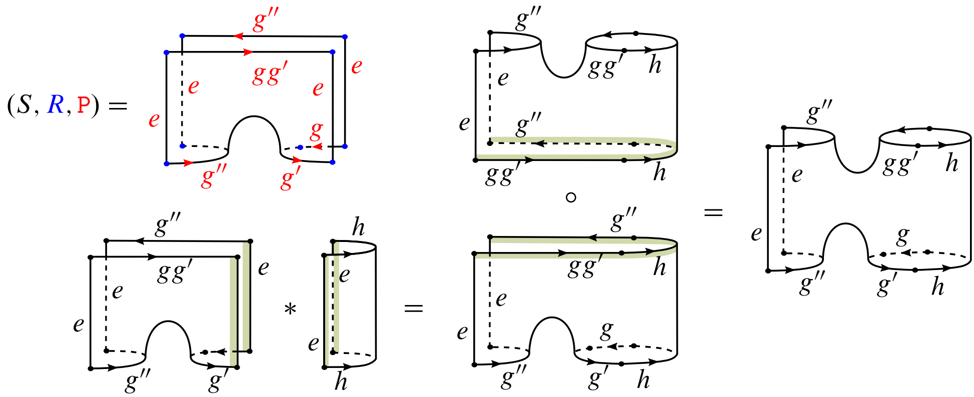


Figure 1: Examples of cobordism type $\langle 2\rangle$–X–surfaces and their compositions.

2–morphisms of $X\mathrm{Bord}_2$ to be X–cobordisms between 1–morphisms, this observation implies that 1–morphisms are more general than 1–dimensional X–cobordisms in that there are possibly extra points in the interior of the underlying 1–dimensional compact manifold.

Definition 2.3 A 1*–dimensional marked X–manifold* is a triple $(M, T, \mathtt{g})$, where M is an oriented compact 1–manifold, $T \subseteq M$ is a finite set with $\partial M \subset T$ such that each connected component of M contains at least two elements of T, and $\mathtt{g} \in [(M, T), (X, x)]$.

According to the arguments above, the bicategory $X\mathrm{Bord}_2$ is expected to have compact oriented 0–manifolds as objects, 1–dimensional marked X–manifolds as 1–morphisms, and cobordism type $\langle 2\rangle$–X–surfaces as 2–morphisms. In this case, for a given cobordism type $\langle 2\rangle$–X–surface, its source and target 1–morphisms are certain components of the horizontal boundary. However, the composition of morphisms is a delicate issue, especially the composition of 1–morphisms.

When we glue two manifolds along their common boundary, the smooth structure on the resulting topological manifold depends on the choice of collar neighborhoods (see [16]). Equivalently, different choices of collars give different smooth structures. However, different choices give diffeomorphic smooth manifolds and diffeomorphisms are noncanonical. Therefore, gluing operation on smooth manifolds is not well defined on the nose, but up to a noncanonical diffeomorphism.

The same results continue to hold for $\langle 2\rangle$–X–surfaces. Let $(S, R, \mathtt{P})$ be a $\langle 2\rangle$–X–surface and $(N, N \cap R, \mathtt{P}|_N)$ be a face with the inclusion map $\iota\colon (N, N \cap R) \hookrightarrow (S, R)$. For a collar neighborhood $U_N \subset S$ of N, a *collar* is a diffeomorphism $\Psi_N \colon U_N \to N \times \mathbb{R}_+$ (see Figure 1). The following proposition implies that 2–morphisms of $X\mathrm{Bord}_2$ must be (relative) diffeomorphism classes of cobordism type $\langle 2\rangle$–X–surfaces in order to have well-defined horizontal and vertical compositions of 2–morphisms:

Proposition 2.4 *Let $(S, R, \mathtt{P})$ and $(S', R', \mathtt{P}')$ be cobordism type $\langle 2\rangle$–X–surfaces with faces and $(N, T, \mathtt{g})$ be a 1–dimensional marked X–manifold together with inclusions $\iota\colon (N, T, \mathtt{g}) \hookrightarrow (S, R, \mathtt{P})$ and $\iota'\colon (N, T, \mathtt{g}) \hookrightarrow (S', R', \mathtt{P}')$ realizing $(N, T, \mathtt{g})$ as a face of both $\langle 2\rangle$–X–surfaces. Then $S \cup_N S'$ is a topological manifold with boundary. If in addition we are given collars $\Psi_+\colon N \times \mathbb{R}_+ \to S$ and $\Psi_-\colon N \times \mathbb{R}_+ \to S'$, then there exists a canonical smooth structure on $S \cup_N S'$ which is compatible with the smooth structures on S and S'. Moreover, different choices of collars produce noncanonically diffeomorphic cobordism type $\langle 2\rangle$–surfaces.*

The proof of this theorem follows from the proofs of Proposition 3.1 and Theorem 3.3 in [22]. Note that gluing $\langle 2\rangle$–X–surfaces vertically along their common horizontal boundary components does not yield a cobordism type $\langle 2\rangle$–X–surface. One needs to choose a diffeomorphism $I \cup I \cong I$ and omit the points on the faces through which $\langle 2\rangle$–X–surfaces are glued. Figure 1 shows examples of vertical and horizontal compositions of $\langle 2\rangle$–X–surfaces, denoted by $\circ$ and $*$, respectively. Following [22], we solve the problem of composition of 1–morphisms by equipping manifolds with germs of neighborhoods. The notion of a germ of neighborhoods was made precise by Schommer-Pries (see Section 3.2.3 in [22]) using halations which are formulated as maps of pro-manifolds.

2.2 Pro-X–manifolds and X–halations

Recall that a directed set is a tuple $(D, \leq)$, where D is a nonempty set and $\leq$ is a reflexive and transitive binary relation such that, for any $x, y \in D$, there exists $z \in D$ with $x \leq z$ and $y \leq z$. We think of a directed set $(D, \leq)$ as a category $\mathcal{D}$ whose objects are elements of D and morphisms are given by the relation $\leq$. Let Man^X be the category of smooth X–manifolds and smooth pointed maps commuting with characteristic maps. A *pro-X–manifold* is a pair $(\mathcal{D}, A)$, where $\mathcal{D}$ is a directed set and $A\colon \mathcal{D} \to \mathrm{Man}^X$ is a functor.

Two directed sets play an important role in describing germs of neighborhoods of X–manifolds: the first one is trivial one, $\mathcal{D}_\bullet = \{\bullet\}$, and the second one is associated to an embedding of X–manifolds as follows. Let $(M, \mathtt{g})$ and $(N, \mathtt{h})$ be X–manifolds possibly with boundary or corners, and let $\iota\colon (M,\mathtt{g}) \hookrightarrow (N,\mathtt{h})$ be an embedding of X–manifolds, ie $\iota(\mathtt{bp}_M) = \mathtt{bp}_N$ and $\mathtt{g} = \mathtt{h}\circ[\iota]$ as elements of $[(M,\mathtt{bp}_M),(X,x)]$. Then the directed set $\mathcal{D}_N$ is given by codimension zero closed X–submanifolds of N containing $\iota(M)$ and the relation is inclusion.

For a given X–manifold $(M,\mathtt{g})$, we denote the pro-X–manifolds corresponding to these directed sets with $(M,\mathtt{g})$ and $(\widehat{M}\subset N,\widehat{\mathtt{g}})$, respectively. There is an obvious inclusion $\iota_M\colon (M,\mathtt{g})\hookrightarrow(\widehat{M}\subset N,\widehat{\mathtt{g}})$ of pro-X–manifolds and an *X–halation* is an inclusion of pro-X–manifolds isomorphic to ι_M. Here morphisms of pro-X–manifolds are the morphisms in Man^X of the corresponding limits and colimits of the diagrams. More precisely, the set of morphisms between two pro-X–manifolds discussed above is

$$\mathrm{Hom}_{\text{pro-Man}^X}(\mathcal{D}_\bullet,\mathcal{D}_N) = \lim_p \operatorname*{colim}_q \mathrm{Hom}_{\mathrm{Man}^X}(\mathcal{D}_\bullet(p),\mathcal{D}_N(q)),$$

where the limit and colimit are taken in the category of sets. The codimension of an X–halation is the codimension of the embedding. We denote an X–halation by $(M,\widehat{M},\widehat{\mathtt{g}})$ and call an X–manifold equipped with an X–halation an *X–haloed manifold*. A map between X–haloed manifolds $(A,\widehat{A},\widehat{\mathtt{a}})$ and $(B,\widehat{B},\widehat{\mathtt{b}})$ is a pair of pro-X–manifold morphisms $A\to B$ and $\widehat{A}\to\widehat{B}$ such that the diagram involving the inclusions $A\hookrightarrow\widehat{A}$ and $B\hookrightarrow\widehat{B}$ commutes. The category of pro-objects in a category $\mathcal{C}$ is generally defined using cofiltered diagrams instead of directed sets. Here we use the results in [22] to simplify arguments and refer reader to [22, Sections 3.2.1 and 3.2.2] for a more detailed exposition on halations.

The solution to the problem of composition of 1–morphisms is to use compatible X–haloed manifolds. The compatibility is given by choosing an orientation for the normal bundle of the embedding which defines X–halation. Such an X–halation is called *cooriented*. Now assume that $(M_0,T_0,\mathtt{g}_0)$ and $(M_1,T_1,\mathtt{g}_1)$ are 1–dimensional marked X–manifolds equipped with cooriented codimension one X–halations $(\widehat{M}_0\subset N_0,\widehat{\mathtt{g}}_0)$ and $(\widehat{M}_1\subset N_1,\widehat{\mathtt{g}}_1)$, respectively. Using the tubular neighborhood theorem and the coorientations of the source and target objects, we write these X–halations as $(M_i\subset M_i\times\mathbb{R}\cup_{\partial M_i\times\mathbb{R}\times\{0\}}\partial M_i\times(\mathbb{R}\times\mathbb{R}_+))$ for $i=0,1$. Using [22, Lemma 3.25], we refine the index of both pro-X–manifolds to natural numbers $\mathbb{N}$ as

$$\begin{aligned}
&i\mapsto M_0\times\{0\}\cup_{\partial_t M_0}\partial_t M_0\times\{0\}\times\Big(-\frac{1}{i},0\Big]\cup_{\partial_s M_0}\partial_s M_0\times\{0\}\times\Big[0,\frac{1}{i}\Big),\\
&i\mapsto M_1\times\{0\}\cup_{\partial_s M_1}\partial_s M_1\times\{0\}\times\Big[0,\frac{1}{i}\Big)\cup_{\partial_t M_1}\partial_t M_1\cup\{0\}\times\Big(-\frac{1}{i},0\Big],\\
&i\mapsto Y\times\Big(-\frac{1}{i},\frac{1}{i}\Big),
\end{aligned}$$

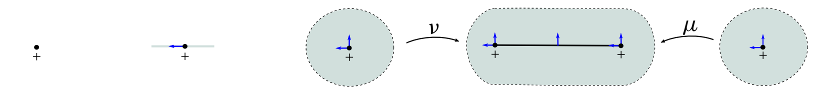


Figure 2: Cooriented X–halations and an X–haloed 1–cobordism.

where ∂_s and ∂_t denote the source and target boundary components and $Y \cong \partial_t M_0 \cong \partial_s M_1$. For each $i \in \mathbb{N}$, the pushout exists and it is the smooth manifold $M_1 \cup_Y M_0$ whose smooth structure is determined by the embedding of $Y \times (-1/i, 1/i)$ and by the smooth structures of M_0 and M_1. The pushout as a cooriented codimension one X–haloed manifold, ie as a 1–morphism of $X\mathrm{Bord}_2$, exists by the results in [11] on commutativity of finite colimits and (cofiltered) limits. Here note that the colimit is taken over the pushout diagram $K = \bullet \leftarrow \bullet \rightarrow \bullet$, which is clearly finite (see [22, Section 3.2.3] for details).

2.3 The X–cobordism bicategory $X\mathrm{Bord}_2$

We are now equipped with the necessary information to define the 2–dimensional X–cobordism bicategory.

Definition 2.5 *The* 2*–dimensional X–cobordism bicategory* $X\mathrm{Bord}_2$ is as follows:

(1) The objects are triples $\{((M, \mathtt{g}), (M, \widehat{M}_1, \hat{\mathtt{g}}_1), (M, \widehat{M}_2, \hat{\mathtt{g}}_2))\}$, where $(M, \mathtt{g})$ is a compact oriented 0–manifold, and $(M, \widehat{M}_1, \hat{\mathtt{g}}_1)$ and $(M, \widehat{M}_2, \hat{\mathtt{g}}_2)$ are cooriented codimension one and codimension two X–halations, respectively, with inclusions $(M, \mathtt{g}) \hookrightarrow (M, \widehat{M}_1, \hat{\mathtt{g}}_1) \hookrightarrow (M, \widehat{M}_2, \hat{\mathtt{g}}_2)$. For brevity we denote such an object as $(M, \widehat{M}_1, \widehat{M}_2, \hat{\mathtt{g}}_2)$.

(2) The 1–morphisms are X–haloed 1–dimensional X–cobordisms; an X–haloed 1–dimensional X–cobordism $(A, \widehat{A}_0, \widehat{A}_1, T, \hat{\mathtt{p}}_1)$ from $(M, \widehat{M}_1, \widehat{M}_2, \hat{\mathtt{g}}_2)$ to $(N, \widehat{N}_1, \widehat{N}_2, \hat{\mathtt{h}}_2)$ consists of

- a 1–dimensional marked X–manifold $(A, T, \mathtt{p})$,
- a codimension zero X–halation $(A, \widehat{A}_0, \hat{\mathtt{p}}_0)$ and a cooriented codimension one X–halation $(A, \widehat{A}_1, \hat{\mathtt{p}}_1)$ with inclusions $(A, \mathtt{p}) \hookrightarrow (A, \widehat{A}_0, \hat{\mathtt{p}}_0) \hookrightarrow (A, \widehat{A}_1, \hat{\mathtt{p}}_1)$,
- a decomposition of the boundary of $(A, T, \mathtt{p})$ as $\partial A = \partial_{\mathrm{in}} A \amalg \partial_{\mathrm{out}} A$ with isomorphisms of X–halations preserving coorientations (see Figure 2)

$$(M, \widehat{M}_1, \widehat{M}_2, \hat{\mathtt{g}}_2) \xrightarrow[\cong]{\mu} (\partial_{\mathrm{in}} A, \widehat{A}_0|_{\partial_{\mathrm{in}}}, \widehat{A}_1|_{\partial_{\mathrm{in}}}, \hat{\mathtt{p}}_1),$$
$$(N, \widehat{N}_1, \widehat{N}_2, \hat{\mathtt{h}}_2) \xrightarrow[\cong]{\nu} (\partial_{\mathrm{out}} A, \widehat{A}_0|_{\partial_{\mathrm{out}}}, \widehat{A}_1|_{\partial_{\mathrm{out}}}, \hat{\mathtt{p}}_1),$$

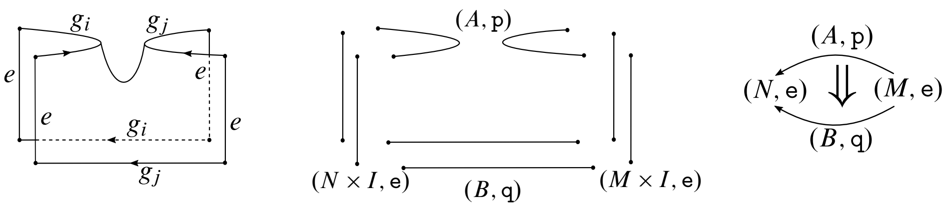


Figure 3: An example of decomposition of a 2–morphism in $X\mathrm{Bord}_2$.

where $\widehat{A}_0|_{\partial_{\mathrm{in}}}$ is cooriented by an inward pointing normal vector and $\widehat{A}_0|_{\partial_{\mathrm{out}}}$ is cooriented by an outward pointing normal vector.

(3) The 2–morphisms are isomorphism classes of X–haloed 2–dimensional X–cobordisms; an X–haloed 2–dimensional X–cobordism $(S, \widehat{S}, R, \widehat{\mathsf{F}})$ from $(A, \widehat{A}_0, \widehat{A}_1, T, \widehat{\mathsf{p}}_1)$ to $(B, \widehat{B}_0, \widehat{B}_1, Q, \widehat{\mathsf{q}}_1)$ consists of a cobordism type $\langle 2\rangle$–X–surface (S, R, F) together with a codimension zero X–halation $(S, \widehat{S}, \widehat{\mathsf{F}})$ and isomorphisms of X–halations (see Figure 3)

$$(A, \widehat{A}_1, \widehat{\mathsf{p}}_1) \amalg (B, \widehat{B}_1, \widehat{\mathsf{q}}_1) \xrightarrow[\cong]{\theta} (\partial_h S, \widehat{S}|_{\partial_h S}, \widehat{\mathsf{F}}|_{\partial_h S}),$$
$$(M \times I, \widehat{M \times I^2}, \widehat{\mathsf{e}}) \amalg (N \times I, \widehat{N \times I^2}, \widehat{\mathsf{e}}) \xrightarrow[\cong]{\eta} (\partial_v S, \widehat{S}|_{\partial_v S}, \widehat{\mathsf{F}}|_{\partial_v S}),$$

where $(A, \widehat{A}_1, \widehat{\mathsf{p}})$ is cooriented by an inward pointing normal vector and $(B, \widehat{B}_1, \widehat{\mathsf{q}})$ is cooriented by an outward pointing normal vector. The X–halations of $M \times I$ and $N \times I$ are induced by their embeddings into $M \times I^2$ and $N \times I^2$ with constant homotopy class $\widehat{\mathsf{e}}$. Coorientations are given by an inward pointing normal vector for $(M \times I, \widehat{M \times I^2}, \widehat{\mathsf{e}})$ and an outward pointing normal vector for $(N \times I, \widehat{N \times I^2}, \widehat{\mathsf{e}})$. For such an X–haloed 2–cobordism we have the notation $\theta(A) = \partial_{h,\mathrm{in}} S$, $\theta(B) = \partial_{h,\mathrm{out}} S$, $\eta(M \times I) = \partial_{v,\mathrm{in}} S$, and $\eta(N \times I) = \partial_{v,\mathrm{out}} S$.

Two X–haloed 2–cobordisms $(S_0, \widehat{S}_0, R_0, \widehat{\mathsf{F}}_0)$ and $(S_1, \widehat{S}_1, R_1, \widehat{\mathsf{F}}_1)$ are *isomorphic* if there is an isomorphism of X–halations $\xi\colon (S_0, \widehat{S}_0, \widehat{\mathsf{F}}_0) \to (S_1, \widehat{S}_1, \widehat{\mathsf{F}}_1)$ which restricts isomorphisms

$$(\partial_{h,\mathrm{in}} S_0, (\widehat{S}_0)|_{\partial_{h,\mathrm{in}} S_0}, (\widehat{\mathsf{F}}_0)|_{\partial_{h,\mathrm{in}} S_0}) \to (\partial_{h,\mathrm{in}} S_1, (\widehat{S}_1)|_{\partial_{h,\mathrm{in}} S_1}, (\widehat{\mathsf{F}}_1)|_{\partial_{h,\mathrm{in}} S_1}),$$
$$(\partial_{h,\mathrm{out}} S_0, (\widehat{S}_0)|_{\partial_{h,\mathrm{out}} S_0}, (\widehat{\mathsf{F}}_0)|_{\partial_{h,\mathrm{out}} S_0}) \to (\partial_{h,\mathrm{out}} S_1, (\widehat{S}_1)|_{\partial_{h,\mathrm{out}} S_1}, (\widehat{\mathsf{F}}_1)|_{\partial_{h,\mathrm{out}} S_1}),$$
$$(\partial_{v,\mathrm{in}} S_0, (\widehat{S}_0)|_{\partial_{v,\mathrm{in}} S_0}, (\widehat{\mathsf{F}}_0)|_{\partial_{v,\mathrm{in}} S_0}) \to (\partial_{v,\mathrm{in}} S_1, (\widehat{S}_1)|_{\partial_{v,\mathrm{in}} S_1}, (\widehat{\mathsf{F}}_1)|_{\partial_{v,\mathrm{in}} S_1}),$$
$$(\partial_{v,\mathrm{out}} S_0, (\widehat{S}_0)|_{\partial_{v,\mathrm{out}} S_0}, (\widehat{\mathsf{F}}_0)|_{\partial_{v,\mathrm{out}} S_0}) \to (\partial_{v,\mathrm{out}} S_1, (\widehat{S}_1)|_{\partial_{v,\mathrm{out}} S_1}, (\widehat{\mathsf{F}}_1)|_{\partial_{v,\mathrm{out}} S_1})$$

such that $\xi|_{\partial S_0}$ is identity, $\xi \circ \eta = \eta'$ and $\xi \circ \theta = \theta'$, where θ' and η' are isomorphisms of cooriented X–halations corresponding to the decomposition $\partial S_1 = \partial_h S_1 \amalg \partial_v S_1$.

Lemma 2.6 *The bicategory* $X\mathrm{Bord}_2$ *is a symmetric monoidal bicategory under disjoint union.*

We skip the proof, which is given in [25] using a method developed by Shulman [24]. Recall that two major goals of this paper are to define 2–dimensional extended homotopy field theories and classify them. The following definition addresses to the first one:

Definition 2.7 Let $\mathcal{C}$ be a symmetric monoidal bicategory. A $\mathcal{C}$*–valued* 2*–dimensional extended homotopy field theory with target* X is a symmetric monoidal 2–functor from $X\mathrm{Bord}_2$ to $\mathcal{C}$.

3 The G–planar decompositions

3.1 G–linear diagrams

Linear diagrams, introduced by Schommer-Pries [22], represent 1–dimensional compact manifolds equipped with a Morse function to $[0,1]$. Briefly speaking, a linear diagram is a triple formed by the set of critical values of a Morse function on a compact 1–manifold, an open cover of $[0,1]$, and combinatorial data describing preimages of a Morse function on open sets. By labeling critical values with cup or cap instead of their indices (see Figure 4) the first ingredient of a linear diagram is defined as follows:

Definition 3.1 [22] A 1*–dimensional graphic* Ψ is a finite subset of $(0,1)$ where each point is labeled with either cup or cap.

For a given 1–dimensional graphic Ψ, an open cover $\mathcal{U}=\{U_\alpha\}_{\alpha\in J}$ of $[0,1]$ having at most double intersections is said to be Ψ*–compatible* if each U_α contains at most one element from Ψ and double intersections are disjoint from Ψ. It is not hard to find such open covers and the second ingredient of a linear diagram is defined as follows:

Definition 3.2 [22] Let Ψ be a 1–dimensional graphic. A *chambering set* Γ for Ψ is a set of isolated points in $(0,1)$ disjoint from μ. *Chambers* of Γ are the connected components of $[0,1]\backslash(\Gamma\cup\mu)$. A chambering set Γ is said to be subordinate to an open cover $\mathcal{U}=\{U_\alpha\}_{\alpha\in J}$ of $[0,1]$ if each chamber is a subset of at least one U_α.

Example 3.3 Figure 5 shows an example of a 1–dimensional oriented compact manifold M equipped with a Morse function $f\colon(M,\partial M)\to([0,1],\{0,1\})$. The

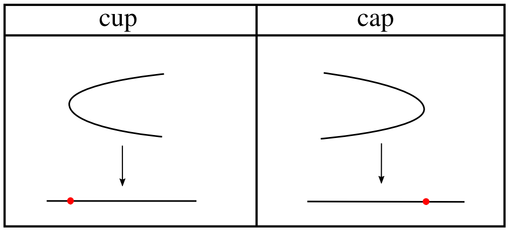


Figure 4: Singularities of a Morse function on a 1–manifold and their images in $\mathbb{R}$.

critical values of f define a 1–dimensional graphic Ψ (see Figure 4). An open cover $\mathfrak{U} = \{U_i\}_{i=1}^4$ of $[0,1]$ is a Ψ–compatible open cover and turquoise points form a chambering set subordinate to $\mathfrak{U}$.

For an oriented compact 1–manifold M, a Morse function of the form $f:(M,\partial M) \to ([0,1],\{0,1\})$ whose critical values are distinct and lie in $(0,1)$ is called a *generic map*. Let Ψ be a 1–dimensional graphic induced from a pair (M,f) of an oriented compact 1–manifold equipped with a generic map. Let Γ be a chambering set subordinate to a Ψ–compatible open cover $\mathfrak{U}$. Since f is a Morse function and chambers are disjoint from μ, the preimage $f^{-1}(V)$ of a chamber V consists of a disjoint union of arcs (possibly empty), each mapping diffeomorphically onto V under f. A *trivialization* of V is an identification of $f^{-1}(V)$ with $\mathbb{N}_{\leq N} \times V$ for some $N \in \mathbb{N}$, where $\mathbb{N}_{\leq N} = \{a \in \mathbb{N} \mid 0 < a \leq N\}$ if $f^{-1}(V)$ is nonempty and an identification with the empty set otherwise. In this case, each $\{i\} \times V$ is called a *sheet* and each sheet is equipped with an orientation.

Trivializations of two neighboring chambers have the same number of sheets if chambers are separated by a point in Γ. If a point in μ separates chambers, then, by the Morse

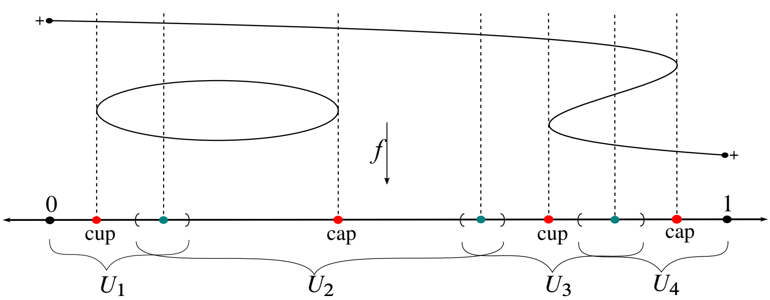


Figure 5: Induced 1–dimensional graphic and a chambering set.

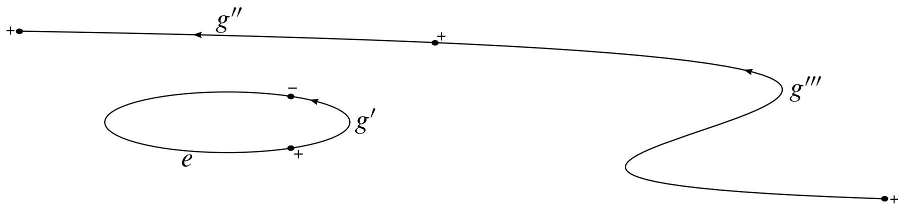


Figure 6: 1–dimensional marked X–manifold (M, T, g).

lemma, the number of sheets differ by two (see Figure 4). A *sheet data* $\mathcal{S}$ [22] for a pair (Ψ, Γ) consists of a trivialization of each chamber and an injection or a permutation between trivializations of neighboring chambers preserving orientations and describing how sheets are glued.

Definition 3.4 [22, Definition 3.45] A *linear diagram* is a triple $(\Psi, \Gamma, \mathcal{S})$ consisting of a 1–dimensional graphic Ψ, a chambering set Γ subordinate to a Ψ–compatible open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of $[0, 1]$, and a sheet data $\mathcal{S}$ associated to the pair (Ψ, Γ).

Any linear diagram yields an oriented compact 1–manifold and a generic map to $[0, 1]$. Our goal is to add extra data of X–manifolds to linear diagrams so that these diagrams produce oriented 1–dimensional marked X–manifolds. Recall that a 1–dimensional marked X–manifold is a triple (M, T, g), where M is an oriented compact 1–manifold, $T \subseteq M$ is a finite set with $\partial M \subset T$ such that each closed connected component of M contains at least two elements of T, and $\mathrm{g} \in [(M, T), (X, x)]$.

We describe the extra data on linear diagrams in an example. Let (M, T, g) be a 1–dimensional marked X–manifold, shown in Figure 6, whose underlying manifold M is the 1–dimensional oriented compact manifold considered in Example 3.3. We consider the same generic map $f : (M, \partial M) \to (I, \partial I)$ and chambering set as in Example 3.3 giving the linear diagram $(\Psi, \Gamma, \mathcal{S})$. First we label elements of Γ. A point in Γ is labeled with β^σ, where $\sigma \in S_N$ is the permutation coming from the sheet data of this point. We then add the elements of $f(T)$ to Γ, which means there are possibly new chambers. Each new chamber has the induced trivialization from the larger chamber, which splits into two. We do not label these added points. After that we equip the boundary components of every sheet, except the critical points of f, with oriented points using the orientation of M (brown points in Figure 7) and label each sheet with a group element using the characteristic map g as shown in Figure 7.

Next, we add labeled points to $[0, 1]$ as follows. If the preimage of a chamber does not have any singularity then the midpoint of that chamber is added. The label of this

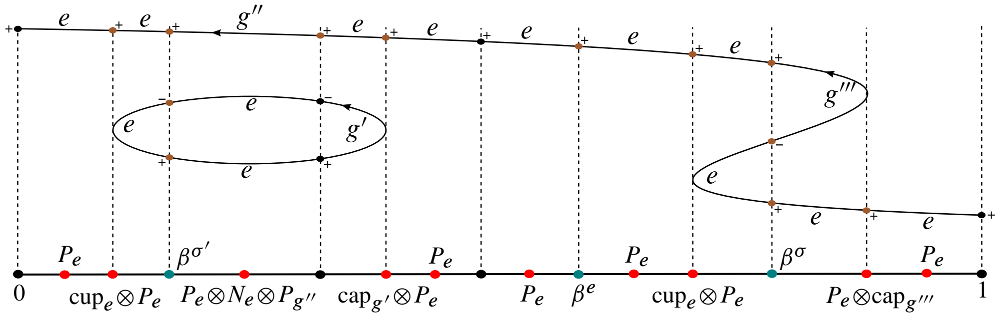


Figure 7: Example of a G–linear diagram without sheet data.

point is given as follows. We assign P_{g_1} for such a sheet with + boundary points and g_1–label, and assign N_{g_2} for a such a sheet with - boundary points and g_2–label. Then the label of the added point is given by $A_1 \otimes A_2 \otimes \cdots \otimes A_n$, where A_i is the assigned label of the i^{th} sheet according to the trivialization of the chamber for $i = 1, \ldots, n$. Similarly, we modify labels cup and cap according to assignments to sheets which are in the same trivialization with these singularities (see Figure 7). Lastly, on sheet data trivialization of sheets involves labeling each sheet with a group element with an arrow as described above and each (unlabeled) point is lifted to boundary of a sheet. We denote this modified linear diagram using the extra data of the 1–dimensional marked X–manifold with $(\Psi^G, \Gamma^G, \mathcal{S}^G)$ and call it a *G–linear diagram*.

It is clear that any G–linear diagram $(\Psi^G, \Gamma^G, \mathcal{S}^G)$ produces a pair $((M', T', \mathrm{g}'), f')$. For any such pair, by choosing a compatible chambering set, we obtain a new pair. These two pairs are related by the following notion. An *X–homeomorphism* between (marked) X–manifolds is a pointed orientation-preserving diffeomorphism commuting with the characteristic maps. An X–homeomorphism between such pairs is called *over* $[0, 1]$ if it commutes with the fixed generic maps.

Proposition 3.5 *Let $(\Psi^G, \Gamma^G, \mathcal{S}^G)$ be a G–linear diagram induced from a pair $((M, T, \mathrm{g}), f)$ of a 1–dimensional oriented marked X–manifold and a generic map $f : (M, \partial M) \to ([0, 1], \{0, 1\})$, and a chambering set Γ for a Ψ–compatible open cover $\mathcal{U}$ of $[0, 1]$. If the pair $((M', T', \mathrm{g}'), f')$ is constructed from $(\Psi^G, \Gamma^G, \mathcal{S}^G)$, then there exists an X–homeomorphism $F : M \to M'$ over $[0, 1]$.*

Proof The diffeomorphism F maps inverse images of chambers to corresponding trivializations. Since corresponding connected components have the same G–labels

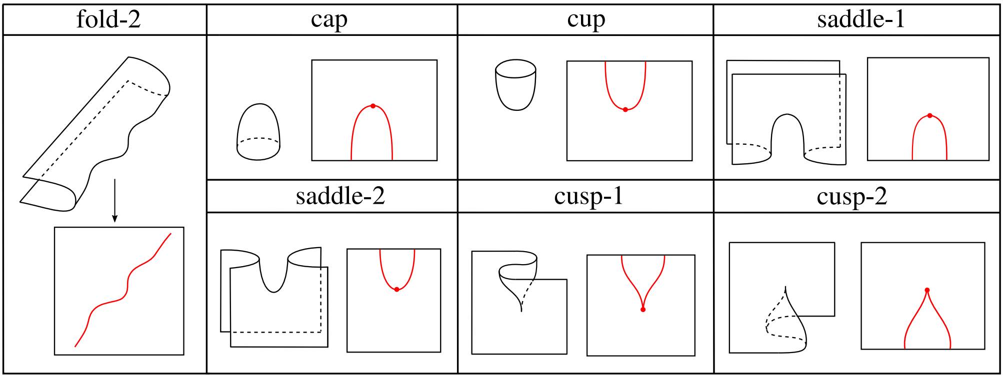


Figure 8: Singularities of Schommer-Pries stratification and their graphics in $\mathbb{R}^2$.

and both f and $f' \circ F$ restrict to the same map on $f^{-1}(V)$ for any chamber V, F is an X–homeomorphism over $[0,1]$. □

3.2 G–planar diagrams

Planar diagrams, introduced by Schommer-Pries [22], represent cobordism type $\langle 2\rangle$–surfaces equipped with a generic map to $I^2 = [0,1] \times [0,1]$. Here generic maps refer to Schommer-Pries' stratification of jet spaces described below. Parallel to linear diagrams, a planar diagram consists of a graphic of a generic map, an open cover of I^2, and a combinatorial data describing preimages of a generic map on open sets.

In his classification of 2–dimensional extended TFTs, Schommer-Pries [22] studied maps from cobordism type $\langle 2\rangle$–surfaces to I^2 and refined the Thom–Boardman stratification of jet spaces. Figure 8 shows the singularities of Schommer-Pries' stratification in normal coordinates and their graphics in I^2. Here, by a *graphic* we mean the image of a singularity under a generic map. In this context, by a generic map we mean a map whose jet sections are transversal to each stratum. In Figure 8, generic maps are projections to the page. The numbers on singularity names indicate their indices. By an index of a singularity, we mean a symmetry of either a singularity or its graphic. For example, fold-1 is obtained from fold-2 by changing the folding direction. Similarly, cap, cup, saddle-1, and saddle-2 are different indices of the Morse singularity. Observe that a cusp singularity has four indices.

For a given cobordism type $\langle 2\rangle$–surface Σ, a generic map for Schommer-Pries stratification has the form

$$f : (\Sigma, \partial_v \Sigma, \partial_h \Sigma) \to (I^2, \partial I \times I, I \times \partial I).$$

Transversality theorems (see [22; 8]) imply that the set of generic maps is dense in $C^\infty((\Sigma, \partial_v\Sigma, \partial_h\Sigma), (I^2, \partial I \times I, I \times \partial I))$. The properties of Schommer-Pries' stratification are listed in the following definition. In particular, the graphic of a generic map for this stratification is a 2–dimensional graphic, which is defined as follows:

Definition 3.6 [22, Definition 1.29] A 2*–dimensional graphic* $\Phi = (\eta, \mu)$ is a diagram in I^2 consisting of a finite number of embedded labeled curves (η) and a finite number of labeled points (μ) satisfying the following conditions:

(i) Elements of η can only have transversal intersections and no three elements intersect at a point. Each element of η is labeled with either fold-1 or fold-2.

(ii) Elements of η are disjoint from $\partial I \times I$ and intersect transversely with $I \times \partial I$. Labeling each of these intersection points on $I \times \partial I$ with cup for fold-1 labeled curves and with cap for fold-2 labeled curves produces 1–dimensional graphics on $I \times \{0\}$ and $I \times \{1\}$.

(iii) Projections of elements of η to the last coordinate of I^2 are local diffeomorphisms.

(iv) Elements of μ are isolated and disjoint from $\partial(I \times I)$. Each element is labeled with one of cup, cap, saddle-1, saddle-2 or cusp-i for $i = 1, 2, 3, 4$.

(v) Each element in μ has a neighborhood in which two elements of η form one of cup, cap, saddle-1, saddle-2 or cusp-i graphic for $i = 1, 2, 3, 4$ (see Figure 8).

We want to extend this definition to cobordism type $\langle 2\rangle$–X–surfaces so that a 2–dimensional graphic additionally contains the X–manifold data. First we consider $\langle 2\rangle$–X–surfaces whose underlying manifolds are singularities of Schommer-Pries' stratification in normal coordinates (see Figure 8). Figure 9 shows their graphics with the X–manifold data. Note that we abbreviate fold-i label to F_i, saddle-i to S_i, and cusp-i to C_i for $i = 1, 2$. Also observe that fold-1 and fold-2 singularities are paths of cup and cap singularities in the previous section. For this reason, henceforth, on any G–linear diagram we replace cup and cap labels with F_1 and F_2 labels, respectively. This implies that the restriction of each diagram in Figure 9 to $I \times \partial I$ yields two partial G–linear diagrams. Later we complete them to G–linear diagrams by adding chambering sets and sheet data.

Compared to graphics of singularities in Figure 8, there are additional arcs connecting graphics of Morse[1] and cusp singularities to the (red) points of the boundary G–linear

[1] Similar to the graphics of saddles, one can add arcs to the graphics of cup and cap and label them with $\varnothing$.

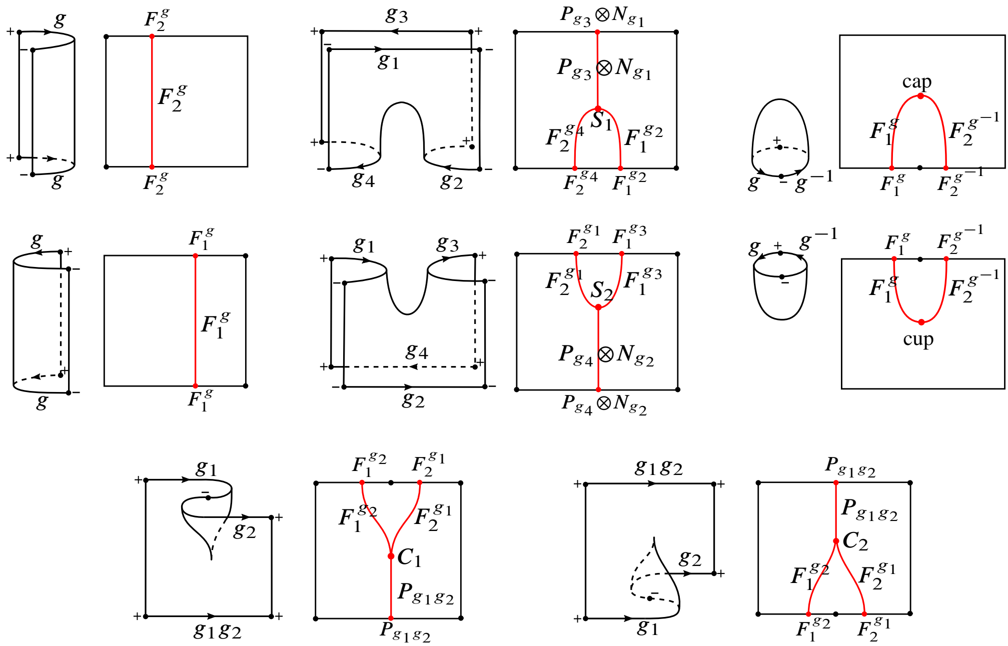


Figure 9: Graphics of singularities with X–manifold data.

diagram. The reason behind the addition of these arcs is the connection between these diagrams and string diagrams, which is the content of Theorem 4.5. We call $\langle 2\rangle$–X–surfaces in Figure 9 *elementary* $\langle 2\rangle$*–*X*–surfaces* since they are building blocks of cobordism type $\langle 2\rangle$–X–surfaces under horizontal and vertical gluing operations. However, this list is not complete. The rest of the elementary $\langle 2\rangle$–X–surfaces and their graphics are given in Figure 10.

We now know the extra data on 2–dimensional graphics of elementary $\langle 2\rangle$–X–surfaces. Using these modified diagrams, for any generic map on a cobordism type $\langle 2\rangle$–X–surface $(\Sigma, R, \mathtt{P})$ we add the X–manifold data $(R, \mathtt{P})$ to the graphic of the generic map in two steps. First we decompose $(\Sigma, R, \mathtt{P})$ into horizontal and vertical compositions of elementary $\langle 2\rangle$–X–surfaces. This is always possible by the nature of Schommer-Pries' stratification and above arguments. Using $\mathtt{P}$ we choose G–labels on each elementary $\langle 2\rangle$–X–surface in the decomposition. We then consider the modified diagrams of these $\langle 2\rangle$–X–surfaces in I^2, as described above. Figure 11 shows an example of this process where the generic map is projection to the page. For a given 2–dimensional graphic $\Phi = (\eta, \mu)$, we denote the union of η and arcs encoding the X–manifold data by η^G and, similarly, μ^G denotes the union of μ and additional labeled points. We call such a

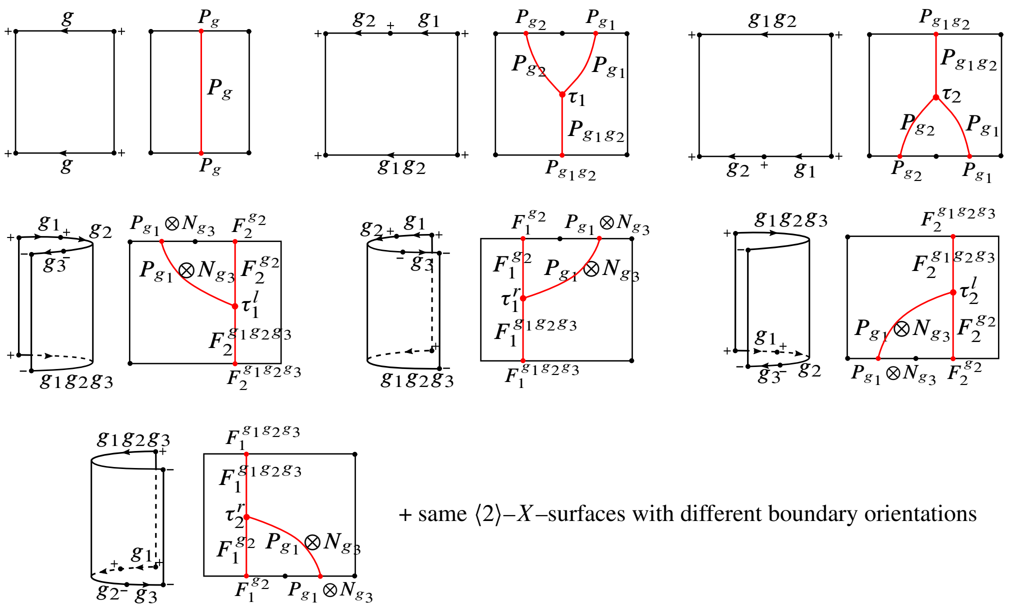


Figure 10: The remaining elementary $\langle 2\rangle$–X–surfaces and their graphics.

2–dimensional graphic Φ equipped with X–manifold data a 2*–dimensional G–graphic* and denote it by $\Phi^G = (\eta^G, \mu^G)$.

Let $\Phi^G = (\eta^G, \mu^G)$ be a 2–dimensional G–graphic; an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of I^2 with at most triple intersections is said to be Φ*–compatible* [22] if each triple intersection is disjoint from μ, each double intersection is disjoint from $\eta \cup \mu$ or contains a single element from η, and the open covers $\{U_\alpha \cap (I \times \{i\})\}_{\alpha \in J}$ of $I \times \{i\}$ for $i = 0, 1$ are

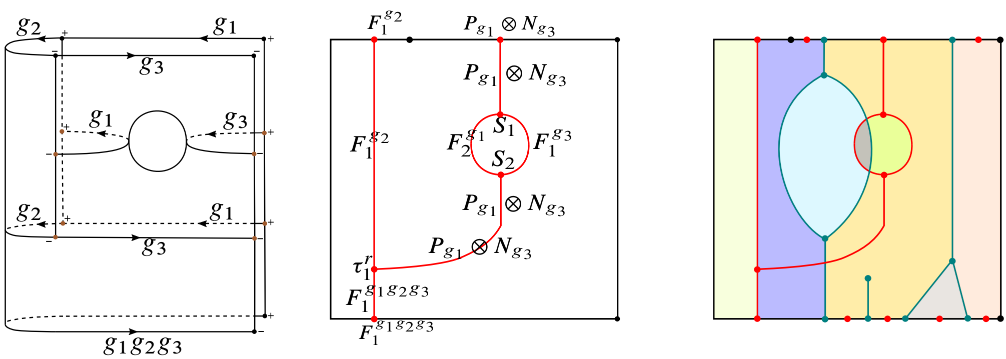


Figure 11: Adding X–manifold data to a graphic and an example of a chambering graph.

compatible with corresponding 1–dimensional graphics obtained from Φ^G. Knowing that $\mathbb{R}^2$ has covering dimension two and sets η and μ are finite, it is not hard to find Φ–compatible open covers for a given graphic Φ.

Definition 3.7 [22, Definition 1.42] Let $\Phi^G = (\eta^G, \mu^G)$ be a 2–dimensional G–graphic. A *chambering graph* Γ for Φ^G is a smoothly embedded graph in I^2 satisfying the following conditions. Vertices of Γ are disjoint from elements of Φ^G and have degree either one or three. Edges of Γ are disjoint from $\partial I \times I$ and transverse to Φ^G and $I \times \partial I$. Furthermore, projection of each edge to the last coordinate is a local diffeomorphism and around each trivalent vertex one of the edges projects to the opposite side of the projection of the other two edges with respect to the image of the vertex.

Definition 3.8 Let Γ be a chambering graph for $\Phi^G = (\eta^G, \mu^G)$. *Chambers* of Γ are the connected components of $I^2 \backslash (\Gamma \cup \eta \cup \mu)$. A chambering graph Γ is said to be subordinate to an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of I^2 if each chamber is a subset of at least one U_α with $\alpha \in J$ and the chambering sets $\Gamma \cap (I \times \{i\})$ are compatible with the restricted open covers $\big\{U_\alpha \cap (I \times \{i\})\big\}_{\alpha \in J}$ for $i = 0, 1$.

Example 3.9 Figure 11 shows an example of a chambering graph Γ where each colored region is a chamber. Note that the chambering graph leads to new red points on $I \times \partial I$ forming two partial G–linear graphs. In this example all new points are labeled with $P_e \otimes N_e$.

Proposition 3.10 *Let Φ^G be a 2–dimensional G–graphic in I^2 and let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ be a Φ–compatible open cover of I^2. Then there exists a chambering graph Γ for Φ^G subordinate to $\mathfrak{U}$.*

The 2–dimensional graphic version of this proposition was proven in Proposition 1.46 of [22]. This version follows from that only using transversality arguments. From now on we assume that all chambering graphs are subordinate to some compatible open cover.

Next, we recall sheet data associated to a pair (Φ, Γ). We know sheet data on the components of $I \times \partial I$ from the previous section. For the other boundary component $\partial I \times I$, sheet data is similar and indeed simpler since vertical boundary components are all identical. Therefore, we consider the open subsets of chambers by removing

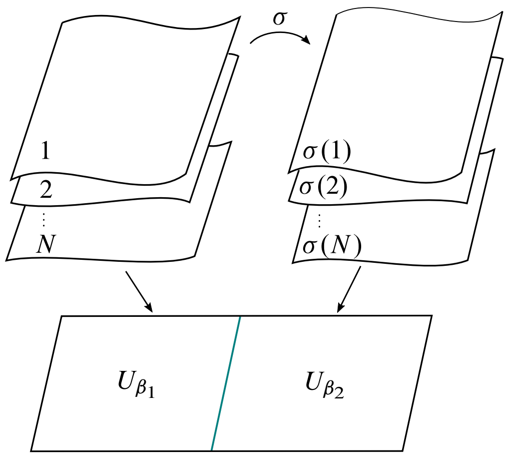


Figure 12

boundary components. That is, for any chamber U_β which intersects with $\partial(I \times I)$ we consider $U'_\beta = U_\beta - (U_\beta \cap \partial(I \times I))$. Since f is generic, the preimage $f^{-1}(U'_\beta)$ consists of a disjoint union of open sets (possibly empty) each mapping diffeomorphically onto U'_β. A *trivialization* of U'_β is an identification of $f^{-1}(U'_\beta)$ with $\mathbb{N}_{\le N} \times U'_\beta$ for some $N \in \mathbb{N}$ if $f^{-1}(U'_\beta)$ is nonempty and an identification with the empty set otherwise. In this case, each $\{i\} \times U'_\beta$ is called a *sheet* and each sheet is oriented. By requiring the same trivializations on U'_β and ∂U_β, we extend these identifications to $\mathbb{N}_{\le N} \times U_\beta$.

Similar to the 1–dimensional case, trivializations of two neighboring chambers have the same number of sheets if chambers are separated by an edge of Γ (see Figure 12). If an element in η separates chambers then the number of sheets differ by two because it is a fold graphic (see Figure 8). *Sheet data* $\mathcal{S}$ [22] for a pair (Φ, Γ) consists of a

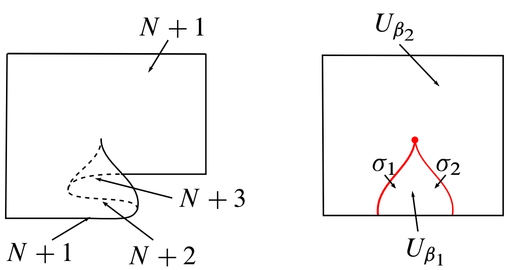


Figure 13

trivialization of each chamber and an injection or a permutation between trivializations of neighboring chambers preserving orientations and describing how sheets are glued (see Figure 12). The gluing description of sheets requires certain conditions on permutations and injections. For example, if three chambers are separated by edges of a trivalent vertex of Γ, then the circular composition of permutations must be the identity. Also, the permutation corresponding to the edge of a univalent vertex must be the trivial permutation. The only nontrivial sheet data is that of a cusp graphic, which we briefly describe. Consider the cusp-2 labeled point in Figure 13. Let $\mathbb{N}_{\leq N+3} \times U_{\beta_1}$ and $\mathbb{N}_{\leq N+1} \times U_{\beta_2}$ be the trivializations such that the sheets $\bigcup_{i=N+1}^{N+3} i \times U_{\beta_1}$ and $(N+1) \times U_{\beta_2}$ belong to a cusp singularity as shown in Figure 13. In this case, restriction of injections to the cusp singularity gives $\sigma_1(N+1) = N+1$ and $\sigma_2(N+1) = N+3$.

Definition 3.11 [22, Definition 1.48] A *planar diagram* is a triple $(\Phi, \Gamma, \mathcal{S})$ consisting of a 2–dimensional graphic Φ, a chambering graph Γ for Φ subordinate to a Φ–compatible open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of I^2, and a sheet data $\mathcal{S}$ associated to the pair (Φ, Γ).

Any planar diagram $(\Phi, \Gamma, \mathcal{S})$ produces a cobordism type $\langle 2 \rangle$–surface Σ with a generic map $f\colon (\Sigma, \partial_v \Sigma, \partial_h \Sigma) \to (I^2, \partial I \times I, I \times \partial I)$. In the case of a tuple (Φ^G, Γ), the associated sheet data can be improved to produce a $\langle 2 \rangle$–X–surface $(\Sigma, R, \mathtt{P})$. We call such sheet data carrying X–manifold data to sheets *G–sheet data* and denote it by $\mathcal{S}^G$. Then, generalizing a planar diagram, we define a *G–planar diagram* as a triple $(\Phi^G, \Gamma, \mathcal{S}^G)$. As an extension of G–linear diagrams, we label edges of a chambering graph with β^σ, where σ is the permutation coming from sheet data. If both sheets are trivialized by the empty set, then the separating edge is labeled with β^ι. We also label vertices of chambering graph as follows. For a fixed trivalent vertex, if two of the edges direct upward, the vertex is labeled with $X^{\sigma,\sigma'}$, and if two of the edges direct downward then it is labeled with $(X^{\sigma,\sigma'})^{-1}$. Here σ and σ' are the permutations corresponding to the sheet data of these edges. A univalent vertex is labeled with X^e if its edge directs upward and it is labeled with $(X^e)^{-1}$ if its edge directs downward. We also label intersections of edges of the chambering graph and 2–dimensional G–graphic. For such an intersection, if an edge of the chambering graph is labeled with β^σ and an arc of the 2–dimensional G–graphic is labeled with A, then the intersection is labeled with β^σ_A.

Example 3.12 Figure 14 shows an example of a cobordism type $\langle 2 \rangle$–X–surface $(\Sigma, R, \mathtt{P})$ and its G–planar diagram with respect to the projection map. We denote the

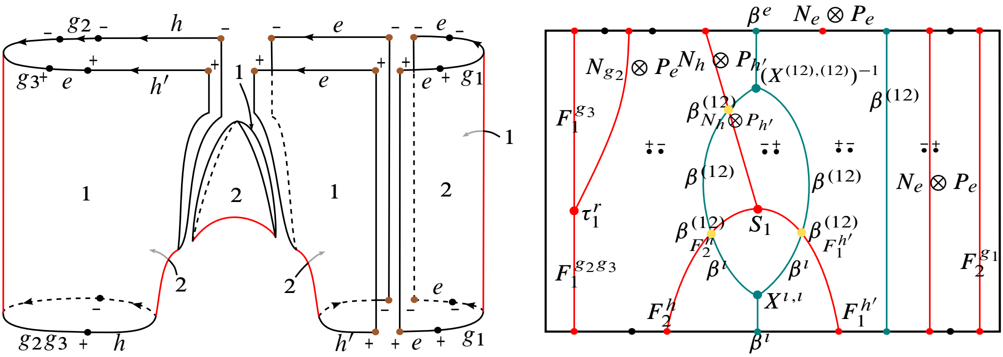


Figure 14: Example of a cobordism type $\langle 2\rangle$–X–surface and its G–planar diagram.

trivializations of sheets with numbers on Σ. Correspondingly, we encode this data on the G–planar diagram by labeling chambers with ordered signed points. Here signs come from the sign of the points on the corner of the corresponding sheet.

Let $(\Sigma, R, \mathtt{P})$ and $(\Sigma', R', \mathtt{P}')$ be $\langle 2\rangle$–X–surfaces endowed with generic maps f and f', respectively. An X–homeomorphism $F\colon (\Sigma, R, \mathtt{P}) \to (\Sigma', R', \mathtt{P}')$ is said to be *over* I^2 if it commutes with the fixed generic maps, ie $f' \circ F = f$.

Proposition 3.13 *Let $f\colon (\Sigma, \partial_v\Sigma, \partial_h\Sigma) \to (I^2, \partial I \times I, I \times \partial I)$ be a generic map on a cobordism type $\langle 2\rangle$–X–surface $(\Sigma, R, \mathtt{P})$ inducing a 2–dimensional graphic Φ^G. Let Γ be a chambering graph for Φ^G subordinate to a Φ–compatible open cover giving a G–planar diagram $(\Phi^G, \Gamma, \mathcal{S}^G)$. If the pair $((\Sigma', R', \mathtt{P}'), f')$ is constructed from $(\Phi^G, \Gamma, \mathcal{S}^G)$, then there exists an X–homeomorphism $F\colon \Sigma \to \Sigma'$ over I^2.*

Proof The diffeomorphism $F\colon \Sigma \to \Sigma'$ maps inverse images of chambers to the corresponding trivializations. Since $F(R) = R'$, $[\mathtt{P}' \circ F] = \mathtt{P}$, and both f and $f' \circ F$ restrict to the same map on $f^{-1}(U_\beta)$ for any chamber U_β, F is an X–homeomorphism over I^2. □

3.3 G–spatial diagrams

Schommer-Pries [22] introduced spatial diagrams to identify planar diagrams which produce diffeomorphic cobordism type $\langle 2\rangle$–surfaces. We extend them to G–spatial diagrams which identify those G–planar diagrams producing X–homeomorphic cobordism type $\langle 2\rangle$–X–surfaces. Then, using these identifications, we define an equivalence relation among G–planar diagrams, and prove the G–planar decomposition theorem.

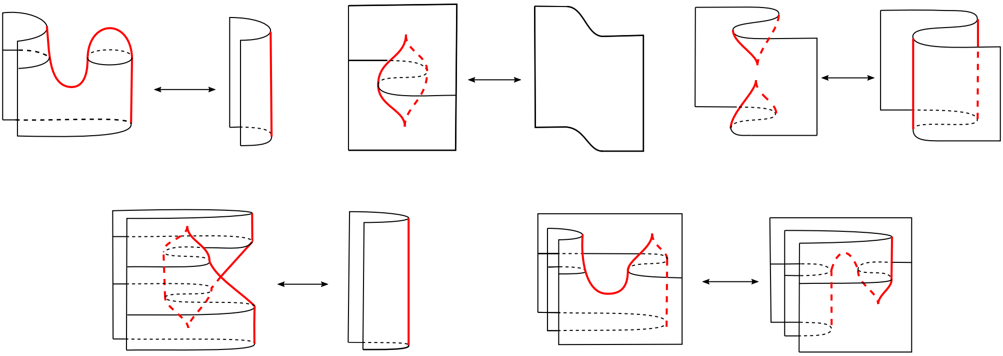


Figure 15: Movie moves coming from codimension 3 singularities.

Different generic maps on a fixed cobordism type $\langle 2\rangle$–surface yield different graphics just as different Morse functions yield different critical values. In the latter case, Cerf theory relates different sets of critical values in terms of isotopies or birth and death of critical values. Similarly, Schommer-Pries [22] related different graphics obtained from different generic maps in terms of isotopies and certain local moves of graphics, called *movie moves* (see Figure 15). These movie moves are obtained from the singularities of certain stratification of jet spaces

$$J^r((\Sigma \times I, \partial_h \Sigma \times I, \partial_v \Sigma \times I), (I^2 \times I, I \times \partial I \times I, \partial I \times I \times I))$$

for a cobordism type $\langle 2\rangle$–surface Σ. Note that in general the map F is not of the form $F(x,t) = (f_t(x), t)$. For our purposes, we consider the subspace of $J^r(\Sigma \times I, I^2 \times I)$ consisting of paths of functions.

Figure 16 shows the graphics of singularities for the Schommer-Pries stratification in normal coordinates. Observe the relation between movie moves in Figure 15 and the horizontal boundary components of the new graphics. The remaining movie moves coming from this stratification are shown in Figure 17. The properties of this stratification are listed in the following definition. In particular, the graphic of a generic map for this stratification is a 3–dimensional graphic, which is defined as follows:

Definition 3.14 [22, Definition 1.30] A 3–*dimensional graphic* $\Delta = (\delta, \eta, \mu)$ is a diagram in $I^2 \times I$ consisting of a finite number of embedded compact labeled surfaces (δ), a finite number of embedded labeled curves (η), and a finite number of embedded labeled points (μ) satisfying the following conditions:

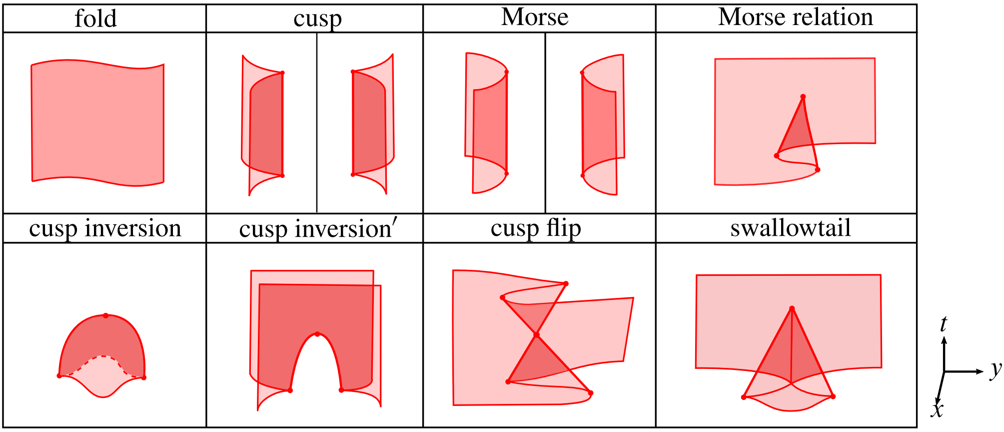


Figure 16: Graphics of the singularities in $I^2 \times I$.

(i) Projections of elements of δ to the last two coordinates are local diffeomorphisms. Elements of δ intersect with $I \times \{i\} \times I$ along vertical line segments $\{(x,i,s)\}_{s\in[0,1]}$ for some $x \in (0,1)$ and for $i = 0, 1$. Each element of δ is labeled with either fold-1 or fold-2.

(ii) Projections of elements of η to the last coordinate are local diffeomorphisms. Every element of η has a neighborhood in which two elements of δ form either Morse or cusp graphics (see Figure 16). Each element of η is labeled with either Morse-i or cusp-i, where $i = 1, 2, 3, 4$ indicates the indices.[2]

(iii) Each element of μ has a neighborhood in which some elements of δ and η form one of the graphics Morse relation-i, cusp inversion-j, cusp inversion′-j, cusp flip-j, or swallowtail-i, where $i = 1, 2, 3, 4$ and $j = 1, 2$ indicate graphics of different indices.

(iv) Elements of μ are labeled with one of the singularities Morse relation-i, cusp inversion-j, cusp inversion′-j, cusp flip-j or swallowtail-j, where $1 \le i \le 8$ and $j = 1, 2, 3, 4$ indicate the indices.

(v) The restriction of the graphic to the components of $I^2 \times \partial I$ gives 2–dimensional graphics.

(vi) Elements of δ, η, and μ are disjoint from $\partial I \times I^2$. They are transversal with respect to each other and to $I^2 \times \partial I$. Moreover, when two surfaces intersect

[2]Morse singularities are paths of cap, cup, saddle-1 and saddle-2 singularities.

along an arc, there can only be finitely many points on the arc with tangent space lying in $\langle \partial_x, \partial_y \rangle$, where (x, y, t) is the coordinate for $I^2 \times I$.

Let Σ be a cobordism type $\langle 2 \rangle$–surface and

$$F\colon (\Sigma \times I, \partial_h \Sigma \times I, \partial_v \Sigma \times I) \to (I^2 \times I, I \times \partial I \times I, \partial I \times I \times I)$$

be a generic map. We know that the restriction of F to the boundary components $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ gives two generic maps on Σ. The converse is also true. That is, for any given two generic maps $f_1, f_2\colon (\Sigma, \partial_v \Sigma, \partial_h \Sigma) \to (I^2, \partial I \times I, I \times \partial I)$, there exists a generic map F on $\Sigma \times I$ with $F|_{\Sigma \times \{0\}} = f_1$ and $F|_{\Sigma \times \{1\}} = f_2$. Therefore, 3–dimensional graphics are designed to identify 2–dimensional graphics obtained from generic maps on $\langle 2 \rangle$–surfaces which are diffeomorphic relative to boundary. Using 2–dimensional G–graphics, we extend this result to $\langle 2 \rangle$–X–surfaces which are X–homeomorphic relative to their boundary. This is useful since, in the cobordism bicategory $X\mathrm{Bord}_2$, the 2–morphisms are X–homeomorphism classes of cobordism type $\langle 2 \rangle$–X–surfaces relative to their boundary.

Movie moves are local relations on 2–dimensional graphics generating the identification of 2–dimensional graphics induced from different generic maps. Figures 18 and 19 show some of the generalized movie moves relating 2–dimensional G–graphics. The remaining generalized movie moves involve singularities with different indices, orientations, and decomposition into elementary $\langle 2 \rangle$–X–surfaces. Similarly, movie moves given in Figure 17 are generalized and their possible versions (different indices, orientations

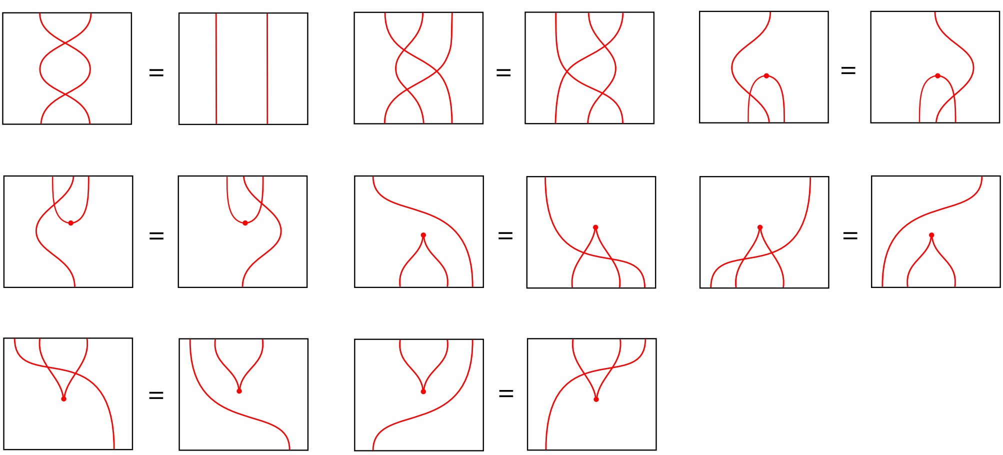


Figure 17: Movie moves coming from intersection of codimension 1 and 2 singularities.

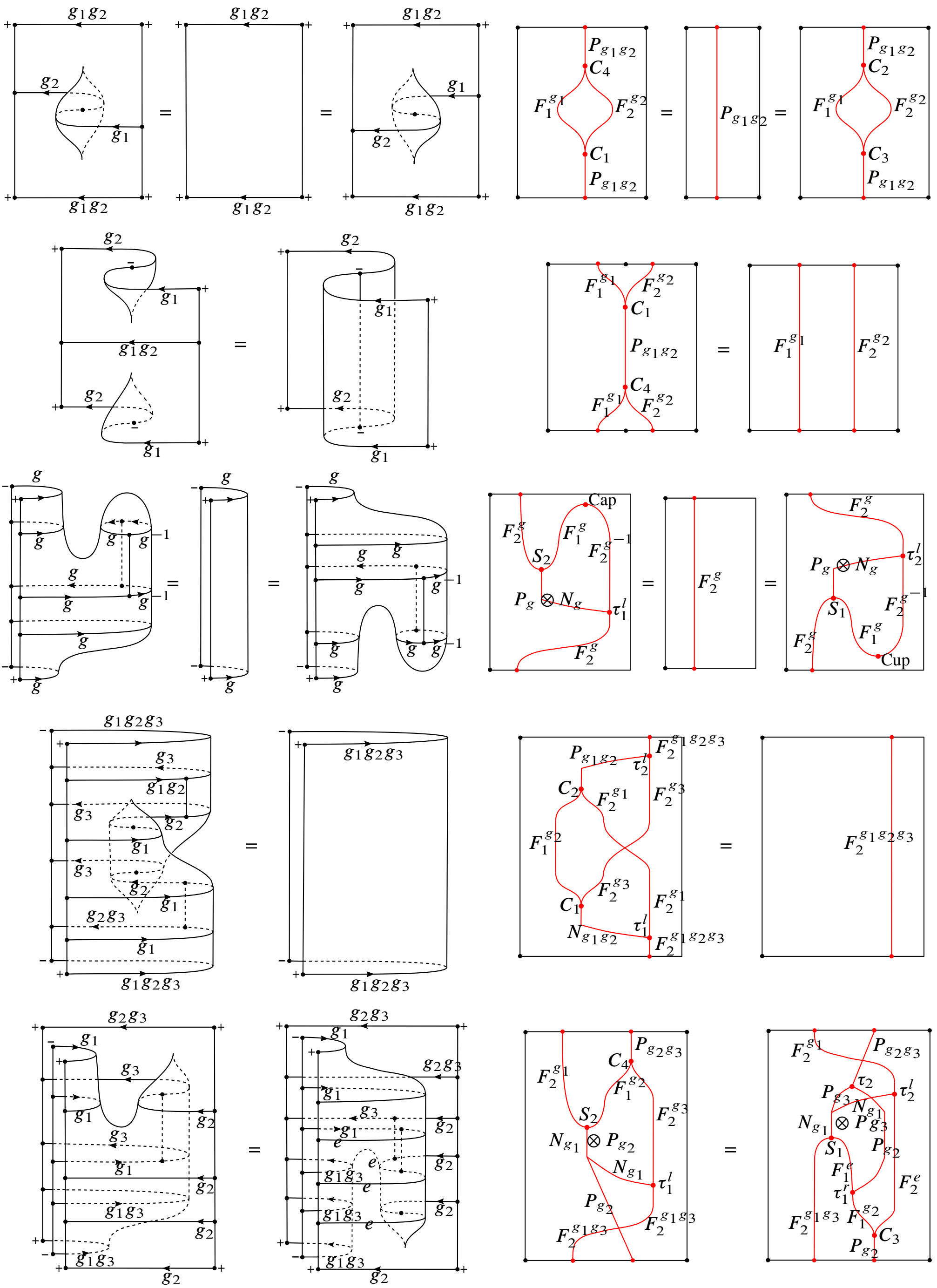


Figure 18: Some of the generalized movie moves.

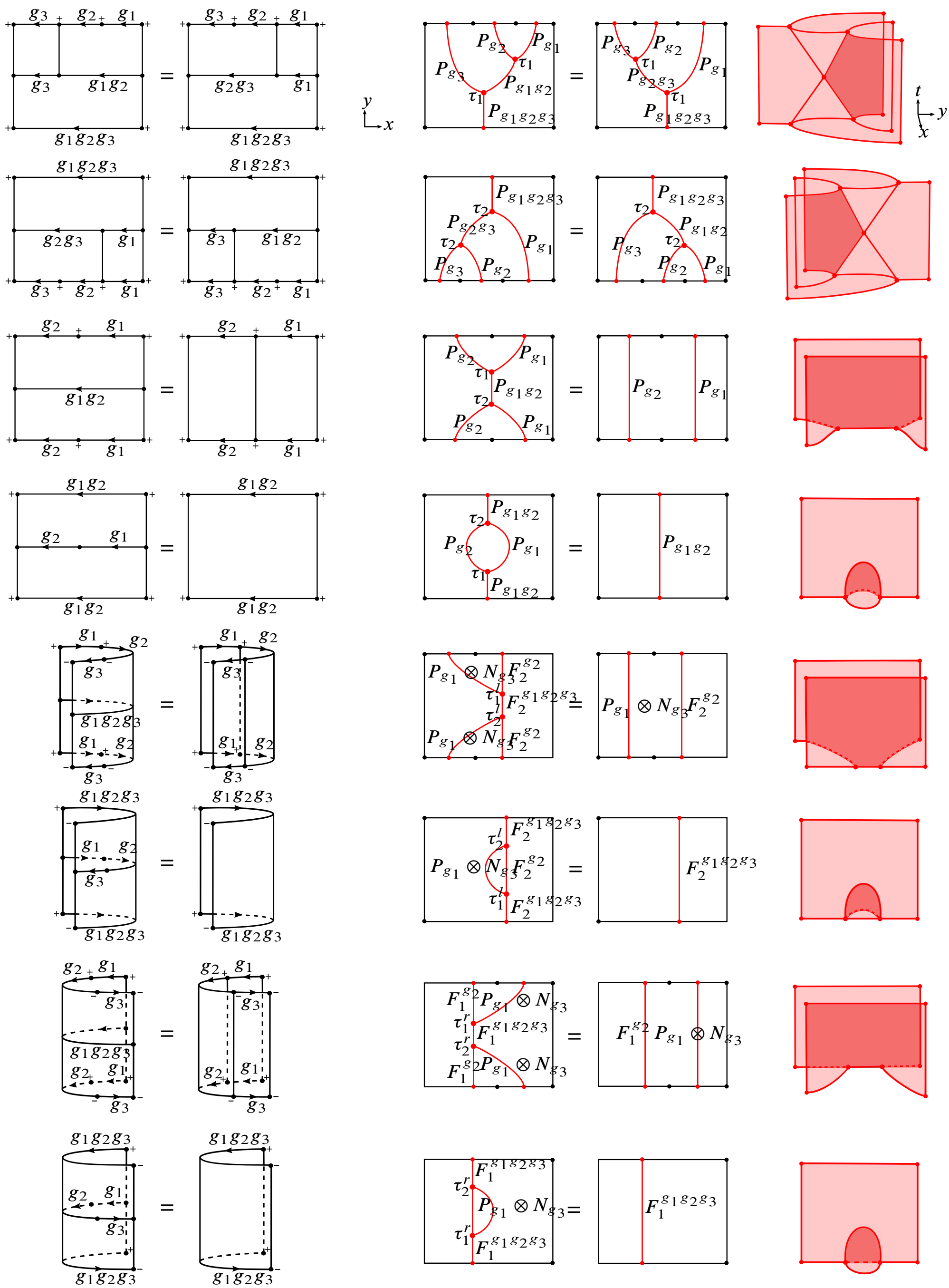


Figure 19: Some of the generalized movie moves obtained by gluing elementary $\langle 2\rangle$–X–surfaces.

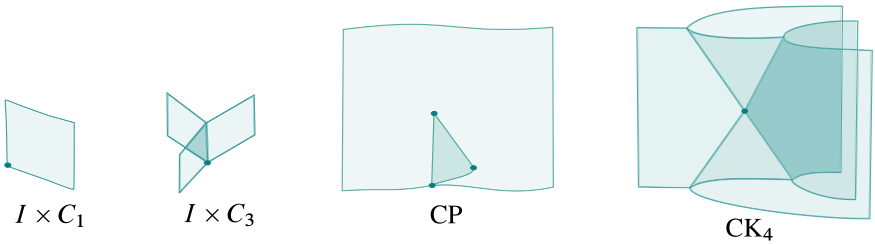


Figure 20: Local models for a chambering foam.

and decompositions) form generalized movie moves. From now on, whenever we refer to these figures we mean the complete list of movie moves.

Similar to the previous two sections, we generalize a 3–dimensional graphic to a G–graphic by adding (labeled) surfaces, arcs, and points. For every new movie move, the corresponding graphic is shown in Figure 19, right. Then we define *a 3–dimensional G–graphic* $\Delta^G = (\delta^G, \eta^G, \mu^G)$ as a 3–dimensional graphic $\Delta = (\delta, \eta, \mu)$ along with a 2–dimensional locally conical stratified space of compact type which is transverse to the graphic and whose local models are given in Figure 19. The notion of locally conical stratified space is also the main ingredient of Definition 3.15 below. Its definition can be found in [22, Definition 1.41]. The reader can think of a locally conical stratified space as a space constructed from given local models just as a manifold is built locally from disks.

Let $\Delta^G = (\delta^G, \eta^G, \mu^G)$ be a 3–dimensional G–graphic. An open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of I^3 with at most 4–fold intersections is said to be *Δ–compatible* [22] if each 4–fold intersection is disjoint from $\delta \cup \eta \cup \mu$, each 3–fold intersection is disjoint from $\mu \cup \eta$ and contains at most a single component of surfaces in δ, each double intersection is disjoint from points in μ, and the open covers $\left\{U_\alpha \cap (I^2 \times \{i\})\right\}_{\alpha \in J}$ of $I^2 \times \{i\}$ for $i = 0, 1$ are compatible with the corresponding 2–dimensional graphics obtained from Δ^G. Since I^3 has covering dimension 3 and there are only finitely many elements in δ, η and μ, Δ–compatible open covers exist.

Definition 3.15 [22, Definition 1.43] Let $\Delta^G = (\delta^G, \eta^G, \mu^G)$ be a 3–dimensional G–graphic. A *chambering foam* Γ for Δ^G is a smooth embedding of a 2–dimensional locally conical stratified space Γ of compact type into $I \times (0,1) \times I$ with the following properties. The space Γ is locally conical with respect to the system of local models I^2, $I \times C_1$, $I \times C_3$, CP, and CK_4 shown in Figure 20. Vertices are disjoint from Δ^G

and lie in the interior. Edges can only intersect with a surface from δ^G. Faces can only intersect with surfaces from δ^G and arcs from η^G. All intersections are transversal and Γ additionally satisfies the following conditions:

(I) The projection $p\colon \Gamma \to I \times I$ to the last two coordinates has no singularity and projection of faces to the last coordinate has no singularity.

(II) For every $t \in I$ satisfying that $(I^2 \times \{t\}) \cap \mu^G = \varnothing$, t is not a critical value of projection $\mathrm{pr}\colon \Gamma \to I$ to the last coordinate, and $(I^2 \times \{t\}) \cap \Gamma$ does not include a vertex of Γ, the graph $(I^2 \times \{t\}) \cap \Gamma$ forms a chambering graph for the 2–dimensional G–graphic $\Delta^G \cap (I^2 \times \{t\})$.

(III) The projection of each one of four edges in the CK_4–model connecting at the cone point to the last coordinate is a local diffeomorphism. Additionally, at least one of them must map to downward of the cone point and at least one of them must map to upward of the cone point.

(IV) The projection of the two edges in the CP–model connecting at the cone point to the last coordinate maps both edges to the same direction with respect to the image of the cone point.

Definition 3.16 Let $\Delta^G = (\delta^G, \eta^G, \mu^G)$ be a 3–dimensional G–graphic and let Γ be a chambering foam for Δ^G. *Chambers* of Γ are the connected components of $I^2 \times I \backslash (\Gamma \cup \delta \cup \eta \cup \mu)$. A chambering foam Γ is said to be *subordinate to an open cover* $\mathcal{O} = \{O_\alpha\}_{\alpha \in J}$ of $I^2 \times I$ if each chamber is a subset of at least one O_α with $\alpha \in J$ and the chambering graphs $\Gamma \cap (I^2 \times \{i\})$ are compatible with the restricted open cover $\{O_\alpha \cap I^2 \times \{i\}\}_{\alpha \in J}$ for $i = 0, 1$.

Lemma 3.17 *Let Γ be a chambering foam for a 3–dimensional G–graphic Δ^G inducing 2–dimensional G–graphics and chambering graphs (Φ_0^G, Γ_0) and (Φ_1^G, Γ_1) on $I^2 \times \{0\}$ and $I^2 \times \{1\}$, respectively. Let $\mathcal{O} = \{O_\alpha\}_{\alpha \in J}$ be a Δ–compatible open cover of I^3 such that Γ_i is subordinate to $\mathcal{O}_i = \mathcal{O}|_{I^2 \times \{i\}}$ for $i = 0, 1$. Then there exists a chambering foam Γ' for Δ^G subordinate to $\mathcal{O}$ whose restrictions to $I^2 \times \{0\}$ and $I^2 \times \{1\}$ yield Γ_0 and Γ_1, respectively.*

In [22], the corresponding statement for a 3–dimensional graphic was proven (see [22, Corollary 1.47]). In the case of nontransversal intersections with new elements encoding X–manifold data, Γ can be slightly modified to make all intersections transversal while being compatible with $\mathcal{O}$.

Just as the movie moves in Figures 18 and 19 generate relations locally between two G–planar diagrams on the boundary of a G–spatial diagram, there are movie moves describing local relations between two chambering graphs on the boundary of a compatible chambering foam. These moves along with corresponding chambering foams are shown in Figure 21. Moreover, there are movie moves coming from an intersection of a 3–dimensional G–graphic with a chambering foam. These local relations are shown in Figure 22, in which the labels are omitted.

Next, we describe the sheet data. Let $\Delta = (\delta, \eta, \mu)$ be a 3–dimensional graphic induced from a generic map $F\colon (\Sigma \times I, \partial_h \Sigma \times I, \partial_v \Sigma \times I) \to (I^2 \times I, I \times \partial I \times I, \partial I \times I \times I)$, where Σ is a cobordism type $\langle 2\rangle$–surface. Let Γ be a chambering foam subordinate to a Δ–compatible open cover. Similar to the previous section, sheet data associated to (Δ, Γ) extends the sheet data of planar diagrams on faces $I^2 \times \{0, 1\}$. Since F is generic, the preimage $F^{-1}(O_\beta)$ of an open chamber consists of a disjoint union of open sets, each mapping diffeomorphically onto O_β. If a chamber O_β is not open then we consider $O'_\beta = O_\beta \backslash (O_\beta \cap \partial(I^2 \times I))$. Then, a *trivialization* of a chamber is the identification of $F^{-1}(O_\beta)$ with $\mathbb{N}_{\leq N} \times O_\beta$ if $F^{-1}(O_\beta)$ is nonempty and an identification with the empty set otherwise. Each $\{i\} \times O_\beta$ is called a *sheet* and each sheet is oriented. For every chamber O_β which is not open, we extend identifications to $\mathbb{N}_{\leq N} \times O_\beta$ by requiring the same trivializations on O'_β and $O_\beta \cap \partial(I^2 \times I)$ coming from sheet data of planar diagrams.

Trivializations of two neighboring chambers have the same number of sheets if chambers are separated by a 2–dimensional stratum of Γ. If an element in δ separates chambers, then the number of sheets differs by two. Sheet data $\mathcal{S}$ [22] for a pair (Δ, Γ) consists of a trivialization of each chamber and an injection or a permutation between trivializations of neighboring chambers preserving orientations and describing how sheets are glued.

The gluing description of sheets requires the following conditions on permutations and injections. In the local models $I \times C_3$, CP, and CK_4, circular compositions of three or four permutations must be the identity. Since fold, cusp, and Morse graphics are paths of the corresponding graphics in the previous section, their sheet data do not change. According to properties of multijet stratification, transversal double and triple fold intersections are possible. There are four chambers for the double and eight for the triple fold intersection. In both cases, different compositions of injections starting from the chamber with the least number of sheets and ending at the chamber with the maximum number of sheets must be the same. The sheet data for the intersection of fold and Morse graphics is the same as double fold intersection and the sheet data

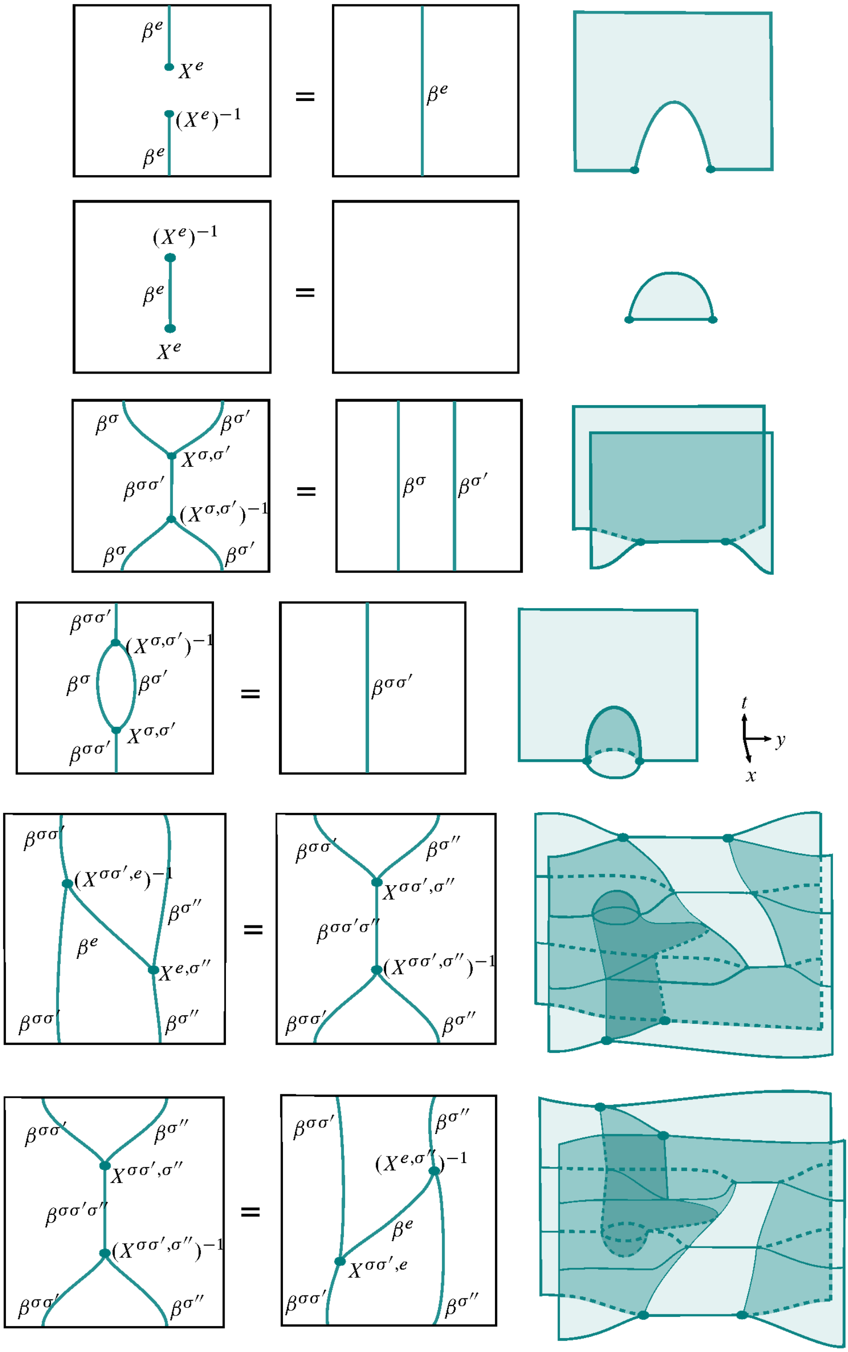


Figure 21: Movie moves for chambering graphs and the corresponding chambering foams.

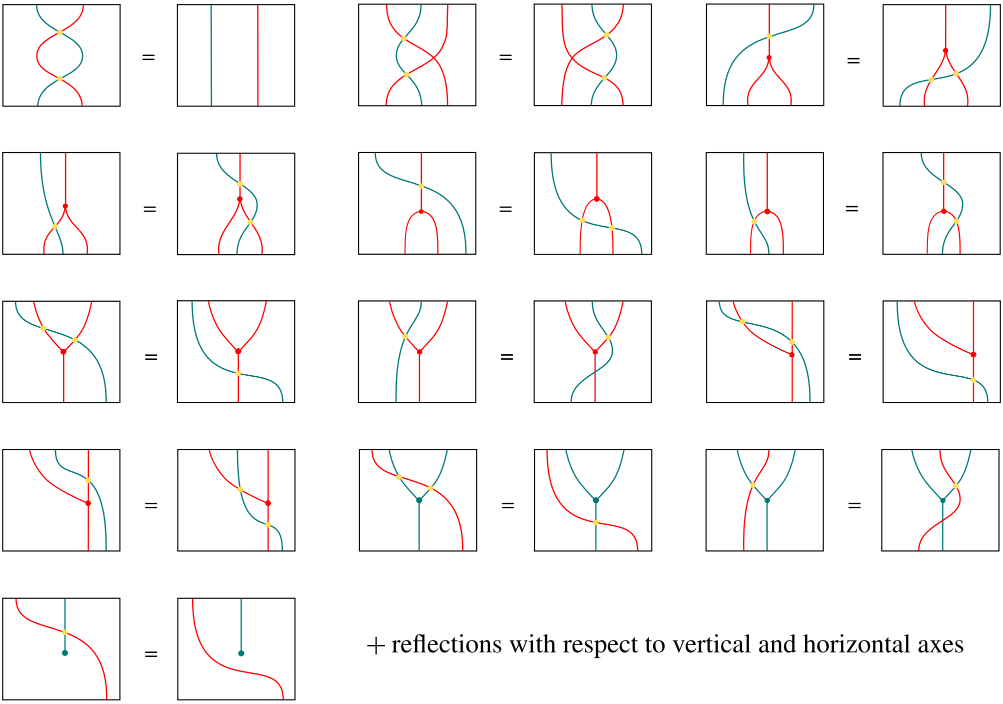


Figure 22: Generating relations $(\mathcal{XR})$ from movie moves of graphics and chambering graphs.

for intersection of fold and cusp graphics follows from the sheet data of cusp graphic. Sheet data of Morse relation, cusp inversion, cusp inversion′, and cusp flip graphics can be interpreted from the corresponding movie moves (see Figure 15). For the details of these sheet data, see [22, Section 1.5.2].

We briefly describe the sheet data of swallowtail-1 graphic shown in Figure 23, where two (blue and green) out of three fold singularities form a double fold crossing. Let $\mathbb{N}_{\leq N} \times U_{\beta_1}$, $\mathbb{N}_{\leq N+2} \times U_{\beta_2}$, and $\mathbb{N}_{\leq N+4} \times U_{\beta_3}$ be trivializations of chambers such that the sheets $\bigcup_{i=N+1}^{N+2} i \times U_{\beta_2}$ and $\bigcup_{j=N+1}^{N+4} j \times U_{\beta_3}$ belong to a swallowtail singularity as shown in Figure 23. Using the sheet data for cusp singularities, restrictions of injections to these sheets give

$$\begin{aligned} \sigma_2(N+1) &= N+3, & \sigma_2(N+2) &= N+4, \\ \sigma_3(N+1) &= N+1, & \sigma_3(N+2) &= N+2, \\ \sigma_5(N+1) &= N+1, & \sigma_5(N+2) &= N+4. \end{aligned}$$

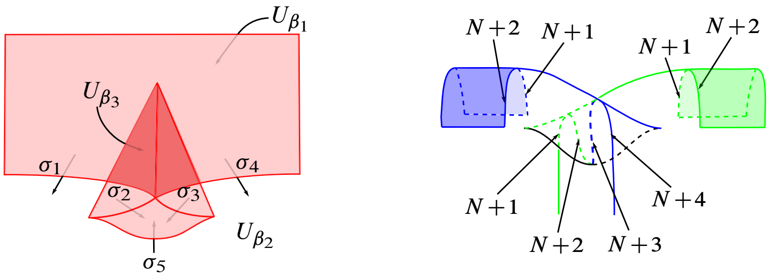


Figure 23: The swallowtail-1 sheet data.

Definition 3.18 [22, Definition 1.49] A *spatial diagram* is a triple $(\Delta, \Gamma, \mathcal{S})$ consisting of a 3–dimensional graphic Δ, a chambering foam Γ for Δ subordinate to a Δ–compatible cover $\mathcal{O} = \{O_\alpha\}_{\alpha \in J}$ of I^3, and a sheet data $\mathcal{S}$ associated to the pair (Δ, Γ).

Any spatial diagram $(\Delta, \Gamma, \mathcal{S})$ produces a compact 3–dimensional manifold with corners M with $\partial M = \Sigma_1 \sqcup \overline{\Sigma}_2$, where Σ_1 and Σ_2 are $\langle 2\rangle$–surfaces. Similar to the previous two sections, in the case of a tuple (Δ^G, Γ), the associated sheet data can be improved to yield a relative homotopy class from M to X and $\langle 2\rangle$–X–surfaces $(\Sigma_1, R_1, \mathtt{P}_1)$ and $(\Sigma_2, R_2, \mathtt{P}_2)$. We call such sheet data carrying X–manifold data to sheets *G–sheet data* and denote it with $\mathcal{S}^G$. Then, generalizing a spatial diagram, we define a *G–spatial diagram* as a triple $(\Delta^G, \Gamma, \mathcal{S}^G)$.

Proposition 3.19 *Let $(\Phi_1^G, \Gamma_1, \mathcal{S}_1^G)$ and $(\Phi_2^G, \Gamma_2, \mathcal{S}_2^G)$ be G–planar diagrams and let $(\Sigma_1, R_1, \mathtt{P}_1)$ and $(\Sigma_2, R_2, \mathtt{P}_2)$ be the constructed cobordism type $\langle 2\rangle$–X–surfaces, respectively. Then $(\Sigma_1, R_1, \mathtt{P}_1)$ is X–homeomorphic to $(\Sigma_2, R_2, \mathtt{P}_2)$ relative to boundary if and only if there exists a G–spatial diagram $(\Delta^G, \Gamma, \mathcal{S}^G)$ which restricts to $(\Phi_1^G, \Gamma_1, \mathcal{S}_1^G)$ and $(\Phi_2^G, \Gamma_2, \mathcal{S}_2^G)$ on the components of $I^2 \times \partial I$.*

Proof $(\Longrightarrow)$ For a given such X–homeomorphism $\mathcal{F}$, we obtain a G–spatial diagram by first taking a generic map on the mapping cylinder of $\mathcal{F}$ and then choosing a compatible chambering foam.

$(\Longleftarrow)$ The properties of the stratification imply that the boundary components of the manifold constructed from the G–spatial diagram are diffeomorphic relative to boundary and the compatibility of G–labels of the arcs on the constructed manifold implies that they are X–homeomorphic. □

We define a relation among G–planar diagrams by $(\Phi_1^G, \Gamma_1, \mathcal{S}_1^G) \sim (\Phi_2^G, \Gamma_2, \mathcal{S}_2^G)$ if there exists a G–spatial diagram $(\Delta^G, \Gamma, \mathcal{S}^G)$ restricting to the given G–planar diagrams on the components of $I^2 \times \partial I$. It is not hard to see that $\sim$ is an equivalence relation. Since generalized movie moves provide (nontrivial) local relations on G–planar diagrams, the equivalence relation $\sim$ can be described using these moves as follows:

Proposition 3.20 *Two G–planar diagrams are equivalent if and only if they can be related by a finite sequence of isotopies or movie moves in Figures 17, 18, 19, 21 and 22.*[3]

Proposition 3.19 implies the following theorem, which is the first main step towards the classification of 2–dimensional extended X–HFTs:

Theorem 3.21 (G–planar decomposition theorem) *The relative X–homeomorphism classes of cobordism type $\langle 2\rangle$–X–surfaces are in bijection with the equivalence classes of G–planar diagrams.*

4 The classification of 2–dimensional extended X–HFTs

4.1 New bicategories arising from diagrams

In this section, we use the G–planar decomposition theorem to introduce symmetric monoidal X–cobordism bicategories with diagrams.

Definition 4.1 An object of an X–cobordism bicategory with diagrams $X\mathrm{Bord}_2^{\mathrm{PD}}$ is a triple $((M, \widehat{M}_1, \widehat{M}_2, \widehat{\mathfrak{g}}_2), \overline{M}, \omega)$, where $(M, \widehat{M}_1, \widehat{M}_2, \widehat{\mathfrak{g}}_2)$ is an object of $X\mathrm{Bord}_2$, $\overline{M}$ is a finite set of ordered oriented points, and $\omega\colon M \to \overline{M}$ is an orientation-preserving bijection.

A 1–morphism is a triple $((A, \widehat{A}_0, \widehat{A}_1, T, \widehat{\mathfrak{p}}_1), L, \nu)$, where $(A, \widehat{A}_0, \widehat{A}_1, T, \widehat{\mathfrak{p}}_1)$ is an X–haloed 1–cobordism, $L = (\Psi^G, \Gamma, \mathcal{S}^G)$ is a G–linear diagram, and $\nu\colon (A, T, p) \to (\overline{A}, \overline{T}, \overline{\mathfrak{p}})$ is an X–homeomorphism over I with $\nu(T) = \overline{T}$, where $(\overline{A}, \overline{T}, \overline{\mathfrak{p}})$ is the pointed 1–cobordism constructed from the G–linear diagram. The composition of two composable triples is componentwise: the composition of 1–morphisms in $X\mathrm{Bord}_2$, the composition of diagrams described below, and the extension of two X–homeomorphisms over I, respectively.

A 2–morphism is a triple $([(S, \widehat{S}, R, \widehat{\mathfrak{F}})], P, \kappa)$, where $[(S, \widehat{S}, R, \widehat{\mathfrak{F}})]$ is an isomorphism class of an X–haloed 2–cobordism, $P = [(\Phi^G, \Gamma, \mathcal{S}^G)]$ is an equivalence class of a

[3] By referencing these figures we mean the list of all generalized movie moves described in this section.

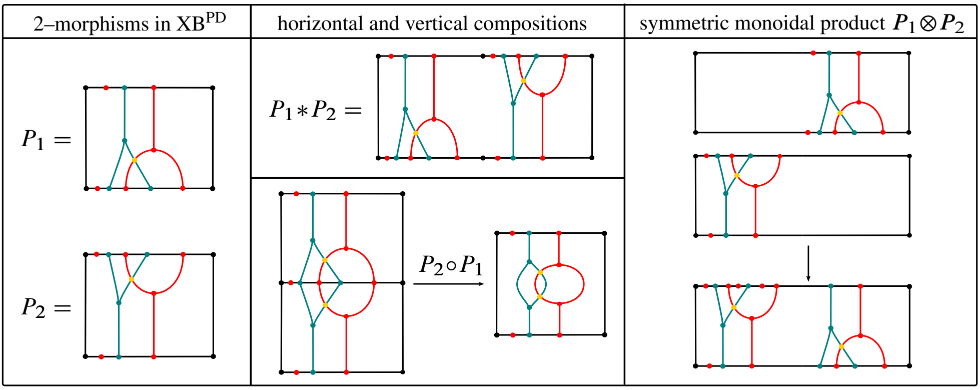


Figure 24: Compositions and symmetric monoidal product of 2–morphisms in $\mathrm{XB}^{\mathrm{PD}}$.

G–planar diagram, and $\kappa\colon (S, R, \mathtt{F}) \to (\bar{S}, \bar{R}, \bar{\mathtt{F}})$ is an X–homeomorphism over I^2, where $(\bar{S}, \bar{R}, \bar{\mathtt{F}})$ is a cobordism type $\langle 2\rangle$–X–manifold constructed from a representative $(\Phi^G, \Gamma, \mathcal{S}^G)$. The composition of two composable triples is componentwise, similar to the composition of 1–morphisms.

The second bicategory $\mathrm{XB}^{\mathrm{PD}}$ is defined by forgetting X–haloed manifolds and cobordisms in $X\mathrm{Bord}_2^{\mathrm{PD}}$ and taking isotopy classes of G–linear diagrams. In order to define isotopic G–linear diagrams, we first need to explain compositions and monoidal products of diagrams.

Horizontal compositions of G–linear and G–planar diagrams are given by the horizontal concatenation of diagrams, where both G–sheet data agree and form new G–sheet data. Vertical composition of equivalence classes of G–planar diagrams is vertical concatenation of diagrams followed by an isomorphism $I \cup_{\mathrm{pt}} I \cong I$ and forgetting the G–linear diagram on the face along which two G–planar diagrams are concatenated. Figure 24 shows an example of horizontal and vertical compositions of 2–morphisms in $\mathrm{XB}^{\mathrm{PD}}$ whose labels are omitted.

A symmetric monoidal structure on $\mathrm{XB}^{\mathrm{PD}}$ is defined as follows. Let $P_1 = (\Phi_1^G, \Gamma_1, \mathcal{S}_1^G)$ and $P_2 = (\Phi_1^G, \Gamma_1, \mathcal{S}_2^G)$ be two G–planar diagrams on $[m, n] \times I$ and on $[a, b] \times I$ for $m, n, a, b \in \mathbb{Z}$, respectively. Let V_{left} be the leftmost chamber of P_1 and V_{right} be the rightmost chamber of P_2. Then $P_1 \otimes P_2$ is defined by stretching V_{left} to the left by $b-a$ units, stretching V_{right} to the right by $n-m$ units, and joining the stretched diagrams (see Figure 24). The G–sheet data and the labels of the resulting diagram are modified accordingly. The symmetric monoidal structure on G–linear diagrams can be deduced from this description. It is not hard to see that the described symmetric monoidal product of diagrams is compatible with the disjoint union of X–haloed manifolds.

Recall that objects of $\mathrm{XB}^{\mathrm{PD}}$ are finite set of ordered oriented points, 1–morphisms are isotopy classes of G–linear diagrams, and 2–morphisms are equivalence classes of G–planar diagrams. The notion of isotopy between G–linear diagrams is generated by the following identifications. Let $L = (\Psi^G, \Gamma, \mathcal{S}^G)$ be any G–linear diagram, $\varnothing$ be the empty G–linear diagram for the empty 1–manifold, and id_a be the identity G–linear diagram of the ordered set a. Then $L = L \otimes \varnothing = \varnothing \otimes L$ and $L = L \circ \mathrm{id}_a = \mathrm{id}_b \circ L$, where $L\colon a \to b$. In this case, it is not hard to see that $\mathrm{XB}^{\mathrm{PD}}$ is a strict 2–category.

Lemma 4.2 *Both $X\mathrm{Bord}_2^{\mathrm{PD}}$ and $\mathrm{XB}^{\mathrm{PD}}$ are symmetric monoidal bicategories under disjoint union of X–haloed manifolds with operation $\otimes$ on diagrams.*

The proof for $\mathrm{XB}^{\mathrm{PD}}$ is very similar to the proof of Lemma 2.6. The case of $X\mathrm{Bord}_2^{\mathrm{PD}}$ follows from the compatibility of symmetric monoidal structures. Considering the G–planar decomposition theorem, a natural question is whether the symmetric monoidal bicategory $\mathrm{XB}^{\mathrm{PD}}$ defined by using diagrams is symmetric monoidally equivalent to X–cobordism bicategory $X\mathrm{Bord}_2$. We give a positive answer using the following theorem:

Theorem 4.3 (Whitehead theorem for symmetric monoidal bicategories [22, Theorem 2.25]) *Let $\mathcal{B}$ and $\mathcal{C}$ be symmetric monoidal bicategories. A symmetric monoidal 2–functor $F\colon \mathcal{B} \to \mathcal{C}$ is a symmetric monoidal equivalence if and only if it is an equivalence of underlying bicategories. That is, F is essentially surjective on objects, essentially full on 1–morphisms, and fully faithful on 2–morphisms.*

Proposition 4.4 *The forgetful 2–functors F and G given by forgetting X–haloed cobordisms and diagrams, respectively,*

$$\mathrm{XB}^{\mathrm{PD}} \xleftarrow[\simeq]{F} X\mathrm{Bord}_2^{\mathrm{PD}} \xrightarrow[\simeq]{G} X\mathrm{Bord}_2,$$

are symmetric monoidal equivalences.

Proof For any given finite set W of ordered oriented points or a compact oriented 0–manifold with cooriented codimension two X–halation $(Y, \widehat{Y}_0, \widehat{Y}_1, \hat{\mathfrak{g}})$, there exist objects in $X\mathrm{Bord}_2^{\mathrm{PD}}$ whose images under F and G are isomorphic to W and $(Y, \widehat{Y}_0, \widehat{Y}_1, \hat{\mathfrak{g}})$, respectively. For any given X–haloed 1–cobordism, there exists a Morse function with distinct critical values leading to a G–linear diagram and any G–linear diagram produces an X–haloed[4] 1–cobordism. Thus, by Proposition 3.5, each 2–functor is (essentially) full on 1–morphisms. Lastly, by the G–planar decomposition theorem, the 2–functors F and G are fully faithful on 2–morphisms. $\square$

Proposition 4.4 implies that the X–cobordism bicategory $X\mathrm{Bord}_2$ is symmetric monoidally equivalent to $\mathrm{XB}^{\mathrm{PD}}$. The advantage of $\mathrm{XB}^{\mathrm{PD}}$ is being a computadic unbiased semistrict symmetric monoidal 2–category. In the appendix, we provide the definition of computadic unbiased semistrict symmetric monoidal 2–category and prove this claim, whose precise statement is given below (see Theorem 4.5). This result is an important step of the classification, which we want to describe here.

The fact that $\mathrm{XB}^{\mathrm{PD}}$ is a computadic symmetric monoidal bicategory roughly means that there exist four sets — namely generating objects $\mathcal{XG}_0$, generating 1–morphisms $\mathcal{XG}_1$, generating 2–morphisms $\mathcal{XG}_2$, and generating relations $\mathcal{XR}$ among 2–morphisms forming the presentation $\mathbb{XP}$ — such that there exists a (canonical) isomorphism of symmetric monoidal bicategories $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP}) \to \mathrm{XB}^{\mathrm{PD}}$, where $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ is constructed from $\mathbb{XP}$. Therefore, we have

$$\mathsf{F}_{\mathrm{uss}}(\mathbb{XP}) \underset{\simeq}{\overset{\exists}{\dashrightarrow}} \mathrm{XB}^{\mathrm{PD}} \underset{\simeq}{\overset{F}{\longleftarrow}} X\mathrm{Bord}_2^{\mathrm{PD}} \underset{\simeq}{\overset{G}{\longrightarrow}} X\mathrm{Bord}_2 \tag{1}$$

and the cofibrancy theorem states that symmetric monoidal 2–functors out of $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ are determined by the images of generating sets subject to the relations. Thus, the classification of 2–dimensional extended X–HFTs up to symmetric monoidal equivalence reduces to understanding images of generators in $\mathbb{XP}$ satisfying relations. The following theorem lists the presentation $\mathbb{XP} = (\mathcal{XG}_0, \mathcal{XG}_1, \mathcal{XG}_2, \mathcal{XR})$ of $\mathrm{XB}^{\mathrm{PD}}$ and its proof is given in Section A.3.

Theorem 4.5 *The symmetric monoidal bicategory* $\mathrm{XB}^{\mathrm{PD}}$ *is a computadic unbiased semistrict symmetric monoidal 2–category with presentation*

$$\mathbb{XP} = (\mathcal{XG}_0, \mathcal{XG}_1, \mathcal{XG}_2, \mathcal{XR})$$

given by the diagram versions[5] *of elements in Figures 25 and 26, and pairs of G–planar diagrams corresponding to equalities in Figure 22, where the labels g_1, g_2, g_3, g_4, g, g' and g'' are indexed over G so that $g_1g_2g_3g_4 = e$.*

[4] Halation can be encoded into G–sheet data by equipping trivializations of chambers with halations.

[5] For generating 2–morphisms, we mean equivalence classes of G–planar diagrams. We consider G–linear and G–planar diagrams whose chambering sets and graphs are trivial, corresponding to covers with single elements.

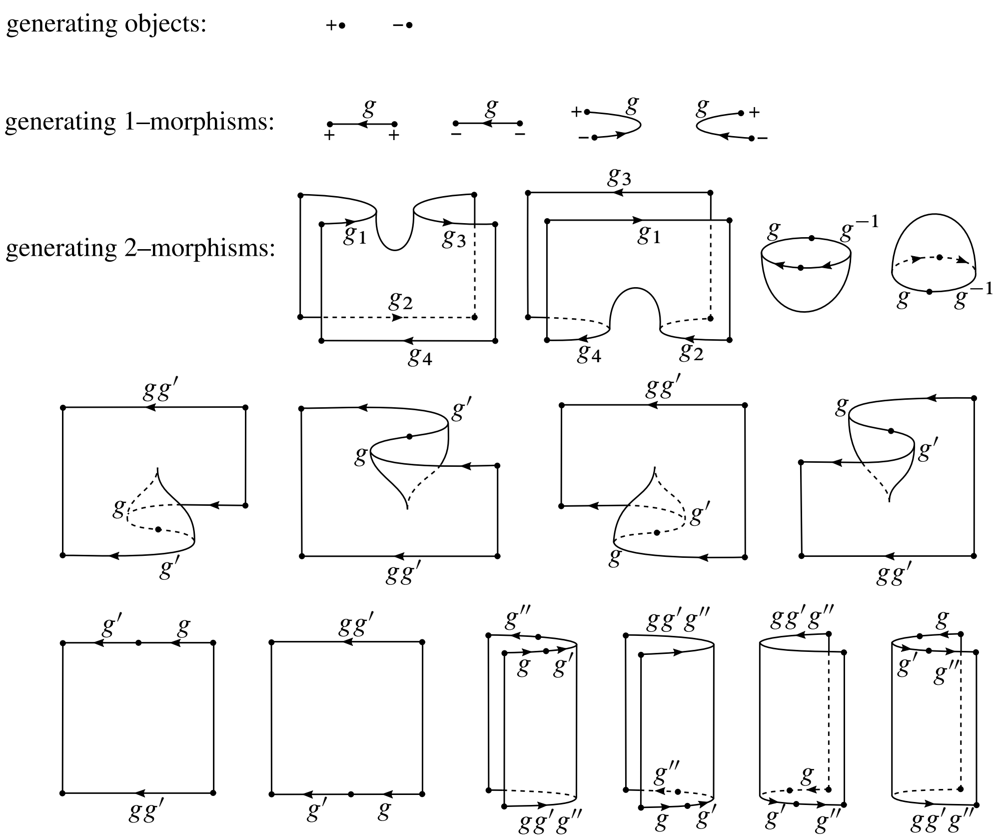


Figure 25: Generating objects ($\mathcal{X}\mathcal{G}_0$), 1–morphisms ($\mathcal{X}\mathcal{G}_1$), and 2–morphisms ($\mathcal{X}\mathcal{G}_2$).

4.2 The cofibrancy theorem

Knowing the equivalence $\mathrm{XB}^{\mathrm{PD}} \simeq X\mathrm{Bord}_2$, the classification of 2–dimensional extended X–HFTs mainly concerns the understanding of symmetric monoidal 2–functors defined on $\mathrm{XB}^{\mathrm{PD}}$. The cofibrancy theorem [22] is a coherence theorem for such 2–functors. More precisely, this theorem allows replacing (weak) symmetric monoidal 2–functors defined on computadic symmetric monoidal bicategories with their strict versions naturally. Furthermore, such strict 2–functors are determined by the images of generating sets of a presentation subject to the relations.

The cofibrancy theorem holds for any computadic monoidal bicategory (see [22; 21]) and specifically for stricter versions of symmetric monoidal bicategories. In this section, we only focus on its version for computadic unbiased semistrict symmetric monoidal 2–categories, which is the key step of the classification. In the following, we denote

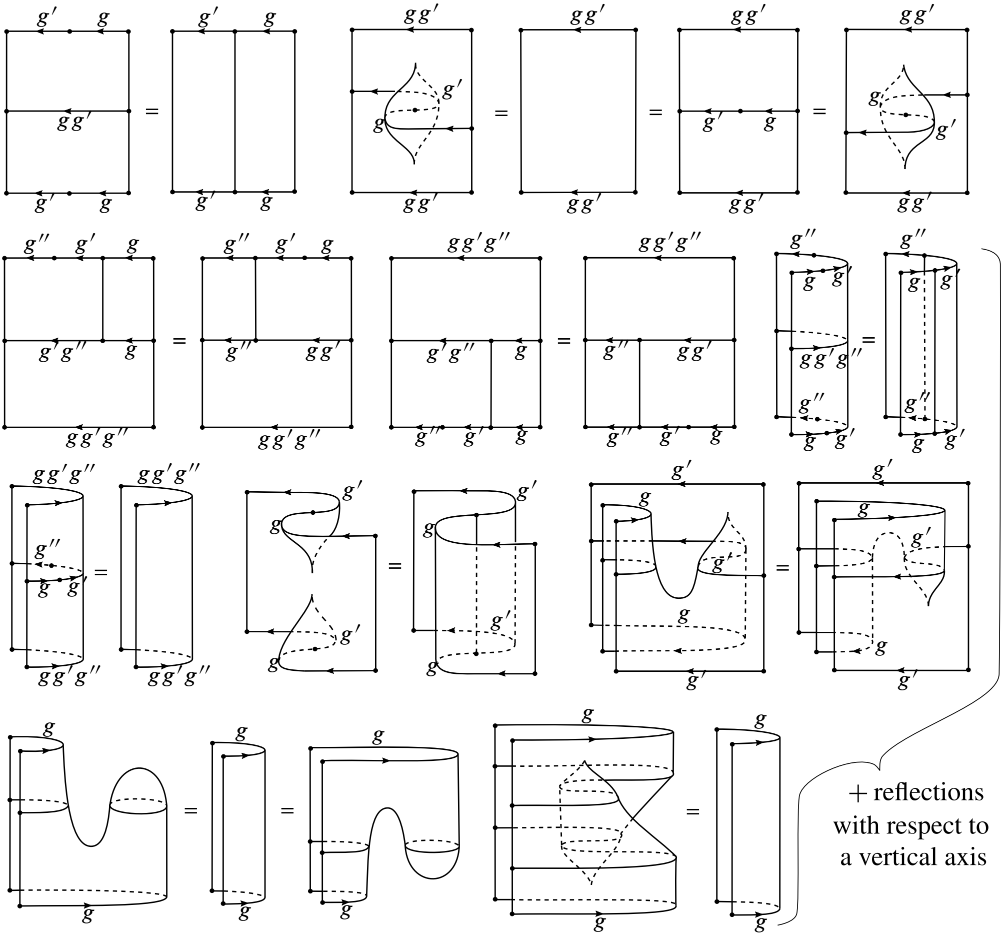


Figure 26: Generating relations ($\mathcal{XR}$) among 2–morphisms.

the collection of objects of a symmetric monoidal bicategory $\mathcal{C}$ by $\mathcal{C}_0$, 1–morphisms by $\mathcal{C}_1$, and 2–morphisms by $\mathcal{C}_2$.

Definition 4.6 Let $(\mathbb{P} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R}), s, t)$ be an unbiased semistrict symmetric presentation and $\mathcal{C}$ be a symmetric monoidal bicategory. *The bicategory* $\mathbb{P}(\mathcal{C})$ *of* $\mathbb{P}$*–data in* $\mathcal{C}$ is defined as follows:

- The objects of $\mathbb{P}(\mathcal{C})$ are triples of assignments (A_0, A_1, A_2) such that $A_i : \mathcal{G}_i \to \mathcal{C}_i$ for $i = 0, 1, 2$ satisfying the following conditions. These assignments extend canonically to $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$, $\mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)/\sim$, and $\mathrm{PG}^{\mathrm{uss}}(\mathcal{G}_2)/\sim$ using the monoidal product and braiding of $\mathcal{C}$ as follows. For any $x = a_1 \dots a_n \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$, we have $A_0(x) = A_0(a_1) \otimes \cdots \otimes A_0(a_n)$, $\mathrm{id}_x = \mathrm{id}_{A_0(x)}$ and $\beta^{\sigma}_{a,\sigma(a)}$ for $\sigma \in S_n$ is given by writing σ as a product of adjacent transpositions first and then applying the braiding of $\mathcal{C}$ to each of them. Composition and monoidal product of 1–morphisms follow similarly by extending A_1 to $\mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)/\sim$. In the extension to $\mathrm{PG}^{\mathrm{uss}}(\mathcal{G}_2)$ interchanger 2–morphisms ϕ are taken as the identity and the extension to the remaining 2–morphisms uses the naturality of the braiding of $\mathcal{C}$ and the corresponding modifications. These extensions are required to be globular, ie $A_0 \circ p = p \circ A_1$ and $A_1 \circ p = p \circ A_2$, where p is used for the source s and target t maps. Among all extensions, only those with $A_2(x) = A_2(y)$ for all $(x, y) \in \mathcal{R}$ are considered.

- The 1–morphisms from $A = (A_0, A_1, A_2)$ to $B = (B_0, B_1, B_2)$ are pairs of assignments (α_0, α_1) such that $\alpha_i : \mathcal{G}_i \to \mathcal{C}_{i+1}$ for $i = 0, 1$, $s(\alpha_0(a)) = A_0(a)$, $t(\alpha_0(b)) = B_0(b)$ and $\alpha_1(f)\colon B_1(f) \circ \alpha_0(a) \xrightarrow{\cong} \alpha_0(b) \circ A_1(f)$ for all $f : a \to b \in \mathcal{G}_1$. These assignments extend canonically to $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ using the monoidal product of $\mathcal{C}$ and to $\mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)/\sim$ using induction on the number of monoidal products and compositions as follows. For $x = a_1 \dots a_n \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$, $\alpha_1(\mathrm{id}_x)$ is the composition

$$B_1(\mathrm{id}_x) \circ \alpha_0(x) = \mathrm{id}_{B_0(x)} \circ \alpha_0(x) \xrightarrow{\ell^{\mathcal{C}}} \alpha_0(x) \xrightarrow{(r^{\mathcal{C}})^{-1}} \alpha_0(x) \circ \mathrm{id}_{A_0(x)} = \alpha_0(x) \circ A_1(\mathrm{id}_x),$$

where ℓ and r are the left and right unitors of the underlying bicategory of $\mathcal{C}$. For $x = a_1 \dots a_n \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ and $\sigma \in S_n$, the 2–morphism $\alpha_1(\beta^{\sigma}_{x,\sigma(x)})$ is given by the components of the braiding (equivalence transformation) of $\mathcal{C}$ on the 1–morphism $\alpha_0(x)$. Next, for elements $f : a \to b$ and $f : a' \to b'$ in $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$, $\alpha_1(f \otimes f')$ is defined as the composition

$$\begin{aligned}(B_1(f) \otimes B_1(f')) \circ (\alpha_0(a) \otimes \alpha_0(a')) &\to (B_1(f) \circ \alpha_0(a)) \otimes (B_1(f') \circ \alpha_0(a'))\\ &\xrightarrow{\alpha_1(f) \otimes \alpha_1(f')} (\alpha_0(b) \circ A_1(f)) \otimes (\alpha_0(b') \circ A_1(f'))\\ &\to (\alpha_0(b) \otimes \alpha_0(b')) \circ (A_1(f) \otimes A_1(f')).\end{aligned}$$

For elements $f : a \to b$ and $g : b \to c$ in $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$, $\alpha_1(g \circ f)$ is defined as the composition

$$\begin{aligned}(B_1(g) \circ B_1(f)) \circ \alpha_0(a) &\xrightarrow{\dot{a}^{\mathcal{C}}} B_1(g) \circ (B_1(f) \circ \alpha_0(a)) \xrightarrow{\mathrm{id} * \alpha_1(f)} B_1(g) \circ (\alpha_0(b) \circ A_1(f))\\ &\xrightarrow{(\dot{a}^{\mathcal{C}})^{-1}} (B_1(g) \circ \alpha_0(b)) \circ A_1(f) \xrightarrow{\alpha_1(g) * \mathrm{id}} (\alpha_0(c) \circ A_1(g)) \circ A_1(f)\\ &\xrightarrow{\dot{a}^{\mathcal{C}}} \alpha_0(c) \circ (A_1(g) \circ A_1(f)),\end{aligned}$$

where $\dot{a}^{\mathcal{C}}$ is the associator of the underlying bicategory of $\mathcal{C}$. These assignments are also required to be natural with respect to equivalence classes of paragraphs. That is, for any $[\beta] \in \mathrm{PG}^{\mathrm{uss}}(\mathcal{G}_2)/\sim$ with $\beta\colon f \to g$, we have $(\mathrm{id}_{\alpha_0(b)} * A_2(\beta)) \circ \alpha_1(f) = \alpha_1(g) \circ (B_2(\beta) * \mathrm{id}_{\alpha_0(a)})$; equivalently,

$$\begin{tikzcd}
A_0(a) \arrow[r, bend left, "A_1(g)"] \arrow[r, bend right, "A_1(f)"'] \arrow[r, phantom, "\Uparrow A_2(\beta)"] \arrow[d, "\alpha_0(a)"'] \arrow[dr, phantom, "\Nearrow \alpha_1(f)"] & A_0(b) \arrow[d, "\alpha_0(b)"] \\
B_0(a) \arrow[r, "B_1(f)"'] & B_0(b)
\end{tikzcd}
=
\begin{tikzcd}
A_0(a) \arrow[r, "A_1(g)"] \arrow[d, "\alpha_0(a)"'] \arrow[dr, phantom, "\Nearrow \alpha_1(g)"] & A_0(b) \arrow[d, "\alpha_0(b)"] \\
B_0(a) \arrow[r, bend left, "B_1(g)"] \arrow[r, bend right, "B_1(f)"'] \arrow[r, phantom, "\Uparrow B_2(\beta)"] & B_0(b)
\end{tikzcd}$$

- The 2–morphisms from $\alpha = (\alpha_0, \alpha_1)$ to $\beta = (\beta_0, \beta_1)$ are assignments $\theta_0\colon \mathcal{G}_0 \to \mathcal{C}_2$ such that $\theta_0(a)\colon \alpha_0(a) \to \beta_0(a)$ for all $a \in \mathcal{G}_0$, where $\alpha, \beta\colon A \to B$ for $A = (A_0, A_1, A_2)$ and $B = (B_0, B_1, B_2)$. These assignments extend canonically to $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ using the monoidal product of $\mathcal{C}$ and they are required to be natural with respect to $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$. That is, for any $f\colon a \to b \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$, we have $\beta_1(f) \circ (\theta_0(a) * \mathrm{id}_{B_1(f)}) = (\mathrm{id}_{A_1(f)} * \theta_0(b)) \circ \alpha_1(f)$; equivalently,

$$\begin{tikzcd}
A_0(a) \arrow[r, "A_1(f)"] \arrow[d, bend right=50, "\alpha_0(a)"'] \arrow[d, bend left=50, "\beta_0(a)"] \arrow[d, phantom, "\xRightarrow{\theta_0(a)}"] \arrow[dr, phantom, "\Nearrow \beta_1(f)"] & A_0(b) \arrow[d, "\beta_0(b)"] \\
B_0(a) \arrow[r, "B_1(f)"'] & B_0(b)
\end{tikzcd}
=
\begin{tikzcd}
A_0(a) \arrow[r, "A_1(f)"] \arrow[d, "\alpha_0(a)"'] \arrow[dr, phantom, "\Nearrow \alpha_1(f)"] & A_0(b) \arrow[d, bend right=50, "\alpha_0(b)"'] \arrow[d, bend left=50, "\beta_0(b)"] \arrow[d, phantom, "\xRightarrow{\theta_0(b)}"] \\
B_0(a) \arrow[r, "B_1(f)"'] & B_0(b)
\end{tikzcd}$$

With the trivial coherence data $(\chi, \iota, \mathcal{W}, \mathcal{G}, \mathcal{R}, \mathcal{U})$, any object of $\mathbb{P}(\mathcal{C})$ considered with the extensions gives rise to a strict symmetric monoidal 2–functor $\mathsf{F}_{\mathrm{uss}}(\mathbb{P}) \to \mathcal{C}$ (see [22; 25]). Similarly, any 1–morphism and 2–morphism of $\mathbb{P}(\mathcal{C})$ considered with their extensions yield a strict[6] symmetric monoidal transformation and a symmetric monoidal modification, respectively (see [22, Section 2.3]).

Definition 4.7 Let $\mathcal{C}$ be a symmetric monoidal bicategory and $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ be a computadic unbiased semistrict symmetric monoidal 2–category for a given presentation

[6] Those monoidal transformations whose monoidal structure 1– and 2–morphisms are identities.

$\mathbb{P} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R})$. The bicategory $\mathrm{SymMon}(\mathsf{F}_{\mathrm{uss}}(\mathbb{P}), \mathcal{C})$ has symmetric monoidal 2–functors as objects, symmetric monoidal transformations as 1–morphisms, and symmetric monoidal modifications as 2–morphisms.

By construction of $\mathbb{P}(\mathcal{C})$, we have a strict inclusion functor

$$\imath : \mathbb{P}(\mathcal{C}) \hookrightarrow \mathrm{SymMon}(\mathsf{F}_{\mathrm{uss}}(\mathbb{P}), \mathcal{C})$$

given by the associated 2–functors, transformations, and modifications. The cofibrancy theorem below states that $\imath$ is an equivalence of bicategories:

Theorem 4.8 (cofibrancy theorem [22, Theorem 2.78]) *Let $\mathcal{C}$ be a symmetric monoidal bicategory and let $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ be a computadic unbiased semistrict symmetric monoidal 2–category constructed from an unbiased semistrict presentation $\mathbb{P} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R})$. Then the inclusion $\imath : \mathbb{P}(\mathcal{C}) \hookrightarrow \mathrm{SymMon}(\mathsf{F}_{\mathrm{uss}}(\mathbb{P}), \mathcal{C})$ is an equivalence of bicategories.*

The following lemma, taken from [22, Lemma 2.15], implies that, for any symmetric monoidal bicategory $\mathcal{C}$, the bicategories $\mathrm{SymMon}(X\mathrm{Bord}_2, \mathcal{C})$ and $\mathrm{SymMon}(\mathrm{XB}^{\mathrm{PD}}, \mathcal{C})$ of symmetric monoidal 2–functors, symmetric monoidal transformations, and modifications are equivalent:

Lemma 4.9 [22] *Let $\mathcal{M}$ and $\mathcal{M}'$ be symmetric monoidal bicategories, and $H : \mathcal{M} \to \mathcal{M}'$ be a symmetric monoidal 2–functor which is an equivalence. Then there is a canonical equivalence $H^* : \mathrm{SymMon}(M', B) \to \mathrm{SymMon}(M, B)$ of bicategories given by composition on the level of objects, and by symmetric monoidal whiskering on the level of 1– and 2–morphisms.*

We denote the bicategory $\mathrm{SymMon}(X\mathrm{Bord}_2, \mathcal{C})$ by $\mathcal{E}\text{–}\mathcal{HFT}(X, \mathcal{C})$ and state the classification of 2–dimensional extended HFTs with target $X \simeq K(G, 1)$ as follows:

Theorem 4.10 *Let $\mathbb{XP}$ be the presentation of $\mathrm{XB}^{\mathrm{PD}}$ given in Theorem 4.5. Then, for any symmetric monoidal bicategory $\mathcal{C}$, there is an equivalence of bicategories $\mathcal{E}\text{–}\mathcal{HFT}(X, \mathcal{C}) \simeq \mathbb{XP}(\mathcal{C})$.*

Proof Theorem 4.5 gives a presentation $\mathbb{XP}$ of $\mathrm{XB}^{\mathrm{PD}}$ as a computadic unbiased semistrict symmetric monoidal 2–category. By the cofibrancy theorem, we have $\mathrm{SymMon}(\mathrm{XB}^{\mathrm{PD}}, \mathcal{C}) \simeq \mathbb{XP}(\mathcal{C})$. Using the symmetric monoidal equivalence between $\mathrm{XB}^{\mathrm{PD}}$ and $X\mathrm{Bord}_2$ in Proposition 4.4 and Lemma 4.9, we obtain the result. □

4.3 $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended X–HFTs

Every 2–dimensional extended X–HFT gives a nonextended one by restricting to oriented X–circles and X–cobordisms between them. A natural question is how the classification of 2–dimensional extended HFTs is related to Turaev's classification of 2–dimensional HFTs by crossed Frobenius G–algebras [27]. To understand this relation, we study extended HFTs taking values in $\mathrm{Alg}^2_{\Bbbk}$, which has $\Bbbk$–algebras as objects, bimodules as 1–morphisms, and bimodule maps as 2–morphisms for a commutative ring $\Bbbk$ with unity.

The symmetric monoidal structure of $\mathrm{Alg}^2_{\Bbbk}$ is given by tensoring over $\Bbbk$. We denote (E,C)–bimodule D by ${}_E D_C$ and omit the symbol $\Bbbk$ when either C or E is $\Bbbk$. We regard ${}_E D_C$ as a 1–morphism from C to E, which is in line with the composition in $X\mathrm{Bord}_2$ (see Figure 3). Composition of 1–morphisms ${}_E D_C$ and ${}_C B_A$ is the bimodule ${}_E(D \otimes_C B)_A$.

Before studying $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended X–HFTs, we recall necessary algebraic notions and introduce quasibiangular G–algebras. Recall that a G–algebra over a commutative ring $\Bbbk$ is an associative $\Bbbk$–algebra K equipped with a decomposition $K = \bigoplus_{g\in G} K_g$ such that $K_g K_h \subseteq K_{gh}$ for any $g,h \in G$. In this case, K_e is the *principal component* and K is called *strongly graded* if $K_g K_h = K_{gh}$ or, equivalently, the natural map $K_g \otimes_{K_e} K_h \to K_{gh}$ is an isomorphism for all $g,h \in G$. The opposite G–algebra of K is defined as $K^{\mathrm{op}} = \bigoplus_{g\in G} K_{g^{-1}}$, where the order of multiplication is reversed.

Definition 4.11 [28] Let $K = \bigoplus_{g\in G} K_g$ be a G–algebra over a commutative ring $\Bbbk$. Recall that an *inner product* on K is a symmetric bilinear form $\eta\colon K \otimes K \to \Bbbk$ satisfying $\eta(ab,c) = \eta(a,bc)$ for any $a,b,c \in K$ such that $\eta|_{K_g \otimes K_h}$ is nondegenerate when $gh = e$ and zero otherwise. A *Frobenius G–algebra* is a G–algebra K with an inner product η and components of K are finitely generated projective $\Bbbk$–modules.

Let $\bigl(K = \bigoplus_{g\in G} K_g, \eta\bigr)$ be a Frobenius G–algebra over $\Bbbk$. Each nondegenerate form $\eta|_{K_g \otimes K_{g^{-1}}}$ yields an element $\eta_g^- = \sum_{i\in I_g} p_i^g \otimes q_i^g \in K_g \otimes K_{g^{-1}}$, called an *inner product element*, where I_g is finite and η_g^- is characterized by $a = \sum_{i\in I_g} \eta(a, q_i^g) p_i^g$ for any $a \in K_g$. Since η is symmetric, we have $\sum_i p_i^{g^{-1}} \otimes q_i^{g^{-1}} = \sum_i q_i^g \otimes p_i^g$ for all $g \in G$.

Recall that an associative $\Bbbk$–algebra A is *separable* if there exists an element $a = \sum_{i=1}^n p_i \otimes q_i \in A \otimes_{\Bbbk} A^{\mathrm{op}}$, called *separability idempotent*, such that $\sum_{i=1}^n p_i q_i = 1$

and $ab = ba$ for all $b \in A$. A separable algebra A is called *strongly separable* if the separability idempotent is symmetric, ie $a = \sum_{i=1}^{n} p_i \otimes q_i = \sum_{i=1}^{n} q_i \otimes p_i$.

Lemma 4.12 *Let $\left(K = \bigoplus_{g\in G} K_g, \eta\right)$ be a Frobenius G–algebra with inner product elements $\left\{\eta_g^- = \sum_i p_i^g \otimes q_i^g\right\}_{g\in G}$ and $z \in K_e$. Then, for any $g, h \in G$ and $b \in K_{g^{-1}}$, we have*

$$\sum_i p_i^h \otimes z q_i^h b = \sum_j b p_j^{gh} \otimes z q_j^{gh}. \tag{2}$$

In particular, for any $b \in K$ and $c \in K_{h^{-1}}$, we have $\sum_j p_j^g b z q_j^g c = \sum_k c p_k^{hg} b z q_k^{hg}$.

Proof Since both sides belong to $K_h \otimes K_{h^{-1}g^{-1}}$, it is enough to check that they give the same functionals on the dual $\Bbbk$–module $K_{h^{-1}} \otimes K_{gh}$. For any $x \in K_{h^{-1}}$ and $y \in K_{gh}$, applying $x \otimes y$ to the left-hand side of (2) and using the cyclic symmetry property of η, we obtain

$$\begin{aligned}\sum_i \eta(p_i^h, x)\eta(z q_i^h b, y) = \sum_i \eta(x, p_i^h)\eta(q_i^h, byz) = \eta\Big(x, \sum_i \eta(byz, q_i^h) p_i^h\Big)\\ = \eta(x, byz).\end{aligned}$$

Similarly, applying $x \otimes y$ to the right-hand side of (2), we have

$$\begin{aligned}\sum_j \eta(b p_j^{gh}, x)\eta(z q_j^{gh}, y) = \sum_j \eta(xb, p_j^{gh})\eta(q_j^{gh}, yz) = \eta\Big(xb, \sum_j \eta(yz, q_j^{gh}) p_j^{gh}\Big)\\ = \eta(xb, yz).\end{aligned}$$

□

We generalize biangular G–algebras, which were introduced by Turaev [27], as follows:

Definition 4.13 A strongly graded Frobenius G–algebra (K, η) is called *quasibiangular* if there exists a central element $z \in K_e$, ie $za = az$ for all $a \in K_e$, such that, for the collection of inner product elements $\left\{\sum_i p_i^g \otimes q_i^g\right\}_{g\in G}$, the equations $\sum_i p_i^g z q_i^g = 1$ hold for all $g \in G$.

Remark By Lemma 4.12, the principal component of a quasibiangular G–algebra is a separable algebra with separability idempotent $\sum_i p_i^e \otimes z q_i^e$. A biangular G–algebra is a quasibiangular G–algebra with $z = 1$. Similarly, the principal component of a biangular G–algebra is strongly separable.

One way of studying an algebra is to study the category of modules over it. Recall that Morita equivalence of algebras is the equivalence of categories of modules. In the case of a graded algebra, one studies the category of graded modules. An equivalence of such categories is called a graded Morita equivalence (see [3, Theorem 3.2]), which was introduced by Boisen [3] as follows:

Definition 4.14 [3] A *G–graded Morita equivalence* ζ between G–algebras $K = \bigoplus_{g\in G} K_g$ and $L = \bigoplus_{g\in G} L_g$ is a quadruple $({}_LU_K, {}_KV_L, \tau, \mu)$, where ${}_LU_K = \bigoplus_{g\in G} U_g$ is a graded (L,K)–bimodule—that is, $L_g U_h K_{g'} \subset U_{ghg'}$—${}_KV_L = \bigoplus_{g\in G} V_g$ is a graded (K,L)–bimodule, and $\tau\colon {}_KK_K \to {}_KV \otimes_L U_K$ and $\mu\colon {}_LU \otimes_K V_L \to {}_LL_L$ are graded (K,K)– and (L,L)–bimodule maps, respectively, such that the compositions

$$\begin{aligned}{}_LU_K \to {}_LU \otimes_K K_K \xrightarrow{\mathrm{id}\otimes\tau} {}_LU \otimes_K (V \otimes_L U_K) \to ({}_LU \otimes_K V) \otimes_L U_K\\ \xrightarrow{\mu\otimes\mathrm{id}} {}_LL \otimes_L U_K \to {}_LU_K,\end{aligned}$$

$$\begin{aligned}{}_KV_L \to {}_KK \otimes_K V_L \xrightarrow{\tau\otimes\mathrm{id}} ({}_KV \otimes_L U) \otimes_K V_L \to {}_KV \otimes_L (U \otimes_K V_L)\\ \xrightarrow{\mathrm{id}\otimes\mu} {}_KV \otimes_L L_L \to {}_KV_L\end{aligned}$$

are id_U and id_V, respectively. When τ and μ are invertible as G–graded bimodule maps, ζ is called a *G–graded Morita context*.

Definition 4.15 Let $\zeta = ({}_LU_K, {}_KV_L, \tau, \mu)$ and $\zeta' = ({}_LU'_K, {}_KV'_L, \tau', \mu')$ be two G–graded Morita equivalences. An *equivalence* of G–graded Morita equivalences ζ and ζ' is a pair of G–graded bimodule maps $\xi\colon {}_LU_K \to {}_LU'_K$ and $\rho\colon {}_KV_L \to {}_KV'_L$ such that $\mu = \mu' \circ (\xi \otimes \rho)$ and $\tau' = (\rho \otimes \xi) \circ \tau$.

Lemma 4.16 [9] *Assume that G–algebras $K = \bigoplus_{g\in G} K_g$ and $L = \bigoplus_{g\in G} L_g$ are G–graded Morita equivalent. Then, if K is strongly graded, then L is also strongly graded.*

Next, we transfer the inner product of one Frobenius G–algebra to another using a graded Morita context between them. As the first step we recall the following lemma:

Lemma 4.17 [22; 10] *Any Morita context $\zeta = ({}_LU_K, {}_KV_L, \tau, \eta)$ between $\Bbbk$–algebras K and L induces a canonical isomorphism of $\Bbbk$–modules $\zeta_*\colon K/[K,K] \to L/[L,L]$.*

An explicit formula for the isomorphism ζ_* in the lemma is provided in [10]. The inner product η of a Frobenius G–algebra (K,η) is determined at its principal component by $\eta(a, b\cdot 1) = \eta(ab, 1)$. This allows us to denote (K,η) by (K,Λ), where $\Lambda\colon K_e \to \Bbbk$ is a nondegenerate trace. Since η is symmetric, Λ factors through $K_e/[K_e,K_e]$.

Lemma 4.12 implies that, for a symmetric Frobenius algebra (K_e, Λ_e), an inner product element $\sum_i p_i^e \otimes q_i^e$ can be considered as the image of $1 \otimes 1$ under a bimodule map $\xi\colon {}_{K_e^1}(K_e)_{K_e^2} \otimes {}_{K_e^3}(K_e)_{K_e^4} \to {}_{K_e^1}(K_e)_{K_e^4} \otimes {}_{K_e^3}(K_e)_{K_e^2}$, where numbers indicate module actions, ie $K_e^i = K_e$ for $i = 1, 2, 3, 4$. In the case of a quasibiangular G–algebra $\bigl(K = \bigoplus_{g \in G} K_g, \Lambda\bigr)$, inner product elements $\bigl\{\sum_i p_i^g \otimes q_i^g\bigr\}_{g \in G \setminus \{e\}}$ are the images of $1 \otimes 1$ under the composition

$$
\begin{aligned}
(3)\quad {}_{K_e^1}(K_e)_{K_e^2} \otimes {}_{K_e^3}(K_e)_{K_e^4} &\to {}_{K_e^1}(K_e)_{K_e^2} \otimes {}_{K_e^3}(K_{g^{-1}} \otimes_{K_e} K_e \otimes_{K_e} K_g)_{K_e^4} \\
&\to {}_{K_e^1}(K_e \otimes_{K_e} K_g)_{K_e^4} \otimes {}_{K_e^3}(K_{g^{-1}} \otimes_{K_e} K_e)_{K_e^2} \to {}_{K_e^1}(K_g)_{K_e^4} \otimes {}_{K_e^3}(K_{g^{-1}})_{K_e^2},
\end{aligned}
$$

where the second homomorphism is the identity on $K_{g^{-1}}$ and K_g, and ξ on $K_e \otimes K_e$. In the following, we consider inner product elements as the images of $1 \otimes 1$ under the above bimodule maps.

Definition 4.18 Let (K, Λ_K) and (L, Λ_L) be quasibiangular G–algebras over $\Bbbk$ with collections of inner product elements $\{\eta_g^K\}_{g \in G}$ and $\{\eta_g^L\}_{g \in G}$, respectively. A G–graded Morita context $\zeta = ({}_L U_K, {}_K V_L, \tau, \mu)$ between K and L is said to be *compatible* if $\Lambda_L = (\zeta_{\{e\}})_* \Lambda_K$ and $\eta_g^L = (\zeta_{\{e\}})_*(\eta_g^K)$ for all $g \in G$, where $(\zeta_{\{e\}})_*(\eta_g^K)$ consists of inner product elements for $(L, (\zeta_e)_* \Lambda_K)$ given by $\xi'(1 \otimes 1)$ under the commutative diagram

$$
\begin{tikzcd}
{}_{L_e}(U_e \otimes K_e \otimes V_e)_{L_e} \otimes_{\Bbbk} {}_{L_e}(U_e \otimes K_e \otimes V_e)_{L_e} \arrow[r, "\mu_{\{e\}}"] \arrow[d, "\mathrm{id} \otimes \xi \otimes \mathrm{id}"'] & {}_{L_e}(L_e)_{L_e} \otimes_{\Bbbk} {}_{L_e}(L_e)_{L_e} \arrow[d, "\xi'"] \\
{}_{L_e}(U_e \otimes K_e \otimes V_e)_{L_e} \otimes_{\Bbbk} {}_{L_e}(U_e \otimes K_e \otimes V_e)_{L_e} \arrow[r, "\mu_{\{e\}}"'] & {}_{L_e}(L_e)_{L_e} \otimes_{\Bbbk} {}_{L_e}(L_e)_{L_e}
\end{tikzcd}
$$

The remaining inner product elements are obtained from ξ' as described above.

Theorem 4.19 *Any* $\mathrm{Alg}_{\Bbbk}^2$*–valued* 2*–dimensional extended* X*–HFT* $Z\colon X\mathrm{Bord}_2 \to \mathrm{Alg}_{\Bbbk}^2$ *whose precomposition* $\mathrm{XB}^{\mathrm{PD}} \xrightarrow{\simeq} X\mathrm{Bord}_2 \xrightarrow{Z} \mathrm{Alg}_{\Bbbk}^2$ *gives a strict symmetric monoidal* 2*–functor determines a triple* (A, B, ζ), *where* A *and* B *are quasibiangular* G*–algebras, and* ζ *is a compatible* G*–graded Morita context between* A *and* B^{op}. *Conversely, for any such triple* (A, B, ζ), *there exists an* $\mathrm{Alg}_{\Bbbk}^2$*–valued* 2*–dimensional extended* X*–HFT.*

Proof Let $Z\colon X\mathrm{Bord}_2 \to \mathrm{Alg}_{\Bbbk}^2$ be such a 2–dimensional extended HFT. The cofibrancy theorem implies that there exists an object Z' in $\mathbb{XP}(\mathrm{Alg}_{\Bbbk}^2)$ such that $\iota(Z')$ is the composition $\mathrm{XB}^{\mathrm{PD}} \xrightarrow{\simeq} X\mathrm{Bord}_2 \xrightarrow{Z} \mathrm{Alg}_{\Bbbk}^2$, where $\iota\colon \mathbb{XP}(\mathrm{Alg}_{\Bbbk}^2) \to \mathrm{SymMon}(\mathrm{XB}^{\mathrm{PD}}, \mathrm{Alg}_{\Bbbk}^2)$ is the equivalence of bicategories.

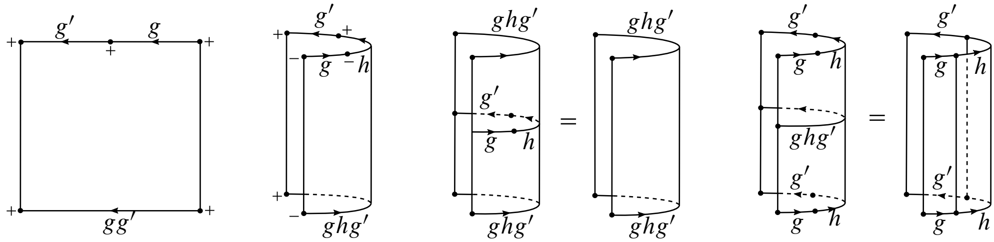


Figure 27: Part of generators and relations giving G–algebra and G–graded module.

We have $\Bbbk$–algebras $Z'(\bullet^+) = A_e$ and $Z'(\bullet^-) = B_e$ in $Z'_0(\mathcal{X}\mathcal{G}_0)$, corresponding to two generating objects of $\mathbb{XP}$. There are four types of generating 1–morphisms and each is indexed by the elements of G. For every $g \in G$, they give the bimodules, in $Z'_1(\mathcal{X}\mathcal{G}_1)$,

$$\begin{aligned} Z'(+\!\overset{g}{\longleftarrow}\!+) &= A_g && (A_e, A_e)\text{–bimodule,} \\ Z'(-\!\overset{g}{\longleftarrow}\!-) &= B_g && (B_e, B_e)\text{–bimodule,} \\ Z'(\tfrac{+}{-}\!\!\rhd g) &= M_g && (B_e \otimes_{\Bbbk} A_e, \Bbbk)\text{–bimodule,} \\ Z'(g\!\lhd\!\tfrac{-}{+}) &= N_g && (\Bbbk, A_e \otimes_{\Bbbk} B_e)\text{–bimodule.} \end{aligned}$$

The first 2–morphism in Figure 27 defines a G–graded product on the bimodule $\bigoplus_{g\in G} A_g$. Associativity of this product is the obvious relation in Figure 26. Denote the corresponding G–algebra by $A = \bigoplus_{g\in G} A_g$. The first relation in Figure 26 shows that the bimodule map

$$(4) \qquad {}_{A_e}(A_{g'}) \otimes_{A_e} (A_g)_{A_e} \xrightarrow{\cong} {}_{A_e}(A_{gg'})_{A_e}$$

is invertible for all $g, g' \in G$. Since multiplication in the G–algebra A is defined using (4), we have $A_g A_{g'} = A_{gg'}$ for all $g, g' \in G$, ie A is strongly graded. Similar arguments for (B_e, B_e)–bimodules $\{B_g\}_{g\in G}$ yield another strongly graded G–algebra $B = \bigoplus_{g\in G} B_g$.

Using the opposite algebra, we can turn algebra actions on bimodules around. More precisely, a left B_e–action on ${}_{A_e\otimes B_e}M_g$ can be turned into a right B_e^{op}–action and the right B_e–action on $(N_g)_{A_e\otimes B_e}$ can be turned into a left B_e^{op}–action. The second 2–morphism in Figure 27 gives

$$(5) \qquad {}_{B_e\otimes A_e}(B_{g^{-1}} \otimes_{\Bbbk} A_{g'}) \otimes_{B_e\otimes A_e} M_h \xrightarrow{\cong} {}_{B_e\otimes A_e}M_{g^{-1}hg'}.$$

Turning B_e–actions on B_g around gives ${}_{A_e}(M_{ghg'})_{B_e^{\mathrm{op}}}$ and the collection of all such bimodule maps turns $\{{}_{A_e}(M_g)_{B_e^{\mathrm{op}}}\}_{g\in G}$ into a G–graded (A, B^{op})–bimodule $M = \bigoplus_{g\in G} M_g$. Similarly, reflections of this 2–morphism and the corresponding relations with respect to a vertical axis show that $N = \bigoplus_{g \in G} N_g$ is a G–graded (B^{op}, A)–bimodule.

There are four types of cusp generators and each is indexed by two elements of G. For every $g, g' \in G$ they give the bimodule maps, in $Z_2'(\mathcal{XG}_2)$,

$$\begin{aligned}
f_1^{gg'} &: {}_{A_e}(A_{gg'})_{A_e} \to {}_{A_e}M_g \otimes_{B_e^{\mathrm{op}}} (N_{g'})_{A_e}, \\
f_2^{gg'} &: {}_{B_e^{\mathrm{op}}}N_g \otimes_{A_e} (M_{g'})_{B_e^{\mathrm{op}}} \to {}_{B_e^{\mathrm{op}}}(B^{\mathrm{op}}_{gg'})_{B_e^{\mathrm{op}}}, \\
f_3^{gg'} &: {}_{B_e^{\mathrm{op}}}(B^{\mathrm{op}}_{gg'})_{B_e^{\mathrm{op}}} \to {}_{B_e^{\mathrm{op}}}N_g \otimes_{A_e} (M_{g'})_{B_e^{\mathrm{op}}}, \\
f_4^{gg'} &: {}_{A_e}M_g \otimes_{B_e^{\mathrm{op}}} (N_{g'})_{A_e} \to {}_{A_e}(A_{gg'})_{A_e},
\end{aligned}$$

given in the order of cusp generators in Figure 25. These bimodule maps are required to satisfy the relations in $\mathcal{XR}$. Relations containing cusp generators indicate that these bimodule maps are two-sided inverses, ie $f_1^{gg'} = (f_4^{gg'})^{-1}$ and $f_2^{gg'} = (f_3^{gg'})^{-1}$. It is not hard to see that for each i the collection $\{f_i^{gg'}\}_{g,g' \in G}$ of bimodule maps forms a G–graded bimodule map f_i. The collection of swallowtail morphisms corresponds to the compositions of graded bimodule maps

$$\begin{gathered}
{}_{B^{\mathrm{op}}}N_A \to {}_{B^{\mathrm{op}}}N \otimes_A A_A \xrightarrow{\mathrm{id} \otimes f_1} {}_{B^{\mathrm{op}}}N \otimes_A M \otimes_{B^{\mathrm{op}}} N_A \xrightarrow{f_2 \otimes \mathrm{id}} {}_{B^{\mathrm{op}}}B^{\mathrm{op}} \otimes_{B^{\mathrm{op}}} N_A \to {}_{B^{\mathrm{op}}}N_A, \\
{}_AM_{B^{\mathrm{op}}} \to {}_AA \otimes_A M_{B^{\mathrm{op}}} \xrightarrow{f_1 \otimes \mathrm{id}} {}_AM \otimes_{B^{\mathrm{op}}} N \otimes_A M_{B^{\mathrm{op}}} \xrightarrow{\mathrm{id} \otimes f_2} {}_AM \otimes_{B^{\mathrm{op}}} B^{\mathrm{op}}_{B^{\mathrm{op}}} \to {}_AM_{B^{\mathrm{op}}}.
\end{gathered}$$

Swallowtail relations imply that both compositions equal to identity bimodule maps of N and M, respectively. In other words, $\zeta = ({}_{B^{\mathrm{op}}}N_A, {}_AM_{B^{\mathrm{op}}}, f_1, f_2)$ is a G–graded Morita context. Using ζ, we can replace B^{op}–module actions with A–module actions as follows. A right (left) B^{op}–module can be turned into a right (left) A–module by tensoring with ${}_{B^{\mathrm{op}}}(N)_A$ (${}_AM_{B^{\mathrm{op}}}$), such as tensoring ${}_AM_{B^{\mathrm{op}}}$ with ${}_{B^{\mathrm{op}}}(N)_A$ yields ${}_AM \otimes_{B^{\mathrm{op}}} N_A$, which is isomorphic to ${}_AA_A$ via f_4.

The remaining generators are Morse generators consisting of saddles, cup, and cap 2–morphisms. The collection of bimodule maps in $Z_2'(\mathcal{XG}_2)$ for the first saddle morphism in Figure 25 yields a graded bimodule map of the form

$${}_{A \otimes B}M \otimes_{\Bbbk} N_{A \otimes B} \to {}_{A \otimes B}(A \otimes_{\Bbbk} B)_{A \otimes B}.$$

By turning the B–module actions around, we obtain

$${}_{A \otimes B^{\mathrm{op}}}(M \otimes_{\Bbbk} N)_{A \otimes B^{\mathrm{op}}} \to {}_{A \otimes B^{\mathrm{op}}}(A \otimes_{\Bbbk} B)_{A \otimes B^{\mathrm{op}}},$$

where the left B^{op}–action is on N and the right B^{op}–action is on M. As pointed out above, B^{op}–module actions can be replaced by A–module actions and we get a graded

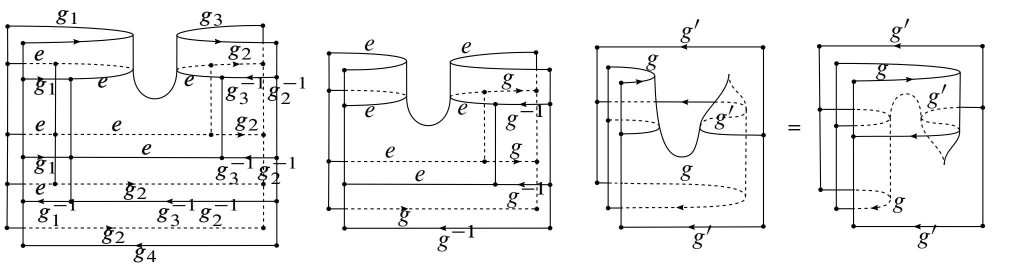


Figure 28: Saddle morphisms and cusp flip relation.

$(A_1 \otimes A_3, A_2 \otimes A_4)$–bimodule map of the form

$$\xi \colon {}_{A_1}A_{A_2} \otimes_{\Bbbk} {}_{A_3}A_{A_4} \to {}_{A_1}A_{A_4} \otimes_{\Bbbk} {}_{A_3}A_{A_2},$$

where numbers indicate module actions, ie $A_i = A$ for $i = 1, 2, 3, 4$. The first morphism in Figure 28 shows that an arbitrary saddle morphism can be obtained from an e–labeled saddle morphism and other generating 2–morphisms. This implies that the graded bimodule map ξ is determined at $1 \otimes 1 \in A_e \otimes A_e$, which we denote by a finite sum $\sum_i p_i^e \otimes q_i^e$, and it satisfies $\sum_i a p_i^e \otimes q_i^e = \sum_i p_i^e \otimes q_i^e a$ for all $a \in A$. Similarly, we denote the image of $1 \otimes 1$ under the second 2–morphism in Figure 28 by $\eta_g^A = \sum_i p_i^g \otimes q_i^g$ for all $g \in G$ (compare with (3)).

In the same way, the collection of bimodule maps in $Z_2'(\mathcal{X}\mathcal{G}_2)$ for the second saddle morphism gives a graded $(A_1 \otimes A_3, A_2 \otimes A_4)$–bimodule map of the form

$$\eta \colon {}_{A_1}A_{A_2} \otimes_{\Bbbk} {}_{A_3}A_{A_4} \to {}_{A_1}A_{A_4} \otimes_{\Bbbk} {}_{A_3}A_{A_2}.$$

The cusp flip relation shown in Figure 28 implies that $\xi = \eta$. Before considering cup and cap generators, note that, using ζ, we can assign the collection of all g– and g^{-1}–labeled circles to $\bigoplus_{g \in G} A_g \otimes_{(A_e \otimes A_e^{\mathrm{op}})} A_{g^{-1}}$. The collections of 2–morphisms

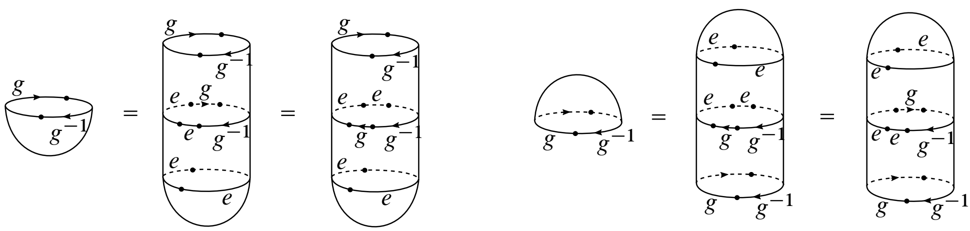


Figure 29: Cup and cap morphisms on nonprincipal components.

in Figure 29 give the bimodule maps

$$\Lambda\colon \bigoplus_{g\in G} A_g \otimes_{A_e\otimes A_e^{\mathrm{op}}} A_{g^{-1}} \to \Bbbk, \quad u\colon \Bbbk \to \bigoplus_{g\in G} A_g \otimes_{A_e\otimes A_e^{\mathrm{op}}} A_{g^{-1}},$$

respectively. Figure 29 implies that cup and cap morphisms are determined on the principal component. Since $A_e \otimes_{A_e\otimes A_e^{\mathrm{op}}} A_e = A_e/[A_e, A_e]$, cup morphism on the principal component can be considered as a symmetric linear map $\Lambda\colon A_e \to \Bbbk$. Additionally, Figure 29 shows that, on nonprincipal components, cup morphism is given by multiplication followed by Λ, leading to a symmetric $\Bbbk$–bilinear map $\eta_g\colon A_g \otimes A_{g^{-1}} \to \Bbbk$. Morse relations involving cup morphism indicate the nondegeneracy of η_g as follows. Assuming $\sum_j \beta_g^j \otimes 1 \otimes \beta_{g^{-1}}^j$ is the image of 1 under $A_e \xrightarrow{\sim} A_g \otimes_{A_e} A_e \otimes_{A_e} A_{g^{-1}}$, the first (left) 2–morphism in Figure 30 corresponds to the composition

$$a \mapsto 1\otimes 1\otimes a \mapsto \sum_j \beta_g^j \otimes \Big(\sum_i p_i^e \otimes q_i^e\Big) \otimes \beta_{g^{-1}}^j \otimes a \mapsto \sum_i p_i^g \otimes q_i^g \otimes a$$
$$\mapsto \sum_i p_i^g \eta(q_i^g, a)$$

and the Morse relation implies that it is equivalent to id_{A_g}. Similarly, reflection of this morphism with label g^{-1} gives $b = \sum_i \eta_g(b, p_i^g) q_i^g$ for any $b \in A_{g^{-1}}$, which shows that η_g is nondegenerate. Thus, (A, η_A) is a Frobenius G–algebra, where $(\eta_A)|_{A_g\otimes A_h}$ is η_g when $h = g^{-1}$ and zero otherwise.

The remaining Morse relations contain cap morphisms which are determined on the principal component. For any $c \in A_e$, assuming $u(1)|_{A_e\otimes A_e} = \sum_j a_j \otimes b_j$, the second 2–morphism in Figure 30 corresponds to the compositions

$$c \otimes \sum_j a_j \otimes b_j \mapsto \sum_{i,j} c p_i^e \otimes a_j q_i^e \otimes b_j \mapsto \sum_{i,j} c p_i^e b_j a_j q_i^e \mapsto c \sum_i p_i^e z q_i^e,$$

where $z = \sum_j b_j a_j \in A_e$. The Morse relation implies that $\sum_i p_i^e z q_i^e = 1$ and consequently $\sum_i p_i^e \otimes z q_i^e$ is a separability idempotent of the algebra A_e. Thus, (A_e, η_e) is a separable symmetric Frobenius algebra, as shown in [22]. Similarly, we have $\sum_i p_i^g z q_i^g = 1$ using the saddle whose image gives η_g^A. Until now we used ζ to replace B^{op}–actions by A–actions. By changing the roles of A and B, we obtain a quasibiangular G–algebra B and ζ is a compatible graded Morita context between B and A^{op}.

Thus, any object in $\mathbb{XP}^{\mathrm{PD}}$ determines a triple (A, B, ζ). Conversely, for any such triple, one constructs an object of $\mathbb{XP}(\mathrm{Alg}_{\Bbbk}^2)$ by assigning values to generating objects, 1–morphisms, and 2–morphisms of $\mathbb{XP}$ satisfying generating relations using

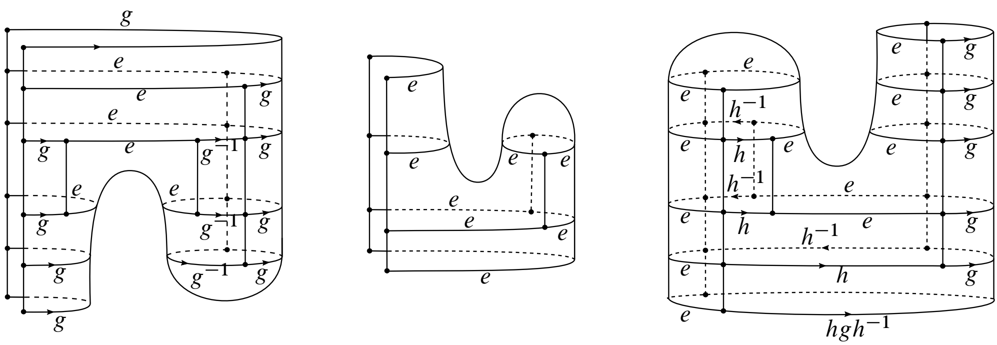


Figure 30: Compositions of generating 2–morphisms forming Morse relations.

the above arguments. Then, by the cofibrancy theorem, this object gives a strict symmetric monoidal 2–functor $\mathrm{XB}^{\mathrm{PD}} \to \mathrm{Alg}^2_{\Bbbk}$ whose composition with the equivalence $X\mathrm{Bord}_2 \xrightarrow{\sim} \mathrm{XB}^{\mathrm{PD}}$ produces the desired extended X–HFT. □

Remark The cofibrancy theorem implies that any symmetric monoidal 2–functor $Z\colon \mathrm{XB}^{\mathrm{PD}} \to \mathrm{Alg}^2_{\Bbbk}$ can be strictified. That is, Z is equivalent to a strict symmetric monoidal 2–functor. From now on, by an $\mathrm{Alg}^2_{\Bbbk}$–valued extended X–HFT giving triple (A, B, ζ) we mean the triple coming from the corresponding strict symmetric monoidal 2–functor.

Turaev [28] defined the G–center of a biangular G–algebra. We extend this notion to a G–center of a quasibiangular G–algebra (K, η) as $Z_G(K) = \bigoplus_{g\in G} \Psi(K_g)$, where $\Psi(a) = \sum_i p_i^e a q_i^e$ for inner product elements $\{\sum_i p_i^g \otimes q_i^g\}_{g\in G}$. In general, the G–center is not commutative and it differs from the usual center of the algebra. However, it has a crossed Frobenius G–algebra structure, which is defined as follows:

Definition 4.20 [28] A Frobenius G–algebra $\bigl(L = \bigoplus_{g\in G} L_g, \eta\bigr)$ is *crossed* if L is endowed with a group homomorphism $\varphi\colon G \to \mathrm{Aut}(L)$ satisfying the following conditions:

(i) φ is conjugation type, ie $\varphi_h(L_g) = L_{hgh^{-1}}$ and $\varphi_h|_{L_h} = \mathrm{id}_{L_g}$ for every $g, h \in G$.

(ii) $ba = \varphi_h(a)b$ for any $a \in L$ and $b \in L_h$.

(iii) $\mathrm{Tr}(\mu\varphi_h\colon L_g \to L_g) = \mathrm{Tr}(\varphi_{g^{-1}}\mu_c\colon L_h \to L_h)$ for all $g, h \in G$ and $c \in L_{ghg^{-1}h^{-1}}$, where $\mu_c\colon L \to L$ is left multiplication by c and Tr is the trace of a map.

(iv) η is invariant under φ.

Lemma 4.21 *Let (K,η) be a quasibiangular G–algebra with a central element $z \in K_e$ and a collection of inner product elements $\{\sum_i p_i^g \otimes q_i^g\}_{g\in G}$. Then $Z_G(K)$ is a unital G–algebra with multiplication $\sum_i p_i^e a q_i^e \cdot \sum_i p_i^e b q_i^e = \sum_{i,j} p_i^e a q_i^e p_j^e b q_j^e z^{-1}$ for all $a,b \in K$ and the triple $(Z_G(K), \eta|_{Z_G(K)}, \{\varphi_g|_{Z_G(K)}\}_{g\in G})$ is a crossed Frobenius G–algebra, where $\varphi_g(a) = \sum_i p_i^g a z q_i^g$ for all $a \in K$ and all $g \in G$.*

Proof The unit of $Z_G(K)$ is $\Psi(z^2) = z$. By Lemma 4.12, we have the equality

$$\sum_i p_i^h b z' q_i^h c = \sum_i c p_i^{gh} b z' q_i^{gh} \tag{6}$$

for all $c \in K_{g^{-1}}$, $z' \in K_e$, $b \in K$ and $g,h \in G$. Taking $z' = 1$ and $g = h = e$ gives

$$\begin{aligned} \Psi(a\Psi(b)z^{-1}) &= \sum_{i,j} p_i^e a p_j^e b q_j^e z^{-1} q_i^e = \sum_{i,j} p_i^e a p_j^e b q_j^e q_i^e z^{-1} \\ &= \sum_{i,j} p_i^e a q_i^e p_j^e b q_j^e z^{-1} = \Psi(a)\cdot\Psi(b), \end{aligned} \tag{7}$$

which implies that $Z_G(K)$ is closed under multiplication. Restriction of η to $Z_G(K)$ is an inner product and hence $(Z_G(K), \eta|_{Z_G(K)})$ is a Frobenius G–algebra. For any $b \in K$ and for all $h \in G$, we have

$$\Psi(z\varphi_h(b)) = \sum_j p_j^e z \Big(\sum_i p_i^h b z q_i^h\Big) q_j^e = \sum_{i,j} p_j^e z q_j^e p_i^h b z q_i^h = \sum_i p_i^h b z q_i^h = \varphi_h(b),$$

which shows that $\varphi_h(K) \subset Z_G(K)$. Similarly, for any $\sum_i p_i^e a q_i^e \in Z_G(K)$, we have

$$\varphi_e\Big(\sum_i p_i^e a q_i^e\Big) = \sum_j p_j^e \Big(\sum_i p_i^e a q_i^e\Big) z q_j^e = \sum_{i,j} p_j^e z q_j^e p_i^e a q_i^e = \sum_i p_i^e a q_i^e,$$

showing $\varphi_e|_{Z_G(K)} = \mathrm{id}_{Z_G(K)}$. Note that, for any $g \in G$ and $a \in K$, we have

$$\varphi_g(\Psi(a)) = \sum_j p_j^g \Big(\sum_i p_i^e a q_i^e\Big) z q_j^g = \sum_j p_j^g z q_j^g \sum_i p_i^g a q_i^g = \sum_i p_i^g a q_i^g$$

and using this we have the equality, for all $\bar a = \Psi(a), \bar b = \Psi(b) \in Z_G(K)$ and $g \in G$,

$$\begin{aligned} \varphi_g(\bar a \cdot \bar b) &= \sum_k p_k^g \Big(\sum_{i,j} p_i^e a q_i^e p_j^e b q_j^e z^{-1}\Big) z q_k^g \\ &= \Big(\sum_i p_i^g a q_i^g\Big)\Big(\sum_j p_j^g b q_j^g\Big)\Big(\sum_k p_k^g q_k^g\Big) = \varphi_g(\bar a)\cdot\varphi_g(\bar b), \end{aligned}$$

showing φ_g is an algebra homomorphism. For the last equality, we have $\sum_k p_k^g q_k^g = z^{-1}$ since $\sum_k p_k^g q_k^g = \sum_{i,j} p_i^e \beta_g^j \beta_{g^{-1}}^j q_i^e = \sum_i p_i^e q_i^e = z^{-1}$, where $\sum_j \beta_g^j \otimes \beta_{g^{-1}}^j \in K_g \otimes_{K_e} K_{g^{-1}}$ is the inverse of $1 \in K_e$ under the product map $K_h \otimes_{K_e} K_{h^{-1}} \to K_e$ (compare the first equality with the bimodule map corresponding to the second 2–morphism in Figure 28). We also have

$$\varphi_g\big(\varphi_h(\Psi(b))\big) = \sum_j p_j^g \Big(\sum_i p_i^h b q_i^h\Big) z q_j^g = \sum_j p_j^g z q_j^g \sum_i p_i^{gh} b q_i^{gh} = \varphi_{gh}(\Psi(b))$$

for all $g, h \in G$ and $b \in K$, which also implies that $\varphi_{g^{-1}}$ is the inverse of φ_g for all $g \in G$. For all $\bar{a} = \Psi(a), \bar{b} = \Psi(b) \in Z_G(K)$ and $g \in G$, using the cyclic symmetry of η, we have

$$\begin{aligned}\eta(\varphi_g(\bar{a}), \bar{b}) &= \eta\Big(\sum_i p_i^g \bar{a} z q_i^g, \sum_j p_j^e b q_j^e\Big) = \eta\Big(\bar{a}, \sum_{i,j} z q_i^g p_j^e b q_j^e p_i^g\Big)\\ &= \eta\Big(\bar{a}, \sum_{i,k} p_k^{g^{-1}} b q_k^{g^{-1}} z q_i^g p_i^g\Big) = \eta(\bar{a}, \varphi_{g^{-1}}(\bar{b})),\end{aligned}$$

showing the inner product η is invariant under $\varphi\colon G \to \mathrm{Aut}(Z_G(K))$. For any $\bar{c} = \Psi(c) \in Z_G(K)_h$, we have

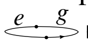
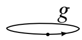
$$\varphi_h(\bar{c}) = \sum_i p_i^h c q_i^h = \sum_{i,j} p_i^e \beta_h^j c \beta_{h^{-1}}^j q_i^e = \sum_i p_i^e c q_i^e = \bar{c},$$

where $\sum_j \beta_h^j \otimes \beta_{h^{-1}}^j \in K_h \otimes K_{h^{-1}}$ is the inverse of $1 \in K_e$ under the product bimodule map. This shows that φ_h acts by the identity on $Z_G(K)_h$ for all $h \in G$. Equation (6) gives $\varphi_g(a)b = b\varphi_{h^{-1}g}(a)$ for $a \in K, b \in K_h$ and $g, h \in G$. In this case, by taking $g = h$, we have $\varphi_h(a)b = ba$. Let $\mu_c\colon K \to K$ be multiplication by $c \in K$; then, for any $g, h \in G$ and $c \in K_{ghg^{-1}h^{-1}}$, we have[7]

$$\begin{aligned}\mathrm{Tr}(\mu_c\varphi_h\colon K_g \to K_g) &= \sum_i \eta(c\varphi_h(p_i^g), q_i^g) = \sum_{i,j} \eta(c p_j^h p_i^g z q_j^h, q_i^g)\\ &= \sum_{i,j} \eta(q_i^g c p_j^h z p_i^g, q_j^h) = \sum_j \eta(\varphi_{g^{-1}}(c p_j^h), q_j^h)\\ &= \mathrm{Tr}(\varphi_{g^{-1}}\mu_c\colon K_h \to K_h).\end{aligned}$$ □

Any 2–dimensional extended HFT produces a nonextended one by restricting it to a symmetric monoidal full subcategory $X\mathbb{Cob}_2$ of $X\mathrm{Bord}_2$, defined as follows. The objects of $X\mathbb{Cob}_2$ are $\{\overset{e\quad g}{\bigcirc}\}_{g\in G}$, the empty 1–morphism in $X\mathrm{Bord}_2$, and disjoint

[7] See [28] for the first equality.

unions of these 1–morphisms. The morphisms of $X\mathbb{C}\mathrm{ob}_2$ are the 2–morphisms of $X\mathrm{Bord}_2$ among these 1–morphisms. We define a symmetric monoidal functor $D\colon X\mathbb{C}\mathrm{ob}_2 \to X\mathrm{Cob}_2$ by  $\mapsto$  for any $g \in G$. On morphisms, D forgets a point on each boundary component and takes the corresponding relative homotopy class. Using the definitions, it is not hard to see that D is an equivalence of categories. By the restriction of $Z\colon X\mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}$ to $X\mathbb{C}\mathrm{ob}_2$ above, we mean precomposing Z with $D^{-1}\colon X\mathrm{Cob}_2 \to X\mathbb{C}\mathrm{ob}_2$.

Corollary 4.22 *Let $Z\colon X\mathrm{Bord}_2 \to \mathrm{Alg}^2_{\Bbbk}$ be an extended HFT giving (A, B, ζ). Then, the nonextended HFT obtained from Z by restricting to $X\mathbb{C}\mathrm{ob}_2$ is the nonextended HFT associated to the G–center of the quasibiangular G–algebra (A, η_A).*

Proof Proceeding with the notation used in the proof of Theorem 4.19, the image of a g–labeled circle under Z is given by

$$A_e \otimes_{A_e \otimes A_e^{\mathrm{op}}} A_g = \{b \in A_g \mid a \cdot b = b \cdot a \text{ for all } a \in A_e\}.$$

The G–center of (A, η_A) is $Z_G(A) = \bigoplus_{g \in G} \Psi(A_g)$. For any $a \in A_e \otimes_{A_e \otimes A_e^{\mathrm{op}}} A_g$, we have

$$a = 1 \cdot a = \Bigl(\sum_i p_i^e z q_i^e\Bigr) a = \sum_i p_i^e a z q_i^e = \Psi(az) \in \Psi(A_g)$$

and, for any $\sum_i p_i^e a q_i^e \in \Psi(A_g)$ and $b \in A_e$, we have

$$\Bigl(\sum_i p_i^e a q_i^e\Bigr) b = \sum_i p_i^e a q_i^e b = \sum_i b p_i^e a q_i^e = b\Bigl(\sum_i p_i^e a q_i^e\Bigr),$$

where the middle equality is the result of Lemma 4.12. Thus, we have $A_e \otimes_{A_e \otimes A_e^{\mathrm{op}}} A_g = \Psi(A_g)$ for all $g \in G$. The third 2–morphism in Figure 30 gives the crossed structure on the restricted HFT and it corresponds to the sequence of compositions

$$\begin{aligned}1 \otimes a \mapsto 1 \otimes \sum_j a_j b_j \otimes a \mapsto \sum_j 1 \otimes \beta_h^j z \beta_{h^{-1}}^j \otimes a \mapsto 1 \otimes \sum_j \beta_h^j \Bigl(\sum_i p_i^e \otimes z q_i^e\Bigr) \beta_{h^{-1}}^j \otimes a \\ \mapsto 1 \otimes \sum_i p_i^h \otimes z q_i^h \otimes a \mapsto 1 \otimes \sum_i p_i^h a z q_i^h,\end{aligned}$$

which coincides with the crossed structure of $Z_G(A)$. $\square$

Example 4.23 Let $\Bbbk$ be an algebraically closed field. Then separable $\Bbbk$–algebras are the same as semisimple $\Bbbk$–algebras. By the Artin–Wedderburn structure theorem, any separable algebra is isomorphic to a product of finitely many matrix algebras over $\Bbbk$. Consider the G–algebra $A = \bigoplus_{g\in G} A_g$ whose principal component is a product $A_e = \prod_{i=1}^n M_{k_i}(\Bbbk)$ of $k_i \times k_i$ matrix algebras over $\Bbbk$ such that each k_i is invertible in $\Bbbk$ and each component is given by $A_g = \ell_g A_e$, where ℓ_g is a basis, ie for any $a \in A_g$, there exists $b \in A_e$ such that $a = \ell_g b$. Define an inner product η on A as

$$\eta(a,b) = \begin{cases} r\,\mathrm{Tr}(L_{ab}\colon A_e \to A_e) & \text{when } ab \in A_e, \\ 0 & \text{otherwise,} \end{cases}$$

where $r \in \Bbbk$ is invertible and L_{ab} is the left multiplication by ab map. We can express the inner product concretely as $\eta\big(\ell_g \prod_{i=1}^n A_i, \ell_{g^{-1}} \prod_{i=1}^n B_i\big) = r \sum_{i=1}^n k_i\,\mathrm{Tr}(A_i B_i)$. For each $g \in G$, an inner product element can be chosen as

$$\eta_g^- = r^{-1} \prod_{i=1}^n k_i^{-1} \sum_{\alpha,\beta=1}^{k_i} \ell_g E_{\alpha,\beta} \otimes \ell_{g^{-1}} E_{\beta,\alpha} \in A_g \otimes A_{g^{-1}},$$

where $E_{\alpha,\beta}$ is the (α,β)–elementary matrix. In this case, the central element $z \in A_e$ is given by $(rI_{k_1}, \dots, rI_{k_n})$, where I_{k_i} denotes the $k_i \times k_i$ identity matrix. Note that $\prod_{i=1}^n k_i^{-1} \sum_{\alpha,\beta=1}^{k_i} E_{\alpha,\beta} \otimes E_{\beta,\alpha}$ is a separability idempotent of A_e. Thus, the map $\Psi\colon A_g \to A_g$ is given by

$$\Psi\Big(\ell_g \prod_{i=1}^n A_i\Big) = r^{-1} \prod_{i=1}^n k_i^{-1} \sum_{\alpha,\beta=1}^{k_i} E_{\alpha,\beta}(\ell_g A_i) E_{\beta,\alpha} = r^{-1} \prod_{i=1}^n k_i^{-1} \ell_g\,\mathrm{Tr}(A_i) I_{k_i},$$

which is a projection onto its center $\ell_g \Bbbk^n$.

4.4 The bicategory of 2–dimensional extended X–HFTs

Until now we have studied the objects of $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$. Theorem 4.10 implies that studying 1– and 2–morphisms of $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ leads us to a bicategory equivalent to $\mathcal{E}$–$\mathcal{HFT}(X, \mathrm{Alg}^2_{\Bbbk})$. Let Z_0 and Z_1 be extended HFTs with target X giving triples (A, B, ζ) and (A', B', ζ'), respectively. A 1–morphism $\alpha\colon Z_0 \to Z_1$ in $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ gives 1–morphisms $\alpha_0(\bullet^+) = {}_{A'_e}R_{A_e}$ and $\alpha_0(\bullet^-) = {}_{B'_e}S_{B_e}$, and 2–morphisms

$$\begin{aligned}
&\alpha_1(+\bullet\xleftarrow{g}\bullet+)\colon {}_{A'_e}A'_g \otimes_{A'_e} R_{A_e} \to {}_{A'_e}R \otimes_{A_e} (A_g)_{A_e}, \\
&\alpha_1(-\bullet\xleftarrow{g}\bullet-)\colon {}_{B'_e}B'_g \otimes_{B'_e} S_{B_e} \to {}_{B'_e}S \otimes_{B_e} (B_g)_{B_e}, \\
&\alpha_1\big(\substack{+\bullet \\ -\bullet}\!\!\supset g\big)\colon {}_{A'_e\otimes B'_e}(M'_g)_{\Bbbk} \to {}_{A'_e\otimes B'_e}(R\otimes S) \otimes_{A_e\otimes B_e} (M_g)_{\Bbbk}, \\
&\alpha_1\big(g \subset\!\!\substack{\bullet - \\ \bullet +}\big)\colon {}_{\Bbbk}N'_g \otimes_{B'_e\otimes A'_e} (S\otimes R)_{B_e\otimes A_e} \to {}_{\Bbbk}(N_g)_{B_e\otimes A_e},
\end{aligned}$$

which are isomorphisms for all $g \in G$ and G–graded bimodules M, M', N and N' are the components of ζ and ζ'. These morphisms are natural with respect to generating 2–morphisms. Naturality with respect to graded multiplication

$$\begin{smallmatrix} + & g' + g & + \\ + & gg' & + \end{smallmatrix}$$

leads to the commutativity of the diagram

$$\begin{tikzcd} {}_{A'_e}(A'_{g'}) \otimes_{A'_e} A'_g \otimes_{A'_e} R_{A_e} \arrow[r, "\alpha_1\left(\begin{smallmatrix} g' & & g \\ + & + & + \end{smallmatrix}\right)"] \arrow[d, "Z_1\left(\begin{smallmatrix} + & g' + g & + \\ + & gg' & + \end{smallmatrix}\right)"'] & {}_{A'_e}R \otimes_{A_e} A_{g'} \otimes_{A_e} (A_g)_{A_e} \arrow[d, "Z_0\left(\begin{smallmatrix} + & g' + g & + \\ + & gg' & + \end{smallmatrix}\right)"] \\ {}_{A'_e}A'_{gg'} \otimes_{A'_e} R_{A_e} \arrow[r, "\alpha_1\left(\begin{smallmatrix} & gg' & \\ + & & + \end{smallmatrix}\right)"'] & {}_{A'_e}R \otimes_{A_e} (A_{gg'})_{A_e} \end{tikzcd}$$

for all $g, g' \in G$. We denote bimodules ${}_{A'_e}A'_g \otimes_{A'_e} R_{A_e}$ and ${}_{A'_e}R \otimes_{A_e} (A_g)_{A_e}$ by ${}_{A'_e}(R'_g)_{A_e}$ and ${}_{A'_e}(R''_g)_{A_e}$, respectively. Commutativity of the above diagram implies that they are naturally isomorphic. Thus, we can use one of them and denote it by R_g. Similarly, S_g denotes a (B'_e, B_e)–bimodule. These assignments and naturality with respect to

$$\left\{\begin{smallmatrix} + & g' + g & + \\ + & gg' & + \end{smallmatrix}\right\}_{g,g' \in G}$$

turn these bimodules into G–graded (A', A)– and (B', B)–bimodules $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$, respectively. Similarly, naturality with respect to G–module generators turns collections $\{\alpha_1(\begin{smallmatrix} + \\ - \end{smallmatrix} g)\}_{g \in G}$ and $\{\alpha_1(g \begin{smallmatrix} - \\ + \end{smallmatrix})\}_{g \in G}$ into G–graded $(A' \otimes B', \Bbbk)$– and $(\Bbbk, B \otimes A)$–bimodule maps, respectively.

Using $\alpha_0(\bullet^-)$, we define a 1–morphism $\alpha'_0(\bullet^+) = {}_{A_e}R'_{A'_e}$ by

$$\begin{aligned} \alpha'_0(\bullet^+) &= [Z_1(e \begin{smallmatrix} + \\ - \end{smallmatrix}) \otimes \mathrm{id}_{Z_0(\bullet+)}] \circ [\alpha(\bullet^-) \otimes \sigma_{Z_0(\bullet+), Z_1(\bullet+)}] \circ [Z_0(\begin{smallmatrix} + \\ - \end{smallmatrix} e) \otimes \mathrm{id}_{Z_1(\bullet+)}], \\ \alpha'_0(\bullet^+) &= [(N'_e)_{B'_e \otimes A'_e} \otimes_{\Bbbk} {}_{A_e}(A_e)_{A_e}] \otimes_{B'_e \otimes A'_e \otimes A_e} [{}_{B'_e}S_{B_e} \otimes_{\Bbbk} \sigma_{A_e, A'_e}] \\ &\qquad \otimes_{B_e \otimes A_e \otimes A'_e} [{}_{B_e \otimes A_e}M_e \otimes_{\Bbbk} {}_{A'_e}(A'_e)_{A'_e}], \end{aligned}$$

where σ is the symmetric braiding of $\mathrm{Alg}^2_{\Bbbk}$. Using $\alpha'_0(\bullet^+)$, we define a 2–morphism

$$\begin{aligned} \alpha'_1(+\overset{g}{\longleftarrow}+) &= Z_0(+\overset{g}{\longleftarrow}+) \circ \alpha'_0(\bullet^+) \to \alpha'_0(\bullet^+) \circ Z_1(+\overset{g}{\longleftarrow}+), \\ \alpha'_1(+\overset{g}{\longleftarrow}+) &= {}_{A_e}A_e \otimes_{A_e} R'_{A'_e} \to {}_{A_e}R' \otimes_{A'_e} (A'_g)_{A'_e}. \end{aligned}$$

Using naturality, R' is turned into a G–graded (A, A')–bimodule $R' = \bigoplus_{g \in G} R'_g$. The 1–morphism $\alpha_0(\bullet^-)$ can be obtained from $\alpha'_0(\bullet^+)$ by applying the composition

$Z_1(\ ) \circ \mathrm{id}_{\alpha_0(\bullet^-)} \circ Z_0(\ )$ to the 1–morphism

$$[Z_1(\overset{e}{\ }{}^{+}_{-}) \otimes Z_0(\overset{e}{\ }{}^{-}_{+}) \otimes \mathrm{id}_{Z_1(\bullet^-)}] \circ [\alpha_0(\bullet^-) \otimes \sigma_{Z_0(\bullet^+),Z_1(\bullet^+)} \otimes \sigma_{Z_1(\bullet^-),Z_0(\bullet^-)}] \circ [Z_0({}^{+}_{-}\overset{e}{\ }) \otimes Z_1({}^{-}_{+}\overset{e}{\ }) \otimes \mathrm{id}_{Z_0(\bullet^-)}]$$

and, similarly, $\alpha_1(-\overset{g}{\bullet\!\longrightarrow\!\bullet}-)$ can be obtained from $\alpha_1'(+\overset{g}{\bullet\!\longrightarrow\!\bullet}+)$. Likewise, using $\alpha_0'(\bullet^+)$ in the images of cusp generators under Z_0, the 2–morphisms $\alpha_1'({}^{+}_{-}\ g)$ and $\alpha_1'(g\ {}^{-}_{+})$ are defined, and $\alpha_1({}^{+}_{-}\ g)$ and $\alpha_1(g\ {}^{-}_{+})$ can be obtained from these 2–morphisms.

As in the proof of Theorem 4.19, using G–graded Morita contexts ζ and ζ', graded bimodules M and M' can be replaced by ${}_{A\otimes A^{\mathrm{op}}}A$ and ${}_{A'\otimes(A')^{\mathrm{op}}}A'$. We can also replace the graded bimodule S by R' using $\alpha_0'(\bullet^+)$. Thus, naturality with respect to G–module generators turns the collection $\{\alpha_1'({}^{+}_{-}\ g)\}_{g\in G}$ into a bimodule map ${}_{A'}A'_{A'} \to {}_{A'}R \otimes_A R'_{A'}$. Similarly, the collection $\{\alpha_1'(g\ {}^{-}_{+})\}_{g\in G}$ is turned into a bimodule map ${}_A R' \otimes_{A'} R_A \to {}_A A_A$. Naturality with respect to cusp generators indicates that the compositions

$$ {}_{A'}R_A \to {}_{A'}A'_{A'} \otimes_{A'} R \xrightarrow{\alpha_1'({}^{+}_{-}\ )\otimes\mathrm{id}} {}_{A'}R \otimes_A R' \otimes_{A'} R_A \xrightarrow{\mathrm{id}\otimes\alpha_1'(\ {}^{-}_{+})} {}_{A'}R \otimes_A A_A \to {}_{A'}R_A,$$

$$ {}_A R'_{A'} \to {}_A R' \otimes_{A'} A'_{A'} \xrightarrow{\mathrm{id}\otimes\alpha_1'({}^{+}_{-}\ )} {}_A R' \otimes_{A'} R \otimes_A R'_{A'} \xrightarrow{\alpha_1'(\ {}^{-}_{+})\otimes\mathrm{id}} {}_A A \otimes_A R'_{A'} \to {}_A R'_{A'}$$

are id_R and $\mathrm{id}_{R'}$, respectively. In other words, α gives a G–graded Morita context between A and A'. Similarly, one can define $\alpha_0'(-\overset{g}{\bullet\!\longrightarrow\!\bullet}-)$ and obtain a G–graded Morita context between B and B'. Naturality with respect to Morse generators indicates that G–graded Morita contexts are compatible. Hence, α leads to two compatible G–graded Morita contexts. In the theory of bicategories, this means that both $\alpha_0(\bullet^+)$ and $\alpha_0(\bullet^-)$ are parts of two adjoint equivalences. Since an adjoint equivalence is the same as an equivalence (see [22, Proposition A.27]), Z_0 and Z_1 are equivalent extended HFTs.

Let $\alpha^1, \alpha^2 : Z_0 \to Z_1$ be 1–morphisms in $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ and $\theta : \alpha^1 \to \alpha^2$ be a 2–morphism in $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$. Assume that Z_0 and Z_1 give triples (A, B, ζ) and (A', B', ζ') as before and 1–morphisms give $\alpha^1_0(\bullet^+) = {}_{A'_e}R_{A_e}$ and $\alpha^2_0(\bullet^+) = {}_{A'_e}P_{A_e}$. Then $\theta_0(\bullet^+) : {}_{A'_e}R_{A_e} \to {}_{A'_e}P_{A_e}$ and the naturality of $\theta_0(\bullet^+)$ with respect to $+\overset{g}{\bullet\!\longrightarrow\!\bullet}+$ is the commutativity of the diagram

$$\begin{tikzcd}
{}_{A'_e}A'_g \otimes_{A'_e} R_{A_e} \arrow[r, "\alpha^1_1(+\overset{g}{\bullet\to\bullet}+)"] \arrow[d, "\theta_0(\bullet^+)"'] & {}_{A'_e}R \otimes_{A_e} (A_g)_{A_e} \arrow[d, "\theta_0(\bullet^+)"] \\
{}_{A'_e}A'_g \otimes_{A'_e} P_{A_e} \arrow[r, "\alpha^2_1(+\overset{g}{\bullet\to\bullet}+)"'] & {}_{A'_e}P \otimes_{A_e} (A_g)_{A_e}
\end{tikzcd}$$

which shows that $\theta_0(\bullet^+)$ is a G–graded bimodule map. Assuming $(\alpha_0')^1(\bullet^+) = {}_{A_e}R'_{A'_e}$ and $(\alpha_0')^2(\bullet^+) = {}_{A'_e}P'_{A_e}$, we similarly have a graded bimodule map $\theta_0'(\bullet^+)\colon {}_{A_e}R'_{A'_e} \to {}_{A_e}P'_{A'_e}$ using $\theta_0(\bullet^-)$ and $(\alpha_0')^i(\bullet^+)$ for $i = 1, 2$. Naturality with respect to and corresponds to the commutativity of these bimodule maps with the unit and counit of the adjunctions. In other words, θ leads to an equivalence of graded Morita contexts. In the same way, using B and B', one gets another equivalence of graded Morita contexts.

Motivated by these observations, we define a bicategory Frob^G and a forgetful 2–functor $\mathcal{F}'\colon \mathbb{XP}(\mathrm{Alg}^2_{\Bbbk}) \to \mathrm{Frob}^G$ as follows. The bicategory Frob^G has quasibiangular G–algebras as objects, compatible G–graded Morita contexts as 1–morphisms, and equivalences of G–graded Morita contexts as 2–morphisms. The forgetful 2–functor $\mathcal{F}'$ maps an object of $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ giving (A, B, ζ) to A. On 1–morphisms, $\mathcal{F}'$ maps $\alpha\colon Z_0 \to Z_1$ to a compatible G–graded Morita context between quasibiangular G–algebras whose principal components are $Z_0(\bullet^+)$ and $Z_1(\bullet^+)$. On 2–morphisms, $\mathcal{F}'$ maps $\theta\colon \alpha^1 \to \alpha^2$ to an equivalence of the compatible G–graded Morita contexts. Composing $\mathcal{F}'$ with the equivalence $\mathcal{E}$–$\mathcal{HFT}(X, \mathrm{Alg}^2_{\Bbbk}) \simeq \mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$, we define $\mathcal{F}$.

Theorem 4.24 *The 2–functor $\mathcal{F}$ is an equivalence of bicategories $\mathcal{E}$–$\mathcal{HFT}(X, \mathrm{Alg}^2_{\Bbbk}) \simeq \mathrm{Frob}^G$.*

Proof It is enough to show that $\mathcal{F}'$ is an equivalence and we use the Whitehead theorem (Theorem 4.3). For a given quasibiangular G–algebra A, the triple $(A, A^{\mathrm{op}}, \mathrm{id})$ gives an object Z of $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ such that $\mathcal{F}'(Z) = A$. Let α be a compatible G–graded Morita context between quasibiangular G–algebras A and A'. Then triples $(A, (A')^{\mathrm{op}}, \alpha)$ and $(A', A^{\mathrm{op}}, \alpha)$ give objects Z_0 and Z_1 in $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ such that $\mathcal{F}'(\alpha') = \alpha$, where $\alpha'\colon Z_0 \to Z_1$.

For any two 1–morphisms $\alpha^1, \alpha^2\colon Z_0 \to Z_1$, we claim that

$$\mathcal{F}'(\alpha^1, \alpha^2)\colon \mathrm{Hom}(\alpha^1, \alpha^2) \to \mathrm{Hom}(\mathcal{F}'(\alpha^1), \mathcal{F}'(\alpha^2))$$

is an injection. Assume that different 2–morphisms $\theta^1, \theta^2\colon \alpha^1 \to \alpha^2$ in $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ give the same equivalence of G–graded Morita contexts. This means that pairs $(\theta_0^1(\bullet^-), (\theta_0')^1(\bullet^-))$ and $(\theta_0^2(\bullet^-), (\theta_0')^2(\bullet^-))$ give different graded bimodule maps while the images of θ^1 and θ^2 under $\mathcal{F}'$, $(\theta_0^1(\bullet^+), (\theta_0')^1(\bullet^+))$ and $(\theta_0^2(\bullet^+), (\theta_0')^2(\bullet^+))$, give the same graded bimodules maps. This is a contradiction because each $(\theta_0')^i(\bullet^-)$ is obtained from $(\theta_0)^i(\bullet^+)$ and each $(\theta_0')^i(\bullet^+)$ is obtained from $(\theta_0)^i(\bullet^-)$ for $i = 1, 2$.

For the surjectivity, let $\theta\colon \mathcal{F}'(\alpha^1) \to \mathcal{F}'(\alpha^2)$ be an equivalence of graded Morita contexts. Then the equivalence of graded Morita contexts $(\theta_0(\bullet^-), \theta_0'(\bullet^-))$ can be obtained from $\theta_0(\bullet^+)$, $\theta_0'(\bullet^+)$, $(\alpha_0')^1(\bullet^-)$, and $(\alpha_0')^2(\bullet^-)$. $\square$

Corollary 4.25 *Two triples (A_1, B_1, ζ_1) and (A_2, B_2, ζ_2) produce equivalent extended X–HFTs if and only if there exists a compatible G–graded Morita context between A_1 and A_2.*

Lastly, we comment on the relation between extended HFTs whose targets are related by covering maps. Let $Y \simeq K(H,1)$ be a pointed CW–complex for a nontrivial subgroup $H \leq G$ and $p\colon (Y,y) \to (X,x)$ be a covering. Then, any Y–manifold can be turned into an X–manifold by postcomposing a representative of the characteristic map with p. This gives a symmetric monoidal 2–functor $\iota_H : Y\mathrm{Bord}_2 \to X\mathrm{Bord}_2$ and precomposing any extended X–HFT with ι_H yields an extended Y–HFT. Moreover, for any symmetric monoidal bicategory $\mathcal{C}$, precomposition of a $\mathcal{C}$–valued extended X–HFT with ι lifts to a 2–functor $\mathrm{SymMon}(X\mathrm{Bord}_2, \mathcal{C}) \to \mathrm{SymMon}(Y\mathrm{Bord}_2, \mathcal{C})$ by forgetting the naturality of transformations with respect to $G\backslash H$–labeled 1–morphisms. Correspondingly, there is a 2–functor $\mathbb{XP}(\mathcal{C}) \to \mathbb{YP}(\mathcal{C})$, where $\mathbb{XP}$ and $\mathbb{YP}$ are the presentations of $X\mathrm{Bord}_2$ and $Y\mathrm{Bord}_2$, respectively. When $\mathcal{C}$ is $\mathrm{Alg}^2_{\Bbbk}$, the functor $\mathrm{Frob}^G \to \mathrm{Frob}^H$ is given by forgetting the $G\backslash H$ components of quasibiangular G–algebras, compatible G–graded Morita contexts, and equivalences of G–graded Morita contexts. In other words, a G–graded Morita context can be considered as a collection of Morita contexts indexed by the subgroups of G (see [3]).

4.5 The $(G\times\mathrm{SO}(2))$–structured cobordism hypothesis

A different approach to categorical classification of (fully) extended oriented HFTs is given by the structured cobordism hypothesis due to Lurie [15]. The cobordism hypothesis [1; 15; 2] was conjectured by Baez and Dolan in their seminal paper [2]. Lurie [15] reformulated the cobordism hypothesis using (∞,n)–categories and generalized it to a structured cobordism hypothesis using homotopy fixed points.

The bordism category involved in the structured cobordism hypothesis consists of manifolds with corresponding structures. For a topological group Γ and a fixed continuous homomorphism $\chi\colon \Gamma \to O(n)$, let $\zeta_\chi\colon \mathbb{R}^n \times_\Gamma E\Gamma$ denote the associated rank n vector bundle over the classified space $B\Gamma$. Then, a Γ–structure on a manifold M of dimension $k \leq n$ consists of a continuous map $f : M \to B\Gamma$ and an isomorphism

$TM \oplus \underline{\mathbb{R}}^{n-k} \to f^*\zeta_\chi$ of vector bundles, where $\underline{\mathbb{R}}^{n-k}$ is a trivial rank $n-k$ vector bundle. In the following theorem the category Bord_n^Γ consists of manifolds equipped with Γ–structures for a fixed homomorphism χ.

(Γ, χ)–structured cobordism hypothesis (Lurie [15]) Let $\mathcal{C}$ be a symmetric monoidal (∞, n)–category (see [5]) and Bord_n^Γ be the symmetric monoidal Γ–structured cobordism (∞, n)–category for a group Γ. Then there is a canonical equivalence of (∞, n)–categories

$$\mathrm{Fun}^\otimes(\mathrm{Bord}_n^\Gamma, \mathcal{C}) \xrightarrow{\sim} ((\mathcal{C}^{\mathrm{fd}})^\sim)^{h\Gamma},$$

where $\mathrm{Fun}^\otimes$ is the (∞, n)–category of symmetric monoidal functors between symmetric monoidal (∞, n)–categories, $\mathcal{C}^{\mathrm{fd}}$ is the (∞, n)–subcategory of fully dualizable objects with duality data, $(\mathcal{C}^{\mathrm{fd}})^\sim$ is the underlying ∞–groupoid and $((\mathcal{C}^{\mathrm{fd}})^\sim)^{h\Gamma}$ is the ∞–groupoid of homotopy Γ–fixed points given by

$$((\mathcal{C}^{\mathrm{fd}})^\sim)^{h\Gamma} = \mathrm{Hom}_\Gamma(E\Gamma, (\mathcal{C}^{\mathrm{fd}})^\sim),$$

where $E\Gamma$ is a weakly contractible ∞–groupoid equipped with a free Γ–action.

Remark A 2–dimensional EHFT with target $X \simeq K(G, 1)$, ie a classifying space BG, is a $(G\times\mathrm{SO}(2))$–structured 2–dimensional ETFT, where $\chi\colon G \times \mathrm{SO}(2) \to O(2)$ is given by $(g, A) \mapsto A$.

When $\Bbbk$ is an algebraically closed field of characteristic zero, Davidovich [6] showed that, for a finite group G, homotopy $(G\times\mathrm{SO}(2))$–fixed points in $(\mathrm{Alg}_\Bbbk^{\mathrm{fd}})^\sim$ are given by G–equivariant algebras. A G–equivariant algebra is a strongly graded Frobenius G–algebra with semisimple principal component. Her methods do not particularly require G to be finite and can be extended to discrete groups directly. Since the notions of separability and semisimplicity for a $\Bbbk$–algebra are equivalent when $\Bbbk$ is an algebraically closed field of characteristic zero, the objects of Frob^G and the objects of $((\mathrm{Alg}_\Bbbk^{\mathrm{fd}})^\sim)^{h(G\times\mathrm{SO}(2))}$ coincide.

Assume that $\Bbbk$ is an algebraically closed field of characteristic zero. The Artin–Wedderburn theorem implies that any separable $\Bbbk$–algebra is isomorphic to a product of matrix algebras over $\Bbbk$. Let $A_e = \mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \cdots \times \mathrm{End}(V_n)$ be such an algebra, where $V_1, V_2, \ldots, V_n$ are finite-dimensional $\Bbbk$–vector spaces. Recall that $A = \bigoplus_{g\in G} A_g$ is strongly graded by the generators, leading to bimodule isomorphisms $\{\tau_{g,g'}\colon A_{g'} \otimes_{A_e} A_g \xrightarrow{\cong} A_{gg'}\}_{g,g\in G}$; that is, each A_g is an invertible (A_e, A_e)–bimodule. Under the above assumption on A_e, these isomorphisms form a function

$\tau\colon G\times G\to(\Bbbk^*)^n$. Moreover, the relations involving these generators give the commutative diagram, for all $g,g',g''\in G$,

$$\begin{tikzcd}
(A_{g''}\otimes_{A_e}A_{g'})\otimes_{A_e}A_g \arrow[r, "\tau(g',g'')\otimes\mathrm{id}"] \arrow[d, "\cong"'] & A_{g'g''}\otimes_{A_e}A_g \arrow[r, "\tau(g,g'g'')"] & A_{gg'g''} \arrow[d, "\mathrm{id}"] \\
A_{g''}\otimes_{A_e}(A_{g'}\otimes_{A_e}A_g) \arrow[r, "\mathrm{id}\otimes\tau(g,g')"'] & A_{g''}\otimes_{A_e}A_{gg'} \arrow[r, "\tau(gg',g'')"'] & A_{gg'g''}
\end{tikzcd}$$

and isotopy classes of G–linear diagrams generate the relations, which can be expressed as the commutative diagram, for all $g\in G$,

$$\begin{tikzcd}
A_g\otimes_{A_e}A_e \arrow[r, "\tau(e,g)"] \arrow[dr, "\cong"'] & A_g \arrow[d, "\mathrm{id}"] & A_e\otimes_{A_e}A_g \arrow[l, "\tau(g,e)"'] \arrow[dl, "\cong"] \\
& A_g &
\end{tikzcd}$$

which imply that τ is a normalized 2–cocycle. Davidovich [6] showed that any invertible (A_e,A_e)–bimodule is isomorphic to one of the form

$$\mathrm{Hom}_{\Bbbk}(V_{\sigma(1)},V_1)\times\mathrm{Hom}_{\Bbbk}(V_{\sigma(2)},V_2)\times\cdots\times\mathrm{Hom}_{\Bbbk}(V_{\sigma(n)},V_n)$$

for some permutation $\sigma\in S_n$; denote this bimodule by A^σ. Since the direct sum $A=\bigoplus_{g\in G}A_g$ forms a G–algebra, permutations indeed form a homomorphism $\sigma\colon G\to S_n$.

It is known that all traces on a matrix algebra are given as some (nonzero) constant multiple of the matrix trace. Thus, in the case of $A_e=\mathrm{End}(V_1)\times\mathrm{End}(V_2)\times\cdots\times\mathrm{End}(V_n)$ there are constants $r_i\in\Bbbk^*$ for $i=1,\dots,n$ and the inner product of the quasibiangular G–algebra $A=\bigoplus_{g\in G}A^{\sigma(g)}$ is given by $\eta(f,g)=\mathrm{Tr}(r\circ(g\circ^\sigma f))$ for any $f\in A_g$ and $g\in A_{g^{-1}}$, where $r=(r_1\,\mathrm{id}_{V_1},\dots,r_n\,\mathrm{id}_{V_n})$ and $\circ^\sigma$ is the composition of morphisms under σ such as $f_i\circ g_{\sigma(i)}$ for $f_i\in\mathrm{Hom}_{\Bbbk}(V_{\sigma(i)},V_i)$ and $g_{\sigma(i)}\in\mathrm{Hom}_{\Bbbk}(V_{\sigma(\sigma(i))},V_{\sigma(i)})$. Since the inner product is invariant under cyclic order, ie $\eta(f,g\cdot h)=\eta(h\cdot f,g)=\eta(h,f\cdot g)$, the vector $r\in(\Bbbk^*)^n$ must satisfy $\mathrm{Im}(\sigma)\subseteq\mathrm{Stab}_{S_n}(r)$, where S_n acts on r by permuting the entries. More explicitly, as an example, consider the products $(h\circ^\sigma g)\circ^\sigma f$ and $g\circ^\sigma(f\circ^\sigma h)$ for $f\in A_g$, $g\in A_{g'}$ and $h\in A_{(gg')^{-1}}$. Then the corresponding traces of these morphisms in $A_e=\mathrm{End}(V_1)\times\mathrm{End}(V_2)\times\cdots\times\mathrm{End}(V_n)$ are related by the permutation $\sigma(gg')\in S_n$.

Using the above arguments, when $\Bbbk$ is an algebraically closed field of characteristic zero, we can conclude that, up to an isomorphism, a quasibiangular G–algebra

$\left(A=\bigoplus_{g\in G} A_g, \eta\right)$ is determined by a Morita class of the principal component ($n\geq 1$), a normalized 2–cocycle $\tau\colon G\times G\to (\Bbbk^*)^n$, a homomorphism $\sigma\colon G\to S_n$, and an element $r\in(\Bbbk^*)^n$ with $\mathrm{Im}(\sigma)\subseteq \mathrm{Stab}_{S_n}(r)$.

Let $({}_{A_2}M_{A_1}, {}_{A_1}N_{A_2}, \kappa, \mu)$ be a graded Morita context between two quasibiangular G–algebras (A_1,η_1) and (A_2,η_2) which are determined by the normalized 2–cocycles τ_i, homomorphisms $\sigma_i : G\to S_n$, and elements $r_i\in(\Bbbk^*)^n$ with $\mathrm{Im}(\sigma_i)\subseteq \mathrm{Stab}_{S_n}(r_i)$ for $i=1,2$. Then M and N are invertible (A_2, A_1)– and (A_1, A_2)–bimodules, respectively, which means there exists $\sigma\in S_n$ such that M_e is isomorphic to

$$M^\sigma = \mathrm{Hom}(V_{\sigma(1)}, W_1)\times \mathrm{Hom}(V_{\sigma(2)}, W_2)\times\cdots\times \mathrm{Hom}(V_{\sigma(n)}, W_n),$$

where $(A_i)_g = \mathrm{Hom}(V_{\sigma_i^g(1)}, V_1)\times\cdots\times \mathrm{Hom}(V_{\sigma_i^g(n)}, V_n)$ for all $g\in G$ and $\sigma_i^g = \sigma_i(g)\in S_n$ for $i=1,2$. Being a graded (A_2, A_1)–bimodule forces σ to satisfy $\sigma_2^g = \sigma\sigma_1^g\sigma^{-1}$ for all $g\in G$. In this case, nonprincipal components are given as $M_g = \mathrm{Hom}(V_{\sigma'_g(1)}, W_1)\times\cdots\times \mathrm{Hom}(V_{\sigma'_g(n)}, W_n)$, where $\sigma'_g=\sigma\sigma_1^g=\sigma_2^g\sigma$ for all $g\in G$. Using the similar arguments for the invertible (A_1, A_2)–bimodule N, we obtain $N_g = \mathrm{Hom}(W_{\sigma''_g(1)}, V_1)\times\cdots\times\mathrm{Hom}(W_{\sigma''_g(n)}, V_n)$ for $\sigma''_g=\sigma_1^{g^{-1}}\sigma^{-1}=\sigma^{-1}\sigma_2^{g^{-1}}$ for all $g\in G$.

Transferring the Frobenius form via the graded Morita context amounts to finding a central element in $(A_2)_e$ corresponding to $r_1\in(\Bbbk^*)^n$. Using the identity component M_e, this element is given by $\sigma(r_1)(\mathrm{id}_{W_1}, \mathrm{id}_{W_2},\dots,\mathrm{id}_{W_n})$. Thus, we have the equality $\sigma(r_1)=r_2\in(\Bbbk^*)^n$. The bimodule isomorphisms $\kappa\colon {}_{A_1}(A_1)_{A_1}\to {}_{A_1}N\otimes_{A_2} M_{A_1}$ and $\mu\colon {}_{A_2}M\otimes_{A_1} N_{A_2}\to {}_{A_2}(A_2)_{A_2}$ lead to a map $\phi\colon G\to(\Bbbk^*)^n$ and the graded Morita context equations produce a map $\phi\colon (A_1)_g\to(A_2)_g$ for all $g\in G$ such that the diagram

$$\begin{tikzcd}
(A_1)_g\otimes(A_1)_h \arrow[r, "\phi_1(g{,}h)"] \arrow[d, "\phi(g)\phi(h)"'] & (A_1)_{gh} \arrow[d, "\phi(gh)"] \\
(A_2)_g\otimes(A_2)_h \arrow[r, "\phi_2(g{,}h)"'] & (A_2)_{gh}
\end{tikzcd}$$

commutes for all $g,h\in G$. This means that normalized 2–cocycles $\phi_1,\phi_2\colon G\times G\to(\Bbbk^*)^n$ differ by a coboundary $\partial\phi$. Thus, we conclude that quasibiangular G–algebras, up to compatible G–graded Morita contexts, are in bijection with

$$\coprod_{r=1}^{\infty}\ \coprod_{[r]\in(\Bbbk^*)^n/S_n} H^2(G;(\Bbbk^*)^n)\times\mathrm{Hom}(G,\mathrm{Stab}_{S_n}(r))/\sim,$$

where the equivalence $\sim$ is given by conjugation. Using Theorem 4.24, we derive the following proposition, which was previously proven by Davidovich [6]:

Proposition 4.26 [6] *The set of equivalence classes of fully extended oriented 2–dimensional G–equivariant TFTs, ie EHFTs with $K(G,1)$ target, with values in $\mathrm{Alg}^2_{\Bbbk}$ is in bijection with*

$$\pi_0\mathrm{Fun}^{\otimes}(\mathrm{Bord}_2^{G\times\mathrm{SO}(2)}, \mathrm{Alg}^2_{\Bbbk}) \cong \coprod_{r=1}^{\infty} \coprod_{[r]\in(\Bbbk^*)^n/S_n} H^2(G;(\Bbbk^*)^n)\times \mathrm{Hom}(G,\mathrm{Stab}_{S_n}(r))/\sim,$$

where the equivalence $\sim$ is given by conjugation.

The fact that $\mathrm{Alg}^2_{\Bbbk}$–valued fully extended oriented 2–dimensional G–equivariant TFTs are classified in two different ways — namely using the structured cobordism hypothesis and without using it — is an important step towards the verification of the $(G\times\mathrm{SO}(2))$–structured cobordism hypothesis for $\mathrm{Alg}^2_{\Bbbk}$–valued such TFTs. In this case, the $(G\times\mathrm{SO}(2))$–structured cobordism hypothesis gives an equivalence of bigroupoids $\mathcal{E}\text{–}\mathcal{HFT}(X,\mathrm{Alg}^2_{\Bbbk}) \simeq ((\mathrm{Alg}^{\mathrm{fd}}_{\Bbbk})^{\sim})^{h(G\times\mathrm{SO}(2))}$. We have $\mathcal{E}\text{–}\mathcal{HFT}(X,\mathrm{Alg}^2_{\Bbbk}) \simeq \mathrm{Frob}^G$ by Theorem 4.24 and Davidovich [6] showed that the fundamental bigroupoid of $((\mathrm{Alg}^{\mathrm{fd}}_{\Bbbk})^{\sim})^{h(G\times\mathrm{SO}(2))}$ is equivalent to the bigroupoid $\mathrm{Grp}_2(BG,\mathcal{G}_{\mathrm{ori}})$ of 2–functors, transformations, and modifications. Here the bigroupoid BG has one object, $|G|$ 1–morphisms, and only identity 2–morphisms, and the bigroupoid $\mathcal{G}_{\mathrm{ori}}$ has semisimple Frobenius algebras as objects, invertible bimodules compatible with Frobenius forms as 1–morphisms, and invertible bimodule maps as 2–morphisms (see Proposition 3.3.2 in [6]).

Davidovich showed that every 2–functor $F\colon BG\to\mathcal{G}_{\mathrm{ori}}$ gives rise to a quasibiangular G–algebra, and vice versa [6, Proposition 3.4.5]. We define a 2–functor $\mathcal{F}\colon\mathrm{Frob}^G\to\mathrm{Grp}_2(BG,\mathcal{G}_{\mathrm{ori}})$ as follows. The image of a quasibiangular G–algebra is the corresponding functor described above. Next, let $({}_{A_2}M_{A_1}, {}_{A_1}N_{A_2}, \kappa, \mu)$ be a G–graded compatible Morita context between two quasibiangular G–algebras (A_1,η_1) and (A_2,η_2) which are determined by two triples (τ_i,σ_i,r_i) for $i=1,2$ such that $M_e\cong M^{\sigma}$, as in the proof of Proposition 4.26 above. Then we define $\mathcal{F}(({}_{A_2}M_{A_1}, {}_{A_1}N_{A_2}, \kappa, \mu))$ to be the natural transformation between the corresponding 2–functors producing the bimodule M^{σ} (see the proof of Proposition 4.26 given in [6]). Lastly, the image of an equivalence of G–graded Morita contexts under $\mathcal{F}$ is the modification producing the invertible bimodule map $M^{\sigma}\to M^{\sigma'}$ between the corresponding bimodules described above.

Proposition 4.26 implies that $\mathcal{F}$ is essentially surjective on objects. The fact that any natural transformation $\eta\colon F_1\to F_2$ between functors $F_1,F_2\colon BG\to\mathcal{G}_{\mathrm{ori}}$ is isomorphic to one producing a bimodule of the form M^{σ} (see the proof of Proposition 4.26 given in [6]) implies that $\mathcal{F}$ is essentially full on 1–morphisms. Using similar arguments as in the proof of Theorem 4.24, one can show that $\mathcal{F}$ is fully faithful on 2–morphisms. Consequently, the 2–functor $\mathcal{F}\colon \mathrm{Frob}^G \to \mathrm{Grp}_2(BG, \mathcal{G}_{\mathrm{ori}})$ is an equivalence by the Whitehead theorem for bicategories (Theorem 4.3).

Corollary 4.27 *For any discrete group G and any algebraically closed field $\Bbbk$ of characteristic zero, the $(G\times \mathrm{SO}(2))$–structured cobordism hypothesis for $\mathrm{Alg}^2_{\Bbbk}$–valued oriented EHFTs with target $X \simeq K(G,1)$ holds true.*

5 Extended unoriented X–HFTs and their classifications

In this section, by allowing X–manifolds and X–cobordisms to be nonorientable, we define 2–dimensional extended unoriented X–HFTs and classify them. The definition and classification of these theories are parallel with the oriented case and we only describe the changes. All of the manifolds and cobordisms in this section do not have any orientation data and they are not necessarily orientable.

5.1 The unoriented X–cobordism bicategory and its presentation

In this section, we define 2–dimensional extended unoriented HFTs with target $X \simeq K(G,1)$, where every element of G has order two. To avoid repetition, from now on, we assume that G is such a group and X is a pointed $K(G,1)$–space. The restriction to such groups is not essential but for convenience.[8] As in the oriented case, the 2–dimensional extended unoriented X–cobordism bicategory plays the essential role and it is defined using X–halations (see Section 2.2) as follows:

Definition 5.1 The *2–dimensional extended unoriented X–cobordism bicategory* $X\mathrm{Bord}_2^{\mathrm{un}}$ has

- quadruples $(M, \widehat{M}_1, \widehat{M}_2, \widehat{\mathrm{g}}_2)$ consisting of compact 0–manifolds equipped with two cooriented X–halations as objects,
- quintuples $(A, \widehat{A}_0, \widehat{A}_1, T, \widehat{\mathrm{p}}_1)$ consisting of 1–dimensional marked X–manifolds equipped with two cooriented X–halations as 1–morphisms,
- isomorphism classes $[(S, \widehat{S}, R, \widehat{\mathrm{F}})]$ of quadruples consisting of cobordism type $\langle 2\rangle$–X–surfaces equipped with a codimension zero X–halation as 2–morphisms.

[8] Order two elements appear as a result of new generators and we avoid keeping track of which elements are required to have order two and which ones are not. See [12] for the case of nonextended HFTs with arbitrary aspherical targets.

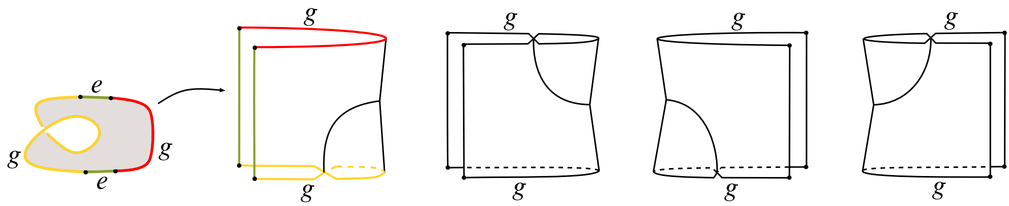


Figure 31: Additional generating 2–morphisms of $\mathcal{X}\mathcal{G}_2^{\mathrm{un}}$.

Similar to $X\,\mathrm{Bord}_2$, the disjoint union operation is the symmetric monoidal product for $X\,\mathrm{Bord}_2^{\mathrm{un}}$.

Definition 5.2 For a symmetric monoidal bicategory $\mathcal{C}$, a $\mathcal{C}$*–valued* 2*–dimensional extended unoriented homotopy field theory with target* X is a symmetric monoidal 2–functor from $X\,\mathrm{Bord}_2^{\mathrm{un}}$ to $\mathcal{C}$.

It is not hard to modify G–linear, G–planar, and G–spatial diagrams for the unoriented setting. We only need to consider sheets with no additional orientation data. In this case, each diagram produces an unoriented version of the corresponding X–manifold. There are new sheet data for the fold singularity coming from $\langle 2\rangle$–X–surface representatives of a Möbius band. These $\langle 2\rangle$–X–surfaces are shown in Figure 31. This extra sheet data produces new relations coming from different possible gluings of these generators with themselves and with the earlier (orientable) generators. Figure 32 shows these relations of 2–morphisms of $X\,\mathrm{Bord}_2^{\mathrm{un}}$ instead of the corresponding unoriented G–planar diagrams. By generalizing the equivalence relations on the set of unoriented G–planar

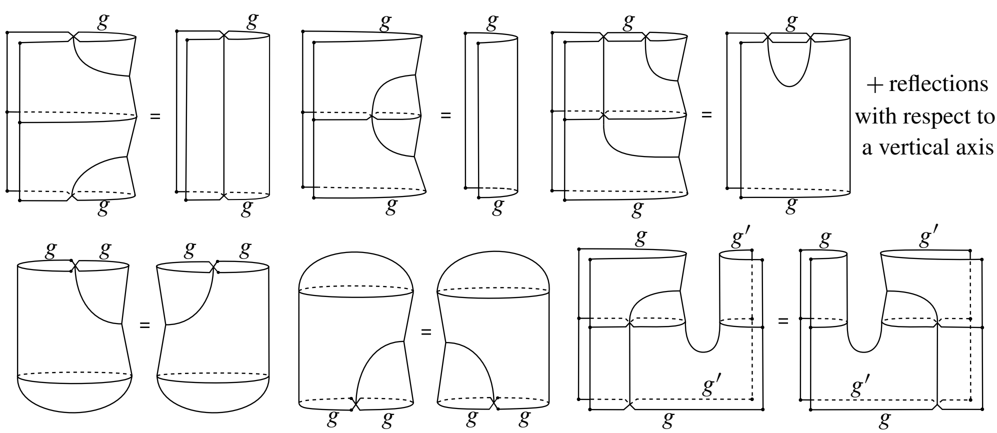


Figure 32: Additional generating relations of $\mathcal{X}\mathcal{R}^{\mathrm{un}}$.

diagrams given by unoriented G–spatial diagrams, we obtain the following unoriented G–planar decomposition theorem:

Theorem 5.3 *The relative X–homeomorphism classes of unoriented cobordism type $\langle 2\rangle$–X–surfaces are in bijection with the equivalence classes of unoriented G–planar diagrams.*

Parallel to the oriented case, we define a new symmetric monoidal bicategory $\mathrm{XB}^{\mathrm{PD,un}}$, which has unoriented points as objects, unoriented G–linear diagrams as 1–morphisms, and equivalence classes of unoriented G–planar diagrams as 2–morphisms. The unoriented G–planar decomposition theorem implies that $\mathrm{XB}^{\mathrm{PD,un}}$ is symmetric monoidally equivalent to $X\mathrm{Bord}_2^{\mathrm{un}}$ via the Whitehead theorem for symmetric monoidal bicategories. The correspondence between G–planar diagrams and the string diagrams for unbiased semistrict symmetric monoidal bicategories gives the following result:

Theorem 5.4 *The symmetric monoidal bicategory $\mathrm{XB}^{\mathrm{PD,un}}$ is a computadic unbiased semistrict symmetric monoidal 2–category with the presentation*

$$\mathbb{XP}^{\mathrm{un}} = (\mathcal{XG}_0^{\mathrm{un}}, \mathcal{XG}_1^{\mathrm{un}}, \mathcal{XG}_2^{\mathrm{un}}, \mathcal{XR}^{\mathrm{un}})$$

which has one generating object, $\{\bullet\}$, G–linear diagrams of $\{\overset{g}{\bullet\!\!-\!\!\bullet}, \supset^{g}, {}^{g}\!\subset\}_{g\in G}$ as the generating 1–morphisms, G–planar diagram versions of elements in Figures 25 and 31 as the generating 2–morphisms, and G–planar diagram versions of pairs in Figures 26 and 32 as the generating relations.

The bicategory $\mathcal{E}$–$\mathcal{HFT}^{\mathrm{un}}(X, \mathcal{C})$ of 2–dimensional extended unoriented X–HFTs has $\mathcal{C}$–valued unoriented extended X–HFTs as objects, symmetric monoidal transformations as 1–morphisms, and symmetric monoidal modifications as 2–morphisms. For any given symmetric monoidal bicategory $\mathcal{C}$, using the cofibrancy theorem, we state the classification of $\mathcal{C}$–valued 2–dimensional extended unoriented X–HFTs as the equivalence of bicategories $\mathcal{E}$–$\mathcal{HFT}^{\mathrm{un}}(X, \mathcal{C}) \simeq \mathbb{XP}^{\mathrm{un}}(\mathcal{C})$.

Remark There is a symmetric monoidal 2–functor $\mathrm{Forget}^{\mathrm{or}}\colon X\mathrm{Bord}_2 \to X\mathrm{Bord}_2^{\mathrm{un}}$ given by forgetting the orientation. In the same way, any oriented or unoriented 2–dimensional extended TFT leads to an oriented or unoriented extended HFT, respectively, by forgetting the X–manifold data. The diagram

$$\begin{tikzcd}
\mathcal{E}\text{–}\mathcal{TFT}^{\mathrm{un}}(\mathcal{C}) \arrow[r, "\mathrm{Forget}^{\mathrm{or}}"] \arrow[d, "\mathrm{Forget}^{X}"'] & \mathcal{E}\text{–}\mathcal{TFT}(\mathcal{C}) \arrow[d, "\mathrm{Forget}^{X}"] \\
\mathcal{E}\text{–}\mathcal{HFT}^{\mathrm{un}}(X,\mathcal{C}) \arrow[r, "\mathrm{Forget}^{\mathrm{or}}"'] & \mathcal{E}\text{–}\mathcal{HFT}(X,\mathcal{C})
\end{tikzcd}$$

indicates the universality of unoriented 2–dimensional extended TFTs in this context, where $\mathcal{E}\text{–}\mathcal{TFT}^{\mathrm{un}}(\mathcal{C})$ and $\mathcal{E}\text{–}\mathcal{TFT}(\mathcal{C})$ are defined similarly using $\mathrm{Bord}_2^{\mathrm{un}}$ and Bord_2, respectively.

5.2 $\mathrm{Alg}^2_{\mathbb{k}}$–valued 2–dimensional extended unoriented X–HFTs

Tagami [26] classified 2–dimensional nonextended unoriented HFTs by extended crossed Frobenius G–algebras (see also [12]). Similar to the oriented case, our goal is to understand the relation between his classification and the restriction of $\mathrm{Alg}^2_{\mathbb{k}}$–valued 2–dimensional extended unoriented HFTs to circles and cobordisms between them.

Firstly, we introduce necessary algebraic notions. Let K be a G–algebra and V be a (K, K^{op})–bimodule. The *conjugate* of V is the (K, K^{op})–bimodule $\underline{V}$ obtained by turning actions around. Similarly, the *conjugate* of a graded Morita context $\zeta = ({}_{K^{\mathrm{op}}}U_K, {}_K V_{K^{\mathrm{op}}}, \tau, \mu)$ is given by $\underline{\zeta} = ({}_{K^{\mathrm{op}}}\underline{U}_K, {}_K\underline{V}_{K^{\mathrm{op}}}, \underline{\tau}, \underline{\mu})$. We generalize stellar algebras introduced in [22] to stellar G–algebras as follows:

Definition 5.5 A *stellar G–algebra* is a G–algebra $K = \bigoplus_{g\in G} K_g$ equipped with a G–graded Morita context $\zeta = ({}_{K^{\mathrm{op}}}U_K, {}_K V_{K^{\mathrm{op}}}, \tau, \mu)$ together with an isomorphism of G–graded Morita contexts $\sigma\colon \zeta \cong \underline{\zeta}$ such that $\sigma \circ \underline{\sigma}$ is the identity isomorphism, where $\underline{\sigma}$ is the induced isomorphism between $\underline{\underline{\zeta}}$ and $\underline{\zeta}$.

The stellar structure on a G–algebra can be transferred along a graded Morita context as follows. Let $\rho = ({}_K U'_L, {}_L V'_K, \kappa, \nu)$ be a G–graded Morita context between G–algebras K and L and let (K, ζ, σ) be a stellar structure on K with $\zeta = ({}_{K^{\mathrm{op}}}U_K, {}_K V_{K^{\mathrm{op}}}, \tau, \mu)$. Then $(L, \rho_*\zeta, \rho_*\sigma)$ is a stellar algebra, where

$$\rho_*\zeta = ({}_{L^{\mathrm{op}}}\underline{U}' \otimes_{K^{\mathrm{op}}} U \otimes_K U'_L, {}_L V' \otimes_K V \otimes_{K^{\mathrm{op}}} \underline{V}'_{L^{\mathrm{op}}}, \underline{\kappa} \otimes \tau \otimes \kappa, \underline{\nu} \otimes \mu \otimes \nu)$$

and $\rho_*\sigma\colon \rho_*\zeta \cong \underline{\rho_*\zeta}$ is given by σ.

Definition 5.6 Let (K,ζ,σ) be a stellar G–algebra with $\zeta=({}_{K^{\mathrm{op}}}U_K, {}_K V_{K^{\mathrm{op}}}, \tau, \mu)$ and let (K,η) be a quasibiangular G–algebra. The stellar structure is said to be *compatible* with the quasibiangular G–algebra if there exists an element $\sum_j a_j \otimes b_j \in K_e \otimes K_e$ giving the central element $z=\sum_j b_j a_j$ such that the diagrams[9]

$$\begin{tikzcd}
K\otimes K \arrow[r, "\tau\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\tau"'] & (V\otimes U)\otimes K \arrow[r, "\sigma\otimes\mathrm{id}"] & (\underline{V}\otimes\underline{U})\otimes K \arrow[r, "\underline{\tau}^{-1}\otimes\mathrm{id}"] & K\otimes K \arrow[d, "\eta"] \\
K\otimes(V\otimes U) \arrow[r, "\mathrm{id}\otimes\sigma"'] & K\otimes(\underline{V}\otimes\underline{U}) \arrow[r, "\mathrm{id}\otimes\underline{\tau}^{-1}"'] & K\otimes K \arrow[r, "\eta"'] & \Bbbk
\end{tikzcd}$$

$$\begin{tikzcd}
\Bbbk \arrow[r, "\iota"] \arrow[d, "\iota"'] & K\otimes K \arrow[r, "\tau\otimes\mathrm{id}"] & (V\otimes U)\otimes K \arrow[r, "\sigma\otimes\mathrm{id}"] & (\underline{V}\otimes\underline{U})\otimes K \arrow[d, "\underline{\tau}^{-1}\otimes\mathrm{id}"] \\
K\otimes K \arrow[r, "\mathrm{id}\otimes\tau"'] & K\otimes(V\otimes U) \arrow[r, "\mathrm{id}\otimes\sigma"'] & K\otimes(\underline{V}\otimes\underline{U}) \arrow[r, "\mathrm{id}\otimes\underline{\tau}^{-1}"'] & K\otimes K
\end{tikzcd}$$

$$\begin{tikzcd}
K\otimes K \arrow[r, "\tau\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\tau"'] & (V\otimes U)\otimes K \arrow[r, "\sigma\otimes\mathrm{id}"] & (\underline{V}\otimes\underline{U})\otimes K \arrow[r, "\underline{\tau}^{-1}\otimes\mathrm{id}"] & K\otimes K \arrow[d, "\xi"] \\
K\otimes(V\otimes U) \arrow[r, "\mathrm{id}\otimes\sigma"'] & K\otimes(\underline{V}\otimes\underline{U}) \arrow[r, "\mathrm{id}\otimes\underline{\tau}^{-1}"'] & K\otimes K \arrow[r, "\xi"'] & K\otimes K
\end{tikzcd}$$

commute, where $\iota(1)|_{K_e\otimes K_e}=\sum_j a_j\otimes b_j$ and $\xi\colon {}_{K_1}K_{K_2}\otimes {}_{K_3}K_{K_4}\to {}_{K_1}K_{K_4}\otimes {}_{K_3}K_{K_2}$ is a graded bimodule map with $K_i=K$ for $i=1,2,3,4$ and $\xi(1)=\sum_i p_i^e\otimes q_i^e$ is an inner product element of the principal component. We call such a compatible quadruple (K,η,ζ,σ) a *quasibiangular stellar G–algebra*.

Definition 5.7 A morphism of quasibiangular stellar G–algebras $(K,\eta^k,\zeta^K,\sigma^K)$ and $(L,\eta^L,\zeta^L,\sigma^L)$ is a compatible G–graded Morita context $\rho=({}_KU_L, {}_LV_K,\tau,\mu)$ together with an equivalence of G–graded Morita contexts $\phi\colon \zeta^L\to\rho_*\zeta^K$ such that $\rho_*\sigma^K\circ\phi=\underline{\phi}\circ\sigma^L$, where $\underline{\phi}\colon\underline{\zeta}^L\to\underline{\rho_*\zeta}^K$. Two such morphisms (ρ,ϕ) and (ρ',ϕ') are isomorphic if there exists an equivalence of G–graded Morita contexts $\alpha\colon\rho\to\rho'$ such that $\phi'=\alpha\circ\phi$ and $\underline{\phi}'=\underline{\alpha}\circ\underline{\phi}$ for $\underline{\alpha}\colon\underline{\rho}\to\underline{\rho}'$.

Theorem 5.8 *Let G be a group with each nonidentity element having order* 2 *and X be a $K(G,1)$–space. Any $\mathrm{Alg}^2_{\Bbbk}$–valued* 2*–dimensional extended unoriented X–HFT $Z\colon X\mathrm{Bord}_2^{\mathrm{un}}\to\mathrm{Alg}^2_{\Bbbk}$ whose precomposition $\mathrm{XB}^{\mathrm{PD,un}}\xrightarrow{\simeq} X\mathrm{Bord}_2^{\mathrm{un}}\xrightarrow{Z}\mathrm{Alg}^2_{\Bbbk}$ gives a*

[9] Tensors in diagrams are taken over K, K^{op} or $K\otimes_{\Bbbk}K^{\mathrm{op}}$.

strict symmetric monoidal 2–functor determines a quasibiangular stellar G–algebra (A, η, ζ, σ). Conversely, for any quasibiangular stellar G–algebra (A, η, ζ, σ), there exists an $\mathrm{Alg}^2_{\Bbbk}$–valued 2–dimensional extended unoriented X–HFT.

Proof Let $Z\colon X\mathrm{Bord}^{\mathrm{un}}_2 \to \mathrm{Alg}^2_{\Bbbk}$ be such a 2–dimensional extended unoriented HFT. The cofibrancy theorem implies that there exists an object Z' in $\mathbb{XP}^{\mathrm{un}}(\mathrm{Alg}^2_{\Bbbk})$ such that $\iota(Z')$ is the composition $\mathrm{XB}^{\mathrm{PD,un}} \xrightarrow{\simeq} X\mathrm{Bord}^{\mathrm{un}}_2 \xrightarrow{Z} \mathrm{Alg}^2_{\Bbbk}$, where $\iota\colon \mathbb{XP}^{\mathrm{un}}(\mathrm{Alg}^2_{\Bbbk}) \to \mathrm{SymMon}(\mathrm{XB}^{\mathrm{PD,un}}, \mathrm{Alg}^2_{\Bbbk})$ is the equivalence of bicategories.

Following the proof of Theorem 4.19, we have a strongly graded G–algebra $A = \bigoplus_{g\in G} A_g$, where $Z'(\bullet) = A_e$. We also have G–graded $(A\otimes A, \Bbbk)$– and $(\Bbbk, A\otimes A)$–bimodules $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$, respectively. By turning actions around, we obtain the (A, A^{op})–bimodule M and (A^{op}, A)–bimodule N.

Bimodule maps in $Z'_2(\mathcal{X}^{\mathrm{un}}\mathcal{G}_2)$ corresponding to cusp generators (subject to relations) yield a G–graded Morita context $\zeta = ({}_{A^{\mathrm{op}}}N_A, {}_AM_{A^{\mathrm{op}}}, f_1, f_2)$ between A and A^{op}, where $f_1\colon {}_AA_A \to {}_AM \otimes_{A^{\mathrm{op}}} N_A$ and $f_2\colon {}_{A^{\mathrm{op}}}N \otimes_A M_{A^{\mathrm{op}}} \to {}_{A^{\mathrm{op}}}A^{\mathrm{op}}_{A^{\mathrm{op}}}$ are invertible G–graded bimodule maps. Bimodule maps in $Z'_2(\mathcal{X}^{\mathrm{un}}\mathcal{G}_2)$ for the Morse generators satisfying the relations imply that (A, η) is a quasibiangular G–algebra. The generators in Figure 31 give the graded bimodule maps, in $Z'_2(\mathcal{X}^{\mathrm{un}}\mathcal{G}_2)$,

$$\sigma_1\colon {}_AM_{A^{\mathrm{op}}} \to {}_A\underline{M}_{A^{\mathrm{op}}}, \quad \sigma_2\colon {}_{A^{\mathrm{op}}}N_A \to {}_{A^{\mathrm{op}}}\underline{N}_A,$$
$$\sigma'_1\colon {}_A\underline{M}_{A^{\mathrm{op}}} \to {}_AM_{A^{\mathrm{op}}}, \quad \sigma'_2\colon {}_{A^{\mathrm{op}}}\underline{N}_A \to {}_{A^{\mathrm{op}}}N_A.$$

These graded bimodule maps are subject to the relations in Figure 32. Thereby, we have $\sigma'_1\circ\sigma_1 = \mathrm{id}_M$, $\sigma_1\circ\sigma'_1 = \mathrm{id}_{\underline{M}}$, $\sigma'_2\circ\sigma_2 = \mathrm{id}_N$, and $\sigma_2\circ\sigma'_2 = \mathrm{id}_{\underline{N}}$. These isomorphisms of bimodules lead to an isomorphism $\sigma\colon \zeta \cong \underline{\zeta}$. Applying σ to $\underline{\underline{\zeta}}$ gives another isomorphism $\underline{\sigma}\colon \underline{\underline{\zeta}} \to \underline{\zeta}$, whose composition with σ gives $\sigma\circ\underline{\sigma}\colon \underline{\underline{\zeta}} \cong \overline{\overline{\zeta}}$. The third relation in the first row of Figure 32 and its reflection indicate that compositions of bimodule maps $\underline{\underline{M}} \to \underline{M} \to M$ and $\underline{\underline{N}} \to \underline{N} \to N$ are identity maps.

Thus, additional generators and relations among them lead to a stellar structure (ζ, σ) on the quasibiangular G–algebra A. The remaining relations imply compatibility, giving the quasibiangular stellar G–algebra (A, η, ζ, σ). For any quasibiangular stellar G–algebra, one constructs an object of $\mathbb{XP}^{\mathrm{un}}(\mathrm{Alg}^2_{\Bbbk})$ by assigning values to generating objects, 1–morphisms, and 2–morphisms of $\mathbb{XP}$ satisfying generating relations using the above arguments. Then, this object gives a strict symmetric monoidal 2–functor $\mathrm{XB}^{\mathrm{PD,un}} \to \mathrm{Alg}^2_{\Bbbk}$ whose composition with the equivalence $X\mathrm{Bord}^{\mathrm{un}}_2 \xrightarrow{\simeq} \mathrm{XB}^{\mathrm{PD,un}}$ produces the desired unoriented extended X–HFT. □

Similar to the oriented case, every 2–dimensional extended unoriented HFT with target X produces a nonextended one by precomposition, $X\mathrm{Cob}_2^{\mathrm{un}} \to X\mathbb{C}\mathbb{o}\mathbb{b}_2^{\mathbb{u}\mathbb{n}} \to \mathrm{Alg}_{\Bbbk}^2$, where $X\mathrm{Cob}_2^{\mathrm{un}}$ and $X\mathbb{C}\mathbb{o}\mathbb{b}_2^{\mathbb{u}\mathbb{n}}$ are defined just as $X\mathrm{Cob}_2$ and $X\mathbb{C}\mathbb{o}\mathbb{b}_2$ using unoriented X–manifolds. In the unoriented case, extended crossed Frobenius G–algebras play an important role in the study of 2–dimensional nonextended unoriented X–HFTs and they are defined as follows:

Definition 5.9 [26] Let (K,η,φ) be a crossed Frobenius G–algebra over $\Bbbk$. An *extended* structure on K consists of a $\Bbbk$–module homomorphism $\Phi\colon K \to K$ and a family of elements $\{\theta_g \in K_e\}_{g\in G}$ satisfying the following conditions:

(1) $\Phi(K_g) \subset K_g$ and $\Phi(\theta_g) = \theta_g$ for all $g \in G$.

(2) $\Phi \circ \varphi_g = \varphi_g \circ \Phi$ for all $g \in G$.

(3) $\Phi(vw) = \Phi(w)\Phi(v)$ for any $v, w \in K$ and $\Phi(1_K) = 1_K$.

(4) $\Phi^2 = \mathrm{id}$.

(5) $\eta \circ (\Phi \otimes \Phi) = \eta$.

(6) For any $g, h, l \in G$ and $v \in K_{gh}$, we have

$$m \circ (\Phi \circ \varphi_l) \circ \Delta_{g,h}(v) = \varphi_l(\theta_{gl}\theta_l v),$$
$$m \circ (\varphi_l \otimes \Phi) \circ \Delta_{g,h}(v) = \varphi_l(\theta_{hl}\theta_l v),$$

where $\Delta_{g,h}\colon K_{gh} \to K_g \otimes K_h$ is defined by $(\mathrm{id}_g \otimes \eta) \circ (\Delta_{g,h} \otimes \mathrm{id}_h) = m$. Such a map $\Delta_{g,h}$ is uniquely determined since η is nondegenerate and each K_g is finitely generated.

(7) $\Phi(\theta_h v) = \varphi_{hg}(\theta_{hg} v)$ for any $g, h \in G$ and $v \in K_g$.

(8) $\varphi_h(\theta_g) = \theta_g$ for any $g, h \in G$.

(9) For any $g, h, l \in G$, we have $\theta_g\theta_h\theta_l = q(1)\theta_{ghl}$, where $q\colon \Bbbk \to K_e$ is defined as follows: let $\{a_i \in K_{gh}\}_{i=1}^n$ and $\{b_i \in K_{gh}\}_{i=1}^n$ be families of elements of K_{gh} satisfying $\sum_i \eta(b_i \otimes v)a_i = \varphi_{hl}(v)$ for any $v \in K_{gh}$. As in (3), such a_i and b_i are uniquely determined and $q(1) = \sum_i a_i b_i$.

Theorem 5.10 (Tagami [26]) *Let G be a group with each nonidentity element having order 2. There is a bijection between the isomorphism classes of 2–dimensional unoriented HFTs with target $X \simeq K(G,1)$ and the isomorphism classes of extended crossed Frobenius G–algebras.*

Corollary 5.11 *Assume that $Z\colon X\mathrm{Bord}_2^{\mathrm{un}} \to \mathrm{Alg}_{\Bbbk}^2$ determines a quasibiangular stellar G–algebra (A,η,ζ,σ). The stellar structure (ζ,σ) gives an extended structure on the crossed Frobenius G–algebra $Z_G(A)$. Moreover, the corresponding 2–dimensional X–HFT is the unoriented X–HFT obtained by restricting Z to $X\mathbb{C}\mathrm{ob}_2^{\mathrm{un}}$.*

Proof We have a crossed Frobenius G–algebra $(Z_G(A), \eta|_{Z_G(A)}, \{\varphi|_{Z_G(A)}\}_{g\in G})$. By Tagami's classification, the 2–dimensional unoriented HFT given by the restriction of Z to circles and cobordisms between them induces an extended structure on $Z_G(A)$. We claim that the homomorphism Φ and elements $\{\theta_g \in Z_G(A)_e\}_{g\in G}$ come from the stellar structure (ζ,σ) on A.

In [26], for each $g\in G$, the restriction $\Phi|_{Z_G(A)_g}\colon Z_G(A)_g \to Z_G(A)_g$ is the involution induced by an orientation-reversing homeomorphism of a g–labeled circle. In the extended case, this morphism is given by additional 2–morphisms (Figure 31). More precisely, $\Phi|_{Z_G(A)_g}\colon A_e \otimes_{A_e\otimes A_e^{\mathrm{op}}} A_g \to A_e \otimes_{A_e\otimes A_e^{\mathrm{op}}} A_g$ is defined by $\Phi(a\otimes b) = a\otimes \Phi_g(b)$, where Φ_g is defined so that the diagram

$$\begin{tikzcd}
{}_{A_e}M_g \otimes_{A_e^{\mathrm{op}}} (N_e)_{A_e} \arrow[r, "\cong"] \arrow[d, "\sigma_1\otimes\mathrm{id}"'] & {}_{A_e}(A_g)_{A_e} \arrow[d, "\Phi_g = Z(\ )"] \\
{}_{A_e}\underline{M_g} \otimes_{A_e^{\mathrm{op}}} (N_e)_{A_e} \arrow[r] & {}_{A_e}(A_g)_{A_e}
\end{tikzcd}$$

commutes. It is not hard to see that Φ reverses the orientation of the oriented (input) circle. In [26], for every $g\in G$, the element θ_g is the image of HFT under the Möbius strip whose boundary is labeled by $g^2=e$, where the Möbius strip is considered as the cobordism from the empty 1–manifold to the boundary circle. In the extended case, $\theta_g \in A_e \otimes_{A_e\otimes A_e^{\mathrm{op}}} A_e$ is the image of $1\in\Bbbk$ under the composition of a $\{g,g\}$–labeled cap morphism followed by new generators (see Figure 32), which is composed with module actions turning boundary labels into $\{e,e\}$ (see Figure 29).

The involution Φ and elements $\{\theta_g\}_{g\in G}$ are defined according to their topological description given in [26]. Hence, $(Z_G(A), \eta|_{Z_G(A)}, \{\varphi_g|_{Z_G(A)}\}_{g\in G}, \Phi, \{\theta_g\}_{g\in G})$ is an extended crossed Frobenius G–algebra, which, by definition, corresponds to the restriction of $Z\colon X\mathrm{Bord}_2^{\mathrm{un}} \to \mathrm{Alg}_{\Bbbk}^2$ to X–circles and unoriented X–cobordisms between them. □

5.3 The bicategory of 2–dimensional extended unoriented X–HFTs

In order to upgrade Theorem 5.8 to an equivalence of bicategories, we study morphisms in the bicategory $\mathbb{XP}^{\mathrm{un}}(\mathrm{Alg}_{\Bbbk}^2)$. Let α be a 1–morphism from Z_0 to Z_1 giving quasibiangular stellar G–algebras (A,η,ζ,σ) and $(A',\eta',\zeta',\sigma')$, respectively. We know from the oriented case that α gives a compatible G–graded Morita context ξ between G–algebras A and A'. Assuming $\alpha_0(\bullet) = {}_{A_e}R_{A'_e}$ and $\xi = ({}_AR_{A'}, {}_{A'}R'_A, \tau, \mu)$, naturality with respect to the first generator in Figure 31 is the commutativity of the diagram

$$\begin{tikzcd}
{}_{A'_e}(M'_g)_{(A'_e)^{\mathrm{op}}} \arrow[r, "\alpha_1(g)"] \arrow[d, "\sigma'=Z_1(g)"'] & {}_{A'_e}R' \otimes_{A_e} M_g \otimes_{A_e^{\mathrm{op}}} \underline{R'}_{(A'_e)^{\mathrm{op}}} \arrow[d, "\xi_*\sigma=Z_0(g)"] \\
{}_{A'_e}(\underline{M'_g})_{(A'_e)^{\mathrm{op}}} \arrow[r, "\alpha_1(g)"'] & {}_{A'_e}R' \otimes_{A_e} \underline{M_g} \otimes_{A_e^{\mathrm{op}}} \underline{R'}_{(A'_e)^{\mathrm{op}}}
\end{tikzcd}$$

where M and M' are components of the graded Morita contexts ζ and ζ', respectively. There are similar commutative diagrams for the remaining three generators. These diagrams indicate that the G–graded Morita context ξ gives an equivalence of G–graded Morita contexts ζ' and $\xi_*\zeta$ with $\underline{\alpha}\circ\sigma' = \xi_*\sigma\circ\alpha$. In other words, α leads to a morphism of stellar G–algebras (see Definition 5.7).

Let $\theta\colon \alpha^1 \to \alpha^2$ be a 2–morphism in $\mathbb{XP}(\mathrm{Alg}^2_{\Bbbk})$ with $\theta_0(\bullet) = {}_{A_e}R_{A'_e} \to {}_{A_e}P_{A'_e}$. In the oriented case we observed that θ induces an equivalence of G–graded Morita contexts $\xi = ({}_AR_{A'}, {}_{A'}R'_A, \tau, \mu)$ and $\rho = ({}_AP_{A'}, {}_{A'}P'_A, \kappa, \nu)$. Naturality of $\theta_0(\bullet)$ with respect to g is the commutativity of the diagram

$$\begin{tikzcd}
{}_{A'_e}(\underline{M'_g})_{(A'_e)^{\mathrm{op}}} \arrow[r, "\alpha^1_1(g)"] \arrow[d, "\mathrm{id}"'] & {}_{A'_e}R' \otimes_{A_e} \underline{M_g} \otimes_{A_e^{\mathrm{op}}} \underline{R'}_{(A'_e)^{\mathrm{op}}} \arrow[d, "\theta_0(\bullet\,\bullet)"] \\
{}_{A'_e}(\underline{M'_g})_{(A'_e)^{\mathrm{op}}} \arrow[r, "\alpha^2_1(g)"'] & {}_{A'_e}P' \otimes_{A_e} \underline{M_g} \otimes_{A_e^{\mathrm{op}}} \underline{P'}_{(A'_e)^{\mathrm{op}}}
\end{tikzcd}$$

and there is a similar diagram for the naturality with respect to g. Naturality for $\{g\}_{g\in G}$ and $\{g\}_{g\in G}$ gives $\alpha^2 = \theta\circ\alpha^1$ and naturality for $\{g\}_{g\in G}$ and $\{g\}_{g\in G}$ gives $\underline{\alpha_2} = \underline{\theta}\circ\underline{\alpha^1}$. In other words, θ gives an isomorphism of stellar G–algebra morphisms (see Definition 5.7).

These observations lead us to define a bicategory Frob^G_*, which has quasibiangular stellar G–algebras as objects, their morphisms as 1–morphisms, and isomorphisms of quasibiangular stellar G–algebra morphisms as 2–morphisms. The above arguments imply that there exists a 2–functor $\mathcal{F}'\colon \mathbb{XP}^{\mathrm{un}}(\mathrm{Alg}^2_{\Bbbk}) \to \mathrm{Frob}^G_*$. Composing $\mathcal{F}'$ with the equivalence $\mathcal{E}$–$\mathcal{HFT}^{\mathrm{un}}(X,\mathrm{Alg}^2_{\Bbbk}) \simeq \mathbb{XP}^{\mathrm{un}}(\mathrm{Alg}^2_{\Bbbk})$, we define the 2–functor $\mathcal{F}$.

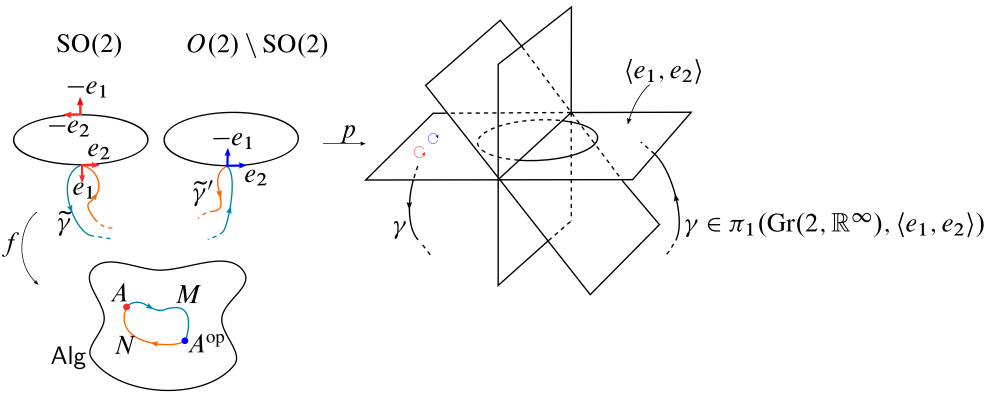


Figure 33: A reflection-invariant map.

Theorem 5.12 *The* 2*–functor* $\mathcal{F}\colon \mathcal{E}\text{–}\mathcal{HFT}^{\mathrm{un}}(X, \mathrm{Alg}^2_{\Bbbk}) \to \mathrm{Frob}^G_*$ *is an equivalence of bicategories.*

Proof The proof follows from the above arguments and the Whitehead theorem for bicategories. □

5.4 The $(G \times O(2))$–structured cobordism hypothesis

Parallel to oriented case, we want to compare Theorem 5.12 with the classification given by the $(G\times O(2))$–structured cobordism hypothesis. To do this, we need to understand homotopy $(G\times O(2))$–fixed points in $(\mathrm{Alg}^{\mathrm{fd}}_{\Bbbk})^{\sim}$, which are given by

$$((\mathrm{Alg}^{\mathrm{fd}}_{\Bbbk})^{\sim})^{h(G\times O(2))} = \mathrm{Map}_G\big(EG, \mathrm{Map}_{O(2)}(EO(2), \mathsf{Alg})\big),$$

where G acts on invariant maps trivially and Alg is the 2–type corresponding to the ∞–groupoid $(\mathrm{Alg}^{\mathrm{fd}}_{\Bbbk})^{\sim}$. Recall that the unoriented Grassmannian $\mathrm{Gr}(2,\mathbb{R}^\infty)$ is a model for $BO(2)$ and the Stiefel manifold $V(2,\mathbb{R}^\infty)$ is one for $EO(2)$. The universal principal $O(2)$–bundle $p\colon V(2,\mathbb{R}^\infty) \to \mathrm{Gr}(2,\mathbb{R}^\infty)$ is given by $p((e_1,e_2)) = \langle e_1, e_2\rangle$, ie the plane generated by the orthonormal 2–frame (e_1,e_2).

Lemma 5.13 *The reflection-invariant maps in* $\mathrm{Map}(V(2,\mathbb{R}^\infty), \mathsf{Alg})$ *determine stellar structures on* $\Bbbk$*–algebras.*

Proof A reflection ω in $O(2)$ acts on Alg by sending a $\Bbbk$–algebra A to its opposite algebra A^{op}. Let f be a reflection-invariant map with $f((e_1,e_2)) = A$. Let γ be a representative of the nontrivial element of $\pi_1(\mathrm{Gr}(2,\mathbb{R}^\infty), \langle e_1,e_2\rangle) \cong \mathbb{Z}/2\mathbb{Z}$. Lift γ to $\widetilde{\gamma}$ starting at (e_1,e_2) and ending at $\omega((e_1,e_2))$ (see Figure 33). Then $f(\widetilde{\gamma})$ is a

(A^{op}, A)–bimodule M and invariance under ω means $f(\omega(\tilde{\gamma})) = {}_{A^{\mathrm{op}}}\underline{M}_A = \omega(M)$. Lifting γ to $\tilde{\gamma}'$ starting at $\omega((e_1, e_2))$ gives a path ending at (e_1, e_2). Similarly, $f(\tilde{\gamma}')$ is an (A, A^{op})–bimodule N and we have $f(\omega(\tilde{\gamma}')) = {}_A\underline{N}_{A^{\mathrm{op}}} = \omega(N)$. Loops $\tilde{\gamma}' * \tilde{\gamma}$ and $\tilde{\gamma} * \tilde{\gamma}'$ bound disks since $V(2, \mathbb{R}^\infty)$ is contractible. In other words, there exist basepoint-fixing homotopies which take these loops to constant loops at (e_1, e_2) and $\omega((e_1, e_2))$, respectively. Images of the second homotopy and the time reversed version of the first homotopy under f yield invertible bimodule maps $\mu\colon {}_{A^{\mathrm{op}}}M \otimes_A N_{A^{\mathrm{op}}} \to {}_{A^{\mathrm{op}}}A^{\mathrm{op}}_{A^{\mathrm{op}}}$ and $\tau\colon {}_AA_A \to {}_AN \otimes_{A^{\mathrm{op}}} M_A$, respectively. The compositions of homotopies corresponding to the conditions on μ and τ to form a Morita context are the constant homotopies of paths $\tilde{\gamma}$ and $\tilde{\gamma}'$. Thus, $\zeta = ({}_{A^{\mathrm{op}}}M_A, {}_AN_{A^{\mathrm{op}}}, \tau, \mu)$ is a Morita context. Similarly, loops $\tilde{\gamma} * \omega(\tilde{\gamma})$ and $\tilde{\gamma}' * \omega(\tilde{\gamma}')$ bound, which implies that there is an equivalence of Morita contexts $\sigma\colon \zeta \cong \underline{\zeta}$. Since the order of reflection is two, we have $\sigma \circ \underline{\sigma} = \mathrm{id}$. Thus, any reflection-invariant map leads to stellar algebra structures on algebras. □

Lemma 5.14 *For an algebraically closed field $\Bbbk$ of characteristic zero, the homotopy $(G\times O(2))$–fixed points of $(\mathrm{Alg}^{\mathrm{fd}}_{\Bbbk})^\sim$ are quasibiangular stellar G–algebras.*

Proof Serre automorphism trivializes the homotopy SO(2)–action (see [6]), which turns the space of homotopy $(G\times \mathrm{SO}(2))$–fixed points into

$$\mathrm{Map}_G\big(EG, \mathrm{Map}(\widetilde{\mathrm{Gr}}(2, \mathbb{R}^\infty), \mathsf{Alg})\big).$$

Davidovich [6] showed that homotopy SO(2)–fixed points are semisimple symmetric Frobenius $\Bbbk$–algebras. Then, understanding homotopy $O(2)$–fixed points means understanding invariance under reflections. Using Lemma 5.13, we conclude that homotopy $O(2)$–fixed points are finite-dimensional semisimple symmetric Frobenius $\Bbbk$–algebras with a stellar structure.

The stellar structure is compatible with the Frobenius form as follows. A Frobenius form on a $\Bbbk$–algebra A is determined by a central element, which is the image of 1 under a bimodule map $z\colon {}_AA_A \to {}_AA_A$. Geometrically, $z(1)$ is an element of

$$\pi_2\big(\mathrm{Map}(B\mathrm{SO}(2), \mathsf{Alg}_r), f\big) \cong (\Bbbk^\times)^r,$$

where the algebra $A \in \mathsf{Alg}_r \subset \mathsf{Alg} = \coprod_{r=1}^\infty \mathsf{Alg}_r$ is isomorphic to

$$\mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \cdots \times \mathrm{End}(V_r)$$

under the Artin–Wedderburn isomorphism for finite-dimensional $\Bbbk$–vector spaces $V_1, \ldots, V_r$.

Compatibility means that (horizontal) composition of z with $\underline{\zeta}$ yields z again. Geometrically, this corresponds to conjugating the representing sphere based at f with loops in Alg_r given by bimodules of $\underline{\zeta}$. Since this loop is contractible, conjugation does not change $z(1)$ in the second homotopy group. Thus, we have a compatible stellar structure, and following Davidovich's methods [6] we obtain that, for a discrete group G, homotopy $(G\times O(2))$–fixed points are quasibiangular stellar G–algebras. $\square$

The above lemma is an important step toward the verification of the $(G\times O(2))$–structured cobordism hypothesis for $\mathrm{Alg}^2_{\mathbb{k}}$–valued 2–dimensional extended unoriented HFTs with $K(G,1)$ target. This version of the cobordism hypothesis states the equivalence of bigroupoids Frob^G_* and $((\mathrm{Alg}^{\mathrm{fd}}_{\mathbb{k}})^\sim)^{h(G\times O(2))}_{\le 2}$, where $((\mathrm{Alg}^{\mathrm{fd}}_{\mathbb{k}})^\sim)^{h(G\times O(2))}_{\le 2}$ is the fundamental bigroupoid of $((\mathrm{Alg}^{\mathrm{fd}}_{\mathbb{k}})^\sim)^{h(G\times O(2))}$. Lemma 5.14 implies that the objects of these bigroupoids coincide. The next step is to give an explicit (algebraic) description of the bigroupoid $((\mathrm{Alg}^{\mathrm{fd}}_{\mathbb{k}})^\sim)^{h(G\times O(2))}_{\le 2}$ and to write down the 2–functor $\mathcal{F}\colon \mathrm{Frob}^G_* \xrightarrow{\simeq} ((\mathrm{Alg}^{\mathrm{fd}}_{\mathbb{k}})^\sim)^{h(G\times O(2))}_{\le 2}$, similar to the oriented case. Here we skip these steps, which may later appear elsewhere.

Appendix Unbiased semistrict symmetric monoidal 2–categories

In this section we recall unbiased semistrict monoidal 2–categories and their computadic versions, and prove Theorem 4.5. Our main reference is [22, Section 2.10]. A similar exposition is given in [21, Appendix].

A.1 String diagrams for bicategories

The symmetric monoidal bicategory $\mathrm{XB}^{\mathrm{PD}}$ is not a fully weak symmetric monoidal bicategory but a certain stricter version. The strict bicategories we are interested in are unbiased semistrict symmetric monoidal 2–categories introduced by Schommer-Pries [22]. To recall their definition, we first review string diagrams for bicategories.

Alternative to pasting diagrams, string diagrams are tools describing morphisms in a bicategory. Instead of arrows between objects and 1–morphisms, a string diagram consists of regions, arcs, and vertices. Each region represents an object and each arc represents a 1–morphism between objects whose corresponding regions share this arc as a common boundary. Each vertex represents a 2–morphism between 1–morphisms

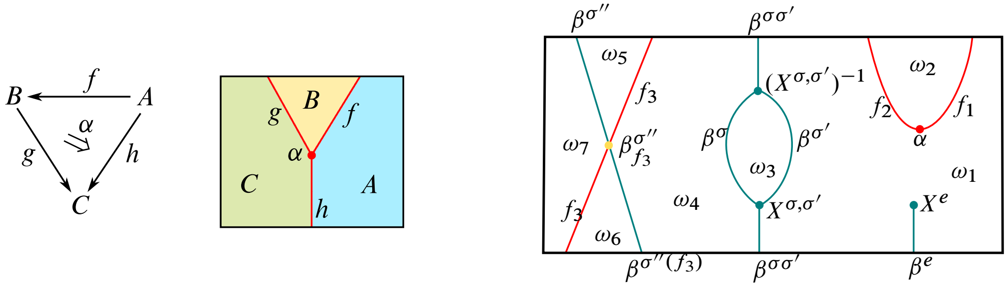


Figure 34: Pasting diagram with the corresponding string diagram and a string diagram for an unbiased semistrict symmetric monoidal 2–category.

whose corresponding arcs are connected with each other via this vertex. In Figure 34, left, a pasting diagram and the corresponding string diagram is shown. Note that we read string diagrams from right to left and from top to bottom.

Unbiased semistrict symmetric monoidal 2–categories are strict enough to admit a version of a string diagram. An example of such a string diagram is shown in Figure 34, in which regions are labeled with objects $\{\omega_i\}_{i=1}^7$, red arcs are labeled with 1–morphisms $\{f_j\}_{j=1}^3$, and a red vertex is labeled with a 2–morphism α. However, there are additional strings and vertices of different colors coming from the coherence morphisms of an unbiased semistrict symmetric monoidal 2–category.

Definition A.1 [22, Definition 2.32] An *unbiased semistrict symmetric monoidal 2–category* is a triple $(\mathcal{C}, \beta, X)$, where $\mathcal{C} = (\mathcal{C}, \otimes, \iota, \alpha, \lambda, \rho, \mathcal{P}, \mathcal{M}, \mathcal{L}, \mathcal{R})$ is a monoidal bicategory (see [25, Appendix]) such that:

(i) The underlying bicategory is a strict 2–category.

(ii) The transformations α, λ and ρ and modifications $\mathcal{P}$, $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{R}$ are identities.

(iii) The monoidal product $\otimes = (\otimes, \phi^{\otimes}_{(f,f'),(g,g')}, \phi^{\otimes}_{(a,a')})\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is cubical. That is, the interchanger

$$\phi^{\otimes}_{(f,f'),(g,g')}\colon (f \otimes f') \circ (g \otimes g') \to (f \circ g) \otimes (f' \circ g')$$

is the identity if either f or g' is the identity 1–morphism and $\phi^{\otimes}_{(a,a')}\colon \mathrm{id}_{a \otimes a'} \to \mathrm{id}_a \otimes \mathrm{id}_{a'}$ is the identity for all objects a and a'.

Secondly, β denotes a collection of transformations (turquoise edges and yellow point in Figure 34)

$$\{\beta^{\sigma}\colon (\mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_n \to \mathcal{C}) \to (\mathcal{C}_{\sigma(1)} \otimes \mathcal{C}_{\sigma(2)} \otimes \cdots \otimes \mathcal{C}_{\sigma(n)} \to \mathcal{C})\}_{\sigma \in S_n, n \geq 0},$$

where $\mathcal{C}_i = \mathcal{C}$ for all $i = 1, \dots, n$ and $S_0 := \{\iota\}$ with $\beta^\iota : (\iota \hookrightarrow \mathcal{C}) \to (\iota \hookrightarrow \mathcal{C})$ being the identity transformation between inclusions of functors. Lastly, X denotes a collection of invertible modifications (turquoise points in Figure 34)

$$X^{\sigma,\sigma'} : (\beta^\sigma * 1) \circ \beta^{\sigma'} \to \beta^{\sigma\sigma'} \quad \text{and} \quad X^e : \mathrm{id} \to \beta^e$$

for every $\sigma, \sigma' \in S_n$ and identity element $e \in S_n$ such that:

(i) The transformations $\{\beta^\sigma\}_{\sigma \in S_n, n \geq 0}$ and modifications $\{X^{\sigma,\sigma'}, X^e\}_{\sigma,\sigma',e \in S_n, n \geq 0}$ satisfy the conditions

$$\beta^{\mathrm{id} \sqcup \sigma} = \mathrm{id} \otimes \beta^\sigma, \qquad \beta^{\sigma \sqcup \mathrm{id}} = \beta^\sigma \otimes \mathrm{id},$$
$$X^{(\mathrm{id} \sqcup \sigma),(\mathrm{id} \sqcup \sigma')} = \mathrm{id} * X^{\sigma,\sigma'}, \quad X^{(\sigma \sqcup \mathrm{id}),(\sigma' \sqcup \mathrm{id})} = X^{\sigma,\sigma'} * \mathrm{id}$$

and the first three conditions on Figure 35 for all $\sigma.\sigma', \sigma'', e \in S_n$ and $n > 0$,

(ii) For a fixed $n > 0$ and a collection of n natural numbers $\{k_i\}_{i=1}^n$, let $\widetilde{\sigma} \in S_N$ be given by the operadic product[10] $\sigma \circ (\tau_i)$, where $N = \sum_i k_i$, $\sigma \in S_n$, and $\tau_i \in S_{k_i}$ for all $i = 1, 2, \dots, n$. Then the 2–morphism $\beta^\sigma_{(\beta^{\sqcup \tau_i})} = \beta^{\sqcup \tau_{\sigma(i)}} \circ \beta^\sigma \to \beta^\sigma \circ \beta^{\sqcup \tau_i}$ satisfies the equality given by the last condition in Figure 35 for all $n > 0$, $\sigma \in S_n$, and $\tau_i \in S_{k_i}$. In particular, when $\tau_i = e$ for all $i = 1, \dots, n$, we have $\beta^{\widetilde{\sigma}} = \beta^\sigma$, $X^{\widetilde{e}} = X^e$, and $X^{\widetilde{\sigma},\widetilde{\sigma}'} = X^{\sigma,\sigma}$ for all $\sigma, \sigma', e \in S_n$,

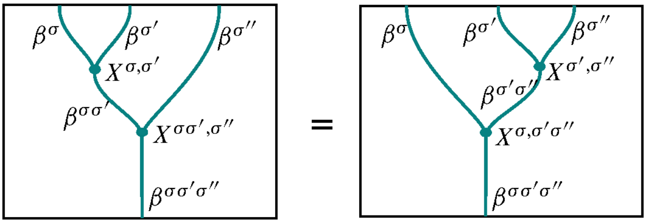
(iii) The transformations $\{\beta^\sigma\}_{\sigma \in S_n, n \geq 0}$ and modifications $\{X^{\sigma,\sigma'}, X^e\}_{\sigma,\sigma',e \in S_n, n \geq 0}$ satisfy the conditions given by the reflections of diagrams in Figure 35 with respect to a horizontal axis.

In order to prove Theorem 4.5, we first need to show that the symmetric monoidal bicategory $\mathrm{XB}^{\mathrm{PD}}$ is an unbiased semistrict symmetric monoidal 2–category. Recall that objects of $\mathrm{XB}^{\mathrm{PD}}$ are finite sets of ordered oriented points, 1–morphisms are isotopy classes of G–linear diagrams, and 2–morphisms are equivalence classes of G–planar diagrams.

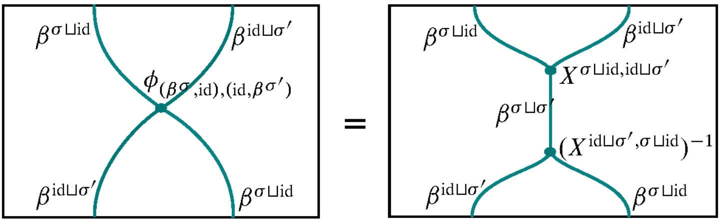
Lemma A.2 *Chambering sets, graphs, and foams equip* $\mathrm{XB}^{\mathrm{PD}}$ *with the structure of an unbiased semistrict symmetric monoidal* 2*–category.*

Proof Recall that compositions of morphisms in $\mathrm{XB}^{\mathrm{PD}}$ are given by the concatenation of diagrams. Since 1–morphisms are isotopy classes of G–linear diagrams and 2–morphisms are equivalence classes of G–planar diagrams, the underlying bicategory is

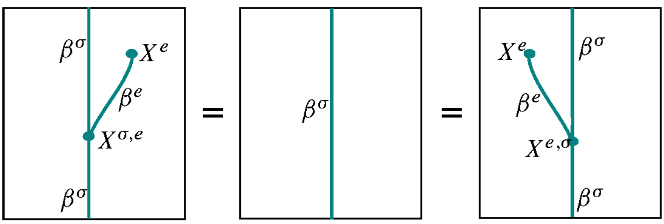
[10] By an operadic product we mean the composition $\sigma(\tau_1, \dots, \tau_n)$.



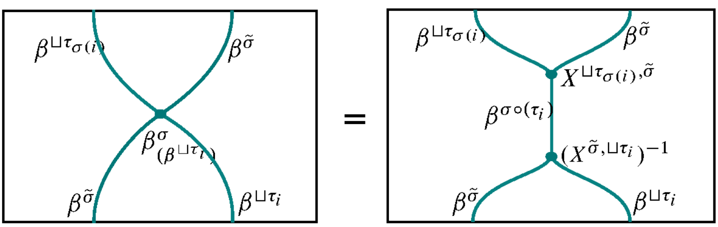
$$X^{\sigma\sigma',\sigma''}\circ(X^{\sigma,\sigma'}*1)=X^{\sigma,\sigma'\sigma''}\circ(1*X^{\sigma',\sigma''})$$



$$\phi_{(\beta^\sigma,\mathrm{id}),(\mathrm{id},\beta^{\sigma'})}=(X^{\mathrm{id}\sqcup\sigma',\sigma\sqcup\mathrm{id}})^{-1}\circ X^{\sigma\sqcup\mathrm{id},\mathrm{id}\sqcup\sigma'}$$



$$X^{\sigma,e}\circ(1*X^e)=\mathrm{id}_{\beta^\sigma}=X^{e,\sigma}\circ(X^e*1)$$



$$\beta^\sigma_{(\beta^{\sqcup\tau_i})}=(X^{\tilde\sigma,\sqcup\tau_i})^{-1}\circ X^{\sqcup\tau_{\sigma(i)},\tilde\sigma}$$

Figure 35: Some of the axioms of an unbiased semistrict symmetric monoidal 2–category.

a strict 2–category. The symmetric monoidal structure of $\mathrm{XB}^{\mathrm{PD}}$ is cubical by definition. The transformations α, λ and ρ and modifications $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{R}$ are identities since 2–morphisms are equivalence classes of G–planar diagrams. The local models CP

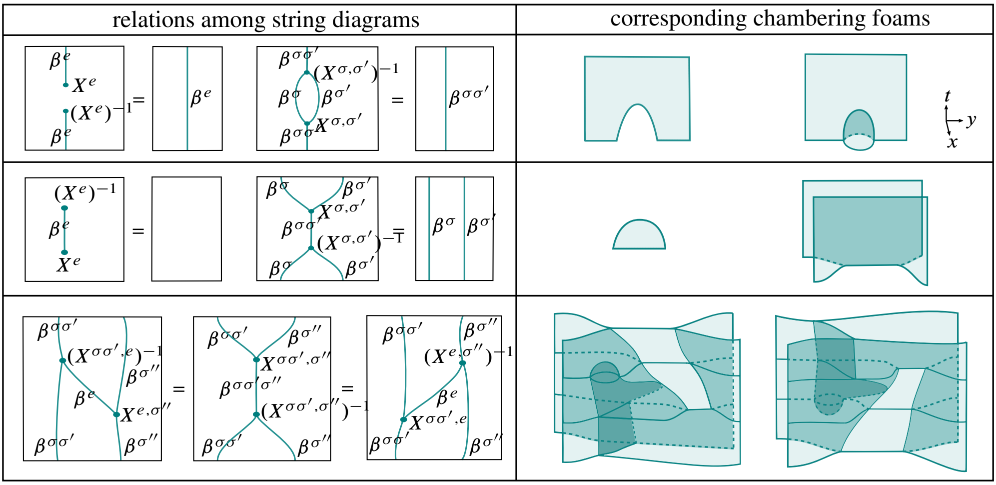


Figure 36: Relations between string diagrams for unbiased semistrict symmetric monoidal 2–categories and the corresponding spatial foams.

and CK_4 of chambering foam shown in Figure 20 give two conditions in Figure 35, left. For the remaining two conditions, recall that a chambering graph can only have univalent and trivalent vertices. □

Let $(\mathcal{C}, \beta, X)$ be an unbiased semistrict symmetric monoidal 2–category. The invertibility of modifications $\{X^{\sigma,\sigma'}, X^e\}_{\sigma,\sigma',e \in S_n, n \geq 0}$ and axioms of unbiased semistrict symmetric monoidal 2–category generate relations between structure morphisms. These relations are given[11] in Figure 36, left, in terms of string diagrams. Since chambering foams are responsible for the relations between boundary chambering graphs, in Figure 36, right, chambering foams corresponding to these relations are shown.

A.2 Computadic unbiased semistrict symmetric monoidal 2–categories

To finish the proof of Theorem 4.5, we need to show that the unbiased semistrict symmetric monoidal 2–category $\mathrm{XB}^{\mathrm{PD}}$ is computadic. As the next step, we review computadic unbiased semistrict symmetric monoidal 2–categories and show that $\mathrm{XB}^{\mathrm{PD}}$ is an example. Such a 2–category is constructed from a certain presentation (an unbiased semistrict symmetric monoidal 3–computad), which we call an *unbiased semistrict presentation*. This type of presentation $\mathbb{P}$ consists of four sets $(\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R})$ together with source and target maps $s, t : \mathcal{G}_1 \to \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ and $s, t : \mathcal{G}_2 \to \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$, which we

[11] Different labelings of string diagrams are possible and each possible labeling is a relation.

describe below. For a given such $\mathbb{P} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R}, s, t)$, the four sets are respectively called *generating objects*, *generating* 1*–morphisms*, *generating* 2*–morphisms*, and *generating relations among* 2*–morphisms*. The following series of definitions start with the ingredients of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ and continue with the definition of each ingredient in the given order.

Definition A.3 For a given unbiased semistrict presentation $\mathbb{P} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{R}, s, t)$, the objects of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ are *binary words* in $\mathcal{G}_0$, the 1–morphisms are *binary sentences* in $\mathcal{G}_1$, and the 2–morphisms are *equivalence classes of paragraphs* in $\mathcal{G}_2$.

Definition A.4 Let $\mathcal{G}_0$ be a set. The set $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ of *binary words in* $\mathcal{G}_0$ contains the symbol $\imath$, the elements of $\mathcal{G}_0$, and $\otimes$ products, ie $a \otimes b \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ for all $a, b \in \mathrm{BW}^{uss}(\mathcal{G}_0)$ such that, for any $a \in \mathcal{G}_0$, the elements $\imath \otimes a$, a, and $a \otimes \imath$ are identified.

Since binary words in $\mathcal{G}_0$ form the objects of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$, the set $\mathcal{G}_1$ of generating 1–morphisms is equipped with source and target maps $s, t \colon \mathcal{G}_1 \to \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$.

Definition A.5 Let $\mathcal{G}_1$ be a set equipped with maps $s, t \colon \mathcal{G}_1 \to \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$. The set $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$ of *binary words in* $\mathcal{G}_1$ contains elements of $\mathcal{G}_1$, id_a and $\beta^{\sigma}_{a,\sigma(a)}$ for any $a \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$ and σS_n, where a is a word of length n. The extension of the source and target maps to these elements are $s(\mathrm{id}_a) = a$, $t(\mathrm{id}_a) = a$, $s(\beta^{\sigma}_{a,\sigma(a)}) = a$ and $t(\beta^{\sigma}_{a,\sigma(a)}) = \sigma(a)$.

Definition A.6 Let $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$ be a set of binary words in $\mathcal{G}_1$ with $s, t \colon \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1) \to \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$. The set $\underline{\mathrm{BS}}^{\mathrm{uss}}(\mathcal{G}_1)$ contains binary words in $\mathcal{G}_1$, compositions $g \circ f$ for any $f, g \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$ with $s(g) = t(f)$, and monoidal products $f \otimes g$ for any $f, g \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$. The source and target maps extend naturally to $\underline{\mathrm{BS}}^{\mathrm{uss}}(\mathcal{G}_1)$ by

- $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$ for any $g \circ f \in \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$,
- $s(f \otimes g) = s(f) \otimes s(g)$ and $t(f \otimes g) = t(f) \otimes t(g)$ for any $f, g \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$.

The set $\mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$ of *binary sentences in* $\mathcal{G}_1$ is the quotient $\underline{\mathrm{BS}}^{\mathrm{uss}}(\mathcal{G}_1)/\sim$, where $\sim$ is the smallest equivalence relation generated by the identifications $f \otimes \mathrm{id}_\imath \sim f$, $f \sim \mathrm{id}_\imath \otimes f$, $f \circ \mathrm{id}_a \sim f \sim \mathrm{id}_b \circ f$, and $f \otimes f' \sim (\mathrm{id}_b \otimes f') \circ (f \otimes \mathrm{id}_{a'})$ for any $f, f' \in \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$ with $s(f) = a$, $t(f) = b$, $s(f') = a'$, and $t(f') = b'$ (see [22, Lemma 2.81]).

symbol	source	target
id_f	f	f
$\phi^{\otimes}_{(f_1,f_2),(f_3,f_4)}$	$(f_1 \otimes f_2) \circ (f_3 \otimes f_4)$	$(f_1 \circ f_3) \otimes (f_2 \circ f_4)$
$\phi^{\otimes}_{a,a'}$	$\mathrm{id}_{a \otimes a'}$	$\mathrm{id}_a \otimes \mathrm{id}_{a'}$
${}_r\beta^{\sigma}_f$	$f \circ \beta^{\sigma}$	$\beta^{\overline{\sigma}} \circ f$
${}_l\beta^{\sigma}_f$	$\beta^{\sigma} \circ f$	$f \circ \beta^{\overline{\sigma}}$
$X^{\sigma,\sigma'}$	$(\beta^{\sigma} * 1) \circ \beta^{\sigma'}$	$\beta^{\sigma\sigma'}$
X^e	id	β^e

Table 1: Binary words in $\mathcal{G}_2$.

Since binary sentences in $\mathcal{G}_1$ form the 1–morphisms of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$, the set $\mathcal{G}_2$ of generating 2–morphisms is equipped with source and target maps $s, t\colon \mathcal{G}_2 \to \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$ satisfying $s \circ s = s \circ t$ and $t \circ s = t \circ t$. Then, an *unbiased semistrict symmetric monoidal* 2*–computad* $\mathbb{P}^{\mathcal{G}}$ consists of generating sets $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$ together with maps $s, t\colon \mathcal{G}_1 \to \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$, and $s, t\colon \mathcal{G}_2 \to \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$ satisfying $s \circ s = s \circ t$ and $t \circ s = t \circ t$.

Definition A.7 Let $\mathcal{G}_2$ be a set equipped with $s, t\colon \mathcal{G}_2 \to \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$ satisfying $s \circ s = s \circ t$ and $t \circ s = t \circ t$. The set $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_2)$ of *binary words in* $\mathcal{G}_2$ contains every element of $\mathcal{G}_2$ and the symbols in Table 1 for every $f \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_1)$ and every $f_1, f_2, f_3, f_4 \in \mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$ with $s(f_1) = t(f_3)$ and $s(f_2) = t(f_4)$, and for all $\sigma, e \in S_n, \overline{\sigma} \in S_m$ for $n, m \geq 0$. Moreover, $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_2)$ contains the inverses of symbols in Table 1 except for symbols containing the β^{σ}. The set of *preparagraphs* $\underline{\mathrm{PG}}^{\mathrm{uss}}(\mathcal{G}_2)$ is constructed from $\mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_2)$ by adding compositions and monoidal products as above. Similar to $\mathrm{BS}^{\mathrm{uss}}(\mathcal{G}_1)$, there are certain identifications on $\underline{\mathrm{PG}}^{\mathrm{uss}}(\mathcal{G}_2)$ generated by $\phi^{\otimes}_{(\mathrm{id}, f_2),(f_3,f_4)} = \mathrm{id}$, $\phi^{\otimes}_{(f_1,f_2),(f_3,\mathrm{id})} = \mathrm{id}$ and $\phi^{\otimes}_{(a,a')} = \mathrm{id}$ for all $f_1, f_2, f_3, f_4 \in B^{\mathrm{uss}}(\mathcal{G}_1)$ and $a, a' \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$. We consider the smallest equivalence relation $\sim$ on $\underline{\mathrm{PG}}^{\mathrm{uss}}(\mathcal{G}_2)$ generated by these identifications along with identifications

$$\{p \circ p^{-1} = \mathrm{id}_{t(p)}, p^{-1} \circ p = \mathrm{id}_{s(p)}\}_{p \in \underline{\mathrm{PG}}^{\mathrm{uss}}(\mathcal{G}_2)}$$

and those for β and X coming from the definition of unbiased semistrict symmetric monoidal 2–category. The quotient set is denoted by $\mathrm{PG}^{\mathrm{uss}}(\mathcal{G}_2)$ and called the set of *paragraphs* in $\mathcal{G}_2$.

Definition A.8 The set $\mathcal{R}$ of generating relations among 2–morphisms for an unbiased semistrict symmetric monoidal 2–computad $\mathbb{P}^{\mathcal{G}} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, s, t)$ consists of pairs

(F, G) of paragraphs in $\mathcal{G}_2$ in $\mathsf{F}(\mathbb{P}^{\mathcal{G}})$ with $s(F) = s(G)$ and $t(F) = t(G)$. An *unbiased semistrict presentation or unbiased semistrict symmetric monoidal* 3*–computad* $\mathbb{P}$ consists of an unbiased semistrict symmetric monoidal 2–computad $\mathbb{P}^{\mathcal{G}}$ and a set $\mathcal{R}$ of generating relations among 2–morphisms for $\mathbb{P}^{\mathcal{G}}$.

The 2–morphisms of the computadic unbiased semistrict symmetric monoidal 2–category $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ are the $\wr$–equivalence classes of paragraphs in $\mathcal{G}_2$, where $\wr$ is the smallest equivalence relation on $\mathrm{PG}^{\mathrm{uss}}(\mathcal{G}_2)$ such that $\wr$ is generated by $\mathcal{R}$ and closed under compositions and monoidal products. An unbiased semistrict symmetric monoidal 2–category $(\mathcal{C}, \beta, X)$ is called *computadic* if there exists a strict symmetric monoidal equivalence $\mathcal{F}\colon \mathsf{F}_{\mathrm{uss}}(\mathbb{P}) \to \mathcal{C}$ for an unbiased semistrict presentation $\mathbb{P}$.

In simpler terms, $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ can be described as follows. The objects of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ are words in $\mathcal{G}_0$. There are two kinds of elementary 1–morphisms, which can be described as

(i) $\beta^{\sigma}_{a,\sigma(a)}\colon a \to \sigma(a)$, where $\sigma \in S_n$ for $n \geq 0$ and a is a word of length n,

(ii) $\mathrm{id}_a \otimes f \otimes \mathrm{id}_b$, where $f \in \mathcal{G}_1$ and $a, b \in \mathrm{BW}^{\mathrm{uss}}(\mathcal{G}_0)$,

so that nonidentity 1–morphisms of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ are given by compositions of elementary 1–morphisms. The 2–morphisms of $\mathsf{F}_{\mathrm{uss}}(\mathbb{P})$ are the equivalence classes of string diagrams, where two string diagrams are equivalent if they are related by finitely many (local) moves which come from the generating relations $\mathcal{R}$, Figures 35 and 36, and the naturality of β and X with generating morphisms. Compositions of morphisms are given by horizontal and vertical concatenations of string diagrams while the (cubical) monoidal product is given by stretching out diagrams from different horizontal directions and merging them (see Figure 24).

Example A.9 Consider an unbiased semistrict presentation

$$\mathbb{XP} = (\mathcal{XG}_0, \mathcal{XG}_1, \mathcal{XG}_2, \mathcal{XR})$$

whose generating sets are given as $\mathcal{XG}_0 = \{\bullet^+, \bullet^-\}$, $\mathcal{XG}_1 = \{F_2^g, F_1^g, P_g, N_g\}$, ie G–linear diagrams of $\{{}^{+}_{-} \to g,\ g \to {}^{-}_{+},\ + \xrightarrow{g} +,\ - \xrightarrow{g} -\}_{g \in G}$ without chambering sets, and $\mathcal{XG}_2$ consists of G–planar diagrams without chambering graphs of generating 2–morphisms in Figure 25. The set of relations $\mathcal{XR}$ consists of pairs of G–planar diagrams corresponding to pairs of $\langle 2\rangle$–X–surfaces in Figure 26.

An object of $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ is either ι or words in $\bullet^+$ and $\bullet^-$. Each 1–morphism is a composition of the following two types of elementary 1–morphisms: $\beta^{\sigma}_{a,\sigma(a)}$, where

$\sigma \in S_n$ and a is a word of length n; and a G–linear diagram whose 1–morphism is labeled with $\mathrm{id}_{a_1} \otimes \cdots \otimes f \otimes \cdots \otimes \mathrm{id}_{a_n}$ for some $0 < k \leq n$, where a_i is either $\bullet^+$ or $\bullet^-$ and $f \in \mathcal{G}_1$. Here we use $\mathrm{id}_{\bullet^+} = \overline{P_e}$, $\mathrm{id}_{\bullet^-} = \overline{N_e}$, and $f \otimes f' = (\mathrm{id}_b \otimes f') \circ (f \otimes \mathrm{id}_{a'})$. The 2–morphisms of $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ are equivalence classes of paragraphs $\mathrm{PG}^{\mathrm{uss}}(\mathcal{XG}_2)$, where the equivalence relation is generated by the set of generating relations $\mathcal{XR}$ (see Figures 17, 18, 19, and 26), and the string diagrams are given in Figures 22, 35, and 36.

A.3 Proof of Theorem 4.5

Note that the string diagram interpretation of elements of $\mathrm{PG}^{\mathrm{uss}}(\mathcal{XG}_2)$ coincides with the string diagram interpretation of 2–morphisms of $\mathrm{XB}^{\mathrm{PD}}$ except for (possible) black points on G–linear diagrams (see Figure 14). More precisely, the sets of labels for regions coincide, in both string diagrams there are two types of 1–morphisms whose sets of labels and possible intersecting patterns coincide, and in both string diagrams there are three types of vertices whose sets of labels coincide for each type of vertex. Lastly, equivalence relations on both string diagrams are generated by the same local moves or, equivalently, by the same movie moves. This observation suggests an isomorphism between $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ and $\mathrm{XB}^{\mathrm{PD}}$, namely a symmetric monoidal equivalence preserving the unbiased semistrict symmetric monoidal structures. The following lemma shows that this is indeed the case and finishes the proof of Theorem 4.5:

Lemma A.10 *There exists a canonical isomorphism $\Theta: \mathsf{F}_{\mathrm{uss}}(\mathbb{XP}) \to \mathrm{XB}^{\mathrm{PD}}$ of unbiased semistrict symmetric monoidal 2–categories.*

Proof Comparing the descriptions of unbiased semistrict symmetric monoidal 2–categories $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ and $\mathrm{XB}^{\mathrm{PD}}$ given above, it is not hard to define the 2–functor Θ. On the level of objects Θ maps ι to the empty set and words in set $\{\bullet^+, \bullet^-\}$ to the finite ordered oriented points given by the words.

On 1–morphisms, it is enough to specify the images of elementary 1–morphisms $\{\beta^\sigma, \mathrm{id}_{a_1} \otimes \cdots \otimes f \otimes \cdots \otimes \mathrm{id}_{a_n}\}$, where $\sigma \in S_n$, $n \geq 0$, $f \in \mathcal{XG}_1$, and $a_i \in \{\bullet^+, \bullet^-\}$ for $1 \leq i \leq n$. For $\sigma \in S_n$ and a word a, the 1–morphism $\Theta(\beta^\sigma_{a,\sigma(a)})$ is a G–linear diagram whose chambering set has only one element labeled by β^σ. The latter 1–morphism is mapped to a G–linear diagram described in Example A.9. Recall that 1–morphisms of $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ are equivalence classes determined by certain identifications and 1–morphisms of $\mathrm{XB}^{\mathrm{PD}}$ are isotopy classes of G–linear diagrams. It is not hard to see that the above assignments are well defined on 1–morphisms.

The 2–functor Θ maps the equivalence class $[P]$ of a paragraph $P \in \mathrm{PG}^{\mathrm{uss}}(\mathcal{XG}_2)/\sim$ to the equivalence class of the string diagram corresponding to P. This assignment makes sense because, as mentioned in Example A.9, any representative string diagram can be interpreted in both 2–categories. Since in both the 2–categories $\mathsf{F}_{\mathrm{uss}}(\mathbb{XP})$ and $\mathrm{XB}^{\mathrm{PD}}$ string diagrams are considered up to the same lists of local moves (movie moves), this assignment is well defined.

We use the Whitehead theorem for symmetric monoidal bicategories (Theorem 4.3) to show that Θ is a symmetric monoidal equivalence. It is clear that Θ is essentially surjective on objects. We claim that Θ is essentially full on 1–morphisms. To prove this, it is enough to show that every G–linear diagram is isomorphic to a composition of 1–morphisms $\{\Theta(\beta^{\sigma}_{a,\sigma(a)}), \Theta(\mathrm{id}_{a_1} \otimes \cdots \otimes \mathrm{id}_{a_n}), \Theta(\mathrm{id}_{a_1} \otimes \cdots \otimes f \otimes \cdots \otimes \mathrm{id}_{a_n})\}$ for some $n \geq 0$, $\sigma \in S_n$, and $f \in \mathcal{XG}_1$. For a given G–linear diagram it is obvious how to write it as a composition of these diagrams except for extra black points. Recall that when we compose G–linear diagrams we do not remove black points along which two diagrams are concatenated. However, there are invertible G–planar diagrams which remove these points (see Figure 10). Therefore, up to invertible 2–morphisms, every G–linear diagram can be written as a composition of the above 1–morphisms.

Recall that G–planar diagrams are formed using generic maps and the X–manifold data of cobordism type $\langle 2\rangle$–X–surfaces. Thus, any G–planar diagram can be obtained from generating 2–morphisms in Figure 25 under horizontal and vertical compositions, and symmetric monoidal product operation. This implies that, for any G–planar diagram, there exists a paragraph such that their equivalence classes are matched by Θ. Consequently, θ is fully faithful on 2–morphisms.

Hence, the Whitehead theorem implies that Θ is an equivalence. By definition, Θ preserves the unbiased semistrict symmetric monoidal structures. That is, $\mathrm{XB}^{\mathrm{PD}}$ is a computadic unbiased semistrict symmetric monoidal 2–category. □

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