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Beta families arising from a v_2^9 self-map on $S/(3, v_1^8)$

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We show that v_2^9 is a permanent cycle in the 3–primary Adams–Novikov spectral sequence computing $\pi_*(S/(3, v_1^8))$, and use this to conclude that the families $\beta_{9t+3/i}$ for $i = 1, 2$, $\beta_{9t+6/i}$ for $i = 1, 2, 3$, $\beta_{9t+9/i}$ for $i = 1, \dots, 8$, $\alpha_1\beta_{9t+3/3}$, and $\alpha_1\beta_{9t+7}$ are permanent cycles in the 3–primary Adams–Novikov spectral sequence for the sphere for all $t \geq 0$. We use a computer program by Wang to determine the additive and partial multiplicative structure of the Adams–Novikov E_2 page for the sphere in relevant degrees. The $i = 1$ cases recover previously known results of Behrens and Pemmaraju and the second author. The results about $\beta_{9t+3/3}$, $\beta_{9t+6/3}$ and $\beta_{9t+9/8}$ were previously claimed by the second author; the computer calculations allow us to give a more direct proof. As an application, we determine the image of the Hurewicz map $\pi_*S \to \pi_*\mathrm{tmf}$ at $p = 3$.

55Q45, 55Q51, 55T25

1 Introduction

Miller, Ravenel and Wilson [9, Theorem 2.6] showed that the 2–line of the Adams–Novikov E_2 page for the sphere is generated by classes $\beta_{i/j,k}$ for i, j, k satisfying certain conditions. At the prime 3, the β elements with $i \leq 9$ are

$$\begin{aligned} &\beta_i \quad \text{for } i = 1, 2, 4, 5, 7, 8, && \beta_{3/j} \text{ and } \beta_{6/j} \quad \text{for } j = 1, 2, 3, \\ &\beta_{9/j} \quad \text{for } j = 1, \dots, 9, && \beta_{9/3,2}, \end{aligned}$$

where we write $\beta_{i/j} := \beta_{i/j,1}$ and $\beta_i := \beta_{i/1}$. They have order 3 except for $\beta_{9/3,2}$, which satisfies $3\beta_{9/3,2} = \beta_{9/3}$. Of these, the permanent cycles are

$$\begin{gathered} \beta_1, \quad \beta_2, \quad \beta_{3/2}, \quad \beta_3, \quad \beta_5, \quad \beta_{6/3}, \quad \beta_{6/2}, \quad \beta_6, \\ \beta_7 + c\beta_{9/9} \quad \text{for some } c \in \{\pm 1\}, \qquad \beta_{9/j} \quad \text{for } 1 \leq j \leq 8. \end{gathered} \tag{1}$$

For $i \le 7$ and $\beta_7 + c\beta_{9/9}$, these assertions can be read off the exhaustive calculation of Ravenel [14, Table A3.4] of $\pi_n S_3^\wedge$ in stems $n \le 108$; see also Oka [11] for many of the survival results and Shimomura [15] for the nonsurvival results. The element $\beta_7 + c\beta_{9/9}$ is an Arf invariant class (an odd-primary analogue of the $p = 2$ Kervaire invariant classes), discussed in Ravenel [13, page 439]; the survival of the Arf invariant classes is not known in general at $p = 3$. The survival of $\beta_{9/j}$ for $j \le 8$ is a consequence of Theorem 5.1, which does not depend on prior knowledge about this element, but we do not claim originality for this result.

These results arise from exhaustive calculations in tractable stems, but it is possible to prove results about β elements outside the range of feasible computation. One strategy is as follows. Suppose $\beta_{i/j}$ is a permanent cycle. It is v_1–power-torsion; that is, there exists a type 2 complex V such that $\beta_{i/j}$ factors as $S \xrightarrow{f_1} V \xrightarrow{f_2} S$ (we omit degree shifts for clarity of notation). If v_2^t is a v_2 self-map on V, then we may construct elements of $\pi_*(S)$ as

$$S \xrightarrow{f_1} V \xrightarrow{v_2^t} V \xrightarrow{f_2} S, \quad \text{with } t \ge 0.$$

For this family to be of interest, one must also show that the elements are nonzero, for example by identifying their Adams–Novikov representatives.

Let $S/3$ denote the mod 3 Moore space, and for $m \ge 1$ let $S/(3, v_1^m)$ denote the cofiber of the m–fold iterate of Adams' v_1 self-map $S/3 \xrightarrow{v_1} S/3$; see Toda [19]. Behrens and Pemmaraju [3] show there is a v_2^9 self-map on $S/(3, v_1)$ and use this to prove the existence of nonzero homotopy classes represented by β_{9t+s} for $s = 1, 2, 5, 6, 9$ and $t \ge 0$. The second author [18] proves the existence of β_{9t+3}. By comparison to L_2–local homotopy, he shows in [15] that the elements

$$\beta_{9t+4}, \quad \beta_{9t+7}, \quad \beta_{9t+8}, \quad \beta_{9t+3/3}, \quad \beta_{9t/3,2}, \quad \beta_{3^i s/3^i}$$

are not permanent cycles for $t \ge 1$, $s \not\equiv 0 \pmod 3$, and $i > 1$. The main goal of this paper is to construct a v_2^9 self-map on $S/(3, v_1^8)$ and show that the remaining β elements in $\pi_s(S)$ for $s \le |v_2^9| = 144$ also give rise to infinite families.

Theorem 5.1 *For all $t \ge 0$, the classes*

$$\begin{aligned} &\beta_{9t+3/j} \quad \text{for } j = 1, 2, && \beta_{9t+6/j} \quad \text{for } j = 1, 2, 3, \\ &\beta_{9t+9/j} \quad \text{for } j = 1, \dots, 8, && \alpha_1\beta_{9t+3/3} \quad \text{and} \quad \alpha_1\beta_{9t+7} \end{aligned}$$

are permanent cycles in the Adams–Novikov spectral sequence for the sphere.

These families are interesting in part because the Hurewicz map $\pi_*(S) \to \pi_*(\mathrm{tmf})$ detects $\beta_{9t+1}, \alpha_1\beta_{9t+3/3}, \beta_{9t+6/3}$ and $\alpha_1\beta_{9t+7}$, as we show in Theorem 6.5. Together with the well-known behavior in the 0– and 1–lines, this completely determines the Hurewicz image of tmf at $p = 3$. Behrens, Mahowald and Quigley [2] calculate the Hurewicz image of tmf at $p = 2$. Since $\pi_*\big(\mathrm{tmf}\big[\frac{1}{6}\big]\big) = \mathbb{Z}\big[\frac{1}{6}, a_4, a_6\big]$ is concentrated on the Adams–Novikov 0–line, our work together with the $p = 2$ case forms the complete determination of the Hurewicz image of tmf at all primes.

Following the strategy outlined above, much of the work involves showing that v_2^9 is a permanent cycle in the Adams–Novikov spectral sequence computing $\pi_*(S/(3, v_1^8))$; this is Theorem 4.6. All of our explicit calculations are in the Adams–Novikov spectral sequence for $S/3$. To relate this to $S/(3, v_1^8)$ we use a lemma due to the second author (Lemma 4.1) that relates v_1^m–extensions in the Adams–Novikov spectral sequence for $S/3$ to differentials in the Adams–Novikov spectral sequence for $S/(3, v_1^m)$. Combined with Oka's result [10] that $S/(3, v_1^m)$ is a ring spectrum for $m \geq 2$, this implies the existence of a v_2^9 self-map.

Corollary 4.7 *For $2 \leq m \leq 8$, there is a nonzero self-map*

$$v_2^9 \colon \Sigma^{144} S/(3, v_1^m) \to S/(3, v_1^m).$$

There is also a similar result for $m = 9$, but correction terms for v_2^9 are needed; see Remark 4.8.

Our proof that v_2^9 is a permanent cycle in the Adams–Novikov spectral sequence computing $\pi_*(S/(3, v_1^8))$ relies on analysis of the 143 stem in the Adams–Novikov spectral sequence for the sphere. This is greatly aided by software written by Wang [21; 20], which computes the E_2 page of the Adams–Novikov spectral sequence for the sphere using the algebraic Novikov spectral sequence. In addition, the software computes multiplication by p, α_1 and arbitrary $\beta_{i/j}$ elements. Wang's software was originally written for use at $p = 2$; the minor modifications we used to change the prime are available at [5] and data, charts and more documentation are available at [4]. The calculations that make use of computer data occur solely in Section 3.

We now comment on the overlap between this work and the preprint [16] by the second author: both works construct v_2^9 and the families $\beta_{9t+3/3}$, $\beta_{9t+6/3}$, and $\beta_{9t+9/8}$, but we find the methods here to be more straightforward. The earlier preprint uses the machinery of infinite descent to control the complexity of the Adams–Novikov spectral sequence, while we opt to work directly with the Adams–Novikov E_2 page, controlling the complexity using Wang's program. In particular, our analysis in Section 3, which is the crucial input for the construction of v_2^9, follows from the β_1–multiplication structure given by the computer calculations, as most of the elements in play are highly β_1–divisible.

We conclude this section by giving an outline of the rest of the paper. In Section 2 we state notational conventions for the rest of the paper and write down some easy facts about the Adams–Novikov spectral sequence that will be used extensively in the remaining sections. Most of the work for proving Theorem 4.6 occurs in Section 3, which makes use of computer calculations to determine the Adams–Novikov spectral sequence for $S/3$ near the 143 stem. Theorem 4.6, which constructs v_2^9, is proved in Section 4. In Section 5 we prove Theorem 5.1, which constructs the promised β families. This involves explicit calculations in a tractable range of stems to prove that $v_1^2 v_2^3$, $v_1 v_2^6$ and $\alpha_1 v_1 v_2^3$ in $E_2(S/(3, v_1^4))$, and $\alpha_1 v_1 v_2^7$ in $E_2(S/(3, v_1^2))$, are permanent cycles. In Section 6 we determine the 3–primary Hurewicz image of tmf (Theorem 6.5).

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2 Notation and preliminaries

At a fixed prime p, the Brown–Peterson spectrum BP has coefficient ring $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ with $|v_i| = 2p^i - 2$, and ring of co-operations $\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*[t_1, t_2, \dots]$ with $|t_i| = 2p^i - 2$. Given a finite p–local spectrum X, the Adams–Novikov spectral sequence

$$E_2 = \mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* X) \Rightarrow \pi_*(X)_{(p)} \tag{2}$$

converges. Henceforth everything will be implicitly localized at the prime $p = 3$. The E_2 page of (2) can be calculated as the cohomology of the normalized cobar complex

$$\mathrm{BP}_* \xrightarrow{\eta_R - \eta_L} \overline{\mathrm{BP}_*\mathrm{BP}} \to \overline{\mathrm{BP}_*\mathrm{BP}}^{\otimes 2} \to \cdots, \tag{3}$$

although this is not an efficient means of computation. For further background on the Adams–Novikov spectral sequence, see [6] and [14, Sections 4.3–4.4].

Let $E_r^{s,f}(X)$ denote the E_r page of (2), restricted to stem s and Adams–Novikov filtration f. We say that an element in $\pi_s X$ is detected in filtration f if it is represented by a nonzero class in $E_\infty^{s,f}(X)$. Throughout, any equality of homotopy or E_2 page elements should be understood to be true up to units (that is, up to signs).

We will make frequent use of the cofiber sequence

$$S \xrightarrow{3} S \xrightarrow{i} S/3 \xrightarrow{j} \Sigma S.$$

We will also consider the cofiber sequences

$$\Sigma^{4m} S/3 \xrightarrow{v_1^m} S/3 \xrightarrow{i_m} S/(3, v_1^m) \xrightarrow{j_m} \Sigma^{4m+1} S/3 \quad \text{for } m \geq 1,$$

eventually focusing primarily on $m = 8$. Henceforth, degree shifts in cofiber and long exact sequences will usually not be shown. The maps i, j, i_m and j_m induce maps of Adams–Novikov spectral sequences, which we will denote with the same letters. We note the effect on degrees: given $x \in E_2^{s,f}(S/3)$, we have $j(x) \in E_2^{s-1,f+1}(S)$; given $x \in E_2^{s,f}(S/(3, v_1^m))$, we have $j_m(x) \in E_2^{s-4m-1,f+1}(S/3)$. The maps i and i_m preserve degrees. Sometimes we omit applications of i or i_m in the notation for brevity; for example, we write $\beta_1 \in E_2^{10,2}(S/3)$ to refer to $i(\beta_1)$. This is justified by regarding $E_2(S/3)$ as a module over $E_2(S)$.

By the geometric boundary theorem (see [14, Theorem 2.3.4]), the maps induced by j and j_m on E_2 pages coincide with the boundary maps in the long exact sequences of Ext groups

$$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/3) \to \mathrm{Ext}^{*+1,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*),$$
$$\mathrm{Ext}^{*,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(3, v_1^m)) \to \mathrm{Ext}^{*+1,*}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/3).$$

Definition 2.1 We will say that an element $x \in E_2^{*,*}(S/3)$ is a *bottom cell element* if it is in the image of $i\colon E_2^{*,*}(S) \to E_2^{*,*}(S/3)$. An element $x \in E_2^{*,*}(S/3)$ is a *top cell element* if its image under the boundary map $j\colon E_2^{*,*}(S/3) \to E_2^{*-1,*}(S)$ is nonzero.

Notation 2.2 (i) If $x \in E_2^{s,f}(S)$ is 3–torsion, we will let $\bar{x} \in E_2^{s+1,f-1}(S/3)$ denote a class such that $j(\bar{x}) = x$. Note that $\bar{x}$ may not always be uniquely determined.

(ii) If $x \in E_2(X)$ is a permanent cycle converging to $y \in \pi_*(X)$, write $y = \{x\}$.

In the rest of this section we present some preliminaries that are important for working with the Adams–Novikov spectral sequences for S, $S/3$ and $S/(3, v_1^m)$. All of these facts are well known, and the rest of this section can be skipped by a knowledgeable reader. First we recall some frequently encountered permanent cycles in the 3–primary Adams–Novikov spectral sequence for the sphere. The comparisons below to the Adams spectral sequence are not needed in the rest of the paper, and are just presented for those readers who are more familiar with the Adams elements; a reference for computational facts about the Adams spectral sequence E_2 page is [14, Section 3.4], and for the corresponding Adams–Novikov elements is [9] or [6, Section 6] for the Greek letter construction and [14, Theorem 4.4.20] for low stems.

- $\alpha_1 \in E_2^{3,1}(S)$ is represented by $[t_1]$ in the cobar complex (3), and is called h_0 in the Adams spectral sequence.
- $\beta_1 \in E_2^{10,2}(S)$ equals the Massey product $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$, and is called $b_0 = b_{10}$ in the Adams spectral sequence.
- $\beta_2 \in E_2^{26,2}(S)$ is called $k = k_0 = \langle h_0, h_1, h_1 \rangle$ in the Adams spectral sequence. (This does not correspond to an Adams–Novikov Massey product since h_1 does not exist in the Adams–Novikov E_2 page.)

The 0–line (generated by just $1 \in E_2^{*,0}(S)$) and the 1–line $E_2^{*,1}(S)$ consist of the image of the J homomorphism. These classes are all permanent cycles; the image of the 1–line under i is $\alpha_1 v_1^m$ for $m \geq 0$. The 0–line of $E_2(S/3)$ is the polynomial algebra on v_1.

Fact 2.3 *Let* $X = S$, $S/3$, *or* $S/(3, v_1^m)$ *for* $m \geq 1$.

(i) $E_2^{s,f}(X) = 0$ *if* $s + f \not\equiv 0 \pmod 4$.

(ii) $E_2^{s,f}(X) = E_5^{s,f}(X)$.

Proof See [14, Proposition 4.4.2] for the statement about $X = S$. This sparseness for the sphere also implies the first statement for $X = S/3$ and $S/(3, v_1^m)$, as can be seen by looking at the degrees of the long exact sequences in Ext groups corresponding to the short exact sequences

$$\mathrm{BP}_* \xrightarrow{3} \mathrm{BP}_* \to \mathrm{BP}_*/3 \quad \text{and} \quad \mathrm{BP}_*/3 \xrightarrow{v_1^m} \mathrm{BP}_*/3 \to \mathrm{BP}_*/(3, v_1^m).$$

The second statement follows from the first. □

Most of our calculations in the Adams–Novikov spectral sequence for $S/(3, v_1^m)$ for $m \geq 2$ implicitly use the following fact.

Theorem 2.4 [10] *For $m \geq 2$, $S/(3, v_1^m)$ is a ring spectrum.*

It is also well known that $S/3$ is a ring spectrum.

Lemma 2.5 (i) *If $x \in E_{10}(S)$, then $\beta_1^6 x$ is zero in $E_{10}(S)$. If $x \in E_{10}(S/3)$, then $\beta_1^6 x$ is zero in $E_{10}(S/3)$.*

(ii) *If $x \in E_6(S)$, then $\alpha_1\beta_1^3 x$ is zero in $E_6(S)$. If $x \in E_6(S/3)$, then $\alpha_1\beta_1^3 x$ is zero in $E_6(S/3)$.*

(iii) *We have $v_1^2 \cdot \beta_1 = 0$ in $E_2(S/3)$.*

Proof For (i), the classical differential $d_9(\alpha_1\beta_4) = \beta_1^6$ (see Table 1) implies that $\beta_1^6 = 0$ in $E_{10}(S)$, and hence $\beta_1^6 = 0$ in $E_{10}(S/3)$. Part (ii) is an analogous consequence of the Toda differential $d_5(\beta_{3/3}) = \alpha_1\beta_1^3$. Part (iii) is [17, Lemma 2.13]. □

Next, we record some basic facts about transferring differentials by naturality across various Adams–Novikov spectral sequences.

Lemma 2.6 *Let $m \geq 1$.*

(i) *If there is a nontrivial differential $d_5(x) = y$ in $E_5(S)$ where y is not 3–divisible, then there is a nontrivial differential $d_5(i(x)) = i(y)$ in $E_5(S/3)$.*

(ii) *If there is a nontrivial differential $d_5(j(x)) = j(y)$ in $E_5(S)$ for $x, y \in E_5(S/3)$, then $d_5(x) \neq 0$, and $d_5(x) \equiv y \pmod{\operatorname{Im}(i)}$.*

Proof For (i), by naturality of i, there is a differential $d_5(i(x)) = i(y)$. We just need to check that $i(y)$ is nonzero in $E_5(S/3)$. This follows from the fact that $E_2(S/3) = E_5(S/3)$ and the assumption that $i(y)$ is nonzero in $E_2(S/3)$. For (ii), since j commutes with the differential and $E_2(S/3) = E_5(S/3)$, we have that $d_5(x) \equiv y \pmod{\ker(j)}$. The long exact sequence

$$\cdots \xrightarrow{3} E_2(S) \xrightarrow{i} E_2(S/3) \xrightarrow{j} E_2(S) \xrightarrow{3} \cdots$$

implies that $\ker(j) = \operatorname{Im}(i)$. □

We use the following lemma without further mention when working with β elements, applying it to the case $x = v_2^i$.

Lemma 2.7 *Let $m \geq 1$. For any $x \in E_2(S/(3, v_1^m))$, we have $j_m(x) = j_{m+k}(v_1^k x)$.*

Proof The map of short exact sequences

$$\begin{tikzcd}
\mathrm{BP}_*/3 \arrow[r, "v_1^m"] \arrow[d, "="'] & \mathrm{BP}_*/3 \arrow[r] \arrow[d, "v_1^k"] & \mathrm{BP}_*/(3, v_1^m) \arrow[d, "v_1^k"] \\
\mathrm{BP}_*/3 \arrow[r, "v_1^{m+k}"] & \mathrm{BP}_*/3 \arrow[r] & \mathrm{BP}_*/(3, v_1^{m+k})
\end{tikzcd}$$

induces a map of long exact sequences after applying $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -)$. In particular, we have a commutative diagram

$$\begin{tikzcd}
\overbrace{\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(3, v_1^m))}^{E_2(S/(3,v_1^m))} \arrow[r, "j_m"] \arrow[d, "v_1^k"] & \overbrace{\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/3)}^{E_2(S/3)} \arrow[d, "="] \\
\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/(3, v_1^{m+k})) \arrow[r, "j_{m+k}"] & \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/3)
\end{tikzcd}$$

□

3 Computer-assisted calculations in the 143 stem

In this section we study the Adams–Novikov spectral sequence for $S/3$ in the 143 stem and nearby stems; this is the main technical input needed for Theorem 4.6. We make use of computer calculations of the Adams–Novikov E_2 page for the sphere; the specific facts from the computer data we use are given in Lemma 3.1. The results from this section that are used later are Lemma 3.2 and Proposition 3.7. The former follows immediately from the $\mathbb{F}_3$–vector space structure of $E_2^{*,*}(S)$. The rest of the section is devoted to proving the latter, which says that every permanent cycle in $\pi_{143}(S/3)$ is detected in filtration ≤ 5. This requires more careful analysis using the multiplicative structure of the E_2 page. Lemmas 3.5 and 3.6 give the differentials responsible for killing higher filtration elements in $E_2^{143,*}(S/3)$.

We encourage the reader to refer to the Adams–Novikov chart in [4] while reading this section. Table 1 is a summary of this data: all of the differentials in the chart in [4] are derived from α_1, β_1 and β_2–multiples of the classes in Table 1. Here x_{57} is the generator of $E_2^{57,3}(S)$, x_{75} is the generator of $E_2^{75,5}(S)$, and x_{96} is the generator of $E_2^{96,4}(S)$. Moreover, the differentials are complete through stem 108.

source					
s	f	source	d_r	target	reason
34	2	$\beta_{3/3}$	d_5	$\alpha_1\beta_1^3$	Toda differential
57	3	x_{57}	d_5	$\beta_1^3\beta_2$	forced by [14, Table A3.4]
58	2	β_4	d_5	$\alpha_1\beta_1^2\beta_{3/3}$	forced by [14, Table A3.4]
61	6	$\alpha_1\beta_4$	d_9	β_1^6	forced by [14, Table A3.4]
89	3	nonzero class	d_5	nonzero class	forced by [14, Table A3.4]
96	4	x_{96}	d_5	$\beta_1^2 x_{75}$	see Lemma 3.4

Table 1: Some classical Adams–Novikov differentials.

Lemma 3.1 [4; 5] (i) *We have* $\dim(E_2^{81,3}(S)) = 2$, $E_2^{81,7}(S) = \mathbb{F}_3\{\alpha_1\beta_1^2\beta_4\}$ *and* $\dim(E_2^{81,f}(S)) = 0$ *if* $f \neq 3, 7$.

(ii) $E_2^{95,9}(S) = \mathbb{F}_3\{\beta_1^2 x_{75}\}$, *where* x_{75} *is the generator of* $E_2^{75,5}(S)$, *and we have* $\alpha_1\beta_1^2 x_{75} \neq 0$. *The only other generator in* $E_2^{95,\geq 9}(S)$ *is* $\alpha_1\beta_1^4\beta_2^2 \in E_2^{95,13}(S)$.

(iii) *We have that* $\dim(E_2^{99,5}(S)) = 2$ *and* $\dim(\alpha_1 E_2^{99,5}) = 1$, *and one of the generators of* $E_2^{99,5}(S)$ *is* $\alpha_1 x_{96}$, *where* x_{96} *is the generator of* $E_2^{96,4}(S)$. *Moreover,* $E_2^{99,17}(S) = \mathbb{F}_3\{\alpha_1\beta_1^7\beta_2\}$, *and* $E_2^{99,f}(S) = 0$ *for* $f \neq 5, 17$.

(iv) $E_2^{135,5}(S) = 0 = E_2^{134,6}(S)$.

(v) $E_2^{141,15}(S) = \beta_1^6 E_2^{81,3}(S)$, *and this group has dimension* 2.

(vi) *Figure 2, left, displays the vector space structure of* $E_2^{s,f}(S)$ *for* $140 \leq s \leq 144$, *as well as selected multiplicative structure.*

In Figure 2, left, the names in $E_2^{141,15}(S)$ follow from the proof of Lemma 3.3; other names are multiplications computed using Wang's program.

Lemma 3.2 *If* $x \in E_2^{143,5}(S/3)$ *is* v_1^2*–divisible, then* $x = 0$.

Proof Lemma 3.1(iv) implies $E_2^{135,5}(S/3) = 0$. □

Lemma 3.3 *We have that* $E_2^{81,3}(S)$ *is* 2*–dimensional, and both generators are permanent cycles.*

Proof By [14, Table A3.4], $\pi_{81}(S)_3^\wedge$ is 2–dimensional, and is generated by γ_2 and $\langle\alpha_1, \alpha_1, \beta_5\rangle$. Lemma 3.1(i) gives the structure of $E_2^{81,*}(S)$. It suffices to show that

$\alpha_1\beta_1^2\beta_4$ supports a nontrivial differential; Table 1 implies $d_9(\alpha_1\beta_1^2\beta_4)=\beta_1^8$. Moreover, it is clear from an $E_2(S)$ chart (see [4]) that β_1^8 cannot be the target of a shorter differential. □

Lemma 3.4 *There is a differential*

$$d_5(x_{96}) = \beta_1^2 x_{75}.$$

The $\mathbb{F}_3$–vector space $\alpha_1 E_2^{99,5}(S)$ is 1–dimensional and is generated by a class $\alpha_1 x_{99}$, where x_{99} is a permanent cycle.

See Lemma 3.1 for element definitions. The second sentence is used implicitly when identifying one of the generators of $E_2^{142,14}(S)$ as $\alpha_1\beta_1^4 x_{99}$ (as seen in Figure 2, left): Wang's program only shows that there is a nonzero element in $E_2^{142,14}(S)$ that is $\alpha_1\beta_1^4$ times an element of $E_2^{99,5}(S)$.

Proof By [14, Table A3.4], $\pi_{96}(S)^\wedge_3 = 0$, and the generator $x_{96} \in E_2^{96,4}(S)$ must support a nontrivial differential as it cannot be a target for degree reasons. We claim this implies a differential $d_5(x_{96}) = \beta_1^2 x_{75}$: by Lemma 3.1(ii) the only other possible target is $\alpha_1\beta_1^4\beta_2^2 \in E_2^{95,13}(S)$ (a possibility for $d_9(x_{96})$), but this is zero in E_6 by Lemma 2.5(ii) as it is $\alpha_1\beta_1^3$ times the permanent cycle $\beta_1\beta_2^2$.

From Lemma 3.1(ii), we have that $\alpha_1 d_5(x_{96}) = d_5(\alpha_1 x_{96}) = \alpha_1\beta_1^2 x_{75}$ is nonzero. By [14, Table A3.4], we have $\pi_{99}(S)^\wedge_3/\operatorname{Im} J \cong \mathbb{F}_3$. We claim this permanent cycle

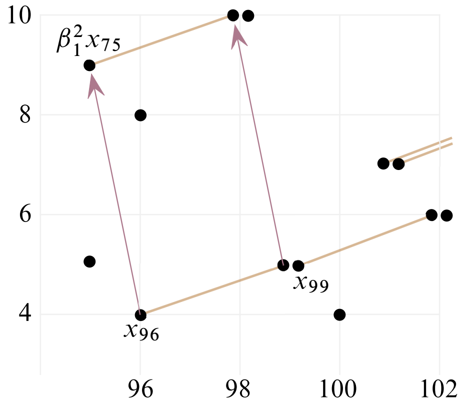

Figure 1: $E_2^{s,f}(S)$ in degrees $95 \le s \le 102$ and $4 \le f \le 10$. Brown lines represent α_1–multiplication. Each dot represents a copy of $\mathbb{F}_3$. The information in this chart used in the proof of Lemma 3.4 is summarized in Lemma 3.1(ii) and (iii).

is detected in filtration 5. By Lemma 3.1(iii), the only other possibility is $\alpha_1\beta_1^7\beta_2 \in E_2^{99,17}(S)$, which is the target of a d_5 differential by Lemma 2.5(ii). Let x_{99} denote the permanent cycle in $E_2^{99,5}(S)$. Since $\alpha_1^2 = 0$ and $\dim(\alpha_1 E_2^{99,5}(S)) = 1$ by Lemma 3.1(iii), we have that $\alpha_1 E_2^{99,5}(S)$ is generated by $\alpha_1 x_{99}$. □

Lemma 3.5 *The generator of $E_2^{142,10}(S)$ supports a nontrivial Adams–Novikov d_5 differential. The generator of $E_2^{142,6}(S)$ supports a nontrivial Adams–Novikov d_9 differential.*

Proof Combining Lemma 3.1(v) with Lemma 3.3, we have that the 2–dimensional vector space $E_2^{141,15}(S)$ is generated by β_1^6–divisible permanent cycles. By Lemma 2.5(i), both classes in $E_2^{141,15}(S)$ are hit by some differential. By the vector space structure of $E_2^{142,*}(S)$ displayed in Figure 2, left, the only possibilities are the indicated d_5 and d_9. □

Lemma 3.6 *The generator of $E_2^{143,9}(S)$ supports a nontrivial Adams–Novikov d_5 differential hitting $\alpha_1\beta_1^4 x_{99}$, where x_{99} is the permanent cycle introduced in Lemma 3.4. One of the two generators of $E_2^{143,5}(S)$ supports a nontrivial Adams–Novikov d_9 differential hitting $\beta_1^6\beta_{6/3}$.*

Proof This proof relies on Figure 2, left, in particular the fact that the elements mentioned are all nonzero. For the first statement, we have $d_5(\beta_{3/3}\cdot\beta_1 x_{99}) = \alpha_1\beta_1^3\cdot\beta_1 x_{99}$ since x_{99} (and hence $\beta_1 x_{99}$) is a permanent cycle. Since $\beta_{6/3} \in E_2^{82,2}(S)$ is a permanent cycle by [14, Table A3.4], we may apply Lemma 2.5(i) to show that $\beta_1^6\beta_{6/3} \in E_2^{142,14}(S)$ is the target of a differential d_r for $r \le 9$. Since the group $E_2^{143,9}(S)$ is one-dimensional and we proved above that the generator supported a nontrivial d_5, the element $\beta_1^6\beta_{6/3}$ must be hit by a d_9. □

Proposition 3.7 *Every element in $\pi_{143}(S/3)$ is detected in Adams–Novikov filtration ≤ 5.*

Proof We list the elements in $E_2^{143,f}(S/3)$ for $f > 5$ in Table 2.

We encourage the reader to refer to Figure 2 alongside the rest of the proof: the diagram on the right is derived from that on the left.

Filtration 9 We claim that both classes $E_2^{143,9}(S/3)$ support d_5 differentials. The bottom cell class does so because of Lemmas 3.6 and 2.6(i), and the top cell class does so because of Lemmas 3.5 and 2.6(ii).

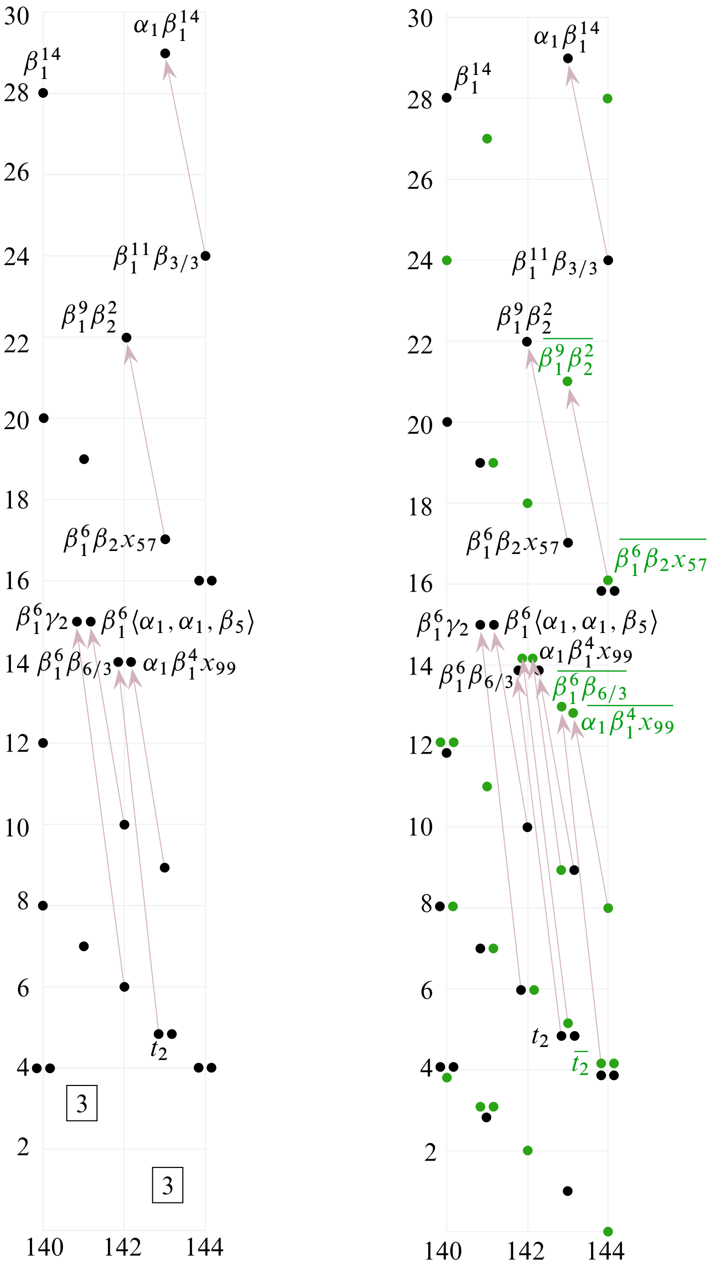

Figure 2: Left: $E_2^{s,f}(S)$ in degrees $140 \le s \le 144$, along with some Adams–Novikov differentials. A box containing "3" denotes a copy of $\mathbb{Z}/27$. Multiplications by α_1 are not shown. Right: $E_2^{s,f}(S/3)$ in degrees $140 \le s \le 144$, along with some Adams–Novikov differentials. Green dots denote top cell classes. Multiplications by α_1 are not shown.

filtration f	# bottom cell generators	# top cell generators
9	1	1
13	0	2
17	1	0
21	0	1
29	1	0

Table 2: Classes in $E_2^{143,f}(S/3)$ for $f \geq 2$.

Filtration 13 We may take the two generators of $E_2^{143,13}(S/3)$ to be classes $\overline{\alpha_1\beta_1^4x_{99}}$ and $\overline{\beta_1^6\beta_{6/3}}$ defined such that their image under j is $\alpha_1\beta_1^4x_{99}$ and $\beta_1^6\beta_{6/3}$, respectively. By Lemma 2.6(ii), the d_5 in Lemma 3.6 induces a d_5 differential hitting $\alpha_1\beta_1^4x_{99}$; note that $\mathrm{Im}(i) = 0$ in this degree.

By Lemma 3.6 we have a class $t_2 \in E_2^{143,5}(S)$ such that $d_9(t_2) = \beta_1^6\beta_{6/3}$. Let $\overline{t_2}$ be the top cell class in $E_2^{144,4}(S/3)$ associated to the 3–torsion element t_2. We wish to show that there is a differential $d_9(\overline{t_2}) = \overline{\beta_1^6\beta_{6/3}}$. First we check that $\overline{t_2}$ survives to the E_9 page. The only possible targets for such a shorter differential are in $E_2^{143,9}(S/3)$, and we showed above that these both support nontrivial d_5 differentials. The map induced by j on E_2 pages shows that $d_9(\overline{t_2}) \equiv \overline{\beta_1^6\beta_{6/3}}$ modulo $\ker(j)$. We have $E_9^{143,13}(S/3) = \mathbb{F}_3\{\overline{\beta_1^6\beta_{6/3}}\}$, and $j(\overline{\beta_1^6\beta_{6/3}}) = \beta_1^6\beta_{6/3}$, which is nonzero in $E_9(S)$. Thus there is a nonzero d_9 differential as claimed.

Filtration 17 The generator of $E_2^{143,17}(S)$ is $\beta_1^6\beta_2x_{57}$, where x_{57} is the generator of $E_2^{57,3}(S)$. Using a differential in Table 1, we have a differential $d_5(\beta_1^6\beta_2x_{57}) = \beta_1^9\beta_2^2$. By Lemma 2.6(i) we have a differential $d_5(i(\beta_1^6\beta_2x_{57})) = i(\beta_1^9\beta_2^2)$.

Filtration 21 By Lemma 2.6(ii), the d_5 differential on $\beta_1^6\beta_2x_{57}$ discussed in the filtration 17 case above gives rise to a differential $d_5(\overline{\beta_1^6\beta_2x_{57}}) = \overline{\beta_1^9\beta_2^2}$ over $S/3$.

Filtration 29 The generator of $E_2^{143,29}(S/3)$ is $i(\alpha_1\beta_1^{14})$; this class is zero in $E_6(S/3)$ by Lemma 2.5(ii). $\square$

Remark 3.8 The dependence of Proposition 3.7 on computer calculations would be reduced if we could make precise the observation that much of the Adams–Novikov E_2–page is β_1–periodic, and classes in high filtrations are highly β_1–divisible. Using [12, Theorem 2.3.1, Remark 2.3.5(c)], one can prove that multiplication by β_1 is an isomorphism on the Adams E_2 page restricted to Adams filtration f_A, stem s, and filtration ν in the algebraic Novikov spectral sequence $\mathrm{Ext}_A^{*,*}(\mathbb{F}_3,\mathbb{F}_3) \Rightarrow E_2^{*,*}(S)$ if

$$f_A > \tfrac{1}{23}s + \tfrac{24}{23}\nu + \tfrac{159}{23}.$$

By keeping track of the effect on the algebraic Novikov spectral sequence, one can derive that β_1 acts injectively (up to higher algebraic Novikov filtration) on the subspace of $E_2^{s,f}(S)$ in algebraic Novikov filtration ν if

$$f > \tfrac{1}{23}s + \tfrac{1}{23}\nu + \tfrac{169}{23}. \tag{4}$$

Surjectivity is harder to prove. Even if we knew that β_1 acted isomorphically on the region (4) (which is often true), this is not enough to prove the β_1–divisibility results we need. For example, in Proposition 3.7 we use the fact that the generator x of $E_2^{143,17}(S)$ is divisible by β_1^6. This element has $\nu = 0$, and $\beta_1^{-1}x$ and $\beta_1^{-2}x$ lie in the region (4) but $\beta_1^{-3}x$ does not. Improving this bound would also be of use more generally to the study of the 3–primary Adams and Adams–Novikov spectral sequences.

4 Survival of v_2^9

In this section, we prove Theorem 4.6, which says that v_2^9 is a permanent cycle in $E_2(S/(3, v_1^8))$. We first explain the choice of exponent of v_1. Since $\eta_R(v_2) \equiv v_2 + v_1 t_1^3 - v_1^3 t_1 \pmod 3$ in the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ (see eg [14, (6.4.16)]), we have that v_2^3 is an element of $E_2(S/(3, v_1^m))$ for $m \le 3$, and v_2^9 is an element of $E_2(S/(3, v_1^m))$ for $m \le 9$. On the other hand, we would like to work with $m \ge 8$, since those are the values of m for which $\beta_{9/8}$ is in the image of the composition of Adams–Novikov E_2 page boundary maps $E_2(S/(3, v_1^m)) \to E_2(S/3) \to E_2(S)$. Trivial modifications to the work in this section show that $v_2^9 \pm v_1^8 v_2^7$ is a self-map on $S/(3, v_1^9)$; see Remark 4.8. However, this slight strengthening is not necessary for our purposes, and we write down our results for $v_2^9 \in \pi_*(S/(3, v_1^8))$ essentially for cosmetic reasons, avoiding the correction term. To obtain the families in Theorem 5.1 other than $\beta_{9t+9/j}$, it suffices to work with $S/(3, v_1^4)$.

The main ingredients for proving Theorem 4.6 are Lemma 3.2 and Proposition 3.7 from the previous section, and the following lemma (below, specialized to our setting) due to the second author. It draws a connection between hidden v_1^8–extensions in $\pi_*(S/3)$, and differentials of the minimum length (ie d_5 differentials) in the Adams–Novikov spectral sequence for $S/(3, v_1^8)$.

Lemma 4.1 [15, Lemma 1.4] *Let $m \geq 1$. Suppose we have $y \in E_5(S/(3,v_1^m))$ such that $j_m(y)$ is a nontrivial permanent cycle in $E_5^{s,f}(S/3)$, and let w denote an element in $E_5(S/3)$ detecting the product $v_1^m \cdot \{j_m(y)\} \in \pi_*(S/3)$. Then there is a differential*

$$d_5(y) = i_m(w)$$

in $E_5(S/(3,v_1^m))$.

In the next lemma we separate out the general strategy used to prove that v_2^9 and other elements in Section 5 are permanent cycles.

Lemma 4.2 (i) *Let $x \in E_2^{s,f}(S/3)$ for $f \leq 3$ be such that $j(x) \in E_2^{s-1,f+1}(S)$ is a permanent cycle and $\{j(x)\} \in \pi_{s-1}(S)$ is an essential element of order 3. Furthermore, suppose that $\mathrm{Im}(i\colon E_2^{s,f}(S) \to E_2^{s,f}(S/3))$ consists of permanent cycles. Then $x \in E_2^{s,f}(S/3)$ is a permanent cycle.*

(ii) *Let $x \in E_2^{s,f}(S/(3,v_1^m))$ for $f \leq 3$ be such that $j_m(x) \in E_2^{s-4m-1,f+1}(S/3)$ is a permanent cycle and $\{j_m(x)\}$ is an essential element with $v_1^m \cdot \{j_m(x)\} = 0$ in $\pi_*(S/3)$. Furthermore, suppose that $\mathrm{Im}(i_m\colon E_2^{s,f}(S/3) \to E_2^{s,f}(S/(3,v_1^m)))$ consists of permanent cycles. Then $x \in E_2^{s,f}(S/(3,v_1^m))$ is a permanent cycle.*

Proof We just prove (i), as (ii) is analogous. Consider the exact sequences

$$\begin{gathered} E_2^{s,f}(S) \xrightarrow{3} E_2^{s,f}(S) \xrightarrow{i} E_2^{s,f}(S/3) \xrightarrow{j} E_2^{s-1,f+1}(S), \\ \pi_s(S) \xrightarrow{3} \pi_s(S) \xrightarrow{i} \pi_s(S/3) \xrightarrow{j} \pi_{s-1}(S) \xrightarrow{3} \pi_{s-1}(S), \end{gathered} \tag{5}$$

associated to the cofiber sequence $S \xrightarrow{3} S \xrightarrow{i} S/3 \xrightarrow{j} S$. (For the first long exact sequence, we are using the fact that j induces the zero map in BP–homology.) Suppose $x \in E_2^{s,f}(S/3)$ is an element such that $j(x)$ is a permanent cycle with $3 \cdot \{j(x)\} = 0$. Then there exists an element $\xi \in \pi_s(S/3)$ such that $j(\xi) = \{j(x)\}$. Since $j\colon S/3 \to \Sigma S$ induces a map of Adams–Novikov spectral sequences, the induced map on homotopy $j\colon \pi_*(S/3) \to \pi_*(\Sigma S) = \pi_{*-1}(S)$ respects Adams–Novikov filtration; thus $j(x)$ being detected in filtration $f+1$ implies ξ is detected in filtration $\leq f$. The assumption $f \leq 3$ combined with Fact 2.3 implies that ξ is detected in filtration f. We may write the detecting element as $x + y$ for some $y \in E_2^{s,f}(S/3)$. By the geometric boundary theorem, $j(x+y)$ converges to $j(\xi)$, and we also have that $j(x)$ converges to $j(\xi)$. So $j(y)$ is a boundary. But $j(y)$ has filtration ≤ 4, so Fact 2.3 implies $j(y) = 0$ in $E_2(S)$. By (5), we have that y is in the image of i. By the assumption about $\mathrm{Im}(i)$, y is a permanent cycle, and we have from above that $x + y$ is a permanent cycle. Therefore, x is a permanent cycle. □

Lemma 4.3 *The element* $\overline{\beta_{9/8}} = j_8(v_2^9) \in E_2^{111,1}(S/3)$ *is a permanent cycle.*

Proof By [14, Table A3.4], there exists $c \in \{\pm 1\}$ such that $x_{106} = \beta_{9/9} + c\beta_7$ in $E_2^{106,2}(S)$ is a 3–torsion permanent cycle. Lemma 4.2(i) applies since $E_2^{*,1}(S)$ consists of permanent cycles; it implies that $\overline{\beta_{9/9}} + c\overline{\beta_7} = j_9(v_2^9) + cj_1(v_2^7) \in E_2^{107,1}(S/3)$ is a permanent cycle. Hence $v_1 \cdot (j_9(v_2^9) + cj_1(v_2^7)) = v_1 \cdot j_9(v_2^9) = j_8(v_2^9)$ is a permanent cycle. □

Lemma 4.4 *We have that* $d_5(v_2^9) = 0$ *in* $E_5^{143,5}(S/(3, v_1^8))$.

Proof Let $x = d_5(v_2^9)$. We first consider the image of this differential along the natural map induced by $i_3' \colon S/(3, v_1^8) \to S/(3, v_1^3)$. Since v_2^3 is an element of $E_2^{48,0}(S/(3, v_1^3))$, by the Leibniz rule and Theorem 2.4, we have $i_3'(x) = i_3'(d_5(v_2^9)) = d_5((v_2^3)^3) = 3v_2^6 d_5(v_2^3) = 0$.

Using Lemma 4.3, we have

$$j_8(x) = j_8(d_5(v_2^9)) = d_5(j_8(v_2^9)) = 0$$

in $E_5(S/3) = E_2(S/3)$. Thus the exact sequence

$$E_2^{143,5}(S/3) \xrightarrow{i_8} E_2^{143,5}(S/(3, v_1^8)) \xrightarrow{j_8} E_2^{111,1}(S/3)$$

gives $x = i_8(y)$ for some $y \in E_2^{143,5}(S/3)$.

Consider the commutative diagram of cofiber sequences obtained using Verdier's axiom:

$$\begin{tikzcd}
S/3 \arrow[r, "v_1^3"] \arrow[d, "i_5"'] & S/3 \arrow[r, "i_3"] \arrow[d, "i_8"] & S/(3, v_1^3) \arrow[d, equal] \\
S/(3, v_1^5) \arrow[r, "v_1^3"] \arrow[d, "j_5"'] & S/(3, v_1^8) \arrow[r, "i_3'"] \arrow[d, "j_8"] & S/(3, v_1^3) \arrow[d] \\
S/3 \arrow[r, equal] & S/3 \arrow[r] & *
\end{tikzcd} \tag{6}$$

This implies that $i_3 = i_3' \circ i_8$; in particular, $i_3(y) = i_3'(i_8(y)) = i_3'(x) = 0$. By the long exact sequence corresponding to the top row of (6), we have that y is v_1^3–divisible. By Lemma 3.2, $y = 0$. □

Lemma 4.5 *The product* $v_1^8 \cdot \{j_8(v_2^9)\}$ *is zero in* $\pi_*(S/3)$.

Proof Let $w \in E_5^{143,f}(S/3)$ be a representative of $v_1^8 \cdot \{j_8(v_2^9)\} \in \pi_{143}(S/3)$. By Proposition 3.7, we have $f \le 5$, and by Fact 2.3, the only possibilities are $f = 1, 5$. The product $v_1^8 \cdot j_8(v_2^9)$ is zero on the E_2 page, so we must have $f = 5$.

By Lemma 4.3, Lemma 4.1 applies to v_2^9; combining this with Lemma 4.4, we have

$$0 = d_5(v_2^9) = i_8(w) \quad \text{in } E_5^{143,5}(S/(3,v_1^8)) = E_2^{143,5}(S/(3,v_1^8)).$$

Thus w is divisible by v_1^8 in $E_2^{143,5}(S/3)$. By Lemma 3.2, we must have $w = 0$ in $E_5^{143,5}(S/3)$. Since w was defined to be an element detecting $v_1^8 \cdot \{j_8(v_2^9)\}$, this product is zero in homotopy. □

Theorem 4.6 *The element $v_2^9 \in E_2^{144,0}(S/(3,v_1^8))$ is a permanent cycle in the Adams–Novikov spectral sequence computing $\pi_*(S/(3,v_1^8))$.*

Proof This will follow from applying Lemma 4.2(ii) to v_2^9. The first two hypotheses of that lemma are satisfied due to Lemmas 4.3 and 4.5. For the last hypothesis, note that $E_2^{144,0}(S/3)$ is generated by v_1^{36}, which is a permanent cycle. □

Corollary 4.7 *For $2 \le m \le 8$, the class v_2^9 in $\pi_{144}(S/(3,v_1^m))$ lifts to a class v_2^9 in $[S/(3,v_1^m), S/(3,v_1^m)]_{144}$.*

Proof Naturality of the map $S/(3,v_1^8) \to S/(3,v_1^m)$ for $m \le 8$ means that Theorem 4.6 directly implies $v_2^9 \in E_2^{144,0}(S/(3,v_1^m))$ is a permanent cycle. By [10, Theorem 6.1], $R = S/(3,v_1^m)$ is a (homotopy) ring spectrum for $m \ge 2$. Thus the desired self-map may be obtained as

$$R \to S \wedge R \xrightarrow{v_2^9 \wedge I} R \wedge R \xrightarrow{\mu} R. \qquad \square$$

Remark 4.8 Essentially the same argument shows that $v_2^9 \pm v_1^8 v_2^7$ is a permanent cycle in $E_2^{144,0}(S/(3,v_1^9))$. In the proof of Lemma 4.3 we show that $j_9(v_2^9) \pm j_1(v_2^7) = j_9(v_2^9 \pm v_1^8 v_2^7)$ is a permanent cycle. The proofs of Lemmas 4.4 and 4.5 go through without modification to show that $d_5(v_2^9 \pm v_1^8 v_2^7) = 0$ in $E_5^{143,5}(S/(3,v_1^9))$ and $v_1^9 \cdot \{j_9(v_2^9 \pm v_1^8 v_2^7)\} = 0$. In the proof of Theorem 4.6, we have $x = c_1(v_2^9 \pm v_1^8 v_2^7) + c_2 v_2^9 + c_3 v_1^{36}$. This time, v_2^9 and $v_1^8 v_2^7$ are not permanent cycles since $\beta_{9/9} = j(j_9(v_2^9))$ and $\beta_7 = j(j_9(v_1^8 v_2^7))$ are not permanent cycles, and v_1^{36} is a permanent cycle. Thus $v_2^9 \pm v_1^8 v_2^7$ is a permanent cycle for some choice of sign.

5 Survival of beta elements

Our goal in this section is to prove that several infinite families of $\beta_{a/b}$ elements are permanent cycles in the Adams–Novikov spectral sequence for the sphere. For indices a and b satisfying the conditions in [9, Theorem 2.6], Miller, Ravenel, and Wilson define cycles $\beta_{a/b}$ in $E_2^{*,2}(S)$ as the image of certain classes in $E_2^{*,0}(S/(3, v_1^b))$ under the composition $j \circ j_b$. In this section we will only consider classes $\beta_{sp^n/b}$ (with $p \nmid s$) such that $b \le p^n$, which enables us to use the equivalent, but simpler, definition

$$\beta_{a/b} = j(j_b(v_2^a)) \in E_2^{16a-4b-2,2}(S)$$

(at $p = 3$). These elements are defined using the boundary maps j and j_b on Ext associated to the short exact sequences

$$\mathrm{BP}_* \xrightarrow{3} \mathrm{BP}_* \to \mathrm{BP}_*/3 \quad \text{and} \quad \mathrm{BP}_*/3 \xrightarrow{v_1^b} \mathrm{BP}_*/3 \to \mathrm{BP}_*/(3, v_1^b);$$

by the geometric boundary theorem these coincide with the maps induced on Adams–Novikov spectral sequences by the maps j and j_b of spectra that we have been considering in this paper. Recall the convention that $\beta_a := \beta_{a/1}$.

Suppose that $\beta_{a/b}$ is a permanent cycle with $b \le 8$ and, in addition, suppose that the corresponding element in homotopy $\beta^h_{a/b} \in \pi_*(S)$ factors as

$$\beta^h_{a/b} \colon S \xrightarrow{B_{a/b}} S/(3, v_1^b) \xrightarrow{j j_b} S \quad \text{for some } B_{a/b} \in \pi_*(S/(3, v_1^b)). \tag{7}$$

In this case, for $t \ge 1$, Corollary 4.7 allows us to define elements in $\pi_*(S)$:

$$\beta^h_{9t+a/b} \colon S \xrightarrow{B_{a/b}} S/(3, v_1^b) \xrightarrow{(v_2^9)^t} S/(3, v_1^b) \xrightarrow{j j_b} S.$$

We warn that existence of a factorization (7) is not automatic, even if $\beta_{a/b}$ is a permanent cycle and such a factorization exists on the level of Adams–Novikov E_2 pages. Our goal is to show the following, proved at the end of the section.

Theorem 5.1 *For all $t \ge 0$, the classes*

$$\begin{array}{llll} \beta_{9t+3/j} & \textit{for } j = 1, 2, & \beta_{9t+6/j} & \textit{for } j = 1, 2, 3, \\ \beta_{9t+9/j} & \textit{for } j = 1, \dots, 8, & \alpha_1\beta_{9t+3/3} & \textit{and} \quad \alpha_1\beta_{9t+7} \end{array}$$

are permanent cycles in the Adams–Novikov spectral sequence for the sphere.

Since $\beta_{3/3}$ and β_7 support Adams–Novikov differentials, none of the families in Theorem 5.1 are trivially multiplicative consequences of a different family. Instead, we have $\alpha_1\beta_{3/3} \in \langle\alpha_1, \alpha_1, \beta_1^3\rangle$ and $\alpha_1\beta_7 \in \langle\alpha_1, \alpha_1, \beta_1^2\beta_{6/3}\rangle$. As we will see in Section 6, the families $\alpha_1\beta_{9t+3/3}$, $\beta_{9t+6/3}$ and $\alpha_1\beta_{9t+7}$ have nontrivial image in π_* tmf, along with the family β_{9t+1} constructed in [3, Corollary 1.2].

Lemma 5.2 *The class $j_4(v_1^2v_2^3) \in E_2^{39,1}(S/3)$ is a permanent cycle such that*

$$j(j_4(v_1^2v_2^3)) = \beta_{3/2} \in E_2^{38,2}(S) \quad \textit{and} \quad v_1^4 \cdot \{j_4(v_1^2v_2^3)\} = 0 \in \pi_*(S/3).$$

Proof We have $j(j_4(v_1^2v_2^3)) = j(j_2(v_2^3)) = \beta_{3/2}$ in $E_2(S)$ by the definition of the β elements along with Lemma 2.7. By classical computations of the Adams–Novikov E_2 page (see eg [14, Figure 1.2.19]), $E_2^{38,f}(S) = 0 = E_2^{37,f+1}(S)$ for $f \geq 3$, so $E_2^{38,f}(S/3) = 0$ for $f \geq 3$. Thus $j_4(v_1^2v_2^3) \in E_2^{39,1}(S/3)$ cannot support a differential of any length.

As $v_1^4 \cdot j_4 = 0$ as a map $E_2(S/(3, v_1^4)) \to E_2(S/3)$, it remains to rule out hidden v_1^4–extensions on $\beta'_{3/2} := \{j_4(v_1^2v_2^3)\} \in \pi_{39}(S/3)$. Using [14, Table A3.4] we have $\pi_{51}(S/3) = \mathbb{F}_3\{\alpha_{13}, \overline{\beta_1^5}\}$, and so $v_1^3 \cdot \beta'_{3/2} = c\overline{\beta_1^5} = c\overline{\beta_1^2} \cdot \beta_1^3$ for some $c \in \mathbb{F}_3$. (If there were an α_{13} component, then the extension would not be hidden.) We have $v_1 \cdot \overline{\beta_1^2} = 0$ for degree reasons, as Ravenel's table implies $\pi_{25}(S/3) = 0$. Thus $v_1 \cdot v_1^3 \cdot \beta'_{3/2} = 0$. □

Lemma 5.3 *The class $j_4(v_1v_2^6) \in E_2^{83,1}(S/3)$ is a permanent cycle such that*

$$j(j_4(v_1v_2^6)) = \beta_{6/3} \in E_2^{82,2}(S) \quad \textit{and} \quad v_1^4 \cdot \{j_4(v_1v_2^6)\} = 0 \in \pi_*(S/3).$$

Proof By Ravenel's table [14, Table A3.4], $\beta_{6/3}$ and β_6 are 3–torsion permanent cycles. Since $j(j_4(v_1v_2^6)) = \beta_{6/3}$ and $j(j_4(v_1^3v_2^6)) = \beta_6$, we apply Lemma 4.2(i) to $j_4(v_1v_2^6) \in E_2^{83,1}(S/3)$ and $j_4(v_1^3v_2^6) \in E_2^{91,1}(S/3)$, noting that $E_2^{*,1}(S)$ consists of permanent cycles. This shows that $j_4(v_1v_2^6)$ and $j_4(v_1^3v_2^6)$ are permanent cycles.

To determine $v_1^4 \cdot \{j_4(v_1v_2^6)\}$, we first consider the possibilities for $v_1^2 \cdot \{j_4(v_1v_2^6)\} \in \pi_{91}(S/3)$: from Ravenel's tables, we have

$$\pi_{91}(S/3) = \mathbb{F}_3\{\alpha_{23}, \beta_1\gamma_2, \beta_1x_{81}, \beta'_6\},$$

where $j(\beta'_6) = \beta_6$. Since $v_1^2 \cdot \beta_1 = 0$ in homotopy by Lemma 2.5(iii), we have $v_1^2 \cdot \{j_4(v_1v_2^6)\} = c_1\alpha_{23} + c_2\beta'_6$ for $c_i \in \mathbb{F}_3$. If $c_1 \neq 0$, then $v_1^4 \cdot \{j_4(v_1v_2^6)\}$ would be detected in filtration 1, contradicting the fact that $v_1^4 \cdot j_4(v_1v_2^6) = 0$ in $E_2(S/3)$. So it suffices to show that $v_1^2 \cdot \beta_6' = 0$. From above, we may write $\beta_6' = \{j_4(v_1^3 v_2^6)\} = \{j_2(v_1 v_2^6)\}$. By [11, Lemma 3], $v_1 v_2^6 \in E_2^{100,0}(S/(3, v_1^2))$ is a permanent cycle, and hence so is $j_2(v_1 v_2^6)$. We have $\{j_2(v_1 v_2^6)\} = j_2(\{v_1 v_2^6\})$ by the geometric boundary theorem, and $v_1^2 \cdot j_2(\{v_1 v_2^6\}) = 0$ by definition of j_2 as a map on homotopy groups. □

Lemma 5.4 *The classes* $v_1^2 v_2^3 \in E_2^{56,0}(S/(3, v_1^4))$ *and* $v_1 v_2^6 \in E_2^{100,0}(S/(3, v_1^4))$ *are permanent cycles in the Adams–Novikov spectral sequence computing* $\pi_*(S/(3, v_1^4))$.

Proof Use Lemma 4.2(ii), with Lemmas 5.2 and 5.3 as input. To check the condition about the image of $i_4 \colon E_2(S/3) \to E_2(S/(3, v_1^4))$ in these degrees, note that $E_2^{*,1}(S/3)$ is generated by the image of $i \colon E_2^{*,1}(S) \to E_2^{*,1}(S/3)$, which consists of permanent cycles (these are all image of J classes), along with elements that map to β elements under j. Standard theory about the β elements [9] implies that $\overline{\beta_{3/2}}$ and $\overline{\beta_{6/3}}$ are the only such elements in the relevant degrees; these are both permanent cycles by Lemmas 5.2 and 5.3. □

Lemma 5.5 *The class* $\alpha_1 v_1 v_2^3 \in E_2^{55,1}(S/(3, v_1^4))$ *is a permanent cycle in the Adams–Novikov spectral sequence.*

Proof There is a Toda bracket $\langle \alpha_1, \alpha_1, \beta_1^3 \rangle \in \pi_{37}(S)$ detected by $\alpha_1 \beta_{3/3}$ in filtration 3, and this class is 3–torsion; see [14, Table A3.4]. In order to apply Lemma 4.2(i) to $j_4(\alpha_1 v_1 v_2^3) \in E_2^{38,2}(S/3)$, we must check that $E_2^{38,2}(S)$ consists of permanent cycles. It follows from standard facts about the Adams–Novikov 2–line [9] that $E_2^{38,2}(S) = \mathbb{F}_3\{\beta_{3/2}\}$. So we may conclude that $j_4(\alpha_1 v_1 v_2^3)$ is a permanent cycle.

Moreover, $v_1^4 \cdot \{j_4(\alpha_1 v_1 v_2^3)\}$ is zero in homotopy: since $\pi_{53}(S) = 0 = \pi_{54}(S)$ by [14, Table A3.4], we have $\pi_{54}(S/3) = 0$. In order to apply Lemma 4.2(ii) to $\alpha_1 v_1 v_2^3 \in E_2^{55,1}(S/(3, v_1^4))$, we must check that $E_2^{55,1}(S/3)$ consists of permanent cycles. This is true because the image of $E_2^{*,1}(S)$ consists of permanent cycles, and analysis of the 2–line reveals that there cannot be a class with nontrivial image in $E_2^{54,2}(S)$. Thus we have that $\alpha_1 v_1 v_2^3$ is a permanent cycle. □

Lemma 5.6 *The class* $\alpha_1 v_1 v_2^7 \in E_2^{119,1}(S/(3, v_1^2))$ *is a permanent cycle in the Adams–Novikov spectral sequence.*

Proof Since $S/(3, v_1^2)$ is a ring spectrum (Theorem 2.4), we may consider this element as a product $v_1 v_2^5 \cdot \alpha_1 v_2^2$. Oka [11, Lemma 2] showed that v_2^5 is a permanent cycle in $E_2^{80,0}(S/(3, v_1))$. This implies that $v_1 v_2^5$ is a permanent cycle in $E_2^{84,0}(S/(3, v_1^2))$.

Next we consider possible differentials on $\alpha_1 v_2^2 \in E_2^{35,1}(S/(3, v_1^2))$. An element in $E_2^{34,f}(S/(3, v_1^2))$ either has nonzero image under j_2 in $E_2^{25,f+1}(S/3)$ or is the image under i_2 of an element of $E_2^{34,f}(S/3)$. From classically known computations of the Adams–Novikov E_2 page (see eg [14, Figure 1.2.19]), we deduce that

$$E_2^{25,*}(S/3) = 0 \quad \text{and} \quad E_2^{35,\geq 3}(S/3) = \mathbb{F}_3\{\overline{\alpha_1\beta_1^3}\}.$$

This implies $E_2^{34,\geq 3}(S/(3, v_1^2))$ is generated by $i_2(\overline{\alpha_1\beta_1^3})$. Observe that $\overline{\alpha_1\beta_1^3} = \overline{\alpha_1}\beta_1^3 = v_1 \cdot \beta_1^3$. Thus the only possible nonzero differential on $v_1 v_2^5 \cdot \alpha_1 v_2^2$ is a d_5 with target $v_1 v_2^5 \cdot v_1 \beta_1^3$. But the target is divisible by v_1^2, hence zero in $E_5(S/(3, v_1^2))$. □

Proof of Theorem 5.1 We show that $\beta_{9t+3/2}$ and $\beta_{9t+3/1}$ are permanent cycles for $t \geq 0$. Since v_2^9 is a permanent cycle in $E_2(S/(3, v_1^8))$ by Theorem 4.6, its image in $E_2(S/(3, v_1^4))$ is a permanent cycle. Lemma 5.4 says that $v_1^2 v_2^3$ is a permanent cycle in $E_2(S/(3, v_1^4))$, so the product $v_1^2 v_2^3 \cdot v_2^{9t} \in E_2(S/(3, v_1^4))$ is a permanent cycle. Recall that $\beta_{9t+3/2} \in E_2(S)$ is defined as $j(j_2(v_2^{9t+3})) = j(j_4(v_1^2 v_2^{9t+3}))$ in $E_2(S/3)$. Since $j_4(v_1^2 v_2^{9t+3})$ is a permanent cycle, so is $j(j_4(v_1^2 v_2^{9t+3}))$. Since $v_1^2 v_2^3$ is a permanent cycle in $E_2(S/(3, v_1^4))$, so is $v_1^3 v_2^3$, so $\beta_{9t+3/1} = j(j_1(v_2^{9t+3})) = j(j_4(v_1^3 v_2^{9t+3}))$ is a permanent cycle in $E_2(S)$.

The family $\beta_{9t+9/8}$ (and hence $\beta_{9t+9/j}$ for $j < 8$) follows directly from the fact that v_2^9 is a permanent cycle in $E_2(S/(3, v_1^8))$. The other families of permanent cycles follow analogously, using Lemma 5.4 again as the input for $\beta_{9t+6/3} = j(j_4(v_1 v_2^{9t+6}))$, Lemma 5.5 as the input for $\alpha_1\beta_{9t+3/3} = j(j_4(\alpha_1 v_1 v_2^{9t+3}))$, and Lemma 5.6 as the input for $\alpha_1\beta_{9t+7} = j(j_2(\alpha_1 v_1 v_2^{9t+7}))$. □

6 3–Primary Hurewicz image of tmf

In this section we determine the image of the Hurewicz map $h\colon \pi_* S \to \pi_* \mathrm{tmf}$ induced by the unit map $S \to \mathrm{tmf}$. The target $\pi_* \mathrm{tmf}$ has been computed via the elliptic spectral sequence (see [1, Section 3]); this is the $Y(4)$–based Adams spectral sequence for tmf, where $Y(4)$ is the Thom spectrum of $\Omega U(4) \to \mathbb{Z} \times BU$. We will denote this spectral sequence by $E_r^{\mathrm{ell}}(\mathrm{tmf})$.

Theorem 6.1 (Hopkins–Mahowald, Bauer [1, Section 6]) *At $p = 3$, $\pi_* \mathrm{tmf}$ is generated by c_4, c_6, Δ, α, β and b, subject to the relation $c_4^3 - c_6^2 = 1728\Delta$ and the relations on the other generators displayed in Figure 3. Multiplication by $\Delta^3 \in \pi_{72}(\mathrm{tmf})$ is injective.*

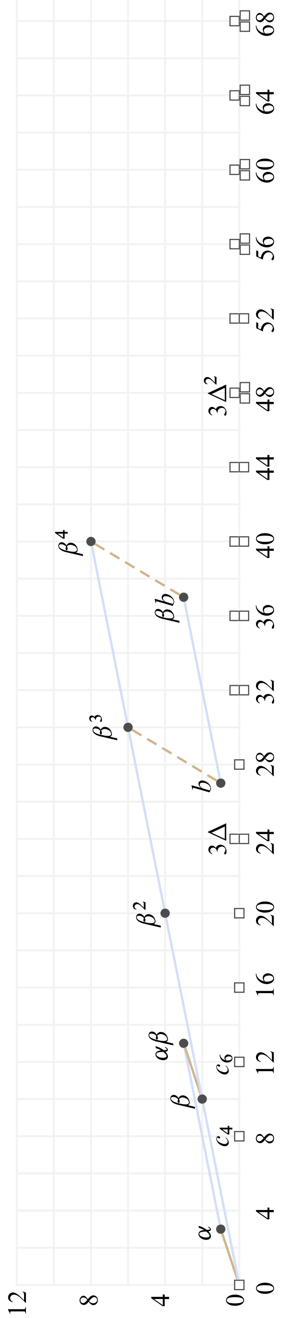

Figure 3: The E_∞ page of the elliptic spectral sequence computing π_s tmf for $0 \leq s \leq 76$. Dashed brown lines represent hidden α–multiples. Squares indicate copies of $\mathbb{Z}_{(3)}$ and dots indicate copies of $\mathbb{F}_3$.

We will show (Theorem 6.5) that all classes in filtration ≥ 2 are in the Hurewicz image, and the only classes in filtrations 0 and 1 in the image are the summands generated by 1 and α. Instead of directly mapping to the elliptic spectral sequence, we use the $K(2)$–local E–based Adams spectral sequence

$$E_2^E(\mathrm{TMF}) = H^*(G_{24}; E_*) \Rightarrow \pi_*(L_{K(2)}\mathrm{TMF})$$

where $E = E_2$ is height 2 Morava E–theory and TMF is the periodic version of tmf. There is a map of spectral sequences $E_r(S) \to E_r^E(\mathrm{TMF})$ induced by the natural maps $\mathrm{BP} \to E$ and $S \to \mathrm{TMF}$. Henn, Karamanov and Mahowald [7, Theorem 1.1] completely determine $E_2^E(\mathrm{TMF}/3)$ and provide formulas that we use to compute the map on E_2 pages $E_2(S) \to E_2^E(\mathrm{TMF})$ in cases of interest; see Lemmas 6.2 and 6.3. For each class in $E_2^E(\mathrm{TMF})$ in filtration ≥ 2, we identify a preimage in $E_2(S)$ that is among the classes proved to be permanent cycles in Theorem 5.1 or [3]; see Proposition 6.4. As we explain in the proof of Theorem 6.5, it suffices to understand the Hurewicz image in $\pi_*(L_{K(2)}\mathrm{TMF})$ because there is an injection $\pi_*(\mathrm{tmf}) \to \pi_*(L_{K(2)}\mathrm{TMF})$; see Lemma 6.6.

First we review some notation and basic facts. We have $E_*/3 = \mathbb{F}_9[\![u_1]\!][u^{\pm 1}]$, and there is a natural map $\mathrm{BP}_* \to E_*$ that sends $v_1 \mapsto u_1 u^{-2}$, $v_2 \mapsto u^{-8}$ and $v_i \mapsto 0$ for $i > 2$. Abusing notation, we will let v_i denote its image in $E_*/3$.

Recall $j : S/3 \to \Sigma S$ denotes the boundary map in the cofiber sequence $S \xrightarrow{3} S \to S/3$. We will also use j to refer to the map $j \wedge \mathrm{TMF} \colon \mathrm{TMF}/3 \to \Sigma\mathrm{TMF}$. Similarly, j_m will denote both boundary maps $S/(3, v_1^m) \to S/3$ and $\mathrm{TMF}/(3, v_1^m) \to \mathrm{TMF}/3$, depending on context.

Lemma 6.2 *In E_* we have*

$$\begin{aligned} v_2^3 &\equiv -\Delta^2 - v_1^2 v_2 \Delta && \bmod (3, v_1^6), \\ v_2^6 &\equiv \Delta^4 - v_1^2 v_2 \Delta^3 && \bmod (3, v_1^3), \\ v_2^{3^n} &\equiv -\Delta^{2\cdot 3^{n-1}} - v_1^{2\cdot 3^{n-1}} v_2^{3^{n-1}} \Delta^{3^{n-1}} && \bmod (3, v_1^{2\cdot 3^n}). \end{aligned}$$

Proof The formula $\Delta \equiv (1 - \omega^2 u_1^2 + u_1^4)\omega^2 u^{-12} \bmod (3, u_1^6)$ from [7, Proposition 5.1] implies

$$\begin{aligned} \Delta^2 &\equiv (1 - 2\omega^2 u_1^2 + u_1^4)(-v_2^3) && \bmod (3, v_1^6), \\ v_1^2 v_2 \Delta &\equiv v_2^3(\omega^2 u_1^2 + u_1^4) && \bmod (3, v_1^6). \end{aligned}$$

where ω denotes an 8th root of unity in $\mathbb{F}_9$. Combining these facts, we obtain the formula for v_2^3; the formulas for v_2^6 and $v_2^{3^n}$ follow from it by squaring and successive cubing, respectively. □

Let

$$\begin{aligned} H\colon E_2(S) &\to E_2^E(\mathrm{TMF}), \\ H'\colon E_2(S/3) &\to E_2^E(\mathrm{TMF}/3), \\ H'_m\colon E_2(S/(3,v_1^m)) &\to E_2^E(\mathrm{TMF}/(3,v_1^m)), \end{aligned}$$

denote the natural maps of spectral sequences.

Lemma 6.3 *We have*

$$\begin{array}{rrr} H(\alpha_1)=\alpha, & H'(j_3(v_2^3))\doteq\Delta\widetilde{\alpha}, & H'(j_3(v_1^2v_2^7))\doteq\Delta^4\widetilde{\alpha}, \\ H(\beta_1)\doteq\beta, & H(\beta_{3/3})\doteq\Delta\beta, & H(\beta_7)\doteq\Delta^4\beta, \\ H'(j_3(v_1^2v_2))\doteq\widetilde{\alpha}, & H'(j_3(v_2^6))\doteq\Delta^3\widetilde{\alpha}, & \\ & H(\beta_{6/3})\doteq\Delta^3\beta, & \end{array}$$

where $j(\widetilde{\alpha})=\beta$. (Here $\doteq$ denotes equality up to multiplication by a unit.)

Proof Following Bauer [1, Section 6], we have $H(\alpha_1)=\alpha$ since they both come from the cobar class $[t_1]$, and $H(\beta_1)=\beta$ because of the Massey products $\beta_1=\langle\alpha_1,\alpha_1,\alpha_1\rangle$ and $\beta=\langle\alpha,\alpha,\alpha\rangle$. We have $j(\widetilde{\alpha})=\beta$ and $j(j_3(v_1^2v_2))=\beta_1$, so

$$j(H'(j_3(v_1^2v_2)))=H(j(j_3(v_1^2v_2)))=\beta.$$

This specifies $H'(j_3(v_1^2v_2))$ up to the image of $E_2^E(\mathrm{TMF})$, but since $E_2^E(\mathrm{TMF}/3)$ is 1–dimensional in the degree of $\widetilde{\alpha}$, there is no ambiguity.

For the next column, we have in $E_2^E(\mathrm{TMF}/(3,v_1^3))$ that

$$H'(j_3(v_2^3))=j_3(H'_3(v_2^3))=j_3(-\Delta^2-v_1^2v_2\Delta)=-\Delta j_3(v_1^2v_2)=-\Delta\cdot\widetilde{\alpha},$$

using Lemma 6.2 and the earlier fact about $H'(j_3(v_1^2v_2))$. Note that $j_3(\Delta^n)=0$ since Δ^n is in the image of $E_2^E(\mathrm{TMF}/3)$. Now apply j to get the statement about $H(\beta_{3/3})$. The remaining facts in this column are analogous, using the fact that $\beta_{6/3}=j(j_3(v_2^6))$. The last column is also proved similarly, using the fact that $\beta_7=j(j_3(v_1^2v_2^7))$. □

By our convention about naming elements in the image of the map $\mathrm{BP}_*\to E_*$, we have $H'_3(v_2)=v_2$.

Proposition 6.4 *For $t \geq 0$ the map $H\colon E_2(S) \to E_2^E(\mathrm{TMF})$ satisfies*

(i) $H(\beta_{9t+1}) \doteq \Delta^{6t}\beta$,

(ii) $H(\beta_{9t+3/3}) \doteq \Delta^{6t+1}\beta$,

(iii) $H(\beta_{9t+6/3}) \doteq \Delta^{6t+3}\beta$,

(iv) $H(\beta_{9t+7}) \doteq \Delta^{6t+4}\beta$.

Proof These statements are all proved the same way; we show (ii). First observe that Lemma 6.2 implies $v_2^9 \equiv -\Delta^6 \pmod{(3, v_1^6)}$. Using Lemmas 6.2 and 6.3 we have

$$\begin{aligned}
H(\beta_{9t+3/3}) &= H(j(j_3(v_2^{9t+3}))) = j(j_3(H_3'(v_2^{9t+3}))) \\
&= j(j_3(H_3'(v_2^3)\cdot H_3'(v_2^{9t}))) = j(j_3((-\Delta^2 - v_1^2 v_2\Delta)\cdot(-1)^t\Delta^{6t})) \\
&= j(j_3((-1)^{t+1}\Delta^{6t+2})) + j(j_3((-1)^{t+1}v_1^2 v_2\Delta^{6t+1})) \\
&= 0 + (-1)^{t+1}\Delta^{6t+1} j(j_3(v_1^2 v_2)) = (-1)^{t+1}\Delta^{6t+1}\beta.
\end{aligned}$$

The last line uses the fact that $j_3(v_1^2 v_2) = \widetilde{\alpha}$ in $E_2^E(\mathrm{TMF}/3)$ from Lemma 6.3. □

In the next theorem, we show that every element in π_* tmf detected in filtration ≥ 2 is in the Hurewicz image. This result is stated without proof in [8, Section 1], but we do not know of any prior proof in the literature.

Theorem 6.5 *The image of the map $h\colon \pi_* S \to \pi_*$ tmf at $p = 3$ consists of the $\mathbb{Z}_{(3)}$ summand generated by 1 and the $\mathbb{F}_3$ summands generated by*

$$\alpha, \quad \Delta^{3t}\beta^i, \quad \Delta^{3t}\alpha\beta, \quad \Delta^{3t}\beta b \quad \text{for } 1 \leq i \leq 4 \text{ and } t \geq 0.$$

More precisely, we have

$$\begin{aligned}
h(\alpha_1) &= \alpha, & h(\beta_1^{i-1}\beta_{9t+1}) &= \Delta^{6t}\beta^i && \text{for } 1 \leq i \leq 4, \\
h(\alpha_1\beta_{9t+3/3}) &= \Delta^{6t}\beta b, & h(\beta_1^{i-1}\beta_{9t+6/3}) &= \Delta^{6t+3}\beta^i && \text{for } 1 \leq i \leq 4, \\
h(\alpha_1\beta_{9t+7}) &= \Delta^{6t+3}\beta b.
\end{aligned}$$

Proof Let $E_2^{\mathrm{ell}}(\mathrm{tmf})$ denote the elliptic spectral sequence for tmf (see [1, Section 6]); recall this is the $Y(4)$–based Adams spectral sequence for tmf. There is a map of spectral sequences $L\colon E_r^{\mathrm{ell}}(\mathrm{tmf}) \to E_r^E(\mathrm{TMF})$ that comes from the map on Adams towers induced by the maps $Y(4) \to MU_P \to E$ (where MU_P denotes periodic MU) and tmf $\to$ TMF. These maps assemble into a diagram of spectral sequences

$$\begin{tikzcd}
E_2^{\mathrm{ell}}(\mathrm{tmf}) \arrow[r, "L"] \arrow[d, Rightarrow] & E_2^E(\mathrm{TMF}) \arrow[d, Rightarrow] & E_2(S) \arrow[l, "H"'] \arrow[d, Rightarrow] \\
\pi_* \,\mathrm{tmf} \arrow[r, "L"] & \pi_* L_{K(2)}\mathrm{TMF} & \pi_* S \arrow[l, "H"'] \arrow[ll, bend left, "h"]
\end{tikzcd} \tag{8}$$

No element $x \in E_\infty^{\mathrm{ell}\,s,0}(\mathrm{tmf})$ for $s \neq 0$ is in the image of h: Lemma 6.6(ii) implies x would be detected in filtration 0 of $E_\infty^E(\mathrm{TMF})$, and $H\colon E_\infty(S) \to E_\infty^E(\mathrm{TMF})$ is zero in filtration 0 for nonzero stems.

Next we turn to elements detected in filtration 1. We have $H(\alpha_1) = \alpha$ by Lemma 6.3; since we have $H = L \circ h$ as maps $\pi_* S \to \pi_* L_{K(2)}\mathrm{TMF}$ and L is injective by Lemma 6.6, this implies $h(\alpha_1) = \alpha \in \pi_*\,\mathrm{tmf}$. The other elements of $E_\infty^{\mathrm{ell}}(\mathrm{tmf})$ in filtration 1 are $\Delta^{3t}\alpha$ for $t \geq 1$ and $\Delta^{3t}b$ for $t \geq 0$; we will show that the permanent cycles they represent are not in the Hurewicz image. By Lemma 6.6(i) they are in the image of h if and only if their images in $\pi_* L_{K(2)}\mathrm{TMF}$ are in the image of H. By Lemma 6.6(ii) they are also detected in filtration 1 in $E_\infty^E(\mathrm{TMF})$, so if they were in the image of H, they would be the image of a class in $E_2(S)$ in filtration 0 or 1. We have $E_2^{s,0}(S) = 0$ for $s > 1$, so it suffices to show that the elements in $E_2^{s,1}(S)$ except for α_1 are in the kernel of h. If $x \in E_2^{s,1}(S)$ with $s > 3$ then $i(x) = \alpha_1 v_1^k$ for some $k \geq 1$. If h' denotes the map $\pi_*(S/3) \to \pi_*(\mathrm{tmf}/3)$ induced by h, we have $h'(\alpha_1 v_1) = 0$ since $\pi_7(\mathrm{tmf}/3) = 0$. Thus $i(h(x)) = h'(i(x)) = 0$ in $\pi_*(S/3)$, which implies that $h(x)$ is 3–divisible. But Figure 3 shows that there are no 3–divisible nonzero targets in Adams–Novikov filtration 1.

We will now show how to use Proposition 6.4 to derive the remaining claims about h; for multiplicative reasons, it suffices to show $i = 1$ in those statements. We will illustrate this with the element $\alpha_1\beta_{9t+3/3}$; the other elements are analogous, using Theorem 5.1 for $\beta_{9t+6/3}$ or [3, Corollary 1.2] for β_{9t+1} in place of Theorem 5.1 below as necessary. By Proposition 6.4, $H(\alpha_1\beta_{9t+3/3}) = \Delta^{6t+1}\alpha\beta$ in $E_2^E(\mathrm{TMF})$. Since $\Delta^{6t+1}\alpha\beta$ is a permanent cycle in $E_2^{\mathrm{ell}}(\mathrm{tmf})$ converging to $\Delta^{6t}\beta b$, we have that $\Delta^{6t+1}\alpha\beta$ is a permanent cycle in $E_2^E(\mathrm{TMF})$ converging to $\Delta^{6t}\beta b$. Theorem 5.1 shows that $\alpha_1\beta_{9t+3/3}$ is a permanent cycle in the Adams–Novikov spectral sequence; write $\alpha_1\beta_{9t+3/3}$ for the (non-α_1–divisible) element in homotopy it converges to. The following diagram summarizes these statements by illustrating (8) applied to these elements:

$$\begin{tikzcd}
\Delta^{6t+1}\alpha\beta \arrow[r, mapsto, "L"] \arrow[d, dotted, mapsto] & \Delta^{6t+1}\alpha\beta \arrow[d, dotted, mapsto] & \alpha_1\beta_{9t+3/3} \arrow[l, mapsto, "H"'] \arrow[d, dotted, mapsto] \\
\Delta^{6t}\beta b \arrow[r, mapsto, "L"] & \Delta^{6t}\beta b & \alpha_1\beta_{9t+3/3} \arrow[l, mapsto, "H"'] \arrow[ll, bend left, "h"]
\end{tikzcd}$$

Thus $H\colon \pi_*S \to \pi_*L_{K(2)}\mathrm{TMF}$ satisfies $H(\alpha_1\beta_{9t+3/3}) = \Delta^{6t}\beta b$. Since H factors through h and $L\colon \pi_*\,\mathrm{tmf} \to \pi_*L_{K(2)}\mathrm{TMF}$ is injective by Lemma 6.6(i), we have that $h(\alpha_1\beta_{9t+3/3}) = \Delta^{6t}\beta b$. □

Lemma 6.6 (i) *The map $L\colon \pi_*\,\mathrm{tmf} \to \pi_*L_{K(2)}\mathrm{TMF}$ is injective on the classes in Theorem 6.5.*

(ii) *The map $L\colon E_\infty^{\mathrm{ell}}(\mathrm{tmf}) \to E_\infty^{E}(\mathrm{TMF})$ is injective in filtrations* 0 *and* 1.

In fact, L is injective on E_∞ pages in all filtrations, but we do not need this fact.

Proof (i) We have

$$\pi_*(L_{K(2)}\mathrm{TMF}) = \bigl(\pi_*(\mathrm{tmf})[(\Delta^3)^{-1}]\bigr)^\wedge_I,$$

where $I = (3, c_4)$; see [8, Section 2]. It is clear from the calculation of $\pi_*\,\mathrm{tmf}$ that the localization map $\pi_*(\mathrm{tmf}) \to (\Delta^3)^{-1}\pi_*(\mathrm{tmf})$ is an injection. It suffices to show that completion at I is injective on the specified classes. This holds because $0 = c_4\cdot\alpha = c_4\cdot\beta = c_4\cdot b$ in $(\Delta^{24})^{-1}\pi_*\,\mathrm{tmf}$ for degree reasons (and these classes are also all 3–torsion).

(ii) Consider an element of $\ker(L\colon E_\infty^{\mathrm{ell}}(\mathrm{tmf}) \to E_\infty^{E}(\mathrm{TMF}))$ represented by $x \in E_2^{\mathrm{ell}}(\mathrm{tmf})$ in filtration 0 or 1. We claim that x is in $\ker(L_2\colon E_2^{\mathrm{ell}}(\mathrm{tmf}) \to E_2^{E}(\mathrm{TMF}))$: since $L_2(x)$ is in filtration 0 or 1, it cannot be the target of a d_r differential for $r \geq 2$. By comparing the calculations of $E_2^{\mathrm{ell}}(\mathrm{tmf})$ and $E_2^{E}(\mathrm{TMF}/3)$ in [1, Section 5] and [7, Theorem 1.1], respectively, it is clear that $L_2'\colon E_2^{\mathrm{ell}}(\mathrm{tmf}/3) \to E_2^{E}(\mathrm{TMF}/3)$ is an injection, so the image of x in $E_2^{\mathrm{ell}}(\mathrm{tmf}/3)$ is zero, which implies (using exactness of the top row in the diagram) $x \in E_2^{\mathrm{ell}}(\mathrm{tmf})$ is 3–divisible:

$$\begin{tikzcd}
E_2^{\mathrm{ell}}(\mathrm{tmf}) \arrow[r, "3"] & E_2^{\mathrm{ell}}(\mathrm{tmf}) \arrow[r, "i"] \arrow[d, "L_2"'] & E_2^{\mathrm{ell}}(\mathrm{tmf}/3) \arrow[d, hook, "L_2'"] \\
& E_2^{E}(\mathrm{TMF}) \arrow[r, "i"] & E_2^{E}(\mathrm{TMF}/3)
\end{tikzcd}$$

Since $E_2^{\mathrm{ell}}(\mathrm{tmf})$ has no 3–divisible classes in filtration 1, we now focus on the filtration 0 case. Let $y = x/3^n \in E_2^{\mathrm{ell}}(\mathrm{tmf})$ be the non-3–divisible generator, which then has nonzero image $i(y)$ in $E_2^{\mathrm{ell}}(\mathrm{tmf}/3)$. Since L_2' is an injection, $L_2'(i(y)) = i(L_2(y)) \neq 0$. We claim that the (nonzero) group generated by $L_2(y)$ is torsion-free: if not, then the corresponding top cell class would be a nonzero class in $E_2^E(\mathrm{TMF}/3)$ in filtration -1, contradicting [7, Theorem 1.1]. So $L_2(x) = 3^n L_2(y) \neq 0$, contradicting the fact above that $x \in \ker(L_2)$. □

Remark 6.7 Our methods are not sufficient to completely determine the image of the map $h' \colon \pi_*(S/3) \to \pi_*(\mathrm{tmf}/3)$. The remaining nontrivial part of this question is to determine which elements $\Delta^n\alpha$ are in the image. Arguments similar to those we have given in this section show that $h'(\overline{\beta_{9t+2}}) = \Delta^{6t+1}\alpha$ and $h'(\overline{\beta_{9t+5}}) = \Delta^{6t+3}\alpha$. However, the families $\Delta^{6t}\alpha$ for $t \geq 1$ and $\Delta^{6t+4}\alpha$ for $t \geq 0$ fit into patterns that are not described by our work in this paper. For example, $\Delta^4\alpha \in \pi_{99}(\mathrm{tmf}/3)$ is not in the image of h' for degree reasons. On the other hand, using the more precise definitions of the β elements in [9, (2.4)] and calculating analogously to Lemma 6.3, we find that the map $E_2(S/3) \to E_2^E(\mathrm{TMF}/3)$ sends $\overline{\beta_{18/11}}$ to $\Delta^{10}\alpha$. As we do not know if $\overline{\beta_{18/11}}$ is a permanent cycle, we are unable to conclude whether $\Delta^{10}\alpha$ is in the image of $\pi_*(S/3)$.

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