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Staircase symmetries in Hirzebruch surfaces

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This paper continues the investigation of staircases in the family of Hirzebruch surfaces formed by blowing up the projective plane with weight b, that was started by Bertozzi, Holm Maw, McDuff, Mwakyoma, Pires and Weiler (2021). We explain the symmetries underlying the structure of the set of b that admit staircases, and show how the properties of these symmetries arise from a governing Diophantine equation. We also greatly simplify the techniques needed to show that a family of steps does form a staircase by using arithmetic properties of the accumulation function. There should be analogous results about both staircases and mutations for the other rational toric domains considered, for example, by Cristofaro-Gardiner, Holm, Mandini and Pires (2020) and by Casals and Vianna (2022).

53D05; 11D99

1 Introduction

1.1 Overview

This paper continues the investigation of the ellipsoidal embedding capacity function for the family of Hirzebruch surfaces H_b that was begun by Bertozzi, Holm, Maw, McDuff, Mwakyoma, Pires and Weiler [1]. Here (H_b,ω) is the one-point blowup $\mathbb{C}P^2(1)\,\#\,\overline{\mathbb{C}P}^2(b)$ of the complex projective plane with line class L of size 1 and exceptional divisor E_0 of size b. The capacity function $c_X\colon [1,\infty)\to\mathbb{R}$ for a general four-dimensional target manifold (X,λ) is defined by

$$c_X(z) := \inf\{\lambda \mid E(1,z) \overset{s}{\hookrightarrow} \lambda X\},$$

where $z\ge 1$ is a real variable, $\lambda X := (X,\lambda\omega)$, an ellipsoid $E(c,d)\subset\mathbb{C}^2$ is the set

$$E(c,d) = \left\{(\zeta_1,\zeta_2)\in\mathbb{C}^2 \;\middle|\; \pi\left(\frac{|\zeta_1|^2}{c}+\frac{|\zeta_2|^2}{d}\right)<1\right\},$$

and we write $E \overset{s}{\hookrightarrow} \lambda X$ if there is a symplectic embedding of E into λX.

It is straightforward to see that $c_{H_b}(z)$ is bounded below by the volume constraint function $V_b(z) = \sqrt{z/(1-b^2)}$, where $1-b^2$ is the appropriately normalized volume of H_b. Further, the function $z \mapsto c_{H_b}(z)$ is piecewise linear when not identically equal to the volume constraint curve. When its graph has infinitely many nonsmooth points (or steps) lying above the volume curve, we say that c_{H_b} has an *infinite staircase*.

It was proven by Cristofaro-Gardiner, Holm, Mandini and Pires [3] that when $b = 1/3$, ie in the case when H_b is monotone, the function c_{H_b} admits a staircase with outer steps at points $z = x_k/x_{k-1}$ that satisfy the recursion $x_{k+1} = 6x_k - x_{k-1}$ and accumulate at the fixed point $3+2\sqrt{2}$ of this recursion.[1] Another key result from this paper is [3, Theorem 1.8], stating that if c_{H_b} has an infinite staircase, then its accumulation point is at the point $z = \mathrm{acc}(b)$, the unique solution >1 of the following quadratic equation involving b:

$$z^2 - \left(\frac{(3-b)^2}{1-b^2} - 2\right) z + 1 = 0. \tag{1.1.1}$$

Further, if H_b has an infinite staircase, then at $z = \mathrm{acc}(b)$ the ellipsoid embedding function must equal the volume:

$$c_{H_b}(\mathrm{acc}(b)) = \sqrt{\frac{\mathrm{acc}(b)}{1-b^2}} =: V_b(\mathrm{acc}(b)). \tag{1.1.2}$$

We say that H_b is *unobstructed* if $c_{H_b}(\mathrm{acc}(b)) = V_b(\mathrm{acc}(b))$. Thus if H_b has a staircase, it is unobstructed. However the converse does not hold: [1, Theorem 6] shows that, although $H_{1/5}$ is unobstructed, it has no staircase. As shown in Figure 1, the function $b \mapsto \mathrm{acc}(b)$ decreases for $b \in [0, 1/3)$, with minimum value $a_{\min} := \mathrm{acc}(1/3) = 3+2\sqrt{2}$, and then increases.

It turns out that the nature of the accumulation function plays a crucial role in our discussion. For example, as shown in Lemma 2.1.1, the properties of the pairs of rational numbers (b, z) with $z = \mathrm{acc}(b)$ are a key to the symmetries of the problem. Other important consequences are collected in Section 2.2. Note also that the case $b = 0$ (that is, the case of $\mathbb{C}P^2$) was fully analyzed by McDuff and Schlenk [6]. Here there is a staircase, which is called the Fibonacci stairs because its numerics are governed by the Fibonacci numbers; see Figure 1. The new staircases that we have found for general H_b are all analogs of that one. However, as we explain in Example 2.3.7, it is perhaps better to consider our staircases to be offshoots of the $1/3$–staircase.

[1] This staircase actually consists of three intertwining strands that each satisfy this recursion; for details see Example 2.3.7. Further, conjecturally $b = 1/3$ is the only rational value of b at which H_b admits a staircase.

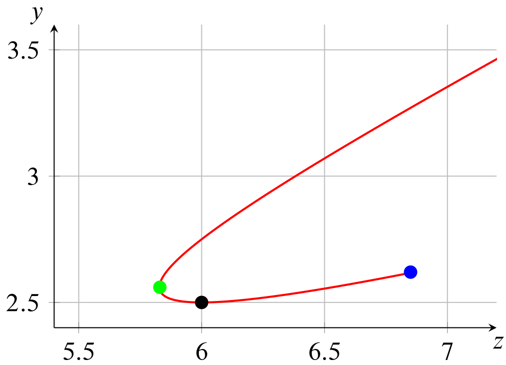

Figure 1: This shows the location of the accumulation point $(z, y) = (\mathrm{acc}(b), V_b(\mathrm{acc}(b)))$ for $0 \le b < 1$. The blue point with $b = 0$ is at (τ^4, τ^2) and is the accumulation point for the Fibonacci stairs. The green point with $z = 3 + 2\sqrt{2}$ and $b = 1/3$ is the accumulation point for the stairs in $H_{1/3}$, and is the minimum of the function $b \mapsto \mathrm{acc}(b)$. The black point with $z = 6$ and $b = 1/5$ is the place where $V_b(\mathrm{acc}(b))$ takes its minimum.

Obstructions to embedding ellipsoids come from certain exceptional divisors in blowups of the target manifold. When $X = H_b$, these divisors live in $\mathbb{C}P^2 \# (N+1)\overline{\mathbb{C}P}^2$ and their homology classes $\boldsymbol{E}$ have the form

$$dL - mE_0 - \sum m_i E_i =: (d, m, \boldsymbol{m}),$$

where $\boldsymbol{m} := (m_1, \dots, m_N)$ with $m_1 \ge m_2 \ge \cdots \ge m_N$. In the most relevant such classes, the tuple of coefficients $\boldsymbol{m}$ consists of the (integral) weight expansion of a center point p/q (see Definition 2.1.5); correspondingly we say that $\boldsymbol{E}$ is *perfect* and write $\boldsymbol{E} := (d, m, p, q)$. For z near p/q, the embedding obstruction is given by

(1.1.3) $$\mu_{\boldsymbol{E},b}(z) = \begin{cases} qz/(d - mb) & \text{if } z \le p/q, \\ p/(d - mb) & \text{if } z \ge p/q; \end{cases}$$

in particular, it has an outer corner (or step) at $z = p/q$. Since, as explained in [1, Section 2.1], $c_{H_b}(z)$ is the maximum over all exceptional classes $\boldsymbol{E}$ of the obstruction functions $\mu_{\boldsymbol{E},b}(z)$, given $\boldsymbol{E} = (d, m, p, q)$ as above, we must have $c_{H_b}(z) \ge \mu_{\boldsymbol{E},b}(z)$ for $z \approx p/q$. We say that the function $\mu_{\boldsymbol{E},b}$ is

- *obstructive* at z if $\mu_{\boldsymbol{E},b}(z) > V_b(z)$, and
- *live* at z if $c_{H_b}(z) = \mu_{\boldsymbol{E},b}(z) > V_b(z)$.

Further, we call $\boldsymbol{E}$ a *center-blocking class* if, for one of the two elements b of $\mathrm{acc}^{-1}(p/q)$, the function $z \mapsto \mu_{\boldsymbol{E},b}(z)$ is obstructive at the center $z = p/q$, since in this case it follows from (1.1.2) that the corresponding surface H_b has no staircase. As explained in [1, Lemma 38], it follows by continuity that for every center-blocking class $\boldsymbol{E}$ there is an open interval $J_{\boldsymbol{E}} \subset [0,1)$ that contains the appropriate[2] point of $\mathrm{acc}^{-1}(p/q)$ and is a component of the set of b–values that are blocked by $\boldsymbol{E}$.

The paper [1] found three families of center-blocking classes, $\boldsymbol{B}^U_n$, $\boldsymbol{B}^L_n$, and $\boldsymbol{B}^E_n$ for $n \geq 0$, together with six associated sequences of staircases. Each of these blocking classes $\boldsymbol{B}$ has an associated maximal blocked interval $J_{\boldsymbol{B}} = (\beta_{\boldsymbol{B},\ell}, \beta_{\boldsymbol{B},u}) \subset (0,1)$ consisting of points b that cannot admit a staircase because $\mu_{\boldsymbol{B},b}(\mathrm{acc}(b)) > V_b(\mathrm{acc}(b))$. However, it turns out that there are staircases at both endpoints of these intervals, which gives three staircase families, $\mathcal{S}^U$, $\mathcal{S}^L$, and $\mathcal{S}^E$, with staircases indexed by $n \geq 0$ and ℓ or u, where the steps of staircases labeled ℓ (for "lower") ascend, while those labeled u (for "upper") descend. The Fibonacci stairs appear as $\mathcal{S}^L_{\ell,\boldsymbol{0}}$.

It was noted in [1, Corollary 60] that the centers p/q of the blocking and step classes for the family $\mathcal{S}^U$ are related to those of $\mathcal{S}^L$ and $\mathcal{S}^E$ by a fractional linear transformation, that we denote by either $p/q \mapsto (ap+bq)/(cp+dq)$ or $(p,q) \mapsto (ap+bq, cp+dq)$. In particular, the two families $\mathcal{S}^U, \mathcal{S}^E$ are related by the *shift*

$$S\colon (p,q) \mapsto (6p-q,\, p) \tag{1.1.4}$$

that implements the recursion underlying the staircase at $1/3$; while the two families $\mathcal{S}^U$ and $\mathcal{S}^L$ are related by the *reflection*

$$R\colon (p,q) \mapsto (6p-35q,\, p-6q), \tag{1.1.5}$$

which fixes the point 7 and takes ∞ to 6.

Our main result verifies a conjecture in [1], and can be informally stated as follows. (For more detail, see Theorems 1.2.4 and 1.2.6.)

Theorem 1.1.1 *For each $i \geq 1$ there are staircase families $(S^i)^\sharp(\mathcal{S}^U)$ and $(S^i)^\sharp(\mathcal{S}^L) = (S^iR)^\sharp(\mathcal{S}^U)$, where the i–fold shift S^i and reflection R act on the centers of the blocking classes and staircase steps as above.*

Moreover, we will see in Proposition 1.2.2 that each staircase family is generated by its blocking classes together with two "seed" classes, a fact that makes it much easier to establish the effect of the symmetries on the staircase families. Note also that although the action of the symmetries S^i and S^iR on the centers p/q of the classes is clear, the

[2] If $m/d > 1/3$ then this will be the larger element in $\mathrm{acc}^{-1}(p/q)$, while if $m/d < 1/3$ it will be the smaller one; see [1, Definition 37]. By Lemma 2.2.13, there is no quasiperfect class with $m/d = 1/3$. If there is no possibility of confusion, we often simply call these classes blocking classes.

action on the other two coordinates (d, m) (that we call the *degree* coordinates) is much less obvious. In particular, this action is not compatible with composition. However, we will see in Section 3.4 that the action on degree can be understood because, as we explain below, the coefficients d and m of all the relevant classes are given in terms of p and q by a general formula (1.2.4).

Remark 1.1.2 In the setting considered by Usher [7], the target manifold $P(1,b)$ is the polydisc $B^2(1)\times B^2(b)$, or equivalently the product of two spheres of areas 1 and b. He finds a doubly indexed family of staircases $\mathcal{S}_{n,k} = (\boldsymbol{E}_{i,n,k})_{i\geq 0}$, where i indexes the staircase steps, $n \geq 0$ indexes the intrinsic recursion $x_{i+1,n,k} = \nu_n x_{i,n,k} - x_{i-1,n,k}$ satisfied by the parameters of the perfect classes $\boldsymbol{E}_{i,n,k}$ for $i \geq 0$ in $\mathcal{S}_{n,k}$, and k indexes a symmetry generated by so-called "Brahmagupta moves" that generate infinitely many families of staircases from a basic family $(\mathcal{S}_{n,0})_{n\geq 0}$. More precisely, in [7, Section 2.2.1], Usher finds a way to encode the parameters of the relevant perfect classes $\boldsymbol{E}$ by means of a triple (x, δ, ε) of integers that satisfy the Diophantine equation $x^2 - 2\delta^2 = 2 - \varepsilon^2$. Here the value of ε is related to the recursion variable n, and, if this is fixed, he shows that a (very!) classically known maneuver that goes from one solution of $x^2 - 2\delta^2 = N$ to another can be implemented in such a way that it preserves the set of perfect classes. Usher expressed this maneuver in arithmetic terms (multiplication by a unit in a number field). However, as we explain in Remark 2.2.6, when expressed in terms of the coordinates (p, q), Usher's basic symmetry is the same as ours, namely the transformation $(p, q) \mapsto (6p - q, p)$.

Usher's setting is simpler than ours in that the function $b \mapsto \operatorname{acc}(b)$ that specifies the accumulation point of any staircase for $P(1, b)$ is injective rather than two-to-one.[3] Also the symmetry between the two classes $[pt\times S^2]$ and $[S^2\times pt]$ allows the arithmetic properties of a general quasiperfect class to be encoded by means of variables that satisfy the equation $x^2 - 2\delta^2 = N$, while the corresponding equation in our setting is $x^2 - 8y^2 = k^2$ (see Lemma 2.1.1). Nevertheless, the two situations are very similar.

The work presented here leads to many interesting questions. Here are some of them.

- The picture developed here seems to make up the first level of an iterative "fractal" kind of structure for the Hirzebruch surfaces H_b. One might consider the family

[3] This statement is oversimplified in that one could well argue that the analog of our family H_b for $b \in [0, 1)$ is the family $P(1, b)$ for $b > 0$ with involution $b \mapsto 1/b$. However, $P(1, b)$ is symplectomorphic to a rescaling of $P(1, 1/b)$, so $c_{P(1,b)}$ is a rescaling of $c_{P(1,1/b)}$ and $\operatorname{acc}(b) = \operatorname{acc}(1/b)$. In our case, if $\operatorname{acc}^{-1}(z) = \{b^+, b^-\}$, the two functions $c_{H_{b^+}}$ and $c_{H_{b^-}}$ can be very different, one with a staircase, and one without; see Figure 2.

of blocking classes $\boldsymbol{B}^U_n$, for $n \geq 0$, extended by two seeds as in Proposition 1.2.2, to be the backbone of the first level of this structure. This level also includes the associated staircase classes. We prove in Proposition 2.2.9 that all the staircase classes are also center-blocking classes. Further, numerical evidence suggests that there are staircases whose steps have centers with 4–periodic continued fractions, indeed it seems with any even period. What seems to be the case is that each pair of adjacent ascending/descending staircases at level one shares a first step, and that this first step is a blocking class with associated 4–periodic staircases. Thus the backbone of the second level should consist of these shared steps, with associated 4–periodic staircases generated by appropriate seeds at level one. For more details, see Magill, McDuff and Weiler [5].

- It also would be very interesting to analyze Usher's results using the current framework, to see if there are analogs of blocking classes, seeds and staircase families. One might be able to build a bridge between the two cases by thinking of a polydisc as a degenerate two-point blowup of $\mathbb{C}P^2$, and then looking at the ellipsoidal capacity function for the family of two-fold blowups of $\mathbb{C}P^2$ that join the two cases. This will also be the subject of future work.

- The recursive patterns behind the staircases for rational target manifolds X are related to almost toric structures and the transformations called mutations that appear for example in [3] and Casals and Vianna [2]. It would be very interesting to know how the symmetries discussed here appear in those contexts.

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1.2 Main results

We now describe our main results in more detail.

In what follows it is important to distinguish purely numerical properties — such as those in (1.2.1) — from geometric properties that are needed to guarantee that a class $\boldsymbol{E}$ gives a live obstruction $\mu_{\boldsymbol{E},b}(z)$ at relevant values of b and z. As above we represent a class $\boldsymbol{E} = dL - mE_0 - \sum_i m_i E_i$, where $(m_1, m_2, \dots)$ is the weight expansion of p/q, by the tuple (d, m, p, q), and say that $\boldsymbol{E}$ is *quasiperfect* if and only if the Diophantine conditions

$$(1.2.1) \qquad 3d = p + q + m, \quad d^2 - m^2 = pq - 1$$

hold. (As explained in [1, Section 2.1], these are equivalent to the conditions $c_1(\boldsymbol{E}) = 1$ and $\boldsymbol{E} \cdot \boldsymbol{E} = -1$, where c_1 is the first Chern class of H_b.) We say that a quasiperfect class $\boldsymbol{E}$ is *perfect* if it is represented by an exceptional curve, which holds if and only if it reduces correctly by Cremona moves (see Lemma 4.1.1). Although a quasiperfect class is obstructive at its center $z = p/q$ for $b = m/d$ by [1, Lemmma 15], the fact that this obstruction is live in a neighborhood of this (b, z) (and hence coincides with the capacity function in some range) follows from "positivity of intersections", namely the fact that the intersection number of two different exceptional divisors is always nonnegative; see [1, Proposition 21]. This implies that the obstruction function $\mu_{\boldsymbol{E},b}(z)$ given by an exceptional class $\boldsymbol{E}$ is strictly larger than all other obstructions for $z \approx p/q$ and $b \approx m/d$. Hence in order to show that a given surface H_b actually has a staircase, we need to prove that the relevant staircase classes are exceptional classes. However, a large part of the following discussion is purely numerical. Notice also that because quasiperfect classes can be obstructive, it makes sense to consider *quasiperfect blocking classes*, ie tuples (d, m, p, q) such that $\mu_{\boldsymbol{B},b}(p/q) > V_b(p/q)$ for the appropriate $b \in \operatorname{acc}^{-1}(p/q)$.

The coefficients $(d_\kappa, m_\kappa, p_\kappa, q_\kappa)$ of the step classes of the staircases we consider always satisfy a recursion of the form

$$(1.2.2) \qquad x_{\kappa+1} = \nu x_\kappa - x_{\kappa-1}, \quad \kappa \ge \kappa_0,$$

for suitable recursion parameter ν and initial value κ_0.[4] Hence, given ν, each sequence of parameters $(x_\kappa)_{\kappa \ge i}$ (for $x = p, q, d, m$) is determined by two initial values x_{κ_0} and x_{κ_0+1} that are called *seeds*. It turns out that the relevant classes (d, m, p, q) can be extended to tuples

$$(1.2.3) \qquad (d, m, p, q, t, \varepsilon), \quad t > 0, \ \varepsilon \in \{\pm 1\},$$

where the integer t is a function of p and q. Further, the tuple (p, q, t, ε) determines the degree variables (d, m) by the formulas

$$(1.2.4) \qquad d := \tfrac{1}{8}(3(p+q) + \varepsilon t), \quad m := \tfrac{1}{8}((p+q) + 3\varepsilon t).$$

[4]Not all infinite staircases satisfy this recursion. The $b = 1/3$ staircase is given by a nonhomogenous recursion. Further, it is not known that all staircases must be given by some recursion.

Hence $|d-3m| = t$, and $\varepsilon = 1$ if and only if $m/d > 1/3$. This point of view is explained in Section 2.2.

It is straightforward to check that both S and R preserve t and hence $\varepsilon(d-3m)$, while they both change the sign of ε. Further, for classes that give obstructions when $b > 1/3$ we have $\varepsilon = 1$, while $\varepsilon = -1$ if the classes are relevant for $b < 1/3$; see Example 3.4.7.

Our first main result is that all the numerical data of a staircase family such as $\mathcal{S}^U$ is determined by a family of quasiperfect classes $(\boldsymbol{B}^n)_{n\ge 0}$ together with two seeds $\boldsymbol{E}_{\ell,\text{seed}}$ and $\boldsymbol{E}_{u,\text{seed}}$. To explain this, we introduce the following language.

- A *prestaircase* $\mathcal{S}$ is a sequence of tuples $\boldsymbol{E}_\kappa := ((d_\kappa, m_\kappa, p_\kappa, q_\kappa, t_\kappa, \varepsilon))_{\kappa\ge 0}$ that is defined recursively with recursion parameter ν, and that satisfy (1.2.4). Given such a sequence, the limits $a_\infty := \lim p_\kappa/q_\kappa$ and $b_\infty := \lim m_\kappa/d_\kappa$ always exist by Corollary 3.1.5. We say that $\mathcal{S}$ is *perfect* if all the classes $\boldsymbol{E}_\kappa$ are perfect, and that $\mathcal{S}$ is *live* if the obstructions $\mu_{\boldsymbol{E}_\kappa, b_\infty}$ are live near $z = p_\kappa/q_\kappa$ for all sufficiently large κ and with b equal to the limiting value b_∞. We will refer to a *staircase* as a live prestaircase.[5] Thus if $\mathcal{S}$ is live, H_{b_∞} has a staircase. This is a slight abuse of notation as not all staircases follow the recursive structure of a prestaircase, but all staircases considered in this paper are indeed prestaircases.

- A prestaircase $\mathcal{S}$ is said to be *associated to a quasiperfect class*
$$\boldsymbol{B} = (d_{\boldsymbol{B}}, m_{\boldsymbol{B}}, p_{\boldsymbol{B}}, q_{\boldsymbol{B}})$$
if the following linear relation is satisfied by its step coefficients:

$$(1.2.5)\qquad (3m_{\boldsymbol{B}} - d_{\boldsymbol{B}})d_\kappa = \begin{cases} (m_{\boldsymbol{B}} - q_{\boldsymbol{B}})p_\kappa + m_{\boldsymbol{B}} q_\kappa & \text{if } \mathcal{S} \text{ ascends,} \\ m_{\boldsymbol{B}} p_\kappa - (p_{\boldsymbol{B}} - m_{\boldsymbol{B}}) q_\kappa & \text{if } \mathcal{S} \text{ descends.} \end{cases}$$

If in addition the prestaircase is perfect, then H_{b_∞} is unobstructed, ie $c_{H_b}(\operatorname{acc}(b)) = V_b(\operatorname{acc}(b))$, and it is shown in [1, Theorem 52] that the limits (b_∞, a_∞) are the parameters (b, z) of the appropriate endpoint of the blocked b–interval $J_{\boldsymbol{B}}$.[6] Moreover, if there is both an ascending and a descending perfect prestaircase associated to $\boldsymbol{B}$ then $\boldsymbol{B}$ is a perfect blocking class. This means in particular that $\boldsymbol{B}$ is obstructive for the b–value corresponding to its center.

[5] We do not insist that the classes in a staircase are perfect; however in all known cases they are perfect. Indeed the only way that we know of to prove that a staircase is live is first to show that it is perfect and then to show there are no "overshadowing classes". See the beginning of Section 4 for more details.

[6] This follows very easily from the calculation in (2.2.8).

- A *prestaircase family* $\mathcal{F}$ consists of a family of quasiperfect classes $(\boldsymbol{B}_n^{\mathcal{F}})_{n\ge 0}$ (called *preblocking classes*) together with ascending prestaircases $\mathcal{S}_{\ell,n}^{\mathcal{F}}$, for $n \neq 1$, and descending prestaircases $\mathcal{S}_{u,n}^{\mathcal{F}}$, for $n \neq 0$, where $\mathcal{S}_{\bullet,n}^{\mathcal{F}}$ is associated with $\boldsymbol{B}_n^{\mathcal{F}}$ for $\bullet = \ell, u$. The family $\mathcal{F}$ is said to be *perfect* if all the classes in $\mathcal{F}$ are perfect, and *live* if all the prestaircases $\mathcal{S}_{\bullet,n}^{\mathcal{F}}$, $\bullet = \ell, u$, are live.

- A prestaircase family is called a *staircase family* if it is live. Then the preblocking classes are perfect by [1, Theorem 52], and in all cases encountered here the step classes are also perfect.

Finally, we make the following definition. It follows from the formulas in Theorems 2.3.1 and 2.3.3 that, because S^i preserves order, prestaircase families that are obtained from $\mathcal{S}^U$ by applying the shift S^i have preblocking classes whose centers ascend, while those that are obtained from $\mathcal{S}^L$ by applying S^i have preblocking classes whose centers descend. It turns out that in all cases the adjacent preblocking class can be considered as part of the appropriate staircase; for example in $\mathcal{S}^U$ the blocking class $\boldsymbol{B}_{n-1}^U$ can be considered as a step in the ascending staircase $\mathcal{S}_{\ell,n}^U$, while $\boldsymbol{B}_{n+1}^U$ can be considered as a step in the descending staircase $\mathcal{S}_{\ell,n}^U$. Clearly the numbering of this adjacent blocking class depends on whether the centers of these classes ascend or descend as n increases.

Definition 1.2.1 A prestaircase family $\mathcal{F}$ is said to be *generated by the quasiperfect classes* $\boldsymbol{B}_n$, *for* $n \ge 0$, *and seeds* $\boldsymbol{E}_{\ell,\text{seed}}$ *and* $\boldsymbol{E}_{u,\text{seed}}$ if the $\boldsymbol{B}_n = (d_n, m_n, p_n, q_n, t_n, \varepsilon)$ are its preblocking classes, and, for all n and $\bullet \in \{\ell, u\}$, the steps in the prestaircase $\mathcal{S}_{\bullet,n}^{\mathcal{F}}$ have recursion parameter $\nu = t_n$ and seeds

- $\boldsymbol{E}_{\ell,\text{seed}}, \boldsymbol{B}_{n-1}$ for $\bullet = \ell$ and $\boldsymbol{E}_{u,\text{seed}}, \boldsymbol{B}_{n+1}$ for $\bullet = u$, if the $\boldsymbol{B}_n$ ascend;
- $\boldsymbol{E}_{\ell,\text{seed}}, \boldsymbol{B}_{n+1}$ for $\bullet = \ell$ and $\boldsymbol{E}_{u,\text{seed}}, \boldsymbol{B}_{n-1}$ for $\bullet = u$, if the $\boldsymbol{B}_n$ descend.

In Propositions 3.2.2 and 3.2.6, we establish the following compact description of the numerical information that determines each of our staircase families. This information will make it much easier to understand the effect of the symmetries.

Proposition 1.2.2 *The staircase families $\mathcal{S}^U$ and $\mathcal{S}^L$ are generated by their blocking classes together with two seeds.*

Remark 1.2.3 As we explain in Example 2.3.7, all the classes (both blocking classes and seeds) that generate $\mathcal{S}^U$ are directly related to the classes that generate the staircase at $b = 1/3$. Since the staircase classes in $H_{1/3}$ satisfy the recursion (1.2.2) with $\nu = 6$

given by S, one can see that the whole structure of the staircases so far discovered in the family of manifolds H_b stems from that of the staircase at $b = 1/3$.

We now explain the action of the family[7] of transformations

$$\mathcal{G} := \{S^i R^\delta \mid \delta \in \{0,1\}, i \geq 0\}, \tag{1.2.6}$$

where S is the shift $p/q \mapsto (6p - q)/p$ and R is the reflection $z \mapsto (6z - 35)/(z - 6)$. The sequence of numbers

$$v_1 = \infty = \tfrac{1}{0}, \quad v_2 = S(v_1) = \tfrac{6}{1}, \quad \dots, \quad v_i = S^{i-1} v_1, \quad \dots$$

has limit $a_{\min} := 3 + 2\sqrt{2}$, which is the accumulation point of the $b = 1/3$ staircase. With $w_1 := 7$ and $w_i = S^{i-1}(w_1)$, we have the following interleaving family of numbers:

$$v_1 = \infty > w_1 = 7 > v_2 = 6 > w_2 = \tfrac{41}{7} > \dots > v_i > w_i > v_{i+1} > \dots \to a_{\min}. \tag{1.2.7}$$

The symmetry S acts as a shift, $S(v_i) = v_{i+1}$, $S(w_i) = w_{i+1}$, while R and its composition with powers of S are reflections. In particular $R_{v_i} := S^{i-2} R S^{-i+1}$ is a reflection that fixes v_i and interchanges w_{i-1} with w_i. As we show in (2.1.3), for each i the two b–values that correspond to the point $z = v_i$ are rational and have simple linear expressions in terms of the numerator and denominator p_i and q_i of $v_i = p_i/q_i$. Hence one might expect them to be relevant to the problem. Note that (except for one or two initial terms) the staircase steps in $\mathcal{S}^U$ all lie in the interval $(7, \infty) = (w_1, v_1)$, where v_i and w_i are as in (1.2.7). Further, $S((w_1, v_1)) = (w_2, v_2)$ while $R((w_1, v_1)) = (v_2, w_1)$. We showed in [1] that R takes the blocking classes and staircase steps of $\mathcal{S}^U$ to $\mathcal{S}^L$ and that S takes $\mathcal{S}^U$ to $\mathcal{S}^E$.

The next theorem gives a numerical description of the image of the staircase families $\mathcal{S}^U$ by $T \in \mathcal{G}$. This extends [1, Corollary 60] where it is shown that S and R take the centers of the classes in $\mathcal{S}^U$ to those of $\mathcal{S}^E$ and $\mathcal{S}^L$ respectively. This extended action is illustrated in Figure 2. Note that S preserves the direction (ascending or descending) of the centers of a family of classes, while R reverses it.

Theorem 1.2.4 *For each $T \in \mathcal{G}$ there is a corresponding prestaircase family $T^\sharp(\mathcal{S}^U)$, consisting of tuples $(d, m, p, q, t, \varepsilon)$ where*

- *(p, q) are the image by T of the corresponding tuple in $\mathcal{S}^U$;*
- *t is unchanged and ε transforms by the factor $(-1)^{i+\delta}$, where $T = S^i R^\delta$;*

[7] $\mathcal{G}$ is not a semigroup because $S^i R S^j = S^{i-j} R \in \mathcal{G}$ only if $i \geq j$; see Lemma 2.1.3.

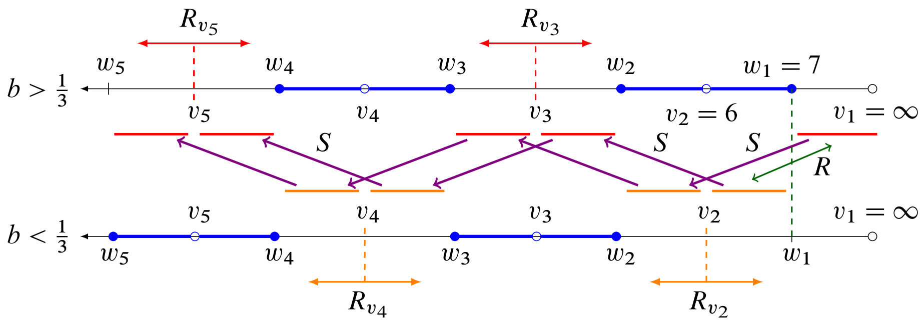

Figure 2: The centers of the staircase family $\mathcal{S}^U$ (resp. $\mathcal{S}^L$) live in the rightmost red (resp. orange) interval. Connecting these, in green, we have the reflection R about $w_1 = 7$ which sends the $\mathcal{S}^U$ to the $\mathcal{S}^L$ family. The shift S, purple, preserves orientation and takes staircase families with b value below (resp. above) $1/3$ to staircases with b value above (resp. below) $1/3$. By applying S iteratively to $\mathcal{S}^U$ and $\mathcal{S}^L$, one obtains staircase families in the other red or orange half-intervals. In each of these half-intervals, there is a family of ascending and descending staircases whose accumulation points converge to some v_i. For each $i \geq 1$, each interval (w_i, w_{i-1}) admits a reflection R_{v_i} that fixes its center point v_i and interchanges the staircases in that interval. Finally, the blue intervals are blocked by a family of principal blocking classes with centers at the v_i.

- (d, m) *are determined by* (p, q, t, ε) *according to* (1.2.4). *In particular,* $\varepsilon(d-3m)$ *remains unchanged.*

This prestaircase family is generated in the sense of Definition 1.2.1 by the images under T *of the seeds and blocking classes of* $\mathcal{S}^U$. *Further,*

(i) *If* $T = S^i$ *for some* $i > 0$, *then the centers of the preblocking classes of* $T^\sharp(\mathcal{S}^U)$ *increase with* n, *and the prestaircase steps lie in the interval* (w_{i+1}, v_{i+1}) *and have* $\varepsilon = (-1)^i$.

(ii) *If* $T = S^i R$ *for some* $i \geq 0$, *then the centers of the preblocking classes decrease with* n, *and the prestaircase steps lie in the interval* (v_{i+1}, w_i) *and have* $\varepsilon = (-1)^{i+1}$.

In particular, $R^\sharp(\mathcal{S}^U) = \mathcal{S}^L$ *while* $S^\sharp(\mathcal{S}^U) = \mathcal{S}^E$.

For the proof see Section 3.3.

Remark 1.2.5 (i) When $b > 1/3$, the accumulation points of the staircases lie in the intervals (w_{2i+1}, w_{2i}), while, for each i, the interval $[w_{2i}, w_{2i-1}]$ is blocked by the *principal blocking class* $\boldsymbol{B}^P_{v_{2i}} := (S^{2i-2})^\sharp(\boldsymbol{B}^U_0)$ with center v_{2i}. Similarly, when $b < 1/3$, the accumulation points of the staircases lie in the intervals (w_{2i}, w_{2i-1}), while, for each i, the interval $[w_{2i+1}, w_{2i}]$ is blocked by the principal blocking class $\boldsymbol{B}^P_{v_{2i+1}} := (S^{2i-1})^\sharp(\boldsymbol{B}^U_0)$ with center v_{2i+1}. See Figures 2 and 3 and Corollary 4.1.5.

(ii) As already noted, the action of a general symmetry $T = S^i R^\delta$ on the degree variable (d, m) of the blocking classes is determined by its action on (p, q) together with (1.2.4). We will see in Lemma 3.4.1 that the action of T on the (d, m) coordinates of the family of blocking classes $\boldsymbol{B}^U_n$ is given by a 2×2 integral matrix $T^*_{\boldsymbol{B}}$. Because, as noted after (1.2.4), the linear function $d - 3m$ is invariant by $T^*_{\boldsymbol{B}}$ modulo sign, this matrix has eigenvector $(3, 1)^\vee$, with eigenvalue $(-1)^{i+\delta}\det(T^*_{\boldsymbol{B}})$. In general, the other eigenvector (with eigenvalue $(-1)^{i+\delta}$) has no obvious interpretation.

However, there are two cases in which it does. Indeed if T is the reflection R_{v_i} that fixes v_i, interchanging w_i with w_{i-1}, then it has two lifts to an action on degree depending on whether we take $b > 1/3$ or $b < 1/3$. As we show in Section 3.4, it turns out that only one of these actions on degree has order two, though they both have clear geometric interpretations. For example, if $i = 2$, so $v_2 = 6$, then $R_{v_2} = SR$ interchanges the centers of the blocking classes and seeds of the staircase families $\mathcal{S}^L$ and $\mathcal{S}^E$. The lift $(R_{v_2})^*_{\boldsymbol{B}}$ that acts on the degree components of the blocking classes has order two and eigenvector $(5, 1)^\vee$, since $1/5 = \mathrm{acc}_L^{-1}(6)$ is the limit of the ratio of the degree components. On the other hand, the matrix $(R_{v_2})^*_{\boldsymbol{P}} = (SR)^*_{\boldsymbol{B}}$ takes the blocking classes of the staircase family $\mathcal{S}^U$ to those of $SR(\mathcal{S}^U)$, fixing the shared principal blocking class B^U_0. Thus it has eigenvector $(3, 2)^\vee$, but, as we calculate in Example 3.4.7, does not have order two. For more details about the action of a general element $T \in \mathcal{G}$ on the degree components of the blocking classes, see Proposition 3.4.3.

Our second main result is that all the new staircases are live.

Theorem 1.2.6 *For each $T \in \mathcal{G}$ the prestaircase family $T^\sharp(\mathcal{S}^U)$ is live, and hence is a staircase family.*

The proof that all the classes involved are perfect is given in Proposition 4.1.4. It is greatly eased by the discovery that their (d, m) components satisfy (1.2.4). The proof that the classes are live is given in Section 4. Although it is based on the methods developed in [1] that we explain at the beginning of Section 4, we have significantly simplified the proof by using new arithmetic arguments. Indeed, in Proposition 4.3.7 we establish a simple, widely applicable criterion for an arbitrary perfect prestaircase family to be live.

The paper [1] considered the following subsets of the b–parameter space $[0,1)$:

$$Stair := \{b \mid H_b \text{ has a staircase}\} \subset [0,1),$$
$$Block := \bigcup\{J_{\boldsymbol{B}} \mid \boldsymbol{B} \text{ is a blocking class}\} \subset [0,1).$$

Each blocking class $\boldsymbol{B}^U_n$ defines an interval $J_{U,n} \subset (1/3,1)$ of b–values that have no staircase because $\mu_{\boldsymbol{B}^U_n,b}(\mathrm{acc}(b)) > V_b(\mathrm{acc}(b))$. Further, the end points of these intervals lie in *Stair*. Similarly, the blocking classes in the staircase family $\mathcal{S}^L = R^\sharp(\mathcal{S}^U)$ define blocked intervals $J_{L,u} \subset (0,1/3)$, whose endpoints lie in *Stair*.

There is an induced action of the symmetries in $\mathcal{G}$ on the b–variable $[0,1)$. This is easiest to describe for the shift S since this gives an injection $[a_{\min},\infty) \to [a_{\min},\infty)$ that fixes $a_{\min} := \mathrm{acc}(1/3) = 3+2\sqrt{2}$. Hence, if we write

$$\mathrm{acc}_L\colon [1/3,1) \to [a_{\min},\infty), \quad \mathrm{acc}_U\colon [0,1/3] \to [a_{\min},\infty)$$

for the appropriate restriction of the function $b \mapsto \mathrm{acc}(b)$ in (1.1.1), for each $k \geq 1$ we can define $(S^k)^*$ by

$$(S^k)^*(b) = \begin{cases} \mathrm{acc}_L^{-1} \circ S^k \circ \mathrm{acc}_L & \text{if } b \leq 1/3, \\ \mathrm{acc}_U^{-1} \circ S^k \circ \mathrm{acc}_U & \text{if } b \geq 1/3. \end{cases} \tag{1.2.8}$$

Since $(S^k)^*$ is conjugate to S^k, the assignment $k \mapsto (S^k)^*$ is a homomorphism; ie $(S^k)^* \circ (S^m)^* = (S^{k+m})^*$ for $k,m \geq 0$.

The reflection symmetries $S^k R$ are not defined on the whole z–interval $[a_{\min},\infty)$ and so do not extend to a global action on b–variable. Instead, in view of Remark 1.2.5(ii), it is most natural to restrict the reflection $R_{v_i} := S^{2i-1}R$ that fixes v_i to the appropriate b–interval corresponding to $(v_{i+1}, v_{i-1}) \supset (w_i, w_{i-1})$. Thus we define

$$(R_{v_i})^*(b) = \begin{cases} \mathrm{acc}_L^{-1} \circ R_{v_i}|_{(v_{i+1},v_{i-1})} \circ \mathrm{acc}_L & \text{if } i \text{ is even}, \\ \mathrm{acc}_U^{-1} \circ R_{v_i}|_{(v_{i+1},v_{i-1})} \circ \mathrm{acc}_U & \text{if } i \text{ is odd}. \end{cases} \tag{1.2.9}$$

The following is an immediate consequence of Theorem 1.2.6 because the endpoints of the blocked b–intervals, $J_{U,n}$ and $J_{L,n}$, are taken by the function $b \mapsto \mathrm{acc}(b)$ to the accumulation points of the corresponding staircases, upon which $\mathcal{G}$ acts geometrically.

Corollary 1.2.7 *Let $J_{U,n}$ (resp. $J_{L,n}$) be the b–interval blocked by $\boldsymbol{B}^U_n$ (resp. $\boldsymbol{B}^L_n$). For $J = J_{U,n}$ with $n \geq 0$, or $J_{L,n}$ with $n \geq 1$ and each $k \geq 0$, the interval $(S^k)^*(J)$ is a component of Block, and its endpoints are in Stair. Moreover, for each $i \geq 2$ the reflection $(R_{v_i})^*$ permutes those intervals $(S^k)^*(J)$ that lie entirely in (v_{i+1}, v_{i-1}).*

Remark 1.2.8 (i) We conjecture that the action of the elements $T \in \mathcal{G}$ described above preserves the sets *Stair* and *Block*. This would hold if, for example, every interval in *Block* was defined by a single blocking class with two associated staircases, and if these, plus the staircases at 0 and $1/3$, were the only staircases. However, the proof of such a result seems out of reach at present.

(ii) Note that the map $(R_{v_i})^*$ has order two since it is defined to be the conjugate of a reflection. The statements in Remark 1.2.5(ii) about the transformations $(R_{v_i})^\sharp$ are rather different, since here we are concerned with the action of R_{v_i} on the degree variables (d,m) of the relevant blocking classes, and not on the conjugate to its action on the z–variable.

2 The accumulation function and its symmetries

In this section, we first discuss the arithmetic properties of the symmetries and related topics. In Section 2.2 we first give an alternative way to understand the coordinates (d,m,p,q) of a quasiperfect class, and then show that all such classes are center-blocking. This second result relies on the particular form of the accumulation function $b \mapsto \operatorname{acc}(b)$. Finally, we describe the staircase families $\mathcal{S}^U$ and $\mathcal{S}^L$ and the staircase at $b = 1/3$ in the language used in [1].

2.1 The fundamental recursion

We have two sets of variables: the z variable on the domain $E(1,z)$ and the b variable on the target. They are related by the equation

$$z^2 - \left(\frac{(3-b)^2}{1-b^2} - 2\right) z + 1 = 0. \tag{2.1.1}$$

Since $b \in [0,1)$, one can check that this equation has two positive solutions that we denote by a and $1/a$, where $a := \operatorname{acc}(b) > 1$. As illustrated in Figure 1, this function is in general two-to-one with a unique minimum $\operatorname{acc}^{-1}(3+2\sqrt{2}) = 1/3$. We denote by[8]

$$\operatorname{acc}_L^{-1} \colon \bigl(3+2\sqrt{2}, \tfrac{1}{2}(7+3\sqrt{5})\bigr] \to \bigl[0, \tfrac{1}{3}\bigr), \quad \operatorname{acc}_U^{-1} \colon (3+2\sqrt{2}, \infty) \to \bigl(\tfrac{1}{3}, 1\bigr)$$

the corresponding inverses to the function $b \mapsto \operatorname{acc}(b)$. Thus, with

$$\tau := a + \frac{1}{a} + 2 = \frac{(3-b)^2}{1-b^2} \ge 8,$$

we have

$$\operatorname{acc}_L^{-1}(a) = \frac{3 - \sqrt{\tau^2 - 8\tau}}{\tau+1}, \quad \operatorname{acc}_U^{-1}(a) = \frac{3 + \sqrt{\tau^2 - 8\tau}}{\tau+1}.$$

[8] Here and elsewhere, L (or ℓ) denotes "lower" while U (or u) denotes "upper".

The minimum $\tau = 8$ is attained when $b = 1/3$. The corresponding equation

$$z^2 - 6z + 1 = 0$$

is therefore very special. It arises by taking the limit $\lim x_{\kappa+1}/x_\kappa$ of the recursion $x_{\kappa+1} = 6x_\kappa - x_{\kappa-1}$ that seems to play a central role in this staircase problem. For example, the steps of the staircase at $b = 1/3$ satisfy this recursion. Moreover, as the following lemma shows, certain properties of the function $b \mapsto \operatorname{acc}(b)$ are invariant with respect to this recursion.

Lemma 2.1.1 (i) *If* $\operatorname{acc}(b) = p/q$ *then*

$$b = \frac{3pq \pm (p+q)\sqrt{\sigma}}{p^2 + q^2 + 3pq}, \quad \text{where } \sigma := (p-3q)^2 - 8q^2. \tag{2.1.2}$$

In particular, b *and* $\operatorname{acc}(b)$ *are both rational if and only if* $(p-3q)^2 - 8q^2 = k^2$ *for some integer* $k \geq 1$.

(ii) *The quantity* $\sigma(p,q) := (p-3q)^2 - 8q^2 = p^2 + q^2 - 6pq$ *is invariant under the transformation* $(p,q) \mapsto (6p - q, p)$.

(iii) *In particular, because* $(p,q) = (6,1)$ *is a solution of* $\sigma(p,q) = 1$, *any successive pair in the sequence*

$$(y_1, y_2, y_3, y_4, \dots) = (1, 6, 35, 204, \dots)$$

gives another solution. Further, these are the only solutions for $\sigma = 1$.

(iv) *If* $p = y_i$ *and* $q = y_{i-1}$ *for some* $i > 1$, *we have*

$$\operatorname{acc}_U^{-1}\left(\frac{p}{q}\right) = \frac{p+q+3}{3p+3q+1}, \quad \operatorname{acc}_L^{-1}\left(\frac{p}{q}\right) = \frac{p+q-3}{3p+3q-1}, \tag{2.1.3}$$

and so

$$\begin{aligned}
&\operatorname{acc}_U^{-1}(6) = \tfrac{5}{11}, \quad \operatorname{acc}_U^{-1}\left(\tfrac{35}{6}\right) = \tfrac{11}{31}, \quad \operatorname{acc}_U^{-1}\left(\tfrac{204}{35}\right) = \tfrac{121}{359}, \quad \dots \searrow \tfrac{1}{3}\\
&\operatorname{acc}_L^{-1}(6) = \tfrac{1}{5}, \quad \operatorname{acc}_L^{-1}\left(\tfrac{35}{6}\right) = \tfrac{19}{61}, \quad \operatorname{acc}_L^{-1}\left(\tfrac{204}{35}\right) = \tfrac{59}{179}, \quad \dots \nearrow \tfrac{1}{3}.
\end{aligned}$$

Proof Let $a = p/q > 1$ so that $a = \operatorname{acc}(b)$ where $b = (3 \pm \sqrt{\tau^2 - 8\tau})/(\tau + 1)$, and $\tau = p/q + q/p + 2$. Since

$$\tau(\tau - 8) = \frac{(p^2 + q^2 + 2pq)(p^2 + q^2 - 6pq)}{p^2 q^2},$$

and $p^2+q^2-6pq=(p-3q)^2-8q^2$, this formula for b simplifies to that in (2.1.2). The rest of (i) is clear. Next, note that (ii) holds because

$$(6p-q)^2+p^2-6(6p-q)p=p^2+q^2-6pq.$$

To prove (iii), note that given any solution (p,q) with $p>q$, one can use the reverse iteration $(p,q)\mapsto(q,6q-p)$ to reduce to a solution with $p>q>0$ and $6q-p\le 0$. But the only such solution is $(6,1)$. Finally, to see that the formula in (iv) for $\mathrm{acc}_U^{-1}(p/q)$ is the same as that in (2.1.2) we must check that

$$(3pq+p+q)(3p+3q+1)=(3pq+p^2+q^2)(p+q+3)=(9pq+1)(p+q+3).$$

One can check that the third order terms on both sides are the same, and that the rest of the identity holds because $p^2+q^2=6pq+1$. The proof for $\mathrm{acc}_L^{-1}(p/q)$ is similar. Thus $\mathrm{acc}_U^{-1}(y_i/y_{i-1})$ decreases with limit $1/3$, while $\mathrm{acc}_L^{-1}(y_i/y_{i-1})$ increases with limit $1/3$. □

Remark 2.1.2 Since $\sigma(p,q)=\sigma(q,p)$, (p,q) is a solution of the equation $\sigma(p,q)=1$ if and only if (q,p) is. We always assume that $p>q$ so that the entries in the pairs $(p,q), S(p,q), S(S(p,q)),\dots$ increase; with the other convention they would decrease. Notice also that the Pell numbers $0,1,2,5,12,29,70,\dots$ that form such a basic element in the polydisc case considered in [4; 7] are closely related to the sequence $0,1,6,35,\dots$ that is fundamental here; indeed the numbers $2y_i$ for $i\ge 0$ are precisely the even-placed Pell numbers.

Let $S:=\begin{pmatrix}6&-1\\1&0\end{pmatrix}$ be the "shift" matrix that implements the recursion

(2.1.4) $$x_{\kappa+1}=6x_\kappa-x_{\kappa-1},\qquad S\begin{pmatrix}x_\kappa\\x_{\kappa-1}\end{pmatrix}=\begin{pmatrix}x_{\kappa+1}\\x_\kappa\end{pmatrix},$$

where the matrix $A:=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ acts on the z variables by the fractional linear transformation

(2.1.5) $$z\mapsto\frac{az+b}{cz+d}=:Az$$

taking (by extension) ∞ to a/c. Starting with $y_0=0$ and $y_1=1$, we get the sequence

$$y_0=0,\quad y_1=1,\quad y_2=6,\quad y_3=35,\quad y_4=204,\quad y_5=1189,\quad y_6=6930,\quad\dots,$$
$$\tfrac{35}{6}=[5;1,5],\quad \tfrac{204}{35}=[5;1,4,1,5],\quad \tfrac{1189}{204}=[5;1,4,1,4,1,5],\quad\dots,$$

with general term $[5;\{1,4\}^k,1,5]$; see Lemma 2.1.6. If we define

(2.1.6) $$R:=\begin{pmatrix}6&-35\\1&-6\end{pmatrix}=\begin{pmatrix}y_2&-y_3\\y_1&-y_2\end{pmatrix},$$

then $R^2 = \mathrm{id}$ and $\det R = -1$, and we will see that the *reflection* R and *shift* S generate symmetries of our problem.

It is convenient to consider the following decreasing sequences of points in the interval $(3+2\sqrt{2}, \infty)$:

$$\begin{gathered} v_1 := \infty, \quad v_2 := 6, \quad v_3 := \tfrac{35}{6}, \quad v_j := \frac{y_j}{y_{j-1}}, \\ w_1 = 7, \quad w_2 := \tfrac{41}{7}, \quad w_k = \frac{y_{k+1}+y_k}{y_k+y_{k-1}}. \end{gathered} \tag{2.1.7}$$

These sequences interweave:

$$3+2\sqrt{2} < \cdots < w_k < v_k < w_{k-1} < \cdots < w_2 < v_2 < w_1 < v_1 = \infty.$$

Lemma 2.1.3 *Let v_j, w_j, S, and R be as above. Then*:

(i) *The following matrix identities hold*:

$$SR = RS^{-1}, \quad S^{-1}R = RS, \quad R \circ R = \mathrm{Id}.$$

(ii) *The matrix*

$$S^k = \begin{pmatrix} y_{k+1} & -y_k \\ y_k & -y_{k-1} \end{pmatrix}$$

has determinant 1; *ie*

$$y_k^2 = y_{k+1}y_{k-1} + 1 = 6y_{k+1}y_k - y_{k+1}^2 + 1 \quad \textit{for all } k \geq 1. \tag{2.1.8}$$

(iii) *With action as in (2.1.5), $S(v_j) = v_{j+1}$ and $S(w_j) = w_{j+1}$, for $j \geq 1$.*

(iv) *The matrices S and R generate the subgroup of $\mathrm{PGL}(2,\mathbb{Z})$ that fixes the quadratic form $p^2 - 6pq + q^2$.*

Proof The proof of (i)–(iii) is straightforward. In particular the formula for S^k holds because S implements the recursion, and we also have $\det(S^k) = (\det(S))^k = 1$. Further, one can check that $A \in \mathrm{GL}(2,\mathbb{Z})$ preserves the form $p^2 - 6pq + q^2$ if and only if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where } c = -b,\ d = 6b + a.$$

Hence, if $\det(A) = 1$ then $a^2 + 6ab + b^2 = 1$, which implies by Lemma 2.1.1(iii) that when $a > -b > 0$ we must have $(a,b) = (y_{k+1}, -y_k)$ for some k, so $A = S^k$ for some $k \geq 1$. It follows similarly that the only other matrices A that preserve the form and have $\det(A) = 1$ have the form $\pm S^k$ for some $k \leq 0$. Further, if $\det(A) = -1$ then because the matrix RA has determinant 1 and preserves the quadratic form, we must have $A = \pm S^k R$ for some $k \in \mathbb{Z}$. Thus (iv) holds. □

Corollary 2.1.4 (i) *For each $i \geq 1$, the restriction of*

$$R_{v_i} := S^{i-2} R S^{-(i-1)} = S^{2i-3} R = \begin{pmatrix} y_{2i-1} & -y_{2i} \\ y_{2i-2} & -y_{2i-1} \end{pmatrix} \tag{2.1.9}$$

to the interval (v_{2i-1}, v_1) is a reflection that fixes v_i and interchanges the points w_{i+k}, w_{i-k-1}, and v_{i+k}, v_{i-k}, for $0 \leq k \leq i-1$.

(ii) *The restriction of R to the interval $(v_2, v_1) = (6, \infty)$ is a reflection that fixes $w_1 = 7$.*

We end this subsection with a brief discussion of the weight expansion and continued fractions.

Definition 2.1.5 The (integral) *weight expansion* of a rational number $p/q \geq 1$ is a recursively defined, nonincreasing sequence of integers $W(p/q) = (W_1, W_2, \dots)$ defined as follows: $W_1 = q$ and $W_n \geq W_{n+1}$ for all n, and if $W_i > W_{i+1} = \cdots = W_n$ (where we set $W_0 := p$), then

$$W_{n+1} = \begin{cases} W_{i+1} & \text{if } W_{i+1} + \cdots + W_{n+1} = (n-i+1) W_{i+1} < W_i, \\ W_i - (n-i) W_{i+1} & \text{otherwise.} \end{cases}$$

Thus $W(p/q)$ starts with $\lfloor p/q \rfloor$ copies of q (where $\lfloor p/q \rfloor$ is the largest integer $\leq p/q$), and ends with some number ≥ 2 of copies of 1. One can check that

$$\sum_{i \geq 1} W_i = p + q - 1, \quad \sum_{i \geq 1} W_i^2 = pq.$$

Using this, it is straightforward to check that equations (1.2.1), for a quasiperfect tuple (d, m, p, q), imply that the corresponding class $\boldsymbol{E} = dL - mE_0 - \sum_i W_i E_i$ satisfies the conditions $c_1(\boldsymbol{E}) = 1$ and $\boldsymbol{E} \cdot \boldsymbol{E} = -1$, as claimed earlier. Moreover, the multiplicities $\ell_0, \dots, \ell_k$ of the entries in $W(p/q)$ are the coefficients of the continued fraction expansion of p/q. Thus, if the distinct weights are $X_0 := q > X_1 > \cdots > X_k = 1$ and we write

$$W(p/q) = (X_0^{\times \ell_0}, X_1^{\times \ell_1}, \dots, 1^{\times \ell_k}),$$

then

$$\frac{p}{q} = [\ell_0; \ell_1, \dots, \ell_k] := \ell_0 + \frac{1}{\ell_1 + \frac{1}{\ell_2 + \cdots + 1/\ell_k}}.$$

As we see from Theorems 2.3.1 and 2.3.3 below, the centers of the staircase steps have very regular continued fraction expansions that, as we now show, behave well under the symmetries.

Lemma 2.1.6 (i) *The shift* $S = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$ *has the following effect on continued fraction expansions, where, for* $x \geq 1$, $CF(x)$ *denotes the continued fraction of* x:

- *If* $z = p/q = [5+k; CF(x)]$ *for some* $k \geq 0, x \geq 1$, *then*
$$Sz = \frac{6p-q}{p} = [5; 1, 4+k, CF(x)].$$
In particular, if $z > 6$ *then* $k \geq 1$ *and the third entry in* $CF(Sz)$ *is at least* 5, *while if* $z < 6$ *then* $k = 0$ *and* $x > 1$ *and we have*
$$S([5; CF(x)]) = [5; 1, 4, CF(x)].$$

(ii) *The reflection* $R = \begin{pmatrix} 6 & -35 \\ 1 & -6 \end{pmatrix}$ *has the following effect on continued fraction expansions*:

- *If* $z = p/q = [6+k; CF(x)]$ *for some* $k \geq 1$ *and* $x \geq 1$, *then*
$$Rz = \frac{6p-35q}{p-6q} = [6; k, CF(x)].$$
Further, $R \circ R = \mathrm{id}$.

(iii) *The quantity* $p^2 - 6pq + q^2$ *is invariant by both* S *and* R.

Proof If $z = [5+k; CF(x)] = 5+k+1/x = ((5+k)x+1)/x$ then
$$Sz = \frac{(29+6k)x+6}{(5+k)x+1} = 5 + \frac{(4+k)x+1}{(5+k)x+1},$$
while
$$[5; 1, 4+k, CF(x)] = 5 + \frac{1}{1+1/(4+k+1/x)} = 5 + \frac{(4+k)x+1}{(5+k)x+1}.$$
This proves (i). The proof of (ii) is similar, and (iii) follows by an easy calculation. □

2.2 Quasiperfect classes

We first explain the action of the symmetries on the quasiperfect classes, and then show in Proposition 2.2.9 that every quasiperfect class with center $> a_{\min} = 3 + 2\sqrt{2}$ is a center-blocking class.

As explained in (1.2.1), a quasiperfect class $\boldsymbol{E} = dL - mE_0 - \sum_i m_i E_i$ is determined by a tuple (d, m, p, q) of positive integers (where $(m_1, m_2, \dots)$ is the weight expansion of p/q as in Definition 2.1.5) that satisfies the conditions

$$3d = p + q + m, \quad d^2 - m^2 = pq - 1. \tag{2.2.1}$$

We will continue to call these the *Diophantine conditions* on $\boldsymbol{E}$ as in [1]. If we use the first equation above to express m as a function of d, p, and q, the second equation is a quadratic in d,

$$8d^2 - 6d(p+q) + p^2 + 3pq + q^2 - 1 = 0, \tag{2.2.2}$$

with solution

$$d = \tfrac{1}{8}\big(3(p+q) \pm \sqrt{p^2 - 6pq + q^2 + 8}\big).$$

Thus, if we define

$$t := \sqrt{p^2 - 6pq + q^2 + 8}, \quad \varepsilon := \pm 1, \tag{2.2.3}$$

the coefficients d and m in $\boldsymbol{E}$ are given by the formulas

$$d := \tfrac{1}{8}(3(p+q) + \varepsilon t), \quad m := \tfrac{1}{8}((p+q) + 3\varepsilon t) \tag{2.2.4}$$

in (1.2.4). In other words, modulo an appropriate choice of ε, we can think of a quasiperfect class as an integer point on the quadratic surface X defined by (2.2.2), where we can use either the coordinates (d, p, q) or (p, q, t).[9] Note, however, that the fact that p, q, and t are integers does not imply that d and m are also.

We now show that a quasiperfect class $(d, m, p, q, t, \varepsilon)$ is uniquely determined by its center p/q.

Lemma 2.2.1 *For each integral solution (p, q, t) of the equation $t^2 = p^2 - 6pq + q^2 + 8$, there are integers (d, m) satisfying*

$$d = \tfrac{1}{8}(3(p+q) + \varepsilon t), \quad m = \tfrac{1}{8}((p+q) + \varepsilon 3t) \tag{2.2.5}$$

for at most one value of $\varepsilon \in \{\pm 1\}$.

Proof If this is false there are positive integers $p, q, t, d_\pm$, and $m_\pm$ such that (d_+, m_+) solve (2.2.5) for $\varepsilon = +1$, while (d_-, m_-) solve it for $\varepsilon = -1$. Then

$$d_+ + d_- = \tfrac{3}{4}(p+q), \quad d_+ - d_- = \tfrac{1}{4}t$$

[9] We are indebted to Peter Sarnak for explaining this point of view to us.

are integers. Since p and q cannot both be even they may written as $4a+1$ and $4b-1$ (in some order), and, with $t := 4s$,

$$s^2 = a^2 - 6ab + b^2 + 2a - 2b + 1.$$

But also we need 8 to divide $3(p+q) + \varepsilon t$, which implies that $a+b+s$ is even. It is now easy to see that there are no integer solutions. □

Corollary 2.2.2 *There is at most one quasiperfect class with center p/q. Conversely, for given (d,m) there is at most one quasiperfect class with these degree variables and $p/q > 1$.*

Proof The first claim follows immediately from Lemma 2.2.1. To prove the second, notice that d and m determine $p+q = 3d - m$ and $pq = d^2 - m^2 + 1$, which uniquely determines p and q modulo order. □

We now discuss the effect of the symmetries on these classes. As always, we write

$$S = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 6 & -35 \\ 1 & -6 \end{pmatrix},$$

where S is the shift and R is the reflection that fixes 7. Note that the action of S on p and q fixes t by Lemma 2.1.1(ii). It is also easy to check that R also fixes t. We now show that the action of these transformations act on the integer points of X extends to an action on the tuples (d,m,p,q,t,ε). It follows from Lemma 2.1.3 that any element of the group generated by S and R can be written $S^i R^\delta$ for $i \in \mathbb{Z}$ and $\delta \in \{0,1\}$.

Definition 2.2.3 Let $T = S^i R^\delta$ for $i \in \mathbb{Z}$ and $\delta \in \{0,1\}$, and suppose that (d,m,p,q,t,ε) is a tuple of integers that satisfy the identities in (2.2.3) and (2.2.4). Then we define

$$T^\sharp(d,m,p,q,t,\varepsilon) := (d',m',p',q',t,\varepsilon') = (d',m',T(p,q),t,(-1)^{i+\delta}\varepsilon) \tag{2.2.6}$$

where d' and m' are given by the formulas

$$d' := \tfrac{1}{8}(3(p'+q') + \varepsilon' t), \quad m' := \tfrac{1}{8}((p'+q') + 3\varepsilon' t), \quad \varepsilon' := (-1)^{i+\delta}\varepsilon.$$

The next lemma shows that this action of T preserves integrality.

Lemma 2.2.4 *For each (d,m,p,q,t,ε) and $T = S^i R^\delta$ as above, $T^\sharp(d,m,p,q,t,\varepsilon)$ is also integral and satisfies the Diophantine conditions (2.2.1). Moreover, for all such T_1 and T_2, $(T_1T_2)^\sharp(d,m,p,q,t,\varepsilon) = (T_1)^\sharp(T_2)^\sharp(d,m,p,q,t,\varepsilon)$.*

Proof The above construction shows that every integral point (p,q,t) of X can be extended to a tuple (d,m,p,q,t,ε) that satisfies the Diophantine conditions. In particular, since t is invariant under the action of S and T, the tuple $(d',m',p',q',t,\varepsilon')$ satisfies these conditions; however we do need to check that it is integral. Because $3(3(p'+q')+\varepsilon' t)-((p'+q')+3\varepsilon' t)$ is divisible by 8, it suffices to check that $(p'+q')+3\varepsilon' t$ is divisible by 8. Thus it suffices to check that if 8 divides $p+q+3\varepsilon t$ for some $\varepsilon\in\{\pm 1\}$, and we set $(p',q')=S(p,q)=(6p-q,p)$, then 8 divides $p'+q'-3\varepsilon t=7p-q-3\varepsilon t$. But this is immediate, since $7p-q-3\varepsilon t=8p-(p+q+3\varepsilon t)$. A similar calculation proves that the action of R preserves integrality.

This proves the first claim. The second follows immediately from the fact that the action of S and R on the coordinates (p,q) is compatible with composition. □

Remark 2.2.5 The tuples (d,m,p,q,t,ε) that correspond to quasiperfect classes have positive entries with $p>q>0$. As we shall see in Example 2.3.7, the classes with $t=1$ belong to the staircase at $b=1/3$, while all other classes of interest have $t\geq 3$ and hence $p/q>a_{\min}:=3+2\sqrt{2}$ (so that $p^2-6pq+q^2>0$). Therefore the full subgroup of $\mathrm{PGL}(2,\mathbb{Z})$ generated by S and R does not act on the staircases. This is why in Theorem 1.2.4 we only consider the restriction of the action of the elements $S^i R^\delta$ for $i\geq 0$ to the classes with centers $p/q>7$.

Remark 2.2.6 We now relate our description of the symmetries to that given by Usher in [7, Section 2.2.1]. He denotes a quasiperfect class $\boldsymbol{E}$ in a blowup of $S^2\times S^2$ by the tuple (a,b,c,d), where a and b are the coefficients of the two lines (each with Chern class 2) and $(c,d):=(p,q)$ are the coordinates of its center. Thus the equations $c_1(\boldsymbol{E})=1$ and $\boldsymbol{E}\cdot\boldsymbol{E}=-1$ become

$$2(a+b)=p+q,\quad 2ab=pq-1.$$

The first equation implies that there are integers x, δ, and ε such that

$$(a,b,p,q)=\big(\tfrac{1}{2}(x+\varepsilon),\tfrac{1}{2}(x-\varepsilon),x+\delta,x-\delta\big).$$

With these variables, the second equation is then

$$x^2-2\delta^2=2-\varepsilon^2. \tag{2.2.7}$$

In terms of the element $x+\delta\sqrt{2}\in\mathbb{Z}[\sqrt{2}]$, this equation simply says that $x+\delta\sqrt{2}$ has norm $2-\varepsilon^2$. He now considers symmetries of the form

$$x+\delta\sqrt{2}\mapsto x'+\delta'\sqrt{2}:=(u+v\sqrt{2})(x+\delta\sqrt{2})=ux+2v\delta+(vx+u\delta)\sqrt{2},$$

where $u + v\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ is an element of norm 1, ie $u^2 - 2v^2 = 1$. Then $(x', \delta', \varepsilon)$ is another solution of (2.2.7). Usher considers the symmetries given by

$$u + v\sqrt{2} = H_{2k} + P_{2k}\sqrt{2}$$

where H_{2k} and P_{2k} are respectively half-Pell and Pell numbers. When $k = 1$, $H_2 = 3$ and $P_2 = 2$, and we get the unit $u + v\sqrt{2} = 3 + 2\sqrt{2}$. Therefore $x' = 3x + 4\delta$ and $\delta' = 2x + 3\delta$, so $(p', q') := (x' + \delta', x' - \delta') = (5x + 7\delta, x + \delta)$. Substituting $x = \frac{1}{2}(p + q)$ and $\delta = \frac{1}{2}(p - q)$, we obtain the transformation

$$(p, q) \mapsto (p', q') = (6p - q, p).$$

More generally, Usher states that the transformation has formula

$$(x, \delta) \mapsto (H_{2k}x + 2P_{2k}\delta,\, P_{2k}x + H_{2k}\delta),$$

which, in terms of the (p, q) coordinates, translates to

$$(p, q) \mapsto \bigl((H_{2k} + \tfrac{3}{2}P_{2k})p - \tfrac{1}{2}P_{2k}q,\, \tfrac{1}{2}P_{2k}p + (H_{2k} - \tfrac{3}{2}P_{2k})q\bigr).$$

To see that this map is the same as $(p, q) \mapsto S^k(p, q)$, notice that by Remark 2.1.2, the entries y_i of S^k have the form $\frac{1}{2}P_{2i}$. Therefore, we have to check a linear identity between the Pell numbers P_n and their half-companions H_n. But because of the recursion, such an identity holds if and only if it holds for two distinct values of n, and when $k = 2$ we have $(H_4, P_4) = (17, 12)$, which gives $(p, q) \mapsto (35p - 6q, 6p - q)$, as required.

Finally, we observe that Usher's symmetries preserve ε which is related to the recursion variable for his staircases, and hence plays much the same role as our variable t.

The next lemma, taken from [1, Example 32] is the key to the proof that every quasiperfect class is center-blocking.

Lemma 2.2.7 *For all $b \in [0, 1)$ the graph of the function $z \mapsto (1 + z)/(3 - b)$ passes through the accumulation point $(\mathrm{acc}(b), V_b(\mathrm{acc}(b)))$. Moreover, for each b, the line lies above the volume curve when $z > \mathrm{acc}(b)$.*

Proof Let $c(b) = (3 - b)^2/(1 - b^2) - 2$. We have

$$\frac{1 + \mathrm{acc}(b)}{3 - b} = \sqrt{\frac{\mathrm{acc}(b)}{1 - b^2}} \tag{2.2.8}$$

$$\iff (1 + \mathrm{acc}(b))^2 = \mathrm{acc}(b)\frac{(3 - b)^2}{1 - b^2} = \mathrm{acc}(b)(c(b) + 2)$$

$$\iff \mathrm{acc}(b)^2 - c(b)\mathrm{acc}(b) + 1 = 0,$$

which holds by the definition of acc(b) in (2.1.1). It follows that the two functions

$$b \mapsto V_b(\mathrm{acc}(b)), \quad b \mapsto \frac{1+\mathrm{acc}(b)}{3-b} \tag{2.2.9}$$

are the same.

It remains to note that for all $z \ge a_{\min} > 5$, the slope

$$\frac{d}{dz}\left(\sqrt{\frac{z}{1-b^2}}\right) = \frac{1}{2}\frac{V_b(z)}{z} = \frac{1+z}{2z(3-b)}, \quad z := \mathrm{acc}(b)$$

of the volume curve $V_b(z)$ at $z = \mathrm{acc}(b)$ is smaller than $1/(3-b)$ which is the slope of the line. □

Remark 2.2.8 (i) This special property of the function $z \mapsto (1+z)/(3-b)$ is the key reason why the linear relations in (1.2.5) imply that the staircase converges to an endpoint of the interval blocked by the associated blocking class; see the proof of [1, Theorem 52].

(ii) By [1, Example 32], when $5 < z < 6$ the function $z \mapsto (1+z)/(3-b)$ that occurs in Lemma 2.2.7 is the obstruction given by the class $\boldsymbol{E} = 3L - E_0 - 2E_1 - E_{2\dots6}$. As noticed by Tara Holm, all the other convex toric domains discussed in [3] seem to have classes that play a similar role.

Proposition 2.2.9 *Every quasiperfect class* $\boldsymbol{E} := (d, m, p, q, t, \varepsilon)$ *with* $p/q > a_{\min} = 3 + 2\sqrt{2}$ *is center-blocking.*

Proof Define $\mathrm{acc}_\varepsilon^{-1}$ to be acc_U^{-1} if $\varepsilon = 1$ and acc_L^{-1} if $\varepsilon = -1$.[10] Then we must check that

$$\mu_{\boldsymbol{E},b}\left(\frac{p}{q}\right) = \frac{p}{d-mb} > V_b\left(\frac{p}{q}\right) = \frac{p+q}{q(3-b)}, \quad b := \mathrm{acc}_\varepsilon^{-1}\left(\frac{p}{q}\right),$$

where we have used (1.1.3) and (2.2.9). Thus we need

$$pq(3-b) > (p+q)(d-mb),$$

or equivalently

$$3pq - d(p+q) > b(pq - m(p+q)).$$

By (2.1.2) we have $b = (3pq + \varepsilon(p+q)\sqrt{\sigma})/((p+q)^2 + pq)$, where $\sigma + 8 = t^2$. Thus we must check that

$$((p+q)^2 + pq)(3pq - d(p+q)) > (3pq + \varepsilon(p+q)\sqrt{\sigma})(pq - m(p+q)).$$

[10] By (2.2.4) we have $m/d > 1/3$ exactly when $\varepsilon = 1$. Moreover the condition $p/q > a_{\min}$ implies that $p/q = \mathrm{acc}(b)$ for at least one b so it is in the domain of $\mathrm{acc}_\varepsilon^{-1}$ for at least one value of ε.

By deleting the term $3p^2q^2$ from both sides, multiplying by 8 and substituting for d and m, we obtain the equivalent inequality

$$\begin{aligned}24pq(p+q)^2-(p+q)((p+q)^2+pq)(3(p+q)+\varepsilon t)\\ &> -3pq(p+q)(p+q+3\varepsilon t)+\varepsilon(p+q)\sqrt{\sigma}(8pq-(p+q+3\varepsilon t)(p+q))\\ &= -3pq(p+q)(p+q+3\varepsilon t)+\varepsilon(p+q)\sqrt{\sigma}(-\sigma-3\varepsilon t(p+q)),\end{aligned}$$

where the last equality uses the identity $8pq-(p+q)^2=-\sigma$. If we take all the terms in this inequality that involve an even power of ε, and put them on the left-hand side, we obtain

$$\begin{aligned}24pq(p+q)^2-3(p+q)^2((p+q)^2+pq)+3pq(p+q)^2+3(p+q)^2t\sqrt{\sigma}\\ &= 3(p+q)^2\bigl(8pq-(p+q)^2-pq+pq+\sigma+(t\sqrt{\sigma}-\sigma)\bigr)\\ &= 3(t\sqrt{\sigma}-\sigma)(p+q)^2 =: A>0,\end{aligned}$$

since $\sigma=p^2-6pq+q^2$ and $t^2=\sigma+8$. If we do the same with the coefficient of ε, we obtain

$$\begin{aligned}-t(p+q)\bigl((p+q)^2+pq\bigr)+9t(p+q)pq+(p+q)\sigma\sqrt{\sigma}\\ &= t(p+q)(-p^2-3pq-q^2+9pq+\sigma)-(t-\sqrt{\sigma})(p+q)\sigma\\ &= (\sqrt{\sigma}-t)(p+q)\sigma =: B.\end{aligned}$$

We need to check that $A>|B|$, which is equivalent to $3(p+q)>\sqrt{\sigma}$. Since this holds, the required inequality is established. □

Remark 2.2.10 Proposition 2.2.9 shows that every class that is defined by a tuple (d,m,p,q,t,ε) with $p/q>a_{\min}$ as in (1.2.4) is in fact a center-blocking class. Thus all our stair steps are center-blocking. However, we have not been able to resolve the question of whether there is a blocking class $\boldsymbol{B}$ of more general type that is not center-blocking. In this case, there would be a point $b_0\in[0,1)$ such that $\mu_{\boldsymbol{B},b_0}(\mathrm{acc}(b_0))>V_{b_0}(\mathrm{acc}(b_0))$. However, if I is the largest interval containing $\mathrm{acc}(b_0)$ on which $\mu_{\boldsymbol{B},b_0}$ is obstructive, and if $a\in I$ is the corresponding break point — see [1, Lemma 14] — then $\mu_{\boldsymbol{B},b}$ would not block a, ie for both elements $b\in\mathrm{acc}^{-1}(a)$ we would have $\mu_{\boldsymbol{B},b}(a)\le V_b(a)$. (For further discussion of blocking classes, see [1, Section 2.3].) We bypass this question here by restricting attention to (quasiperfect) center-blocking classes.

The following fact about blocking classes was pointed out to us by Morgan Weiler. It is somewhat surprising since we know from [1, Lemmma 15(iii)] that every quasiperfect class $\boldsymbol{B}=(d,m,p,q)$ is obstructive when $b=m/d$ and $z=p/q$, ie we have

$\mu_{\boldsymbol{B},m/d}(p/q) > V_{m/d}(p/q)$. Thus it is natural to think that m/d would lie in the blocked interval $J_{\boldsymbol{B}}$, which is defined to be the maximal interval containing $\mathrm{acc}_\varepsilon^{-1}(p/q)$ consisting of parameters b such that $\mu_{\boldsymbol{B},b}(\mathrm{acc}(b)) > V_b(\mathrm{acc}(b))$. However, we now show that this never happens.

Lemma 2.2.11 *Every quasiperfect class $\boldsymbol{B}=(d,m,p,q,t,\varepsilon)$ with $\varepsilon=1$ (resp. $\varepsilon=-1$) has the property that $m/d > b$ (resp. $m/d < b$), for all b in the closure of the blocked b–interval $J_{\boldsymbol{B}}$.*

Proof We first argue by contradiction to show that $m/d \notin \mathrm{cl}(J_{\boldsymbol{B}})$, and then finish the argument by considering the cases $b < 1/3$ and $b > 1/3$ separately.

Thus suppose that $m/d \in \mathrm{cl}(J_{\boldsymbol{B}})$, and let $z_0 := \mathrm{acc}(m/d)$. We first claim that $z_0 > p/q$. To see this note that by (2.1.1) z_0 is the unique solution >1 of the equation

$$z_0 + \frac{1}{z_0} = \frac{(3-m/d)^2}{1-(m/d)^2} - 2,$$

and the function $z \mapsto z + 1/z$ increases when $z > 1$. Therefore, $z_0 > p/q$ exactly when $z_0 + 1/z_0 > p/q + q/p$. But

$$z_0 + \frac{1}{z_0} = \left(\frac{(3-m/d)^2}{1-(m/d)^2} - 2\right) = \frac{p^2+q^2+2}{pq-1} > \frac{p}{q} + \frac{q}{p},$$

where the second equality uses the identities $d^2 - m^2 = pq - 1$ and $3d - m = p + q$. Thus $z_0 > p/q$.

If $m/d \in \mathrm{cl}(J_{\boldsymbol{B}})$ then $z_0 = \mathrm{acc}(m/d) \in \mathrm{cl}(I_{\boldsymbol{B}}) = \mathrm{acc}(\mathrm{cl}(J_{\boldsymbol{B}}))$, so, by [1, Lemma 38(ii)] and [1, Lemma 16], $\mu_{\boldsymbol{B},m/d}(z_0)$ is given by the formula in (1.1.3). Thus we must have

$$\mu_{\boldsymbol{B},m/d}(z_0) = \frac{p}{d - m\cdot m/d} \geq V_{m/d}(z_0) = \frac{1+z_0}{3-m/d},$$

where the first equality holds by (1.1.3) and the fact that $z_0 > p/q$, the inequality holds because we assume $\mu_{\boldsymbol{B},m/d}$ is at least as large as the volume at z_0, and the last equality holds by Lemma 2.2.7 and the fact that $z_0 = \mathrm{acc}(m/d)$. Therefore $z_0 \leq z_1$, where z_1 satisfies the equation $p/(d - m\cdot m/d) = (1+z_1)/(3-m/d)$.

Next note that, by solving for z_1 and using the identities $d^2 - m^2 = pq - 1$ and $3d - m = p + q$, we obtain $z_1 = (p^2+1)/(pq-1)$. It is now straightforward to verify that

$$z_0 + \frac{1}{z_0} = \frac{p^2+q^2+2}{pq-1} = z_1 + \frac{q^2+1}{pq-1} > z_1 + \frac{pq-1}{p^2+1} = z_1 + \frac{1}{z_1}.$$

But this is impossible since we saw above that $z_0 \leq z_1$. This completes the first step.

To finish the argument, notice that we proved above that $z_0 = \mathrm{acc}(m/d) > p/q$, while Proposition 2.2.9 shows that p/q is always blocked by $\boldsymbol{B}$. Thus if $I_{\boldsymbol{B}} := \mathrm{acc}(J_{\boldsymbol{B}})$ denotes the set of blocked z–values, we have $\mathrm{acc}(m/d) > p/q$ where $p/q \in I_{\boldsymbol{B}}$. Therefore $\mathrm{acc}(m/d)$ must be greater than all points in $\mathrm{cl}(I_{\boldsymbol{B}})$. Since $b \mapsto \mathrm{acc}(b)$ preserves orientation exactly *if* $b > 1/3$, this implies that $m/d > b$ for all $b \in \mathrm{cl}(J_{\boldsymbol{B}})$ when $m/d > 1/3$ and $m/d < b$ for all $b \in \mathrm{cl}(J_{\boldsymbol{B}})$ when $m/d < 1/3$. □

We end this subsection with a few remarks about the case $b = 1/3$, which is the focal point of the shift S and separates the two regimes, $b > 1/3$ and $b < 1/3$. As far as we know, this is the unique rational value of b with a staircase.[11] Example 2.3.7 below describes the ascending staircase at $b = 1/3$. Although we have not managed to resolve the question of whether there is also a descending staircase at $b = 1/3$, we can make the following observation. Note that the proof uses the same idea as in Lemma 4.3.3.

Lemma 2.2.12 *If there is no descending staircase when $b = 1/3$, then there is an $\varepsilon > 0$ such that $c_{1/3}(z) = 3(1+z)/8$ for $3 + 2\sqrt{2} = a_{\min} < z < a_{\min} + \varepsilon$.*

Proof It follows from [1, Example 32] (also see Remark 2.3.8(ii)) that the obstruction $\mu_{\boldsymbol{E}_1,b}$ given by the class $\boldsymbol{E}_1 := 3L - E_0 - 2E_1 - E_{2\dots6}$ is precisely $z \mapsto (1+z)/(3-b)$ when $5 < z < 6$. Thus for $z \in [a_{\min}, 6]$ we know that $c_{H_{1/3}}(z) \geq 3(1+z)/8$. If we do not have equality for $z \in (a_{\min}, a_{\min} + \varepsilon)$ and there is no staircase, then there must be a different obstruction curve $z \mapsto (A + Cz)/(d - m/3)$ that goes through the accumulation point $(a_{\min}, 3\sqrt{a_{\min}/8})$. Because $a_{\min}$ is irrational, the equation $(A + Ca_{\min})/(d - m/3) = 3(1 + a_{\min})/8$ can hold only if $A = C$. But then the graphs of the two obstructions are lines of the form $z \mapsto \lambda(1+z)$, and hence coincide. □

This following observation is also relevant because, for example, $d_{\boldsymbol{B}} - 3m_{\boldsymbol{B}}$ appears as the coefficient of d in the linear staircase relation (1.2.5), so that it would be awkward if it were ever zero.

Lemma 2.2.13 *There is no quasiperfect class $\boldsymbol{E} = (d, m, p, q)$ with $m/d = 1/3$.*

Proof By (1.2.1), given any such $\boldsymbol{E}$ the positive integers m, p, and q would have to satisfy $8m = p + q$ and $8m^2 = pq - 1$. Since $m \in \mathbb{Z}$ we must have $p + q \equiv_8 0$, so the second equation gives $p^2 \equiv_8 -1$. This is impossible because $(4k+1)^2$ and $(4k+3)^2$ are both congruent to 1 mod 8. □

[11] In particular, there should be no staircases at the points $b \in \mathrm{acc}^{-1}(v_i)$. This was proved in [1, Theorem 6] for the case $b = 1/5 \in \mathrm{acc}^{-1}(6)$, but the proof seems too elaborate to be easily generalized.

2.3 The known staircases

The paper [1] found three different families of blocking classes $\boldsymbol{B}^U$, $\boldsymbol{B}^E$, and $\boldsymbol{B}^L$ and their associated staircases. For our purposes, $\mathcal{S}^U$ and $\mathcal{S}^L$ are the essential families, since applying powers of S to these staircases generate all the staircase families discussed here.

We now review the theorems in [1, Theorems 56 and 58] that define these two staircase families. In all cases the staircase steps $\boldsymbol{E}_{n,k}$ are quasiperfect classes given by tuples $(d_{n,k}, m_{n,k}, p_{n,k}, q_{n,k})$ that for $x = d, m, p, q$ satisfy the recursion described below. We call $p_{n,k}/q_{n,k}$ the center of the step, and use the staircase relation[12] (together with the linear Diophantine condition $3d = m + p + q$) to determine the entries d and m from knowledge of p and q.

Theorem 2.3.1 *The classes* $\boldsymbol{B}^U_n = (n+3, n+2, 2n+6, 1)$, *for* $n \geq 0$, *with increasing centers, are perfect blocking classes, with the following associated staircases* $\mathcal{S}^U_{\ell,n}$ *and* $\mathcal{S}^U_{u,n}$, *where* $\sigma_n = (2n+5)(2n+1)$ *and* $\mathrm{end}_n = (2n+4)$ *or* $(2n+5, 2n+2)$:

- *For each* $n \geq 1$, $\mathcal{S}^U_{\ell,n}$ *has limit point* $a^U_{\ell,n,\infty} = [2n+5; 2n+1, \{2n+5, 2n+1\}^\infty]$, *and its*
  - ***centers*** $[\{2n+5, 2n+1\}^k, \mathrm{end}_n]$, *for* $k \geq 0$, *where* $\mathrm{end}_n = 2n+4$ *or* $(2n+5, 2n+2)$,
  - ***recursion*** $x_{n,k+1} = (\sigma_n + 2)x_{n,k} - x_{n,k-1}$,
  - ***relation*** $(2n+3)d_{n,k} = (n+1)p_{n,k} + (n+2)q_{n,k}$.
- *For each* $n \geq 0$, $\mathcal{S}^U_{u,n}$ *has limit point* $a^U_{u,n,\infty} = [2n+7; \{2n+5, 2n+1\}^\infty]$, *and has*
  - ***centers*** $[2n+7; \{2n+5, 2n+1\}^k, \mathrm{end}_n]$,
  - ***recursion*** $x_{n,k+1} = (\sigma_n + 2)x_{n,k} - x_{n,k-1}$,
  - ***relation*** $(2n+3)d_{n,k} = (n+2)p_{n,k} - (n+4)q_{n,k}$.

The limit points $a^U_{\bullet,n,\infty}$ *form increasing unbounded sequences in* $(6, \infty) = (v_2, v_1)$, *while the corresponding* b*–values lie in* $(5/11, 1)$, *where* $5/11 = \mathrm{acc}_U^{-1}(6)$, *and increase with limit* 1.

Remark 2.3.2 (i) If v is an integer ≥ 3, the recursion $x_{k+1} = vx_k - x_{k-1}$ has a unique solution > 1 of the form $x_k = \alpha^k$ where $\alpha^2 - v\alpha + 1 = 0$. If the recursion has seeds x_0 and x_1, the general rational solution can be written $X\alpha^k + \overline{X}\bar{\alpha}^k$ where $\bar{\alpha} := 1/\alpha$ is the other solution, $X \in \mathbb{Q}[\sqrt{v^2-4}]$, and for $X = a + b\sqrt{v^2-4}$ we define $\overline{X} := a - b\sqrt{v^2-4}$. Hence, if $x_\bullet$ and $y_\bullet$ both satisfy this recursion, the ratio $x_\bullet/y_\bullet$

[12] In [1] we did not yet realize the role of the variable t.

converges to the quantity X/Y. It is thus straightforward to calculate quantities such as $a^U_{u,n,\infty}$ from knowledge of the recursion plus its seeds; see Lemma 3.1.4.

(ii) The staircases $\mathcal{S}^U_{\bullet,n}$ are described above as having two intertwined strands, one for each end_n. We show in Lemma 3.2.1 that these classes may be combined into a single family with recursion variable $2n+3$. This simpler description of the staircases clarifies their essential structure.

(iii) Notice that there is no ascending staircase in this family with $n=0$ since the centers of its steps would be < 6, the center of $\boldsymbol{B}^U_0$. Of course, there is a staircase of this kind, but we view it here as the image of $\mathcal{S}^U_{u,0}$ by the reflection $R_{v_2} = SR$, and so consider it a member of the staircase family $(SR)^\sharp(\mathcal{S}^U)$.

(iv) Finally, note that staircases that are associated to a blocking class $\boldsymbol{B}$ and labeled with the subscript ℓ are always ascending, and converge to the *lower* end of the z–interval blocked by $\boldsymbol{B}$, while those labeled u descend and converge to the *upper* endpoint of this z–interval.

There is a corresponding definition of the staircase family $\mathcal{S}^L$.

Theorem 2.3.3 *The classes* $\boldsymbol{B}^L_n = (5n, n-1, 12n+1, 2n)$, *for* $n \geq 1$, *with decreasing centers, are perfect and center-blocking, and have the following associated staircases* $\mathcal{S}^L_{\ell,n}$ *and* $\mathcal{S}^L_{u,n}$ *for* $n \geq 1$, *with* σ_n *and* end_n *as in Theorem 2.3.1:*

- $\mathcal{S}^L_{\ell,n}$ *is ascending, with limit point* $a^L_{\ell,n,\infty} = [6; 2n+1, \{2n+5, 2n+1\}^\infty]$, *and has*
  - ***centers*** $[6; 2n+1, \{2n+5, 2n+1\}^k, \text{end}_n]$,
  - ***recursion*** $x_{n,k+1} = (\sigma_n + 2)x_{n,k} - x_{n,k-1}$, *where* $\sigma_n := (2n+1)(2n+5)$,
  - ***relation*** $(2n+3)d_{n,k} = (n+1)p_{n,k} - (n-1)q_{n,k}$.
- $\mathcal{S}^L_{u,n}$ *is descending, with limit point* $a^L_{u,n,\infty} = [6; 2n-1, 2n+1, \{2n+5, 2n+1\}^\infty]$ *and has*
  - ***centers*** $[6; 2n-1, 2n+1, \{2n+5, 2n+1\}^k, \text{end}_n]$,
  - ***recursion*** $x_{n,k+1} = (\sigma_n + 2)x_{n,k} - x_{n,k-1}$,
  - ***relation*** $(2n+3)d_{n,k} = -(n-1)p_{n,k} + (11n+2)q_{n,k}$.

The limit points $a^L_{\bullet,n,\infty}$ *(with* $\bullet = \ell$ *or* u*) form a decreasing sequence in* $(6,7) = (v_2, w_1)$ *with limit* 6, *while the corresponding* b–*values lie in* $\left(0, \frac{1}{5}\right)$ *and increase with limit* $\frac{1}{5}$.

Remark 2.3.4 (i) It follows from Lemma 2.1.6(i)–(ii) that the symmetry R takes the centers both of the blocking classes and of the staircase steps for the family $\mathcal{S}^U$ into those for $\mathcal{S}^L$. Notice that in the case of $\mathcal{S}^U_{\ell,n}$ it takes the step with label k to the step in $\mathcal{S}^U_{u,n}$ with label $k-1$. Note that, with this choice of labeling, the image by R of the step with center $[2n+7;2n+4]$ (which appears both as $\mathcal{S}^U_{\ell,n+1,0}$ and as $\mathcal{S}^U_{u,n,1}$) has no counterpart in the staircase $\mathcal{S}^L_{u,n+1}$, though it could be added to it.

(ii) **The Fibonacci stairs** The Fibonacci stairs are the ascending stairs that should be associated to $\boldsymbol{B}^L_0 = R^\sharp(\boldsymbol{B}^U_0)$. However, such a class would have center at $R(6)=\infty$, and so it does not exist as a geometric obstruction. Nevertheless, if we ignore the first few steps, the steps of the Fibonacci stairs have precisely the form predicted by putting $n=0$ in the formulas for $\mathcal{S}^L_{\ell,n}$; namely, they have

- **centers** $[6;1,\{5,1\}^k,\mathrm{end}_0]$,
- **recursion** $x_{k+1}=5x_k-x_{k-1}$,
- **relation** $3d_k=p_k+q_k$.

Moreover, although the class $\boldsymbol{E}' := 3L(-0E_0)-2E_1-E_2-\cdots-E_7$ is not perfect, its obstruction $\mu_{\boldsymbol{E}',0}(z)$ for $z\in(6,7]$ is the function $z\mapsto(1+z)/3$, which goes through the point $(a_{0,\infty},V_0(a_{0,\infty}))$ (where $a_{0,\infty}=\tau^4$) and equals $c_{H_0}(z)$ for $z\in[\tau^4,7]$. Therefore this class $\boldsymbol{E}'$ plays the geometric role of the blocking class, and we consider these stairs as part of the family $\mathcal{S}^L=R^\sharp(\mathcal{S}^U)$. As we explain before Lemma 3.2.5, there is a different tuple with negative entries that plays the numeric role of the missing blocking class; below we denote this by $\boldsymbol{B}^L_0$.

Notice that all the other families $(S^i)^\sharp(\mathcal{S}^L)$, for $i\ge 1$, have an ascending staircase for $n=0$ that is associated with the blocking class $(S^i)^\sharp(\boldsymbol{B}^L_0)$, which now has positive entries and so is geometric.

Lemma 2.2.11 shows that for every $n\ge 0$ the ratio $(n+2)/(n+3)$ does not lie in the b–interval $J_{\boldsymbol{B}^U_n}$ blocked by $\boldsymbol{B}^U_n$. The next lemma locates this point more precisely.

Lemma 2.3.5 *Let $(d_n,m_n)=(n+3,n+2)$ and $a_n=2n+6$ be the degree variables and center of the blocking class $\boldsymbol{B}^U_n=(n+3,n+2,2n+6,1)$. Then for all $n\ge 0$, we have $\mathrm{acc}(m_n/d_n)<a_{n+1}$ and $m_n/d_n\in J_{\boldsymbol{B}^U_{n+1}}$.*

Proof To see that $z_n := \mathrm{acc}(m_n/d_n) < a_{n+1} = 2n+8$, it suffices to check that $z_n+1/z_n < 2n+8+1/(2n+8)$. But (2.1.1) implies that

$$z_n+\frac{1}{z_n}=\frac{(3-(n+2)/(n+3))^2}{1-((n+2)/(n+3))^2}-2=\frac{4n^2+24n+39}{2n+5}$$
$$=2n+8-\frac{2n+1}{2n+5}<2n+8+\frac{1}{2n+8}.$$

A similar argument shows that $z_n > 2n+7$. Therefore, [1, Lemma 16] implies that $\mu_{\boldsymbol{B}^U_{n+1},m_n/d_n}(z_n)$ is given by the formula (1.1.3), so

$$\mu_{\boldsymbol{B}^U_{n+1},m_n/d_n}(z_n) = \frac{z_n}{d_{n+1} - (m_n/d_n)m_{n+1}} = \frac{z_n}{2}.$$

On the other hand, it follows from [1, Lemma 38] that $m_n/d_n \in J_{\boldsymbol{B}^U_{n+1}}$ exactly when $\mu_{\boldsymbol{B}^U_{n+1},m_n/d_n}(z_n) > V_{m_n/d_n}(z_n)$. Since $z_n = \mathrm{acc}(m_n/d_n)$, we have

$$V_{m_n/d_n}(z_n) = \frac{1+z_n}{3-m_n/d_n}.$$

Thus we need $z_n/2 > (1+z_n)(n+3)/(2n+7)$, which holds because $z_n > 2n+6$. □

Remark 2.3.6 We show in Corollary 4.2.6 that for all our staircases, whether ascending or descending, the ratios $(m_k/d_k)_{k\geq 1}$ decrease when $m_k/d_k > 1/3$ and increase when $m_k/d_k < 1/3$. (Since the staircase steps are defined recursively, Corollary 3.1.5 shows that this follows from the structure of the first two steps.) Further, Proposition 3.2.2 shows that the blocking class $\boldsymbol{B}^U_n$ can be considered as a step in the ascending stair $\mathcal{S}^U_{\ell,n+1}$ associated to $\boldsymbol{B}^U_{n+1}$. Hence m_n/d_n is part of a decreasing sequence that limits on the lower endpoint of $J_{\boldsymbol{B}^U_{n+1}}$. Thus, once we have shown that $m_n/d_n < a_{n+1}$, this reasoning implies that $m_n/d_n \in J_{\boldsymbol{B}^U_{n+1}}$. More generally, an analog of Lemma 2.3.5 holds for any prestaircase family.

Example 2.3.7 (the staircase at $b = 1/3$) This staircase, discovered in [3], behaves in a different way from all other known staircases in H_b. It has three interwoven sequences,

$$\boldsymbol{E}_{k,i} = (d_{k,i},\, m_{k,i},\, p_{i,k} = g_{k,i},\, q_{i,k} = g_{k-1,i}), \quad i = 0,1,2,$$

where for each i the numbers $g_{k,i}$ satisfy the recursion

$$g_{k+1,i} = 6g_{k,i} - g_{k-1,i}$$

with seeds (ie initial values)

$$g_{0,0} = 1, \quad g_{1,0} = 2, \quad g_{0,1} = 1, \quad g_{1,1} = 4, \quad g_{0,2} = 1, \quad g_{1,2} = 5.$$

The centers $p_{i,k}/q_{i,k} := g_{k,i}/g_{k-1,i}$ have continued fractions $[5; \{1,4\}^k, \mathrm{end}_i]$, where the possible ends are $\mathrm{end}_0 = 2$, $\mathrm{end}_1 = (1,3)$ and $\mathrm{end}_2 = \varnothing$.

Normally one needs two seeds, say x_0 and x_1, to fix an iteration of the form $x_{k+1} = \nu x_k - x_{k-1}$. However the variables p and q for the $1/3$–staircase have the form x_k

and x_{k-1}, and so are part of a single recursive sequence. Thus we can consider that each of the three strands of the $1/3$–staircase has a single seed $\boldsymbol{E}_{\mathrm{seed}} = (d, m, p, q)$, namely

$$\boldsymbol{E}_{\mathrm{seed},0} = (1, 0, 2, 1), \quad \boldsymbol{E}_{\mathrm{seed},1} = (2, 1, 4, 1), \quad \boldsymbol{E}_{\mathrm{seed},2} = (2, 0, 5, 1),$$

and that the rest of the staircase comes from the images of the (p, q) components of these three classes under the action of $S\colon (p, q) \mapsto (6p - q, p)$, with (d, m) determined by a modification of (2.2.4) as follows. We have

$$(2.3.1) \quad d_{k,i} = \tfrac{1}{8}(3(g_{k,i} + g_{k-1,i}) + \varepsilon_{k,i} t_i), \quad m_{k,i} = \tfrac{1}{8}(g_{k,i} + g_{k-1,i} + 3\varepsilon_{k,i} t_i),$$

where $\varepsilon_{k,i} = (-1)^{k+i}$, while t_i is constant with respect to k and equal to

$$t_i = \sqrt{g_{1,i}^2 + g_{0,i}^2 - 6 g_{1,i} g_{0,i} + 8} = \begin{cases} 1 & \text{if } i = 0, 1, \\ 2 & \text{if } i = 2. \end{cases}$$

Note that $\varepsilon_{k,i} = 1$ if $m_{k,i}/d_{k,i} > 1/3$ and $= -1$ otherwise.

Note that, in contrast with the case of $\mathcal{S}^U$ and $\mathcal{S}^L$ discussed in Remark 2.3.2(ii) above, it is *not* possible to combine these three interwoven sequences into a single recursive sequence; for example, there is no constant c such that for all k

$$g_{k,2} = c g_{k,1} - g_{k,0}.$$

Further, the relation between the d, p, and q variables in this staircase is significantly different from the homogenous linear relation satisfied by the other staircases. By [1, Proposition 49] the latter relation implies that the ratios $m_{k,i}/d_{k,i}$ are monotonic and converge as $k \to \infty$ so quickly that the step classes are themselves center-blocking classes. On the other hand, classes with centers $< a_{\min}$ cannot be center-blocking, and the ratios $m_{k,i}/d_{k,i}$ lie on both sides of $1/3$. Another important distinction between this staircase and the others is the fact that the t–variable is fixed for each strand. The point here is that in all cases the variable called t is fixed by the shift S. This shift implements the recursion of the $1/3$–staircase, while it takes every other staircase to a different one.

Finally, we remark that the three seed classes $\boldsymbol{E}_{\mathrm{seed},i}$, for $i = 0, 1, 2$, have rather different natures. The first two have the form of the blocking classes

$$\boldsymbol{B}_n^U := (n + 3, n + 2, 2n + 6, 1), \quad n \geq 0,$$

and indeed are given by taking $n = -2, -1$ in this formula. One might consider that the $\boldsymbol{B}_n^U$, for $n \geq -2$, form a single family of classes, that behave differently according as to where the center $2n+6$ lies in relation to $a_{\min}$. If the center is $< a_{\min}$ then iteration by S gives a staircase for $b = 1/3 = \mathrm{acc}^{-1}(a_{\min})$, while if the center is $> a_{\min}$ then we get a family of blocking classes that block b–intervals that converge to $1/3$, but do not form a staircase since they are not live when $b = 1/3$. (To see this, note that $t = |d-3m|$ is fixed by S, while by [1, Lemma 15] a perfect class $\boldsymbol{E}$ with center p/q is obstructive at its center for a given value of b if and only if $|bd-m| < \sqrt{1-b^2}$. Hence a class that is live at $b = 1/3$ must have $|t| < 3$, while $\boldsymbol{B}_n^U$ has $t = 2n+3$.)

The third seed $\boldsymbol{E}_{\mathrm{seed},2} = (2,0,5,1)$ with $(t,\varepsilon) = (2,-1)$ is significantly different. Both it and $\boldsymbol{E}_{\mathrm{seed},0} = (1,0,2,1)$ are steps in the Fibonacci staircase at $b = 0$, but the iteration that gives this sequence is not S but rather $x_{\kappa+1} = 3x_\kappa - x_{\kappa-1}$. All the other steps in the Fibonacci staircase have centers $> a_{\min}$, and are blocking classes by [1, Lemma 41]. On the other hand, because $S(1,1) = (5,1)$, the strand formed by $\boldsymbol{E}_{\mathrm{seed},2}$ and its iterates under S can be extended one place backwards by the tuple

$$\boldsymbol{E}_{-1,2} := (1,1,1,1) \quad \text{with } (t,\varepsilon) = (2,1). \tag{2.3.2}$$

It turns out that the sequence of classes formed by $\boldsymbol{E}_{-1,2}$, $\boldsymbol{E}_{\mathrm{seed},2}$, and the images of $\boldsymbol{E}_{\mathrm{seed},2}$ by S^i have an important role to play in generating the staircase families; see for example Corollaries 3.2.3 and 3.2.7 and Lemma 3.3.3.

Remark 2.3.8 (the role of low degree classes) (i) The three seeds of the $1/3$–staircase are given by the only exceptional curves in H_b with degree at most two. These classes have centers $< a_{\min} = 3+2\sqrt{2}$, while all the other classes of interest here have centers $> a_{\min}$.

(ii) There is one exceptional divisor of degree three, that gives rise to three potentially interesting exceptional divisors in H_b that we will label by the coefficient of E_0, namely[13]

$$\boldsymbol{E}_0 := 3L - 2E_1 - E_{2\ldots7}, \quad \boldsymbol{E}_1 := 3L - E_0 - 2E_1 - E_{2\ldots6}, \quad \boldsymbol{E}_2 := 3L - 2E_0 - E_{1\ldots6}.$$

Each of these classes has a special role to play:

- $\boldsymbol{E}_0$ substitutes geometrically (but not numerically) for the blocking class for the Fibonacci stairs (see Remark 2.3.4(ii)).
- $\boldsymbol{E}_1$ is a (nonperfect) class with breakpoint 6. This class is discussed extensively in [1, Example 32]. It is obstructive at $z = 6$ for

$$\mathrm{acc}_L^{-1}(6) = \tfrac{1}{5} < b < \tfrac{5}{11} = \mathrm{acc}_U^{-1}(6),$$

[13] Here we use the shorthand of [6], where $E_{i\ldots j} := \sum_{k=i}^{j} E_k$.

and for this range of b we have

$$\mu_{\boldsymbol{E},b}(z) = \frac{1+z}{3-b} \quad \text{for } z \in (5,6), \qquad \mu_{\boldsymbol{E},b}(z) = \frac{7}{3-b} \quad \text{for } z \geq 6.$$

The obstruction given by this class is discussed further in Lemma 2.2.7. Its properties turn out to be a crucial element in our proofs.

- $\boldsymbol{E}_2$ is the perfect blocking class $\boldsymbol{B}_0^U$ with center 6.

3 Structure of the prestaircase families

We begin by explaining the recursion that underlies our staircases. In Section 3.2, we then describe the staircase families $\mathcal{S}^U$ and $\mathcal{S}^L$ in the new language, and show that they are indeed generated by their blocking classes together with two seeds, as claimed in Proposition 1.2.2.

In Section 3.3, we prove Theorem 1.2.4. This establishes that for each $T = S^i R^\delta$, $T^\sharp(\mathcal{S}^U)$ is a prestaircase family. This entails examining how the seeds and blocking classes are transformed under S. Furthermore, this implies that each prestaircase in $T^\sharp(\mathcal{S}^U)$ is associated to a blocking class, an essential feature of a prestaircase family. In Section 3.4, we compute how $S^\sharp$ and $R^\sharp_{v_i}$ act on the degree coordinates (d,m) of the blocking classes.

3.1 The single recursion

The following lemma shows that we can consider each staircase in $\mathcal{S}^U$ and $\mathcal{S}^L$ to be a single family of classes satisfying the recursion $x_{\kappa+1} = (2n+3)x_\kappa - x_{\kappa-1}$ with parameter $2n+3$ instead of a double family of classes as in Theorems 2.3.1 and 2.3.3 that each satisfy the recursion $x_{k+1} = ((2n+1)(2n+5)+2)x_k - x_{k-1}$ with parameter $(2n+1)(2n+5)$. Note that an increase of k by 1 corresponds to an increase of κ by 2.

Lemma 3.1.1 (i) *Let $m \geq 1$ be any integer, and consider*

$$\frac{p_1}{q_1} = [m; 2n+4], \qquad \frac{p_2}{q_2} = [m; 2n+5, 2n+2],$$
$$\frac{p_3}{q_3} = [m; 2n+5, 2n+1, 2n+4], \qquad \frac{p_4}{q_4} = [m; 2n+5, 2n+1, 2n+5, 2n+2].$$

Then for $x_\bullet = p_\bullet, q_\bullet$ we have $x_{\kappa+1} = (2n+3)x_\kappa - x_{\kappa-1}$ for $\kappa = 1, 2$.

(ii) *If $x_\bullet$ is a sequence such that $x_{\kappa+1} = \nu x_\kappa - x_{\kappa-1}$ for all κ, then*

$$x_{\kappa+2} = (\nu^2 - 2)x_\kappa - x_{\kappa-2} \quad \text{for all } \kappa.$$

In particular, if $\nu = 2n+3$ then $\nu^2 - 4 = (2n+1)(2n+5)$.

(iii) *If y_κ and z_κ satisfy the recursion $x_{\kappa+1} = \nu x_\kappa - x_{\kappa-1}$, then*

$$y_{\kappa+2} z_\kappa - z_{\kappa+2} y_\kappa = \nu(y_{\kappa+1} z_\kappa - z_{\kappa+1} y_\kappa).$$

Proof Item (i) by direct calculation. Item (ii) holds because $\nu x_{\kappa-1} = x_\kappa + x_{\kappa-2}$, so

$$x_{\kappa+2} = \nu x_{\kappa+1} - x_\kappa = \nu(\nu x_\kappa - x_{\kappa-1}) - x_\kappa = (\nu^2-1)x_\kappa - \nu x_{\kappa-1} = (\nu^2-2)x_\kappa - x_{\kappa-2}.$$

Finally, (iii) holds because

$$y_{\kappa+2} z_\kappa - z_{\kappa+2} y_\kappa = \det\begin{pmatrix} z_\kappa & y_\kappa \\ z_{\kappa+2} & y_{\kappa+2}\end{pmatrix} = \det\begin{pmatrix} z_\kappa & y_\kappa \\ \nu z_{\kappa+1} & \nu y_{\kappa+1}\end{pmatrix}. \qquad \square$$

We next show that this recursion extends naturally to the triples $(p_\kappa, q_\kappa, t_\kappa)$ that parametrize quasiperfect classes as in Definition 2.2.3, provided that the initial seeds satisfy the compatibility condition (3.1.3) that is given below. Further, (3.1.2) below states that the tuple (p, q, t) is the coordinate of a point on the surface X defined by (2.2.2). Thus the next lemma gives compatibility conditions on the seeds (p_0, q_0, t_0) and (p_1, q_1, t_1) under which the recursion with parameter ν extends to the integral points of X.

It is convenient to use matrix notation with

(3.1.1) $$A := \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{x} := \begin{pmatrix} p \\ q \\ t \end{pmatrix}.$$

Thus the matrix A is symmetric, and $X = \{\boldsymbol{x} \mid \boldsymbol{x}^T A \boldsymbol{x} = 8\}$.

Lemma 3.1.2 *Suppose that $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are integral vectors that satisfy the following conditions for some integer $\nu > 0$:*

(3.1.2) $$\boldsymbol{x}_i^T A \boldsymbol{x}_i = 8, \quad i = 0, 1,$$

(3.1.3) $$\boldsymbol{x}_1^T A \boldsymbol{x}_0 = 4\nu.$$

Then the vectors $\boldsymbol{x}_2 := \nu \boldsymbol{x}_1 - \boldsymbol{x}_0, \boldsymbol{x}_1$ also satisfy these conditions for the given ν.

Proof Since A is symmetric,

$$(\nu \boldsymbol{x}_1 - \boldsymbol{x}_0)^T A (\nu \boldsymbol{x}_1 - \boldsymbol{x}_0) = \nu^2 \boldsymbol{x}_1^T A \boldsymbol{x}_1 - 2\nu \boldsymbol{x}_1^T A \boldsymbol{x}_0 + \boldsymbol{x}_0^T A \boldsymbol{x}_0 = 8$$

exactly when $8\nu^2 = 2\nu \boldsymbol{x}_1^T A \boldsymbol{x}_0$, which holds by (3.1.3). Further, (3.1.3) holds for $\nu \boldsymbol{x}_1 - \boldsymbol{x}_0, \boldsymbol{x}_1$ because, by (3.1.3),

$$(\nu \boldsymbol{x}_1 - \boldsymbol{x}_0)^T A \boldsymbol{x}_1 = 8\nu - (\boldsymbol{x}_0)^T A \boldsymbol{x}_1 = 4\nu. \qquad \square$$

Corollary 3.1.3 *Any two integral triples $\boldsymbol{x}_i = (p_i, q_i, t_i)$ for $i = 0, 1$ that satisfy (3.1.2) and (3.1.3) for a given ν can be extended to a sequence $\boldsymbol{x}_i$ for $i \geq 0$ using recursion parameter ν, where each successive pair satisfies these conditions. Further, for fixed $\varepsilon \in \{\pm 1\}$, the corresponding quantities*

$$d_i = \tfrac{1}{8}(3(p_i + q_i) + \varepsilon t_i), \quad m_i = \tfrac{1}{8}(p_i + q_i + 3\varepsilon t_i)$$

of (2.2.4) also satisfy this recursion and hence are integers, provided that they are integers for $i = 0, 1$.

We end this section with some useful formulas about recursively defined sequences. The following is an adaptation of [1, Lemma 47].

Lemma 3.1.4 *Let x_κ for $\kappa \geq 0$ be a sequence of integers that satisfy the recursion*

(3.1.4) $$x_{\kappa+1} = \nu x_\kappa - x_{\kappa-1}, \quad \nu \geq 3,$$

and let

$$\lambda = \tfrac{1}{2}(\nu + \sqrt{\sigma}) \in \mathbb{Q}[\sqrt{\sigma}]$$

be the larger root of the equation $x^2 - \nu x + 1 = 0$, where $\sigma = \nu^2 - 4 > 1$. Then there is a number $X \in \mathbb{Q}[\sqrt{\sigma}]$ such that

(3.1.5) $$x_\kappa = X\lambda^\kappa + \bar{X}\bar{\lambda}^\kappa,$$

where $\overline{a + b\sqrt{\sigma}} := (a - b\sqrt{\sigma})$, so that $\lambda\bar{\lambda} = 1$. Further, if we write $X = X' + X''\sqrt{\sigma}$, then

(3.1.6) $$X' = \frac{x_0}{2}, \quad X'' = \frac{2x_1 - \nu x_0}{2\sigma}.$$

Proof If the monomials $x_\kappa = c^\kappa$ satisfy the recursion then we must have $c^2 - \nu c + 1 = 0$, so that $c = \frac{1}{2}(\nu \pm \sqrt{\nu^2 - 4}) = \frac{1}{2}(\nu \pm \sqrt{\sigma})$. Let λ be the larger solution so that $\bar{\lambda}$ is the smaller one, and we have $\lambda\bar{\lambda} = 1$. Since (3.1.4) has a unique solution once given the seeds x_0 and x_1, it follows that for each choice of constants A and B, the numbers

(3.1.7) $$x_\kappa := A\lambda^\kappa + B\bar{\lambda}^\kappa$$

form the unique solution with

$$x_0 = A + B, \quad x_1 = A\lambda + B\bar{\lambda}.$$

Then $A, B \in \mathbb{Q}[\sqrt{\sigma}]$. Notice also that $\sqrt{\sigma}$ is irrational since $\sigma = \nu^2 - 4$ is never a perfect square when $\nu \geq 3$. It follows easily that $x_0, x_1 \in \mathbb{Q}$ only if we also have $B := \bar{A}$.

Thus we have

$$x_0 = X + \overline{X} = X' + X''\sqrt{\sigma} + X' - X''\sqrt{\sigma} = 2X',$$

which gives $X' = x_0/2$, and

$$x_1 = (X' + X''\sqrt{\sigma}) \cdot \tfrac{1}{2}(\nu + \sqrt{\sigma}) + (X' - X''\sqrt{\sigma}) \cdot \tfrac{1}{2}(\nu - \sqrt{\sigma}) = X'\nu + X''\sigma,$$

which gives $X'' = (2x_1 - \nu x_0)/(2\sigma)$, as claimed □

Corollary 3.1.5 *Let x_κ and y_κ, for $\kappa \geq 0$, be increasing sequences that both satisfy (3.1.4) for some $\nu \geq 3$ and are such that x_κ and y_κ are positive for $\kappa \geq 1$. Suppose further that at most one of x_0 and y_0 are zero. Then*:

(i) *The ratios $(x_\kappa/y_\kappa)_{\kappa \geq 1}$ form a monotonic sequence that is strictly increasing if $x_1 y_0 - x_0 y_1 > 0$ and is strictly decreasing if $x_1 y_0 - x_0 y_1 < 0$.*

(ii) *In all cases, $\lim_{\kappa \to \infty} x_\kappa/y_\kappa$ exists and equals X/Y, where X and Y are the constants defined in (3.1.6).*

(iii) *Provided that $x_1 y_0 - x_0 y_1 \neq 0$, the limit X/Y is irrational.*

Proof Note first that

$$x_{\kappa+1} y_\kappa - x_\kappa y_{\kappa+1} = (6x_\kappa - x_{\kappa-1}) y_\kappa - x_\kappa(6y_\kappa - y_{\kappa-1}) = x_\kappa y_{\kappa-1} - x_{\kappa-1} y_\kappa$$

is independent of κ. Hence the sequence x_κ/y_κ for $\kappa \geq 1$ is monotonic,[14] and the quantity $x_1 y_0 - x_0 y_1$ determines whether it is increasing or decreasing. Part (ii) is an immediate consequence of the formula in (3.1.7), and the fact that $\lambda > 1$.

To prove the third claim, suppose first that $2x_1 > \nu x_0$. Then $X = X' + X''\sqrt{\sigma}$ is irrational since the coefficient X'' of $\sqrt{\sigma}$ does not vanish, and $\sigma = \nu^2 - 4$ is not a perfect square. Similarly, if $2y_1 > \nu y_0$ then Y is irrational. Further,

$$\frac{X}{Y} = \frac{(X' + X''\sqrt{\sigma})(Y' - Y''\sqrt{\sigma})}{(Y' + Y''\sqrt{\sigma})(Y' - Y''\sqrt{\sigma})} = \frac{(X'Y' - X''Y''\sigma) + (X''Y' - X'Y'')\sqrt{\sigma}}{(Y')^2 - (\sigma Y'')^2}$$

is irrational unless $X''Y' - X'Y'' = 0$. But formula (3.1.6) implies that $X''Y' - X'Y''$ is a multiple of $x_1 y_0 - x_0 y_1$ and so is nonzero by hypothesis.

[14] If $x_0/y_0 \leq 0$, then the sequence is only monotonic for $\kappa \geq 1$. This will be the case for the seeds of $\mathcal{S}^U_u$ computed in Lemma 3.2.1(ii)

To deal with the possibility that X'' or Y'' equals 0, notice that the limit ratio X/Y does not depend on which pair of terms k_0, k_0+1 we take as the initial terms (though the values of X and Y do depend on this choice). Further, because by hypothesis $\nu \geq 3$ and $x_0 < x_1$, we have $x_2 = \nu x_1 - x_0 > \nu x_1/2$, so $2x_2 > \nu x_1$. Similarly, $2y_2 > \nu y_1$. Therefore if we define the quantities X and Y as in (3.1.6) but starting with $\kappa = 1$ then the above argument shows that the ratio X/Y is irrational. □

3.2 Generating the known staircase families

We now prove Proposition 1.2.2; its proof is contained in Corollaries 3.2.3 and 3.2.7. In all cases considered in this paper, the recursion variable $\nu = 2n+3$ for the prestaircase $\mathcal{S}^{\mathcal{F}}_{\bullet,n}$ in a staircase family $\mathcal{F}$ is a linear function of n, and we will enumerate the terms x_κ in our recursively defined sequences such that the value of x_κ is a polynomial of degree κ in n. In particular, x_0 is a constant. In contrast, the staircase sequences were enumerated in Theorems 2.3.1 and 2.3.3 via a number k that counted the iterations of the repeated pair $\{2n+5, 2n+1\}$ in the continued fraction expansion of p_k/q_k.

Lemma 3.2.1 (i) *If* $(p_0, q_0, t_0) = (1, 1, 2)$ *and* $(p_1, q_1, t_1) = (\nu+1, 1, \nu-2)$ *for any* $\nu \geq 3$, *then the identities in Lemma 3.1.2 hold. Further, with* $(d_0, m_0) = (1, 1)$ *and* $(d_1, m_1) = \big(\frac{1}{2}(\nu+1), \frac{1}{2}(\nu-1)\big)$, *the identities in (2.2.4) hold with* $\varepsilon = 1$.

(ii) *If* $(p_0, q_0, t_0) = (-5, -1, 2)$ *and* $(p_1, q_1, t_1) = (\nu+5, 1, \nu+2)$ *for any* $\nu \geq 1$, *then the identities in Lemma 3.1.2 hold. Further, with* $(d_0, m_0) = (-2, 0)$ *and* $(d_1, m_1) = \big(\frac{1}{2}(\nu+5), \frac{1}{2}(\nu+3)\big)$, *the identities in (2.2.4) hold with* $\varepsilon = 1$.

(iii) *If we define the triple* $(p_\kappa, q_\kappa, t_\kappa)$ *by the recursion* $x_{\kappa+1} = \nu x_\kappa - x_{\kappa-1}$ *for* $\kappa \geq 1$, *then the ratios* p_κ/q_κ *form an increasing sequence in case* (i) *and a decreasing sequence in case* (ii).

Proof The claims in (i) and (ii) hold by an easy computation. To prove (iii), it suffices by Corollaries 3.1.3 and 3.1.5 to check the first two terms. But in case (i) we have $p_1/q_1 > p_0/q_0$, while in case (ii), $p_1/q_1 = \nu+5 > p_2/q_2 = (\nu^2+5\nu+5)/(\nu+1)$. □

Proposition 3.2.2 *All the classes involved in the staircase family* $\mathcal{S}^U$ *can be extended to tuples* $(d, m, p, q, t, \varepsilon)$ *with* $\varepsilon = 1$. *Moreover, the staircase* $\mathcal{S}^U_{\bullet,n}$ *has recursion parameter* $2n+3$, *which is the* t*–coefficient in* $\boldsymbol{B}^U_n$, *and it can be extended to have the seeds described in Lemma 3.2.1. More precisely*:

- *The ascending staircase* $\mathcal{S}^U_{\ell,n}$ *for* $n \geq 1$ *has initial step given by the tuple* $(d, m, p, q, t) = (1, 1, 1, 1, 2)$ *and next step (at* $\kappa = 1$*) given by the (*t*–extended) blocking class*

$$\boldsymbol{B}^U_{n-1} = (n+2, n+1, 2n+4, 1, 2n+1).$$

- *The descending staircase* $\mathcal{S}^U_{u,n}$ *for* $n \geq 1$ *has initial step given by the tuple* $(d, m, p, q, t) = (-2, 0, -5, -1, 2)$ *and next step (at* $\kappa = 1$*) given by the (*t*–extended) blocking class*

$$\boldsymbol{B}^U_{n+1} = (n+4, n+3, 2n+8, 1, 2n+5).$$

Corollary 3.2.3 *The family* $\mathcal{S}^U$ *is generated in the sense of Definition 1.2.1 by its blocking classes* $\boldsymbol{B}^U_n = (n+3, n+2, 2n+6, 1, 2n+3)$ *together with the seeds*

$$\boldsymbol{E}^U_{\ell,\mathrm{seed}} = (1, 1, 1, 1, 2), \quad \boldsymbol{E}^U_{u,\mathrm{seed}} = (-2, 0, -5, -1, 2), \quad \varepsilon = 1.$$

Moreover, for all staircases in this family, whether ascending or descending, the ratios m_κ/d_κ *decrease.*

Proof of Proposition 3.2.2 It is straightforward to check that the d and m coordinates in $\boldsymbol{B}^U_n := (n+3, n+2, 2n+6, 1)$ for $n \geq 0$ are given by the formulas in (2.2.4) with

$$(p, q, t) = (2n+6, 1, 2n+3), \quad \varepsilon = 1. \tag{3.2.1}$$

By Lemma 3.1.1, if for each $n \geq 1$ we enumerate the ascending staircase $\mathcal{S}^U_{\ell,n}$ as a single staircase that is indexed by the degree κ of $p_\bullet$ as a function of n, then this staircase has

- recursion parameter $\nu = 2n+3$,
- initial steps with centers $p_1/q_1 = (2n+4)/1$ and

$$\frac{p_2}{q_2} = [2n+5; 2n+2] = \frac{(2n+5)(2n+2)+1}{2n+2},$$

- linear relation $(2n+3)d_\kappa = (n+1)p_\kappa + (n+2)q_\kappa$.

The linear relation implies that the d values of the first two steps are $d_1 = n+2$ and $d_2 = (n+1)(2n+5)$. Note also that the class with center at p_1/q_1 is precisely $\boldsymbol{B}^U_{n-1}$.

By the definition of t in (2.2.3), we have that $t_1 = 2n+1 = \nu - 2$ and that d_1 is as predicted by (2.2.4) with $\varepsilon = 1$. Further if we take $(p_0, q_0, t_0) = (1, 1, 2)$ as in Lemma 3.2.1(i), then

$$(p_2, q_2) = (\nu p_1 - p_0, \nu q_1 - q_0).$$

Therefore we may think of the tuple $(d_2, m_2, p_2, q_2, t_2)$ as given by a recursion with $\nu = 2n+3$ and $\varepsilon = 1$, with initial terms

$$(p_0, q_0, t_0) = (1, 1, 2), \quad (d_0, m_0) = (1, 1). \tag{3.2.2}$$

Note that the entries d_0, p_0, and q_0 also satisfy the linear relation

$$(2n+3)d_\kappa = (n+1)p_\kappa + (n+2)q_\kappa.$$

Therefore, as in Corollary 3.1.3, all subsequent terms in this staircase must have degree coefficients (d, m) given by (2.2.4). Thus, as claimed, for each n this sequence is generated by the tuple $(d, m, p, q, t) = (1, 1, 1, 1, 2)$ together with the appropriate blocking class.

Similarly, by Lemma 3.1.1, the descending staircase classes for $\mathcal{S}^U_{u,n}$ with $n \geq 0$, when indexed by κ have

- recursion parameter $\nu = 2n+3$,
- initial steps with centers
  $$\frac{p_2}{q_2} = [2n+7; 2n+4] = \frac{(2n+7)(2n+4)+1}{2n+4},$$
  $$\frac{p_3}{q_3} = [2n+7; 2n+5, 2n+2] = \frac{(2n+7)((2n+5)(2n+2)+1)+2n+2}{(2n+5)(2n+2)+1},$$
- linear relation $(2n+3)d_\kappa = (n+2)p_\kappa - (n+4)q_\kappa$.

The linear relation implies that the corresponding d values are $d_2 = 2n^2+11n+14$ and $d_3 = 4n^3+28n^2+60n+38$. Note that we can add the blocking class $\boldsymbol{B}^U_{n+1} = (n+4, n+3, 2n+8, 1)$ to the staircase as the step for $\kappa = 1$ because

$$p_1 := (2n+3)p_2 - p_3 = 2n+8, \quad q_1 := (2n+3)q_2 - q_3 = 1,$$

and the appropriate linear relation $(2n+3)(n+4) = (n+2)(2n+8) - (n+4)$ holds.

In fact, this staircase has the form of Lemma 3.2.1(ii), with initial tuples

$$(p_0, q_0, t_0) = (-5, -1, 2), \quad (p_1, q_1, t_1) = (2n+8, 1, 2n+5). \tag{3.2.3}$$

The next entry in the recursive sequence is then

$$(p_2, q_2, t_2) = \big((2n+8)(2n+3)+5, 2n+4, (2n+5)(2n+3)-2\big),$$

which has the same values for p_2 and q_2, as does the first staircase step given above. The formulas in (2.2.4) give

$$(d_0, m_0) = (-2, 0), \quad (d_1, m_1) = (n+4, n+3) \tag{3.2.4}$$

with $\varepsilon = 1$. Further, the linear relation $(2n+3)d_\kappa = (n+2)p_\kappa - (n+4)q_\kappa$ holds in both cases. Thus, again, this staircase is generated by the initial tuple $(d,m,p,q,t) = (-2,0,-5,-1,2)$ together with the appropriate blocking class. □

Proof of Corollary 3.2.3 The first claim is an immediate consequence of Proposition 3.2.2. By Corollary 3.1.5 to check the second claim we must check that $m_1 d_0 - m_0 d_1 < 0$ for all staircases. For the descending staircases, this is immediate because $m_0 = 0$ and $d_0 < 0$ while $m_1 > 0$. For the ascending staircases, this holds because $m_0 = d_0 = 1$ while $m_1 < d_1$. □

Remark 3.2.4 (i) The parameters $(d,m,p,q) = (1,1,1,1)$ of the initial step in the ascending staircases do correspond to those of an exceptional class in H_b, albeit one that is not live for the relevant z values (which are all $> 3+2\sqrt{2} = a_{\min}$). Thus its parameters are both geometrically and numerically meaningful. (See also the discussion concerning (2.3.2).) On the other hand, the parameters $(-2,0,-5,-1)$ of the initial step in the descending staircases are negative and so, though numerically meaningful, have no immediate interpretation in terms of a point $p/q \in (1,\infty)$. Instead, they parametrize the ray $\{(-5,-1)\lambda \mid \lambda > 0\}$ in $\mathbb{R}^2$. Though we do not do this here, one could think of the symmetries in terms of their action on these rays; see Remark 3.3.6.

(ii) For each $n \geq 1$, the ascending staircase $\mathcal{S}^U_{\ell,n+1}$ with recursion parameter $t_{\boldsymbol{B}} = 2n+5$ has the same second step as the descending staircase $\mathcal{S}^U_{u,n}$ with $t_{\boldsymbol{B}} = 2n+3$. Indeed, the formulas given above show that this step has center $(d,m,p,q,t,1)$ with p and q given by

$$\frac{p}{q} = [2n+7; 2n+4] = [2(n+1)+5; 2(n+1)+2]$$

and with $t = (2n+3)(2n+5) - 2$.

There is a similar story for the staircase family $\mathcal{S}^L$, except that now there is no geometric blocking class for $n = 0$, and the tuple $(d,m,p,q,t) = (0,-1,1,0,3)$ that replaces it has no obvious geometric meaning. Nevertheless, we define $\boldsymbol{B}^L_0 := (0,-1,1,0,3)$ for the current purposes. (See Remark 2.3.4(ii).)

Here is the appropriate numerical lemma.

Lemma 3.2.5 (i) *If $(p_0,q_0,t_0) = (5,1,2)$ and $(p_1,q_1,t_1) = (6\nu-5, \nu-1, \nu+2)$ for any $\nu > 1$, then the identities in Lemma 3.1.2 hold. Further, with $(d_0,m_0) = (2,0)$ and $(d_1,m_1) = \big(\frac{5}{2}(\nu-1), \frac{1}{2}(\nu-3)\big)$, the identities in (2.2.4) hold with $\varepsilon = -1$.*

(ii) *If* $(p_0,q_0,t_0)=(-29,-5,2)$ *and* $(p_1,q_1,t_1)=(6\nu-29,\nu-5,\nu-2)$ *for any* $\nu\geq 3$, *then the identities in Lemma 3.1.2 hold*, *Further, with* $(d_0,m_0)=(-13,-5)$ *and* $(d_1,m_1)=\left(\frac{5}{2}(\nu-5),\frac{1}{2}(\nu-7)\right)$, *the identities in (2.2.4) hold with* $\varepsilon=-1$.

(iii) *If we define the triple* $(p_\kappa,q_\kappa,t_\kappa)$ *by the recursion* $x_{\kappa+1}=\nu x_\kappa-x_{\kappa-1}$ *for* $\kappa\geq 1$, *then the ratios* p_κ/q_κ *form an increasing sequence in case* (i) *and a decreasing sequence (for* $\kappa\geq 2$*) in case* (ii).

Proof This is left to the reader. □

Proposition 3.2.6 *All the classes involved in the staircase family* $\mathcal{S}^L$ *can be extended to tuples* (d,m,p,q,t,ε) *with* $\varepsilon=-1$. *Moreover, the staircase* $\mathcal{S}^L_{\bullet,n}$ *has recursion parameter* $2n+3$, *which is the* t*–coefficient in* $\boldsymbol{B}^L_n$, *and it can be extended to have the seeds described in Lemma 3.2.5. More precisely*:

- *The ascending staircase* $\mathcal{S}^L_{\ell,n}$ *for* $n\geq 0$ *has initial step given by* $(d,m,p,q,t)=(2,0,5,1,2)$ *and with the next step (at* $\kappa=1$*) given by the blocking class* $\boldsymbol{B}^L_{n+1}$ *with* $t=2n+5$.
- *The descending staircase* $\mathcal{S}^L_{u,n}$ *for* $n\geq 2$ *has initial step given by* $(d,m,p,q,t)=(-13,-5,-29,-5,2)$ *and with the next step (at* $\kappa=1$*) given by the blocking class* $\boldsymbol{B}^L_{n-1}$ *with* $t=2n+1$. *When* $n=1$ *there is the same initial step, and the step at* $\kappa=1$ *is given by the tuple* $\boldsymbol{B}^L_0:=(0,-1,1,0,3)$.

Corollary 3.2.7 *The family* $\mathcal{S}^L$ *is generated in the sense of Definition 1.2.1 by its blocking classes* $\boldsymbol{B}^L_n:=(5n,n-1,12n+1,2n,2n+3)$ *for* $n\geq 1$, *together with the seeds*

$$\boldsymbol{E}^L_{\ell,\mathrm{seed}}=(2,0,5,1,2),\quad \boldsymbol{E}^L_{u,\mathrm{seed}}=(-13,-5,-29,-5,2),\quad \varepsilon=-1,$$

where in the definition of $\mathcal{S}^L_{u,1}$ *we take the tuple* $\boldsymbol{B}^L_0$ *to be* $(0,-1,1,0,3)$. *Further, for all staircases in the family with* $n>0$, *both ascending and descending, the ratios* m_κ/d_κ *increase.*

Proof of Proposition 3.2.6 The blocking classes $\boldsymbol{B}^L_n:=(5n,n-1,12n+1,2n)$ for $n\geq 1$ are given by the formulas in (2.2.4), with

$$(p,q,t)=(12n+1,2n,2n+3),\quad \varepsilon=-1. \tag{3.2.5}$$

By Lemma 3.1.1, if for each $n\geq 0$ we enumerate the ascending staircase $\mathcal{S}^L_{\ell,n}$ as a single staircase that is indexed by the degree κ of $p_\bullet$ as a function of n, then this staircase has

- recursion parameter $\nu = 2n+3$,
- initial steps p_κ/q_κ with centers

$$\frac{p_2}{q_2} = [6; 2n+1, 2n+4] = \frac{24n^2+62n+34}{4n^2+10n+5},$$
$$\frac{p_3}{q_3} = [6; 2n+1, 2n+5, 2n+2] = \frac{(24n^2+98n+121)2n+89}{(4n^2+16n+19)2n+13},$$

- linear relation $(2n+3)d_\kappa = (n+1)p_\kappa - (n-1)q_\kappa$.

The linear relation implies that $(d_2, m_2) = (10n^2+25n+13, 2n^2+3n)$.

One can easily check that the values (p_2, q_2) and (p_3, q_3) given above agree with those in the recursive sequence with $\nu = 2n+3$ and initial terms

$$(d_0, m_0, p_0, q_0, t_0) = (2, 0, 5, 1, 2),$$
$$(d_1, m_1, p_1, q_1, t_1) = (5(n+1), n, 12n+13, 2n+2, 2n+5).$$

Hence because the tuple for $\kappa = 0, 1$ satisfies (2.2.4) with $\varepsilon = -1$, all classes in the staircase have this form. This establishes the claims concerning $\mathcal{S}^L_{\ell,n}$.

By Lemma 3.1.1, if for each $n \geq 1$ we enumerate the descending staircase $\mathcal{S}^L_{u,n}$ as a single staircase that is indexed by the degree κ of $p_\bullet$ as a function of n, then this staircase has

- recursion parameter $\nu = 2n+3$,
- initial steps with centers

$$\frac{p_3}{q_3} = [6; 2n-1, 2n+1, 2n+4] = \frac{48n^3+100n^2+22n-1}{8n^3+16n^2+2n-1},$$
$$\frac{p_4}{q_4} = [6; 2n-1, 2n+1, 2n+5, 2n+2] = \frac{96n^4+344n^3+320n^2+50n+1}{16n^4+56n^3+48n^2+2n-2},$$

- linear relation $(2n+3)d_\kappa = -(n-1)p_\kappa + (11n+2)q_\kappa$.

Again one can check by direct computation that this sequence is generated by the tuples

$$(p_0, q_0, t_0) = (-29, -5, 2),$$
$$(p_1, q_1, t_1) = (12n-11, 2n-2, 2n+1),$$
$$(p_2, q_2, t_2) = (24n^2+14n-4, 4n^2+2n-1, 4n^2+8n+1),$$

which have the form described in Lemma 3.2.5(ii) with $\nu = 2n+3, \varepsilon = -1$. Moreover, formula (2.2.4) gives the following values for d and m with $\varepsilon = -1$:

$$(d_0, m_0) = (-13, -5), \quad (d_1, m_1) = (5n-5, n-2). \qquad \square$$

Proof of Corollary 3.2.7 The first claim is a consequence of Proposition 3.2.6. By Corollary 3.1.5, to check the second claim we must check that $m_1 d_0 - m_0 d_1 > 0$. For the descending staircases, we have $m_0 = -5$, $d_0 = -13$, $m_1 = n-2$, $d_1 = 5n-5$ and $n \geq 1$, so $m_1 d_0 = -13(n-2) > -5(5n-5) = -25n+5 = m_0 d_1$. For the ascending staircases with $n \geq 1$, this holds because $m_0 = 0$ while $m_1, d_0 > 0$. □

3.3 Proof of Theorem 1.2.4

We now prove Theorem 1.2.4 stating that each symmetry $T \in \mathcal{G}$ transforms the staircase family $\mathcal{S}^U$ into another prestaircase family $T^\sharp(\mathcal{S}^U)$; in particular $T = R$ interchanges the families $\mathcal{S}^U$ and $\mathcal{S}^L$.

We know from Definition 2.2.3 and Lemma 2.2.4 how $T = S^i R^\delta$ acts on quasiperfect classes. The corresponding definition for staircase families is as follows.

Definition 3.3.1 Given $T = S^i R^\delta \in \mathcal{G}$, we define $T^\sharp(\mathcal{S}^U)$ to be the collection of preblocking classes $(T^\sharp(\boldsymbol{B}^U_n))_{n\geq 0}$ together with the seeds $T^\sharp(\boldsymbol{E}^U_{\mathrm{seed},\ell})$ and $T^\sharp(\boldsymbol{E}^U_{\mathrm{seed},u})$.

We first prove our earlier claim that the reflection R takes the family $\mathcal{S}^U$ to the family $\mathcal{S}^L$. Recall from Propositions 3.2.2 and 3.2.6 that all the classes in $\mathcal{S}^U$ have $\varepsilon = 1$ while all those in $\mathcal{S}^L$ have $\varepsilon = -1$.

Lemma 3.3.2 (i) *The map $R^\sharp$ takes the blocking classes and seeds of the staircase family $\mathcal{S}^U$ together with all the associated staircase steps to the corresponding elements in the family $\mathcal{S}^L$. Further, $\boldsymbol{E}^U_{u,\mathrm{seed}} = -\boldsymbol{E}^L_{\ell,\mathrm{seed}}$, where for a class $\boldsymbol{E} = (d, m, p, q, t, \varepsilon)$ we define $-\boldsymbol{E} := (-d, -m, -p, -q, t, -\varepsilon)$.*

(ii) *Moreover, $S^\sharp(\boldsymbol{E}^U_{\ell,\mathrm{seed}}) = \boldsymbol{E}^L_{\ell,\mathrm{seed}}$ and $S^\sharp(\boldsymbol{E}^U_{u,\mathrm{seed}}) = \boldsymbol{E}^L_{u,\mathrm{seed}}$.*

Proof We already noted in [1, Corollary 60] that the reflection R takes the step classes in $\mathcal{S}^U$ to those of $\mathcal{S}^L$. By Corollaries 3.2.3 and 3.2.7, the seeds of these families are

$$\boldsymbol{E}^U_{\ell,\mathrm{seed}}, \boldsymbol{E}^U_{u,\mathrm{seed}}: \quad (d, m, p, q, t, \varepsilon) = (1, 1, 1, 1, 2, 1), (-2, 0, -5, -1, 2, 1)$$
$$\boldsymbol{E}^L_{\ell,\mathrm{seed}}, \boldsymbol{E}^L_{u,\mathrm{seed}}: \quad (d, m, p, q, t, \varepsilon) = (2, 0, 5, 1, 2, -1), (-13, -5, -29, -5, 2, -1)$$

Therefore $\boldsymbol{E}^U_{u,\mathrm{seed}} = -\boldsymbol{E}^L_{\ell,\mathrm{seed}}$ as claimed. Moreover, because $R(1,1) = (-29,-5)$ and $R(5,1) = (-5,-1)$, we find that

$$R^\sharp(\boldsymbol{E}^U_{\ell,\mathrm{seed}}) = \boldsymbol{E}^L_{u,\mathrm{seed}}, \quad R^\sharp(\boldsymbol{E}^U_{u,\mathrm{seed}}) = \boldsymbol{E}^L_{\ell,\mathrm{seed}}.$$

i	-1	0	1	2	3	4	5	6
g_i	1	1	5	29	169	985	5741	33 461
m_i		1	0	5	24	145	840	4901
d_i		1	2	13	74	433	2522	14 701

Table 1

Since R is a reflection that interchanges the ascending and descending staircases, this reversal of seeds is to be expected. Note also that the centers of the blocking classes $\boldsymbol{B}_n^L$ descend rather than ascend, so this reversal is also consistent with Definition 1.2.1 that explains how the blocking classes and seeds generate a staircase. Thus R takes the full structure of the family $\mathcal{S}^U$ to that of $\mathcal{S}^L$.

The proof of (ii) is straightforward, and is left to the reader. □

In order to show that for arbitrary $T \in \mathcal{G}$ the seeds and preblocking classes $T^\sharp(\mathcal{S}^U)$ define a staircase family, we must show that the appropriate linear relations (1.2.5) hold. This proof is largely based on analyzing the seed classes.

As already noted, modulo sign, the seeds $\boldsymbol{E}^U_{\bullet,\mathrm{seed}}$ and $\boldsymbol{E}^L_{\bullet,\mathrm{seed}}$ for the staircase families $\mathcal{S}^U$ and $\mathcal{S}^L$ are classes that appear in the third strand (the one with $i = 2$) of the staircase at $b = 1/3$; see Example 2.3.7. This strand is generated by the recursion S with action on (d, m) given by (2.3.1), and hence consists of the classes

$$\begin{gathered}\boldsymbol{E}_i := (d_i, m_i, g_i, g_{i-1}, 2, (-1)^i), \quad i \geq 0, \\ d_i = \tfrac{1}{8}(3(g_i + g_{i-1}) + 2(-1)^i), \quad m_i = \tfrac{1}{24}(3(g_i + g_{i-1}) + 18(-1)^i),\end{gathered} \tag{3.3.1}$$

with values given in Table 1. Note that $g_i = y_{i+1} - y_i$ for all i. Further, the ε coordinate alternates between the value $+1$ and -1, while $t = 2$ for all i.

Lemma 3.3.3 *For all $i \geq 0$, the families $(S^{i+1})^\sharp(\mathcal{S}^U)$ and $(S^i R)^\sharp(\mathcal{S}^U)$ have the same lower seeds and the same upper seeds; namely*

$$\boldsymbol{E}_{\ell,\mathrm{seed}}^{(S^{i+1})^\sharp(\mathcal{S}^U_\ell)} = \boldsymbol{E}_{\ell,\mathrm{seed}}^{(S^i R)^\sharp(\mathcal{S}^U_u)} = \boldsymbol{E}_{i+1}, \quad \boldsymbol{E}_{u,\mathrm{seed}}^{(S^{i+1})^\sharp(\mathcal{S}^U_u)} = \boldsymbol{E}_{u,\mathrm{seed}}^{(S^i R)^\sharp(\mathcal{S}^U_\ell)} = -\boldsymbol{E}_{i+2},$$

where $\boldsymbol{E}_i$ is as in (3.3.1). Furthermore, the extended action $R^\sharp_{v_{i+2}}$ of the reflection $R_{v_{i+2}}$ that is defined in (2.2.6) interchanges the lower and upper seeds of these families.

Proof The first statement is an easy consequence of Lemma 3.3.2(ii) and the fact that $S^\sharp(\boldsymbol{E}_i) = \boldsymbol{E}_{i+1}$ for all i.

For the second statement, note that by Corollary 2.1.4(i),

$$(R_{v_{i+2}})^\sharp = (S^{2i+1}R)^\sharp = (S^i RS^{-(i+1)})^\sharp.$$

The upper seed of $\mathcal{S}^U_{\boldsymbol{u}}$ is $-\boldsymbol{E}_1$; thus to compute the lower seed of $(R^\sharp_{v_{i+2}})((S^{i+1})^\sharp(\mathcal{S}^U_{\boldsymbol{u}}))$ we are considering

$$(S^i RS^{-(i+1)})^\sharp((S^{i+1})^\sharp(-\boldsymbol{E}_1)) = (S^i)^\sharp R^\sharp(-\boldsymbol{E}_1) = (S^i)^\sharp(\boldsymbol{E}_1) = \boldsymbol{E}_{i+1},$$

where the first equality follows from Lemma 2.2.4. The upper seed can be computed similarly. □

Recall from the discussion concerning (1.2.5) that, in order for a staircase $\mathcal{S}$ with steps $\boldsymbol{E} := (d, m, p, q, t, \varepsilon)$ to be associated to the preblocking class

$$\boldsymbol{B} = (d_{\boldsymbol{B}}, m_{\boldsymbol{B}}, p_{\boldsymbol{B}}, q_{\boldsymbol{B}}, t_{\boldsymbol{B}}, \varepsilon),$$

we require that for each step the following linear relation holds:

$$(3m_{\boldsymbol{B}} - d_{\boldsymbol{B}})d = \begin{cases} (m_{\boldsymbol{B}} - q_{\boldsymbol{B}})p + m_{\boldsymbol{B}}q & \text{if } \mathcal{S} \text{ ascends,} \\ m_{\boldsymbol{B}}p - (p_{\boldsymbol{B}} - m_{\boldsymbol{B}})q & \text{if } \mathcal{S} \text{ descends.} \end{cases} \tag{3.3.2}$$

Lemma 3.3.4 (i) *The identities in (3.3.2) may be rewritten as*

$$m_{\boldsymbol{B}}m = d_{\boldsymbol{B}}d - q_{\boldsymbol{B}}p \quad \text{if } \mathcal{S} \text{ ascends,} \tag{3.3.3}$$

$$m_{\boldsymbol{B}}m = d_{\boldsymbol{B}}d - p_{\boldsymbol{B}}q \quad \text{if } \mathcal{S} \text{ descends.} \tag{3.3.4}$$

(ii) *Given any two classes $\boldsymbol{E}'$ and $\boldsymbol{E}''$, (3.3.3) holds for the pair $(\boldsymbol{E}, \boldsymbol{B}) = (\boldsymbol{E}', \boldsymbol{E}'')$ if and only if (3.3.4) holds for the pair $(\boldsymbol{E}, \boldsymbol{B}) = (\boldsymbol{E}'', \boldsymbol{E}')$.*

(iii) *Equation (3.3.3) holds for the pair $(\boldsymbol{E}, \boldsymbol{B})$ if and only if (3.3.4) holds for the pair $(S^\sharp(\boldsymbol{E}), \boldsymbol{B})$. Further, (3.3.4) holds for the pair $(\boldsymbol{E}, \boldsymbol{B})$ if and only if (3.3.3) holds for the pair $(\boldsymbol{E}, S^\sharp(\boldsymbol{B}))$.*

(iv) *If both (3.3.3) and (3.3.4) hold for $(\boldsymbol{E}, \boldsymbol{B})$ then they both hold for $(S^\sharp(\boldsymbol{E}), S^\sharp(\boldsymbol{B}))$.*

Proof To prove (i), rewrite the term $3m_{\boldsymbol{B}}d$ on the left-hand side as $m_{\boldsymbol{B}}(p+q+m)$ and simplify.

Proving (ii) is also straightforward: if $\boldsymbol{E}' := (d', m', p', q')$ and $\boldsymbol{E}'' := (d'', m'', p'', q'')$, then both equations reduce to the claim that $m'm'' = d'd'' - p'q''$.

Now consider (iii). Let $\boldsymbol{E} = (d, m, p, q, t, \varepsilon)$, so that

$$S^\sharp(\boldsymbol{E}) = (d', m', (6p-q), p, t, -\varepsilon),$$

$$d' = \tfrac{1}{8}(3(7p-q) - \varepsilon t), \quad m' = \tfrac{1}{8}((7p-q) - 3\varepsilon t).$$

To prove the claim about the pair $(S^\sharp(\boldsymbol{E}), \boldsymbol{B})$, we want to show that the equation $m_{\boldsymbol{B}}(p+q+3t\varepsilon) = d_{\boldsymbol{B}}(3(p+q)+t\varepsilon) - 8q_{\boldsymbol{B}}\,p$ holds if and only if

$$m_{\boldsymbol{B}}((7p-q)-3\varepsilon t) = d_{\boldsymbol{B}}(3(7p-q)-\varepsilon t) - 8p_{\boldsymbol{B}}\,p$$

also holds. But by adding the two equations, we obtain the linear Diophantine identity

$$m_{\boldsymbol{B}} 8p = d_{\boldsymbol{B}} 24p - (q_{\boldsymbol{B}} + p_{\boldsymbol{B}})8p,$$

which always holds. This proves the first claim in (iii).

The second claim in (iii) now follows from the symmetry relation in (ii). We have

$$\begin{aligned}
\text{(3.3.4) holds for } (\boldsymbol{E}, \boldsymbol{B}) &\iff \text{(3.3.3) holds for } (\boldsymbol{B}, \boldsymbol{E}) && \text{(by (ii))}\\
&\iff \text{(3.3.4) holds for } (S^\sharp(\boldsymbol{B}), \boldsymbol{E})\\
&\iff \text{(3.3.3) holds for } (\boldsymbol{E}, S^\sharp(\boldsymbol{B})) && \text{(by (ii)).}
\end{aligned}$$

Thus (iii) holds, and a similar argument proves (iv). □

Proof of Theorem 1.2.4 We must show that for each $T \in \mathcal{G}$ the seeds and preblocking classes in $T^\sharp(\mathcal{S}^U)$ form a prestaircase family in the sense of Definition 1.2.1. Here, we define the preblocking classes and seeds of $T^\sharp(\mathcal{S}^U)$ to be the images of the preblocking classes and seeds in $\mathcal{S}^U$ by the formula in (2.2.6). Lemma 2.2.4 shows that these are quasiperfect classes. Further, Lemma 3.3.3 shows that the lower and upper seeds of both families $(S^{i+1})^\sharp(\mathcal{S}^U)$ and $(S^i R)^\sharp(\mathcal{S}^U)$ are $\boldsymbol{E}_{i+1}$ and $-\boldsymbol{E}_{i+2}$.

Therefore, it remains to check that each preblocking class $\boldsymbol{B}_n$ is associated both to the ascending prestaircase with seeds $\boldsymbol{E}_{\ell,\mathrm{seed}}$, $\boldsymbol{B}_{n-1}$ and recursion parameter $t_{\boldsymbol{B}_n}$, and to the descending prestaircase with seeds $\boldsymbol{E}_{u,\mathrm{seed}}$, $\boldsymbol{B}_{n+1}$ and recursion parameter $t_{\boldsymbol{B}_n}$. Thus each prestaircase must satisfy the appropriate linear relation (1.2.5). Because the steps in the staircases of a staircase family $\mathcal{F}$ with seeds $\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}$, $\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}$ and preblocking classes $\boldsymbol{B}^{\mathcal{F}}_n$ are defined recursively by Corollary 3.1.3, it suffices to check this in the cases

$$\begin{aligned}
(\boldsymbol{E}, \boldsymbol{B}) &= (\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n), (\boldsymbol{B}^{\mathcal{F}}_{n-1}, \boldsymbol{B}^{\mathcal{F}}_n), && n \geq 1, && \text{if steps } \uparrow,\ \boldsymbol{B}^{\mathcal{F}}_n \uparrow\\
&= (\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n), (\boldsymbol{B}^{\mathcal{F}}_{n+1}, \boldsymbol{B}^{\mathcal{F}}_n), && n \geq 0, && \text{if steps } \downarrow,\ \boldsymbol{B}^{\mathcal{F}}_n \uparrow,\\
(\boldsymbol{E}, \boldsymbol{B}) &= (\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n), (\boldsymbol{B}^{\mathcal{F}}_{n+1}, \boldsymbol{B}^{\mathcal{F}}_n), && n \geq 1, && \text{if steps } \uparrow,\ \boldsymbol{B}^{\mathcal{F}}_n \downarrow\\
&= (\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n), (\boldsymbol{B}^{\mathcal{F}}_{n-1}, \boldsymbol{B}^{\mathcal{F}}_n), && n \geq 0, && \text{if steps } \downarrow,\ \boldsymbol{B}^{\mathcal{F}}_n \downarrow.
\end{aligned}$$

It is shown in [1] that the prestaircases in the families $\mathcal{S}^U$ and $\mathcal{S}^L$ are staircases, and their seeds, blocking classes and steps are described in Corollaries 3.2.3 and 3.2.7.

Therefore the needed identities hold when $\mathcal{F} = \mathcal{S}^U$ (with ascending blocking classes) and $\mathcal{S}^L$ (with descending blocking classes). Thus it suffices to show that if $\mathcal{F}$ satisfies the appropriate set of identities, then its image $S^\sharp(\mathcal{F})$ does as well.

For clarity, let us first suppose that $\mathcal{F}$ has ascending blocking classes, and denote its seeds by $\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}, \boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}} = -S^\sharp(\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}})$ and its preblocking classes by $\boldsymbol{B}^{\mathcal{F}}_n$ for $n \ge 0$. Then we know that for all n,

$$(\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n) \text{ and } (\boldsymbol{B}^{\mathcal{F}}_{n-1}, \boldsymbol{B}^{\mathcal{F}}_n) \text{ satisfy (3.3.3)},$$
$$(\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n) \text{ and } (\boldsymbol{B}^{\mathcal{F}}_{n+1}, \boldsymbol{B}^{\mathcal{F}}_n) \text{ satisfy (3.3.4)}.$$

In particular, adjacent blocking classes $\boldsymbol{B}^{\mathcal{F}}_n$ and $\boldsymbol{B}^{\mathcal{F}}_{n+1}$ satisfy both relations (3.3.3) and (3.3.4). Hence Lemma 3.3.4(iv) shows that their images by $S^\sharp$ also satisfy both relations. Further, since

$$\boldsymbol{E}_\ell := S^\sharp(\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}) = \boldsymbol{E}^{S^\sharp(\mathcal{F})}_{\ell,\mathrm{seed}} = -\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}},$$

Lemma 3.3.4(iii) shows that the pair

$$(S^\sharp(\boldsymbol{E}^{\mathcal{F}}_{\ell,\mathrm{seed}}), S^\sharp(\boldsymbol{B}^{\mathcal{F}}_n)) = (-\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}, S^\sharp(\boldsymbol{B}^{\mathcal{F}}_n))$$

satisfies (3.3.3) because (3.3.4) holds for $(\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}, \boldsymbol{B}^{\mathcal{F}}_n)$. (Note that the $-$ sign in front of $\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}$ is immaterial.) In turn, applying Lemma 3.3.4(ii), we find that (3.3.3) for $(\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}, S^\sharp(\boldsymbol{B}^{\mathcal{F}}_n))$ implies (3.3.4) for $(S^\sharp(\boldsymbol{B}^{\mathcal{F}}_n), \boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}})$, and hence, by Lemma 3.3.4(iii), (3.3.3) for $(S^\sharp(\boldsymbol{B}^{\mathcal{F}}_n), S^\sharp(\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}))$ and therefore (3.3.4) for $(S^\sharp(\boldsymbol{E}^{\mathcal{F}}_{u,\mathrm{seed}}), S^\sharp(\boldsymbol{B}^{\mathcal{F}}_n))$. This completes the proof in this case.

The argument when $\mathcal{F}$ has descending blocking classes is essentially the same, and is left to the reader. Finally note that claims (i) and (ii) in the theorem follow from the known behavior of $\mathcal{S}^U$ and $\mathcal{S}^L$, and the fact that $S(v_i) = v_{i+1}$ and $S(w_i) = w_{i+1}$. □

Remark 3.3.5 (i) As the elements $T \in \mathcal{G}$ bring the seeds to seeds and the blocking classes to blocking classes by formula (2.2.6), Lemma 2.2.4(ii) implies composition behaves nicely on the staircase families, namely $T_1^\sharp T_2^\sharp(\mathcal{S}^U) = (T_1T_2)^\sharp(\mathcal{S}^U)$. This is simply because the formulas for $T^\sharp(p,q,t)$ are compatible with composition, and the degree coordinates (d,m) are determined by the values of p and q. We will see in Section 3.4 that while $T^\sharp$ acts linearly on (p,q) (namely, via products of S and R), there is no general linear map on the degree coordinates (d,m) that respects composition.

(ii) The paper [1] established the existence of a third set of staircases, there called $\mathcal{S}^E$, and showed that the centers of its blocking classes and staircase steps are the images via S of the centers of the corresponding classes in $\mathcal{S}^U$. The interested reader can check that, just as in the proofs of Propositions 3.2.2 and 3.2.6, $\mathcal{S}^E$ forms a staircase family with seeds $S^\sharp(\boldsymbol{E}^U_{\ell,\text{seed}})$, $S^\sharp(\boldsymbol{E}^U_{u,\text{seed}})$ and blocking classes $S^\sharp(\boldsymbol{E}^U_n)$. Thus $\mathcal{S}^E = S^\sharp(\mathcal{S}^U)$, and so has the same seeds as $\mathcal{S}^L$ by Lemma 3.3.2.

Now the E–family has an ascending family of blocking classes in the interval $(w_2, v_2) = (41/7, 6)$, while the L–family has a descending family of blocking classes in the interval (v_2, w_1). We showed in [1, Corollary 60] that the centers of the blocking classes and steps in $\mathcal{S}^E$ are mapped to those in $\mathcal{S}^L$ by the reflection $R_{v_2} = RS^{-1} = SR$ that fixes v_2 and interchanges $w_2 = 41/7$ and $w_1 = 7$. Moreover, by Lemma 3.3.3, $R^\sharp_{v_2}$ interchanges the seeds of these two staircase families $\mathcal{S}^E$ and $\mathcal{S}^L$. We show in Lemma 3.4.5 below that the action of $R^\sharp_{v_2}$ on the (d, m, p, q) components of those blocking classes of $\mathcal{S}^E$ and $\mathcal{S}^L$ with centers in (w_2, w_1) decomposes as the product $(R_{v_2})^*_{\boldsymbol{B}} \times R_{v_2} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2 \times \mathbb{Z}^2$ of two matrices of order 2. This is a rather special situation that, we explain in the discussion before Proposition 3.4.3, does have a nice geometric interpretation.

Remark 3.3.6 The sequence g_i is recursively generated by $g_{i+1} = 6g_i - g_{i-1}$ giving us terms in the sequence as i increases. Rewriting this as $g_{i-1} = 6g_i - g_{i+1}$, we can solve for terms in the sequence as i decreases. In particular, from the data in Table 1, we can solve for terms in this sequence with negative indices. Geometrically, we want to consider the sequence $\{g_i/g_{i-1}\}$, the center of these classes. Computing a few terms in these sequences, we get

i	-4	-3	-2	-1	0	1	2	3
g_i	169	29	5	1	1	5	29	169
g_i/g_{i-1}	$\frac{169}{985}$	$\frac{29}{169}$	$\frac{5}{29}$	$\frac{1}{5}$	1	5	$\frac{29}{5}$	$\frac{169}{29}$

Thus the sequence $\{g_i\}$ reflects on itself, namely $g_i = g_{-(i+1)}$. Note that both the centers of $\boldsymbol{E}_i$ and $-\boldsymbol{E}_i$ are elements of this sequence $\{g_i/g_{i-1}\}$ when $i \geq 0$. In particular, only terms with $i \geq 0$ appear as centers of the seed classes.

As S implements the recursion by 6, $S(g_i/g_{i-1}) = g_{i+1}/g_i$, so S always shifts this sequence one step to the right. Now, R is the reflection of this sequence about $5 = g_1/g_0$; indeed we can compute that $R(g_i/g_{i-1}) = g_{2-i}/g_{1-i}$. (Note: R is not the reflection about 1 which would map each element to its reciprocal.) Hence, R takes centers of seeds to terms in the sequence, but not necessarily to valid centers since $g_{2-i}/g_{1-i} < 1$ when $i > 2$. But applying S will shift g_{2-i}/g_{1-i} to the right. In particular, the reflection $R_{v_i} = S^{2i-3}R$ has just the right number of shifts to move the image under R of the seeds of $S^{i-1}(\mathcal{S}^U)$ to the seeds of $S^{i-2}(\mathcal{S}^L)$. In other words, this reflection interchanges the lower and upper seeds of these families. As we show in Lemma 3.4.5 it also interchanges their blocking classes.

3.4 Action of the symmetries on blocking class degree

Although the results in this section are not needed for the proof of our main results, they throw light on the geometric nature of the symmetries. To simplify the language needed, we will assume it is already known that all classes in $T^\sharp(\mathcal{S}^U)$ are perfect, a result that is proved in Proposition 4.1.4. In particular, this means that all the preblocking classes in $T^\sharp(\mathcal{S}^U)$ are in fact blocking classes, and so we will talk about blocking classes rather than preblocking classes.

The first step is to derive a formula for the action of T on the degree coordinates (d,m) of the blocking classes in $\mathcal{S}^U$. Because the formula (2.2.4) for d and m in terms of p and q is affine, we might expect this action to be affine and to depend on the t–variable (or equivalently on n). However, it turns out to be linear and independent of t and n, and in fact is given by a 2×2 matrix (with integer entries) that we denote by $T^*_{\boldsymbol{B}}$. Note that $T^*_{\boldsymbol{B}}$ gives the action on the degree coordinates of blocking classes, and in particular does not describe the action on the degrees of the staircase steps. For convenience, we denote the vector with components d and m by $(d,m)^\vee$.

Lemma 3.4.1 (i) *For each $T = S^iR^\delta \in \mathcal{G}$, there is a 2×2 matrix $T^*_{\boldsymbol{B}}$ such that for each blocking class $\boldsymbol{B}^U_n = (d_n, m_n, p_n, q_n, t, 1)$ with $4n \geq 0$, in $\mathcal{S}^U$, the corresponding blocking class in $T^\sharp(\boldsymbol{B}^U_n)$ has degree coefficients $T^*_{\boldsymbol{B}}((d_n,m_n)^\vee)$.*

(ii) *In all cases, $T^*_{\boldsymbol{B}}$ has eigenvector $(3,1)^\vee$ with eigenvalue $(-1)^{i+\delta}\det(T^*_{\boldsymbol{B}})$.*

Proof Let $T^\sharp(d,m,p,q,t,1) = (d',m',p',q',t,\varepsilon')$, where $\boldsymbol{B}^U_n = (d,m,p,q,t,1)$. By (2.2.4), $3m'-d' = t\varepsilon'$ where t is constant under $T^\sharp$ and

$$\varepsilon' t = (-1)^{i+\delta} t = (-1)^{i+\delta}(3m-d) = (-1)^{i+\delta}(p-3q), \tag{3.4.1}$$

where the last equality holds because $\boldsymbol{B}^U_n = (n+3, n+2, 2n+6, 1, 2n+3, 1)$. Therefore,

$$\begin{aligned} d' &= \tfrac18(3(p'+q')+\varepsilon' t) = \tfrac18(3(p'+q')+(-1)^{i+\delta}(p-3q)), \\ m' &= \tfrac18((p'+q')+3\varepsilon' t) = \tfrac18((p'+q')+3(-1)^{i+\delta}(p-3q)). \end{aligned} \tag{3.4.2}$$

Since p' and q' are linear functions of p and q, and the coefficients of the $\boldsymbol{B}_n^U$ satisfy $p = 2d$ and $q = d - m$, the degree coefficients d' and m' are linear functions of d and m. This proves (i).

To prove (ii), let $A = T_{\boldsymbol{B}}^* = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Because the transformation A preserves the linear function $d - 3m$ modulo the sign factor $(-1)^{i+\delta}$, the transpose matrix A^T has eigenvector $(1,-3)^\vee$ with eigenvalue $(-1)^{i+\delta}$. This implies that the transformation A preserves the subspace orthogonal to $(1,-3)^\vee$, and hence has eigenvector $(3,1)^\vee$ with some eigenvalue λ. Further, the matrices A and A^T have the same eigenvalues, λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \mathrm{Tr}(A) = a + d$. We know that one eigenvalue, say λ_1, is $(-1)^{i+\delta}$. If $\lambda_1 = \lambda = (-1)^{i+\delta}$ then the identities $A(3,1)^\vee = \lambda(3,1)^\vee$ and $A^T(1,-3)^\vee = \lambda(1,-3)^\vee$ imply that $a + d = 2\lambda$. The identity $\mathrm{Tr}(A) = a + d = 2\lambda$ shows that the other eigenvalue is also $\lambda = (-1)^{i+\delta}$. Therefore $\det(A) = 1$ and $\lambda = (-1)^{i+\delta}\det(T_{\boldsymbol{B}}^*)$ as claimed. On the other hand, if $\lambda \neq \lambda_1 = (-1)^{i+\delta}$, then $\det(T_{\boldsymbol{B}}^*) = \lambda_1\lambda_2$, so $\lambda = \lambda_2 = (-1)^{i+\delta}\det(T_{\boldsymbol{B}}^*)$ in this case as well. □

Remark 3.4.2 It would be more correct (though also more cumbersome) to denote the map $T_{\boldsymbol{B}}^*$ by $T_{U,\boldsymbol{B}}^*$ since its formula depends on the domain staircase $\mathcal{S}^U$ via the identity $3m - d = p - 3q$ in (3.4.1) used to express $\varepsilon' t$ as a function of p and q. Each staircase family has an analogous, but different, identity of this kind. For example, the blocking classes $(5n, n-1, 12n+1, 2n)$ satisfy $3m - d = -3p + 17q$. Correspondingly, the assignment $T \to T_{\boldsymbol{B}}^*$ is not compatible with composition in general, though it is in a few special cases. For example, even though the reflection R in $w_1 = 7$ has order 2, $R_{\boldsymbol{B}}^*$ does not have determinant ± 1 and $(R_{\boldsymbol{B}}^*)^2 \neq \mathrm{id}$; see Proposition 3.4.3(ii). Further, we show in Example 3.4.4 that $(S^4)_{\boldsymbol{B}}^* \neq ((S^2)_{\boldsymbol{B}}^*)^2$.

We now show that the formula for $T_{\boldsymbol{B}}^*$ does have understandable features. Observe that the blocking classes $\boldsymbol{B}_n^U$ have centers at $(p_n, q_n) = (2n+6, 1)$ and degree components (d, m),

$$(d_n, m_n) = (3+n, 2+n) = (3,2) + (1,1)n, \qquad \lim_{n\to\infty} \frac{m_n}{d_n} = 1.$$

This is no accident: we should expect the limits of the sequences (m_n/d_n) and (p_n/q_n) to correspond via the function acc_U^{-1}. However, this is a degenerate case since both limits lie on the boundary of the domain of this function. Let us apply the same analysis to the staircase families $\mathcal{S}^L = R^\sharp(\mathcal{S}^U)$ and $\mathcal{S}^E = S^\sharp(\mathcal{S}^U)$. As illustrated in Figures 2 and 3, both these families have steps in the interval $(w_2, w_1) = (41/7, 7)$. Further, the centers of the blocking classes in $\mathcal{S}^E$ increase with limit $v_2 = 6$, while those of $\mathcal{S}^L$ decrease, also with limit v_2. Thus, in both cases, the ratios (m_n/d_n) of the degree coefficients of the blocking classes converge to $\mathrm{acc}_L^{-1}(6) = 1/5$. Correspondingly, we show below that in both cases $T = R, S$ the matrix $T^*_{\boldsymbol{B}}$ takes the vector $(1,1)^\vee$ that gives the coefficient of n in (d,m) to $(5,1)^\vee$. This observation generalizes as follows.

Proposition 3.4.3 (i) *The matrix $(S^i)^*_{\boldsymbol{B}}$ has integer entries and is determined by the following properties*:

- $(S^i)^*_{\boldsymbol{B}}((3,2)^\vee) = \frac{1}{8}\big(3(y_{i+2}+y_{i+1}) + 3(-1)^i, y_{i+2}+y_{i+1}+9(-1)^i)^\vee\big)$.
- $(S^i)^*_{\boldsymbol{B}}((1,1)^\vee) = ((s_i, r_i)^\vee)$, *where*

$$s_i = \tfrac{1}{4}(3y_{i+1} + 3y_i + (-1)^i), \quad r_i = \tfrac{1}{4}(y_{i+1}+y_i+3(-1)^i). \tag{3.4.3}$$

In particular $r_i/s_i = \mathrm{acc}_\varepsilon^{-1}(v_{i+1})$, *where* $\varepsilon = (-1)^i$ *and we define*

$$\mathrm{acc}_\varepsilon^{-1} := \begin{cases} \mathrm{acc}_U^{-1} & \textit{if } \varepsilon = +1, \\ \mathrm{acc}_L^{-1} & \textit{if } \varepsilon = -1. \end{cases}$$

Further, $\det((S^i)^*_{\boldsymbol{B}}) = (-1)^i(y_{i+1}-y_i)$.

(ii) *When $T = R$ we have* $R^*_{\boldsymbol{B}} = \begin{pmatrix} -10 & 15 \\ -3 & 4 \end{pmatrix}$.

(iii) *The matrix $(S^iR)^*_{\boldsymbol{B}}$ has integer entries and is determined by the following properties*:

- $(S^iR)^*_{\boldsymbol{B}}((3,2)^\vee) = \frac{1}{8}\big(3(y_{i+1}+y_i) - 3(-1)^i, y_{i+1}+y_i-9(-1)^i)^\vee\big)$.
- $(S^iR)^*_{\boldsymbol{B}}((1,1)^\vee) = ((s_{i+1}, r_{i+1})^\vee)$, *where s_{i+1} and r_{i+1} are as in* (3.4.3).

Further, $\det((S^iR)^*_{\boldsymbol{B}}) = (-1)^i(y_{i+2}-y_{i+1})$.

Proof The matrix $(S^i)^*_{\boldsymbol{B}}$ is obviously determined by the images of the vectors $(3,2)^\vee$ and $(1,1)^\vee$, and we first check that these images are as stated. Recall the sequence

$$y_0, y_1, y_2, y_3, \ldots = 0, 1, 6, 35, \ldots, \quad \text{where } S(y_i, y_{i-1}) = (y_{i+1}, y_i),$$

and write

$$(p_n, q_n)^\vee = (6,1)^\vee + 2n(1,0)^\vee = (y_2, y_1)^\vee + 2n(y_1, y_0)^\vee.$$

Then

$$S^i((p_n, q_n)^\vee) = (y_{i+2}, y_{i+1})^\vee + 2n(y_{i+1}, y_i)^\vee,$$

and because $t = 2n+3$ is fixed by S, we can derive the formula for $(S^i)^*_{\boldsymbol{B}}((3,2)^\vee)$ and $(S^i)^*_{\boldsymbol{B}}((1,1)^\vee)$ by looking at the constant term and coefficient of n in (2.2.6). Thus, if X is the matrix with columns $(3,2)^\vee$ and $(1,1)^\vee$, we have $(S^i)^*_{\boldsymbol{B}}\, X = \frac{1}{8}A$, where

$$A = \begin{pmatrix} 3w_{i+1} + 3\varepsilon & 6w_i + 2\varepsilon \\ w_{i+1} + 9\varepsilon & 2w_i + 6\varepsilon \end{pmatrix}, \quad w_i = y_{i+1} + y_i, \quad \varepsilon := (-1)^i.$$

It follows from Lemma 2.2.4 that the entries of $(S^i)^*_{\boldsymbol{B}}\, X$ are integers. Hence, because the matrix X with columns $(3,2)^\vee$ and $(1,1)^\vee$ has determinant $+1$, the entries of $(S^i)^*_{\boldsymbol{B}}$ are also integers. Further,

$$\begin{aligned} \det A &= \varepsilon(18w_{i+1} + 6w_i - 54w_i - 2w_{i+1}) \\ &= 16\varepsilon(y_{i+2} + y_{i+1} - 3(y_{i+1} + y_i)) \\ &= 64\varepsilon(y_{i+1} - y_i), \end{aligned}$$

where the last equality holds because $y_{i+2} = 6y_{i+1} - y_i$. Therefore $\det((S^i)^*_{\boldsymbol{B}})$ is as claimed. This proves (i).

Claim (ii) follows from the fact that $R^*_{\boldsymbol{B}}$ takes the degree components $(4,3)$ and $(5,4)$ of $\boldsymbol{B}^U_1$ and $\boldsymbol{B}^U_2$ to the corresponding components $(5,0)$ and $(10,1)$ of $\boldsymbol{B}^L_1$ and $\boldsymbol{B}^L_2$; see (3.2.5). Note that

$$R^*_{\boldsymbol{B}} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 15 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -1 & 1 \end{pmatrix}.$$

Finally, (iii) follows by arguing as in (i), noting that R interchanges the pairs $(6,1) = (y_2, y_1)$ and $(1,0) = (y_1, y_0)$. Therefore

$$S^i R((6,1)^\vee)) = ((y_{i+1}, y_i)^\vee), \quad S^i R((1,0)^\vee) = ((y_{i+2}, y_{i+1})^\vee),$$

so, as before, (2.2.6) implies that the action on the corresponding degree coordinates $(3,2)^\vee$ and $(1,1)^\vee$ is

$$\begin{aligned} (S^i R)^*_{\boldsymbol{B}}((3,2)^\vee) &= \tfrac{1}{8}\big((3(y_{i+1} + y_i) + 3(-1)^{i+1}, y_{i+1} + y_i + 9(-1)^{i+1}))^\vee\big), \\ (S^i R)^*_{\boldsymbol{B}}((1,1)^\vee) &= \tfrac{1}{4}\big((3(y_{i+2} + y_{i+1}) + (-1)^{i+1}, y_{i+2} + y_{i+1} + 3(-1)^{i+1}))^\vee\big). \end{aligned}$$

These formulas are consistent with the fact that by Proposition 3.2.6 the blocking classes $\boldsymbol{B}^L_0 = R^\#(\boldsymbol{B}^U_0)$ and $\boldsymbol{B}^L_1 = R^\#(\boldsymbol{B}^U_1)$, are $(0,-1,1,0)$ and $(5,0,13,2)$. The determinant calculation can be checked as before. □

Example 3.4.4 As examples of the above formulas, we have

$$S^*_{\boldsymbol{B}} = \begin{pmatrix} 5 & 0 \\ 2 & -1 \end{pmatrix}, \quad (S^2)^*_{\boldsymbol{B}} = \begin{pmatrix} 28 & 3 \\ 9 & 2 \end{pmatrix}, \quad (S^3)^*_{\boldsymbol{B}} = \begin{pmatrix} 164 & 15 \\ 55 & 4 \end{pmatrix}, \quad (S^4)^*_{\boldsymbol{B}} = \begin{pmatrix} 955 & 90 \\ 318 & 31 \end{pmatrix}.$$

Note that the second column of these matrices coincides with the degree components of a corresponding principal[15] blocking class $(S^{i-2})^*(\boldsymbol{B}_0^U)$. However, these matrices do *not* give the action of S^i on the seed classes, even when i is even. For example, the lower seeds of $\mathcal{S}^U$ and $(S^2)^\sharp(\mathcal{S}^U)$ are

$$\boldsymbol{E}^U_{\ell,\text{seed}} = (1,1,1,1,2,1), \quad \boldsymbol{E}^{(S^2)^\sharp(\mathcal{S}^U)}_{\ell,\text{seed}} = (13,5,29,5,2,1).$$

Because these matrices $(S^i)^*_{\boldsymbol{B}}$ do not give the action on the seeds, they also do not act on the degrees of the staircase steps.

We could compute similar matrices $(T)^*_{\bullet,\text{seed}}$ that would take the degrees of $\boldsymbol{E}^U_{\bullet,\text{seed}}$ to the degrees of $T^\sharp(\boldsymbol{E}^U_{\bullet,\text{seed}})$. However, as above, $(T)^*_{\bullet,\text{seed}}$ would also not respect composition because there is no analog of the identity $t = 3m-d = ap+bq$ in (3.4.1) for the seeds. (This is easy to check using the fact that $t = 2$ by (2.3.1).) Thus we only present the formulas for the degree coordinates generated by the recursion S by (3.3.1); also see Lemma 3.3.3 and Remark 3.3.6.

By Lemma 2.1.3 and Corollary 2.1.4 we can also write

$$S^{i-1}R = (S^{i-1}RS^{-i})S^i = R_{v_{i+1}}S^i,$$

where $R_{v_{i+1}}$ is a reflection that fixes the point v_{i+1} and interchanges the steps of the two staircase families $(S^i)^\sharp(\mathcal{S}^U)$ and $(S^{i-1}R)^\sharp(\mathcal{S}^U)$ with steps in the short interval (w_{i+1}, w_i) around v_{i+1}; see Figure 2. Let us denote by $(R_{v_{i+1}})^*_{\boldsymbol{B}}$ the matrix that takes the blocking class degrees in $(S^{i-1}R)^\sharp(\mathcal{S}^U)$ to those of $(S^i)^\sharp(\mathcal{S}^U)$; see Figure 3. Note that to be consistent with our interpretation of R that takes $\mathcal{S}^U$ to $\mathcal{S}^L$ we take the p/q coordinates of the domain staircase family of this reflection to lie *above* the fixed point.[16] Further, when i is even (resp. odd), $(R_{v_{i+1}})^*_{\boldsymbol{B}}$ acts on blocking classes of staircases with $b_\infty > 1/3$ (resp. $b_\infty < 1/3$). For example, $(R_{v_2})^*_{\boldsymbol{B}}$ takes the blocking classes of $\mathcal{S}^L$ to those of $\mathcal{S}^E = S^\sharp(\mathcal{S}^U)$.

Lemma 3.4.5 *The matrix*

$$(R_{v_{i+1}})^*_{\boldsymbol{B}} := (S^i)^*_{\boldsymbol{B}} \circ ((S^{i-1}R)^*_{\boldsymbol{B}})^{-1}$$

has eigenvectors $(s_i, r_i)^\vee$ *and* $(3,1)^\vee$ *with corresponding eigenvalues* 1 *and* -1. *Thus it has order two, and hence also takes the degree components of the blocking classes in* $(S^i)^\sharp(\mathcal{S}^U)$ *to those of* $(S^{i-1}R)^\sharp(\mathcal{S}^U)$.

[15] See Lemma 3.4.6; these are well defined for $i-2 \geq 0$ and need appropriate interpretation when $i = 1$.

[16] This choice is relevant because in general the degree component $T^*_{\boldsymbol{B}}$ of a reflection $T^\sharp$ does not have order two; see Lemma 3.4.6.

Proof Proposition 3.4.3 shows that

$$(S^{i-1}R)^*_{\boldsymbol{B}}((1,1)^\vee) = (s_i, r_i) = (S^i)^*_{\boldsymbol{B}}((1,1)^\vee)$$

and $\det((S^{i-1}R)^*) = -\det((S^i)^*)$. The result now follows from Lemma 3.4.1(ii). □

There turns out to be a second natural action of the reflections R_{v_j} for $j \ge 0$ on blocking class degree corresponding to its action on the other component of $[0, 1/3) \cup (1/3, \infty)$. Indeed, as illustrated in Figure 3 and Remark 1.2.5(i), this reflection acts in two ways on our staircases depending on whether one takes $b > 1/3$ or $b < 1/3$. The action discussed above (with $j = i + 2 \ge 2$) interchanges the blocking classes of the two staircase families $(S^{i+1})^\sharp(\mathcal{S}^U)$ and $(S^i R)^\sharp(\mathcal{S}^U) = (S^i)^\sharp(\mathcal{S}^L)$, the centers of whose blocking classes converge to v_{i+2} while, as we will see, the second action of $R_{v_{i+2}}$ fixes the center v_{i+2} of the blocking class $(S^i)^\sharp(\boldsymbol{B}^U_0)$ and takes the ascending blocking classes in $(S^i)^\sharp(\mathcal{S}^U)$ with centers in $[w_{i+1}, v_{i+1}]$ to the descending blocking classes in $(S^{i+1}R)^\sharp(\mathcal{S}^U) = (S^{i+1})^\sharp(\mathcal{S}^L)$ with centers in $[v_{i+3}, w_{i+2}]$. Note, when i is odd (resp. even), the principal blocking class $(S^i)^\sharp(\boldsymbol{B}^U_0)$ blocks a b–region with $b < 1/3$ (resp. $b > 1/3$).

We will denote the matrix that gives this second action on degree by $(R_{v_{i+1}})^*_{\boldsymbol{P}}$. It is again defined by its action on relevant blocking classes, but the family used is different than before.

Notice that the matrix $(R_{v_{i+1}})^*_{\boldsymbol{P}}$ obtained this way does not have order two, so that it matters that we define it via (2.2.6) applied to its action on the blocking classes in $(S^i)^\sharp(\mathcal{S}^U)$. Besides the eigenvector $(3,1)^\vee$, its second eigenvector (with eigenvalue 1) must be given by the degree components of the class $(S^i)^\sharp(\boldsymbol{B}^U_0)$. Note also that the resulting matrix $(R_{v_{i+2}})^*_{\boldsymbol{P}}$ is not in general integral.

We call the blocking classes $(S^i)^\sharp(\boldsymbol{B}^U_0)$, $i \ge 0$, *principal blocking classes*. The next result spells out their main properties.

Lemma 3.4.6 (i) *The principal blocking class $(S^i)^\sharp(\boldsymbol{B}^U_0)$ has components*

$$\left(\tfrac{3}{8}(y_{i+2} + y_{i+1} + (-1)^i), \tfrac{1}{8}(y_{i+2} + y_{i+1} + 9(-1)^i), y_{i+2}, y_{i+1}, 3, (-1)^i\right).$$

(ii) *The transformation $(R_{v_{i+2}})^\sharp_{\boldsymbol{P}} : (S^i)^\sharp(\mathcal{S}^U) \to (S^{i+1})^\sharp(\mathcal{S}^L)$ acts on the degrees of the blocking classes by the matrix*

$$(R_{v_{i+2}})^*_{\boldsymbol{P}} := (S^{i+1}R)^*_{\boldsymbol{B}}((S^i)^*_{\boldsymbol{B}})^{-1}$$

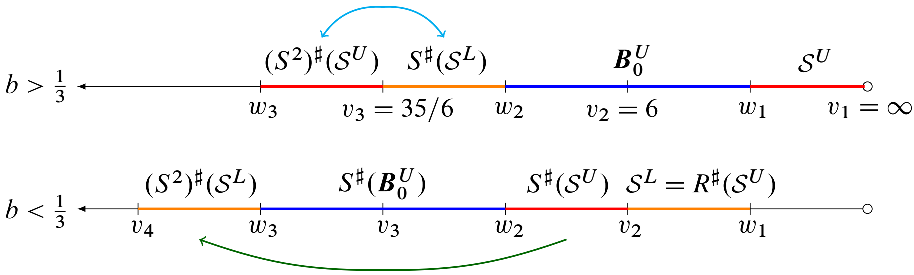

Figure 3: Here we illustrate the different actions of the reflection R_{v_3} that acts on the z axis by fixing v_3. The reflection $(R_{v_3})^\sharp$, represented by the light blue arrow, interchanges the upper and lower staircase families of $(S^2)^\sharp(\mathcal{S}^U)$ and $S^\sharp(\mathcal{S}^L)$, and the corresponding matrix $(R_{v_3})^*_{\boldsymbol{B}}$ acts as a reflection on the degrees of the blocking classes associated to these staircases. On the other hand, on the lower number line, we have the principal blocking class $S^\sharp(\boldsymbol{B}_0^U)$ with center v_3 blocking the corresponding blue interval with $b < 1/3$. On either side of this blocked region live the centers of the blocking classes of $(S^2)^\sharp(\mathcal{S}^L)$ and $S^\sharp(\mathcal{S}^U)$. The matrix $(R_{v_3})^*_{\boldsymbol{P}}$ maps the degree coordinates of $S^\sharp(\mathcal{S}^U)$ to those of $(S^2)^\sharp(\mathcal{S}^L)$. This is represented by the green arrow. Note that this is not a reflection, so the arrow only goes in one direction.

with determinant given by $-(y_{i+3}-y_{i+2})/(y_{i+2}-y_{i+1})$. *Its eigenvalues are* $\det((R_{v_{i+2}})^*_{\boldsymbol{P}})$ *and* 1, *with corresponding eigenvectors given by* $(3,1)^\vee$ *and the degree components of* $(S^i)^\sharp(\boldsymbol{B}_0^U)$.

Proof The formula for the degree components of the principal blocking class $\boldsymbol{B} := (S^i)^\sharp(\boldsymbol{B}_0^U)$ follows from (2.2.6). The claim in (ii) follows from Proposition 3.4.3. □

Example 3.4.7 The reflection R_{v_2} that fixes 6 has two extensions to an action on blocking classes,

$$(R_{v_2})^*_{\boldsymbol{B}} = \begin{pmatrix} 4 & -15 \\ 1 & -4 \end{pmatrix}, \quad (R_{v_2})^*_{\boldsymbol{P}} = \begin{pmatrix} -59 & 90 \\ -20 & 31 \end{pmatrix}. \tag{3.4.4}$$

Here $(R_{v_2})^*_{\boldsymbol{B}}$, with determinant -1, interchanges the blocking classes of $\mathcal{S}^E = S(\mathcal{S}^U)$ with those of $\mathcal{S}^L$. Both of these families have $b_\infty < 1/3$ and their accumulation points limit to v_2. On the other hand, $(R_{v_2})^*_{\boldsymbol{P}}$, with determinant -29, fixes the degree components $(3,2)$ of $\boldsymbol{B}_0^U$ and takes the blocking classes of $\mathcal{S}^U$ to those of $(S^2R)^\sharp(\mathcal{S}^U)$. Both of these staircases have $b_\infty > 1/3$ and lie on different sides of the interval blocked by the principal blocking class B_0^U; see Figure 3. Thus $(R_{v_2})^*_{\boldsymbol{B}} = (S)^*_{\boldsymbol{B}} \circ ((R)^*_{\boldsymbol{B}})^{-1}$, and one can check that $(R_{v_2})^*_{\boldsymbol{P}} = (SR)^*_{\boldsymbol{B}}$.

4 The prestaircases are live

This section completes the proof of Theorem 1.2.6. We first show that the prestaircase families defined in Section 3 are perfect prestaircase families. Establishing that all staircase classes are perfect implies that each such class $\boldsymbol{E}_\kappa = (d_\kappa, m_\kappa, p_\kappa, q_\kappa, t_\kappa, \varepsilon)$ is live at its center p_κ/q_κ when $b = m_\kappa/d_\kappa$. Therefore, by (1.1.3), the capacity function $c_{H_{m_\kappa/d_\kappa}}$ takes the value $p_\kappa d_\kappa/(d_\kappa^2 - m_\kappa^2)$ at the point p_κ/q_κ, and this implies by continuity[17] that the limiting value b_∞ is unobstructed, ie $c_{H_{b_\infty}}(a_\infty) = V_{b_\infty}(a_\infty)$, where $a_\infty = \lim(p_\kappa/q_\kappa)$.

What we have to show is that, at least for sufficiently large κ, the class $\boldsymbol{E}_\kappa$ remains live at the limiting b value $b_\infty := \lim m_k/d_k$. In [1] we established this in two steps, first showing that if the ratios m_κ/d_κ satisfy a bound such as that in (4.2.1) below then, by the positivity of the intersections of exceptional classes, the degree of any class $\boldsymbol{E}$ such that $\mu_{\boldsymbol{E},b_\infty}(p_\kappa/q_\kappa) \geq \mu_{\boldsymbol{E}_\kappa,b_\infty}(p_\kappa/q_\kappa)$ is bounded above by a constant that is independent of κ. This means that there can be only finitely many such classes. In particular, there must be one class $\boldsymbol{E}_{\mathrm{ov}}$ that dominates infinitely many of the steps. This is possible only if the obstruction $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}(z)$ given by this class goes through the accumulation point $(a_\infty, V_{b_\infty}(a_\infty))$ of the staircase.[18] We call such a class an *overshadowing class* because its obstruction $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}(z)$ overshadows the staircase steps so that they cease to be visible at $b = b_\infty$.

In [1] we were able to find rather good bounds for the degree of a potentially overshadowing class, and hence could show that they do not exist by a case by case analysis. This method is not feasible here since the bounds on the degree of any overshadowing class of a staircase in the family $(S^i R^\delta)^\sharp(\mathcal{S}^U)$ increase too rapidly with i. However it turns out that we can exploit the fact that the obstruction $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}(z)$ goes through the accumulation point to obtain powerful arithmetic information about the degree components d_{ov} and m_{ov} of $\boldsymbol{E}_{\mathrm{ov}}$, which is enough to rule out the existence of such a class. This argument hinges on the results of Lemma 2.2.7, namely, that the following two functions are the same:

$$b \mapsto V_b(\mathrm{acc}(b)), \quad b \mapsto \frac{1 + \mathrm{acc}(b)}{3 - b} \tag{4.0.1}$$

[17] This holds because, for any quasiperfect class $\boldsymbol{E} = (d, m, p, q)$, we have that $\mu_{\boldsymbol{E},b}(p/q) \leq V_b(p/q)(\sqrt{1 + 1/(d^2 - m^2)})$ and here $d_\kappa \to \infty$; see [1, Lemma 15].

[18] Indeed, we know that the accumulation point is unobstructed, so $\boldsymbol{E}_{\mathrm{ov}}$ cannot be obstructive at this point, and if (for a descending staircase) the obstruction $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}$ crossed the volume curve to the right of the accumulation point then it could at best overshadow only a finite number of steps.

4.1 The prestaircase classes are perfect

We first prove that all the classes in the families $T^\sharp(\mathcal{S}^U)$ are perfect, that is, they are exceptional classes, and then use this fact in Corollary 4.1.5 to gain information about the z–intervals that are blocked by the blocking classes.

We use the following recognition principle, which is explained for example in [6, Proposition 1.2.12].

Lemma 4.1.1 *An integral class*

$$\boldsymbol{E} := dL - \sum_{i=1}^{N} n_i E_i$$

in the N–fold blowup $\mathbb{C}P^2 \# N\overline{\mathbb{C}P}^2$ represents an exceptional divisor if and only if it may be reduced to E_1 by repeated application of Cremona transformations.

Here, a *Cremona transformation* is a composition of the transformation

$$c_{xyz}\Bigl(dL - \sum_{i=1} n_i E_i\Bigr) = (d + \delta_{xyz})L - \sum_{i\in\{x,y,z\}} (n_i + \delta_{x,y,z})E_i - \sum_{i\notin\{x,y,z\}} n_i E_i,$$

where $\delta_{xyz} = d - n_x - n_y - n_z$, and a reordering operation. Writing $\boldsymbol{E}$ in coordinates $(d; n_1, \dots, n_N)$, c_{xyz} adds δ_{xyz} to the coordinates d, n_x, n_y and n_z, and the reordering can reorder any of the n_i. Because Cremona moves are reversible, to verify $\boldsymbol{E}_k$ is exceptional, we just need to show it reduces to some other $\boldsymbol{E}_j$ that we know to be exceptional. We say $\boldsymbol{E}_k$ and $\boldsymbol{E}_j$ are *Cremona equivalent* if one can be reduced to the other.

Our staircase classes are quasiperfect and hence have the form $(d, m, W(p/q))$ where $W(p/q) = (W_1, \dots, W_N)$ is the integral weight expansion of p/q. We denote such a class by the tuple (d, m, p, q), and consider the Cremona moves as acting on the corresponding sequence $(d, m, W_1, W_2, W_3, \dots)$. Thus, for example,

$$\begin{aligned} &c_{012}((d, m, W_1, W_2, W_3, \dots)) \\ &\quad = (2d - m - W_1 - W_2, d - (W_1 + W_2), d - (m + W_2), d - (m + W_1), W_3, \dots). \end{aligned}$$

Here is the key lemma.

Lemma 4.1.2 *Suppose (d, m, p, q) with $p/q > 5$ satisfies $3d = m + p + q$, and that d and m are defined by (2.2.4). Then (d, m, p, q) is Cremona equivalent to $S^\sharp(d, m, p, q)$.*

Proof Let $S^\sharp(d,m,p,q,\varepsilon)=(D,M,P,Q,-\varepsilon)$. By Lemma 2.1.6, when $p/q=[5+k,CF(x)]$ we have $S(p/q)=[5;1,4+k,CF(x)]$. Thus, the integral weight expansion of $S(p/q)$ will always have five more integers than the integral weight expansion of p/q. As $S(p/q)=P/Q=(6p-q)/p$, we have that

$$\begin{aligned} W(P/Q)=W((6p-q)/p)&=(p^{\times 5},p-q)\sqcup W((p-q)/q)\\ &=(Q^{\times 5},P-5Q)\sqcup W((p-q)/q), \end{aligned}$$

where $W((p-q)/q)$ by definition is $W(p/q)$ with the first entry q removed. Thus, to reduce the class (D,M,P,Q) to the class (d,m,p,q), it suffices to show that we can reduce

$$(D,M,Q^{\times 5},P-5Q)\sqcup W((p-q)/q)\quad\text{to}\quad (d,m,q)\sqcup W((p-q)/q).$$

Note that in this reduction we must get rid of five terms because as mentioned $W(S(p/q))$ has five more terms than $W(p/q)$.

Next, observe that

$$\begin{aligned} &c_{256}c_{234}c_{016}c_{345}c_{012}(D;Q^{\times 5},M,P-5Q)\\ &\qquad=(8D-3(M+P);0^{\times 2},3D-M-2P+5Q,0^{\times 3},3D-2M-P). \end{aligned}$$

This can be seen by direct computation, where the zeros come from the linear Diophantine condition $3D-M-P-Q=0$. The first three steps give the two zeros in positions 1 and 2, the fourth step results in the two zeros in positions 4 and 5, and the fifth step results in one zero in position 6. Furthermore, since $P=6p-q$ and $Q=p$, it follows from (2.2.4) that

$$D=\tfrac{1}{8}(3(7p-q)-\varepsilon t),\quad M=\tfrac{1}{8}((7p-q)-3\varepsilon t).$$

Performing these substitutions, we get

$$\begin{aligned} 8D-3(M+P)&=\tfrac{1}{8}(3(p+q)+\varepsilon t)=d,\\ 3D-M-2P+5Q&=q,\\ 3D-2M-P&=\tfrac{1}{8}(p+q+3\varepsilon t)=m. \end{aligned}$$

We conclude that

$$c_{256}c_{234}c_{016}c_{345}c_{012}(D;Q^{\times 5},M,P-5Q)=(d;0^{\times 2},q,0^{\times 3},m).$$

Thus these five Cremona moves and an appropriate reordering reduces

$$(D,M,Q^{\times 5},P-5Q)\sqcup W((p-q)/q)\quad\text{to}\quad (d,m,q)\sqcup W((p-q)/q).$$

Hence, the class (D,M,P,Q) is Cremona equivalent to (d,m,p,q), as claimed. □

In [1], it was shown that both $\mathcal{S}^U$ and $\mathcal{S}^L$ are perfect, so it is enough to show that $S^\sharp$ preserves Cremona equivalence, but there is an equally nice argument that $R^\sharp$ preserves Cremona equivalence.

Lemma 4.1.3 *Suppose (d,m,p,q) are such that d and m are defined by (2.2.4), and $p/q>7$. Then (d,m,p,q) is Cremona equivalent to $R^\sharp(d,m,p,q)$.*

Proof Let $R^\sharp(d,m,p,q)=(D,M,P,Q)$. Assume $p/q=[6+k;CF(x)]$ for some $k\geq 1$ and $x\geq 1$. Then

$$W(p/q)=(q^{6+k},p-(6+k)q,\dots).$$

By Lemma 2.1.6(ii), $R(p/q)=(6p-35q)/(p-6q)=[6,k,CF(x)]$, and thus

$$W(P/Q)=(Q^{\times 6},P-6Q^{\times k},Q-k(P-6Q),\dots)=(p-6q^{\times 6},q^{\times k},p-(6+k)q,\dots).$$

Only the first 6 terms of the weight expansions $W(P/Q)$ and $W(p/q)$ differ. Thus, to show that (D,M,P,Q) is Cremona equivalent to (d,m,p,q), we need to consider the degree coordinates and the first 6 terms of the weight sequence for each.

We use the notation $\mapsto_{ijk}$ to represent applying c_{ijk} to the previous tuple. Applying two Cremona moves to (D,M,P,Q) gives

$$\begin{aligned}(D,M,Q^{\times 6})&\mapsto_{123}(2D-3Q,M,D-2Q^{\times 3},Q^{\times 3})\\&\mapsto_{456}(4D-9Q,M,D-2Q^{\times 3},2D-5Q^{\times 3}).\end{aligned}\tag{4.1.1}$$

Applying three Cremona moves to (d,m,p,q) gives:

$$\begin{aligned}(d,m,q^{\times 6})&\mapsto_{012}(2d-m-2q,d-2q,d-m-q^{\times 2},q^{\times 4})\\&\mapsto_{034}(3d-2(m+2q),2d-m-4q,d-m-q^{\times 4},q^{\times 2})\\&\mapsto_{156}(5d-3(m+3q),2d-m-4q,3d-2(m+3q),d-m-q^{\times 3},2d-m-4q^{\times 2}),\end{aligned}$$

which we can reorder to get

$$(5d-3(m+3q),3d-2(m+3q),2d-m-4q^{\times 3},d-m-q^{\times 3}).\tag{4.1.2}$$

We claim this reordered tuple is precisely (4.1.1). To see this, we write each term in each of the tuples in terms of p and q. We are going to assume that for (d,m,p,q), $\varepsilon=1$. By (2.2.4), Lemma 2.1.6(iii), and the definition of R, we can write (d,m,p,q) and (D,M,P,Q) completely in terms of p and q,

$$\begin{aligned}(d,m,p,q)&=\bigl(\tfrac{1}{8}(3(p+q)+t),\tfrac{1}{8}(p+q+3t),p,q\bigr),\\(D,M,P,Q)&=\bigl(\tfrac{1}{8}(21p-123q-t),\tfrac{1}{8}(7p-41q-3t),6p-35q,p-6q\bigr).\end{aligned}$$

Now, we expand all of the entries in (4.1.2) and (4.1.1) in terms of p and q to get the equalities

$$\begin{aligned} 5d-3(m+3q) &= \tfrac{1}{2}(3p-15q-t) = 4D-9Q, \\ 3d-2(m+3q) &= \tfrac{1}{8}(7p-41q-3t) = M, \\ 2d-m-4q &= \tfrac{1}{8}(5p-27q-t) = D-2Q, \\ d-m-q &= \tfrac{1}{4}(p-3q-t) = 2D-5Q. \end{aligned}$$ □

Proposition 4.1.4 *For each $T \in \mathcal{G}$, the classes in the prestaircase family $T^\sharp(\mathcal{S}^U)$ are perfect, that is, they are exceptional classes.*

Proof By [1, Section 3.4], this holds for $\mathcal{S}^U$ and $\mathcal{S}^L$. Hence this is an immediate consequence of Lemma 4.1.2. □

Here is a typical corollary. For simplicity we only consider the principal blocking classes mentioned in Lemma 3.4.6, but a similar argument applies to all blocking classes that have associated perfect staircases.

Corollary 4.1.5 *For each $i \geq 0$, the z–interval blocked by the principal blocking class $(S^i)^\sharp(\boldsymbol{B}_0^U)$ contains $[w_{i+2}, w_{i+1}]$.*

Proof It follows from Remark 2.3.4(i) and Theorem 1.2.4 that the prestaircases $(S^{i+1}R)^\sharp(\mathcal{S}_0^U)$, $(S^i)^\sharp(\mathcal{S}_0^U)$ are associated to $\boldsymbol{B} = (S^i)^\sharp(\boldsymbol{B}_0^U)$. Because they consist of perfect classes, we know from [1, Lemma 27 and Theorem 52] that their z–limit points $\alpha_{\boldsymbol{B},\ell}$ and $\alpha_{\boldsymbol{B},u}$ are unobstructed and that the interval $(\alpha_{\boldsymbol{B},\ell}, \alpha_{\boldsymbol{B},u})$ lying between them is precisely the z–interval blocked by $\boldsymbol{B}$. Thus it suffices to show that

$$\alpha_{\boldsymbol{B},\ell} < w_{i+2} < w_{i+1} < \alpha_{\boldsymbol{B},u}.$$

Since S preserves order and takes w_i to w_{i+1} for all i, we only need check this for $i = 0$, and this can be done either by direct evaluation or by comparing the continued fraction expansions of these quantities. □

4.2 Recognizing staircases

We proved the following staircase recognition theorem in [1, Theorem 51].

Theorem 4.2.1 *Let $\mathcal{S} = (\boldsymbol{E}_\kappa)$ be a perfect prestaircase, let λ be as in Lemma 3.1.4, and denote by D, M, P, Q the constants X defined by (3.1.6), where $x_\kappa = d_\kappa, m_\kappa, p_\kappa, q_\kappa$ respectively. Suppose in addition that at least one of the following conditions holds:*

(i) *There is an* $r/s > 0$ *such that* $M/D < r/s$,

$$\frac{m_\kappa^2 - 1}{d_\kappa m_\kappa} < \frac{M}{D} < \frac{s + m_\kappa(rd_\kappa - sm_\kappa)}{r + d_\kappa(rd_\kappa - sm_\kappa)} \quad \text{for all } \kappa \geq \kappa_0,$$

and there is no overshadowing class at $(z, b) = (P/Q, M/D)$ *of degree* $d' < s/(r - sb_\infty)$ *and with* $m'/d' > r/s$.

(ii) *There is an* $r/s > 0$ *such that* $M/D > r/s$,

$$\frac{m_\kappa(sm_\kappa - rd_\kappa) - s}{d_\kappa(sm_\kappa - rd_\kappa) - r} < \frac{M}{D} < \frac{m_\kappa}{d_\kappa} \quad \text{for all } \kappa \geq \kappa_0,$$

and there is no overshadowing class at $(z, b) = (P/Q, M/D)$ *of degree* $d' < s/(sb_\infty - r)$ *and with* $m'_\kappa/d' < r/s$.

Then $\mathcal{S}$ *is live, and it is a staircase for* $H_{M/D}$ *that accumulates at* P/Q.

The next result states the estimates that we must establish in order to apply the above theorem.

Lemma 4.2.2 *Consider a prestaircase with classes* $(d_\kappa, m_\kappa; q_\kappa \boldsymbol{w}(p_\kappa/q_\kappa))$, *where the ratios* $b_\kappa := m_\kappa/d_\kappa$ *have limit* b_∞, *and let the constants* D, D', D'', M, M', M'' *and* σ *be as in* (3.1.6), *with* $x_\kappa = d_\kappa, m_\kappa$ *respectively.*

(i) *Suppose that* $M\overline{D} - \overline{M}D \neq 0$. *Then the* b_κ *are strictly increasing if and only if*

$$M\overline{D} - \overline{M}D = 2\sqrt{\sigma}(M''D' - M'D'') = \frac{m_1 d_0 - m_0 d_1}{\sqrt{\sigma}} > 0,$$

and otherwise they are strictly decreasing.

(ii) *If* $m_1 d_0 - d_1 m_0 > 0$, $b_\infty < r/s \leq 1$, *and*

$$|m_1 d_0 - d_1 m_0| \leq \sqrt{\sigma} \frac{sD - rM}{|rD - sM|}, \tag{4.2.1}$$

then there is a κ_0 *such that*

$$\frac{m_\kappa}{d_\kappa} \leq b_\infty = \frac{M}{D} \leq \frac{s + m_\kappa(rd_\kappa - sm_\kappa)}{r + d_\kappa(rd_\kappa - sm_\kappa)} \quad \text{for } \kappa \geq \kappa_0.$$

(iii) *If* $m_1 d_0 - d_1 m_0 < 0$, $b_\infty > r/s > 0$, *and* (4.2.1) *holds, then there is a* κ_0 *such that*

$$\frac{m_\kappa(sm_\kappa - rd_\kappa) - s}{d_\kappa(sm_\kappa - rd_\kappa) - r} \leq b_\infty = \frac{M}{D} \leq \frac{m_\kappa}{d_\kappa} \quad \text{for } \kappa \geq \kappa_0.$$

Proof This is a reformulation of [1, Lemma 67] in which (i) incorporates the calculation

$$M''D' - M'D'' = \frac{m_1 d_0 - d_1 m_0}{2\sigma}$$

that follows from (3.1.6). □

In [1, Section 3], an r/s was carefully chosen in order to reduce the number of potential overshadowing classes that needed to be ruled out. For our purposes, we simply need to know an r/s exists since we use an arithmetic argument to rule out overshadowing classes. The following corollary shows that some r/s does indeed exist.

Corollary 4.2.3 *There exists r/s such that either condition (ii) or (iii) in Lemma 4.2.2 is satisfied.*

Proof For condition (ii), assume $m_1 d_0 - d_1 m_0$ is positive. As r/s approaches b_∞ from the right,

$$\frac{sD - rM}{|rD - sM|} = \frac{1 - (r/s)b_\infty}{r/s - b_\infty}$$

approaches infinity. Hence, there always exists some $b_\infty < r/s$ such that condition (ii) is satisfied. A similar argument applies for condition (iii). □

Remark 4.2.4 The statement in Lemma 4.2.2 is not identical to that in [1, Lemma 67] because we have changed notation, now indexing by κ rather than by k. We now explain the relation between the two statements. In [1, Remark 68(i)], we calculated that

$$M''D' - M'D'' = \frac{1}{2(2n+3)\sqrt{\sigma}}(m_1 d_0 - m_0 d_1).$$

However, there the staircases were divided into two parts according to the different endings, and (as in Lemma 3.1.1) the recursion parameter was $\sigma + 2$, where

$$\sigma = (2n+1)(2n+5) = \nu^2 - 4.$$

This reformulation should not change the limiting values M and D, and hence M', M'', D', and D''. However in our current notation $m_{k=1}$ and $d_{k=1}$ are denoted by $m_{\kappa=2}$ and $d_{\kappa=2}$, since the staircases in [1] are indexed by k while in the current paper we combine the two strands and index via κ, which is (approximately) $2k$. Thus if we index by κ and take $2n+3 = \nu$ we have

$$m_2 d_0 - m_0 d_2 = (\nu m_1 - m_0)d_0 - m_0(\nu d_1 - d_0) = \nu(m_1 d_0 - m_0 d_1),$$

$$\frac{1}{2(2n+3)\sqrt{\sigma}}(m_2 d_0 - m_0 d_2) = \frac{1}{2\sqrt{\sigma}}(m_1 d_0 - m_0 d_1),$$

which is consistent with the identities in Lemma 4.2.2.

By Corollary 4.2.3, we do not need to estimate r/s by computing $m_1d_0 - d_1m_0$. However, Lemma 4.2.2(i) shows that the sign of $m_1d_0 - d_1m_0$ determines whether the terms $b_\kappa = m_\kappa/d_\kappa$ strictly increase or decrease for a prestaircase, which is important for the overshadowing argument.

While the quantities m_κ and d_κ do depend on both i and n, we suppress those indices for simplicity.

Lemma 4.2.5 (i) *For $S^i(\mathcal{S}^U_{\ell,n})$, $m_1d_0 - m_0d_1 = -\varepsilon(2ny_i + y_{i+1})$, where $\varepsilon = (-1)^i$.*

(ii) *For $S^i(\mathcal{S}^U_{u,n})$, $m_1d_0 - m_0d_1 = -\varepsilon(2ny_{i+1} + y_{i+2})$, where $\varepsilon = (-1)^i$.*

(iii) *For $S^i(\mathcal{S}^L_{\ell,n})$, $m_1d_0 - m_0d_1 = -\varepsilon(2ny_{i+1} + y_i)$, where $\varepsilon = (-1)^{i+1}$.*

(iv) *For $S^i(\mathcal{S}^L_{u,n})$, $m_1d_0 - m_0d_1 = -\varepsilon(2ny_{i+2} + y_{i+1})$, where $\varepsilon = (-1)^{i+1}$.*

Proof Let $\bullet$ be one of (U,ℓ), (U,u), (L,ℓ), or (L,u). For two seeds $(p_0, q_0, t_0, \varepsilon)$ and $(p_1, q_1, t_1, \varepsilon)$ of $S^i(\mathcal{S}_{\bullet,n})$, the degree formulas from (2.2.4) imply that

$$m_1d_0 - m_0d_1 = \tfrac{1}{8}\varepsilon((p_0 + q_0)t_1 - (p_1 + q_1)t_0). \tag{4.2.2}$$

Denote the seeds of the initial staircase in $\mathcal{S}^U$ or $\mathcal{S}^L$ by $(p_0^\bullet, q_0^\bullet, t_0^\bullet)$ and $(p_1^\bullet, q_1^\bullet, t_1^\bullet)$, and note that

$$S^i = \begin{pmatrix} y_{i+1} & -y_i \\ y_i & -y_{i-1} \end{pmatrix},$$

where $y_{i-1} = 6y_i - y_{i+1}$. Using the invariance of $t_0^\bullet$ and $t_1^\bullet$ under S, we obtain that

$$\begin{aligned} &m_1d_0 - m_0d_1 \\ &= \tfrac{1}{8}\varepsilon\big((t_1^\bullet(p_0^\bullet + q_0^\bullet) - t_0^\bullet(p_1^\bullet + q_1^\bullet))y_{i+1} + (t_1^\bullet(p_0^\bullet - q_0^\bullet) - t_0^\bullet(p_1^\bullet - q_1^\bullet) + 6(t_0^\bullet q_1^\bullet - t_1^\bullet q_0^\bullet))y_i\big). \end{aligned}$$

Then, by substituting the relevant formulas for the seeds from Lemma 3.2.1 and Lemma 3.2.5, we get the desired results.

For example, for (ii), we have

$$(p^U_{0,u}, q^U_{0,u}, t^U_{0,u}) = (-5, -1, 2), \quad (p^U_{1,u}, q^U_{1,u}, t^U_{1,u}) = (2n + 8, 1, 2n + 5) \tag{4.2.3}$$

giving

$$m_1d_0 - m_0d_1 = -\varepsilon((2n + 6)y_{i+1} - y_i) = 2ny_{i+1} + y_{i+2},$$

as claimed. □

Corollary 4.2.6 *For a prestaircase $T^\sharp(\mathcal{S}^U)$, if $b_\infty > 1/3$, then m_κ/d_κ strictly decreases. If $b_\infty < 1/3$, then m_κ/d_κ strictly increases.*

Proof Lemma 4.2.5 implies that $m_1d_0 - m_0d_1$ always has the opposite sign to ε. As ε is positive if and only if $b_\infty > 1/3$, it follows that $m_1d_0 - m_0d_1 < 0$ if and only if $b_\infty > 1/3$. Now use Lemma 4.2.2(i). □

Finally, we prove an estimate that will be useful below.

Lemma 4.2.7 *Let $\mathcal{S}$ be one of the descending stairs $S^i(\mathcal{S}^U_{u,n})$ for all $i, n \geq 0$ except $(i,n) = (0,0)$ or one of the stairs $S^i(\mathcal{S}^L_{u,n})$ for all $i \geq 0$ and $n \geq 1$. Denote the steps of $\mathcal{S}$ by $(d_\kappa, m_\kappa, p_\kappa, q_\kappa, t_\kappa, \varepsilon)$ for $\kappa \geq 0$, and let $b_\infty = \lim m_\kappa/d_\kappa$. Then there is a κ_0 such that*

$$\frac{p_\kappa q_\kappa}{p_\kappa + q_\kappa} > \frac{d_\kappa - m_\kappa b_\infty}{3 - b_\infty}, \quad \kappa \geq \kappa_0. \tag{4.2.4}$$

Proof Since $(3d_\kappa - m_\kappa)p_\kappa q_\kappa = (p_\kappa + q_\kappa)(d_\kappa^2 - m_\kappa^2 + 1)$, we have

$$\begin{aligned}\frac{p_\kappa + q_\kappa}{d_\kappa} &= \left(3 - \frac{m_\kappa}{d_\kappa}\right)p_\kappa q_\kappa - (p_\kappa + q_\kappa)\left(d_\kappa - \frac{m_\kappa^2}{d_\kappa}\right) \\ &= (3 - b_\infty)p_\kappa q_\kappa - (p_\kappa + q_\kappa)(d_\kappa - m_\kappa b_\infty) - \left(\frac{m_\kappa}{d_\kappa} - b_\infty\right)(p_\kappa q_\kappa - m_\kappa(p_\kappa + q_\kappa)).\end{aligned}$$

The inequality (4.2.4) is equivalent to the claim that the sum of the first two terms on the right-hand side is positive. Therefore (4.2.4) will hold provided that

$$\frac{p_\kappa + q_\kappa}{d_\kappa} > \left(\frac{m_\kappa}{d_\kappa} - b_\infty\right)(m_\kappa(p_\kappa + q_\kappa) - p_\kappa q_\kappa), \quad \kappa \geq \kappa_0.$$

Since

$$m_\kappa(p_\kappa + q_\kappa) - p_\kappa q_\kappa = m_\kappa(3d_\kappa - m_\kappa) - (d_\kappa^2 - m_\kappa^2 + 1),$$

it suffices to show that

$$\frac{3d_\kappa - m_\kappa}{d_\kappa} > \left(\frac{m_\kappa}{d_\kappa} - b_\infty\right)(m_\kappa(3d_\kappa - m_\kappa) - (d_\kappa^2 - m_\kappa^2 + 1)), \quad \kappa \geq \kappa_0.$$

Since by Lemma 3.1.4 we have $m_\kappa = M\lambda^\kappa + \overline{M}\lambda^{-\kappa}$ and $d_\kappa = D\lambda^\kappa + \overline{D}\lambda^{-\kappa}$ for some $\lambda > 1$, one easily checks that this will hold if

$$\frac{3D - M}{|3M - D|} > |\overline{M}D - \overline{D}M| = \frac{|m_1d_0 - d_1m_0|}{\sqrt{\sigma}}, \quad \sigma = \nu^2 - 4, \tag{4.2.5}$$

where ν is the recursion parameter of $\mathcal{S}$.

Claim 1 *Equation (4.2.5) holds when $\mathcal{S} = (S^i)^\sharp(\mathcal{S}^U_{u,n})$ for all $i, n \geq 0$ except $i = n = 0$.*

Proof First consider the case $\mathcal{S} = \mathcal{S}^U_{u,n}$. When $X = P, Q, T$, Lemma 3.1.4 implies that $X = (1/2\sqrt{\sigma})(2x_1 - x_0(\nu - \sqrt{\sigma}))$, where $\nu = 2n + 3$ is the recursion variable, and $\sigma = \nu^2 - 4 = (2n+1)(2n+5)$. The seeds for this staircase are given in (4.2.3). Since $\nu - \sqrt{\sigma} > 0$ and the seed class has $p_0 + q_0 < 0$ while $t_0 > 0$, we have

$$\begin{aligned}\frac{3D-M}{|3M-D|} = \frac{P+Q}{T} &= \frac{2(p_1+q_1)-(p_0+q_0)(\nu-\sqrt{\sigma})}{2t_1 - t_0(\nu-\sqrt{\sigma})}\\ &> \frac{p_1+q_1}{t_1} = \frac{2n+9}{2n+5}\\ &> \frac{m_0 d_1 - m_1 d_0}{\sqrt{\sigma}} = \frac{2n+6}{\sqrt{(2n+1)(2n+5)}}\end{aligned}$$

provided that $(2n+9)^2(2n+1) > (2n+6)^2(2n+5)$. But this holds unless $n = 0, 1, 2, 3$. If $n = 1, 2, 3$ then one can check by direct calculation that (4.2.5) holds. Moreover, although it does not hold when $n = 0$ one can check that in this case we have

$$\text{(4.2.6)} \qquad \frac{3D-M}{|3M-D|} > \frac{34}{35}\,\frac{m_0 d_1 - m_1 d_0}{\sqrt{\sigma}} = \frac{y_4}{y_2 y_3}\,\frac{6}{\sqrt{\sigma}}.$$

Now suppose that $\mathcal{S} = (S^i)^\sharp(\mathcal{S}^U_{u,n})$. Notice that σ and ν are invariant under the shift. The quantity $|3M - D|$ is also invariant under the shift since $3M - D$ is the limit of $(3m_\kappa - d_\kappa)\lambda^{-\kappa} = \varepsilon t_\kappa \lambda^{-\kappa}$. To consider how much the right-hand side of (4.2.5) increases under iterations of the shift, we again use Lemma 4.2.5. Because $y_{i+2} < 6y_{i+1}$ for all $i > 0$, we can estimate

$$\frac{2ny_{i+1} + y_{i+2}}{2ny_1 + y_2} = \frac{2ny_{i+1} + y_{i+2}}{2n+6} < y_{i+1} \quad \text{for all } i > 0,\ n \geq 0.$$

Therefore the result will hold in the case $n > 0$ if we show that when we apply S^i the quantity $3D - M$ increases by a factor of at least y_{i+1}. But by Lemma 2.1.3, $3D - M = P + Q$ is taken by S^i to $(y_{i+1} + y_i)P - (y_i + y_{i-1})Q$. Therefore we need $(y_{i+1} + y_i)P - (y_i + y_{i-1})Q > y_{i+1}(P + Q)$, or equivalently

$$y_i P > (y_{i+1} + y_i + y_{i-1})Q = 7y_i Q.$$

But this holds because P/Q is the accumulation point of a staircase in $\mathcal{S}^U$ and so satisfies $P/Q > 7$. Indeed $w_1 = 7$ is blocked by $\boldsymbol{B}^U_0$ by Corollary 4.1.5.

In the case $n = 0$, the right-hand side of (4.2.5) increases by the factor $y_{i+2}/6$. Therefore it suffices to show that when we apply S^i the quantity $3D - M$ increases by a factor of at least $(35/34)(y_{i+2}/6)$. Since $(35/34)(y_{i+2}/6) \leq y_{i+1}$ when $i \geq 2$, the previous argument applies to show that this holds. In the case $i = 1$, we must check that

$$7P - Q > \tfrac{35}{34}\tfrac{35}{6}(P + Q),$$

which holds because $P/Q = [7; \{5, 1\}^\infty] > [7; 6] = 43/6$. □

Claim 2 *Equation (4.2.5) holds when* $\mathcal{S} = (S^i)^\sharp(\mathcal{S}^L_{u,n})$, $n \geq 1$ *and* $i \geq 0$.

Proof Let $\mathcal{S} = \mathcal{S}^L_{u,n}$. The initial seeds are now $(p^L_{0,u}, q^L_{0,u}, t^L_{0,u}) = (-29, -5, 2)$ and $(p^L_{1,u}, q^L_{1,u}, t^L_{1,u}) = (12n - 11, 2n - 2, 2n + 1)$. Further $\nu = 2n + 3$ so that $\sigma = \sqrt{(2n+1)(2n+5)}$ as before. If we simplify the inequality as before and use the result in Lemma 4.2.5(iv), it follows that it suffices to check that

$$\frac{p_1 + q_1}{t_1} > \frac{m_1 d_0 - m_0 d_1}{\sqrt{\sigma}} = \frac{2ny_2 + y_1}{\sqrt{\sigma}} = \frac{12n+1}{\sqrt{\sigma}}.$$

But this holds for all $n \geq 4$. For the cases $n = 1, 2, 3$, we directly compute both sides of (4.2.5) to verify the inequality holds.

Next consider the case $i > 0$. We argue as before, noting that for each n the accumulation point P/Q is larger than the center of the blocking class $\boldsymbol{B}^L_n$, which is $(12n+1)/2n$ by Theorem 2.3.3. By Lemma 4.2.5(iv), when we apply S^i the right-hand side of (4.2.5) grows by the factor

$$\frac{2ny_{i+2} + y_{i+1}}{2ny_2 + y_1} = \frac{2ny_{i+2} + y_{i+1}}{12n+1}, \quad n \geq 1.$$

Therefore it suffices to check that

$$\begin{aligned}(y_{i+1} + y_i)P - (y_i + y_{i-1})Q &> \frac{2ny_{i+2} + y_{i+1}}{12n+1}(P+Q) \\ &= \Bigl(y_{i+1} - \frac{2n}{12n+1}y_i\Bigr)(P+Q).\end{aligned}$$

Because $y_{i+1} + y_i + y_{i-1} = 7y_i$, this simplifies to

$$\frac{14n+1}{12n+1} y_i P > y_i\Bigl(7 - \frac{2n}{12n+1}\Bigr)Q.$$

Thus we need $P/Q > (82n+7)/(14n+1)$. But this holds for all $n \geq 1$ since $P/Q > (12n+1)/2n$. □

The proof of the lemma is now complete. □

4.3 Overshadowing classes

To utilize the staircase recognition Theorem 4.2.1, it remains to show there are no overshadowing classes. Here, we not only show that there are no overshadowing classes for $S^i R^\delta(\mathcal{S}^U)$, but prove a more general result about overshadowing classes. Namely, given a general perfect prestaircase family with recursion parameter ≥ 3, if $b_\infty < 1/3$

and m_κ/d_κ strictly increases (resp. $b_\infty > 1/3$ and m_κ/d_κ strictly decreases), then the staircase family is live provided only that the staircase is not overshadowed by the obstruction from the exceptional class $\boldsymbol{E}_1 := 3L - E_0 - 2E_1 - E_{2\dots6}$. Recall from Remark 2.3.8(ii) and Lemma 2.2.7 that when $b \in (1/5, 5/11)$ this obstruction is given by the formula $z \mapsto (1+z)/(3-b)$, and hence always passes through the accumulation point $(\mathrm{acc}(b), V_b(\mathrm{acc}(b)))$. Hence, it could overshadow the descending staircases for these b. Since the $\mathcal{S}^U$ staircases have $b > 5/11$ while the $\mathcal{S}^L$ staircases have $b < 1/5$, we only need be concerned about their images under a shift S^i. The next lemma shows that these staircases are not overshadowed in this way, because the slope of any overshadowing class must be greater than the slope of the obstruction $\mu_{\boldsymbol{E}_1,b}$.

Lemma 4.3.1 *Let $\mathcal{S}$ be any descending prestaircase in one of the families $(S^iR^\delta)^\sharp(\mathcal{S}^U)$ where $\delta \in \{0,1\}$ and $i > 0$ or $i = 0$ and $n \geq 1$. Then the slope of an overshadowing class must be larger than $1/(3-b_\infty)$.*

Proof The slope of an overshadowing class must be larger than the slope of the line segments from the accumulation point (z_∞, b_∞) to the outer corners

$$\left(\frac{p_\kappa}{q_\kappa}, \frac{p_\kappa}{d_\kappa - m_\kappa b_\infty}\right)$$

of the staircase. Therefore we must check that

$$\frac{p_\kappa/(d_\kappa - m_\kappa b_\infty) - (1+z_\infty)/(3-b_\infty)}{p_\kappa/q_\kappa - z_\infty} > \frac{1}{3-b_\infty}.$$

When we simplify z_∞ cancels from the inequality, and we get

$$(3-b_\infty)p_\kappa q_\kappa > (p_\kappa + q_\kappa)(d_\kappa - m_\kappa b_\infty),$$

which was proved in Lemma 4.2.7. □

We now use an arithmetic argument to rule out the existence of any other overshadowing classes. The following lemma establishes properties about the common divisors of (d, m, p, q) needed for the argument.

Lemma 4.3.2 *Given a quasiperfect class (d, m, p, q), assume there is an integer k such that km/p and kd/p (resp. km/q and kd/q) are both integers. Then $p \mid k$ (resp. $q \mid k$).*

Proof Assume km/p and kd/p are both integers. Since $g := \gcd(m, d)$ is an integral combination of m and d, this implies $kg/p \in \mathbb{Z}$. But because $q = 3d - p - m$ and $\gcd(p, q) = 1$, we must have $\gcd(p, d, m) = \gcd(p, g) = 1$. Therefore $p \mid k$ and, similarly, $q \mid k$. □

Suppose that at (z_∞, b_∞) there is an overshadowing class $\boldsymbol{E}_{\mathrm{ov}} = (d_{\mathrm{ov}}, m_{\mathrm{ov}}, \boldsymbol{m})$ for some prestaircase. Recall from Corollary 3.1.5 that the limit points z_∞ and b_∞ are irrational. By [6, Proposition 2.3.2], there are integers $A \ge 0$ and $C \ge 0$ such that

$$\mu_{\boldsymbol{E}_{\mathrm{ov}},b}(z) = \frac{A + Cz}{d_{\mathrm{ov}} - m_{\mathrm{ov}} b}, \quad z \approx z_\infty.$$

Lemma 4.3.3 *Let $\mathcal{S}$ be a descending prestaircase with irrational accumulation point z_∞ that is associated to a perfect blocking class $\boldsymbol{B} = (d_{\boldsymbol{B}}, m_{\boldsymbol{B}}, p_{\boldsymbol{B}}, q_{\boldsymbol{B}}, t_{\boldsymbol{B}})$. Suppose that $\boldsymbol{E}_{\mathrm{ov}} = (d_{\mathrm{ov}}, m_{\mathrm{ov}}, \boldsymbol{m})$ is an overshadowing class, and denote by*

$$\mu_{\boldsymbol{E}_{\mathrm{ov}},b}(z) = \frac{A + Cz}{d_{\mathrm{ov}} - m_{\mathrm{ov}} b}$$

the corresponding obstruction. Assume the slope $C/(d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty)$ of $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}(z)$ is $> 1/(3 - b_\infty)$. Then there is a positive integer k such that

(i) *If $b_\infty > 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} > 1/3$, then $m_{\mathrm{ov}}/d_{\mathrm{ov}} < 1/3$ and $d_{\mathrm{ov}} - 3m_{\mathrm{ov}} = k t_{\boldsymbol{B}}$.*

(ii) *If $b_\infty < 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} < 1/3$, then $m_{\mathrm{ov}}/d_{\mathrm{ov}} > 1/3$ and $3m_{\mathrm{ov}} - d_{\mathrm{ov}} = k t_{\boldsymbol{B}}$.*

Proof Note first that the function $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}$ must obstruct an interval $(z_\infty, z_\infty + \varepsilon)$ to the right of the limit point and hence have break point $a_{\mathrm{ov}} > z_\infty$. Further, the condition on the slope implies that $C > A$.

Next note that $V_{b_\infty}(z_\infty)$ is given by the expressions

$$\frac{1 + z_\infty}{3 - b_\infty} = \frac{p_{\boldsymbol{B}}}{d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty} = \frac{A + C z_\infty}{d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty},$$

where the first equality holds by (4.0.1) and the second holds by [1, Lemma 16]. The first equality implies

$$\begin{aligned}&(1 + z_\infty)(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty) = p_{\boldsymbol{B}}(3 - b_\infty)\\ &\qquad\implies z_\infty = \frac{p_{\boldsymbol{B}}(3 - b_\infty) - (d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)}{d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty} = \frac{b_\infty(m_{\boldsymbol{B}} - p_{\boldsymbol{B}}) + 3p_{\boldsymbol{B}} - d_{\boldsymbol{B}}}{d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty},\end{aligned}$$

while the second gives

$$\begin{aligned}&(A + C z_\infty)(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty) = p_{\boldsymbol{B}}(d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty)\\ &\implies z_\infty = \frac{p_{\boldsymbol{B}}(d_{\mathrm{ov}} - b_\infty m_{\mathrm{ov}}) - A(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)}{C(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)} = \frac{b_\infty(A m_{\boldsymbol{B}} - p m_{\mathrm{ov}}) + p_{\boldsymbol{B}} d_{\mathrm{ov}} - A d_{\boldsymbol{B}}}{C(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)}.\end{aligned}$$

Therefore,

$$b_\infty(A m_{\boldsymbol{B}} - p_{\boldsymbol{B}} m_{\mathrm{ov}}) + p_{\boldsymbol{B}} d_{\mathrm{ov}} - A d_{\boldsymbol{B}} = C(b_\infty(m_{\boldsymbol{B}} - p_{\boldsymbol{B}}) + 3p_{\boldsymbol{B}} - d_{\boldsymbol{B}}).$$

All quantities here are integers except for b_∞ which is irrational because z_∞ is. Therefore the coefficients of b_∞ must be equal. Thus we have

$$Am_{\boldsymbol{B}} - p_{\boldsymbol{B}} m_{\mathrm{ov}} = C(m_{\boldsymbol{B}} - p_{\boldsymbol{B}}), \quad p_{\boldsymbol{B}} d_{\mathrm{ov}} - A d_{\boldsymbol{B}} = C(3p_{\boldsymbol{B}} - d_{\boldsymbol{B}}).$$

We can solve these equations for m_{ov} and d_{ov} to get

$$\begin{aligned} m_{\mathrm{ov}} &= \frac{(A-C)m_{\boldsymbol{B}} + Cp_{\boldsymbol{B}}}{p_{\boldsymbol{B}}} = C + (A-C)m_{\boldsymbol{B}}/p_{\boldsymbol{B}}, \\ d_{\mathrm{ov}} &= \frac{3p_{\boldsymbol{B}}C + (A-C)d_{\boldsymbol{B}}}{p_{\boldsymbol{B}}} = 3C + (A-C)d_{\boldsymbol{B}}/p_{\boldsymbol{B}}. \end{aligned} \tag{4.3.1}$$

As m_{ov} and d_{ov} are integers, $(A-C)m_{\boldsymbol{B}}/p_{\boldsymbol{B}}$ and $(A-C)d_{\boldsymbol{B}}/p_{\boldsymbol{B}}$ are both integers. Then, by Lemma 4.3.2, $p_{\boldsymbol{B}} \,|\, (A-C)$. This proves $k = (C-A)/p_{\boldsymbol{B}}$ is an integer.

Since $C - A > 0$, we also have $k > 0$. Since $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} > 1/3$ by assumption, the formulas in (4.3.1) then imply that $3m_{\mathrm{ov}} < d_{\mathrm{ov}}$.

We can compute

$$d_{\mathrm{ov}} - 3m_{\mathrm{ov}} = 3C - kd_{\boldsymbol{B}} - 3C + 3km_{\boldsymbol{B}} = k(3m_{\boldsymbol{B}} - d_{\boldsymbol{B}}) = kt_{\boldsymbol{B}}.$$

This proves (i). The proof for (ii) follows similarly. □

We prove a similar result for ascending staircases.

Lemma 4.3.4 *Let $\mathcal{S}$ be an ascending prestaircase with irrational accumulation point z_∞ that is associated to a perfect blocking class $\boldsymbol{B} = (d_{\boldsymbol{B}}, m_{\boldsymbol{B}}, p_{\boldsymbol{B}}, q_{\boldsymbol{B}}, t_{\boldsymbol{B}})$. Suppose that $\boldsymbol{E}_{\mathrm{ov}} = (d_{\mathrm{ov}}, m_{\mathrm{ov}}, \boldsymbol{m})$ is an overshadowing class, and denote by*

$$\mu_{\boldsymbol{E}_{\mathrm{ov}},b}(z) = \frac{A + Cz}{d_{\mathrm{ov}} - m_{\mathrm{ov}} b}$$

the corresponding obstruction. Then there is a positive integer k such that

- (i) *If $b_\infty > 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} > 1/3$, then $m_{\mathrm{ov}}/d_{\mathrm{ov}} < 1/3$ and $d_{\mathrm{ov}} - 3m_{\mathrm{ov}} = kt_{\boldsymbol{B}}$.*
- (ii) *If $b_\infty < 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} < 1/3$, then $m_{\mathrm{ov}}/d_{\mathrm{ov}} > 1/3$ and $3m_{\mathrm{ov}} - d_{\mathrm{ov}} = kt_{\boldsymbol{B}}$.*

Proof Notice first that we must have $A - C > 0$ since if $A \le C$ the slope of the obstruction $\mu_{\boldsymbol{E}_{\mathrm{ov}},b_\infty}$ is at least that of the line $z \mapsto (1+z)/(3-b_\infty)$ which is $< V_{b_\infty}(z)$ for $z < z_\infty$, and so is not obstructive for $z < z_\infty$, while the overshadowing class is obstructive.

Next, as in Lemma 4.3.3 we consider the equalities

$$\frac{1+z_\infty}{3-b_\infty} = \frac{q_{\boldsymbol{B}} z_\infty}{d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty} = \frac{A + C z_\infty}{d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty},$$

where the second equality follows from [1, Lemma 16]. This implies

$$(1+z_\infty)(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty) = q_{\boldsymbol{B}} z_\infty (3-b_\infty) \implies z_\infty = \frac{d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty}{q_{\boldsymbol{B}}(3-b_\infty) - (d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)},$$

$$(A + C z_\infty)(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty) = q_{\boldsymbol{B}} z_\infty (d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty)$$
$$\implies z_\infty = \frac{A(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)}{q_{\boldsymbol{B}}(d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty) - C(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)}.$$

Hence we must have

$$A(q_{\boldsymbol{B}}(3-b_\infty) - (d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty)) = q_{\boldsymbol{B}}(d_{\mathrm{ov}} - m_{\mathrm{ov}} b_\infty) - C(d_{\boldsymbol{B}} - m_{\boldsymbol{B}} b_\infty).$$

As before, the coefficients of b_∞ on both sides must agree, which gives

$$d_{\mathrm{ov}} = 3A - (A-C)d_{\boldsymbol{B}}/q_{\boldsymbol{B}}, \quad m_{\mathrm{ov}} = A - (A-C)m_{\boldsymbol{B}}/q_{\boldsymbol{B}}. \tag{4.3.2}$$

Since $A - C > 0$ and d_{ov} and m_{ov} are integers, Lemma 4.3.2 implies $k = (A-C)/q_{\boldsymbol{B}}$ is a positive integer.

Furthermore, these formulas prove that if $b_\infty > 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} > 1/3$ then $d_{\mathrm{ov}} > 3m_{\mathrm{ov}}$. We can again compute

$$d_{\mathrm{ov}} - 3m_{\mathrm{ov}} = k\, t_{\boldsymbol{B}}.$$

This proves (i). The proof for (ii) follows similarly. □

We use the results of Lemmas 4.3.3 and 4.3.4 to rule out overshadowing classes for a large set of prestaircases, which implies that none of the prestaircases considered in this paper have overshadowing classes.

Lemma 4.3.5 *Let $\mathcal{S}$ be a prestaircase with irrational accumulation point z_∞ that is associated to a perfect blocking class $\boldsymbol{B} = (d_{\boldsymbol{B}}, m_{\boldsymbol{B}}, p_{\boldsymbol{B}}, q_{\boldsymbol{B}}, t_{\boldsymbol{B}})$ such that $t_{\boldsymbol{B}} \geq 3$. Suppose that either $b_\infty > 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} > 1/3$ and the m_κ/d_κ strictly decrease or $b_\infty < 1/3$ and $m_{\boldsymbol{B}}/d_{\boldsymbol{B}} < 1/3$ and the m_κ/d_κ strictly increase. Assume further that if $\mathcal{S}$ is descending any overshadowing class must have slope $> 1/(3-b_\infty)$. Then the prestaircase $\mathcal{S}$ has no overshadowing classes at all.*

Proof These conditions, with Lemma 4.3.1, ensure that the conditions in either Lemma 4.3.3 or Lemma 4.3.4 hold for $\mathcal{S}$. For a class $\boldsymbol{E}_{\mathrm{ov}} = (d_{\mathrm{ov}}, m_{\mathrm{ov}}, \boldsymbol{m}_{\mathrm{ov}})$ to be overshadowing, it must be obstructive for $b = b_\infty$ at its break point $a_{\rm ov}$. By [1, Lemma 15] this is possible only if $|d_{\rm ov} b_\infty - m_{\rm ov}| < 1$. Further, for each staircase there is a positive integer k such that $|d_{\rm ov} - 3m_{\rm ov}| = t_{\boldsymbol{B}} k$. Thus we have

$$3 > 3|b_\infty d_{\rm ov} - m_{\rm ov}| > |d_{\rm ov} - 3m_{\rm ov}| = t_{\boldsymbol{B}} k \geq 3k,$$

where the second inequality holds because Lemmas 4.3.3 and 4.3.4 imply that either

$$\frac{m_{\rm ov}}{d_{\rm ov}} < \frac{1}{3} < b_\infty \quad \text{or} \quad b_\infty < \frac{1}{3} < \frac{m_{\rm ov}}{d_{\rm ov}}.$$

As k must be positive, no such $d_{\rm ov}$ and $m_{\rm ov}$ can exist. □

Corollary 4.3.6 *For each $T \in \mathcal{G}$, the prestaircase family $T^\sharp(\mathcal{S}^U)$ has no overshadowing classes.*

Proof Because $t_{\boldsymbol{B}} = 2n+3$ in all cases, the results of Corollary 4.2.6 and Lemma 4.3.1 show that all the staircases in $T^\sharp(\mathcal{S}^U)$ except for $\mathcal{S}^U_{u,0}$ satisfy the conditions in Lemma 4.3.5 and hence have no overshadowing classes. The proof for $\mathcal{S}^U_{u,0}$ is given in [1, Example 70]. □

Now, we establish a straightforward way to check if a perfect prestaircase is live based on the above overshadowing arguments.

Proposition 4.3.7 *Let $\mathcal{S}$ be a perfect prestaircase with irrational accumulation point z_∞ associated to a blocking class $\boldsymbol{B}$ with recursion parameter $t_{\boldsymbol{B}} \geq 3$. Suppose that either $b_\infty > 1/3$ and the m_κ/d_κ strictly decrease, or $b_\infty < 1/3$ and the m_κ/d_κ strictly increase. Assume further that if $\mathcal{S}$ is descending the slope of any overshadowing class must be $> 1/(3-b_\infty)$. Then $\mathcal{S}$ is a staircase, namely it is perfect and live.*

Proof By Corollary 4.2.3 and Lemma 4.3.5, the perfect prestaircase satisfies the conditions of Theorem 4.2.1 implying that $\mathcal{S}$ is live. □

A consequence of this proposition is Theorem 1.2.6.

Corollary 4.3.8 *For each $T \in \mathcal{G}$, $T^\sharp(\mathcal{S}^U)$ is live.*

Proof For all staircases except $\mathcal{S}^U_{u,0}$ (which was shown to be live in [1]), the conditions of Proposition 4.3.7 are satisfied since $t_{\boldsymbol{B}} = 2n+3$, and Corollary 4.2.6 and Lemma 4.3.1 hold. □

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