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Bigrading the symplectic Khovanov cohomology

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We construct a well-defined relative second grading on symplectic Khovanov cohomology from holomorphic disc counting. We use a version of symplectic Khovanov cohomology defined for bridge diagrams rather than braids. We show that our second grading recovers the Jones grading of Khovanov homology up to an overall grading shift over any characteristic-zero field, through proving that the isomorphism of Abouzaid and Smith can be refined as an isomorphism between bigraded cohomology theories. The central idea of the proof is to construct an exact triangle for symplectic Khovanov cohomology that behaves similarly to the unoriented skein exact triangle for Khovanov homology.

53D40, 57K18, 57K10, 57R58

1 Introduction

In [11], Seidel and Smith defined a singly graded link invariant, *symplectic Khovanov cohomology* $\mathrm{Kh}^*_{\mathrm{symp}}(L)$. It is the Lagrangian intersection Floer cohomology of two Lagrangians in a symplectic manifold $\mathcal{Y}_n$. The manifold $\mathcal{Y}_n$ is built through taking a fiber of the restriction of the adjoint quotient map $\chi\colon \mathfrak{sl}_{2n}(\mathbb{C}) \to \mathrm{Conf}^0_{2n}(\mathbb{C})$ to a nilpotent slice $\mathcal{S}_n$. The Lagrangians are determined by the link L as follows. A given link L in S^3 can be realized as a braid closure of $\beta_L \in \mathrm{Br}_n$ for some n depending on L. The braid $\beta_L \times \mathrm{id} \in \mathrm{Br}_{2n}$ gives a path in the configuration space $\mathrm{Conf}^0_{2n}(\mathbb{C})$. Parallel transport along $\beta_L \times \mathrm{id}$ induces a symplectomorphism of $\mathcal{Y}_n$ to itself (precisely speaking, from an arbitrarily large compact subset of $\mathcal{Y}_n$ to itself). There is a distinguished Lagrangian submanifold $\mathcal{K}$ given by iterated vanishing cycles. Let $(\beta_L \times \mathrm{id})(\mathcal{K})$ be its image under the parallel transport. $\mathrm{Kh}^*_{\mathrm{symp}}(L)$ is defined to be the Floer cohomology group

$$\mathrm{Kh}^*_{\mathrm{symp}}(L) = \mathrm{HF}^{*+n+w}(\mathcal{K}, (\beta_L \times \mathrm{id})(\mathcal{K})), \tag{1-1}$$

where w is the writhe of β_L. There is a conjectural relation between $\mathrm{Kh}^*_{\mathrm{symp}}$ and the original Khovanov homology $\mathrm{Kh}^{*,*}$ defined by Khovanov in [5].

Conjecture 1.1 [11, Conjecture 2] For any link $L \subset S^3$,

$$\mathrm{Kh}^k_{\mathrm{symp}}(L) \cong \bigoplus_{i-j=k} \mathrm{Kh}^{i,j}(L).$$

This conjecture is true over any characteristic-zero field, proved by Abouzaid and Smith in [2, Theorem 7.6]. For the cases of characteristic-nonzero fields, in the very few examples that have been computed, such as the case of the trefoil in [11, Proposition 55], the theories are isomorphic.

Our results rely on the theorem of Abouzaid and Smith that Conjecture 1.1 is true over any characteristic-zero field, so we assume the characteristic of the base field $\Bbbk$ to be zero unless noted otherwise.

Note that both Khovanov homology and symplectic Khovanov cohomology are the homology of graded or relatively graded cochain complexes in which the differential raises the grading by one.

We work with the framework of Manolescu's Hilbert scheme reformulation of $\mathrm{Kh}^*_{\mathrm{symp}}$, in which $\mathcal{Y}_n$ is viewed as a symplectically embedded open subscheme of $\mathrm{Hilb}^n(A_{2n-1})$, the n^{th} Hilbert scheme of the Milnor fiber of A_{2n-1}–surface singularity.

One of the advantages of Manolescu's reformulation [7] is that we can work with *bridge diagrams* instead of braid closures. A bridge diagram is a decorated link diagram obtained by breaking the link diagrams into n pairwise disjoint α arcs and n pairwise disjoint β arcs so that β arcs overcross α arcs at any intersection. These arcs give rise to two Lagrangians $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ in $\mathcal{Y}_n \subset \mathrm{Hilb}^n(A_{2n-1})$. The main theorem of [7] implies that $\mathrm{Kh}^*_{\mathrm{symp}}(L) \cong \mathrm{HF}^{*+n+w}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ for a specific type of bridge diagram, called a *flattened braid diagram*, where n is the number of strands and w is the writhe of the corresponding braid. All braids can give rise to flattened braid diagrams, but not all bridge diagrams are isotopic to flattened braid diagrams.

Attempts were made to generalize symplectic Khovanov cohomology to arbitrary bridge diagrams, an $\mathbb{F}_2$–coefficients version by Hendricks, Lipshitz and Sarkar [4, Section 7], and a relatively graded version by Waldron [13, Section 6]. In this paper, we give an absolute grading for Waldron's construction. Recall that the writhe w of the diagram is the number of positive crossings minus the number of negative crossings, and the rotation number rot of the diagram is the number of counterclockwise minus the number of clockwise Seifert circles. We can conclude:

Theorem 1.2 *For any oriented bridge diagram representing a link L, let x_0 be the generator whose coordinates are the starting point of each β arc. Then the Floer cohomology groups*

$$\mathrm{Kh}^*_{\mathrm{symp}}(L) = \mathrm{HF}^{*+\mathrm{gr}(x_0)+w+\mathrm{rot}}(\mathcal{K}_\alpha, \mathcal{K}_\beta) \tag{1-2}$$

are link invariants.

As Waldron proved in the relative case [13, Theorem 1.1], the absolutely graded invariant defined with bridge diagram is also canonical, ie the Floer groups from two equivalent bridge diagrams are related through a canonical isomorphism.

The orientation of the diagram, especially for a link diagram, is crucial in computing the correction terms and locating x_0. Throughout this paper, we always assume that our bridge diagrams are oriented.

Abouzaid and Smith [1, Equation 2.31] constructed an endomorphism ϕ of $\mathrm{CF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ from the linear part of an *nc-vector field*, which was introduced in [1, Definition 2.3]. This endomorphism is a chain map and thus induces a generalized eigenspace decomposition of $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$. The eigenvalues will equip the Floer cohomology group $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ with an additional grading, called the *weight grading*.

We will only use a relative version of the weight grading because an absolute grading relies on choices of auxiliary data for each Lagrangian, called *equivariant structures*. Equivariant structures are defined in [1, Definition 2.10] and in Definition 3.3 of this paper; the choice of equivariant structure changes the grading by a shift, as in [1, Lemma 2.1.2(2)], so we may obtain our relative grading without addressing this choice. As a result, we will skip the discussion of equivariant structures in this paper. We prove that for any bridge diagram, the relative weight grading recovers the Jones grading (called the quantum grading in some contexts) of Khovanov homology.

Theorem 1.3 *Symplectic Khovanov cohomology, graded by* $(\mathrm{gr}, \mathrm{wt})$*, and Khovanov homology, graded by* (i, j)*, are isomorphic as bigraded vector spaces over any field of characteristic zero, where the gradings are related by* $\mathrm{gr} = i - j$ *and* $\mathrm{wt} = -j + c$*, where c is a correction term for the relative weight grading.*

By definition, the weight grading lives in $\overline{\mathbb{k}}$, the algebraic closure of the base field $\mathbb{k}$. The theorem implies that the weight grading is integral. Defining an absolute weight grading requires us to make a specific choice on the equivariant structures on the

Lagrangians depending on the writhe, crossing number, and other properties of the bridge diagram. It is natural for us to ask the following question.

Question 1.4 How do we specify the choice of the equivariant structure such that the weight grading is an absolute grading?

As for the purpose of this paper, we are satisfied with a relative grading. Thus, we will only fix an arbitrary choice of equivariant structures throughout this paper.

To prove Theorem 1.3, we show that Abouzaid and Smith's long exact sequence of the unoriented skein relation for symplectic Khovanov cohomology groups [2, Equation 7.9] decomposes with respect to the weight grading. In other words, if we choose any weight grading wt_1 of the first group, there exists a weight grading wt_2 of the second group and wt_3 of the third group such that the map connecting the first and second group changes the weight grading by a constant $\mathrm{wt}_2 - \mathrm{wt}_1$, and constant weight grading changes $\mathrm{wt}_3 - \mathrm{wt}_2$ and $\mathrm{wt}_1 - \mathrm{wt}_3$ for the other two maps:

$$\cdots \to \mathrm{HF}^{*,\mathrm{wt}_1}(L_+) \to \mathrm{HF}^{*,\mathrm{wt}_2}(L_0) \to \mathrm{HF}^{*+2,\mathrm{wt}_3}(L_\infty) \to \mathrm{HF}^{*+1,\mathrm{wt}_1}(L_+) \to \cdots, \tag{1-3}$$

where L_+ is a link with a positive crossing at p, and L_0 and L_∞ are the two resolutions at p.

In the singly graded case, the isomorphism of Abouzaid and Smith between symplectic Khovanov cohomology and Khovanov homology can be used to build a commutative diagram between the exact triangle above and the exact triangle for the unoriented skein relation in Khovanov homology. We show the maps between the exact triangles given by the isomorphism of Abouzaid and Smith are bigraded by induction on the number of crossings. A purity result of Abouzaid and Smith [1, Theorem 1.1] leads to a computation for crossingless diagrams of an unlink that every element $x \in \mathrm{Kh}^k_{\mathrm{symp}}(L)$ will satisfy $\mathrm{wt}(x) = k$ for some choice of equivariant structures.

This finishes the proof of Theorem 1.3. Our argument so far is diagrammatic: we have not proved that the relative weight grading is independent of bridge diagrams yet. Now that we know the relative weight grading recovers the Jones grading up to an overall grading shift for any diagram, and with the fact that Jones grading is independent of link diagrams, we prove a conjecture of Abouzaid and Smith.

Theorem 1.5 *The relative weight grading on* $\mathrm{Kh}^*_{\mathrm{symp}}(L)$ *is independent of the choice of link diagram.*

It is worth noting that our proof of Theorem 1.5 is not internal to symplectic geometry, and the invariance of the relative weight grading relies on the well-definedness of the Jones grading in combinatorial Khovanov homology. We conclude this introduction with the following question.

Question 1.6 Is there a proof of invariance of the weight grading from pure symplectic geometry?

Organization

The paper is organized as follows. In Section 2, we review the definition of symplectic Khovanov cohomology and construct an absolute grading on the symplectic Khovanov cohomology for bridge diagrams. In Section 3, we give a precise definition of the weight grading and construct a bigraded unoriented skein exact triangle of symplectic Khovanov cohomology. In Section 4, we prove the main theorem by showing that Abouzaid and Smith's isomorphism between symplectic Khovanov cohomology and combinatorial Khovanov homology preserves the second grading.

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2 A review of symplectic Khovanov cohomology $\mathbf{Kh}^*_{\mathbf{symp}}(L)$

We will briefly review the original definition of symplectic Khovanov cohomology and give a formal definition of symplectic Khovanov cohomology of a bridge diagram in Section 2.1. We will discuss the homological grading in Section 2.2. The construction of this section is not restricted to characteristic-zero fields.

2.1 Symplectic Khovanov cohomology for bridge diagrams

The link invariant $\mathrm{Kh}^*_{\mathrm{symp}}(L)$ was first introduced by Seidel and Smith in [11] as the Lagrangian intersection Floer cohomology of two Lagrangians in $\mathcal{Y}_n$, constructed as a nilpotent slice in $\mathfrak{sl}_{2n}(\mathbb{C})$. In this original formulation, the first Lagrangian is created by a technique called the *iterated vanishing cycle*, while the second Lagrangian is the image of the first Lagrangian under the symplectomorphism induced by the braid whose closure is the link L. Manolescu in [7] introduced a reformulation using Hilbert schemes, which is easier to visualize and more similar to other low-dimensional invariants, such as Heegaard Floer homology. In this subsection, Lagrangian Floer cohomology groups are relatively graded; see Remark 2.4.

Let us start with the complex surface of Milnor fiber A_{2n-1}. Consider the complex surface

$$A_{2n-1} = \{(u, v, z) \in \mathbb{C}^3 \mid u^2 + v^2 + p(z) = 0\} \in \mathbb{C}^3, \tag{2-1}$$

where $p(z) = (z - p_1) \cdots (z - p_{2n})$.

Recall that the Hilbert scheme of n points on A_{2n-1}, $\mathrm{Hilb}^n(A_{2n-1})$, is the space of closed zero-dimensional subschemes of A_{2n-1} of length n. The Hilbert scheme $\mathrm{Hilb}^n(A_{2n-1})$ is closely related to the n–fold symmetric product $\mathrm{Sym}^n(A_{2n-1})$ of A_{2n-1}, through the *Hilbert–Chow morphism*:

Proposition 2.1 [8, Chapter 9] *The Hilbert–Chow morphism π is a natural morphism from $\mathrm{Hilb}^n(X)$ to $\mathrm{Sym}^n(X)$ such that*

$$\pi(Z) = \sum_{x \in X} \mathrm{length}(Z_x)[x]. \tag{2-2}$$

Moreover, if X is complex one-dimensional, then π is an isomorphism. If X is smooth and complex two-dimensional, π is a crepant resolution of singularities and $\mathrm{Hilb}^n(X)$ is smooth.

Let $\Delta = \{(x_1, \dots, x_n) \in \mathrm{Sym}^n(A_{2n-1}) \mid x_i = x_j \text{ for some } i \neq j\}$ be the *diagonal* of the symmetric product. The proposition above implies that π is one-to-one away from the diagonal Δ, so we can think of $\mathrm{Hilb}^n(A_{2n-1})$ as $\mathrm{Sym}^n(A_{2n-1})$ when away from Δ. Explicitly, a point in $\mathrm{Hilb}^n(A_{2n-1})$ corresponds to an ideal

$$I \subset \mathcal{O} = \mathbb{C}[u, v, z]/(u^2 + v^2 + p(z))$$

with $\dim_{\mathbb{C}}(\mathcal{O}/I) = n$. Thus, given n distinct points (u_i, v_i, z_i) in A_{2n-1}, we can construct an ideal by taking the product of $(u-u_i, v-v_i, z-z_i)$, which gives a point in $\mathrm{Hilb}^n(A_{2n-1})$.

Now we proceed to the definition of the manifold $\mathcal{Y}_n$. First, we consider a projection $i : A_{2n-1} \to \mathbb{C}$ such that $i(u,v,z) = z$. It induces a map $\mathbb{C}[z] \hookrightarrow \mathbb{C}[u,v,z] \to \mathcal{O}$. As $\mathbb{C}[z]$ intersects with the ideal $(u^2+v^2+p(z))$ in $\mathbb{C}[u,v,z]$ trivially, if we denote by R_1 the image of $\mathbb{C}[z]$ in $\mathcal{O}$, we have $R_1 \cong \mathbb{C}[z]$. For any ideal $I \in \mathrm{Hilb}^n(\mathbb{C})$, we then define the projection $i(I) = I \cap R_1$ and the manifold

$$\mathcal{Y}_n = \{I \in \mathrm{Hilb}^n(A_{2n-1}) \mid i(I) \text{ has length } n\}. \tag{2-3}$$

The complement of $\mathcal{Y}_n$ defines a divisor in the Hilbert scheme $\mathrm{Hilb}^n(A_{2n-1})$,

$$D_r = \{I \in \mathrm{Hilb}^n(A_{2n-1}) \mid i(I) \text{ has at most length } n-1\}. \tag{2-4}$$

Manolescu proved [7, Proposition 2.7] that $\mathcal{Y}_n$ is biholomorphic to the space Seidel and Smith considered in [11].

Now, we construct the Lagrangians from a link diagram. A *bridge diagram* D for a link L is a triple $(\vec{\alpha}, \vec{\beta}, \vec{p})$, where $\vec{p} = (p_1, p_2, \dots, p_{2n})$ are $2n$ distinct points in $\mathbb{R}^2$, $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ are n pairwise disjoint embedded arcs and $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ are also pairwise disjoint embedded arcs such that $\partial\big(\bigcup \alpha_i\big) = \partial\big(\bigcup \beta_i\big) = \{p_1, \dots, p_{2n}\}$ and if we let the β arcs overcross the α arcs at the intersections in $\mathbb{R}^2$, we obtain a diagram for L.

For each arc α_i or β_i, we can associate a Lagrangian sphere Σ_{α_i} or Σ_{β_i} in A_{2n-1} through the equations

$$\Sigma_{\alpha_i} = \{(u,v,z) \in A_{2n-1} \mid z \in \alpha_i, u, v \in \sqrt{-p(z)}\mathbb{R}\}, \tag{2-5}$$

$$\Sigma_{\beta_i} = \{(u,v,z) \in A_{2n-1} \mid z \in \beta_i, u, v \in \sqrt{-p(z)}\mathbb{R}\}. \tag{2-6}$$

There are natural projection maps $\Sigma_{\alpha_i} \to \alpha_i$ and $\Sigma_{\beta_i} \to \beta_i$. Each interior point of the arc lifts to a copy of the circle, whereas each endpoint lifts to a single point. Therefore Σ_{α_i} and Σ_{β_i} are copies of S^2. With appropriate choice of Kähler form, they are Lagrangian spheres; see [7, Section 4].

The spheres Σ_{α_i} are pairwise disjoint in A_{2n-1} because their projection arcs α_i are pairwise disjoint in $\mathbb{C}$. Similarly, the spheres Σ_{β_i} are also pairwise disjoint. With these

spheres, we can build two Lagrangians in $\mathrm{Hilb}^n(A_{2n-1})$ and, in fact, in $\mathcal{Y}_n$ by

$$\mathcal{K}_\alpha = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \cdots \times \Sigma_{\alpha_n}, \tag{2-7}$$

$$\mathcal{K}_\beta = \Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \cdots \times \Sigma_{\beta_n}. \tag{2-8}$$

It is worth pointing out that $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ do not intersect transversely. The intersections of the spheres Σ_{α_i} and Σ_{β_j} in A_{2n-1} are points in the fibers of the endpoints of the α arcs and β arcs and circles S^1 in the fibers of interior intersections of those arcs. Thus, we could have some tori in $\mathcal{K}_\alpha \cap \mathcal{K}_\beta$. To deal with this, we can either perturb one of the Lagrangians (see [7, Section 6.1]), or use Floer theory with clean intersections, as in [9]. We can use either method when we compute Floer cohomology groups, because both methods result in isomorphic homology groups.

For example, we can perturb $\mathcal{K}_\beta$ as follows. As $\mathcal{K}_\beta$ is a product $\Sigma_{\beta_1} \times \Sigma_{\beta_2} \times \cdots \times \Sigma_{\beta_n}$, it will be easier to perturb each Σ_{β_i} in the complex surface A_{2n-1}. Let N be a circle in the intersection $\Sigma_{\alpha_j} \cap \Sigma_{\beta_i} \subset A_{2n-1}$ and V be a small neighborhood of N such that V intersects no other Lagrangian spheres. Following Weinstein [14, Theorem 6.5], we use the standard height function on N to isotope Σ_{β_i} into Σ'_{β_i} so that Σ'_{β_i} is identical to Σ_{β_i} outside of V and Σ'_{β_i} intersects $\Sigma_{\beta_i} \cap V$ only at the maximum and minimum of the height function. We repeat this process for all the S^1 intersections on Σ_{β_i} and for all the β Lagrangian spheres Σ_{β_i}. With a slight abuse of notation, we call the resulting Lagrangian spheres Σ'_{β_i} and let $\mathcal{K}'_\beta = \Sigma'_{\beta_1} \times \Sigma'_{\beta_2} \times \cdots \times \Sigma'_{\beta_n}$. The resulting Floer cochain complex $\mathrm{CF}^*(\mathcal{K}_\alpha, \mathcal{K}'_\beta)$ will be quasi-isomorphic to $\mathrm{CF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$.

For simplicity, we treat $\mathcal{K}_\beta$ as the original Lagrangian perturbed, unless mentioned otherwise, so that it intersects transversely with $\mathcal{K}_\alpha$ at isolated points. Each intersection at the interior of arcs α_i and β_j now gives the intersection of Σ_{α_i} and Σ_{β_j} at two points instead of a circle.

Proposition 2.2 [7, Theorem 1.2; 13, Theorem 4.12] *For any bridge diagram D representing a link L, the Floer cohomology $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ in $\mathcal{Y}_n = \mathrm{Hilb}^n(A_{2n-1}) \backslash D_r$ is canonically isomorphic to Seidel and Smith's symplectic Khovanov homology $\mathrm{Kh}^*_{\mathrm{symp}}(L)$.*

With Proposition 2.2 in mind, we finally define:

Definition 2.3 The symplectic Khovanov cohomology $\mathrm{Kh}^*_{\mathrm{symp}}(D)$ of a bridge diagram D is defined to be $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ in $\mathcal{Y}_n = \mathrm{Hilb}^n(A_{2n-1}) \backslash D_r$.

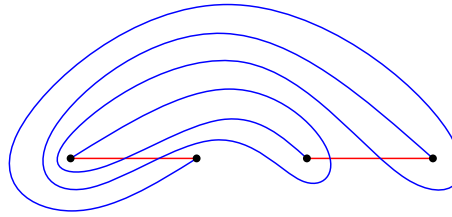

Figure 1: A flattened braid diagram for the trefoil.

Remark 2.4 We have not yet given an absolute grading for this definition. In fact, Proposition 2.2 is proved in the relatively graded case. Manolescu discussed an absolute grading in [7, Section 6.2], but it only applies in the case of flattened braid diagrams, where explicit choices can be made on Lagrangians to construct an absolute Maslov grading. Waldron's construction in [13] works for any bridge diagram, but only in the relatively graded case.

In the rest of the paper, whenever we mention $\mathrm{Kh}^*_{\mathrm{symp}}(L)$, we will always be working with Definition 2.3 rather than the original definition of Seidel and Smith, unless noted otherwise.

2.2 An absolute grading for bridge diagrams

In this subsection, we prove Theorem 1.2 by showing that the homological grading shifted by $\mathrm{gr}(x_0) + \mathrm{rot} + w$ is invariant under isotopy, handlesliding and stabilization. We reiterate our conventions here that are crucial in defining the absolute grading. Our link is oriented. At each intersection, the β arc overcrosses the α arc. The distinguished generator x_0 has the coordinates at the starting points of all β arcs.

The idea of this correction term is from Droz and Wagner in [3], in which the authors considered grid diagrams obtained from deforming flattened braid diagrams. A flattened braid diagram of a braid $b \in \mathrm{Br}_n$ is a special bridge diagram associated to the braid b whose marked points p_i are placed on the real line ordered by their indices, the α arcs are segments on the real line connecting p_{2i-1} and p_{2i}, and the β arcs are the images of α arcs after applying the braid b action on the odd-numbered strands.

Remark 2.5 An interesting example is that if we take a bridge diagram representing a braid closure, the rotation number is exactly the number of strands (because all Seifert circles are going counterclockwise), and the writhe of the diagram matches the writhe

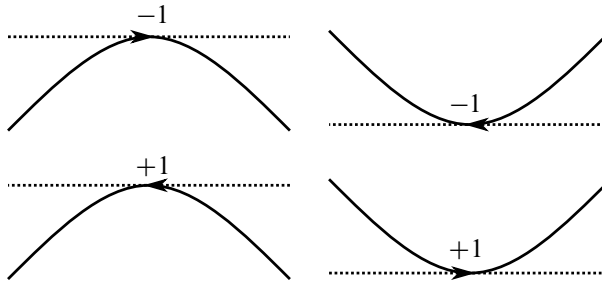

Figure 2: Local picture near the marked points with horizontal tangent lines and their sign assignments.

of the braid. The distinguished generator has homological grading 0 in any flattened braid diagram. Thus the correction term $\mathrm{gr}(x_0)+\mathrm{rot}+w$ for the braid closure diagram agrees with Manolescu's correction term $n+w$ of the flattened braid diagram, where these two diagrams are isotopic as link diagrams.

Equipping the Lagrangian Floer homology with an absolute grading requires us to grade our Lagrangians by choosing some sections of the canonical bundle. In the braid closure setup, the two Lagrangians are related by some fibered Dehn twists and thus the choice on the second Lagrangian can be induced from the first one canonically. In the general bridge diagram case (not a flattened braid diagram), there is no easy way to assign a choice to the second Lagrangian, so instead, we assign to the generator x_0 the grading 0.

Manolescu in [7, Section 6.2] pointed out a combinatorial method to compute the relative homological grading. In short, we replace each β arc β_j with an oriented figure-eight γ_j (and the orientation does not matter) in a small neighborhood of β_j. We arrange our diagram so that each α arc is horizontal and each figure-eight is vertical wherever they intersect. Each intersection of Lagrangian spheres Σ_{α_i} and Σ_{β_j} corresponds to an intersection of the arc α_i and the figure-eight γ_j. We travel along the figure-eight γ_j and mark the points that have horizontal tangent lines. We assign $+1$ to those marked points if the part of the figure-eight near the marked point is locally oriented counterclockwise, and -1 if clockwise; see Figure 2.

Now, we label each point on γ_j with a number, starting with the quantity 0 at the intersection representing the starting point of β_j. Note that this is an arbitrary choice, but we will correct for it later. We move along the figure-eight γ_j following its orientation. We change the quantity we assign only when we move past the marked

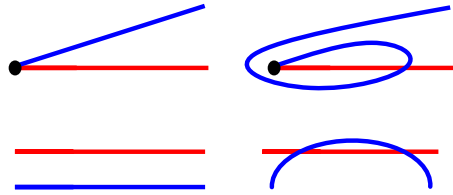

Figure 3: Two possible bridge diagram isotopies corresponding to Reidemeister I and II moves.

points with horizontal tangent lines. The new quantity will be the sum of the previous one and the number we put on the marked points, ie $+1$ or -1; see Figure 4 for an illustration of the computation. An easy combinatorial argument implies that the sum of the numbers on such marked points is 0 and thus we get back to 0 when we reach back to the starting point. Each of the generators of the Floer cohomology group corresponds to n intersection points and the sum of the labeled numbers will be its relative grading. In the construction, we made a choice that the coordinate at each starting point of the β arc has grading 0, and thus the distinguished generator x_0 has grading 0. In other words, the grading shifted by $\mathrm{gr}(x_0)$ will be independent of the grading we assign to the starting points of the β arc. We now prove the following proposition about the absolute grading.

Proposition 2.6 $\mathrm{HF}^{*+\mathrm{gr}(x_0)+\mathrm{rot}+w}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ *is invariant under bridge diagram isotopy.*

Proof An isotopy of a bridge diagram induces Hamiltonian-isotopic Lagrangians and thus keeps the relatively graded group $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ unchanged. If any crossing arises, it must come from one of the cases shown in Figure 3, which correspond to a Reidemeister I move and a Reidemeister II move (whereas the Reidemeister move III is from a combination of (de-)stabilizations and handleslides).

In the first case (see top row of Figure 3), there are two subcases: the α arc oriented to the left, and to the right. The easier subcase is the right-going α curve. The crossing is a positive crossing and the bigon region gives an additional clockwise Seifert circle. The distinguished generator x_0 has no coordinate in this region and thus its grading does not change. The total change of $\mathrm{gr}(x_0) + \mathrm{rot} + w$ is 0.

The other subcase is more subtle. The crossing is still positive but the Seifert circle is now counterclockwise, which increases the correction term by 2. The distinguished generator x_0 indeed takes a coordinate at the black dot in the top row of Figure 3.

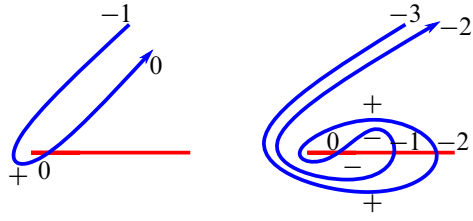

Figure 4: Isotopy corresponding to Reidemeister I move. We label each intersection and the endpoint of the figure-eight with its grading and we abbreviate the ± 1 labels on the marked points with horizontal tangent lines to $\pm$ signs to avoid confusion.

With the illustration of figure-eights in Figure 4, we can see that the relative grading is changed by 2, which cancels the contribution of the other two terms to $\mathrm{gr}(x_0)+\mathrm{rot}+w$. In fact, we can explicitly describe the new cochain complex. The bigon region in the complex plane lifts to topological disks that have holomorphic representatives in the complex surface A_{2n-1} and thus in $\mathcal{Y}_n$ when multiplied with constant discs, connecting one of the two generators at the interior intersection to the endpoint. One of the moduli spaces has dimension one and contributes to differentials which cancel pairs of generators having identical coordinates except one coordinate in the picture. At the cochain level, the isotopy created three copies of the original cochain complex with generators that have a coordinate at the black dot, but two of the three copies are canceled on the homology level. The surviving copy should have the same grading as the original cochain. It is not hard to see that the other moduli space of the bigon region has dimension two; thus, locally changing coordinate from the lower grading coordinate at the intersection to the black dot decreases the grading by 2. So in the new diagram, the grading of the distinguished generator decreases by 2 relative to other generators, which cancels the contribution by the sum of writhe and rotation number.

In the second case which corresponds to a Reidemeister II move, the two new crossings always have different signs so the writhe remains unchanged regardless of the orientation. If the two strands are oriented in the same direction, the Seifert surface remains unchanged and thus the rotation number is also the same. If the two strands are oriented in the opposite direction, there will be two subcases, whether the two strands are from the same Seifert circle or not. In the first subcase (the top row of Figure 5), locally one Seifert circle breaks into three circles, but the middle one is of the opposite direction. In the second subcase (the bottom row of Figure 5), there are two Seifert circles of the same direction in each picture. In either case, the rotation number remains the same as

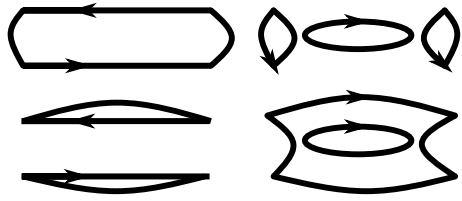

Figure 5: Two cases for Reidemeister II isotopy when two strands are oriented in the opposite direction. The first row corresponds to the case when both strands are oriented in the same direction and the second row corresponds to the case when strands are oriented in the opposite direction.

well. Apart from the two new intersections in the new diagram, all other generators are from the diagram before the isotopy and it is not hard to see that the gradings of such generators remain unchanged. Thus $\mathrm{gr}(x_0)$ remains unchanged as well, relative to other generators. □

Proposition 2.7 $\mathrm{HF}^{*+\mathrm{gr}(x_0)+\mathrm{rot}+w}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ *is invariant under stabilization.*

Proof Stabilizations will not change the writhe or the rotation number of the diagram. It suffices to show that the grading change of the homology is the same as the grading change of the distinguished generator x_0, when we apply stabilizations. This follows because Waldron's proof of invariance of the relative grading under stabilization establishes an isomorphism on the cochain level; see [13, Lemma 5.19]. □

Handleslide invariance is the hardest among the three. Before the proof, we make some topological observations. With isotopy invariance, we should arrange our diagram to be as simple as possible.

We explicitly describe the process of the handleslide of arc α_2 over α_1, as shown in Figure 6. First, we choose the boundary of a tubular neighborhood of α_1, so that each midpoint intersection of any β arc locally intersects the circle exactly twice, and the β arcs connecting the endpoints of α_1 locally intersect the circle exactly once. Then pick any path γ from α_2 to the circle that does not intersect any α arc in its interior, and perform a connected sum of α_2 and the circle along γ so that each intersection between the path and the β arc creates exactly two intersections on the connected sum.

Proposition 2.8 $\mathrm{HF}^{*+\mathrm{gr}(x_0)+\mathrm{rot}+w}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ *is invariant under handleslides.*

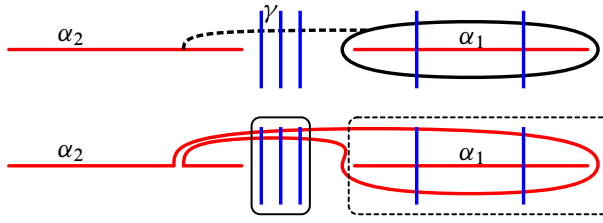

Figure 6: Handlesliding α_2 over α_1 along the arc γ. In the diagram after handlesliding, as shown in the second row, the left (solid) box corresponds to Figure 7 and the right (dashed) box corresponds to Figure 8.

Proof It is clear that new intersections are created in pairs, and each pair contains exactly one negative and one positive crossing. Thus, the writhe of the diagram remains unchanged.

Next, we show that the rotation number remains unchanged as well. The proof breaks into three steps.

First, we study the intersections created around the path γ, ie the intersections in the solid box in Figure 6. Locally, there is a series of parallel β arcs intersecting two α arcs. The two α arcs are parallel to the path γ and always oriented in the opposite directions.

Let us label a β arc with positive sign if it goes from bottom to top, and with negative sign if it goes from top to bottom. For convenience, we order those parallel β arcs from left to right. We study the local picture with only two adjacent β arcs. There are four cases in total shown in Figure 7. In the case $++$, the Seifert circle containing the left arc remains unchanged. The Seifert circle containing the right arc goes to the local picture involving the right β arc and the next one to the right outside the figure. The Seifert circle will either be unchanged, if the next β arc is also $+$, or it will be discussed in the case $+-$ below, if the next β arc is $-$. In the case $--$, the Seifert circle containing the right arc remains the same and the other Seifert circle will either be unchanged or will be discussed in the case $+-$ below.

In the case $+-$, there are two subcases, whether the two β arcs belong to the same Seifert circle or different ones, similar to the Reidemeister II move argument as shown in Figure 5. In this case either a clockwise circle is broken into two or two clockwise circles are merged into one. In either case, the rotation number is reduced by 1.

In the case $-+$, one counterclockwise circle is created. The other two Seifert circles are studied already in other cases. Thus, the rotation number is increased by 1.

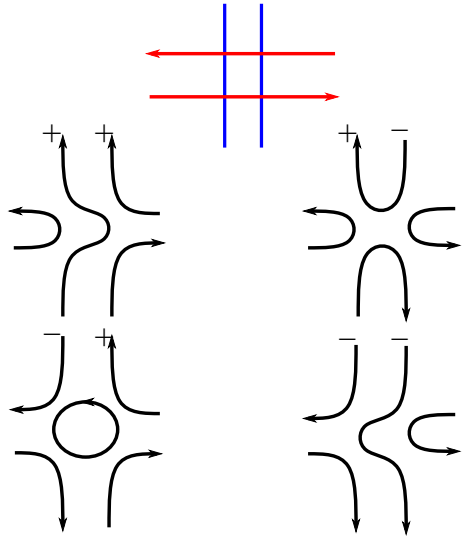

Figure 7: All four possible Seifert circles at the intersections near the path γ, depending on the orientation of β arcs.

To sum this up, each time the sign changes from $+$ to $-$, the rotation number decreases by 1, and each time the sign changes from $-$ to $+$, the rotation number increases by 1. At the intersection of the path γ and α_1, α_1 is oriented to the right to match the orientation of the α arcs in the handleslide picture, and the circle will remain the same if the first β is positive, and decreases the rotation number by 1 if the first β is negative. Thus, if the last β arc is positive, the rotation number remains the same, and if the last one is negative, the rotation number is reduced by 1. The last β arc is also related to the last step of the argument.

The second step is to compute the contribution of new intersections created at the circle near α_1, ie the intersections in the dashed box of Figure 6. Without loss of generality, let us assume α_1 is oriented from left to right. The part in the dashed box of Figure 8 is exactly the same as the picture before the handleslide. The rest of the region is two parallel α arcs oriented in the opposite direction, intersecting a series of parallel β arcs. The study of these new intersections is similar to the first half of the argument and the conclusion is similar, except now the sign depends on the first β arc. We claim that the rotation number remains the same if the first arc is positive, and increases by 1 if the first arc is negative.

Combing the results from the first two steps, the change of the rotation number only depends on the direction of the last β arc in Figure 7 and the first β arc in Figure 8.

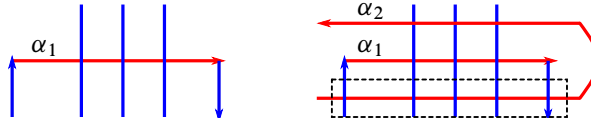

Figure 8: Diagrams near α_1. The left one is before the handleslide and the right one is after the handleslide.

If both are of the same direction, the total contribution to the rotation number is 0. If the β from the first half is $-$ and the β from the second half is $+$, then we have a counterclockwise Seifert circle while all the other Seifert circles remain unchanged; see Figure 9. This Seifert circle cancels the effect of the second arc being negative, ie increases the rotation number by 1. In the symmetric case, the creation of a clockwise Seifert circle between the two arcs cancels the effect of the first β arc being negative, ie decreases the rotation number by 1. In any of the cases above, the total change of rotation number is 0.

Lastly, we need to check the relative grading of the homology remains the same relative to the distinguished generator x_0. First of all, the isomorphism between symplectic Khovanov homology groups induced by handlesliding is induced by a continuation map on the cochain level. It is also easy to see that there is an injection from the generators of the original cochain complex to the ones of the cochain complex after the handleslide. It is easy to see that the homological gradings of the corresponding generators are not changed by considering the formulation of the relative grading in terms of tangencies of figure-eights, because we only changed one of the arcs and we essentially only applied an isotopy to one of the α arcs if we allow the isotopy to move across other marked points and α arcs. Moreover, the distinguished generator is obviously one of the generators that are preserved in the correspondence, and thus the homological grading relative to the distinguished generator x_0 remains the same. □

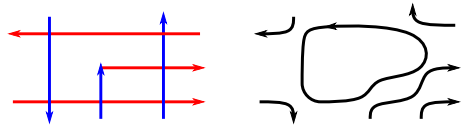

Figure 9: The figure on the left is the region connecting Figures 7 and 8 with the case $-+$. The left β arc is the rightmost β arc of Figure 7, the middle short β arc is the leftmost β arc of Figure 8, and the right β arc is the second leftmost β arc of Figure 8. The figure on the right is its Seifert circles and there is a counterclockwise Seifert circle in the middle.

Combining the propositions in this subsection, we complete a proof for Theorem 1.2, that we obtain an absolutely graded version of symplectic Khovanov cohomology for bridge diagrams.

3 Weight grading on $\mathrm{Kh}^*_{\mathrm{symp}}(L)$

In this section, we discuss the construction of the weight grading in Section 3.1, following the idea of Abouzaid and Smith in [1]. We then turn to its behavior under certain Floer products in Section 3.2. Lastly, we construct a long exact sequence of the unoriented skein relation for bigraded symplectic Khovanov cohomology groups in Section 3.3.

3.1 Construction of weight grading

In this subsection, we define the weight grading wt on $\mathrm{Kh}^*_{\mathrm{symp}}$ following the idea of Abouzaid and Smith in [1, Section 3]. The key point of constructing such a grading is to build an automorphism, more precisely, the linear term of a *noncommutative vector field*, or an *nc-vector field* [1, Definition 2.3],

$$\phi\colon \mathrm{CF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta) \to \mathrm{CF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta), \tag{3-1}$$

which preserves the homological grading and commutes with the differential. Thus, it induces an automorphism Φ on homology. If x is an eigenvector of eigenvalue λ, we define the weight grading $\mathrm{wt}(x) = \lambda$. Most results of this section hold for fields of any characteristic but we will restrict our discussion to fields of characteristic zero; see Remark 3.4 for more detail. In this subsection only, we use L for Lagrangians rather than links, as we will not use link diagrams in this subsection.

The idea of defining the map ϕ is to study certain moduli spaces of holomorphic maps in a partially compactified space $\overline{\mathcal{Y}}_n$ of $\mathcal{Y}_n$. We will begin by introducing the partial compactification $\overline{\mathcal{Y}}_n$ before we set up the moduli spaces. We leave some of the technical details to [1]. Abouzaid and Smith give a hypothesis under which it is possible to define an nc-vector field on an exact Fukaya category [1, Hypothesis 3.1], and show that a certain partial compactification of $\mathcal{Y}_n$ satisfies their hypothesis [1, Section 6].

Remark 3.1 It is worth noting that for some geometric hypotheses to hold — see [1, Lemma 6.5], for example — Abouzaid and Smith considered Lagrangians in fibered position, which are the Lagrangians that are fibered over α or β arcs, throughout their

papers [1; 2]. This will not hold if we perturb one of the Lagrangians to make the Lagrangians intersect transversely. As a result, Abouzaid and Smith worked with Lagrangian Floer theory with clean intersections. From now on, we no longer perturb our Lagrangians but assume that the Lagrangians intersect cleanly. We do so without further comment.

Recall that the Milnor fiber A_{2n-1} admits a Lefschetz fibration structure over $\mathbb{C}$, with regular fibers cones and singular fibers cylinders. We partially compactify A_{2n-1} into $\bar{A}_{2n-1}$ by adding two sections so that each regular fiber is compactified to two 2–spheres, while each singular fiber is compactified to a 2–sphere. Then we can partially compactify $\mathcal{Y}_n$ as $\bar{\mathcal{Y}}_n = \mathrm{Hilb}^n(\bar{A}_{2n-1})\backslash D_r$, where D_r is defined analogously to equation (2-4) for the partially compactified surface. Following the notation of Abouzaid and Smith, we let s_0 and s_∞ be the two sections we add to A_{2n-1}. Let D_0 be the divisor of subschemes whose support meets $s_0 \cup s_\infty$.

Now we turn to the moduli space. Let $\mathcal{R}^{k+1}_{(0,1)}$ be the moduli space of holomorphic classes of closed unit discs with the additional data of

- two interior marked points $z_0 = 0$ and $z_1 \in (0,1)$;
- $k+1$ boundary punctures at $q_0 = 1$, and k others $q_1, \dots q_k$ placed counterclockwise.

Following [1, Section 3.7], we define

$$\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \dots, x_1) \tag{3-2}$$

as the moduli space of finite-energy holomorphic maps $u\colon \mathbb{D} \to \bar{\mathcal{Y}}_n$ such that

(M-1) $u^{-1}(D_r) = \varnothing$,

(M-2) $u^{-1}(D_0) = z_0$,

(M-3) $u^{-1}(D_0') = z_1$, where D_0' is a divisor linearly equivalent to D_0 but shares no irreducible component with D_0,

(M-4) the relative homology class satisfies $[u]\cdot[D_0] = 1$,

(M-5) $u(q_i) = x_i$,

(M-6) the boundary segment between q_i and q_{i+1} maps to L_i.

This is the moduli space originally used by Seidel and Solomon in [12] for the q-intersection number, and it was later used by Abouzaid and Smith in [1, Section 3] for the nc-vector field.

Now we are ready to introduce our automorphism Φ, which only involves the case of $k = 1$. When $k = 1$, the virtual dimension of $\mathcal{R}^2_{(0,1)}(x, y)$ is the difference of the homological grading $\mathrm{gr}(x) - \mathrm{gr}(y)$; see [1, Lemma 3.16]. It makes sense to consider a map of degree 0 that counts holomorphic discs in $\mathcal{R}^2_{(0,1)}(x, y)$:

$$b^1(x) = \sum_{y\,|\,\mathrm{gr}(y)=\mathrm{gr}(x)} \#\mathcal{R}^2_{(0,1)}(x, y)\, y. \tag{3-3}$$

In fact, b^1 is the linear part of a *Hochschild cochain* $b \in \mathrm{CC}^*(\mathcal{F}(\mathcal{Y}_n), \mathcal{F}(\mathcal{Y}_n))$ if we allow multiple inputs instead of one single input. A precise construction of the Hochschild cochain b can be found at [1, Equation 3.89].

Remark 3.2 The construction of an nc-vector field requires a closed Hochschild cochain, but the cochain b is not closed; see [1, Proposition 3.20]. However, it can be made closed by adding some terms involving sphere bubbles; see [1, Equation 3.90]. To keep this section short, we will not introduce these additional terms here, because the degenerations involving sphere bubbles can be excluded analogously to the argument in last three paragraphs of [1, Section 3.9].

We can also denote by b^k the part of b with k inputs, which can be defined by counting discs in $\mathcal{R}^{k+1}_{(0,1)}(x_0; x_k, \ldots, x_1)$, similarly to the definition of b^1. The terms b^k for general k are not needed in this paper, except for the case $k = 0$, which will only be used in the definition below about equivariant structures. As we do not intend to expand the discussion of the equivariant structures, we refer the readers to [1, Section 3.4] for more details. The term b^0 is defined by counting holomorphic discs in $\mathcal{R}^1_{(0,1)}(x)$ with one output and no input such that the entire boundary is mapped to some Lagrangian L. The virtual dimension of $\mathcal{R}^1_{(0,1)}(y)$ is $\mathrm{gr}(y) - 1$ and thus $b^0|_L \in \mathrm{CF}^1(L, L)$ for each Lagrangian submanifold L.

Definition 3.3 An equivariant object is a pair (L, c), with $L \in \mathrm{Ob}(\mathcal{F}(M))$ and $c \in \mathrm{CF}^0(L, L)$, with $dc = b^0|_L$.

By definition, given an exact Lagrangian L, the obstruction to the existence of an equivariant structure c is given by $b^0|_L \in \mathrm{HF}^1(L, L) \cong H^1(L)$, and the set of choices when this vanishes is an affine space equivalent to $H^0(L)$. In our case, the Lagrangians are products of spheres, thus $H^1(L) \cong \{0\}$ and $H^0(L) \cong \Bbbk$.

Since we do not assume that b^0 vanishes, for a cocycle $b \in \mathrm{CC}^1(\mathcal{F}(M), \mathcal{F}(M))$ and equivariant objects $(\mathcal{K}_\alpha, c_\alpha)$ and $(\mathcal{K}_\beta, c_\beta)$, the linear term b^1 is not always a chain map. However, we can define a chain map following [1, Equation 2.31],

$$\phi(x) = b^1(x) - \mu^2(c_\alpha, x) + \mu^2(x, c_\beta). \tag{3-4}$$

It induces an endomorphism Φ on $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$. If we consider the generalized eigenspace decomposition, the eigenvalue of the generalized eigenvector x will be its weight grading, denoted as $\mathrm{wt}(x)$.

Remark 3.4 The construction above applies to fields $\Bbbk$ of any characteristic. The weight grading is a priori indexed by elements of the algebraic closure $\bar{\Bbbk}$. If $\mathrm{char}(\Bbbk) = 0$, there exist additive subgroups $\mathbb{Z}$ of $\Bbbk$, and thus of its algebraic closure $\bar{\Bbbk}$. As for finite fields, if $\Bbbk = \mathbb{F}_p$ with a prime number p for simplicity, we can then compute $\bar{\Bbbk} = \bigcup_n \mathbb{F}_{p^{n!}}$; it is no longer possible to find an additive subgroup $\mathbb{Z}$ of $\mathbb{F}_p$ or of its algebraic closure. As a result, working with a characteristic-zero field will not only enable us to use the result of Abouzaid and Smith that identifies symplectic Khovanov cohomology and Khovanov homology, but also make it easier to obtain an integral weight grading, in contrast to finite fields.

We learn from [1, Lemma 2.12] that since $\mathrm{HF}^0(\mathcal{K}, \mathcal{K}) \cong \Bbbk$, changing equivariant structures shifts the overall weight gradings (ie the eigenvalues) by a constant

$$\Phi_{(\mathcal{K}_\alpha, c_\alpha), (\mathcal{K}_\beta, c_\beta)} = \Phi_{(\mathcal{K}_\alpha, c_\alpha + s_\alpha), (\mathcal{K}_\beta, c_\beta + s_\beta)} + (s_\alpha - s_\beta)\,\mathrm{id}\,. \tag{3-5}$$

So we have the following definition of the relative weight grading:

Definition 3.5 Let Φ be the endomorphism constructed above on $\mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$ and x be an eigenvector of Φ. The relative weight grading $\mathrm{wt}(x)$ is defined to be the eigenvalue of x. This construction relies on auxiliary choices of equivariant structures on $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, but different choices of such structures will only change all gradings by a fixed number. Thus $\mathrm{wt}(x)$ as a relative grading is independent of choices of equivariant structures.

Remark 3.6 With given equivariant structures on $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$, we can compute an absolute weight grading. But at the time of writing this paper, the author does not know the choices that would give a correction term independent of the link diagram.

Then we can rephrase Abouzaid and Smith's purity result as follows.

Proposition 3.7 [1, Proposition 6.11] *Let D be a bridge diagram without any crossings. Then we can choose c_α on $\mathcal{K}_\alpha$ and c_β on $\mathcal{K}_\beta$ such that for any $x \in \mathrm{HF}^*(\mathcal{K}_\alpha, \mathcal{K}_\beta)$, we have $\mathrm{wt}(x) = \mathrm{gr}(x)$.*

Recall the Khovanov homology of an unlink of k components U_k is

$$\mathrm{Kh}^{*,*}(U_k) = \bigotimes^{k} (\Bbbk_{(0,1)} \oplus \Bbbk_{(0,-1)}). \tag{3-6}$$

With the choice of the equivariant structures by Abouzaid and Smith, we have

$$\mathrm{Kh}^{*,*}_{\mathrm{symp}}(U_k) = \bigotimes^{k} (\Bbbk_{(1,1)} \oplus \Bbbk_{(-1,-1)}). \tag{3-7}$$

This proposition is a special case of our main theorem with crossing number equal to 0, if we relate gradings (gr, wt) on symplectic Khovanov cohomology and (i, j) on Khovanov homology with the formula

$$i = \mathrm{gr} - \mathrm{wt}, \tag{3-8}$$

$$j = -\mathrm{wt}. \tag{3-9}$$

3.2 Floer product and weight grading

In [1, Section 3], Abouzaid and Smith pointed out an important fact that the weight grading is compatible with Floer products, without actually phrasing and proving the precise statement. Also in [12, Equation 4.9], Seidel and Solomon discussed the derivation property in a similar setup. We prove the following proposition:

Proposition 3.8 *Let $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be compact Lagrangians given by crossingless matchings in $\mathcal{Y}_n$. Then for any eigenvector $\alpha \in \mathrm{HF}^*(\mathcal{K}_1, \mathcal{K}_2)$ and $\beta \in \mathrm{HF}^*(\mathcal{K}_0, \mathcal{K}_1)$,*

$$\mathrm{wt}(\mu^2(\alpha, \beta)) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta). \tag{3-10}$$

Proof Consider the boundary strata of the moduli space $\overline{\mathcal{R}}^3_{(0,1)}(x_0; x_1, x_2)$, where we have three boundary marked points and two interior marked points. As before, we exclude sphere bubbles. We can also exclude the cases when two interior marked points are split into two different components by the assumption (M-4). This is because both components will have positive intersection with D_0 and thus $[u] \cdot [D_0] \geq 2$, which contradicts (M-4).

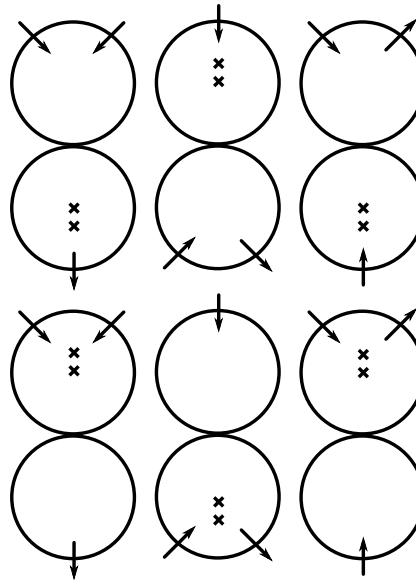

Figure 10: Six possible boundary bubbles. The incoming arrows are inputs and the outgoing ones are outputs.

With sphere bubbles excluded as discussed in Remark 3.2, there are still six possible types of degeneration; see Figure 10. Three degenerations shown in the first row compute

$$\phi(\mu^2(x_1, x_2)), \tag{3-11}$$

$$\mu^2(x_1, \phi(x_2)), \tag{3-12}$$

$$\mu^2(\phi(x_1), x_2). \tag{3-13}$$

The three degenerations shown in the second row compute

$$\mu^1\phi^2(x_1, x_2), \tag{3-14}$$

$$\phi^2(\mu^1(x_1), x_2), \tag{3-15}$$

$$\phi^2(x_1, \mu^1(x_2)). \tag{3-16}$$

If we pass to homology, those terms with μ^1 will vanish and thus we have the following relation by counting all the boundary components of the one-dimensional moduli space $\overline{\mathcal{R}}^2_{(0,1)}(x_0; x_1, x_2)$,

$$\phi(\mu^2(x_1, x_2)) = \mu^2(x_1, \phi(x_2)) + \mu^2(\phi(x_1), x_2). \tag{3-17}$$

This is equivalent to

$$\begin{aligned}\text{(3-18)}\qquad \mathrm{wt}(\mu^2(x_1,x_2))\mu^2(x_1,x_2) &= \mu^2(x_1,\mathrm{wt}(x_2)x_2)+\mu^2(\mathrm{wt}(x_1)x_1,x_2)\\ &= (\mathrm{wt}(x_1)+\mathrm{wt}(x_2))\mu^2(x_1,x_2),\end{aligned}$$

which proves the result. □

Remark 3.9 By studying a similar setup with more boundary marked points, we can generalize the result above to higher Floer products. Specifically, when we consider the product of three elements, we have $\mathrm{wt}(\mu^3(x_1,x_2,x_3)) = \mathrm{wt}(x_1)+\mathrm{wt}(x_2)+\mathrm{wt}(x_3)$.

3.3 A long exact sequence of $\mathrm{Kh}^{*,*}_{\mathrm{symp}}(L)$

Abouzaid and Smith [2, Equation 7.9] constructed a long exact sequence from an exact triangle of bimodules over the Fukaya category of $\mathcal{Y}_n$,

$$\text{(3-19)}\qquad \cdots \to \mathrm{Kh}^{*}_{\mathrm{symp}}(L_+) \to \mathrm{Kh}^{*}_{\mathrm{symp}}(L_0) \to \mathrm{Kh}^{*+2}_{\mathrm{symp}}(L_\infty) \to \cdots,$$

where L_+ is a link diagram with a positive crossing and L_0 and L_∞ are diagrams given by 0 or ∞ resolutions at the positive crossing. The goal of this chapter is to give an explicit construction of such a long exact sequence within the framework of bridge diagrams that preserves the weight grading, just like the combinatorial Khovanov homology.

We use the following local diagrams for the computation: we name the blue curves β, the green curves γ and the yellow curves δ, respectively, in Figure 11, and the other components of γ and δ are small isotopies of the components of β such that there are no intersections in the interiors of the arcs. Thus, the Lagrangians $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta$ intersect pairwise transversely. (Here we moved the actual arcs rather than perturbing the Lagrangians to avoid issues we discussed in Remark 3.2.) Pairing α with β gives L_+, pairing α with γ gives L_0, and pairing α with δ gives L_∞. We have the following exact sequence:

Proposition 3.10 [2, Proposition 7.4] *If we have α, β, γ and δ curves presented locally as in Figure 11 and β, γ and δ are the same apart from in this region, then we have the following exact sequence*

$$\text{(3-20)}\qquad \cdots \xrightarrow{c_1} \mathrm{HF}^*(\mathcal{K}_\alpha,\mathcal{K}_\beta) \xrightarrow{c_2} \mathrm{HF}^*(\mathcal{K}_\alpha,\mathcal{K}_\gamma) \xrightarrow{c_3} \mathrm{HF}^{*+2}(\mathcal{K}_\alpha,\mathcal{K}_\delta) \xrightarrow{c_1} \cdots.$$

In particular, there are elements

$$c_1 \in \mathrm{CF}^*(\mathcal{K}_\beta,\mathcal{K}_\delta), \quad c_2 \in \mathrm{CF}^*(\mathcal{K}_\gamma,\mathcal{K}_\beta) \quad \text{and} \quad c_3 \in \mathrm{CF}^*(\mathcal{K}_\delta,\mathcal{K}_\gamma)$$

such that the maps above are Floer products with the corresponding elements.

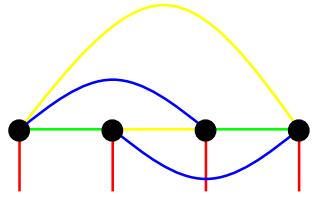

Figure 11: Exact triangle of Lagrangians. The red, blue, green and yellow curves are α, β, γ and δ curves, respectively.

Proof Abouzaid and Smith proved that there is an exact triangle of bimodules of the Fukaya category of $\mathcal{Y}_n$ among the identity bimodule, the cup–cap bimodule and the bimodule representing a half-twist τ. Evaluating these bimodules at $\mathcal{K}_\gamma$ as the second object, we have an exact triangle of one-sided modules between $\mathcal{K}_\gamma$, $\mathcal{K}_\delta$ and a one-sided module that is equivalent to $\mathrm{HF}^*(\bullet, \mathcal{K}_\beta)$, such that the maps connecting those terms are Floer products with some elements $c_1 \in \mathrm{CF}^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in \mathrm{CF}^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $c_3 \in \mathrm{CF}^*(\mathcal{K}_\delta, \mathcal{K}_\gamma)$. Evaluating these one-sided modules at $\mathcal{K}_\alpha$, we have the desired long exact sequence. □

Now we need to show that this long exact sequence preserves the relative weight grading in the sense that each element summing to c_i has the same weight grading so that c_i has a well-defined weight grading, and moreover the weight gradings of c_1, c_2 and c_3 sum to 0. With a closer look into the diagram, we have the following observation:

Lemma 3.11 *Each pair of β, γ and δ forms a bridge diagram for an unlink of $n-1$ components without any crossings.*

Proof In the region shown in Figure 11, each pair of β, γ and δ forms a crossingless unknot. In the other regions not shown in the figure, each pair of arcs forms a crossingless unknot component as well. Thus we have one crossingless unknot component in Figure 11 and $n-2$ crossingless unknot components outside that figure. Together, each pair of β, γ and δ forms a bridge diagram for an unlink of $n-1$ components without any crossings. □

From the computations of [1, Proposition 5.12], we have:

Lemma 3.12 *With some grading shifts, we compute*

$$\mathrm{CF}^*(\mathcal{K}_\beta, \mathcal{K}_\gamma) \cong \mathrm{CF}^*(\mathcal{K}_\gamma, \mathcal{K}_\delta) \cong \mathrm{CF}^*(\mathcal{K}_\delta, \mathcal{K}_\beta) \cong \bigotimes^{n-1} H^*(S^2), \tag{3-21}$$

and thus all generators are cocycles.

Now, we prove the following proposition about the weight grading on those groups:

Proposition 3.13 *For a fixed choice of equivariant structures on the Lagrangians $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta$, there are well-defined weight gradings for the elements $c_1 \in \mathrm{CF}^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$, $c_2 \in \mathrm{CF}^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $c_3 \in \mathrm{CF}^*(\mathcal{K}_\delta, \mathcal{K}_\gamma)$.*

Proof From Lemma 3.12, we know that each of the Floer cochain complexes and the Floer cohomology groups above can be made so that weight grading equals homological grading. Together with the observation that choices of equivariant structures will only apply overall grading shifts to all weight gradings, thus we know the weight grading of any element in CF^n must be the same. If we fix a set of equivariant structures on $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta$, we have a well-defined weight grading for each c_i from its homological grading plus the effect a grading shift from changing the equivariant structure from the standard one. □

Before we prove $\mathrm{wt}(c_1)+\mathrm{wt}(c_2)+\mathrm{wt}(c_3)=0$, recall the following lemma from Seidel:

Lemma 3.14 [10, Lemma 3.7] *A triple $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta$ forms an exact triangle of Lagrangians if and only if there exist*

$$c_1 \in \mathrm{CF}^1(\mathcal{K}_\beta, \mathcal{K}_\delta), \quad c_2 \in \mathrm{CF}^0(\mathcal{K}_\gamma, \mathcal{K}_\beta), \quad c_3 \in \mathrm{CF}^0(\mathcal{K}_\delta, \mathcal{K}_\gamma),$$
$$h_1 \in \mathrm{HF}^0(\mathcal{K}_\gamma, \mathcal{K}_\beta), \quad h_2 \in \mathrm{HF}^0(\mathcal{K}_\beta, \mathcal{K}_\delta), \quad k \in \mathrm{HF}^{-1}(\mathcal{K}_\beta, \mathcal{K}_\beta)$$

such that

$$\mu^1(h_1) = \mu^2(c_3, c_2), \tag{3-22}$$

$$\mu^1(h_2) = -\mu^2(c_1, c_3), \tag{3-23}$$

$$\mu^1(k) = -\mu^2(c_1, h_1) + \mu^2(h_2, c_2) + \mu^3(c_1, c_3, c_2) - e_{\mathcal{K}_\beta}, \tag{3-24}$$

where $e_{\mathcal{K}_\beta} \in \mathrm{HF}^0(\mathcal{K}_\beta, \mathcal{K}_\beta)$ is the identity element of the Floer product

$$\mu^2 \colon \mathrm{HF}^*(\mathcal{K}_\beta, \mathcal{K}) \otimes \mathrm{HF}^*(\mathcal{K}_\beta, \mathcal{K}_\beta) \to \mathrm{HF}^*(\mathcal{K}_\beta, \mathcal{K}).$$

Lemma 3.15 *For any choice of equivariant structures on Lagrangians, the sum of weight gradings satisfies $\mathrm{wt}(c_1)+\mathrm{wt}(c_2)+\mathrm{wt}(c_3)=0$.*

Proof We know μ^1 vanishes on all the Floer groups above. If we prove that $h_1 = 0$ and $h_2 = 0$, then the equation (3-23) becomes $\mu^3(c_1, c_3, c_2) = e_{\mathcal{K}_\beta}$. Together with the fact that weight grading is compatible with Floer product, see Remark 3.9, we know that $\mathrm{wt}(c_1)+\mathrm{wt}(c_2)+\mathrm{wt}(c_3) = \mathrm{wt}(e_{\mathcal{K}_\beta})$. From [1, Lemma 4.10], no matter which equivariant structure we choose on $\mathcal{K}_\beta$, the identity always has the weight grading 0.

To see that $h_1 = 0$ and $h_2 = 0$, we need to look into the absolute grading. We claim that $\mathrm{HF}^*(\mathcal{K}_\gamma, \mathcal{K}_\beta)$ and $\mathrm{HF}^*(\mathcal{K}_\beta, \mathcal{K}_\delta)$ are supported in odd degrees. This is because pairing γ with β gives a flattened braid diagram for a braid σ with writhe 1. The number of strands is even and thus this group is supported in odd degrees. Similarly, pairing β with δ gives σ^{-1}, which also gives an odd-degree supported group. Notice that the third map c_3 should have degree 2 in the absolute grading case, thus the exact triangle is actually between $\mathcal{K}_\beta$, $\mathcal{K}_\gamma$ and $\mathcal{K}_\delta[2]$. But shifting the grading of Floer groups by 2 does not change the fact that h_1 and h_2 have even degrees. Thus, they must be 0. □

Thus we conclude the following:

Proposition 3.16 *The long exact sequence of equation (3-20),*

$$\cdots \xrightarrow{c_1} \mathrm{HF}^{*,\mathrm{wt}_1}(\mathcal{K}_\alpha, \mathcal{K}_\beta) \xrightarrow{c_2} \mathrm{HF}^{*,\mathrm{wt}_2}(\mathcal{K}_\alpha, \mathcal{K}_\gamma) \xrightarrow{c_3} \mathrm{HF}^{*+2,\mathrm{wt}_3}(\mathcal{K}_\alpha, \mathcal{K}_\delta) \xrightarrow{c_1} \cdots, \tag{3-25}$$

decomposes with respect to weight gradings.

Proof From Proposition 3.13, we know that there is a well-defined weight grading for c_1, c_2 and c_3. If we start with the first group $\mathrm{HF}^{*,\mathrm{wt}_1}(\mathcal{K}_\alpha, \mathcal{K}_\beta)$, the only nontrivial map will be at weight grading $\mathrm{wt}_1 + \mathrm{wt}(c_2)$ because the weight grading is compatible with Floer products. The same goes for $\mathrm{wt}_3 = \mathrm{wt}_1 + \mathrm{wt}(c_3) + \mathrm{wt}(c_2)$. The next weight grading should be $\mathrm{wt}_1 + \mathrm{wt}(c_2) + \mathrm{wt}(c_3) + \mathrm{wt}(c_1) = \mathrm{wt}_1$, which is exactly where we started with the first group. □

4 Proof of the main theorem via a bigraded isomorphism

In this section, we prove our main theorem, Theorem 1.3, through showing that our isomorphism is a bigraded refinement of the isomorphism in [2]. We start this section by rephrasing the main theorem of [2] as follows:

Proposition 4.1 [2, Theorem 7.5] *For any bridge diagram L, we have an isomorphism H between symplectic Khovanov cohomology and Khovanov homology,*

$$H\colon \mathrm{Kh}^*_{\mathrm{symp}}(L) \to \mathrm{Kh}^*(L). \tag{4-1}$$

In [2], we can only conclude from the original argument of Abouzaid and Smith that H is canonical for braid closures. With Proposition 2.2, that symplectic Khovanov cohomology is defined canonically (and the same for combinatorial Khovanov homology), we can claim that H is canonical for any link diagram. This is an isomorphism with only information on the homological grading. But Abouzaid and Smith's proof of the isomorphism between symplectic and ordinary Khovanov homology implies

that the long exact sequence in equation (3-20) commutes with the corresponding long exact sequence for Khovanov homology, as illustrated in the following lemma.

Lemma 4.2 *Fix a link diagram L_+ and its unoriented resolutions L_0 and L_∞ at one of the crossings. We represent their corresponding bridge diagrams with (α,β), (α,γ) and (α,δ) arcs respectively such that α, β, γ and δ are locally shown in Figure 11. The isomorphism H is compatible with the exact sequence (3-20), ie the following diagram commutes*:

$$\begin{array}{ccccc}
\mathrm{HF}^*(\mathcal{K}_\alpha,\mathcal{K}_\gamma) & \longrightarrow & \mathrm{HF}^{*+2}(\mathcal{K}_\alpha,\mathcal{K}_\delta) & \longrightarrow & \mathrm{HF}^{*+1}(\mathcal{K}_\alpha,\mathcal{K}_\beta) \\
\downarrow H & & \downarrow H & & \downarrow H \\
\mathrm{Kh}^*(L_0) & \longrightarrow & \mathrm{Kh}^{*+2}(L_\infty) & \longrightarrow & \mathrm{Kh}^{*+1}(L_+)
\end{array} \tag{4-2}$$

$$\begin{array}{ccccc}
\longrightarrow \mathrm{HF}^{*+1}(\mathcal{K}_\alpha,\mathcal{K}_\gamma) & \longrightarrow & \mathrm{HF}^{*+3}(\mathcal{K}_\alpha,\mathcal{K}_\delta) \\
\downarrow H & & \downarrow H \\
\longrightarrow \mathrm{Kh}^{*+1}(L_0) & \longrightarrow & \mathrm{Kh}^{*+3}(L_\infty)
\end{array}$$

where the upper row is the exact sequence of equation (3-20), and the lower row is the exact sequence for combinatorial Khovanov homology with grading $i-j$.

Proof In the proof of the Abouzaid–Smith isomorphism, they show that the cup functors of Khovanov and symplectic Khovanov are identified with the isomorphism in the arc algebra; see [2, Corollary 6.16]. Moreover, we know that the cap functors in both cases are adjoint to the corresponding cup functors from [2, Proposition 7.4]. The horizontal maps in the first squares are given by applying the same cap–cup functor in each case, and thus they commute with the isomorphisms in the first square.

For the other squares, the diagram naturally commutes if we replace the third group of each row with the mapping cone of the other two, given the fact that the first square is already commutative. Moreover, in the proof of [2, Theorem 7.6], the isomorphism H and the long exact sequences are constructed via the mapping cones, and thus factor through the cones of horizontal maps of the first square. Together with the fact that the third group of each row is isomorphic to a mapping cone of the other two (see [2, Proposition 7.4]), we conclude that the second and third squares are also commutative. □

Corollary 4.3 *The diagram involving the exact sequences of resolving a negative crossing is also commutative.*

Recall that we compared the bigradings of two theories at the end of Section 3.1 for unlinks.

Proposition 4.4 [1, Proposition 6.11] *The isomorphism H preserves the weight grading if D is a crossingless diagram, with grading correspondence* $\mathrm{gr} = i - j$ *and* $\mathrm{wt} = -j$.

We are now ready to prove the main theorem.

Proof of Theorem 1.3 We only need to prove that for any fixed Jones grading j_0, H is also an isomorphism with $\mathrm{wt} = -j_0 + c$ with some grading shift c,

$$H\colon \mathrm{Kh}^{*,-j_0}_{\mathrm{symp}}(L) \to \mathrm{Kh}^{*,j_0}(L). \tag{4-3}$$

We prove that this statement is true for any link (bridge) diagram by induction on the number of crossings. The base case for unlinks is proved with Proposition 4.4.

Now we assume that L_+ is a link diagram with n crossings, whereas its resolutions L_0 and L_∞ have $n-1$ crossings. Let us assume we are resolving at a positive crossing. If all crossings are negative, a similar argument can be applied for the exact sequences induced by resolving at a negative crossing. By the inductive hypothesis, the maps H are bigraded isomorphisms for L_0 and L_∞. Let us fix a Jones grading j_1 on $\mathrm{Kh}(L_0)$. The maps in the long exact sequence will be trivial unless the Jones grading j_2 on $\mathrm{Kh}(L_\infty)$ is $j_1 - 3v - 2$, where v is the signed count of crossings between the arc that ends at the left-most endpoint in Figure 11 and other components, and j_3 on $\mathrm{Kh}(L_+)$ is $j_1 + 1$. Thus we can decompose our commutative diagram with respect to the Jones grading:

$$\begin{array}{ccccc}
\mathrm{HF}^{*,-j_1}(\mathcal{K}_\alpha,\mathcal{K}_\gamma) & \xrightarrow{c_2} & \mathrm{HF}^{*+2,-j_2}(\mathcal{K}_\alpha,\mathcal{K}_\delta) & \xrightarrow{c_3} & \mathrm{HF}^{*+1,j_3'}(\mathcal{K}_\alpha,\mathcal{K}_\beta) \\
\downarrow H & & \downarrow H & & \downarrow H \\
\mathrm{Kh}^{*,j_1}(L_0) & \longrightarrow & \mathrm{Kh}^{*+2,j_2}(L_\infty) & \longrightarrow & \mathrm{Kh}^{*+1,j_3}(L_+)
\end{array} \tag{4-4}$$

$$\begin{array}{ccccc}
\xrightarrow{c_1} \mathrm{HF}^{*+1,-j_1}(\mathcal{K}_\alpha,\mathcal{K}_\gamma) & \xrightarrow{c_2} & \mathrm{HF}^{*+3,-j_2}(\mathcal{K}_\alpha,\mathcal{K}_\delta) \\
\downarrow H & & \downarrow H \\
\longrightarrow \mathrm{Kh}^{*+1,j_1}(L_0) & \longrightarrow & \mathrm{Kh}^{*+3,j_2}(L_\infty)
\end{array}$$

where the weight grading j_3' is given by $-j_2 + \mathrm{wt}(c_3)$. The first, second, fourth and fifth columns are all isomorphisms, so by the five lemma, we conclude that the third column is also an isomorphism and thus we know that the map H is also a bigraded

isomorphism for L_+. As for the grading correspondence, because the c_i all have fixed weight grading after specifying the choice of equivariant structures, if we change j_1 by any number k, we change j_3 also by k. As for the first row, if $-j_1$ is changed into $-j_1-k$, this will result in j_3' shifting by $-k$ as well. This is enough to show that the weight grading recovers the Jones grading as relative gradings. □

As a corollary of the theorem above, we also conclude Theorem 1.5, that the relative weight grading is independent of the choice of link diagrams.

Proof of Theorem 1.5 For any two bridge diagrams D and D' representing a link L, relative weight gradings wt and wt$'$ can be defined on D, and respectively D'. Theorem 1.3 indicates that both wt and wt$'$ coincide with $-j$ with as relative gradings. Thus wt and wt$'$ are the same as relative gradings. □

Lastly, we provide some insight into Question 1.6. At the writing of this paper, the author cannot provide a pure symplectic proof of Theorem 1.5 without referring to the invariance of the bigrading on Khovanov homology. Such a proof would consist of the invariance of weight grading under isotopy, handleslide and stabilization.

Isotopy and handleslide invariance could be potentially proved via Proposition 3.8, if we look closely enough into the bridge diagrams and the Floer products. Isotopy and handleslide invariance can also be deduced from Hamiltonian isotopy invariance. But general Hamiltonian isotopy invariance will require virtual perturbations (see [1, Remark 3.21]) because some of the transversality assumptions will not be preserved under general isotopies — the Lagrangian might bound some Maslov zero discs after isotopy, say.

The proof of stabilization invariance will be different from the other two invariance proofs. It requires some degeneration arguments in the Hilbert scheme setup, relating holomorphic discs in $\overline{\mathcal{Y}}_n$ and discs in $\overline{\mathcal{Y}}_{n-1}$ so that differentials in the stabilized diagram can be identified with differentials in the original diagram. One could extend the existing degeneration arguments for $\mathcal{Y}_n$ before the partial compactification. In the original nilpotent slice setup, Seidel and Smith gave such an argument in [11, Section 5.4]. However, the weight grading is defined using Hilbert schemes, but a stabilization invariance is never fully established in the Hilbert scheme setup. The author expects that one could potentially adapt the recent work of Mak and Smith [6] into the degeneration argument (or the so-called *neck-stretching argument*) of Hendricks, Lipshitz and Sarkar in [4, Section 7.4.1], fixing the issue mentioned in their correction, and then generalize the argument to the partial compactification $\overline{\mathcal{Y}}_n$.

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