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Realization of Lie algebras and classifying spaces of crossed modules

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The category of complete differential graded Lie algebras provides nice algebraic models for the rational homotopy types of nonsimply connected spaces. In particular, there is a realization functor, $\langle - \rangle$, of any complete differential graded Lie algebra as a simplicial set. In a previous article, we considered the particular case of a complete graded Lie algebra, L_0, concentrated in degree 0 and proved that $\langle L_0 \rangle$ is isomorphic to the usual bar construction on the Maltsev group associated to L_0.

Here we consider the case of a complete differential graded Lie algebra, $L = L_0 \oplus L_1$, concentrated in degrees 0 and 1. We establish that the category of such two-stage Lie algebras is equivalent to explicit subcategories of crossed modules and Lie algebra crossed modules, extending the equivalence between pronilpotent Lie algebras and Maltsev groups. In particular, there is a crossed module $\mathscr{C}(L)$ associated to L. We prove that $\mathscr{C}(L)$ is isomorphic to the Whitehead crossed module associated to the simplicial pair $(\langle L \rangle, \langle L_0 \rangle)$. Our main result is the identification of $\langle L \rangle$ with the classifying space of $\mathscr{C}(L)$.

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Introduction

In this text, we pursue the study of the rational homotopy type of spaces with models in the category **cdgl** of complete differential graded Lie algebras, as developed by the authors with Buijs and Murillo [4]. We emphasize that in this approach, there are no requirements concerning simple connectivity or nilpotency. In particular, to any finite simplicial complex is associated a cdgl M_X whose homology in degree 0 is the Maltsev completion of $\pi_1(X)$ [4, Theorem 10.5].

One of the main tools in this theory is a cosimplicial cdgl $\mathfrak{L}_\bullet = \{\mathfrak{L}_n\}_{n \geq 0}$, where $\mathfrak{L}_0$ is the free Lie algebra on a Maurer–Cartan element in degree -1, and $\mathfrak{L}_1$ is the Lawrence–Sullivan interval (see below for more details). This cosimplicial cdgl plays a role similar to the simplicial algebra of PL–forms on $\underline{\Delta}^\bullet$. It enables us to construct a realization functor from the category of complete differential graded Lie algebras to the category of simplicial sets, $\langle - \rangle \colon \mathbf{cdgl} \to \mathbf{Sset}$, defined by

$$\langle L \rangle_\bullet := \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

If a Lie algebra L is concentrated in degree 0, we proved in [6, Theorem 0.1] that its realization $\langle L \rangle$ is *isomorphic* to the usual bar construction on the group $\exp L$, constructed on the set L with the Baker–Campbell–Hausdorff product.

Here we consider the next step: L is a connected cdgl with nontrivial homology only in degrees 0 and 1. Geometrically, this corresponds to the notion of homotopy 2–types and, by analogy, a connected cdgl L such that $H_*L = H_0L \oplus H_1L$ is called a *2–type cdgl*. First of all, if $L = L_{\geq 0}$ and $H_{\geq 2}L = 0$, then the Lie subalgebra $I = L_{\geq 2} \oplus dL_2$ is an ideal because if $a \in L_0$ and $b \in L_2$, then $da = 0$ and $[a, d(b)] = d[a, b]$. Moreover I is acyclic, and the quotient map is a quasi-isomorphism,

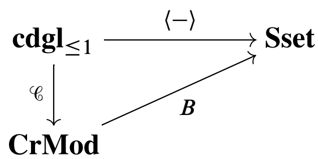
$$\varphi\colon (L, d) \xrightarrow{\simeq} (L/I, \bar{d}).$$

Since the realization functor $\langle - \rangle$ preserves quasi-isomorphisms of connected cdgls [4, Corollary 8.2 and Remark 8.6], we get a weak homotopy equivalence

$$\langle\varphi\rangle\colon \langle L, d\rangle \xrightarrow{\simeq} \langle L/I, \bar{d}\rangle.$$

We have thus reduced the problem to considering only cdgls L of the form $L = L_0 \oplus L_1$ and denote by $\mathbf{cdgl}_{\leq 1}$ the corresponding subcategory of **cdgl**. We associate to such L a natural crossed module $\mathscr{C}(L)$ and denote by **CrMod** the category of crossed modules. Our main result, which extends [6, Theorem 0.1], can be formulated as follows.

Theorem 1 *If L is a complete differential graded Lie algebra such that $L = L_0 \oplus L_1$, then its geometric realization $\langle L\rangle$ is naturally isomorphic to the classifying simplicial set $B\mathscr{C}(L)$; ie the diagram*



commutes up to natural isomorphisms.

This theorem shows that the functor $\langle - \rangle$ generalizes many classical constructions.

Geometrically, crossed modules appear in the work of Whitehead [14]. If (X, A) is a pair of topological spaces, based in A, Whitehead proved that the boundary map $d\colon \pi_2(X, A) \to \pi_1(A)$, together with the action of $\pi_1(A)$ on $\pi_2(X, A)$, defines a crossed module. Then, in [11], Mac Lane and Whitehead showed that the spaces X with $\pi_q(X) = 0$, for $q \geq 2$, are determined by the crossed module of the pair (X, X_1), where X_1 is the 1–dimensional skeleton of X. For any cdgl $L = L_0 \oplus L_1$, the geometric realization $\langle L\rangle$ is determined by the crossed module associated to the pair $(\langle L\rangle, \langle L_0\rangle)$. Our second main result identifies this crossed module with $\mathscr{C}(L)$.

Theorem 2 *The Whitehead crossed module associated to the simplicial pair $(\langle L\rangle, \langle L_0\rangle)$ is isomorphic to the crossed module $\mathscr{C}(L)$ introduced above.*

In short, these two theorems unify the geometric realizations of complete differential graded Lie algebras of the form $L = L_0 \oplus L_1$ and of crossed modules. In the last section, we extend the correspondence between Maltsev groups and pronilpotent Lie algebras to crossed modules. We introduce the categories of Maltsev crossed modules and of pronilpotent Lie algebra crossed modules and prove an isomorphism of categories.

Theorem 3 *The following three categories are isomorphic*:

(1) *the category of pronilpotent differential graded Lie algebras of the form* $L = L_0 \oplus L_1$,

(2) *the category of pronilpotent Lie algebra crossed modules*,

(3) *the category of Maltsev crossed modules.*

Moreover, the equivalence between (1) *and* (3) *is given by the functor* $\mathscr{C}$.

As a next step for the future, we can consider a connected cdgl L such that $H_{\geq n+1} L = 0$ for some $n \geq 1$. Using the ideal $J = L_{\geq n+1} \oplus dL_{n+1}$, the same argument used above gives a weak homotopy equivalence

$$\langle \varphi \rangle \colon \langle L, d \rangle \xrightarrow{\simeq} \langle L/J, \bar{d} \rangle.$$

We conjecture that the differential d defines an n–cat-group structure on $\mathscr{C}(L)$ — in the sense of Loday in [10] — and that the geometric realization $\langle L/J, \bar{d} \rangle$ is isomorphic to the realization of this n–cat-group.

Our program is carried out in Sections 1–7 below, whose headings are self-explanatory.

Conventions and notation

In a graded Lie algebra L, the group of elements of degree i is denoted by L_i. A Lie algebra differential decreases the degree by 1, ie $dL_i \subset L_{i-1}$. If $x \in L$, we denote by ad_x the Lie derivation of L defined by $\mathrm{ad}_x(y) = [x, y]$.

If there is no ambiguity, the product of two elements m and m' of a group M is denoted by mm'. Sometimes, if several laws are involved, we can use some specific notation, such as $m \perp m'$ or $m * m'$, to avoid confusion. An action of a group N on a group M is always a left action and is denoted by $(n, m) \mapsto {}^n m$. We denote then by $M \rtimes N$ the semidirect product whose multiplication law is defined by

$$(m, n)(m', n') = (m\, {}^n m', nn').$$

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1 Background on Lie models

A *complete differential graded Lie algebra* (henceforth cdgl) is a differential graded Lie algebra L equipped with a decreasing filtration of differential Lie ideals such that $F^1 = L$, $[F^p L, F^q L] \subset F^{p+q} L$ and

$$L = \varprojlim_n L/F^n L.$$

If no filtration is specified, it is understood that we consider the lower central series.

Let $V=\bigoplus_{i\in\mathbb{Z}} V_i$ be a rational graded vector space. We denote by $\mathbb{L}(V)$ the free graded Lie algebra on V, and by $\mathbb{L}^{\geq n}(V)$ the ideal of $\mathbb{L}(V)$ generated by the brackets of length greater than or equal to n. The *completion of* $\mathbb{L}(V)$ is the inverse limit

$$\widehat{\mathbb{L}}(V)=\varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V).$$

This is a cdgl for the filtration given by the ideals $G^n=\ker(\widehat{\mathbb{L}}(V)\to\mathbb{L}(V)/\mathbb{L}^{>n}(V))$. The correspondence $V\to\widehat{\mathbb{L}}(V)$ gives a left adjoint to the forgetful functor to graded rational vector spaces [4, Proposition 3.10]. We call $\widehat{\mathbb{L}}(V)$ the *free complete graded Lie algebra on* V.

If θ is a derivation of degree 0 on a cgl L, the exponential map e^θ is a cgl automorphism of L defined by

$$e^\theta=\sum_{i\geq 0}\frac{\theta^i}{i!}.$$

In particular, for any $x\in L_0$, e^{ad_x} is a cgl automorphism of L. Therefore, in any cgl L, the Lie subalgebra L_0 admits a group structure whose multiplication law $*$ is given by the Baker–Campbell–Hausdorff product [1, Chapter II.6, Proposition 4; 13, Section 3.4] and characterized by

$$e^{\mathrm{ad}_{x*y}}=e^{\mathrm{ad}_x}\circ e^{\mathrm{ad}_y}.$$

Now we recall the first properties of the cosimplicial cdgl $\mathfrak{L}_\bullet$ [4, Chapter 6]. Denote as usual by $\underline{\Delta}^n$ the simplicial set in which $\underline{\Delta}^n_p$ is the set of $(p+1)$–tuples of integers $(j_0,\dots,j_p)$ such that $0\leq j_0\leq\cdots\leq j_p\leq n$. We also denote by Δ^n the simplicial complex formed by the nonempty subsets of $\{0,\dots,n\}$. The subcomplex $\dot{\Delta}^n$ of Δ^n is the simplicial complex containing the proper nonempty subsets of $\{0,\dots,n\}$.

Finally $s^{-1}C_*\underline{\Delta}^n$ denotes the desuspension of the simplicial chain complex on $\underline{\Delta}^n$ and $s^{-1}C_*\Delta^n$ the desuspension of the complex of simplicial chains on Δ^n, which is isomorphic to $s^{-1}N_*\underline{\Delta}^n$, the complex of nondegenerate chains on $\underline{\Delta}^n$. Then, as a graded Lie algebra (without differential), we set

$$\mathfrak{L}_n=\widehat{\mathbb{L}}(s^{-1}C_*\Delta^n).$$

In other words, $\mathfrak{L}_n$ is the free complete graded Lie algebra on elements $a_{i_0\dots i_k}$ of degree $|a_{i_0\dots i_k}|=k-1$, for all $0\leq i_0<\cdots<i_k\leq n$. For instance, we have $|a_i|=-1$ and $|a_{i_0i_1}|=0$.

The family $\underline{\Delta}^\bullet=\{\underline{\Delta}^n\}_{n\geq 0}$ is a cosimplicial object in the category of simplicial sets. It follows that the family $s^{-1}N_*\underline{\Delta}^\bullet$ is a cosimplicial object in the category of chain complexes. The identification $s^{-1}C_*\Delta^n\cong s^{-1}N_*\underline{\Delta}^n$ makes $s^{-1}C_*\Delta^n$ a cosimplicial object in the category of chain complexes. The extension of the cofaces and codegeneracies as morphisms of Lie algebras gives morphisms of complete graded Lie algebras $\delta^i\colon\mathfrak{L}_n\to\mathfrak{L}_{n+1}$ and $\sigma^i\colon\mathfrak{L}_n\to\mathfrak{L}_{n-1}$. More precisely,

$$\delta^i(a_{j_0\dots j_p})=a_{r_0\dots r_p}\qquad\text{with } r_k=\begin{cases} j_k & \text{if } j_k<i,\\ j_k+1 & \text{if } j_k\geq i,\end{cases}$$

$$\sigma^i(a_{j_0\dots j_p})=a_{r_0\dots r_p}\qquad\text{with } r_k=\begin{cases} j_k & \text{if } j_k\leq i,\\ j_k-1 & \text{if } j_k>i,\end{cases}$$

if $r_0<\cdots<r_p$. Otherwise, $\sigma^i(a_{j_0\dots j_p})=0$.

Proposition 1.1 [4, Theorem 6.1] *Each $\mathfrak{L}_n$ can be endowed with a differential d satisfying the following properties.*

(i) *The linear part d_1 of d is given by*

$$d_1 a_{i_0 \dots i_p} = \sum_{j=0}^{p} (-1)^j a_{i_0 \dots \hat{i}_j \dots p}.$$

(ii) *The generators a_i are Maurer–Cartan elements; ie $da_i = -1/2[a_i, a_i]$.*

(iii) *The cofaces δ^i and the codegeneracies σ^i are cdgl morphisms.*

(iv) *For $n \geq 2$,*

$$da_{0\dots n} = [a_0, a_{0\dots n}] + \Phi,$$

with $\Phi \in \widehat{\mathbb{L}}(s^{-1} C_* \dot{\Delta}^n)$.

Thus, in particular, the family $\mathfrak{L}_\bullet$ is a cosimplicial cdgl.

Let us specify the cdgl $\mathfrak{L}_n$ in low dimensions.

- $\mathfrak{L}_0 = (\mathbb{L}(a_0), d)$ is the free Lie algebra on a Maurer–Cartan element a_0.
- $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$ is the Lawrence–Sullivan interval — see [9] — with

$$da_{01} = [a_{01}, a_1] + \frac{\mathrm{ad}_{a_{01}}}{e^{\mathrm{ad}_{a_{01}}} - 1}(a_1 - a_0).$$

- $\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d)$ is a model of the triangle — see [4, Proposition 5.14] — with the differential

$$d(a_{012}) = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}]. \tag{1-1}$$

The cosimplicial cdgl $\mathfrak{L}_\bullet$ leads naturally to the definition of cdgl models for any simplicial set and to a geometric realization for any given cdgl; see [4, Chapter 7]. For our purpose, we only need the realization of a cdgl L, defined as the simplicial set

$$\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L),$$

which satisfies properties of the classical Quillen realization. For instance, for any $n \geq 1$, $\pi_n \langle L \rangle = H_{n-1} L$, where the group law of $H_0 L$ is the BCH product; see [4, Section 4.2] or [1, Chapter II.6.4].

2 Crossed modules and cdgls

For general background on crossed modules, we refer the reader to the historical papers of Whitehead [11; 14] or to more modern presentations, such as [2; 3; 10]. We recall only the basics we need.

Definition 2.1 A crossed module $\mathscr{C} = (d : M \to N)$ is a morphism of groups d together with an action of N on M, given by group automorphisms $n \mapsto (m \mapsto {}^n m)$ satisfying two conditions:

(1) For all $m \in M$ and $n \in N$, $d({}^n m) = n d(m) n^{-1}$.

(2) For all $m \in M$, $m' \in M$, ${}^{d(m)} m' = m m' m^{-1}$.

If the group N acts on itself by conjugation, the first property means that d is compatible with the N–action. It also implies that the group $d(M)$ is a normal subgroup of N and that $\ker d$ is an N–submodule of M.

On the other hand, we remark that if $d(m) = 1$, the second property implies $mm' = m'm$ which means that $\ker d$ is included in the center of M. The same property shows that $\operatorname{Im} d$ acts trivially on $\ker d$ and induces thus an action of $\operatorname{coker} d$ on $\ker d$.

Now let $L = L_0 \oplus L_1$ be a cdgl. In what follows L_0 is always considered as a group equipped with the BCH product denoted by $*$. We will prove that $d\colon L_1 \to L_0$ is a crossed module. The first step consists in defining a group structure on L_1. This construction was originally carried out in [4, Definition 6.14].

Proposition 2.2 *For any cdgl (L, d) such that $L = L_0 \oplus L_1$, L_1 admits a natural product $\perp$ for which the differential $d\colon (L_1, \perp) \to (L_0, *)$ is a group morphism. Moreover, $a \perp b = a + b$ if a and b are cycles.*

Proof The different possibilities for a definition of this law are described in [4, Section 6.5]. We recall here the construction for the convenience of the reader, beginning with the "universal" example, the cdgl $L' = \widehat{\mathbb{L}}(u_1, u_2, du_1, du_2)$, with u_i in degree 1. Since $HL' = 0$ there is an element ω in L'_1 such that

$$d\omega = du_1 * du_2. \tag{2-1}$$

Of course such an element is not unique. If ω' is another element satisfying (2-1), the difference $\omega - \omega'$ is a boundary since $H_{\geq 1}L' = 0$. This shows that the class of ω is well defined in the cdgl quotient $(L'/(L'_{\geq 2} \oplus dL'_2), \bar{d})$. We denote this class by $u_1 \perp u_2$. By construction, it satisfies

$$\bar{d}(u_1 \perp u_2) = du_1 * du_2.$$

Among all the different possible choices for ω, one starts with the Baker–Campbell–Hausdorff series for $du_1 * du_2$. Replacing in each term one and only one du_i by u_i, we get an element ω with $d\omega = du_1 * du_2$. This gives

$$\omega = u_1 + u_2 + \tfrac{1}{2}[u_1, du_2] + \tfrac{1}{12}[du_1, [du_1, u_2]] - \tfrac{1}{12}[du_2, [du_1, u_2]] + \cdots . \tag{2-2}$$

Now, let L be a cdgl with $L = L_0 \oplus L_1$, $e_1, e_2 \in L_1$, and $f : L' \to L$ the unique cdgl map sending u_i to e_i. Then the element $e_1 \perp e_2 := f(u_1 \perp u_2)$ is a well-defined element in L_1. By construction, if e_1 and e_2 are cycles, using the image of the formula (2-2) in L, we have $e_1 \perp e_2 = e_1 + e_2$.

For the associativity of $\perp$, we consider $L'' = \widehat{\mathbb{L}}(u_1, u_2, u_3, du_1, du_2, du_3)$ and observe that in L''_1/dL''_2 we have $(u_1 \perp u_2) \perp u_3 = u_1 \perp (u_2 \perp u_3)$ because both have the same boundary. The same is thus true in L_1. □

With this group structure on L_1 we can now prove that $L = L_0 \oplus L_1$ is a crossed module.

Proposition 2.3 *Let (L, d) be a connected complete differential graded Lie algebra such that $L = L_0 \oplus L_1$. Then $d\colon (L_1, \perp) \to (L_0, *)$ is a crossed module.*

Proof Recall from [4, Definition 12.40] that the group L_0 acts on L_1 by

$$ {}^{x}z = e^{\mathrm{ad}_x}(z), \quad \text{for all } x \in L_0, z \in L_1. $$

From [4, Corollary 4.12] it follows that, for any $x \in L_0$, $y \in L_0$ and $z \in L_1$,

$$ {}^{(x*y)}z = e^{\mathrm{ad}_{x*y}}(z) = e^{\mathrm{ad}_x}(e^{\mathrm{ad}_y}z) = {}^{x}({}^{y}z). $$

To prove that the function $y \mapsto {}^{x}y$ is a group homomorphism, as in Proposition 2.2, we consider a universal example. Let $E = \widehat{\mathbb{L}}(x, z, t, dz, dt)$ with x in degree 0, z and t in degree 1, and $dx = 0$. Since the injection $\mathbb{L}(x) \to E$ is a quasi-isomorphism, we have $H_{\geq 1}(E) = 0$. Observe that in $E/(E_{\geq 2} \oplus dE_2)$,

$$ \begin{aligned} d({}^{x}(z \perp t)) = e^{\mathrm{ad}_x}(d(z \perp t)) &= e^{\mathrm{ad}_x}(dz * dt) \\ &= x * dz * dt * x^{-1} \\ &= x * dz * x^{-1} * x * dt * x^{-1} \\ &= e^{\mathrm{ad}_x}(dz) * e^{\mathrm{ad}_x}(dt) = d(e^{\mathrm{ad}_x}z) * d(e^{\mathrm{ad}_x}t) = d({}^{x}z \perp {}^{x}t). \end{aligned} $$

Thus, in E_1/dE_2, we get

$$ {}^{x}(z \perp t) = {}^{x}z \perp {}^{x}t. $$

The same is therefore also true in L_1.

As x is a cycle, by [4, Propositions 4.10 and 4.13],

$$ d({}^{x}z) = e^{\mathrm{ad}_x}(dz) = x * dz * x^{-1}, $$

and property (1) of Definition 2.1 is satisfied. For property (2), we use once again the universal example $L' = \widehat{\mathbb{L}}(u_1, u_2, du_1, du_2)$ already considered in the proof of Proposition 2.2. Since in L'_1/dL'_2 we have

$$ d({}^{du_1}u_2) = du_1 * du_2 * du_1^{-1} = d(u_1 \perp u_2 \perp u_1^{-1}), $$

we deduce that

$$ {}^{du_1}u_2 = u_1 \perp u_2 \perp u_1^{-1}, $$

and thus the same is true in L_1. □

Remark 2.4 By Proposition 2.2, under the hypotheses of Proposition 2.3, we deduce that the group structures $\perp$ and $+$ coincide on $H_1 L = \ker d$.

We have thus defined a functor $\mathscr{C}\colon \mathbf{cdgl}_{\leq 1} \to \mathbf{CrMod}$.

3 The crossed module of a realization and Theorem 2

In this section, in the case $L = L_0 \oplus L_1$, we establish the isomorphism between $\mathscr{C}(L)$ and the Whitehead crossed module of $(\langle L \rangle, \langle L_0 \rangle)$.

Proof of Theorem 2 The realization $\langle L \rangle = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$ of a cdgl $L = L_0 \oplus L_1$ is a Kan complex [4, Proposition 7.13]. We first compute $\pi_1(\langle L_0 \rangle)$ and $\pi_2(\langle L \rangle, \langle L_0 \rangle)$, and for that we use the homotopy relation introduced in [12, Section 3].

Since $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$, the map $f \mapsto f(a_{01})$ induces an isomorphism of sets

$$\langle L_0 \rangle_1 = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_1, L_0) \xrightarrow{\cong} L_0.$$

Since $\partial_i f = 0$, for $i = 0, 1$, each element of L_0 defines an element of $\pi_1(\langle L_0 \rangle)$. Now, two such 1–simplices, g and f, are homotopic in $\langle L_0 \rangle$ if there exists a map $h\colon \mathfrak{L}_2 \to L_0$ such that $\partial_1 h = g$, $\partial_2 h = f$ and $\partial_0 h = 0$. The simplex h is called a homotopy from f to g.

In the particular case $g = 0$, from the simplicial structure of the realization, we get $h(a_{02}) = h(a_{12}) = 0$ and $h(a_{01}) = f(a_{01})$. Since $h(a_{012}) = 0$, we have an equivalence

$$f \sim 0 \iff 0 = dh(a_{012}) = h(a_{01} * a_{12} * a_{02}^{-1}) = f(a_{01}).$$

Therefore $\pi_1 \langle L_0 \rangle = L_0$.

To compute the relative homotopy group $\pi_2(\langle L \rangle, \langle L_0 \rangle)$, we consider the set

$$K = \{ f \in \langle L \rangle_2 = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_2, L) \mid \partial_i f = 0 \text{ for } i = 1,\, 2 \text{ and } \partial_0 f \in \langle L_0 \rangle \}.$$

If $f \in K$, we have $\partial_0 f(a_{01}) = f(\delta^0(a_{01})) = f(a_{12}) = f(da_{012}) = df(a_{012})$ and thus the correspondence $K \to L_1$ which maps f to $f(a_{012})$ is an isomorphism. By [12, Definitions 3.3 and 3.6], two simplices, f and g, of K are homotopic rel $\langle L_0 \rangle$ if $\partial_0 f \sim \partial_0 g$ in $\langle L_0 \rangle$ by a homotopy h, and there exists a 3–simplex $\omega\colon \mathfrak{L}_3 \to L$ such that $\partial_0 \omega = h$, $\partial_2 \omega = f$, $\partial_3 \omega = g$ and $\partial_1 \omega = 0$.

For getting an expression of these conditions at the level of cdgls, we recall [4, Proposition 6.16] the differential d of $\mathfrak{L}_3$, which uses the operation $\perp$ introduced in the proof of Proposition 2.2,

$$d(a_{0123}) = e^{\mathrm{ad}_{a_{01}}} a_{123} - (a_{012} \perp a_{023} \perp a_{013}^{-1}). \tag{3-1}$$

From $L_{\geq 2} = 0$, we deduce $\omega(a_{0123}) = 0$. By the definition of K, $\omega(a_{123}) = \partial_0 \omega(a_{012}) = h(a_{012}) = 0$ since L_0 has no element of degree 1. We also have

$$\omega(a_{012}) = \partial_3 \omega(a_{012}) = g(a_{012}), \quad \omega(a_{013}) = \partial_2 \omega(a_{012}) = f(a_{012}), \quad \omega(a_{023}) = \partial_1 \omega(a_{012}) = 0.$$

Thus, by applying ω to both sides of (3-1), we obtain

$$0 = 0 - g(a_{012}) \perp 0 \perp f(a_{012})^{-1},$$

ie $0 = g(a_{012}) \perp f(a_{012})^{-1}$. This implies $0 = dg(a_{012}) * df(a_{012})^{-1}$ and $df(a_{012}) = dg(a_{012})$.

It remains to describe $g(a_{012}) \perp f(a_{012})^{-1}$. From the compatibility of the differential with Lie bracket and the fact that L_1 is an abelian Lie algebra, we get $[g(a_{012}), dg(a_{012})] = -\frac{1}{2} d[g(a_{012}), g(a_{012})] = 0$. In the BCH product $df(a_{012}) * dg(a_{012})$, all terms except the linear ones contain a bracket $[df(a_{012}), dg(a_{012})]$ which becomes $[df(a_{012}), g(a_{012})] = [dg(a_{012}), g(a_{012})] = 0$ in the formula (2-2). We thus obtain

$$g \perp f^{-1} = g - f.$$

We have proven $\pi_2(\langle L\rangle,\langle L_0\rangle)\cong L_1$ and $\pi_1(\langle L_0\rangle)\cong L_0$. We also showed that the connecting map $\partial\colon \pi_2(\langle L\rangle,\langle L_0\rangle)\to\pi_1(\langle L_0\rangle)$, given by $[f]\mapsto[\partial_0 f]$, corresponds to $df(a_{012})$ in the previous isomorphisms.

Consider now the action of $\pi_1(\langle L_0\rangle)=L_0$ on $\pi_2(\langle L\rangle,\langle L_0\rangle)=L_1$. Let $a\in L_0$, $b\in L_1$ and ${}^ab$ the element of L_1 corresponding to this action. Recall [4, Lemma 4.23] that $y=e^{\mathrm{ad}_a}b$ is also an element of L_1 such that $dy=a*db*a^{-1}$. Both constructions, ${}^ab$ and $e^{\mathrm{ad}_a}b$, are natural, so to prove ${}^ab=e^{\mathrm{ad}_a}b$, we have only to prove it for the cdgl L'' quotient of $L'=\widehat{\mathbb{L}}(a,u,du)$, with $\deg u=1$, by the ideal $L'_{\geq 2}\oplus dL'_2$. The required identification follows from $d({}^au)=d(e^{\mathrm{ad}_a}u)$ and the injectivity of $d\colon L''_1\to L''_0$. We have thus recovered the crossed module $\mathscr{C}(L)$. □

4 The classifying space of a crossed module

By definition, the classifying space of a crossed module $\mathscr{C}$ is the classifying space of the nerve of the categorical group associated to $\mathscr{C}$. Let us specify this association.

Recall that a categorical group is a group object in the category of groups (see [10, Section 1.1]),

$$G \underset{t}{\overset{s}{\rightrightarrows}} N,$$

where N is a subgroup of G, s and t are homomorphisms such that $s|_N=t|_N=\mathrm{id}_N$ and $[\ker s,\ker t]=1$.

In [10], J L Loday defines a categorical group associated to a crossed module $\mathscr{C}=(d\colon(M,\perp)\to(N,*))$ as follows:

- $G=M\rtimes N$ is the product $M\times N$ with the semidirect product given by the action of N on M. Thus, the product of (m',n') and (m,n) in G is

$$(m',n')\bullet(m,n)=(m'\perp{}^{n'}m,\,n'*n).$$

- An element (m,n) of G has for source and target, respectively,

$$s(m,n)=dm*n\quad\text{and}\quad t(m,n)=n.$$

Thus, the group N is interpreted as the group of objects viewed in G as $\{1\}\times N$. The group $G=M\rtimes N$ is the group of arrows with the morphisms s and t giving the source and the target. Two elements (m',n') and (m,n) are composable if

$$n'=t(m',n')=s(m,n)=dm*n.$$

In this case the composition is defined by

$$(m',n')\circ(m,n)=(m'\perp m,n).$$

We deduce easily from property (1) of Definition 2.1 that s and t are group homomorphisms. We also verify that the source of a composite is the source of the first factor and the target is the target of the second factor:

$$s(m'\perp m,n)=d(m'\perp m)*n=dm'*dm*n=dm'*n'=s(m',n'),\quad t(m'\perp m,n)=n=t(m,n).$$

Finally, composition is a group homomorphism; see [10, Lemma 2.2].

The usual nerve of a category is a simplicial set. When the category is a categorical group, we obtain naturally a simplicial group. Let us describe the nerve of the categorical group associated to a crossed module $\mathscr{C}=(d\colon(M,\perp)\to(N,*))$. We have

$$\mathrm{Ner}_1=M\rtimes N \overset{d_1}{\underset{d_0}{\rightrightarrows}} \mathrm{Ner}_0=N$$

with $d_0(m,n)=t(m,n)=n$, $d_1(m,n)=s(m,n)=dm*n$ and $s_0\colon \mathrm{Ner}_0\to\mathrm{Ner}_1$ is the canonical injection $N\to M\rtimes N$.

An element of Ner_k is a sequence $(m_i,n_i)_{1\le i\le k}$ such that

$$n_i=t(m_i,n_i)=s(m_{i-1},n_{i-1})=dm_{i-1}*n_{i-1}.$$

As the n_i, for $i\geq 2$, are determined by n_1 and the family $(m_i)_{1\le i\le k}$, the sequence $(m_i,n_i)_{i\le k}$ can be identified with the sequence

$$(m_k,m_{k-1},\dots,m_1,n_1)\in M^k\times N.$$

In particular,

$$\mathrm{Ner}_k=M^k\times N. \tag{4-1}$$

Each Ner_k is a group, the multiplication being given component wise. With the identification (4-1), this product is given by

$$((m_i)_{1\le i\le k},n)\bullet((m_i')_{1\le i\le k},n')=((m_i\perp^{d(\perp_{j=1}^{i-1}m_j)*n}m_i')_{1\le i\le k},n*n').$$

The boundary and degeneracy maps of Ner_* are morphisms of groups defined as usual by

$$\begin{aligned}
d_0(m_k,\dots,m_1,n)&=(m_k,\dots,m_2,d(m_1)*n),\\
d_i(m_k,\dots,m_1,n)&=(m_k,\dots,m_{i+1}\perp m_i,\dots,m_1,n),\quad 0<i<k,\\
d_k(m_k,\dots,m_1,n)&=(m_{k-1},\dots,m_1,n),\\
s_i(m_k,\dots,m_1,n)&=(m_k,\dots,m_i,1,m_{i-1},\dots,m_1,n),\quad 0\le i\le k.
\end{aligned}$$

The identity $e_k\in\mathrm{Ner}_k$ is the element $(1,\dots,1,1)$.

Recall from [5, Definition 3.20] or [7, page 255] the classifying functor $\overline{W}$ which goes from the category of simplicial groups to the category of reduced simplicial sets. The classifying space $B\mathscr{C}$ of the crossed module $\mathscr{C}$ is the space obtained by composing Ner_* with $\overline{W}$,

$$B\mathscr{C}=\overline{W}(\mathrm{Ner}_*).$$

By definition of $\overline{W}$,

$$(B\mathscr{C})_k = \{(h_{k-1}, \dots, h_0) \mid h_i \in \mathrm{Ner}_i\}.$$

The boundaries and degeneracies are given by

$$\begin{aligned}
d_0(h_{k-1}, \dots, h_0) &= (h_{k-2}, \dots, h_0),\\
d_i(h_{k-1}, \dots, h_0) &= (d_{i-1}h_{k-1}, \dots, d_0 h_{k-i} \bullet h_{k-i-1}, h_{k-i-2}, \dots, h_0), \quad 0 < i < k,\\
d_k(h_{k-1}, \dots, h_0) &= (d_{k-1}h_{k-1}, \dots, d_1 h_1),\\
s_0(h_{k-1}, \dots, h_0) &= (1, h_{k-1}, \dots, h_0),\\
s_i(h_{k-1}, \dots, h_0) &= (s_{i-1}h_{k-1}, \dots, s_0 h_{k-i}, 1, h_{k-i-1}, \dots, h_0), \quad 0 < i \le k.
\end{aligned}$$

In particular, in low dimensions,

$$B\mathscr{C}_0 = 1, \quad B\mathscr{C}_1 = N, \quad B\mathscr{C}_2 = (M \rtimes N) \times N, \quad B\mathscr{C}_3 = (M^2 \rtimes N) \times (M \rtimes N) \times N.$$

5 The classifying space functor $\overline{W}$ and twisting functions

Let A_* be a simplicial set. By [12, Corollary 27.2], there is a bijective correspondence between morphisms of simplicial sets $\varphi\colon A_* \to \overline{W} \circ \mathrm{Ner}_* = B\mathscr{C}$ and twisting functions

$$\tau = \{\tau_k \colon A_k \to \mathrm{Ner}_{k-1}\}_{k \ge 1}.$$

Recall [12, Definition 18.3] that a twisting function τ is a family of maps $\tau_k \colon A_k \to \mathrm{Ner}_{k-1}$ satisfying, for $x \in A_k$,

$$\begin{aligned}
d_0 \tau x &= \tau d_1 x \bullet (\tau d_0 x)^{-1},\\
d_i \tau x &= \tau d_{i+1} x, \quad i > 0,\\
s_i \tau x &= \tau s_{i+1} x, \quad i \ge 0,\\
\tau s_0 x &= e_k \in \mathrm{Ner}_k.
\end{aligned}$$

The simplicial map $\varphi_k \colon A_k \to (B\mathscr{C})_k$ associated to the twisting function τ is given by

$$\varphi_k x = (\tau x, \tau d_0 x, \dots, \tau d_0^{k-1} x). \tag{5-1}$$

Conversely [12, page 88], the twisting function τ associated to a simplicial morphism $\varphi\colon A_* \to \overline{W}(\mathrm{Ner}_*)$ is defined by

$$\tau = \tau(\mathrm{Ner}_*) \circ \varphi,$$

where $\tau(\mathrm{Ner}_*)$ is the twisting function associated to the identity on $\overline{W}(\mathrm{Ner}_*)$,

$$\tau(\mathrm{Ner}_*) \colon \overline{W}(\mathrm{Ner}_*)_k \to \mathrm{Ner}_{k-1},$$

defined by

$$\tau(\mathrm{Ner}_*)(g_{n-1}, \dots, g_0) = g_{n-1}.$$

6 Proof of Theorem 1

First we compute the simplicial set $\langle L\rangle_\bullet = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$ in the case $L = L_0 \oplus L_1$. By $L_{\geq 2} = 0$ and [4, Corollary 6.5], we have isomorphisms

$$\mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) \cong \mathrm{Hom}_{\mathbf{cdgl}}((\widehat{\mathbb{L}}((s^{-1}\Delta^k)_{\leq 2}), d), L) \cong \mathrm{Hom}_{\mathbf{cdgl}}((\widehat{\mathbb{L}}((s^{-1}\Lambda^k_0)_{\leq 2}), d), L).$$

Since any morphism of codomain L vanishes on elements of negative degree, we can quotient by the differential ideal generated by the generators of degree -1. This gives as free cgl

$$\overline{\mathfrak{L}}_k = (\widehat{\mathbb{L}}(a_{ij}, a_{0st}), d) \quad \text{with } 0 \leq i < j \leq k \text{ and } 0 < s < t \leq k.$$

Finally, in view of the differential in $\mathfrak{L}_2$, recalled in (1-1), the differential of $\overline{\mathfrak{L}}_k$ satisfies

$$da_{ij} = 0 \quad \text{and} \quad da_{0st} = a_{0s} * a_{st} * a_{0t}^{-1}.$$

In the rest of this text, we will use that, for all k, there exists an isomorphism

$$\langle L\rangle_k = \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) = \mathrm{Hom}_{\mathbf{cdgl}}(\overline{\mathfrak{L}}_k, L).$$

Proposition 6.1 *If $L = L_0 \oplus L_1$, then the morphism*

$$\Psi\colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) \to L_0^k \times L_1^{k(k-1)/2}$$

given by $\Psi(f) = \big((f(a_{r\,r+1}))_{0\leq r<k}, (f(a_{r,r+1,s}))_{r+1<s\leq k}\big)$ is an isomorphism.

Proof For the sake of simplicity write for $i < j$, $a_{ji} = a_{ij}^{-1}$, and for $0 \leq i < j < r \leq k$,

$$\begin{aligned} a_{irj} &= a_{ijr}^{-1}, \\ a_{rij} &= {}^{a_{ri}}a_{ijr} = {}^{a_{ir}^{-1}}a_{ijr}, \\ a_{jir} &= {}^{a_{ji}}a_{irj} = {}^{a_{ij}^{-1}}a_{ijr}^{-1}, \\ a_{jri} &= {}^{a_{ji}}a_{ijr} = {}^{a_{ij}^{-1}}a_{ijr}, \\ a_{rji} &= {}^{a_{ri}}a_{irj} = {}^{a_{ir}^{-1}}a_{ijr}^{-1}. \end{aligned}$$

With this notation, when the integers i, j and r are all different from each other and between 0 and k,

$$da_{ijr} = a_{ij} * a_{jr} * a_{ri}.$$

Suppose that the elements $f(a_{r,r+1})$ and $f(a_{r,r+1,t})$, with $r+1<t$, are defined. Then the other elements, $f(a_{r,r+s})$ and $f(a_{r,r+s,t})$ with $r+s<t$, can be derived by induction on s from the formulas

$$\begin{aligned} f(a_{r,r+s+1}) &= df(a_{r,r+1,r+s+1})^{-1} * f(a_{r,r+1}) * f(a_{r+1,r+s+1}), \\ f(a_{r,r+s+1,t}) &= {}^{f(a_{r,r+1})}\big(f(a_{r+1,r,r+s+1}) \perp f(a_{r+1,r+s+1,t}) \perp f(a_{r+1,t,r})\big). \end{aligned}$$

This shows that Ψ is injective. The same construction process shows that Ψ is also surjective. □

The isomorphism of our main theorem is based on a family τ of maps

$$\tau_k \colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_k, L) \to \mathrm{Ner}_{k-1}, \quad k \geq 1,$$

defined by

$$\tau_k f = (m_{k-1}, \dots, m_1, n) \in M^{k-1} \times N,$$

with $n = f(a_{01})$, $m_1 = (f(a_{012}))^{-1}$ and $m_i = (f(a_{01(i+1)}))^{-1} \perp f(a_{01i})$, for $i \geq 2$. In low dimensions, this gives

$$\begin{aligned} \tau_1 f &= f(a_{01}) \in N, \\ \tau_2 f &= (f(a_{012})^{-1}, f(a_{01})) \in M \times N, \\ \tau_3 f &= (f(a_{013})^{-1} \perp f(a_{012}), f(a_{012})^{-1}, f(a_{01})) \in M^2 \times N. \end{aligned}$$

Proposition 6.2 *The family τ is a twisting function.*

Proof Observe that $m_{i+1} \perp m_i = f(a_{01(i+2)})^{-1} \perp f(a_{01i})$. Thus, the index $i+1$ disappears in the expression of $d_i \tau_k f$ and we get $d_i \tau_k f = \tau_{k-1} d_{i+1} f$ for $0 < i < k-1$. A similar argument gives also the result for d_{k-1}. We have reduced the problem to proving the more subtle equality involving d_0. We use an induction, supposing the result is true for τ_j, with $j < k$, and considering τ_k. Due to the inductive step, we can concentrate the computations on the left-hand factor. From the definitions,

$$\begin{aligned} \tau_{k-1} d_1 f &= (f(a_{02k}))^{-1} \perp f(a_{02(k-1)}), \dots, f(a_{02})), \\ \tau_{k-1} d_0 f &= ((f(a_{12k})^{-1} \perp f(a_{12(k-1)}), \dots, f(a_{12})), \\ d_0 \tau_k f &= ((f(a_{01k})^{-1} \perp f(a_{01(k-1)}), \dots, (df(a_{012}))^{-1} * f(a_{01})). \end{aligned}$$

We determine the product of the two last terms,

$$d_0 \tau_k f \bullet \tau_{k-1} d_0 f = (f(a_{01k})^{-1} \perp f(a_{01(k-1)}) \perp {}^{\gamma}(f(a_{12k})^{-1} \perp f(a_{12(k-1)})), \dots),$$

where $\gamma = dm_{k-2} * dm_{k-1} * \cdots * dm_1 * n = f(a_{0(k-1)}) * (f(a_{1(k-1)}))^{-1}$. To obtain the equality with $\tau_{k-1} d_1 f$, we consider the computation in $\overline{\mathfrak{L}}_k$,

$$\begin{aligned} d(a_{01k}^{-1} \perp a_{01(k-1)} \perp {}^{a_{0(k-1)} * a_{1(k-1)}^{-1}}(a_{12k}^{-1} \perp a_{12(k-1)})) &= a_{0k} * a_{2k}^{-1} * a_{2(k-1)} * a_{0(k-1)}^{-1} \\ &= d(a_{02k}^{-1} \perp a_{02(k-1)}). \end{aligned}$$

Similar computations give the corresponding equalities for degeneracy maps. □

Denote by φ the morphism of simplicial sets induced by the previous twisting function τ,

$$\varphi \colon \mathrm{Hom}_{\mathbf{cdgl}}(\mathfrak{L}, L) \to B\mathscr{C}(L).$$

The following result finishes the proof of the theorem.

Proposition 6.3 *The morphism φ is an isomorphism of simplicial sets.*

Proof Recall from (5-1) that

$$\varphi_k f = (\tau f, \tau d_0 f, \ldots, \tau d_0^{k-1} f).$$

Moreover, using $d_0 f = f\delta^0$, we get $\tau d_0 f = (m'_{k-2}, \ldots, m'_1, n')$, with $n' = f(a_{12})$, $m'_1 = f(a_{123})^{-1}$ and for $i > 1$, $m'_i = f(a_{12(i+2)})^{-1} \perp f(a_{12(i+1)})$. By iteration from $(d_0)^\ell f = f(\delta^0)^\ell$, we deduce that the image of φ_k is the linear subspace generated by the elements $f(a_{r,r+1})$, for $0 \le r < k$, and $f(a_{r,r+1,s})$, for $r+1 < s \le k$. The result thus follows from Proposition 6.1. □

7 Maltsev crossed modules and Theorem 3

In this section, we establish an isomorphism of categories between $\mathbf{cdgl}_{\le 1}$ and a subcategory of crossed modules. We use the Lie algebra crossed modules introduced by Kassel and Loday in [8]. We begin with a reminder of [8].

In Definition 2.1, the group action of N on M corresponds to a homomorphism from N in the group of automorphisms of M. For Lie algebras, $\mathfrak{n}$ and $\mathfrak{m}$, an action of $\mathfrak{n}$ on $\mathfrak{m}$ corresponds to a Lie morphism $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ in the Lie algebra of derivations of $\mathfrak{m}$. The action of $n \in \mathfrak{n}$ on $m \in \mathfrak{m}$ is denoted $\mathfrak{v}(n).m$. We can now state [8, Définition A.1].

Definition 7.1 A Lie algebra crossed module is a morphism of Lie algebras, $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$, together with an action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$, satisfying two conditions:

(1) For all $m \in \mathfrak{m}$ and $n \in \mathfrak{n}$, $\mathfrak{u}(\mathfrak{v}(n).m) = [n, \mathfrak{u}(m)]$.

(2) For all $m \in \mathfrak{m}$, $m' \in \mathfrak{m}$, $\mathfrak{v}(\mathfrak{u}(m)).m' = [m, m']$.

We now introduce the "rational" versions of crossed modules. If G is a group, $G^k = [G, G^{k-1}]$ denotes the lower central series of G.

Definition 7.2 (1) A group G is a *Maltsev group* (or prounipotent rational group) if each G^k/G^{k+1} is a $\mathbb{Q}$–vector space, $\dim G/G^2 < \infty$ and $G = \varprojlim_k G/G^k$.

(2) A crossed module $d\colon M \to N$ is a *Maltsev crossed module* if M and N are Maltsev groups and the action of N on M satisfies $({}^n m)m^{-1} \in M^{k+1}$ for all $m \in M^k$ and $n \in N$.

If $\mathfrak{m}$ is a Lie algebra, $\mathfrak{m}^k = [\mathfrak{m}, \mathfrak{m}^{k-1}]$ denotes the lower central series of $\mathfrak{m}$.

Definition 7.3 (1) A Lie algebra $\mathfrak{m}$ is *pronilpotent* if $\dim \mathfrak{m}/\mathfrak{m}^2 < \infty$ and $\mathfrak{m} = \varprojlim_k \mathfrak{m}/\mathfrak{m}^k$.

(2) A Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ is *pronilpotent* if $\mathfrak{m}$ and $\mathfrak{n}$ are pronilpotent Lie algebras and the action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ satisfies $\mathfrak{v}(\mathfrak{n}).\mathfrak{m}^k \subset \mathfrak{m}^{k+1}$.

Remark 7.4 The completion of a Lie algebra $\mathfrak{m}$ satisfying $\dim \mathfrak{m}/\mathfrak{m}^2 < \infty$ is the Lie algebra

$$\widehat{\mathfrak{m}} = \lim_{k} \mathfrak{m}/\mathfrak{m}^k.$$

This is a pronilpotent Lie algebra since $\widehat{\mathfrak{m}} = \lim_k \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^k$.

If a Lie algebra $\mathfrak{m}$ acts on a vector space V, we denote by V^k the sequence of subspaces $V^0 = V$, $V^k = \mathfrak{m}.V^{k-1}$.

Definition 7.5 The action of $\mathfrak{m}$ on V is *pronilpotent* if $V = \varprojlim_k V^k$. In particular, a cdgl $L = L_0 \oplus L_1$ is *pronilpotent* if the Lie algebra L_0 is pronilpotent and if the adjoint action of L_0 on L_1 is pronilpotent.

Proof of Theorem 3 We only define the correspondences for objects, the extension to morphisms being immediate.

To show that (1) implies (2), we start with a pronilpotent cdgl $L = L_0 \oplus L_1$ and we construct a pronilpotent Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ with action $\mathfrak{v}\colon \mathfrak{n} \to \operatorname{Der}(\mathfrak{m})$. We denote d the differential of L and $[-,-]$ its bracket.

We set $\mathfrak{n} = L_0$, $\mathfrak{m} = L_1$. The bracket on $\mathfrak{n}$ is the bracket of L_0 and the bracket on $\mathfrak{m}$ is defined by

$$[a,b]' = [da,b] \quad \text{for } a,b \in L_1.$$

We check that $[-,-]'$ is an (ungraded) Lie bracket. Since $[a,b] = 0$, the antisymmetry follows from

$$0 = d[a,b] = [da,b] + [db,a] = [a,b]' + [b,a]'.$$

The proof is similar for the Jacobi identity. The morphism $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ is the differential d; this is a Lie algebra morphism,

$$\mathfrak{u}([a,b]') = d[da,b] = [da,db] = [\mathfrak{u}(a),\mathfrak{u}(b)] \quad \text{for all } a,b \in \mathfrak{m}.$$

The action $\mathfrak{v}\colon \mathfrak{n} \to \operatorname{Der}(\mathfrak{m})$ is given by the adjoint action, $\mathfrak{v}(x) = \operatorname{ad}_x$. The formulas (1) and (2) of Definition 7.1 also follow immediately: letting $a,b \in \mathfrak{m} = L_1$ and $x \in \mathfrak{n} = L_0$,

$$\mathfrak{u}(\mathfrak{v}(x).a) = d(\operatorname{ad}_x(a)) = d[x,a] = [x,da] = [x,\mathfrak{u}(a)], \quad \mathfrak{v}(\mathfrak{u}(a)).b = \operatorname{ad}_{da}(b) = [da,b] = [a,b]'.$$

By definition, since L is pronilpotent the associated Lie algebra crossed module is also pronilpotent.

To show that (2) implies (1), we start with a pronilpotent Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ with action $\mathfrak{v}\colon \mathfrak{n} \to \operatorname{Der}(\mathfrak{m})$ and we construct a pronilpotent cdgl $L = L_0 \oplus L_1$. We define $L_0 = \mathfrak{n}$ as a Lie algebra and $L_1 = \mathfrak{m}$ as a vector space. For $a \in L_1$ and $x \in L_0$, we set $[x,a] = \mathfrak{v}(x).a$ and $d = \mathfrak{u}$. We check easily that d is a derivation and $L = L_0 \oplus L_1$ is pronilpotent.

The associations (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) give the desired isomorphism of categories for the two first points of the statement.

To show that (2) implies (3), we start with a pronilpotent Lie algebra crossed module $\mathfrak{u}\colon \mathfrak{m} \to \mathfrak{n}$ with action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ and we construct a Maltsev crossed module $d\colon M \to N$. We define M and N to be the vector spaces $\mathfrak{m}$ and $\mathfrak{n}$ respectively, with the group structure given by the Baker–Campbell–Hausdorff product, and set $d = \mathfrak{u}$. The action $\mathfrak{v}$ extends in an action by $e^{\mathfrak{v}}$: for $n \in N = \mathfrak{n}$ and $m \in M = \mathfrak{m}$, we set

$$^{n}m = e^{\mathfrak{v}(n)}(m).$$

As $\mathfrak{v}$ is a morphism of Lie algebras, we have $\mathfrak{v}[n,n'] = [\mathfrak{v}(n), \mathfrak{v}(n')]$ for all $n, n' \in N$, and so the Baker–Campbell–Hausdorff formula implies $\mathfrak{v}(n * n') = \mathfrak{v}(n) * \mathfrak{v}(n')$ and

$$^{(n*n')}m = e^{\mathfrak{v}(n*n')}(m) = e^{\mathfrak{v}(n)}(e^{\mathfrak{v}(n')}(m)).$$

Thus, we have a group action. The two additional properties of Maltsev crossed modules are easily deduced from the corresponding properties of Lie algebra crossed modules as well as the pronilpotency conditions.

To show that (3) implies (2), as we do for the cases (1) and (2), the previous process are reversed. We associate a pronilpotent Lie algebra to a Maltsev group, replacing the exponential by the functor $L \mapsto \log(1+L)$. The only significant point is the construction of the Lie algebra action $\mathfrak{v}\colon \mathfrak{n} \to \mathrm{Der}(\mathfrak{m})$ from the group action $\nu\colon N \to \mathrm{Aut}(M)$; this is done by

$$\mathfrak{v}(n).m = \log(1+\nu(n))(m).$$

We end with the study of the composition (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). We start with $L = L_0 \oplus L_1$ and in step (2) we define a bracket on L_1 by $[a,b]' = [da,b]$. Then, in the second implication, we endow L_1 with a group law coming from the Baker–Campbell–Hausdorff formula, $a * b = \log(e^a e^b)$. This formula can be written as

$$\begin{aligned} a * b &= a + b + \tfrac{1}{2}[a,b]' + \tfrac{1}{12}[a,[a,b]']' - \tfrac{1}{12}b,[a,b]']' + \cdots \\ &= a + b + \tfrac{1}{2}[da,b] + \tfrac{1}{12}[da,[da,b]] - \tfrac{1}{12}db,[da,b]] + \cdots . \end{aligned}$$

This is exactly the expression of $a \perp b$ given in the formula (2-2). We recover the group law on L_1 in $\mathscr{C}(L)$. The rest of the verification is straightforward. □

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