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Models for knot spaces and Atiyah duality

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Let $\mathrm{Emb}(S^1, M)$ be the space of smooth embeddings from the circle to a closed manifold M. We introduce a new spectral sequence converging to $H^*(\mathrm{Emb}(S^1, M))$ for a simply connected closed manifold M of dimension 4 or more, which has an explicit E_1–page and a computable E_2–page. As applications, we compute some part of the cohomology for $M = S^k \times S^l$ with some conditions on the dimensions k and l, and prove that the inclusion $\mathrm{Emb}(S^1, M) \to \mathrm{Imm}(S^1, M)$ to the immersions induces an isomorphism on π_1 for some simply connected 4–manifolds. This gives a restriction on a question posed by Arone and Szymik. The idea to construct the spectral sequence is to combine a version of Sinha's cosimplicial model for the knot space and a spectral sequence for a configuration space by Bendersky and Gitler. The cosimplicial model consists of configuration spaces of points (with a tangent vector) in M. We use Atiyah duality to transfer the structure maps on the configuration spaces to maps on Thom spectra of the quotient of a direct product of M by the fat diagonal. This transferred structure is the key to defining our spectral sequence, and is also used to show that Sinha's model can be resolved into simpler pieces in a stable category.

18M75, 55P43, 55T99, 57R40; 18N40

1 Introduction

In [36; 37] Sinha constructed cosimplicial models of spaces of knots in a manifold of dimension ≥ 4, based on Goodwillie–Weiss embedding calculus; see Goodwillie and Klein [17], Goodwillie and Weiss [18], and Weiss [41]. The model was crucially used in the affirmative solution to Vassiliev's conjecture for a spectral sequence for the space of long knots in $\mathbb{R}^d$ (with $d \geq 4$) for rational coefficient by Lambrechts, Turchin and Volić in [25] (see Boavida de Brito and Horel [5] for other coefficients). We study a version of Sinha's model in stable categories.

Let $\mathrm{Emb}(S^1, M)$ be the space of smooth embeddings from the circle S^1 to a manifold M (without any basepoint condition) endowed with the C^∞–topology. The space $\mathrm{Emb}(S^1, M)$ is studied by Arone and Szymik [1] and Budney [8], and study of embedding spaces including the knot space is a motivation of Campos and Willwacher [10] and Idrissi [22]

In the rest of the paper, M denotes a connected closed smooth manifold of dimension d. Our knot space $\mathrm{Emb}(S^1, M)$ is slightly different from the one considered by Sinha, but we can construct a cosimplicial model similar to Sinha's, which is called *Sinha's cosimplicial model* and denoted by $\mathcal{C}^\bullet(M)$. Its n^{th} space is homotopy equivalent to the configuration space of $n+1$ ordered points in M with a unit tangent vector.

To state our first main theorem, we need some notation. Let SM be the tangent sphere bundle of M. Fix an embedding $e_0\colon SM \to \mathbb{R}^K$, and a tubular neighborhood ν of the image $e_0(SM)$ in $\mathbb{R}^K$. Let $\mathcal{D}$ be the little interval operad. We use a notion of a $\mathcal{D}$–comodule, which plays a role similar to a simplicial object but is homotopically more flexible. We work with the category of symmetric spectra $\mathcal{SP}$. For a manifold N and an integer $n \geq 1$, N^n denotes the direct product of n copies of N. The fat diagonal of M^n is by definition the union of all the diagonals of M^n. We regard the product ν^n as a disk bundle over SM^n via the obvious identification $(e_0(SM))^n = SM^n$. The following theorem gives a dual equivalence between the configuration spaces and quotients by a fat diagonal, which preserves structure necessary to recover (some part of) the knot space.

Theorem 1.1 (Theorem 4.4 and Lemma 4.7) *Under the above notation, there exists a zigzag of weak equivalences of left $\mathcal{D}$–comodules of nonunital commutative symmetric ring spectra*

$$(\mathcal{C}_M)^\vee \simeq \mathcal{T}_M,$$

where $(\mathcal{C}_M)^\vee$ is a comodule whose n^{th} object is the Spanier–Whitehead dual of the configuration space of n points with a tangent vector in M, and $\mathcal{T}_M$ is a comodule whose n^{th} object is a natural model of the Thom spectrum

$$\Sigma^{-nK}\,\mathrm{Th}(\nu^n)/\mathrm{Th}(\nu^n|_{\mathrm{FD}_n}).$$

Here

- Σ *denotes the suspension equivalence and* $\mathrm{Th}(-)$ *denotes the associated Thom space*,
- FD_n *is the preimage of the fat diagonal by (the product of) the projection* $SM^n \to M^n$, *and*
- $\nu^n|_{\mathrm{FD}_n}$ *denotes the restriction of the base to* FD_n.

See Section 2.1 and Definitions 2.10, 4.1, 4.3 and 4.5 for details of the notation. Theorem 1.1 is a structured version of the Poincaré–Lefschetz duality

$$H^*(\mathcal{C}^{n-1}(M)) \cong H_*(SM^n, \mathrm{FD}_n), \tag{1-1}$$

deduced from a homotopy equivalence $\mathcal{C}^{n-1}(M) \simeq SM^n - \mathrm{FD}_n$. (We are loose on degrees.) If we do not consider the (nonunital) commutative multiplications, an analogue of Theorem 1.1 holds in the category of prespectra (in the sense of Mandell, May, Schwede and Shipley [28]), a more naive, nonsymmetric monoidal category of spectra, and it is enough to prove Theorem 1.2, but the multiplications may be useful for future study and our construction hardly becomes easier for prespectra.

To state the second main theorem, we need additional notation. For a positive integer n, let $\mathsf{G}(n)$ be the set of graphs G with set of vertices $V(G) = \underline{n} = \{1, \dots, n\}$ and set of edges $E(G) \subset \{(i,j) \mid i, j \in \underline{n} \text{ with } i < j\}$. Let D_G be the subspace of SM^n consisting of elements whose image by the projection to M^n has the same i^{th} and j^{th} components if i and j are connected by an edge of G ($i, j \in \underline{n}$). The space FD_n in Theorem 1.1 is the union of the spaces D_G whose graph G has at least one edge. D_G is a rather comprehensible space compared to the space $\mathcal{C}^{n-1}(M)$. For example, its cohomology ring is computed in Lemmas 6.5 and 6.6 under some assumptions. Throughout this paper, we fix a coefficient ring k and suppose k is either of a subring of the rationals $\mathbb{Q}$ or the field $\mathbb{F}_\mathfrak{p}$ of $\mathfrak{p}$ elements for a prime $\mathfrak{p}$. All normalized singular (co)chains C^* and C_* and singular (co)homology H^* and H_* are supposed to have coefficients in k, unless otherwise stated. As an application of Theorem 1.1, we introduce a new spectral sequence converging to $H^*(\mathrm{Emb}(S^1, M))$.

Theorem 1.2 (Theorems 5.16, 5.17 and 6.11) *Suppose M is simply connected and of dimension $d \geq 4$. There exists a second-quadrant spectral sequence $\{\check{\mathbb{E}}_r^{pq}\}_r$ converging to $H^{p+q}(\mathrm{Emb}(S^1, M))$ such that:*

(1) *Its E_2–page is isomorphic to the total homology of the normalization of a simplicial commutative differential bigraded algebra $A_\bullet^{\star *}(M)$ which is defined in terms of the cohomology ring $H^*(D_G)$ for various graphs G and maps between them,*

$$\check{\mathbb{E}}_2^{pq} \cong H(NA_\bullet^{\star *}(M)) \Rightarrow H^{p+q}(\mathrm{Emb}(S^1, M)),$$

where the bidegree is given by $* = p$ and $\star - \bullet = q$.

(2) *If $H^*(M)$ is a free k–module, and the Euler number $\chi(M)$ is zero or invertible in k, the object $A_\bullet^{\star *}(M)$ is determined by the ring $H^*(M)$.*

We call this spectral sequence the *Čech spectral sequence*, or in short, the *Čech s.s.* A feature of this spectral sequence is that its E_1 page and differential d_1 are explicitly determined by the cohomology of M. As spectral sequences for $H^*(\mathrm{Emb}(S^1, M))$ we have the Bousfield–Kan type cohomology spectral sequence converging to $H^*(\mathrm{Emb}(S^1, M))$, see Definition 2.7, and Vassiliev's spectral sequence [40] converging to the relative cohomology $H^*(\Omega_f(M), \mathrm{Emb}(S^1, M))$, where $\Omega_f(M)$ is the space of smooth maps $S^1 \to M$. But no small (ie degreewise finite-dimensional) page of these spectral sequences has been computed in general. The E_1–page of the Bousfield–Kan type s.s. is described by the cohomology of the ordered configuration spaces of points with a vector in M, which is difficult to compute; Vassiliev's first term is also interesting but complicated. By this feature, we can compute examples; see Section 7. We obtain new computational results in the case of the product of two spheres. While we only do elementary computation in the present paper, one of potential merits of Čech s.s. is that computation of higher differentials will be relatively accessible since we deal with the fat diagonals and Čech complex instead of configuration spaces. The other is that we will be able to enrich it with operations such as the cup product and square, and relate them to those on $H^*(M)$. We will deal with these subjects in future work. Precisely speaking, we can also construct the Čech spectral sequence in the 3–dimensional or nonsimply connected case, where it does not converge to $H^*(\mathrm{Emb}(S^1, M))$ but might have some information about the knot space; see Remark 5.18.

Arone and Szymik studied $\mathrm{Emb}(S^1, M)$ for the case of dimension $d = 4$ in [1]. Let $\mathrm{Imm}(S^1, M)$ be the space of smooth immersions $S^1 \to M$ with the C^∞–topology and $i_M : \mathrm{Emb}(S^1, M) \to \mathrm{Imm}(S^1, M)$ be the inclusion. Among other results, they proved that i_M is 1–connected, so in particular surjective on π_1 in general. (They proved interesting results for the nonsimply connected case $M = S^1 \times S^3$; see also Budney and Gabai [9].) They asked whether there is a simply connected 4–manifold M such that i_M has nontrivial kernel on π_1. Using Theorem 1.2, we give a restriction to this question:

Corollary 1.3 *Suppose that M is simply connected, of dimension 4 and satisfies $H_2(M;\mathbb{Z}) \neq 0$, and that the intersection form on $H_2(M;\mathbb{F}_2)$ is represented by a matrix whose inverse has at least one nonzero diagonal component. Let $i_M : \mathrm{Emb}(S^1, M) \to \mathrm{Imm}(S^1, M)$ be the inclusion to the space of immersions. Then the map i_M induces an isomorphism on π_1. In particular, $\pi_1(\mathrm{Emb}(S^1, M)) \cong H_2(M;\mathbb{Z})$.*

The assumption does not depend on the choice of matrix. For example, $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, the connected sum of complex projective planes, satisfies the assumption, while $M = S^2 \times S^2$ does not. For the case of $H_2(M) = 0$, by Proposition 5.2 of [1], $\mathrm{Emb}(S^1, M)$ is simply connected. We can also prove this similarly to Corollary 1.3. The case of all of the diagonal components of the matrix being zero is unclear by our method.

Remark 1.4 In the recent preprint [23], Kosanović gave a proof of a complete answer to the question, which states that the inclusion i_M induces an isomorphism of π_{d-1} if M is simply connected and of dimension $d \geq 4$ by an independent method.

Sinha's cosimplicial model can be considered as a resolution of $\mathrm{Emb}(S^1, M)$ into simpler spaces. We resolve it into further simpler pieces in the category of chain complexes as an application of Theorems 1.1 and 1.2. To state the result, we need additional notation. We consider a category Ψ of planar rooted trees and edge contractions. It is equipped with a functor $\mathcal{G} \circ \mathcal{F} : \Psi \to \Delta$, where Δ is the category of the standard simplices. We also use a category $\mathsf{G}(n)^+$. Roughly speaking, the objects of $\mathsf{G}(n)^+$ are a symbol $*$ and the graphs in $\mathsf{G}(n)$, and the morphisms are the inclusions (of edge sets) and formal arrows $* \to G$ to the graphs having at least one edge. Let $\widetilde{\Psi}$ be the Grothendieck construction of a functor from Ψ sending a tree T to the category $\mathsf{G}(|v_r| - 1)^+$, where $|v_r|$ denotes the valence of the root vertex of T. So an object of $\widetilde{\Psi}$ is a pair (T,G) of a tree T and a graph G with exactly $|v_r|-1$ vertices (or the symbol $*$). Let $\eta\colon \widetilde{\Psi}\to\Psi$ be the projection given by $\eta(T,G)=T$.

Theorem 1.5 (Theorem 8.4) *Under the above notation, there exists a functor $\mathsf{T}_M\colon \widetilde{\Psi}^{\mathrm{op}}\to\mathcal{SP}$ satisfying the following conditions:*

(1) *Its value on $(T,G)\in\widetilde{\Psi}$ is a natural model of the Thom spectrum*

$$\Sigma^{-mK}\,\mathrm{Th}(\nu^m|_{D_G})\quad\text{with } m=|v_r|-1$$

if G is a graph, and the basepoint if $G=*$.

(2) *There exists a zigzag of weak equivalences of functors*

$$(\mathcal{G}\circ\mathcal{F})^*(\mathcal{C}^\bullet(M)^\vee)\simeq\mathbb{L}\eta_!\mathsf{T}_M\colon\Psi^{\mathrm{op}}\to\mathcal{SP}.$$

Here the dual of the cosimplicial model is regarded as a functor from Δ^{op} and $\mathbb{L}\eta_!$ is the (derived) left Kan extension along η.

(3) *Suppose M is simply connected and of dimension $d\geq 4$. There exists a zigzag of quasi-isomorphisms of chain complexes*

$$C^*(\mathrm{Emb}(S^1,M))\simeq\operatorname*{hocolim}_{\widetilde{\Psi}^{\mathrm{op}}}C_*\circ\mathsf{T}_M.$$

Here hocolim *denotes the homotopy colimit, and C_* on the right-hand side is a certain singular chain functor from spectra to chain complexes.*

See Section 2.1 and Definitions 5.1 and 8.1 for details of the notation. We give an intuitive explanation for this theorem. We regard $\mathsf{G}(n)$ as the full subcategory of $\mathsf{G}(n)^+$. Let $\varnothing$ denote the graph with no edges. There is a standard quasi-isomorphism $C_*(\mathrm{FD}_n)\simeq\operatorname{hocolim}_{G\in C_1}C_*(D_G)$, where $C_1=\mathsf{G}(n)^{\mathrm{op}}-\{\varnothing\}$. Since the relative complex $C_*(SM^n,\mathrm{FD}_n)$ is the homotopy cofiber of the inclusion $C_*(\mathrm{FD}_n)\to C_*(SM^n)=C_*(D_\varnothing)$, we have quasi-isomorphisms

$$C^*(\mathcal{C}^{n-1}(M))\simeq C_*(SM^n,\mathrm{FD}_n)\simeq\operatorname*{hocolim}_{G\in C_2}C_*(D_G),$$

where we set $C_2=(\mathsf{G}(n)^+)^{\mathrm{op}}$ and $C_*(D_G)=0$ for $G=*$. We regard this presentation as a resolution of $C^*(\mathcal{C}^{n-1}(M))$. A category of planar rooted trees is a lax analogue of the category of the standard simplices. Actually, homotopy limits over these categories are weakly equivalent. So, intuitively speaking, existence of the functor T_M means potential compatibility of the resolution and the cosimplicial structure.

We shall explain why we use spectra, which also serves as an outline of our arguments. Our motivation is to derive a new spectral sequence from Sinha's cosimplicial model. The idea is to combine the cosimplicial model and a procedure of constructing a spectral sequence for the cohomology of the configuration space due to Bendersky and Gitler [3]. So we consider the above duality (1-1), and describe the chain complex $C_*(SM^n,\mathrm{FD}_n)$ by an augmented Čech complex as follows. Consider

$$C_*(D_\varnothing)\xleftarrow{\partial}\bigoplus_{G\in\mathsf{G}(n,1)}C_*(D_G)\xleftarrow{\partial}\bigoplus_{G\in\mathsf{G}(n,2)}C_*(D_G)\xleftarrow{\partial}\bigoplus_{G\in\mathsf{G}(n,3)}C_*(D_G)\xleftarrow{\partial}\cdots,$$

where $\mathsf{G}(n,p) \subset \mathsf{G}(n)$ denotes the subset of graphs with exactly p edges. We want to extend this to the following commutative diagram of semisimplicial chain complexes by defining suitable face maps d_i:

$$\begin{tikzcd}
C^*(\mathcal{C}^n(M)) \arrow[d, "(d^i)^*"] & C_*(D_\varnothing) \arrow[l, "\mathrm{PD}"'] \arrow[d, "d_i"] & \bigoplus_{G\in\mathsf{G}(n+1,1)} C_*(D_G) \arrow[l] \arrow[d, "d_i"] & \bigoplus_{G\in\mathsf{G}(n+1,2)} C_*(D_G) \arrow[l] \arrow[d, "d_i"] & \cdots \arrow[l] \\
C^*(\mathcal{C}^{n-1}(M)) & C_*(D_\varnothing) \arrow[l, "\mathrm{PD}"'] & \bigoplus_{G\in\mathsf{G}(n,1)} C_*(D_G) \arrow[l] & \bigoplus_{G\in\mathsf{G}(n,2)} C_*(D_G) \arrow[l] & \cdots \arrow[l]
\end{tikzcd} \tag{1-2}$$

Here d^i is the coface map of $\mathcal{C}^\bullet(M)$, and PD actually denotes the zigzag

$$C^*(\mathcal{C}^n(M)) \to C_*(D_\varnothing, \mathrm{FD}_n) \leftarrow C_*(D_\varnothing)$$

of the cap product with the fundamental class and the quotient map. If we could construct a semisimplicial double complex in the right-hand side of PD in (1-2), by taking the total complex, we would have a certain triple complex $C_{\bullet\star *}$, where $\bullet$ (resp. $\star$, $*$) denotes the cosimplicial (resp. Čech, singular) degree. Then by filtering with $\star+\bullet$, we would obtain a spectral sequence as in Theorem 1.2.

Unfortunately, it is difficult to define degeneracy maps d_i fitting into (1-2). This difficulty is essentially analogous to the one in the construction of a certain chain-level intersection product on $C_*(M)$. We shall explain this point more precisely. The coface map $d^i : \mathcal{C}^n(M) \to \mathcal{C}^{n+1}(M)$ is a deformed diagonal, and the usual diagonal induces the intersection product on homology. So the maps d_i should be something like a deformed intersection product. The simplicial identities for d_i are analogous to the associativity of an intersection product. In addition, the map $(d^i)^*$ on the cochain is analogous to the cup product. So construction of d_i is analogous to construction of a chain-level intersection product which is associative and compatible with the cup product through the duality. We could not find such a product in the literature.

A nice solution is found in a construction due to R Cohen and Jones [11; 12] in string topology. They used spectra to give a homotopy theoretic realization of the loop product, which led to a proof of an isomorphism between the loop product and a product on Hochschild cohomology (see Moriya [30] for a detailed account). Their key notion is the Atiyah duality, which is an equivalence between the Spanier–Whitehead dual $M^\vee$ and the Thom spectrum $M^{-TM} = \Sigma^{-K}\,\mathrm{Th}(\nu)$. To prove their isomorphism, Cohen [11] introduced a model of M^{-TM} in the category $\mathcal{SP}$, and refined the duality to an equivalence of (nonunital) commutative symmetric ring spectra. This equivalence can be regarded as a multiplicative version of the Poincaré duality. In fact, the multiplication on the model of M^{-TM} works as an analogue of a chain level intersection product in their theory. So is efficient to construct necessary semisimplicial objects and their equivalence in $\mathcal{SP}$, then take chain complexes of them, and derive a spectral sequence. This is why we use spectra.

Even if we use spectra, the (co)simplicial object is too rigid, and we use a laxer notion of a left comodule over an A_∞–operad.

As we demonstrate, the duality is very useful to transfer structures on the configuration space to the Thom spectrum of the quotient by the fat diagonal, which is homotopically more accessible, and may be applied in much research on configuration spaces. In future work, we will study collapse of Sinha's (or Vassiliev's) spectral sequence for the space of long knots in $\mathbb{R}^d$ [36] using the duality.

The organization of the paper is as follows. In Section 2, we introduce basic notions. We define a version of Sinha's cosimplicial model and show that its homotopy limit is equivalent to the space $\mathrm{Emb}(S^1, M)$. We define the notions of a (co)module and Hochschild complex of a comodule over the associahedral operad. These notions are minor variations of ones given by others. Section 3 is the technical heart of this paper. We introduce a version of Cohen's model of Thom spectra and use it to construct the comodule $\mathcal{T}_M$ in Theorem 1.1. We take care about definitions of parameters such as the radius of tubular neighborhoods to make structure maps of a comodule compatible with the diagonals. In Section 4, we prove Theorem 1.1. In Sections 5 and 6, we prove Theorem 1.2. These two sections have a homotopical and algebraic nature compared to the previous sections, where we give detailed space level constructions. In Section 5, we define a chain functor for symmetric spectra and construct the spectral sequence filtering Hochschild complex of the chains of a resolution of the comodule $\mathcal{T}_M$. We prove that the E_1–page of the Čech spectral sequence is quasi-isomorphic to the total complex of a simplicial differential bigraded algebra, and prove the convergence of the Čech spectral sequence. In Sections 3–5 we mainly deal with comodules, but we need the cosimplicial model in the proof of convergence since we deduce it from a theorem of Bousfield. In Section 6, we compute the cohomology rings $H^*(D_G)$ and maps between them, and give a description of the simplicial algebra in terms of the cohomology ring $H^*(M)$ under some assumptions. The computation is standard work based on Serre spectral sequences. In Section 7, we compute examples and prove Corollary 1.3. In Section 8, we prove Theorem 1.5.

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2 Preliminaries

In this section, we fix notation and introduce basic notions. Nothing is essentially new.

2.1 Notation and terminology

- We denote by Δ the category of standard simplices. Its objects are the finite ordered sets $[n]=\{0,\dots,n\}$ for $n\geq 0$ and its morphisms are the weakly order-preserving maps. We denote by Δ_n the full subcategory of Δ that consists of the objects $[k]$ with $k\leq n$. We define a category (or poset) P_n as follows. The objects are the nonempty subsets S of $\underline{n}$, and there is a unique morphism $S\to S'$ if and only if $S\subset S'$.

$\mathcal{G}_n : \mathrm{P}_{n+1} \to \Delta_n$ denotes the functor given in [37, Definition 6.3]. It sends a set S to $[\#S-1]$ and an inclusion $S \subset S'$ to the composition $[\#S-1] \cong S \subset S' \cong [\#S'-1]$, where $\cong$ denotes the order-preserving bijection.

- For a category $\mathcal{C}$, a morphism of $\mathcal{C}$ is also called a *map of* $\mathcal{C}$. A *symmetric sequence in* $\mathcal{C}$ is a sequence $\{X_k\}_{k\geq 0}$ (or $\{X(k)\}_{k\geq 1}$) of objects in $\mathcal{C}$ equipped with an action of the k^{th} symmetric group Σ_k on X_k (or $X(k)$) for each k. The group Σ_k acts from the right throughout this paper.

- Let $\mathsf{G}(n)$ be the set of graphs defined in Section 1. For a graph $G \in \mathsf{G}(n)$, we regard $E(G)$ as an ordered set with the lexicographical order. To ease notation, we write (i,j) with $i > j$ to denote the edge (j,i) of a graph in $\mathsf{G}(n)$. For a map $f : \underline{n} \to \underline{m}$ of finite sets, we denote by the same symbol f the map $\mathsf{G}(n) \to \mathsf{G}(m)$ defined by

$$E(f(G)) = \{(f(i), f(j)) \mid (i,j) \in E(G) \text{ with } f(i) \neq f(j)\}.$$

Also, f denotes the natural map $\pi_0(G) \to \pi_0(f(G))$ between the connected components.

- Our notion of a *model category* is that of [21]. $\mathbf{Ho}(\mathcal{M})$ denotes the homotopy category of a model category $\mathcal{M}$.

- We will denote by $\mathcal{CG}$ the category of all compactly generated spaces and continuous maps (see [21, Definition 2.4.21]), by $\mathcal{CG}_*$ the category of pointed compactly generated spaces and pointed maps, and by $\wedge$ the smash product of pointed spaces.

- For a category $\mathcal{C}$, a *cosimplicial object* $X^\bullet$ *in* $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. A map of cosimplicial objects is a natural transformation. X^n denotes the object of $\mathcal{C}$ at $[n]$. We define maps

$$d^i : [n] \to [n+1] \quad \text{for } 0 \leq i \leq n+1 \qquad \text{and} \qquad s^i : [n] \to [n-1] \quad \text{for } 0 \leq i \leq n-1$$

by

$$d^i(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \geq i, \end{cases} \qquad \text{and} \qquad s^i(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i. \end{cases}$$

Here $d^i, s^i : X^n \to X^{n\pm 1}$ denote the maps corresponding to the same symbols. As is well known, a cosimplicial object $X^\bullet$ is identified with a sequence of objects $X_0, X_1, \dots, X_n, \dots$ equipped with a family of maps $\{d^i, s^i\}$ satisfying the cosimplicial identity; see [16]. We call a cosimplicial object in $\mathcal{CG}$ a *cosimplicial space*. Similarly, a simplicial object $X_\bullet$ in $\mathcal{C}$ is a functor $\Delta^{\text{op}} \to \mathcal{C}$. We denote by $d_i, s_i : X_{n\pm 1} \to X_n$ the maps corresponding to d^i and s^i.

- Our notion of a *symmetric spectrum* is that of Mandell, May, Schwede and Shipley [28]. A symmetric spectrum consists of a symmetric sequence $\{X_k\}_{k\geq 0}$ in $\mathcal{CG}_*$ and a map $\sigma_X : S^1 \wedge X_k \to X_{k+1}$ for each $k \geq 0$ which is subject to certain conditions. The category of symmetric spectra is denoted by $\mathcal{SP}$. We denote by $\wedge = \wedge_S$ the canonical symmetric monoidal product on $\mathcal{SP}$ given in [28], and by $\mathbb{S}$ the sphere spectrum, the unit for $\wedge$. Henceforth the term "spectrum" means symmetric spectrum. For a spectrum, we refer to the numbering of the underlying sequence as the *level*.

- For $K \in \mathcal{CG}$ and $X \in \mathcal{SP}$, we define a tensor $K \mathbin{\hat{\otimes}} X \in \mathcal{SP}$ by $(K \mathbin{\hat{\otimes}} X)_k = (K_+) \wedge X_k$, where K_+ is K with disjoint basepoint. This tensor is extended to a functor $\mathcal{CG} \times \mathcal{SP} \to \mathcal{SP}$ in an obvious manner. For $K, L \in \mathcal{CG}$ and $X, Y \in \mathcal{SP}$, we call the natural isomorphisms

$$K \mathbin{\hat{\otimes}} (L \mathbin{\hat{\otimes}} X) \cong (K \times L) \mathbin{\hat{\otimes}} X \quad \text{and} \quad K \mathbin{\hat{\otimes}} (X \wedge Y) \cong (K \mathbin{\hat{\otimes}} X) \wedge Y,$$

the *associativity isomorphisms*. A natural isomorphism $(K \times L) \mathbin{\hat{\otimes}} (X \wedge Y) \cong (K \mathbin{\hat{\otimes}} X) \wedge (L \mathbin{\hat{\otimes}} Y)$ is defined by successive compositions of the associativity isomorphisms and the symmetry one for $\wedge$. We define a mapping object $\mathrm{Map}(K, X) \in \mathcal{SP}$ by $\mathrm{Map}(K, X)_k = \mathrm{Map}_*(K_+, X_k)$, where the right-hand side is the usual internal hom object (mapping space) of $\mathcal{CG}_*$. This defines a functor $(\mathcal{CG})^{\mathrm{op}} \times \mathcal{SP} \to \mathcal{SP}$. The functors $K \mathbin{\hat{\otimes}} (-)$ and $\mathrm{Map}(K, -)$ form an adjoint pair. We set $K^\vee = \mathrm{Map}(K, \mathbb{S})$ for $K \in \mathcal{CG}$.

- We use the *stable model structure on* $\mathcal{SP}$; see [28]. This is only used in Section 5.1 and Section 8. Weak equivalences in this model structure are called *stable equivalences*. *Level equivalences* and *π_*–isomorphisms* are more restricted classes of maps in $\mathcal{SP}$; see [28]. The former are the levelwise weak homotopy equivalences and the latter are the maps which induce an isomorphism between (naive) homotopy groups defined as the colimit of the sequence of canonical maps $\iota_k : \pi_*(X_k) \to \pi_{*+1}(X_{k+1})$.

- We say a spectrum X is *semistable* if there exists a number $\alpha > 1$ such that, for any sufficiently large l, the map $\iota_l : \pi_k(X_l) \to \pi_{k+1}(X_{l+1})$ is an isomorphism for each $k \leq \alpha l$. Semistability in this sense implies semistability in the sense of [34], so a stable equivalence between semistable spectra (in our sense) is a π_*–isomorphism.

- A *nonunital commutative symmetric ring spectrum* (in short, *NCRS*) is a spectrum A with a commutative associative multiplication $A \wedge A \to A$ (but possibly without a unit). A map of NCRS is a map of spectra preserving the multiplication.

- $\mathcal{CH}_{\mathsf{k}}$ denotes the category of (possibly unbounded) chain complexes over k and chain maps. Differentials raise the degree (see the next item for our degree rule). We endow $\mathcal{CH}_{\mathsf{k}}$ with the model structure where weak equivalences are quasi-isomorphisms and fibrations are surjections. We denote by $\otimes = \otimes_k$ the standard tensor product of complexes.

- We deal with modules with multiple degrees (or gradings). For modules having superscript(s) and/or subscript(s), their total degree is given by the formula

$$(\text{total degree}) = (\text{sum of superscripts}) - (\text{sum of subscripts}).$$

For example, singular chains in $C_p(M)$ have degree $-p$, and the total degree of a triply graded module $A_\bullet^{\star *}$ is $* + \star - \bullet$. We denote by $|a|$ the (bi)degree of a. We sometimes omit super- or subscripts if unnecessary.

- For a simplicial chain complex $C_\bullet^*$ (ie a functor $\Delta^{\mathrm{op}} \to \mathcal{CH}_{\mathsf{k}}$), the *normalized complex* (or *normalization*) $NC_\bullet^*$ is a double complex defined by taking the normalized complex of a simplicial k–module in each chain degree.

- For a small category C and a cofibrantly generated model category $\mathcal{M}$ (in the sense of [21]), we denote by $\mathcal{F}un(C, \mathcal{M})$ the category of functors $C \to \mathcal{M}$ and natural transformations, which is endowed with the projective model structure; see [20]. The colimit functor $\mathrm{colim}_C : \mathcal{F}un(C, \mathcal{M}) \to \mathcal{M}$ is a left Quillen functor. Its left derived functor is denoted by $\mathrm{hocolim}_C$ and called the *homotopy colimit* over C.

• A *commutative differential bigraded algebra* (in short, *CDBA*) is a bigraded module $A^{\star *}$ equipped with a unital multiplication which is graded commutative for the total degree and preserves the bigrading, and a differential $\partial\colon A^{\star *} \to A^{\star+1,*}$ which satisfies the Leibniz rule for the total degree. A *map* of CDBA is a map of differential graded algebras preserving bigrading.

2.2 Čech complex and homotopy colimit

Definition 2.1 Let $\mathcal{M}$ be a cofibrantly generated model category. We define a functor

$$\check{\mathrm{C}}\colon \mathcal{F}un(\mathrm{P}^{\mathrm{op}}_{n+1}, \mathcal{M}) \to \mathcal{F}un(\Delta^{\mathrm{op}}, \mathcal{M}) \quad \text{by } \check{\mathrm{C}}X[k] = \coprod_{f\colon [k]\to \underline{n+1}} X_{f([k])},$$

where f runs through the weakly order-preserving maps. For an order-preserving $\alpha : [l] \to [k] \in \Delta$, the map $\check{\mathrm{C}}X[k] \to \check{\mathrm{C}}X[l]$ is the sum of the maps $X_{f([k])} \to X_{f\circ\alpha([l])}$ induced by the inclusion $f\circ\alpha([l]) \subset f([k])$.

Lemma 2.2 *We use the notation of Definition 2.1. Let $X \in \mathcal{F}un(\mathrm{P}^{\mathrm{op}}_{n+1}, \mathcal{M})$ be a functor.*

(1) *There exists an isomorphism* $\mathrm{hocolim}_{\mathrm{P}^{\mathrm{op}}_{n+1}} X \cong \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \check{\mathrm{C}}X$ *in* $\mathbf{Ho}(\mathcal{M})$ *which is natural for* X.

(2) *X is cofibrant in $\mathcal{F}un(\mathrm{P}^{\mathrm{op}}_{n+1}, \mathcal{M})$ if the following canonical map is a cofibration in $\mathcal{M}$ for each $S \in \mathrm{P}_{n+1}$:*

$$\operatorname*{colim}_{S' \supsetneq S} X_{S'} \to X_S.$$

Proof Let $(i_n \circ \mathcal{G}_n)^* : \mathcal{F}un(\Delta^{\mathrm{op}}, \mathcal{M}) \to \mathcal{F}un(\mathrm{P}^{\mathrm{op}}_{n+1}, \mathcal{M})$ be the pullback by the composition of $\mathcal{G}_n$ and the inclusion $i_n\colon \Delta_n \to \Delta$. Clearly the pair $(\check{\mathrm{C}}, (i_n \circ \mathcal{G}_n)^*)$ is a Quillen adjoint pair, and it is also clear that $\mathrm{colim}_{\mathrm{P}^{\mathrm{op}}_{n+1}} X$ and $\mathrm{colim}_{\Delta^{\mathrm{op}}} \check{\mathrm{C}}X$ are naturally isomorphic. Part (1) follows from these observations. Part (2) is a special case of [21, Theorem 5.1.3]. □

2.3 Goodwillie–Weiss embedding calculus and Sinha's cosimplicial model

In this subsection, we give the definition of the cosimplicial space $\mathcal{C}^\bullet(M)$ modeling $\mathrm{Emb}(S^1, M)$, and state its property. This is a minor variation of the model given in [37]. In [37], models of a space of embeddings from the interval $[0, 1]$ to a manifold with some endpoint condition, while we consider embeddings $S^1 \to M$ without any basepoint condition. The difference which needs care is that the homotopy limit of our cosimplicial model on the subcategory Δ_n need not to be weak homotopy equivalent to the n^{th} stage of the corresponding Taylor tower, while Sinha's original one is. At the ∞–stage, they are equivalent, which is sufficient for our purpose. We begin with an analogue of the punctured knot model in [37, Definition 3.4], which is an intermediate object between $\mathrm{Emb}(S^1, M)$ and $\mathcal{C}^\bullet(M)$.

Definition 2.3 • Let $S^1 = [0, 1]/0 \sim 1$ and $J_i \subset S^1$ be the image of the interval $(1-1/2^i-1/10^i, 1-1/2^i)$ by the quotient map $[0, 1] \to S^1$.

- We fix an embedding $M \to \mathbb{R}^{N+1}$ for sufficiently large N. We endow M with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^{N+1}$ via this embedding. Let SM denote the total space of the unit tangent sphere bundle of M.
- For a subset $S \subset \underline{n+1}$, let $E_S(M)$ be the space of embeddings $S^1 - \bigcup_{i \in S} J_i \to M$ of constant speed.
- Define a functor $\mathcal{E}_n(M)\colon \mathrm{P}_{n+1} \to \mathcal{CG}$ by assigning to a subset S the space $E_S(M)$, and set

$$P_n \operatorname{Emb}(S^1, M) := \operatorname*{holim}_{\mathrm{P}_{n+1}} \mathcal{E}_n(M).$$

Let $\alpha_n\colon \operatorname{Emb}(S^1, M) \to P_n \operatorname{Emb}(S^1, M)$ be the map induced by restriction of the domain. The category P_n is regarded as a subcategory of P_{n+1} via the standard inclusion $\underline{n} \to \underline{n+1}$. By our choice of J_i, we have a canonical restriction map $r_n\colon P_n \operatorname{Emb}(S^1, M) \to P_{n-1} \operatorname{Emb}(S^1, M)$. The maps α_n induce a map

$$\alpha_\infty\colon \operatorname{Emb}(S^1, M) \to \operatorname*{holim}_n P_n \operatorname{Emb}(S^1, M),$$

where the right side is the homotopy limit of the tower $\cdots \xrightarrow{r_{n+1}} P_n \operatorname{Emb}(S^1, M) \xrightarrow{r_n} P_n \operatorname{Emb}(S^1, M) \xrightarrow{r_{n-1}} \cdots \xrightarrow{r_2} P_1 \operatorname{Emb}(S^1, M)$.

Remark 2.4 Our choice of J_i is different from [37], since we adopt the reverse labeling of coface and codegeneracy maps of the cosimplicial model to [37], for the author's preference. This does not cause any new problem.

Lemma 2.5 *Suppose $d \geq 4$. The map $\alpha_n\colon \operatorname{Emb}(S^1, M) \to P_n \operatorname{Emb}(S^1, M)$ is $(n-1)(d-3)$–connected. In particular, α_∞ is a weak homotopy equivalence.*

Proof Let $p\colon \operatorname{Emb}(S^1, M) \to SM$ be the evaluation of value and tangent vector at $0 \in S^1$. As is well known, p is a fibration. Let D be a closed subset on M diffeomorphic to a closed d–dimensional disk. Let $\operatorname{Emb}([0,1], M - \operatorname{Int}(D))$ be the space of embeddings $[0,1] \to M - \operatorname{Int}(D)$ whose value and tangent vector at endpoints are a fixed value in ∂D and vector. If we take a point of SM, for some choice of the disk D, fixed endpoints and embedded path between the points in D, we have the inclusion from $\operatorname{Emb}([0,1], M - \operatorname{Int}(D))$ to the fiber of p at the point. This inclusion is a weak homotopy equivalence. Its homotopy inverse is given by shrinking the disk D to the point. Thus, we have a homotopy fiber sequence

$$\operatorname{Emb}([0,1], M - \operatorname{Int}(D)) \to \operatorname{Emb}(S^1, M) \to SM.$$

Restricting the domain, we have a similar fiber sequence $E_S(M - \operatorname{Int}(D)) \to E_S(M) \to SM$, where the left-hand side is the space defined in [37, Definition.3.1] with the obvious modification for J_i. (In [37], M denotes a manifold with boundary, so we apply the definitions to $M - \operatorname{Int}(D)$ instead of our closed M.) Passing to homotopy limits, we have the diagram

$$\begin{tikzcd}
\operatorname{Emb}([0,1], M - \operatorname{Int}(D)) \arrow[r] \arrow[d] & \operatorname{Emb}(S^1, M) \arrow[r] \arrow[d, "\alpha_n"] & SM \arrow[d, "\mathrm{id}"] \\
P_n \operatorname{Emb}([0,1], M - \operatorname{Int}(D)) \arrow[r] & P_n \operatorname{Emb}(S^1, M) \arrow[r] & SM
\end{tikzcd}$$

where both horizontal sequence are homotopy fiber sequences and the left bottom corner is the punctured knot model in [37, Definition.3.4] (with the obvious modification for J_i). As in [37, Theorem.3.5], by theorems of Goodwillie, Klein, and Weiss, the left vertical arrow is $(n-1)(d-3)$–connected, and so is the middle. □

Remark 2.6 Let $T_n \operatorname{Emb}(S^1, M)$ be the n^{th} stage of the Taylor tower (or polynomial approximation). Restriction of the domain induces a map $P_n \operatorname{Emb}(S^1, M) \to T_n \operatorname{Emb}(S^1, M)$ which is compatible with canonical maps from $\operatorname{Emb}(S^1, M)$, but the author does not know whether this map is a weak homotopy equivalence.

Our cosimplicial space is analogous to the well-known cosimplicial model of a free loop space, just like Sinha's original space is analogous to that of a based loop space. So the space $\mathcal{C}^n(M)$ is related to a configuration space of $n+1$ points (not n points).

Definition 2.7 Let $\|-\|$ denote the standard Euclidean norm in $\mathbb{R}^{N+1}$.

- Let $C_n(M) = \{(x_0, \dots, x_{n-1}) \in M^n \mid x_k \neq x_l \text{ if } k \neq l\}$ be the ordered configuration space of n points in M. Similarly, we set $C_2([n]) = \{(k, l) \in [n]^{\times 2} \mid k \neq= l\}$.
- Let $\overline{C}_n(M)$ be the closure of the image of the map

$$C_n(M) \to M^n \times (S^N)^{\times C_2([n-1])}, \quad (x_k)_k \mapsto (x_k, u_{kl})_{kl},$$

where $u_{kl} = (x_l - x_k)/\|x_l - x_k\|$. $\overline{C}_n(M)$ is the same as the space in Definition 4.1(6) of [37], though our labeling of points begins with 0. Define a space $\mathcal{C}^n(M)$ by the following pullback diagram:

$$\begin{tikzcd}
\mathcal{C}^n(M) \arrow[r] \arrow[d] & SM^{n+1} \arrow[d] \\
\overline{C}_{n+1}(M) \arrow[r] & M^{n+1}
\end{tikzcd}$$

Here the right vertical arrow is the product of standard projection and the bottom horizontal one is the composition of the canonical inclusion $\overline{C}_{n+1}(M) \to M^{\times n+1} \times (S^N)^{\times C_2([n])}$ and the projection.

- Let $\tau \colon T_x M \to \mathbb{R}^{N+1}$ be the linear monomorphism from the tangent space induced by the differential of the embedding fixed in Definition 2.3 and the identification $T_x\mathbb{R}^{N+1} \cong \mathbb{R}^{N+1}$ by the standard basis. Set $A'_{n+1}(M) := M^{\times n+1} \times (S^N)^{\times ([n]^{\times 2})}$. Let $\beta'_{n+1} \colon \mathcal{C}^n(M) \to A'_{n+1}(M)$ be the map given by

$$\beta'_{n+1}(x_k, u_{kl}, y_k) = (x_k, u'_{kl}) \qquad \text{and} \qquad u'_{kl} = \begin{cases} u_{kl} & \text{if } k \neq l, \\ \tau(y_k) & \text{if } k = l, \end{cases}$$

where y_k is a unit tangent vector at x_k. This is clearly a monomorphism. For an integer i with $0 \le i \le n+1$, we define a map $d_i \colon [n+1] \to [n]$ by

$$d_i(k) = \begin{cases} k & \text{if } k \le i, \\ k-1 & \text{if } k > i, \end{cases} \quad \text{for } 0 \le i \le n \qquad \text{and} \qquad d_{n+1} = d_0 \circ \sigma,$$

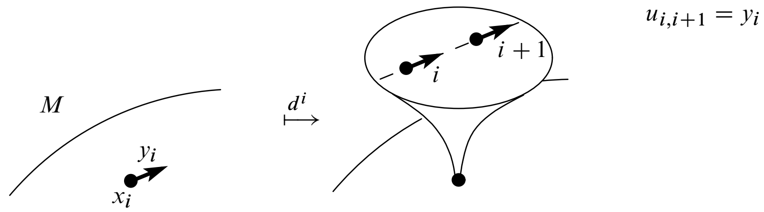

Figure 1: Intuition of the coface map d^i. Here y_i is the vector at x_i.

where σ is the cyclic permutation $\sigma(k) = k+1 \pmod{n+2}$. (This d_i is the same as s^i in Section 2.1, but we use the different notation to avoid confusion.) We define a map $d^i : A'_{n+1}(M) \to A'_{n+2}(M)$ by

$$d^i(x_k, u_{kl})_{0\le k,l\le n} = (x_{f(k)}, u_{f(k),f(l)})_{0\le k,l\le n+1} \quad \text{with } f = d_i.$$

This map restricts to the map $d^i : \mathcal{C}^n(M) \to \mathcal{C}^{n+1}(M)$ via β'_{n+1}, β'_{n+2}. Similarly, we define a map $s^i : \mathcal{C}^n(M) \to \mathcal{C}^{n-1}(M)$ for $0 \le i \le n-1$ as the pullback by the map

$$s_i : [n-1] \to [n], \qquad s_i(k) = \begin{cases} k & \text{if } k \le i, \\ k+1 & \text{if } k > i. \end{cases}$$

The collection $\mathcal{C}^\bullet(M) = \{\mathcal{C}^n(M), d^i, s^i\}$ forms a cosimplicial space. Well-definedness of this is verified in Lemma 2.8.

- We call the Bousfield–Kan type cohomology spectral sequence associated to $\mathcal{C}^\bullet(M)$ the *Sinha spectral sequence for M*, in short, the *Sinha s.s.*, and denote it by $\{\mathbb{E}_r\}_r$.

Intuitively, an element of $\overline{C}_n(M)$ is a configuration of n points in M, some points of which are allowed to collide, or in other words, to be infinitesimally close, and the direction of collision is recorded as the unit vector u_{kl} if the k^{th} and l^{th} points collide. An element of $\mathcal{C}^n(M)$ is an element of $\overline{C}_{n+1}(M)$, each point of which has a unit tangent vector. For $0 \le i \le n$, the map d^i replaces the i^{th} point in a configuration with the two points colliding at the point along its vector. These points are labeled by i and $i+1$. Their vectors are copies of the original vector (see Figure 1). The map d^{n+1} replaces the 0^{th} points with two points similarly, and labels them by $n+1$ and 0 (and slides other labels). The map s^i forgets the $(i+1)^{\text{th}}$ point and vector.

Lemma 2.8 (1) *The map $C_n(M) \to M^n \times (S^N)^{\times C_2([n-1])}$ given in Definition 2.7 restricts to a homotopy equivalence $C_n(M) \to \overline{C}_n(M)$.*

(2) *The cosimplicial space $\mathcal{C}^\bullet(M)$ is well defined.*

Proof Part (1) is proved in [35, Corollary 4.5 and Theorem. 5.10]. For (2), by [35, Proposition 6.6] the image of d^i and s^i is contained in $\mathcal{C}^{n\pm1}(M)$—$C'_n\langle[M]\rangle$ in the proposition is the same as $\mathcal{C}^{n-1}(M)$ in our notation. Confirmation of the cosimplicial identities is routine work. For example, to confirm $d^{n+2}d^i = d^i d^{n+1} : \mathcal{C}^n(M) \to \mathcal{C}^{n+2}(M)$ for $i < n+2$, it is enough to confirm the dual identity

$d_i d_{n+2} = d_{n+1} d_i : [n+2] \to [n]$. Both sides are equal to the map

$$k \mapsto \begin{cases} k & \text{if } k \le i, \\ k-1 & \text{if } i < k < n+2, \\ 0 & \text{if } k = n+2, \end{cases} \quad \text{if } i < n+1, \qquad k \mapsto \begin{cases} k & \text{if } k \le n, \\ 0 & \text{if } k = n+1, n+2, \end{cases} \quad \text{if } i = n+1.$$

□

Lemma 2.9 *Let $\mathcal{G}_n^* \mathcal{C}^\bullet(M)$ be the composition functor* $\mathrm{P}_{n+1} \xrightarrow{\mathcal{G}_n} \Delta_n \xrightarrow{\mathcal{C}^\bullet(M)} \mathcal{CG}$.

(1) *The homotopy limits of $\mathcal{E}_n(M)$ and $\mathcal{G}_n^* \mathcal{C}^\bullet(M)$ are connected by a zigzag of weak homotopy equivalences which are compatible with the inclusion $\underline{n} \to \underline{n+1}$.*

(2) *The homotopy limit of $\mathcal{C}^\bullet(M)$ over Δ_n and that of $\mathcal{G}_n^* \mathcal{C}^\bullet(M)$ over P_{n+1} are connected by a zigzag of weak homotopy equivalences which are compatible with the inclusion $\underline{n} \to \underline{n+1}$.*

(3) *If $d \ge 4$, the homotopy limit of $\mathcal{C}^\bullet(M)$ over Δ and $\mathrm{Emb}(S^1, M)$ are connected by a zigzag of weak homotopy equivalences.*

Proof The proof of (1) is completely analogous to the proof of [37, Lemma 5.19] so we omit details. The idea of the proof is to consider the two space $\mathcal{C}^{\#S-1}(M)$ and $E_S(M)$ as subspaces of a common space, where one can "shrink components of embeddings until they become tangent vectors", as in [37, Definition 5.14]. The space is a subspace of the space of compact subspaces of $\mathcal{C}^{\#S-1}(M)$ with the Hausdorff metric. This space and the inclusions can be chosen to be compatible with maps in P_{n+1}. For example, the restriction $E_S(M) \to E_{S'}(M)$ corresponding to the inclusion $S = \underline{n+1} \subset S' = \underline{n+2}$ divides the component including the image of $0 \in S^1$ into two components, since the image of J_{n+2} is removed. At the limit of shrinking components, this is consistent with the coface map d^{n+1}. These inclusions to the common space give rise to a zigzag of natural transformations which is a weak homotopy equivalence at each set $S \subset \underline{n+1}$. This induces the claimed zigzag. Part (2) follows from the fact that the functor $\mathcal{G}_n$ is left cofinal; see Theorem 6.7 of [37]. Part (3) follows from (1), (2) and Lemma 2.5. □

2.4 Operads, comodules and the Hochschild complex

The term *operad* means *nonsymmetric* (or *non–Σ*) operad; see [24; 31]. An operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n \ge 1}$ in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is a sequence of objects equipped with maps

$$(- \circ_i -) \colon \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m+n-1) \quad \text{for } 1 \le i \le m$$

in $\mathcal{C}$, called *partial compositions*, which are subject to certain conditions. $\mathcal{O}(n)$ is called the object at *arity n*. More precisely, our notion of an operad is different from the one in [24; 31] only in that we do not consider the object at arity 0, so conditions on partial compositions given in [24; 31] are imposed only in the ranges of all involved arities being 1 or more. We mainly consider operads in $\mathcal{CG}$ (resp. in $\mathcal{CH}_{\mathrm{k}}$), which are called topological operads (resp. chain operads), where the monoidal product is the standard cartesian product (resp. tensor product). Let $\mathcal{O}$ be a topological operad. $C_*(\mathcal{O})$ denotes the chain operad given by $C_*(\mathcal{O})(n) = C_*(\mathcal{O}(n))$ with the induced structure. We equip the sequence $\{\mathcal{O}(n) \mathbin{\hat{\otimes}} \mathbb{S}\}_n$ of spectra with a structure of an operad in $\mathcal{SP}$ as follows. The i^{th} partial composition is given by

$$(\mathcal{O}(m)\,\hat{\hat{\otimes}}\,\mathbb{S})\wedge(\mathcal{O}(n)\,\hat{\hat{\otimes}}\,\mathbb{S})\cong(\mathcal{O}(m)\times\mathcal{O}(n))\,\hat{\hat{\otimes}}\,(\mathbb{S}\wedge\mathbb{S})\cong(\mathcal{O}(m)\times\mathcal{O}(n))\,\hat{\hat{\otimes}}\,\mathbb{S}\xrightarrow{(-\circ_i-)\hat{\hat{\otimes}}\mathrm{id}}\mathcal{O}(m+n-1)\,\hat{\hat{\otimes}}\,\mathbb{S}.$$

See Section 2.1 for the isomorphisms. The action of Σ_n is the naturally induced action. We denote this operad by the same symbol, $\mathcal{O}$. We let $\mathcal{A}$ denote both of the (discrete) topological and $\Bbbk$–linear versions of the associative operad by abuse of notation. For the $\Bbbk$–linear version, we fix a generator $\mu \in \mathcal{A}(2)$ throughout this paper. $\mathcal{K}$ denotes the Stasheff associahedral operad, and $\mathcal{A}_\infty$ the cellular chain operad of $\mathcal{K}$. Precisely speaking, $\mathcal{A}_\infty$ is generated by a set $\{\mu_k \in \mathcal{A}_\infty(k)\}_{k\geq 2}$ with $|\mu_k| = -k+2$, with partial compositions. The differential is given by the formula

$$d\mu_k = \sum_{\substack{l,p,q\\ l+q=k+1}} (-1)^\zeta \mu_l \circ_{p+1} \mu_q,$$

where $\zeta = \zeta(l,p,q) = p + q(l-p-1)$.

In the following definition, we adopt the point–set description, as if a category $\mathcal{C}$ were the category of sets, for simplicity.

Definition 2.10 • Let $\mathcal{O}$ be an operad over a symmetric monoidal category $\mathcal{C}$. A *(left) $\mathcal{O}$–comodule in* $\mathcal{C}$ is a symmetric sequence $X = \{X(n)\}_{n\geq 1}$ in $\mathcal{C}$ equipped with maps

$$(-\circ_i -)\colon \mathcal{O}(m)\otimes X(m+n-1)\to X(n)\in\mathcal{C}$$

for $m\geq 1$, $n\geq 1$ and $1\leq i\leq n$, called *partial compositions*, which satisfy the following conditions:

(1) For $a\in\mathcal{O}(m)$, $b\in\mathcal{O}(l)$ and $x\in X(l+m+n-2)$,

$$a\circ_i (b\circ_j x) = \begin{cases} b\circ_j (a\circ_{i+l-1} x) & \text{if } j<i,\\ (a\circ_{j-i+1} b)\circ_i x & \text{if } i\leq j\leq i+m-1,\\ b\circ_{j-m+1}(a\circ_i x) & \text{if } i+m-1<j.\end{cases}$$

(2) For the unit $1\in\mathcal{O}(1)$ and $x\in X(n)$, we have $1\circ_i x = x$.

(3) For $a\in\mathcal{O}(m)$, $x\in X(m+n-1)$ and $\sigma\in\Sigma_n$,

$$(a\circ_i x)^\sigma = a\circ_{\sigma^{-1}(i)}(x^{\sigma_1}),$$

where $\sigma_1\in\Sigma_{m+n-1}$ is the permutation induced by σ, replacing the letter $\sigma^{-1}(i)$ with the m letters $\sigma^{-1}(i),\dots,\sigma^{-1}(i)+m-1$. In other words,

$$\sigma_1(k) = \begin{cases} \sigma(k) & \text{if } k<\sigma^{-1}(i) \text{ and } \sigma(k)<i,\\ \sigma(k)+m-1 & \text{if } k<\sigma^{-1}(i) \text{ and } \sigma(k)>i,\\ i+k-\sigma^{-1}(i) & \text{if } \sigma^{-1}(i)\leq k\leq\sigma^{-1}(i)+m-1,\\ \sigma(k-m+1) & \text{if } k>\sigma^{-1}(i)+m-1 \text{ and } \sigma(k-m+1)<i,\\ \sigma(k-m+1)+m-1 & \text{if } k>\sigma^{-1}(i)+m-1 \text{ and } \sigma(k-m+1)>i.\end{cases}$$

A *map* $f\colon X_1\to X_2$ *of $\mathcal{O}$–comodules* is a sequence of maps in $\mathcal{C}$ $\{f_n\colon X_1(n)\to X_2(n)\}_n$ which is compatible with the actions of symmetric groups and the partial compositions.

- A (*right*) $\mathcal{O}$–*module in* $\mathcal{C}$ is a symmetric sequence $Y = \{Y(n)\}_{n\geq 1}$ equipped with a set of partial compositions $Y(n)\otimes\mathcal{O}(m)\to Y(m+n-1)$ which satisfy the following conditions:
  (1) For $a\in\mathcal{O}(m)$, $b\in\mathcal{O}(l)$ and $y\in y(n)$,
  $$(y\circ_j a)\circ_i b = \begin{cases}(y\circ_i b)\circ_{j+l-1} a & \text{if } i<j,\\ y\circ_j (a\circ_{i-j+1} b) & \text{if } j\leq i\leq j+m-1,\\ (y\circ_{i+m-1} b)\circ_j a & \text{if } i>j+m-1.\end{cases}$$
  (2) For the unit $1\in\mathcal{O}(1)$ and $y\in X(n)$, we have $y\circ_i 1 = y$.
  (3) For $a\in\mathcal{O}(m)$, $y\in X(n)$ and $\sigma\in\Sigma_n$,
  $$y^\sigma\circ_i a = (y\circ_{\sigma(i)} a)^{\sigma_2},$$
  where $\sigma_2\in\Sigma_{m+n-1}$ is the permutation induced by σ, replacing the letter i with the m letters $i,\dots,i+m-1$. In other words,
  $$\sigma_2(k) = \begin{cases}\sigma(k) & \text{if } k<i \text{ and } \sigma(k)<\sigma(i),\\ \sigma(k)+m-1 & \text{if } k<i \text{ and } \sigma(k)>\sigma(i),\\ \sigma(i)+k-i & \text{if } i\leq k\leq i+m-1,\\ \sigma(k-m+1) & \text{if } k>i+m-1 \text{ and } \sigma(k-m+1)<\sigma(i),\\ \sigma(k-m+1)+m-1 & \text{if } k>i+m-1 \text{ and } \sigma(k-m+1)>\sigma(i).\end{cases}$$

A *map of modules* is defined similarly to that of comodules.

- For a topological operad $\mathcal{O}$ (regarded as an operad in $\mathcal{SP}$), an $\mathcal{O}$–*comodule of NCRS* is an $\mathcal{O}$–comodule X in $\mathcal{SP}$ such that each $X(n)$ is equipped with a structure of an NCRS and the action of Σ_n on $X(n)$ and the partial composition $(a\circ_i -)\colon X(n+m-1)\to X(n)$ is a map of NCRS for each $a\in\mathcal{O}(m)$. A *map of comodules of NCRS* is a map of comodules which is also a map of NCRS at each arity.

- For a topological operad $\mathcal{O}$ and an $\mathcal{O}$–module Y, we define an $\mathcal{O}$–comodule $Y^\vee$ of NCRS as follows:
  (1) We set $Y^\vee(n) = Y(n)^\vee$ (see Section 2.1).
  (2) For $f\in Y^\vee(n)$ and $\sigma\in\Sigma_n$, we define an action f^σ by $f^\sigma(y) = f(y^{\sigma^{-1}})$ for each $y\in Y(n)$.
  (3) For $a\in\mathcal{O}(m)$ and $f\in Y^\vee(m+n-1)$, we define a partial composition $a\circ_i f$ by $a\circ_i f(y) = f(y\circ_i a)$ for each $y\in Y(n)$.
  (4) We define a multiplication $Y^\vee(n)\wedge Y^\vee(n)\to Y^\vee(n)$ as the pushforward by the multiplication of $\mathbb{S}$. (This is actually unital.)

This construction is natural for maps of $\mathcal{O}$–modules.

- An $\mathcal{A}$–*comodule* X *of CDBA* is an $\mathcal{A}$–comodule (in $\mathcal{CH}_{\mathsf{k}}$) such that each $X(n)$ is a CDBA, and the partial composition $\mu\circ_i(-)\colon X(n)\to X(n-1)$ — with the fixed generator $\mu\in\mathcal{A}(2)$ — and action of $\sigma\in\Sigma_n$ preserve the differential, bigrading, multiplication and unit.

The axioms for the partial compositions of modules (Definition 2.10) are the standard ones, which are naturally interpreted in terms of concatenation of trees. The action of $\sigma\in\Sigma_n$ is interpreted as replacement of labels i on leaves with labels $\sigma^{-1}(i)$, and the axiom is the natural one with this interpretation. The axioms for a comodule are simply dual to those for a module. The comodule in Example 2.14 may give some intuition for it.

Remark 2.11 The notion of a right module in Definition 2.10 is similar to the one in [26]. A right $\mathcal{O}$–module is also essentially the same as a topological contravariant functor from the PROP of $\Sigma\mathcal{O}$ to spaces (or spectra), and a left $\mathcal{O}$–comodule is a covariant functor. Here $\Sigma\mathcal{O}$ is the standard symmetrization of $\mathcal{O}$, ie $\Sigma\mathcal{O}(n) = \mathcal{O}(n) \times \Sigma_n$; see [29].

Composing the unity and associativity isomorphisms, we get a natural isomorphism $K \mathbin{\hat{\otimes}} X \cong (K \mathbin{\hat{\otimes}} \mathbb{S}) \wedge X$ in $\mathcal{SP}$. Let $\mathcal{O}$ be a topological operad. Via this isomorphism, a structure of an $\mathcal{O}$–comodule in $\mathcal{SP}$ on a symmetric sequence X is equivalent to a set of maps

$$\mathcal{O}(m) \mathbin{\hat{\otimes}} X(m+n-1) \to X(n)$$

which satisfy conditions completely similar to those given in Definition 2.10. We also call these maps partial compositions, and henceforth will define comodules in $\mathcal{SP}$ with these maps.

Remark 2.12 Precisely speaking, comodules in Definition 2.10 should be called contracomodules, because our comodules are to modules as contramodules are to comodules in [32], but for simplicity we adopt our terminology.

The following definition is essentially due to [16], though we adopt a different sign rule.

Definition 2.13 Let X^* be an $\mathcal{A}_\infty$–comodule in $\mathcal{CH}_{\mathsf{k}}$. We define a chain complex $(\mathrm{CH}_\bullet\, X^*, \tilde{d})$, called the *Hochschild complex of* X, as follows. Set $\mathrm{CH}_n\, X^* = X^*(n+1)$. By our convention, the total degree is $* - \bullet$. The differential $\tilde{d}$ is given as a map

$$\tilde{d} = d - \delta\colon \bigoplus_{a-n=k} \mathrm{CH}_n\, X^a \to \bigoplus_{a-n=k+1} \mathrm{CH}_n\, X^a.$$

Here d is the internal (original) differential on $X^a(n+1)$ and δ is given by

$$\delta(x) = \sum_{i=0}^{n} \sum_{k=2}^{n-i+1} (-1)^\epsilon \mu_k \circ_{i+1} x + \sum_{s=1}^{n} \sum_{k=s+1}^{n+1} (-1)^\theta \mu_k \circ_1 x^s$$

for $x \in X^a(n+1)$, where $\epsilon = \epsilon(a,i,k) = (a+i)(k+1)$, $\theta = \theta(s,n,k,a) = sn + (k+1)a$ and x^s denotes the image of x by the action of the permutation in Σ_{n+1} which transposes the first $n-s+1$ letters and the last s letters.

The following example gives some intuition for the definitions of a comodule and the Hochschild complex, but is not used later.

Example 2.14 Let $\mathcal{C}$ be the category of k–modules and A be a k–algebra. Let $m_n \in \mathcal{A}(n)$ be the element defined by successive partial compositions of the generator $\mu \in \mathcal{A}(2)$. Define an $\mathcal{A}$–comodule X_A by

$$X_A(n) = A^{\otimes n}, \quad m_k \circ_i (x_1 \otimes \cdots \otimes x_{k+n-1}) = x_1 \otimes \cdots \otimes x_{i-1} \otimes (x_i \cdots x_{i+k-1}) \otimes x_{i+k} \otimes \cdots \otimes x_{k+n-1},$$

where $x_i \cdots x_{i+k-1}$ is the product in A. We regard X_A as an $\mathcal{A}_\infty$–comodule via a map $\mathcal{A}_\infty \to \mathcal{A}$ of operads. The Hochschild complex of X_A is the usual (unnormalized) Hochschild complex of the associative algebra A.

Lemma 2.15 *With the notation of Definition 2.13,* $(\tilde{d})^2 = 0$.

Proof Roughly,

$$\begin{aligned}
(\tilde{d})^2(x) &= \tilde{d}(dx - \delta x) = ddx - d\delta x - \delta dx - \delta\delta x \\
&= d(\mu_k \circ_{i+1} x + \mu_k \circ_1 x^s) + (\mu_k \circ_{i+1} dx + \mu_k \circ_1 dx^s) \\
&\qquad - \mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) + \mu_l \circ (\mu_k \circ_1 x^s) + \mu_l \circ_1 (\mu_k \circ_{i+1} x)^t + \mu_l \circ_1 (\mu_k \circ_1 x^s)^t \\
&= (d\mu_k) \circ_{i+1} x + (d\mu_k) \circ_1 x^s \\
&\qquad - \mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) + \mu_l \circ (\mu_k \circ_1 x^s) + \mu_l \circ_1 (\mu_k \circ_{i+1} x)^t + \mu_l \circ_1 (\mu_k \circ_1 x^s)^t.
\end{aligned}$$

(Here we already canceled the terms containing dx, since the cancellation of signs is obvious.) So we have six types of terms. To see which terms cancel with each other, we divide these terms into the following smaller classes:

(1) $(d\mu_k) \circ_{i+1} x$, $d\mu_k = \sum \mu_l \circ_{p+1} \mu_q$,

(2) $(d\mu_k) \circ_1 x^s$, $d\mu_k = \sum \mu_l \circ_{p+1} \mu_q$:
 - (a) $s < p+1$,
 - (b) $p+q \le s$,
 - (c) $p = 0$ and $q > s$,
 - (d) $p > 0$ and $p+q > s \ge p+1$,

(3) $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x)$:
 - (a) $i < j$,
 - (b) $j+l-1 < i$,
 - (c) $j \le i \le j+l-1$,

(4) $\mu_l \circ_{j+1} (\mu_k \circ_1 x^s)$:
 - (a) $j = 0$,
 - (b) $j > 0$,

(5) $\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t$:
 - (a) $i+1 < n-k-t+3$ and $l < s+i+1$,
 - (b) $i+1 < n-k-t+3$ and $l \ge s+i+1$,
 - (c) $i+1 \ge n-k-t+3$,

(6) $\mu_l \circ_1 (\mu_k \circ_1 x^s)^t$.

Now we claim that the terms in (1) cancel with the terms in (3c), (2a) with (5b), (2b) with (5c), (2c) with (4a), (2d) with (6), (3a) with (3b) and (4b) with (5a).

We shall verify the first and third parts of the claim. Other verification is similar and omitted. For the first one, the coefficient of a term $(\mu_l \circ_{p+1} \mu_q) \circ_{i+1} x$ in (1) is $(-1)^{\alpha_1}$, where

$$\alpha_1 = \zeta(l,p,q) + \epsilon(a,i,l+q+1) + 1.$$

For a term in (3-c), by the rules of the partial composition, $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) = (\mu_l \circ_{i-j+1} \mu_k) \circ_{j+1} x$. In order to match this term with a term in (1), we set $q' = k,\ p'+1 = i-j+1$ and $i'+1 = j+1$. This change of subscripts implies $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x) = (\mu_l \circ_{p'+1} \mu_{q'}) \circ_{i'+1} x$. Clearly $j = i'$ and $i = p'+i'$. The coefficient of $\mu_l \circ_{j+1} (\mu_k \circ_{i+1} x)$ in (3-c) is $(-1)^{\alpha_2}$, where

$$\alpha_2 = \epsilon(a,i,k) + 1 + \epsilon(a+k-2,j,l) + 1 = \epsilon(a,p'+i',q') + \epsilon(a-q'+2,i',l) + 2.$$

When we substitute $q' = q,\ p' = p$ and $i' = i$ in the last expression, elementary computation shows $\alpha_1 + \alpha_2 \equiv 1 \pmod 2$. Thus the terms in (1) cancel with the terms in (3-c).

For the third part, the coefficient of a term $(\mu_l \circ_{p+1} \mu_q) \circ_1 x^s$ in (2-b) is $(-1)^{\beta_1}$, where

$$\beta_1 = \zeta(l,p,q) + \theta(s,n,l+q-1,a) + 1.$$

For a term in (5-c), the condition $i+1 \geq n-k-t+3$ implies that μ_k acts on a part of the last t letters. By this, and the rule of the partial composition, we have

$$\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t = \mu_l \circ_1 (\mu_k \circ_{i-n+k+t-1} (x^{t+k-1})) = (\mu_l \circ_{i-n+k+t-1} \mu_k) \circ_1 x^{t+k-1}.$$

In order to match this term with a term in (2-b), we set $p'+1 = i-n+k+t-1$, $q' = k$ and $s' = t+k-1$. This change of subscripts implies $\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t = (\mu_l \circ_{p'+1} \mu_{q'}) \circ_1 x^{s'}$. Clearly $t = s'-q'+1$ and $i = p'+n-s'+1$. The coefficient of $\mu_l \circ_1 (\mu_k \circ_{i+1} x)^t$ is $(-1)^{\beta_2}$, where

$$\begin{aligned}\beta_2 &= \epsilon(a,i,k) + 1 + \theta(t,a-k+2,n-k+1,l) + 1 \\ &= \epsilon(a,p'+n-s'+1,q') + \theta(s'-q'+1,n-q'+1,a-q'+2,l) + 2.\end{aligned}$$

When we substitute $q' = q,\ p' = p$ and $s' = s$ in the last expression, elementary computation shows $\beta_1 + \beta_2 \equiv 1 \pmod 2$. Thus the terms in (2-b) cancel with the terms in (5-c). $\square$

3 The comodule $\mathcal{T}_M$

The purpose of this section is to define the comodule $\mathcal{T}_M$.

3.1 A model of a Thom spectrum

We introduce a model of a Thom spectrum in the category of symmetric spectra. This model is essentially due to Cohen [11], and is slightly different from Cohen's original nonunital model, mainly in that we use expanding embeddings.

Definition 3.1 Let N be a closed manifold. We fix a Riemannian metric on N and denote by $d_N(-,-)$ the distance on N induced by the metric. The standard Euclidean norm on $\mathbb{R}^k$ is denoted by $\|-\|$. The distance in $\mathbb{R}^k$ is induced by $\|-\|$.

- For a smooth embedding $e\colon N \to L$ to a Riemannian manifold L, we set a number
$$r(e) = \inf\left\{ \frac{d_L(e(x), e(y))}{d_N(x,y)} \;\middle|\; x, y \in N \text{ with } x \neq y \right\}.$$
It is easy to see $r(e) > 0$. We say e is *expanding* if the inequality $r(e) \geq 1$ holds. $\mathrm{Emb}^{\mathrm{ex}}(N, L)$ denotes the space of all expanding embeddings from N to L with the topology induced by the C^∞–topology.

- For a smooth embedding $e\colon N \to \mathbb{R}^k$, we define a number $|e|$ by
$$|e| = \sum_{i=1}^{k} \max\{|e^i(y)| \mid y \in N\},$$
where $e^i : N \to \mathbb{R}$ is the i^{th} component of e and $|-|$ is the absolute value.

- Let $e\colon N \to \mathbb{R}^k$ be a smooth embedding. For $\epsilon > 0$, we denote by $\nu_\epsilon(e)$ the open subset of $\mathbb{R}^k$ consisting of the points whose Euclidean distance from $e(N)$ is smaller than ϵ. Let $L(e)$ denote the minimum of 1 and the least upper bound of $\epsilon > 0$ such that there exists a retraction $\pi_e\colon \nu_\epsilon(e) \to e(N)$ satisfying the following conditions:
  - For any $u \in \nu_\epsilon(e)$ and any $y \in N$ we have $\|\pi_e(u) - u\| \leq \|e(y) - u\|$, and equality holds if and only if $\pi_e(u) = e(y)$.
  - For any $y \in N$ we have $\pi_e^{-1}(\{e(y)\}) = B_\epsilon(e(y)) \cap (e(y) + (T_yN)^\perp)$. Here $B_\epsilon(e(y))$ is the open ball with center $e(y)$ and radius ϵ.
  - The closure $\bar{\nu}_\epsilon(e)$ of $\nu_\epsilon(e)$ is a smooth submanifold of $\mathbb{R}^k$ with boundary.

(Such a retraction exists for a sufficiently small $\epsilon > 0$ by a version of the tubular neighborhood theorem; see [27].) The retraction π_e satisfying the above conditions is unique. We regard the map $\pi_e : \nu_\epsilon(e) \to e(N)$ as a disk bundle over N, identifying N and $e(N)$ via e.

- Let $\widetilde{N}_k^{-\tau}$ be the subspace of $\mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k) \times \mathbb{R} \times \mathbb{R}^k$ consisting of the triples (e, ϵ, u) with $0 < \epsilon < L(e)$. Define a subspace $\partial\widetilde{N}_k^{-\tau} \subset \widetilde{N}_k^{-\tau}$ by $(e, \epsilon, u) \in \partial\widetilde{N}_k^{-\tau}$ if and only if $u \notin \nu_\epsilon(e)$. We put
$$N_k^{-\tau} = \widetilde{N}_k^{-\tau} / \partial\widetilde{N}_k^{-\tau}.$$
We define a structure of a symmetric spectrum on $N^{-\tau}$ as follows:
  - We let Σ_k act on $\mathbb{R}^k$ and $\mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k)$ by the standard permutation on components. The action of Σ_k on $N_k^{-\tau}$ is given by $[e, \epsilon, u]^\sigma = [e^\sigma, \epsilon, u^\sigma]$.
  - The map $S^1 \wedge N_k^{-\tau} \to N_{k+1}^{-\tau}$ is given by $t \wedge [e, \epsilon, u] \mapsto [0 \times e, \epsilon, (t, u)]$, where we regard S^1 as $\mathbb{R} \cup \{\infty\}$, and $0 \times e\colon M \to \mathbb{R}^{k+1}$ is given by $(0 \times e)(x) = (0, e(x))$.

- We shall define a structure of NCRS on $N^{-\tau}$. An element of $(N^{-\tau} \wedge N^{-\tau})_k$ is represented by data $\langle [e_1, \epsilon_1, u_1], [e_2, \epsilon_2, u_2]; \sigma \rangle$ consisting of $[e_i, \epsilon_i, u_i] \in N^{-\tau}_{k_i}$ for $i = 1, 2$ and $k_1 + k_2 = k$, and $\sigma \in \Sigma_k$. We define a commutative associative multiplication $\mu\colon N^{-\tau} \wedge N^{-\tau} \to N^{-\tau}$ by

$$\mu(\langle [e_1, \epsilon_1, u_1], [e_2, \epsilon_2, u_2]; \sigma \rangle) = [e_{12}, \epsilon_{12}, (u_1, u_2)]^{\sigma}.$$

Here we set $e_{12} = (e_1 \times e_2) \circ \Delta$, where $\Delta\colon N \to N \times N$ is the diagonal map, and set

$$\epsilon_{12} = \min\left\{ \frac{\epsilon_1}{8^{|e_2|}}, \frac{\epsilon_2}{8^{|e_1|}}, L(e_{12}), \frac{L(e'_1)}{8^{|e_{12}|-|e'_1|}}, \dots, \frac{L(e'_m)}{8^{|e_{12}|-|e'_m|}} \;\middle|\; m \geq 2, e'_1\colon N \to \mathbb{R}^{l_1}, \dots, e'_m\colon N \to \mathbb{R}^{l_m} \right\},$$

where the finite sequence $(e'_1, \dots, e'_m)$ runs through the sequences of expanding embeddings satisfying $(e'_1 \times \cdots \times e'_m) \circ \Delta^m = (e_{12})^{\tau}$ for a permutation $\tau \in \Sigma_{k_1+k_2}$ and the diagonal map $\Delta^m\colon N \to N^m$.

Lemma 3.2 *The structure of NCRS on $N^{-\tau}$ given in Definition 3.1 is well defined*

Proof Most of the proof is the same as the proof of [11, Theorem 3]. We shall only verify the associativity of the number ϵ_{12}. Let $[e_i, \epsilon_i, u_i]$ be an element of $N^{-\tau}_{k_i}$ for $i = 1, 2, 3$. We denote by $\epsilon_{(12)3}$ (resp. $\epsilon_{1(23)}$) the number in the second entry of the product of the three elements where the elements labeled by $i = 1, 2$ (resp. $i = 2, 3$) are multiplied at first. By definition,

$$\epsilon_{(12)3} = \min\left\{ \frac{\epsilon_{12}}{8^{|e_3|}}, \frac{\epsilon_3}{8^{|e_{12}|}}, L(e_{123}), \frac{L(e'_1)}{8^{|e_{123}|-|e'_1|}}, \dots, \frac{L(e'_m)}{8^{|e_{123}|-|e'_m|}} \;\middle|\; m \geq 2, e'_1, \dots, e'_m \right\},$$

where $e_{123} = (e_1 \times e_2 \times e_3) \circ \Delta^3$, and the finite sequence $(e'_1, \dots, e'_m)$ runs through the sequences of expanding embeddings satisfying $(e'_1 \times \cdots \times e'_m) \circ \Delta^m = (e_{123})^{\tau}$ for some $\tau \in \Sigma_{k_1+k_2+k_3}$. By the obvious equality $|e_{12}| = |e_1| + |e_2|$, we have

$$\epsilon_{(12)3} = \min\left\{ \frac{\epsilon_1}{8^{|e_2|+|e_3|}}, \frac{\epsilon_2}{8^{|e_1|+|e_3|}}, \frac{\epsilon_3}{8^{|e_1|+|e_2|}}, L(e_{123}), \frac{L(e'_1)}{8^{|e_{123}|-|e'_1|}}, \dots, \frac{L(e'_m)}{8^{|e_{123}|-|e'_m|}} \;\middle|\; m \geq 2, e'_1, \dots, e'_m \right\},$$

where the finite sequence $(e'_1, \dots, e'_m)$ runs through the same set as above. The number $\epsilon_{1(23)}$ is also seen to be equal to the value of the right-hand side. □

3.2 Construction of a comodule $\widetilde{\mathcal{T}}_M$

Definition 3.3 • For a closed interval $c = [a, b]$, we set $|c| = b - a$, and call the point $\frac{1}{2}(a + b) \in c$ the *center of c*.

- We define a version of the little interval operad, denoted by $\mathcal{D}$, as follows. For $n \geq 1$, let $\mathcal{D}(n)$ be the set of n–tuples $(c_1, c_2, \dots, c_n)$ of closed subintervals $c_i \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ such that $c_1 \cup \cdots \cup c_n = [-\frac{1}{2}, \frac{1}{2}]$ and $c_i \cap c_j$ is a one-point set, or empty if $i \neq j$, and the labeling of $1, \dots, n$ is consistent with the usual order of the real line $\mathbb{R}$ (so $-\frac{1}{2} \in c_1$ and $\frac{1}{2} \in c_n$). $\mathcal{D}(1)$ is understood as the one-point set consisting of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. We topologize $\mathcal{D}(n)$ as a subspace of $\mathbb{R}^n$ by the inclusion sending each interval to its center. The partial composition is given in a way that is completely analogous to the usual little interval operad.

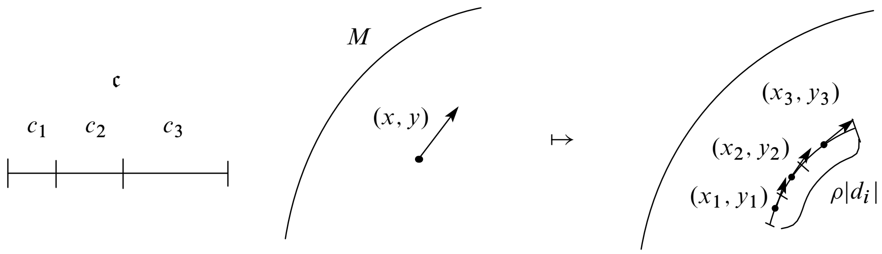

Figure 2: The map Δ'. The geodesic segment is divided into the pieces of rate of length $|c_1| : |c_2| : |c_3|$.

- We identify $H_0(\mathcal{D}(2))$ with $\mathcal{A}(2)$ by sending the generator represented by a topological point to the generator μ.

Recall that we fixed a Riemannian metric on M in Definition 2.3. Henceforth we equip the space SM with the Sasaki metric, and the product SM^n of n copies of SM with the product metric. We assume the maximum of the distance between two points in SM is larger than 1. This is clearly possible by modifying the embedding used in the definition of the metric on M. This assumption is used in the proof Lemma 3.11(2). We fix a positive number ρ small enough that a geodesic of length ρ exists for any initial value in M. After Lemma 3.7, we impose an additional assumption on ρ.

Definition 3.4 We define a map

$$\Delta' = \Delta[\mathfrak{d}, \mathfrak{c}; i] \colon SM \to SM^m$$

for each $\mathfrak{d} = (d_1, \dots, d_n) \in \mathcal{D}(n)$, $\mathfrak{c} = (c_1, \dots, c_m) \in \mathcal{D}(m)$ and $1 \le i \le n$. Let (x, y) denote a point of SM with $x \in M$ and $y \in S_xM$, where S_xM denotes the fiber of the sphere bundle over x. Let $s \colon \left[-\frac{1}{2}\rho, \frac{1}{2}\rho\right] \to M$ denote the geodesic segment with length parameter such that $s(0) = x$ and the tangent vector of s at 0 is y. Let $t_j \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ be the center of c_j, put $x_j = s(\rho \cdot |d_i| \cdot t_j)$ and set y_j to be the tangent vector of s at $\rho \cdot |d_i| \cdot t_j$. We set $\Delta'(x, y) = ((x_1, y_1), \dots, (x_m, y_m))$; see Figure 2.

The following lemma is clear from the definition of $\Delta[\mathfrak{d}, \mathfrak{c}; i]$.

Lemma 3.5 *For any configurations $\mathfrak{d}$, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ and numbers i and j, the following equality holds:*

$$\Delta[\mathfrak{d}, \mathfrak{c}_1 \circ_j \mathfrak{c}_2; i] = (1_{j-1} \times \Delta[\mathfrak{d} \circ_i \mathfrak{c}_1, \mathfrak{c}_2; i + j - 1] \times 1_{m-j}) \circ \Delta[\mathfrak{d}, \mathfrak{c}_1; i].$$

Here m is the arity of $\mathfrak{c}_1$, and 1_l is the identity on SM^l. □

Lemma 3.6 *For any sufficiently small positive number ρ, the map $\Delta[\mathfrak{d}, \mathfrak{c}; i]$ is expanding for any numbers $n \ge 1$, $m \ge 1$ and i with $1 \le i \le n$, and elements $\mathfrak{d} \in \mathcal{D}(n)$ and $\mathfrak{c} \in \mathcal{D}(m)$.*

Proof It is enough to prove the case of $m = 2$, since for $m \ge 3$, Δ' is equal to a successive composition of copies of Δ' of arity 2 by Lemma 3.5. We set $\rho_0 = |d_i|\rho$. We shall consider the case that M is a metric vector space V as a local model. Take points $(x, y), (v, w) \in \widehat{V} = V \times SV$, where SV is the unit sphere in V. Put $\mathfrak{c} = (c_1, c_2)$. Let $-s$ and t be the centers of c_1 and c_2, respectively, with $0 < s, t < \frac{1}{2}$

and $s+t=\frac{1}{2}$. By definition, $\Delta'(x,y)=[(x-\rho_0 sy,y),(x+\rho_0 ty,y)]$. When we set $a=\|x-v\|$ and $b=\|y-w\|$, we easily see

$$\begin{aligned}\|\Delta'(x,y)-\Delta'(v,w)\|^2 &\geq 2a^2-\rho_0|s-t|ab+\left\{\tfrac{1}{4}\rho_0^2(s^2+t^2)+2\right\}b^2\\ &\geq 2a^2-\tfrac{1}{2}\rho_0|s-t|(a^2+b^2)+\left\{\tfrac{1}{4}\rho_0^2(s^2+t^2)+2\right\}b^2.\end{aligned}$$

So

$$\frac{\|\Delta'(x,y)-\Delta'(v,w)\|}{\|(x,y)-(v,w)\|}\geq\frac{\sqrt{7}}{2}\quad\text{for }\rho<1. \tag{3-1}$$

We shall consider the case of a general manifold M. There exists a number $r>0$ such that, for sufficiently small ρ, for any point $p\in M$ and any pair $(x,y),(v,w)\in T_pM\times ST_pM$ with $\|x\|,\|v\|\leq r$, we have the inequality

$$\frac{d(\Delta'_M(\exp x,\exp' y),\Delta'_M(\exp v,\exp' w))}{d(\Delta'_{T_pM}(x,y),\Delta'_{T_pM}(v,w))}>1-\tfrac{1}{100}, \tag{3-2}$$

where exp is the exponential map at p and $\exp'$ is its differential. Combining (3-1) and (3-2), for $(x,y),(v,w)\in SM$, we see $d_{SM^2}(\Delta'(x,y),\Delta'(v,w))>d_{SM}((x,y),(v,w))$ if $d_M(x,v)\leq r$. For the case of $d_M(x,v)>r$, if we take ρ sufficiently small relative to r, the following inequality holds:

$$\frac{d(\Delta'(x,y),\Delta'(v,w))}{d(\Delta(x,y),\Delta(v,w))}>1-\tfrac{1}{100}\quad\text{for }(x,y),(v,w)\in SM\text{ with }d(x,v)>r.$$

Here $\Delta\colon SM\to SM^{\times 2}$ is the usual diagonal. Then, if $d_M(x,v)>r$, we have the inequality

$$d(\Delta'(x,y),\Delta'(v,w))>\left(1-\tfrac{1}{100}\right)\sqrt{2}\,d((x,y),(v,w)).$$

Thus, we have shown the lemma. □

The following lemma is an exercise of Riemannian geometry:

Lemma 3.7 *For any sufficiently small positive number ρ, the following condition holds. For any $n\geq 2$, $G\in\mathsf{G}(n)$ and set of positive numbers $\{\epsilon_{ij}\mid i<j\text{ for }(i,j)\in E(G)\}$ satisfying $\sum_{(i,j)\in E(G)}\epsilon_{ij}<\rho$, the inclusion of subspaces of M^n*

$$\{(x_1,\dots,x_n)\mid\forall(i,j)\in E(G),x_i=x_j\}\to\{(x_1,\dots,x_n)\mid\forall(i,j)\in E(G),d(x_i,x_j)\leq\epsilon_{ij}\}$$

is a homotopy equivalence. □

Assumption In the rest of paper, we fix the number ρ so that Lemmas 3.6 and 3.7 hold.

We define a $\mathcal{D}$–comodule $\widetilde{\mathcal{T}}_M$ of NCRS. We set

$$SM^{-\tau}(n)=(SM^n)^{-\tau};$$

see Definition 3.1. We first define a subspectrum $\widetilde{\mathcal{T}}_M(\mathfrak{c})\subset SM^{-\tau}(n)$ as follows:

$$\widetilde{\mathcal{T}}_M(\mathfrak{c})_k=\left\{[e,\epsilon,u]\in SM^{-\tau}(n)_k\mid\epsilon<\tfrac{1}{2}\rho\min\{|c_1|,\dots,|c_n|\}\right\}.$$

We define a subspectrum $\widetilde{\mathcal{T}}_M(n) \subset \mathrm{Map}(\mathcal{D}(n), SM^{-\tau}(n))$ as follows:

$$\phi \in \widetilde{\mathcal{T}}_M(n)_k \iff \phi(\mathfrak{c}) \in \widetilde{\mathcal{T}}_M(\mathfrak{c})_k \quad \text{for all } \mathfrak{c} \in \mathcal{D}(n).$$

It is clear that the inclusion $\widetilde{\mathcal{T}}_M(n) \to \mathrm{Map}(\mathcal{D}(n), SM^{-\tau}(n))$ is a level-equivalence for any $n \geq 1$. We denote the sequence $\{\widetilde{\mathcal{T}}_M(n)\}$ by $\widetilde{\mathcal{T}}_M$.

We shall define an action of Σ_n on $\widetilde{\mathcal{T}}_M(n)$, with which we regard $\widetilde{\mathcal{T}}_M$ as a symmetric sequence. For $\mathfrak{c} = (c_1, \dots, c_n) \in \mathcal{D}(n)$ and $\sigma \in \Sigma_n$, we define $\mathfrak{c}^\sigma \in \mathcal{D}(n)$ to be the configuration of the subintervals of length $|c_{\sigma(1)}|, |c_{\sigma(2)}|, \dots, |c_{\sigma(n)}|$ placed from the side of $-\frac{1}{2}$ to the side of $\frac{1}{2}$. For $[e, \epsilon, u] \in SM^{-\tau}(n)_k$ and $\sigma \in \Sigma_n$, we set $[e, \epsilon, u]_\sigma = [e \circ \underline{\sigma}, \epsilon, u]$ where $\underline{\sigma}\colon SM^n \to SM^n$ is given by $(z_1, \dots, z_n) \mapsto (z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(n)})$. (To distinguish the action of Σ_k which is a part of the structure of the spectrum, we use the subscript $[-]_\sigma$.)

Definition 3.8 With the above notation, for $\phi \in \widetilde{\mathcal{T}}_M(n)_k$ and $\sigma \in \Sigma_n$ we define an element $\phi^\sigma \in \widetilde{\mathcal{T}}_M(n)_k$ by

$$\phi^\sigma(\mathfrak{c}) = \{\phi(\mathfrak{c}^{\sigma^{-1}})\}_\sigma.$$

Clearly $\phi \mapsto \phi^\sigma$ gives a Σ_n–action on $\widetilde{\mathcal{T}}_M(n)$.

In order to define a partial composition on $\widetilde{\mathcal{T}}_M$, we shall define a map

$$\Xi = \Xi[\mathfrak{d}, \mathfrak{c}; i]\colon SM^{-\tau}(n+m-1) \to SM^{-\tau}(n).$$

For an element $[e, \epsilon, u] \in SM^{-\tau}(n+m-1)_k$, we put

- $e' = e \circ (1_{i-1} \times \Delta' \times 1_{n-i})\colon SM^n \to \mathbb{R}^k$, where $\Delta' = \Delta[\mathfrak{d}, \mathfrak{c}; i]$ and 1_l is the identity on SM^l, and
- $\epsilon' = (1/8^{m-1}) \min\{\epsilon, L(e, \mathfrak{d} \circ_i \mathfrak{c})\}$, where $L(e, \mathfrak{c}')$ is the minimum of the numbers $L(e \circ \Delta[\mathfrak{c}_1, \mathfrak{c}_2; j])$ over all triples $(\mathfrak{c}_1, \mathfrak{c}_2, j)$ satisfying $\mathfrak{c}' = (\mathfrak{c}_1 \circ_j \mathfrak{c}_2) \circ_l \mathfrak{c}_3$ for some configuration $\mathfrak{c}_3$ and number l.

By Lemma 3.6, e' is expanding. We set $\Xi([e, \epsilon, u]) = [e', \epsilon', u]$. Clearly Ξ is a well-defined map of spectra.

Definition 3.9 Using the above notation:

- We define a partial composition

$$(- \circ_i -)\colon \mathcal{D}(m) \hat{\otimes} \widetilde{\mathcal{T}}_M(n+m-1) \to \widetilde{\mathcal{T}}_M(n)$$

  on $\widetilde{\mathcal{T}}_M$ by setting

$$(\mathfrak{c} \circ_i \phi)(\mathfrak{d}) = \Xi(\phi(\mathfrak{d} \circ_i \mathfrak{c})) \quad \text{where } \Xi = \Xi[\mathfrak{d}, \mathfrak{c}; i],$$

  for elements $\phi \in \widetilde{\mathcal{T}}_M(n+m-1)$, $\mathfrak{c} \in \mathcal{D}(m)$ and $\mathfrak{d} \in \mathcal{D}(n)$.
- We define a multiplication $\tilde{\mu}\colon \widetilde{\mathcal{T}}_M(n) \wedge \widetilde{\mathcal{T}}_M(n) \to \widetilde{\mathcal{T}}_M(n)$ by

$$\tilde{\mu}(\langle \phi_1, \phi_2; \sigma \rangle)(\mathfrak{d}) = \mu(\langle \phi_1(\mathfrak{d}), \phi_2(\mathfrak{d}); \sigma \rangle),$$

  where μ denotes the multiplication given in Definition 3.1.

With these operations and the action of Σ_n in Definition 3.8, we regard $\widetilde{\mathcal{T}}_M$ as a $\mathcal{D}$–comodule of NCRS.

Lemma 3.10 *The structure of a $\mathcal{D}$–comodule of NCRS on $\widetilde{\mathcal{T}}_M$ given in Definition 3.9 is well defined.*

Proof By Lemma 3.5, we see the equality in Definition 2.10(1) holds. The equality in (2) in the same definition is clear.

We shall prove the equality in (3). Take elements $\mathfrak{c} \in \mathcal{D}(m)$, $\mathfrak{d} \in \mathcal{D}(n)$, $\phi \in \widetilde{\mathcal{T}}_M(m+n-1)$ and $\sigma \in \Sigma_n$. By definition,

$$(\mathfrak{c} \circ_i \phi)^\sigma(\mathfrak{d}) = \{\mathfrak{c} \circ_i \phi(\mathfrak{d}^{\sigma^{-1}})\}_\sigma = \{\Xi_1(\phi(\mathfrak{d}^{\sigma^{-1}} \circ_i))\}_\sigma,$$

$$\mathfrak{c} \circ_{\sigma^{-1}(i)} (\phi^{\sigma_1})(\mathfrak{d}) = \Xi_2\{\phi((\mathfrak{d} \circ_{\sigma^{-1}(i)} \mathfrak{c})^{\sigma_1^{-1}})_{\sigma_1}\},$$

where $\Xi_1 = \Xi[\mathfrak{d}^{\sigma^{-1}}, \mathfrak{c}; i]$ and $\Xi_2 = \Xi[\mathfrak{d}, \mathfrak{c}; \sigma^{-1}(i)]$. It is easy to check the equalities

$$\mathfrak{d}^{\sigma^{-1}} \circ_i \mathfrak{c} = (\mathfrak{d} \circ_{\sigma^{-1}(i)} \mathfrak{c})^{\sigma_1^{-1}} \quad \text{and} \quad \{\Xi_1(x)\}_\sigma = \Xi_2(x_{\sigma_1}).$$

These verify the desired equality. Compatibility of the multiplication with the partial composition is obvious. □

3.3 Construction of the comodule $\mathcal{T}_M$

Let p and q be two different integers with $1 \le p, q \le n$, and $\mathfrak{c} \in \mathcal{D}(n)$ be an element. We set a number $\delta_{pq}(\mathfrak{c}, \epsilon)$ by

$$\delta_{pq}(\mathfrak{c}, \epsilon) = \tfrac{1}{2}\rho(|c_p| + |c_q|) - \epsilon$$

for a number ϵ. We define a subspectrum $\mathcal{T}_{pq}(\mathfrak{c}) \subset \widetilde{\mathcal{T}}_M(\mathfrak{c})$ by the following equivalence. For each $k \ge 0$,

$$[e, \epsilon, u] \in \mathcal{T}_{pq}(\mathfrak{c})_k \iff [e, \epsilon, u] = * \quad \text{or} \quad d_M(x_p, x_q) \le \delta_{pq}(\mathfrak{c}, \epsilon),$$

where $x_i \in M$ is the image of the i^{th} component of $\pi_e(u)$ by the standard projection $SM \to M$ for $i = p, q$. On the right-hand side, $\delta_{pq}(\mathfrak{c}, \epsilon)$ is positive by the definition of $\widetilde{\mathcal{T}}_M(\mathfrak{c})$. Define a subspectrum $\mathcal{T}_{pq}(n) \subset \widetilde{\mathcal{T}}_M(n)$ by

$$\phi \in \mathcal{T}_{pq}(n)_k \iff \phi(\mathfrak{c}) \in \mathcal{T}_{pq}(\mathfrak{c})_k \quad \text{for all } \mathfrak{c} \in \mathcal{D}(n).$$

Clearly we have $\mathcal{T}_{pq}(n) = \mathcal{T}_{qp}(n)$. The following lemma is the key to defining the comodule $\mathcal{T}_M$. Most of the preceding technical definitions are necessary to make this lemma hold.

Lemma 3.11 (1) *For any numbers $n \ge 1$ and $m \ge 2$ and element $\mathfrak{c} \in \mathcal{D}(m)$, let $\mathfrak{c} \circ_i \mathcal{T}_{pq}(n+m-1) \subset \widetilde{\mathcal{T}}_M(n)$ denote the image of $\mathcal{T}_{pq}(n+m-1)$ by the map $\mathfrak{c} \circ_i (-)$. We have the following inclusion at each level k:*

$$\mathfrak{c} \circ_i \mathcal{T}_{pq}(n+m-1) \subset \begin{cases} \{*\} & \text{if } i \le p < q \le i+m-1, \\ \mathcal{T}_{pi}(n) & \text{if } p < i \le q \le i+m-1, \\ \mathcal{T}_{p,q-m+1}(n) & \text{if } p < i, i+m-1 < q, \\ \mathcal{T}_{i,q-m+1}(n) & \text{if } i \le p \le i+m-1 < q, \\ \mathcal{T}_{p-m+1,q-m+1}(n) & \text{if } i+m-1 < p < q. \end{cases}$$

More precisely, for example, the second inclusion means $\mathfrak{c} \circ_i \mathcal{T}_{pq}(n+m-1)_k \subset \mathcal{T}_{pi}(n)_k$ for each k.

(2) *The image of $\mathcal{T}_{pq}(n) \wedge \widetilde{\mathcal{T}}_M(n)$ by the multiplication $\tilde{\mu}$ given in Definition 3.9 is contained in $\mathcal{T}_{pq}(n)$.*

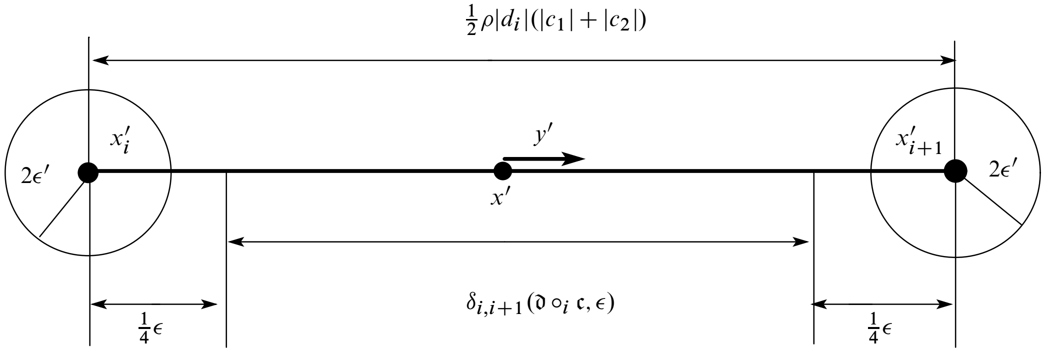

Figure 3: The first inclusion of Lemma 3.11(1) with $n=2$. The bold line is a part of the geodesic segment used to define Δ', (x', y') is the i^{th} component of $\pi_{e'}(u)\in SM^n$, and x_i and x_{i+1} exist in the interior of the disks at x'_i and x'_{i+1} if $(\mathfrak{c}\circ_i\phi)(\mathfrak{d})\neq *$.

Proof We shall show (1). Let $\mathfrak{c}\in\mathcal{D}(m)$, $\mathfrak{d}\in\mathcal{D}(n)$ and $\phi\in\mathcal{T}_{pq}(n+m-1)_k$ be elements. Let (e,ϵ,u) be a representative of $\phi(\mathfrak{d}\circ_i\mathfrak{c})$. Write

$$\pi_e(u)=((x_1,y_1),\dots,(x_{n+m-1},y_{n+m-1})),$$
$$\{(1_{i-1})\times\Delta'\times(1_{n-i})\}(\pi_{e'}(u))=((x'_1,y'_1),\dots,(x'_{n+m-1},y'_{n+m-1})),$$

with $x_j,x'_j\in M$, $y_j\in S_{x_j}M$ and $y'_j\in S_{x'_j}M$. Here we use the notation given in the paragraph above Definition 3.9. We shall show the first inclusion, the case of $i\le p<q\le i+m-1$.

The situation of the case $n=2$ is as in Figure 3 (so $p=i$ and $q=i+1$). We first give a sketch of the proof for $n=2$. We suppose $(\mathfrak{c}\circ_i\phi)(\mathfrak{d})\neq *$ and will show a contradiction. Since the map Δ' arranges points along a geodesic and the length of the geodesic segment between x'_i and x'_{i+1} is $\frac{1}{2}\rho|d_i|(|c_1|+|c_2|)$, we have $d_M(x'_i,x'_{i+1})>\delta(\mathfrak{d}\circ_i\mathfrak{c},\epsilon)$. As we have taken ϵ' in the definition of Ξ sufficiently small, x_i and x'_i (resp. x_{i+1} and x'_{i+1}) are sufficiently close. These observations imply $d_M(x_i,x_{i+1})>\delta(\mathfrak{d}\circ_i\mathfrak{c},\epsilon)$, or, equivalently, $\phi(\mathfrak{d}\circ_i\mathfrak{c})\notin\mathcal{T}_{pq}(\mathfrak{d}\circ_i\mathfrak{c})$.

We shall give the formal proof. We assume $(\mathfrak{c}\circ_i\phi)(\mathfrak{d})\neq *$. Since the image of e' is contained in the image of e and the map π_e sends u to its closest point in $e(M)=M$, we have

$$\|u-e(\pi_e(u))\|\le\|u-e'(\pi_{e'}(u))\|<\epsilon'.$$

So

$$\|e'(\pi_{e'}(u))-e(\pi_e(u))\|\le\|e'(\pi_{e'}(u))-u\|+\|u-e(\pi_e(u))\|<2\epsilon'.$$

As $e'=e\circ(1_{i-1})\times\Delta'\times(1_{n-i})$ and e is expanding,

$$d\{((x'_1,y'_1),\dots,(x'_{n+m-1},y'_{n+m-1})),((x_1,y_1),\dots,(x_{n+m-1},y_{n+m-1}))\}<2\epsilon',$$

where d denotes the distance in SM^{n+m-1}. So

$$d_M(x_j,x'_j)\le d_{SM}((x_j,y_j),(x'_j,y'_j))<2\epsilon'\quad\text{for } j=1,\dots,n+m-1.$$

By this inequality, and the definition of the map Δ', we have the inequality

$$\begin{aligned} d_M(x_p, x_q) &\geq d_M(x'_p, x'_q) - d_M(x_p, x'_p) - d_M(x_q, x'_q) \geq d_M(x'_p, x'_q) - 4\epsilon' \\ &\geq \tfrac{1}{2}\rho|d_i|(|c_{p-i+1}| + |c_{q-i+1}|) - 4\epsilon' = \tfrac{1}{2}\rho(|(\mathfrak{d} \circ_i \mathfrak{c})_p| + |(\mathfrak{d} \circ_i \mathfrak{c})_q|) - 4\epsilon' \\ &\geq \tfrac{1}{2}\rho(|(\mathfrak{d} \circ_i \mathfrak{c})_p| + |(\mathfrak{d} \circ_i \mathfrak{c})_q|) - \epsilon/2 > \delta_{pq}(\mathfrak{d} \circ_i \mathfrak{c}, \epsilon). \end{aligned}$$

This inequality implies $\phi(\mathfrak{d} \circ_i \mathfrak{c}) \notin \mathcal{T}_{pq}(\mathfrak{d} \circ_i \mathfrak{c})$, which is a contradiction. So $(\mathfrak{c} \circ_i \phi)(\mathfrak{d}) = *$, and we have proved the first inclusion.

We shall show the second inclusion, the case of $p < i \leq q \leq i + m - 1$. Let $(x', y') \in SM$ be the i^{th} component of $\pi_{e'}(u)$. Clearly,

$$((x'_i, y'_i), \dots, (x'_{i+m-1}, y'_{i+m-1})) = \Delta'(x', y').$$

By an argument similar to the above, we have the inequality

$$\begin{aligned} d_M(x'_p, x') &\leq d_M(x'_p, x_p) + d_M(x_p, x_q) + d_M(x_q, x'_q) + d_M(x'_q, x') \\ &\leq 2\epsilon' + \delta_{pq}(\mathfrak{d} \circ_i \mathfrak{c}, \epsilon) + 2\epsilon' + \tfrac{1}{2}\rho|d_i|(1 - |c_{q-i+1}|) \\ &= \tfrac{1}{2}\rho(|d_p| + |d_i||c_{q-i+1}|) - \epsilon + 4\epsilon' + \tfrac{1}{2}\rho|d_i|(1 - |c_{q-i+1}|) \leq \tfrac{1}{2}\rho(|d_p| + |d_i|) - \tfrac{1}{2}\epsilon < \delta_{pq}(\mathfrak{d}, \epsilon'). \end{aligned}$$

This implies the second inclusion. The other cases are similar to the first and second cases. The proof of (2) is similar in view of the assumption on the metric given in the paragraph after Definition 3.3, and so is omitted. □

Let $\mathcal{T}_{\text{fat}}(n)$ be the subspectrum of $\widetilde{\mathcal{T}}_M(n)$ whose space at level k is given by

$$\mathcal{T}_{\text{fat}}(n)_k = \bigcup_{1 \leq p < q \leq n} \mathcal{T}_{pq}(n)_k.$$

Since $\{\mathcal{T}_{pq}(n)\}^\sigma = \mathcal{T}_{\sigma^{-1}(p), \sigma^{-1}(q)}(n)$, we have that $\mathcal{T}_{\text{fat}}(n)$ is stable under the action of Σ_n. By Lemma 3.11, the sequence $\{\mathcal{T}_{\text{fat}}(n)\}_{n \geq 0}$ is stable under partial compositions and is an ideal for the multiplication $\tilde{\mu}$. So the sequence $\{\mathcal{T}_{\text{fat}}(n)\}_{n \geq 0}$ inherits a structure of a comodule from $\widetilde{\mathcal{T}}_M$, and we can define the quotient comodule as follows:

Definition 3.12 We define a spectrum $\mathcal{T}_M(n)$ by the quotient (collapsing to $*$)

$$\mathcal{T}_M(n)_k = \widetilde{\mathcal{T}}_M(n)_k / \mathcal{T}_{\text{fat}}(n)_k$$

for each $k \geq 0$ and $n \geq 2$, and by $\mathcal{T}_M(1) = \widetilde{\mathcal{T}}_M(1)$. We regard the sequence $\mathcal{T}_M = \{\mathcal{T}_M(n)\}_{n \geq 1}$ as a comodule of NCRS with the structure induced by that on $\widetilde{\mathcal{T}}_M$.

4 Atiyah duality for comodules

Definition 4.1 We define the following zigzag consisting of $\mathcal{D}$–comodules of NCRS and maps between them:

$$(\mathcal{C}_M)^\vee \xleftarrow{(i_0)^\vee} (\widetilde{F}_M)^\vee \xrightarrow{(i_1)^\vee} (F_M)^\vee \xleftarrow{q_*} F'_M \xrightarrow{p_*} F^\dagger_M \xleftarrow{\Phi} \mathcal{T}_M.$$

- Set $\mathcal{C}_M(n) = \mathcal{C}^{n-1}(M)$. When we regard a configuration as an element of $\mathcal{C}_M(n)$, we label its points by $1, \dots, n$ instead of $0, \dots, n-1$. We give the sequence $\mathcal{C}_M = \{\mathcal{C}_M(n)\}_{n \geq 1}$ a structure of an $\mathcal{A}$–module as follows. For the unique element $\mu \in \mathcal{A}(2)$ and an element $x \in \mathcal{C}_M(n)$, we set $x \circ_i \mu = d^{i-1}(x)$, where d^{i-1} is the coface operator of $\mathcal{C}^\bullet(M)$. The action of Σ_n on $\mathcal{C}_M(n)$ is given by permutation of labels and $(\mathcal{C}_M)^\vee$ is the $\mathcal{A}$–comodule of NCRS given in Definition 2.10. By pulling back the action by the unique operad morphism $\mathcal{D} \to \mathcal{A}$, we also regard $(\mathcal{C}_M)^\vee$ as a $\mathcal{D}$–comodule.

- Let $F_M(n)$ be the subspace of $\mathcal{D}(n) \times SM^n$ defined by the following condition. For an element $(\mathfrak{c}; (x_1, y_1), \dots, (x_n, y_n)) \in \mathcal{D}(n) \times SM^n$ with $x_i \in M$ and $y_i \in S_{x_i}M$,

$$(\mathfrak{c}; (x_1, y_1), \dots, (x_n, y_n)) \in F_M(n) \iff d(x_i, x_j) \geq \tfrac{1}{2}\rho(|c_i| + |c_j|) \quad \text{for each pair } (i, j) \text{ with } i \neq j,$$

where ρ is the number fixed in Section 3.2.

- The sequence $\{F_M(n)\}$ has a structure of a $\mathcal{D}$–module. For $\mathfrak{c} \in \mathcal{D}(n)$ and $(\mathfrak{d}; z_1, \dots, z_n) \in F_M(n)$, we set $(\mathfrak{d}; z_1, \dots, z_n) \circ_i \mathfrak{c} = (\mathfrak{d} \circ_i \mathfrak{c}; z_1, \dots, \Delta'(z_i), \dots, z_n)$, where $\Delta' = \Delta[\mathfrak{d}, \mathfrak{c}; i]$ is given in Definition 3.4. The symmetric group acts on $F_M(n)$ by permutation of little intervals and components. The $\mathcal{D}$–comodule of NCRS $(F_M)^\vee$ is the one induced by F_M.

- We shall define a symmetric sequence of spectra $\{\mathbb{S}_M(n)\}_n$. Set $\widetilde{\mathbb{S}}_M(n)_k = \widetilde{N}_k^{-\tau}$ for $N = SM^n$ (see Definition 3.1). Define a subspace $\partial(\widetilde{\mathbb{S}}_M(n))_k \subset \widetilde{\mathbb{S}}_M(n)_k$ by $(e, \epsilon, v) \in \partial\widetilde{\mathbb{S}}_M(n)_k$ if and only if $\|v\| \geq \epsilon$. We put

$$\mathbb{S}_M(n)_k = \widetilde{\mathbb{S}}_M(n)_k / \partial\widetilde{\mathbb{S}}_M(n)_k.$$

We regard $\mathbb{S}_M(n)$ as an NCRS by a multiplication defined similarly to that of $N^{-\tau}$, given in Definition 3.1.

- Set $F_M^\dagger(n) := \mathrm{Map}(F_M(n), \mathbb{S}_M(n))$. We give the sequence $\{F_M^\dagger(n)\}_n$ a structure of a $\mathcal{D}$–comodule as follows. For $\mathfrak{c} \in \mathcal{D}(n)$ and $f \in F_M^\dagger(n+m-1)$, set $\mathfrak{c} \circ_i f$ to be the composition

$$F_M(m) \xrightarrow{(-\circ_i \mathfrak{c})} F_M(n+m-1) \xrightarrow{f} \mathbb{S}_M(n+m-1) \xrightarrow{\alpha} \mathbb{S}_M(n).$$

Here α is given by

$$\alpha([e, \epsilon, v]) = [e', \epsilon', v],$$

where e' and ϵ' are as defined in the paragraph above Definition 3.9. Similarly to $(\mathcal{C}_M)^\vee$, we define a multiplication on $F_M^\dagger(n)$ as the pushforward by the multiplication on $\mathbb{S}_M(n)$.

- We define a map $\widetilde{\Phi}_n : \widetilde{\mathcal{T}}_M(n) \to F_M^\dagger(n)$ of spectra by

$$\widetilde{\Phi}_n(\phi)((\mathfrak{c}; z_1, \dots, z_n)) = [e, \bar{\epsilon}, u - e(z_1, \dots, z_n)].$$

Here we write $\phi(\mathfrak{c}) = [e, \epsilon, u]$ and we set $\bar{\epsilon} = \frac{1}{4}\epsilon$. Lemma 4.2 proves that $\widetilde{\Phi}_n$ induces a morphism $\Phi_n : \mathcal{T}_M(n) \to F_M^\dagger(n)$ which forms a morphism of comodules.

- We shall define a $\mathcal{D}$–module $\widetilde{F}_M$. Set

$$\widetilde{F}_{M,1}(n) = [0, 1] \times \mathcal{D}(n) \times \mathcal{C}_M(n) / \sim,$$

where the equivalence relation is generated by the relation $(t,\mathfrak{c},z)\sim(s,\mathfrak{d},z')$ if and only if $s=t=0$ and $z=z'$. $\widetilde{F}_M(n)$ is the subspace of $\widetilde{F}_{M,1}(n)$ consisting of elements $(t,\mathfrak{c},z)$ with $z=(x_k,u_{kl},y_k)$ satisfying

$$t\neq 0 \quad\Longrightarrow\quad z\in \mathrm{Int}(\mathcal{C}_M(n)) \quad\text{and}\quad d_M(x_i,x_j)\geq t\cdot \tfrac{1}{2}\rho(|c_i|+|c_j|).$$

Here $\mathrm{Int}(\mathcal{C}_M(n))$ is the subspace consisting of the elements (x_k,u_{kl},y_k) such that $x_k\neq x_l$ if $k\neq l$, or equivalently, (x_k,u_{kl}) belongs to $C_n(M)$ via the canonical inclusion $C_n(M)\subset \overline{C}_n(M)$. We endow the sequence $\{\widetilde{F}_M(n)\}_n$ with a structure of a $\mathcal{D}$–module analogous to that of F_M. The difference is that we use the number $t\rho$ instead of ρ in the definition of Δ' for $t>0$, and use the module structure on $\mathcal{C}_M$ for $t=0$. The obvious inclusions $i_0\colon \mathcal{C}_M(n)\to\widetilde{F}_M(n)$ and $i_1\colon F_M(n)\to\widetilde{F}_M(n)$ to $t=0,1$ give rise to morphisms of $\mathcal{D}$–modules $i_0\colon\mathcal{C}_M\to\widetilde{F}_M$ and $i_1\colon F_M\to\widetilde{F}_M$.

- To define F'_M, p_* and q_*, we shall define a symmetric sequence of spectra $\{\mathbb{S}'_M(n)\}_n$. Let $\widetilde{\mathbb{S}}'_M(n)$ be the subspace of $\mathrm{Emb}((SM)^n,\mathbb{R}^k)\times\mathbb{R}\times S^k$ consisting of triples (e,ϵ,v) with $0<\epsilon<L(e)$. We put

$$\mathbb{S}'_M(n)_k=\widetilde{\mathbb{S}}'_M(n)_k/\{(e,\epsilon,\infty)\mid e,\epsilon \text{ arbitrary}\},$$

where we regard $S^k=\mathbb{R}^k\cup\{\infty\}$. We regard $\mathbb{S}'_M(n)$ as a spectrum analogously to $\mathbb{S}_M(n)$. Let $p\colon\mathbb{S}'_M(n)\to\mathbb{S}_M(n)$ be the map induced by the collapsing map $S^k\to\mathbb{R}^k/\{v\mid \|v\|\geq\epsilon\}$ and $q\colon\mathbb{S}'_M\to\mathbb{S}$ be the map forgetting the data (e,ϵ). Set $F'_M(n)=\mathrm{Map}(F_M(n),\mathbb{S}'_M(n))$. We regard $\{F'_M(n)\}$ as a $\mathcal{D}$–comodule of NCRS analogously to $F^\dagger_M$. The pushforwards p_* and q_* are clearly morphisms of comodules of NCRS.

Verification of well-definedness of the objects defined in Definition 4.1 is routine work. For example, the associativity of the composition of $\mathcal{C}_M$ follows from the cosimplicial identities of $\mathcal{C}^\bullet(M)$, and that of F_M can be verified similarly to the associativity of little cubes operads. We omit details.

Remark Right modules similar to F_M are used in [2; 6].

Lemma 4.2 *The map $\widetilde{\Phi}_n$ uniquely factors through a map $\Phi_n\colon\mathcal{T}_M(n)\to F^\dagger_M(n)$, and the sequence $\{\Phi_n\}$ is a map of $\mathcal{D}$–comodules of NCRS.*

Proof We shall show that $\widetilde{\Phi}_n(\phi)=*$ for any element $\phi\in\mathcal{T}_{pq}(n)$. Suppose that there exists an element $(\mathfrak{c};z_1,\dots,z_n)\in F_M(n)$ such that $\widetilde{\Phi}_n(\phi)(\mathfrak{c};z_1,\dots,z_n)\neq *\in\mathbb{S}_M(n)$. If we put $\phi(\mathfrak{c})=[e,\epsilon,u]$, the inequality $\|u-e(z_1,\dots,z_n)\|<\frac{1}{4}\epsilon$ holds. So $\|u-e(\pi_e u)\|<\frac{1}{4}\epsilon$. Thus,

$$\|e(\pi_e u)-e(z_1,\dots,z_n)\|\leq\|e(\pi_e u)-u\|+\|u-e(z_1,\dots,z_n)\|<\tfrac{1}{2}\epsilon.$$

As e is expanding, we have $d(\pi_e(u),(z_1,\dots,z_n))<\frac{1}{2}\epsilon$ where d denotes the distance in SM^n. If we write $z_i=(x_i,y_i)$ and $\pi_e(u)=((\bar{x}_1,\bar{y}_1),\dots,(\bar{x}_n,\bar{y}_n))$ as pairs of a point of M and a tangent vector, it follows that $d_M(\bar{x}_i,x_i)<\frac{1}{2}\epsilon$, and

$$d(\bar{x}_p,\bar{x}_q)\geq d(x_p,x_q)-d(x_p,\bar{x}_p)-d(x_q,\bar{x}_q)>\tfrac{1}{2}\rho(|c_p|+|c_q|)-\epsilon=\delta_{pq}(\mathfrak{c},\epsilon).$$

This inequality contradicts the assumption $\phi\in\mathcal{T}_{pq}(n)$. Thus we have proved $\widetilde{\Phi}_n(\mathcal{T}_{pq}(n))=*$. This implies the former part of the lemma. The latter part is obvious. □

Definition 4.3 A $\mathcal{D}$–comodule of NCRS is *semistable* if the spectrum $X(n)$ is semistable for each n. A map $f : X \to Y$ of $\mathcal{D}$–comodules of NCRS is a *π_*–isomorphism* if each map $f_n : X(n) \to Y(n)$ is a π_*–isomorphism (see Section 2.1).

The notion of a π_*–isomorphism in Definition 4.3 is what we call "weak equivalence" in Theorem 1.1. Since a π_*–isomorphism of spectra is a stable equivalence, a π_*–isomorphism of $\mathcal{D}$–comodules gives a stable equivalence at each arity. The following is a version of Atiyah duality which respects our comodules. We devote the rest of this section to its proof.

Theorem 4.4 *As $\mathcal{D}$–comodules of nonunital commutative symmetric ring spectra, $(\mathcal{C}_M)^\vee$ and $\mathcal{T}_M$ are π_*–isomorphic. Precisely speaking, all the comodules in the zigzag in Definition 4.1 are semistable and all the maps in the same zigzag are π_*–isomorphisms.*

Definition 4.5 • For $G \in \mathsf{G}(n)$ and $\mathfrak{c} \in \mathcal{D}(n)$, we define two subspectra $\mathcal{T}_G(\mathfrak{c}), \mathcal{T}_{\mathrm{fat}}(\mathfrak{c}) \subset \widetilde{\mathcal{T}}_M(\mathfrak{c})$ by

$$\mathcal{T}_G(\mathfrak{c}) = \begin{cases} \bigcap_{(p,q)\in E(G)} \mathcal{T}_{pq}(\mathfrak{c}) & \text{if } G \neq \varnothing, \\ \widetilde{\mathcal{T}}_M(\mathfrak{c}) & \text{if } G = \varnothing, \end{cases} \qquad \text{and} \qquad \mathcal{T}_{\mathrm{fat}}(\mathfrak{c}) = \bigcup_{1\le p<q\le n} \mathcal{T}_{pq}(\mathfrak{c}).$$

Similarly, we define a subspectrum $\mathcal{T}_G \subset \widetilde{\mathcal{T}}_M(n)$ by

$$\mathcal{T}_G = \begin{cases} \bigcap_{(p,q)\in E(G)} \mathcal{T}_{pq} & \text{if } G \neq \varnothing, \\ \widetilde{\mathcal{T}}_M(n) & \text{if } G = \varnothing. \end{cases}$$

Here the union and intersections are taken in the levelwise manner.

• We fix an expanding embedding $e_0 : SM \to \mathbb{R}^K$, a positive number $\epsilon_0 < L(e_0)$ and a configuration $\mathfrak{c}_0 \in \mathcal{D}(n)$ such that $\epsilon_0 < \frac{1}{4}\min\{|c_1|, \dots, |c_n|\}$. We set $\nu = \nu_{\epsilon_0}(e_0)$. We impose an additional condition on ϵ_0 in Definition 5.8, which is satisfied by any sufficiently small ϵ_0, and we will assume K is a multiple of 4 in the proof of Theorem 5.16. (We may impose the assumption on K from the beginning, but for the convenience of verification of signs we do not do so.)

• For a graph $G \in \mathsf{G}(n)$, let $M^{\pi_0(G)}$ be the space of maps $\pi_0(G) \to M$ with the product topology, where $\pi_0(G)$ is the set of connected components of G. Let D_G be the pullback of the diagram

$$SM^n \xrightarrow{\text{projection}} M^n \leftarrow M^{\pi_0(G)},$$

where the right arrow is the pullback by the quotient map $\underline{n} \to \pi_0(G)$. D_G is naturally regarded as a subspace of SM^n via the projection of the pullback. This subspace is the same as the one given in Section 1. We define the subspace $\mathrm{FD}_n \subset SM^n$ as the unions of the spaces D_G whose graph G has at least one edge.

• Consider $\nu^n \subset \mathbb{R}^{nK}$ as a disk bundle over SM^n and denote by ν_G be the preimage of D_G by the projection $\nu^n \to SM^n$. Let $\lambda_G : \mathrm{Th}(\nu_G) \to \mathcal{T}_G(\mathfrak{c}_0)_{nK}$ be the map $[u] \mapsto [(e_0)^n, \epsilon_0, u]$. Then λ_G induces a morphism $\lambda_G : \Sigma^{nK}\,\mathrm{Th}(\nu_G) \to \mathcal{T}_G(\mathfrak{c}_0)$ in $\mathbf{Ho}(\mathcal{SP})$, where Σ denotes the suspension.

Lemma 4.6 *For a closed smooth manifold N and $k \geq 1$, the inclusion $I : \mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k) \to \mathrm{Emb}(N, \mathbb{R}^k)$ is a homotopy equivalence.*

Proof Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be a C^∞–function which satisfies the following inequalities:

$$f(x) > \frac{1}{x} \quad \text{for } x < 1, \qquad f(x) \geq 1 \quad \text{for } x \geq 1.$$

We define a continuous map $F\colon \mathrm{Emb}(N, \mathbb{R}^k) \to \mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k)$ by $e \mapsto f(r(e)) \cdot e$, where $r(e)$ is the number given in Definition 3.1, and $\cdot$ denotes componentwise scalar multiplication. A homotopy from $F \circ I$ to id is given by $(t, e) \mapsto \{t + (1-t) f(r(e))\} \cdot e$, and a homotopy from $I \circ F$ to id is also given by the same formula. □

Lemma 4.7 *We use the notation in Definition 4.5. For each $n \geq 1$ and $G \in \mathsf{G}(n)$, $\mathcal{T}_M(n)$ and $\mathcal{T}_G$ are semistable, and each map in the following zigzags in $\mathbf{Ho}(\mathcal{SP})$ is an isomorphism.*

$$\Sigma^{nK}\,\mathrm{Th}(\nu_G) \xrightarrow{\lambda_G} \mathcal{T}_G(\mathfrak{c}_0) \leftarrow \mathcal{T}_G,$$

$$\Sigma^{nK}\{\mathrm{Th}(\nu^n)/\mathrm{Th}(\nu^n|_{\mathrm{FD}_n})\} \xrightarrow{\lambda_G} \mathcal{T}_\varnothing(\mathfrak{c}_0)/\{\mathcal{T}_{\mathrm{fat}}(\mathfrak{c}_0)\} \leftarrow \mathcal{T}_M(n).$$

Here, see Section 1 for FD_n, and the right maps are the evaluations at $\mathfrak{c}_0$.

Proof For simplicity, we shall prove the claim for the maps in the first line for the case of $G = \varnothing$. The same proof works for general G thanks to the assumptions on ρ given in Section 3.2. Set $N = (SM)^n$. The evaluation at $\mathfrak{c}_0$ and the inclusion $\mathcal{T}_\varnothing(\mathfrak{c}_0) \subset N^{-\tau}$ are clearly level equivalences. So all we have to prove is that $\mathcal{T}_\varnothing$ is semistable and that the composition of λ_G and the inclusion, which is also denoted by $\lambda_G\colon \Sigma^{nK}\,\mathrm{Th}(\nu_G) \to N^{-\tau}$, is an isomorphism in $\mathbf{Ho}(\mathcal{SP})$. We define a space $\mathcal{E}_k$ by

$$\mathcal{E}_k = \{(e, \epsilon) \mid e \in \mathrm{Emb}^{\mathrm{ex}}(N, \mathbb{R}^k) \text{ and } 0 < \epsilon < L(e)\}.$$

By Lemma 4.6 and Whitney's theorem, $\mathcal{E}_k$ is $\left(\frac{1}{2}k - n(2d-1) - 1\right)$–connected. Let $P\colon \overline{N}_k^{-\tau} \to \mathcal{E}_k$ be the fiber bundle obtained from the obvious projection $\widetilde{N}_k^{-\tau} \to \mathcal{E}_k$ by collapsing the complements of the $\nu_\epsilon(e)$ in a fiberwise manner (see Definition 3.1). So each fiber of the map P is a Thom space homeomorphic to $\mathrm{Th}(\nu_G)$. P has a section $s\colon \mathcal{E}_k \to \overline{N}_k^{-\tau}$ to the basepoints, and there is an obvious homeomorphism

$$\overline{N}_k^{-\tau}/s(\mathcal{E}_k) \cong N_k^{-\tau}.$$

With this, by observing the Serre spectral sequence for P, we see that the composition

$$S^{k-nK} \wedge \mathrm{Th}(\nu_G) \xrightarrow{\lambda_G} S^{k-nK} \wedge N_{nK}^{-\tau} \xrightarrow{\text{action of } \mathbb{S}} N_k^{-\tau}$$

is $\left(\frac{3}{2}k - 2n(2d-1) - 2\right)$–connected. This implies $N^{-\tau}$ is semistable and λ_G is an isomorphism. □

Proof of Theorem 4.4 Similarly to the proof of Lemma 4.7, it is easy to see $\mathbb{S}_M$ and $\mathbb{S}'_M$ are semistable, which implies each comodule in the zigzag in Definition 4.1 is semistable, combined with the fact that the spaces $F_M(n)$, $\widetilde{F}_M(n)$ and $\mathcal{C}_M(n)$ have homotopy types of finite CW complexes. It is clear that p and q are π_*–isomorphisms, and so are p_* and q_*. Then i_0 and i_1 are homotopy equivalences for each n, since $\widetilde{F}_M(n)$ is homotopy equivalent to the mapping cylinder of the inclusion $C_n(M) \subset \overline{C}_n(M)$, which is also a homotopy equivalence. So $(i_0)^\vee$ and $(i_1)^\vee$ are π_*–isomorphisms. Finally Φ_n is a π_*–isomorphism since it reduces to the equivalence of the original Atiyah duality in the (homotopy) category of classical spectra via Lemma 4.7; see [7]. □

5 Spectral sequences

5.1 A chain functor

Definition 5.1 • For a chain complex C_*, $C[k]_*$ is the chain complex given by $C[k]_l = C_{k+l}$ with the same differential as C_* (without extra sign).

• Fix a fundamental cycle $w_{S^1} \in C_1(S^1)$. Let $\overline{C}_*(U)$ denote the reduced singular chain complex of a pointed space U. We shall define a chain complex $C_*(X)$ for a spectrum X. Define a chain map $i_k^X : \overline{C}_*(X_k)[k] \to \overline{C}_*(X_{k+1})[k+1]$ by $i_k^X(x) = (-1)^l \sigma_*(w_{S^1} \times x)$ for $x \in \overline{C}_l(X_k)$, where $\sigma : S^1 \wedge X_k \to X_{k+1}$ is the structure map of X. We define $C_*(X)$ as the colimit of the sequence $\{\overline{C}_*(X_k)[k]; i_k^X\}_{k \geq 0}$. Clearly the procedure $X \mapsto C_*(X)$ is extended to a functor $\mathcal{SP} \to \mathcal{CH}_{\mathsf{k}}$ in an obvious manner.

• For a spectrum X, we denote by $H_*(X)$ the homology group of $C_*(X)$.

• Let $f\mathcal{CW}$ denote the full subcategory of $\mathcal{CG}$ spanned by finite CW complexes. We define a functor $C_S^* : (f\mathcal{CW})^{\mathrm{op}} \to \mathcal{CH}_{\mathsf{k}}$ by $C_S^q(X) = C_{-q}(X^\vee)$.

The proofs of the following two lemmas are very standard, so we omit them.

Lemma 5.2 *If $f : X \to Y$ is a stable equivalence between semistable spectra, the induced map $f_* : C_*(X) \to C_*(Y)$ is a quasi-isomorphism.* □

Lemma 5.3 *There exists a zigzag of natural transformations between C^* and $C_S^* : (f\mathcal{CW})^{\mathrm{op}} \to \mathcal{CH}_{\mathsf{k}}$, in which each natural transformation is an objectwise quasi-isomorphism.* □

Remark 5.4 The functor C_* does not have any compatibility with symmetry isomorphisms of the monoidal products $\wedge$ in $\mathcal{SP}$ and $\otimes_{\mathsf{k}}$ in $\mathcal{CH}_{\mathsf{k}}$, so the multiplication on $\mathcal{T}_M(n)$ defined in Section 3 does not straightforwardly induce a multiplication on $C_*(\mathcal{T}_M(n))$. To enrich the Čech spectral sequence with multiplicative operations, we will need some extra work as in [33], which is not dealt with here. The E_2–term of the spectral sequence has a multiplication induced by a simplicial CDBA given in Definition 5.14, but its topological meaning is unclear at present.

The functor $C_* : \mathcal{SP} \to \mathcal{CH}_{\mathsf{k}}$ has some compatibility with the tensor $\hat{\otimes}$ with a space.

Lemma 5.5 (1) *For $U \in \mathcal{CG}$ and $X \in \mathcal{SP}$, the collection of Eilenberg–Zilber shuffle maps*

$$\{EZ : C_*(U) \otimes \overline{C}_*(X_k)[k] \to \overline{C}_*((U_+) \wedge X_k)[k]\}_k$$

induces a quasi-isomorphism

$$C_*(U) \otimes C_*(X) \to C_*(U \hat{\otimes} X).$$

(2) *Let $\mathcal{O}$ be a topological operad and Y be an $\mathcal{O}$–comodule in $\mathcal{SP}$. A natural structure of a chain $C_*(\mathcal{O})$–comodule on the collection $C_*Y = \{C_*(Y(n))\}_n$ is defined as follows. The partial composition is given by the composition*

$$C_*(\mathcal{O}(m)) \otimes C_*(Y(m+n-1)) \to C_*(\mathcal{O}(m) \mathbin{\hat{\otimes}} Y(m+n-1)) \to C_*(Y(n)),$$

where the left map is the one defined in (1) *and the right map is induced by the partial composition on Y. The action of Σ_n on $C_*(Y)(n)$ is the one induced naturally.*

Proof The cross product $w_{S^1} \times x$ is equal to $EZ(w_{S^1} \otimes x)$ by definition, and the shuffle maps are associative and compatible with the symmetry isomorphisms of monoidal products without any chain homotopy for normalized singular chains, so the maps EZ are compatible with the maps i_k^X in Definition 5.1 (the sign commuting an element of $C_*(U)$ and w_{S^1} is canceled with the sign attached in the definition of i_k^X). This implies the first part. The second part follows from commutativity of the following diagram, which is clear from the property of the shuffle map mentioned above:

$$\begin{tikzcd} C_*(U) \otimes C_*(V) \otimes C_*(X) \arrow[r] \arrow[d] & C_*(U) \otimes C_*(V \mathbin{\hat{\otimes}} X) \arrow[d] \\ C_*(U \times V) \otimes C_*(X) \arrow[r] & C_*((U \times V) \mathbin{\hat{\otimes}} X) \end{tikzcd}$$

Here $U, V \in \mathcal{CG}$, $X \in \mathcal{SP}$, the left vertical arrow is induced by the EZ shuffle map and other arrows are given by (1). ◻

5.2 Construction of the Čech spectral sequence

Definition 5.6 We define a $C_*(\mathcal{D})$*–comodule* $\check{\mathrm{T}}^M_{\star\star}$ *of double complexes* consisting of the following data:

- a sequence of double complexes $\{\check{\mathrm{T}}^M_{\star\star}(n)\}_{n\geq 1}$ with two differentials d and ∂ of degree $(0,1)$ and $(1,0)$, respectively,
- an action of Σ_n on $\check{\mathrm{T}}^M_{\star\star}(n)$ which preserves the bigrading, and
- a partial composition $(- \circ_i -)\colon C_k(\mathcal{D}(m)) \otimes \check{\mathrm{T}}^M_{\star\star}(m+n-1) \to \check{\mathrm{T}}^M_{\star,*+k}(n)$.

These satisfy the following compatibility conditions in addition to the conditions in Definition 2.10:

$$d\partial = \partial d, \quad d(\alpha \circ_i x) = d\alpha \circ_i x + (-1)^{|\alpha|}\alpha \circ_i dx, \quad \partial(\alpha \circ_i x) = \alpha \circ_i \partial x.$$

We define the double complex $\check{\mathrm{T}}^M_{\star\star}(n)$ by

$$\check{\mathrm{T}}^M_{p*}(n) = \bigoplus_{G \in \mathsf{G}(n,p)} C_*(\mathcal{T}_G)$$

for $p \geq 0$ and $\check{\mathrm{T}}^M_{p,*}(n) = 0$ for $p < 0$, where $\mathsf{G}(n,p) \subset \mathsf{G}(n)$ is the set of graphs with exactly p edges (see Definition 4.5 for $\mathcal{T}_G$). The differential d is the original differential of $C_*(\mathcal{T}_G)$. The other differential ∂ is given by the signed sum

$$\partial = \sum_{t=1}^{p} (-1)^{t+1} \partial_t,$$

where ∂_t is the standard pushforward by the inclusion $\mathcal{T}_G \to \mathcal{T}_{G_t}$ where the graph G_t is defined by removing the t^{th} edge from G (in the lexicographical order). The action of σ on $\widetilde{\mathcal{T}}_M(n)$ restricts to a map $\sigma : \mathcal{T}_G \to \mathcal{T}_{\sigma^{-1}(G)}$; see Section 2.1 for $\sigma^{-1}(G)$. This map induces a chain map $\sigma_* : C_*(\mathcal{T}_G) \to C_*(\mathcal{T}_{\sigma^{-1}(G)})$ by the pushforward of chains. For $G \in G(n,p)$, let $\sigma_G \in \Sigma_p$ denote the composition

$$\underline{p} \cong E(\sigma^{-1}(G)) \to E(G) \cong \underline{p},$$

where $\cong$ denotes the order-preserving bijection and the middle map is given by $(i,j) \mapsto (\sigma(i),\sigma(j))$. We define the action of σ on $\check{\mathrm{T}}^M(n)$ as $\mathrm{sgn}(\sigma_G)\cdot\sigma_*$ on each summand. We now define the partial composition. Let $f_i : \underline{m+n-1} \to \underline{n}$ be the order-preserving surjection which satisfies $f_i(i+t) = i$ for $t = 1,\dots,m-1$. For elements $\alpha \in C_*(\mathcal{D}(m))$ and $x \in C_*(\mathcal{T}_G)$ with $G \in \mathsf{G}(n+m-1)$, if $\#E(f_i(G)) = \#E(G)$ then the partial composition $\alpha \circ_i x \in C_*(\mathcal{T}_{f_i G})$ is defined similarly to Lemma 5.5 with the map $(-\circ_i -) : \mathcal{D}(m)\hat{\otimes}\mathcal{T}_G \to \mathcal{T}_{f_i G}$, and if $\#E(f_i(G)) < \#E(G)$ then $\alpha \circ_i x$ is zero. This partial composition is well defined by Lemma 3.11. The compatibility between d, ∂ and $(- \circ_i -)$ is obvious. We have completed the definition of $\check{\mathrm{T}}^M$.

Let $\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *}(n)$ denote the total complex. Its differential is given by $d + (-1)^q \partial$ on $\check{\mathrm{T}}^M_{\star q}(n)$. We regard the sequence $\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *} = \{\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *}(n)\}_n$ as a chain $C_*(\mathcal{D})$–comodule with the induced structure. We fix an operad map $f : \mathcal{A}_\infty \to C_*(\mathcal{D})$, and regard $\mathrm{Tot}\,\check{\mathrm{T}}^M$ as an $\mathcal{A}_\infty$–comodule by pulling back the partial compositions by f. We consider the Hochschild complex $\mathrm{CH}_\bullet(\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *})$ associated to this $\mathcal{A}_\infty$–comodule; see Definition 2.13. The total degree of elements of $\mathrm{CH}_\bullet(\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *})$ is $-*-\star-\bullet$. We define two filtrations $\{F^{-p}\}$ and $\{\bar{F}^{-p}\}$ on this complex as follows. F^{-p} (resp. $\bar{F}^{-p}$) is generated by the homogeneous parts whose degree satisfies $\star + \bullet \leq p$ (resp. $\bullet \leq p$). We call the spectral sequence associated to $\{F^{-p}\}$ the *Čech spectral sequence*, in short, *Čech s.s.*, and denote it by $\{\check{\mathbb{E}}_r^{-p,q}\}_r$. The spectral sequence associated to $\{\bar{F}^{-p}\}$ is denoted by $\{\bar{\mathbb{E}}_r^{-p,q}\}_r$.

Lemma 5.7 *The spectral sequence $\bar{\mathbb{E}}_r$ in Definition 5.6 and Sinha spectral sequence $\mathbb{E}_r$ in Definition 2.7 are isomorphic after the E_1–page.*

Proof Put $N_0 = \#\{(i,j) \mid i,j \in \underline{n} \text{ with } i<j\}$ and let $X : \mathrm{P}_{N_0} = \mathsf{G}(n) - \{\varnothing\} \to \mathcal{SP}$ be the functor given by $X_G = \mathcal{T}_G$. By applying Lemma 2.2 to this functor, we see that the map $\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *}(n) \to \mathcal{C}_*(\mathcal{T}_M(n))$ induced by the collapsing (quotient) map $\widetilde{\mathcal{T}}_M(n) \to \mathcal{T}_M(n)$ is a quasi-isomorphism. Combining this with Theorem 4.4 and Lemma 5.2, the two comodules $C_*(\mathcal{C}_M^\vee)$ and $\mathrm{Tot}\,\check{\mathrm{T}}_{\star *}$ are quasi-isomorphic. Clearly $\mathrm{CH}_\bullet\, C_*(\mathcal{C}_M)$ is quasi-isomorphic to the normalized complex of $C_*(\mathcal{C}^\bullet(M)^\vee)$, which is quasi-isomorphic to the normalized total complex of $C^*(\mathcal{C}^\bullet(M))$ by Lemma 5.3. Thus, $\mathrm{CH}_\bullet\,\mathrm{Tot}\,\check{\mathrm{T}}^M_{\star *}$ and the normalized total complex of $C^*(\mathcal{C}^\bullet(M))$ are connected by a zigzag of quasi-isomorphisms which preserve the filtration. This zigzag induces a zigzag of morphisms of spectral sequences which are isomorphisms after the E_1–page because the homology of $\mathrm{Tot}\,\check{\mathrm{T}}_{\star *}(n+1)$ is isomorphic to $H^*(\mathcal{C}^n(M))$ under the zigzag. □

5.3 Convergence

In this subsection, we assume M is orientable. We shall prepare some notation and terminology which is necessary to analyze the E_1–page of the Čech s.s.

Definition 5.8 • We fixed an embedding $e_0\colon SM \to \mathbb{R}^K$ and a number ϵ_0 in Definition 4.5. We also fix an isotopy $\iota_t\colon SM \to \mathbb{R}^{2K}$ with $\iota_0 = 0\times e_0$ and $\iota_1 = \Delta_{\mathbb{R}^K}\circ e_0$, where $0\times e_0\colon SM \to \mathbb{R}^{2K}$ is given by $(0\times e_0)(z) = (0, e_0(z))$ and $\Delta_{\mathbb{R}^K}$ is the diagonal map on $\mathbb{R}^K$. We impose the additional condition that ϵ_0 is smaller than $\min\{L(\iota_t) \mid 0 \le t \le 1\}$. We also fix a 1–parameter family of bundle maps $\kappa_t\colon \nu_{\epsilon_0}(0\times e_0) \to \nu_{\epsilon_0}(\iota_t)$ with $\kappa_0 = \mathrm{id}$.

- We fix the following classes:

$$\hat{w} \in H_{2d-1}(SM),\quad \omega_\Delta \in H^{2d-1}(SM\times SM, \Delta(SM)^c),\quad w_{S^K} \in H_K(S^K),\quad \omega_{S^K} \in H^K(S^K),$$
$$\omega_\nu \in H^{K-2d+1}(\mathrm{Th}(\nu)),\quad \omega(n) \in H^{n(K-2d+1)}(\mathrm{Th}(\nu^n)),\quad \gamma \in H^d(SM\times SM, (SM\times_M SM)^c).$$

Here $\hat{w}$ is a fundamental class of SM, $\Delta(SM)^c$ is the complement of the tubular neighborhood of the (standard, nondeformed) diagonal, ω_Δ is the diagonal class satisfying the equality

$$(\hat{w}\times\hat{w}) \cap \omega_\Delta = \Delta_*(\hat{w}) \in H_{2d-1}(SM^2),$$

w_{S^K} is the cross product $(w_{S^1})^{\times n}$ of K copies of the class w_{S^1} fixed in Definition 5.1, ω_{S^K} is the class such that $w_{S^K}\cap\omega_{S^K}$ is the class represented by a point, and ω_ν is the Thom class satisfying the equality

$$\kappa_1^*(\omega_\Delta\cdot(\omega_\nu\times\omega_\nu)) = \omega_{S^K}\times\omega_\nu.$$

Here $\omega_\Delta\cdot(\omega_\nu\times\omega_\nu)$ is naturally regarded as a Thom class for the bundle $\nu_{\epsilon_0}(\Delta_{\mathbb{R}^K}\circ e_0)$. We set $\omega(n) = \omega_\nu^{\times n}$. The class γ is a Thom class of a tubular neighborhood of $SM\times_M SM$ in $SM\times SM$.

- We call a graph in $\mathsf{G}(n)$ which does not contain a cycle (a closed path) a *tree*. For a graph $G \in \mathsf{G}(n)$, vertices i and j are said to be *disconnected* in G if i and j belong to different connected components of G.

- For $i<j$, let $\pi_{ij}\colon SM^n \to SM^{\times 2}$ be the projection given by $\pi_{ij}(z_1,\dots,z_n) = (z_i, z_j)$. Set $D_{ij} = D_G$ for $E(G) = \{(i,j)\}$, and

$$\gamma_{ij} = \pi_{ij}^*(\gamma) \in H^d(SM^n, (D_{ij})^c).$$

For a tree $G \in \mathsf{G}(n)$, write $E(G)$ as $\{(i_1,j_1) < \cdots < (i_r,j_r)\}$ with $i_t < j_t$ for $t = 1,\dots,r$. We put

$$w_G = \hat{w}^{\times n} \cap \gamma_{i_1,j_1}\cdots\gamma_{i_r,j_r} \in H_{n(2d-1)-rd}(D_G).$$

Clearly w_G is a fundamental class of D_G.

- Let $G \in \mathsf{G}(n,r)$ be a tree. Suppose i and $i+1$ are disconnected in G. Let $d_i\colon \underline{n} \to \underline{n-1}$ be the map given by

$$d_i(j) = \begin{cases} j & \text{if } j \le i,\\ j-1 & \text{if } j \ge i+1,\end{cases}$$

and set $H = d_i(G) \in \mathsf{G}(n-1)$. We define maps

$$\phi_G\colon \overline{H}_*(\mathrm{Th}(\nu_G)) \to H_{*-nK}(\mathcal{T}_G),\quad \zeta_G\colon H_*(\mathcal{T}_G) \to H^{-*-dr}(D_G),$$
$$\mu_i\colon H_*(\mathcal{T}_G) \to H_*(\mathcal{T}_H),\qquad m_i\colon H^*(D_G) \to H^*(D_H).$$

The map ϕ_G is the composition

$$\overline{H}_*(\mathrm{Th}(\nu_G)) \xrightarrow{(\lambda_G)_*} \overline{H}_*(\mathcal{T}_G(\mathfrak{c}_0)_{nK}) \to H_{*-nK}(\mathcal{T}_G(\mathfrak{c}_0)) \to H_{*-nK}(\mathcal{T}_G),$$

where λ_G is the map defined in Definition 4.5, the second map is the canonical one and the third is the inverse of evaluation at $\mathfrak{c}_0$. Clearly ϕ_G is an isomorphism. The map ζ_G is the composition $(w_G \cap -)^{-1} \circ (- \cap \omega(n)) \circ \phi_G^{-1}$ consisting of

$$H_*(\mathcal{T}_G) \xrightarrow{\phi_G^{-1}} \overline{H}_{*+nK}(\mathrm{Th}(\nu_G)) \xrightarrow{-\cap\omega(n)} H_{*+n(2d-1)}(D_G) \xrightarrow{(w_G\cap -)^{-1}} H^{-*-dr}(D_G).$$

The map μ_i is induced by the partial composition $\mu \circ_i -$, where $\mu \in H_0(\mathcal{D}(2)) = \mathcal{A}(2)$ is the fixed generator. The map m_i is given by $(-1)^A \Delta_i^*$, where $A = * + dr + n$ with $r = \#E(G)$, and Δ_i^* denotes the pullback by the restriction to D_H of the diagonal

$$\Delta_i : SM^{n-1} \to SM^n, \quad (z_1, \dots, z_{n-1}) \mapsto (z_1, \dots, z_i, z_i, \dots, z_{n-1}).$$

- We denote by $H\check{\mathrm{T}}^M_{\star *}(n)$ the bigraded chain complex obtained by taking the homology of $\check{\mathrm{T}}^M_{\star *}(n)$ for the differential d; see Definition 5.6. Its differential is induced by the differential $(-1)^q \partial$ on $\check{\mathrm{T}}^M_{\star q}(n)$. We regard the collection $H\check{\mathrm{T}}^M = \{H\check{\mathrm{T}}^M(n)\}$ as an $\mathcal{A}$–comodule with the structure induced by $\check{\mathrm{T}}^M$. As a k–module, $H\check{\mathrm{T}}^M(n)$ is the direct sum $\bigoplus_{G \in \mathsf{G}(n)} H_*(\mathcal{T}_G)$. We denote by aG the element of $H\check{\mathrm{T}}^M(n)$ corresponding to $a \in H_*(\mathcal{T}_G)$.

- The homology of the Hochschild complex $\mathrm{CH}_\bullet(H\check{\mathrm{T}}^M_{\star *})$ has the bidegree $(-\bullet-\star, -*)$. We denote the homogeneous part of bidegree (p, q) by $H_{-p,-q}(\mathrm{CH}(H\check{\mathrm{T}}^M))$.

- For two graphs $G, H \in \mathsf{G}(n)$ with $E(G) \cap E(H) = \varnothing$, the product $GH \in \mathsf{G}(n)$ denotes the graph with $E(GH) = E(G) \cup E(H)$. Let $i, j, k \in \underline{n}$ be distinct vertices, and $[ijk] \in \mathsf{G}(n)$ denote the graph with $E([ijk]) = \{(i, j), (j, k)\}$. For a graph $G \in \mathsf{G}(n)$, the products $G[ijk]$, $G[jki]$ and $G[kij]$ have the same connected component (if they are defined), so $\nu_{G[ijk]} = \nu_{G[jki]} = \nu_{G[kij]}$. Using these equalities, and the isomorphisms $\phi_{G'}$ for $G' = G[ijk], G[jki]$ and $G[kij]$, we identify the three groups $H_*(\mathcal{T}_{GH[ijk]})$, $H_*(\mathcal{T}_{G[jki]})$ and $H_*(\mathcal{T}_{G[kij]})$ with one another. Under this identification, let $I(n) \subset H\check{\mathrm{T}}^M(n)$ be the submodule generated by

  - summands of graphs which are not trees, and
  - elements of the form $aG[jki] + (-1)^s aG[ijk] + (-1)^{s+t} aG[kij]$ for $(i, j), (j, k), (i, k) \notin E(G)$, where $a \in H_*(\mathcal{T}_{G[ijk]})$, $s + 1$ is the number of edges of G between (i, j) and (i, k), and $t + 1$ is the number of edges between (i, k) and (j, k).

- We say a graph $G \in \mathsf{G}(n)$ with an edge set $E(G) = \{(i_1, j_1) < \cdots < (i_r, j_r)\}$ is *distinguished* if the following inequalities hold:

$$i_1 < j_1, \dots, i_r < j_r, \quad i_1 < \cdots < i_r.$$

We denote by $\mathsf{G}(n)^{\mathrm{dis}} \subset \mathsf{G}(n)$ the subset of the distinguished graphs.

The following lemma is obvious by the definition of the Čech s.s.

Lemma 5.9 *With the notation in Definition 5.8, the E_2–page of Čech s.s. is isomorphic to the homology of the Hochschild complex of $H\check{\mathrm{T}}^M_{\star\star}$. More precisely, there exists an isomorphism of* k*–modules*

$$\check{\mathbb{E}}_2^{pq} \cong H_{-p,-q}(\mathrm{CH}(H\check{\mathrm{T}}^M)) \quad \textit{for each} \quad (p,q). \qquad \square$$

Lemma 5.10 *With the notation in Definition 5.8, $I(n)$ is acyclic, ie $H_\partial(I(n)) = 0$, and the sequence $\{I(n)\}_n$ is closed under the partial compositions and symmetric group actions.*

Proof Since $\mathsf{G}(n)^{\mathrm{dis}}$ is stable under removing edges, the submodule $\bigoplus_{G\in\mathsf{G}(n)^{\mathrm{dis}}} H_*(\mathcal{T}_G)$ of $H\check{\mathrm{T}}^M(n)$ is a subcomplex. By an argument similar to (the dual of) [14], the inclusion

$$\check{\mathrm{T}}(\mathsf{G}(n)^{\mathrm{dis}}) := \bigoplus_{G\in\mathsf{G}(n)^{\mathrm{dis}}} H_*(\mathcal{T}_G) \subset H\check{\mathrm{T}}^M(n)$$

is a quasi-isomorphism. We easily see that the map $\check{\mathrm{T}}(\mathsf{G}(n)^{\mathrm{dis}}) \to \check{\mathrm{T}}(n)/I(n)$ induced by the inclusion is an isomorphism (see the proof of Lemma 6.9). $\square$

Lemma 5.11 *Let $\bar{e}_t : SM \to \mathbb{R}^{2K}$ be an isotopy with $\bar{e}_0 = 0\times e_0$ and $\bar{e}_1 = e_0\times 0$, and $F_t : \nu_{\epsilon_0}(\bar{e}_0) \to \nu_{\epsilon_0}(\bar{e}_t)$ be an isotopy which is also a bundle map covering $\bar{e}_t$ satisfying $F_0 = \mathrm{id}$. Then*

$$(F_1)^*(\omega_\nu \times \omega_{S^K}) = (-1)^K \omega_{S^K} \times \omega_\nu.$$

Here $\omega_\nu \times \omega_{S^K}$ is considered as a class of $H^{2K-2d+1}(\mathrm{Th}(\nu_{\epsilon_0}(\bar{e}_1)))$ via the map collapsing the subset $\nu_{\epsilon_0}(e) \times \mathbb{R}^K - \nu_{\epsilon_0}(\bar{e}_1)$, and $\omega_{S^K} \times \omega_\nu$ is similarly understood.

Proof Since the only problem is the orientation, it is enough to see a variation of a basis via a local model. Let $e_0 \colon \mathbb{R}^{2d-1} \to \mathbb{R}^K$ be the inclusion to the subspace of elements with the last $K-2d+1$ coordinates being zero. A covering isotopy is given by $F_t(u,v) = ((1-t)u - tv, tu + (1-t)v)$ for $u, v \in \mathbb{R}^K$. Since $F_1(u,v) = (-v,u)$, the derivative $(F_1)_*$ maps a basis $\{\boldsymbol{a}, \boldsymbol{b}\}$ of the tangent space of $\mathbb{R}^{2K}$ to $\{\boldsymbol{b}, -\boldsymbol{a}\}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ denote bases of $T\mathbb{R}^K \times 0$ and $0 \times T\mathbb{R}^K$, respectively. This implies $(F_1)^*(\omega_\nu \times \omega_{S^K}) = (-1)^K (-1)^{K(K-2d+1)} \omega_{S^K} \times \omega_\nu = (-1)^K \omega_{S^K} \times \omega_\nu$. $\square$

Lemma 5.12 *We use the notation in Definition 5.8. Let $G \in \mathsf{G}(n,r)$ be a tree whose vertices i and $i+1$ are disconnected in G. Set $H = d_i(G) \in \mathsf{G}(n-1)$. Then the diagram*

$$\begin{tikzcd} H_*(\mathcal{T}_G) \arrow[r, "\mu_i"] \arrow[d, "\zeta_G"] & H_*(\mathcal{T}_H) \arrow[d, "\varepsilon_1 \zeta_H"] \\ H^{-*-dr}(D_G) \arrow[r, "m_i"] & H^{-*-dr}(D_H) \end{tikzcd}$$

is commutative, where $\varepsilon_1 = (-1)^B$ with $B = K\big(* + 1 + \tfrac{1}{2}(K-1)\big)$.

Proof The claim follows from the commutativity of the following diagram:

$$
\begin{tikzcd}
H_*(\mathcal{T}_G) \arrow[r, "\mu_i"] & H_*(\mathcal{T}_H) & \\
\overline{H}_{*+nK}(\mathrm{Th}(\nu_G)) \arrow[u, "\phi_G"] \arrow[r, "\mu'"] \arrow[d, "\omega(n)"] & \overline{H}_{*+nK}(\mathrm{Th}(\nu')) \arrow[u, "\phi'_H"'] \arrow[d, "\omega'"] & \overline{H}_{*+(n-1)K}(\mathrm{Th}(\nu_H)) \arrow[l, "\alpha"'] \arrow[ul, "\phi_H"'] \arrow[dl, "\varepsilon_1\omega(n-1)"] \\
H_{*+n(2d-1)}(D_G) \arrow[r, "\mu''"] & H_{*+(n-1)(2d-1)}(D_H) & \\
H^{-*-dr}(D_G) \arrow[u, "w_G"] \arrow[r, "m_i"] & H^{-*-dr}(D_H) \arrow[u, "w_H"'] &
\end{tikzcd}
\tag{5-1}
$$

Here:

- ν' is the disk bundle over D_H of fiber dimension $nK-(n-1)(2d-1)$ defined by
$$\nu' = \nu_{\epsilon_0}(e_0^n \circ \Delta_i)|_{D_H},$$
where the restriction is taken as a disk bundle over SM^{n-1}; see Definition 5.8 for Δ_i.
- $\omega' \in \overline{H}^{nK-(n-1)(2d-1)}(\mathrm{Th}(\nu'))$ is given by
$$\omega' = (-1)^C(\omega_\nu)^{\times i-1} \times \omega_\Delta \cdot (\omega_\nu \times \omega_\nu) \times (\omega_\nu)^{\times n-i-1} \quad \text{with } C = (n+i+1)K.$$
- ϕ'_H is defined by using the following map λ'_H similarly to ϕ_H:
$$\lambda'_H : \nu^n \ni u \mapsto (e_0^n \circ \Delta_i, \epsilon_0, u) \in \mathcal{T}(\mathfrak{c}_0)_{nK}.$$
- μ' is the map collapsing the subset $\nu_G - \nu'$, where ν' and ν_G are regarded as subsets in $\mathbb{R}^{nK}$.
- μ'' is the composition
$$H_*(D_G) \to H_*(D_G, \Delta_i(D_H)^c) \to H_{*-2d+1}(\Delta_i(D_H)) \cong H_{*-2d+1}(D_H),$$
where the first map is the standard quotient map, the third is the inverse of the diagonal and the second is the cap product with the class
$$(-1)^{i+1+n} 1 \times \cdots \times \omega_\Delta \times \cdots \times 1 \quad \text{with } \omega_\Delta \text{ in the } i^{\text{th}} \text{ factor}.$$
- α is the composition $(1 \times \kappa_1 \times 1)_* \circ T \circ (\varepsilon_2 w_{S^K} \times -)$ of the maps
$$\overline{H}_{*'}(\mathrm{Th}(\nu_H)) \xrightarrow{\varepsilon_2 w_{S^K} \times -} \overline{H}_{*'+K}(S^K \wedge \mathrm{Th}(\nu_H)) \xrightarrow{T} \overline{H}_{*'+K}(\mathrm{Th}(\nu'')) \xrightarrow{(1\times\kappa_1\times 1)_*} \overline{H}_{*'+K}(\mathrm{Th}(\nu')),$$
where ν'' is the disk bundle over D_H of the same fiber dimension as ν' given by
$$\nu'' = \nu_{\epsilon_0}(e'')|_{D_H} \quad \text{with } e'' = e_0^{\times i-1} \times (0 \times e) \times e_0^{\times n-i} : SM^{n-1} \to \mathbb{R}^{nK},$$
T is the composition of the transposition of S^K from the first to the i^{th} component with the map induced by the map collapsing the subset $(\nu^{\times i-1} \times \mathbb{R}^K \times \nu^{\times n-i+1})|_{D_H} - \nu''$,
$$\varepsilon_2 = (-1)^D, \quad D = K\bigl(*' + \tfrac{1}{2}(K-1) + i + 1\bigr),$$

and $1\times\kappa_1\times 1$ is induced by the restriction of the product map

$$1\times\kappa_1\times 1\colon \mathbb{R}^{(i-1)K}\times\nu_{\epsilon_0}(0\times e_0)\times\mathbb{R}^{(n-i-1)K}\to\mathbb{R}^{(i-1)K}\times\nu_{\epsilon_0}(\Delta_{\mathbb{R}^K}\circ e_0)\times\mathbb{R}^{(n-i-1)K}$$

with κ_1 in the i^{th} component.

- The arrows with a (co)homology class denote the map given by taking the cap product with the class. For example, the right vertical arrow of the middle square denotes the map $x\mapsto x\cap\omega'$.

Our sign rules for graded products are the usual graded commutativity, except for the compatibility of cross and cap products, for which we use the rule

$$(a\times b)\cap(x\times y)=(-1)^{(|a|-|x|)|y|}(a\cap x)\times(b\cap y).$$

These are the rules based on the definitions in [19]. More precisely, we use the homology cross product induced by the simplicial cross product in [19, page 277] (or equivalently, the Eilenberg–Zilber shuffle map) and the cohomology cross product defined by $a\times b=p_1^*a\cup p_2^*b$ where p_i is the projection to the i^{th} component of the product and the cup product is given in [19, page 215]. We also use the cap product given in [19, page 239]. (This irregular sign rule is caused by absence of sign in the definition of cup product, as is standard.) With these rules, the commutativity of the squares in (5-1) is clear since the map Δ' defined in Section 3.2 is isotopic to the usual diagonal. We shall prove commutativity of the two triangles. The commutativity of the upper triangle follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\overline{H}_l(\mathrm{Th}(\nu)_H) \arrow[r, "T\circ(\varepsilon_2 w_{S^K}\times -)"] \arrow[d, "(\lambda_G)_*"] & \overline{H}_{l+K}(\mathrm{Th}(\nu'')) \arrow[r, "(1\times\kappa_1\times1)_*"] \arrow[dr, "(\lambda''_H)_*"] & \overline{H}_{l+K}(\mathrm{Th}(\nu')) \arrow[d, "(\lambda'_H)_*"] \\
\overline{H}_l(\mathcal{T}_H(\mathfrak{c}_0)_{(n-1)K}) \arrow[rr, "\varepsilon_2 w_{S^K}\times"] & & H_{l+K}(\mathcal{T}_H(\mathfrak{c}_0)_{nK})
\end{tikzcd}$$

Here λ''_H is given by $u\mapsto(e_0^{\times i-1}\times(0\times e_0)\times e_0^{\times n-i},\epsilon_0,u)$. Commutativity of the left trapezoid follows from Lemma 5.11 (the sign ε_2 is the product of the sign in i_k^X in Definition 5.1 and the sign in Lemma 5.11), and that of the right triangle follows from the homotopy between $\lambda'_H\circ\kappa_1$ and λ''_H constructed from the isotopy κ_t in Definition 5.6. We shall show that the lower triangle is commutative. We see

$$\begin{aligned}
\varepsilon_1\alpha(x)\cap\omega' &= \{(\kappa_1)_*T_*(w_{S^K}\times x)\}\cap(\omega\times\cdots\times\omega_\Delta(\omega\times\omega)\times\cdots\times\omega)\\
&= \{(\kappa_1)_*T_*(w_{S^K}\times x)\}\cap(\omega\times\cdots\times(\kappa_1^{-1})^*(\omega_{S^K}\times\omega)\times\cdots\times\omega)\\
&= (\kappa_1)_*\{T_*(w_{S^K}\times x)\cap(\omega\times\cdots\times(\omega_{S^K}\times\omega)\times\cdots\times\omega)\}\\
&= (\kappa_1)_*T_*\{(w_{S^K}\times x)\cap T^*(\omega\times\cdots\times(\omega_{S^K}\times\omega)\times\cdots\times\omega)\}\\
&= (\kappa_1)_*T_*\{(w_{S^K}\times x)\cap\omega_{S^K}\times\omega\times\cdots\times\omega)\}\\
&= (w_{S^K}\times x)\cap(\omega_{S^K}\times\omega\times\cdots\times\omega)=x\cap\omega(n-1).
\end{aligned}$$

Here $(\kappa_1)_*$ is an abbreviation of $(1\times\kappa_1\times 1)_*$ and ω of ω_ν. All the capped classes are considered as elements of the homology of the base space D_H of involved disk bundles by projections. The second equality follows from the definition of ω_ν. As endomorphisms on the base space, T_* and $(1\times\kappa_1\times 1)_*$ are the identity, and hence the sixth equality holds. $\square$

The following lemma is easily verified and a proof is omitted.

Lemma 5.13 *Let $G \in \mathsf{G}(n,r)$ be a tree and $K \in \mathsf{G}(n,r-1)$ be the tree made by removing the t^{th} edge (i,j) from G. Under the notation in Definition 5.8, the diagram*

$$\begin{tikzcd} H_*(\mathcal{T}_G) \arrow[r] \arrow[d, "\zeta_G"] & H_*(\mathcal{T}_K) \arrow[d, "\zeta_K"] \\ H^{-*-dr}(D_G) \arrow[r] & H^{-*-d(r-1)}(D_K) \end{tikzcd}$$

is commutative, where the top horizontal arrow is induced by the inclusion and the bottom one is given by $(-1)^{(r-t)d}\Delta^!_{ij}$ with $\Delta^!_{ij}(x) = \gamma_{ij}\cdot x$. □

Definition 5.14 • In the following, for a module X with a decomposition $X = \bigoplus_{G\in\mathsf{G}(n)} X_G$, we denote by $X^{\text{tr}} \subset X$ the direct sum of the summands X_G labeled by a tree G.

• We define an $\mathcal{A}$–comodule $A^{\star *}_M$ of CDBA (see Definition 2.10). Put $H^*_G = H^*(D_G)$. Let $\wedge(g_{ij})$ be the free bigraded commutative algebra generated by elements g_{ij} for $1 \le i < j \le n$, with bidegree $(-1,d)$. For notational convenience, we set $g_{ij} = (-1)^d g_{ji}$ for $i > j$ and $g_{ii} = 0$. For $G \in \mathsf{G}(n)$ with $E(G) = \{(i_1,j_1) < \cdots < (i_r,j_r)\}$, we set $g_G = g_{i_1,j_1}\cdots g_{i_r,j_r}$. Put

$$\widetilde{A}^{\star *}_M(n) = \bigoplus_{G\in\mathsf{G}(n)} H^*_G g_G.$$

Here $H^*_G g_G$ is a copy of H^*_G with degree shift. For $G \in \mathsf{G}(n,r)$ and $a \in H^l_G$, the bidegree of the element $a g_G \in \widetilde{A}_M(n)$ is $(-r, l+dr)$. We give a graded commutative multiplication on $\widetilde{A}_M(n)$ as follows. For $a \in H^l_G$ and $b \in H^m_H$, we set

$$(a g_G)\cdot(b g_H) = \begin{cases} (-1)^{mr(d-1)+s}(a\cdot b) g_{GH} \in H^{l+m}_{GH} g_{GH} & \text{if } E(G)\cap E(H) = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Here we set $r = \#E(G)$, a is regarded as an element of H^*_{GH} by pulling back by the map $i_G : \Delta_{GH} \to D_G$ induced by the quotient map $\pi_0(G) \to \pi_0(GH)$, and similarly for b, and the product $a\cdot b$ is taken in H^*_{GH}. Also, s is the number determined by the equality $g_G\cdot g_H = (-1)^s g_{GH}$ for the product in $\wedge(g_{ij})$.

Let $J(n) \subset \widetilde{A}_M(n)$ be the ideal generated by the elements

$$a(g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij})g_G \quad \text{and} \quad b g_K,$$

where $G, K \in \mathsf{G}(n)$, $a \in H^*_{G[ijk]}$ and $b \in H^*_K$ are elements such that $(i,j), (j,k), (k,i) \notin E(G)$, and K is not a tree. Here by definition, D_G depends only on $\pi_0(G)$, so $\Delta_{G[ijk]} = \Delta_{G[jki]} = \Delta_{G[kij]}$. With these identities, we regard a as an element of $H_{G[jki]} = H_{G[kij]}$, and the first type of generators as elements of

$$H_{G[ijk]} g_{G[ijk]} \oplus H_{G[jki]} g_{G[jki]} \oplus H_{G[kij]} g_{G[kij]}.$$

We define an algebra $A^{\star *}_M(n)$ as the following quotient:

$$A^{\star *}_M(n) = \widetilde{A}^{\star *}_M(n)/J(n).$$

Since the restriction of the quotient map $\widetilde{A}_M(n)^{\mathrm{tr}} \to A_M(n)$ is surjective, we may define a differential, a partial composition and an action of Σ_n on the sequence $A_M = \{A_M(n)\}_n$ through $\widetilde{A}_M(n)^{\mathrm{tr}}$. We define a map $\tilde{\partial}\colon \widetilde{A}_M(n)^{\mathrm{tr}} \to \widetilde{A}_M(n)^{\mathrm{tr}}$ by

$$\tilde{\partial}(ag_G) = \sum_{t=1}^{r} (-1)^{(l+t-1)(d-1)} \Delta^!_{i_t,j_t}(a) g_{i_1,j_1} \cdots \check{g}_{i_t,j_t} \cdots g_{i_r,j_r} \quad \text{for } G \in \mathsf{G}(n) \text{ and } a \in H^l_G,$$

where $\Delta^!_{ij}(a) = \gamma_{ij} \cdot a$ and $\check{g}_{ij}$ means removing g_{ij}. It is easy to see $\tilde{\partial}(\widetilde{A}_M(n)^{\mathrm{tr}} \cap J(n)) \subset \widetilde{A}_M(n)^{\mathrm{tr}} \cap J(n)$. We define the differential ∂ on $A_M(n)$ to be the map induced by $\tilde{\partial}$. For the generator $\mu \in \mathcal{A}(2)$ fixed in Definition 5.8 and an element $ag_G \in \widetilde{A}_M(n)^{\mathrm{tr}}$, we define the partial composition $\mu \circ_i (ag_G)$ by

$$\mu \circ_i (ag_G) = \begin{cases} \Delta^*_i(a) g_H & \text{if } i \text{ and } i+1 \text{ are disconnected in } G, \\ 0 & \text{otherwise,} \end{cases}$$

where $H = d_i(G)$; see Definition 5.8. The action of $\sigma \in \Sigma_n$ on $\widetilde{A}_M(n)^{\mathrm{tr}}$ is given by $(ag_G)^\sigma = a^\sigma (g_G)^\sigma$, where a^σ is the pullback of a by $(\sigma_G)^{-1}$ (see Definition 5.6) and $(g_G)^\sigma$ denotes $g_{\tau(i_1)\tau(j_1)} \cdots g_{\tau(i_r)\tau(j_r)}$ with $\tau = \sigma^{-1}$. The partial composition and the action of Σ_n on $\{\widetilde{A}_M(n)^{\mathrm{tr}}\}_n$ are easily seen to preserve the submodule $\{J(n) \cap \widetilde{A}_M(n)^{\mathrm{tr}}\}_n$ and induce a structure of an $\mathcal{A}$–comodule on A_M.

- Let $s_i : \underline{n} \to \underline{n+1}$ denote the order-preserving monomorphism skipping $i+1$ for $1 \le i \le n$. Then s_i naturally induces a monomorphism $s_i : \pi_0(G) \to \pi_0(s_i G)$ (see Section 2.1), which in turn induces $(s_i)^* \colon D_{s_i G} \to D_G$. Let s_i also denote the induced map $(s_i^*)^* \colon H^*(D_G) \to H^*(D_{s_i G})$. By further abuse of notation, we also denote by s_i the map $A_M(n) \to A_M(n+1)$ given by $s_i(ag_G) = s_i(a) g_{s_i G}$.
- Define a simplicial CDBA $A^{\star *}_\bullet(M)$ (a functor from Δ^{op} to the category of CDBAs) as follows. We set

$$A^{\star *}_n(M) = A^{\star *}_M(n+1).$$

If we consider an element of $A_M(n+1)$ as an element of $A_n(M)$, we relabel its subscripts with $0, 1, \dots, n$ instead of $1, 2, \dots, n+1$. For example, $g_{01} \in A_n(M)$ corresponds to $g_{12} \in A_M(n+1)$. The partial compositions and the maps s_i (defined in the previous item) are also considered as beginning with $(- \circ_0 -)$ and s_0 (originally written as $(- \circ_1 -)$ and s_1). The face map $d_i \colon A_n(M) \to A_{n-1}(M)$ for $0 \le i \le n$ is given by $d_i = \mu \circ_i (-)$ for $i < n$ and $d_n = \mu \circ_0 (-)^\sigma$, where $\sigma = (n, 0, 1, \dots, n-1)$. The degeneracy map $s_i \colon A_n(M) \to A_{n+1}(M)$ for $0 \le i \le n$ is the map defined in the previous item.

Lemma 5.15 *Let i, j and k be numbers with $i < j < k$. The equalities $\gamma_{ij}\gamma_{ik} = \gamma_{ij}\gamma_{jk} = \gamma_{ik}\gamma_{jk}$ hold.*

Proof The three classes are Thom classes in $H^*(SM^n, \Delta^c_{[ijk]})$. So to prove the equality, it is enough to identify the corresponding orientations. This is easily done by observing the corresponding bases. □

Theorem 5.16 *Suppose M is orientable.*

(1) *The two $\mathcal{A}$–comodules $H\check{\mathrm{T}}^M_{\star *}$ and $A^{\star *}_M$ of differential bigraded k–modules are quasi-isomorphic in a manner where $H\check{\mathrm{T}}^M_{-p,-q}$ and $A^{p,q}_M$ correspond for integers p and q. (For $H\check{\mathrm{T}}^M$, see Definition 5.8.)*

(2) *The E_2–page of the Čech s.s. in Definition 5.6 is isomorphic to the total homology of the normalized complex $NA^{\star *}_\bullet(M)$. Under this isomorphism, the homogeneous part $\check{\mathbb{E}}^{pq}_2$ consists of the classes*

represented by a sum of elements in the complex, whose triple degree $(-\bullet, \star, *)$ *satisfies* $p = \star - \bullet$ *and* $q = *$.

The latter part of (2) of this theorem may need some care. It does not mean that the E_2–page is generated by the classes which are represented by elements of $NA(M)$ which are homogeneous for each of the three degrees, since the differential of the E_1–page of the Čech s.s. corresponds to the total differential of $NA(M)$ and changes both of the degrees $\star$ and $\bullet$.

Proof For (1), we consider the composition

$$H_{-*}(\mathcal{T}_G) \xrightarrow{\zeta_G} H_G^{*-dr} \to H_G^{*-dr} g_G.$$

The right map is given by $a \mapsto \varepsilon_3 a g_G$ with the sign

$$\varepsilon_3 = (-1)^E \quad \text{where } E = E(*', n, r) = *'(n + dr) + drn + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}dr(r+1)$$

on $H_G^{*'}$. This composition defines an isomorphism as bigraded $\Bbbk$–modules between $H\check{\mathrm{T}}^M(n)^{\mathrm{tr}}$ and $\widetilde{A}_M(n)^{\mathrm{tr}}$. By Lemma 5.15, this isomorphism maps $H\check{\mathrm{T}}^M_{-\star,-*}(n)^{\mathrm{tr}} \cap I(n)$ into $\widetilde{A}_M^{\star *}(n)^{\mathrm{tr}} \cap J(n)$ isomorphically. A quasi-isomorphism $H\check{\mathrm{T}}^M(n) \to A_M(n)$ is defined by the composition

$$\begin{aligned} H\check{\mathrm{T}}^M_{-\star,-*}(n) \to H\check{\mathrm{T}}^M_{-\star,-*}(n)/I(n) &\cong H\check{\mathrm{T}}^M_{-\star,-*}(n)^{\mathrm{tr}}/\{H\check{\mathrm{T}}^M_{-\star,-*}(n)^{\mathrm{tr}} \cap I(n)\} \\ &\cong \widetilde{A}_M^{\star *}(n)^{\mathrm{tr}}/\{\widetilde{A}_M^{\star *}(n)^{\mathrm{tr}} \cap J(n)\} \cong A_M^{\star *}(n), \end{aligned}$$

where the first map is the quotient map, which is a quasi-isomorphism by Lemma 5.10, the second and fourth maps are induced by inclusions, and the third map is the isomorphism defined above. For the above number E, we see

$$E(*', n-1, r) - E(*', n, r) \equiv *' + dr + n \quad \text{and} \quad E(*' + d, n, r-1) - E(*', n, r) \equiv (*' + 1)d$$

modulo 2. Now we may assume the integer K is a multiple of 4. With this assumption and the above equalities for E, compatibility of the quasi-isomorphism with the partial composition follows from Lemma 5.12 as $\varepsilon_1 = 1$. Compatibility with the (Čech) differentials follows from Lemma 5.13. Compatibility with the actions of Σ_n is clear. The sign $\mathrm{sgn}(\sigma_G)$ in Definition 5.6, the sign occurring in permutations of γ_{ij} and the sign occurring in permutations of g_{ij} are canceled. Thus the isomorphism is a morphism of $\mathcal{A}$–comodules. For (2), by (1), the E_2–page is isomorphic to the homology of the Hochschild complex $\mathrm{CH}_\bullet(A_M)$, which is isomorphic to the unnormalized total complex of $A_\bullet(M)$, and so is quasi-isomorphic to the normalized complex. □

Sinha proved the convergence of his spectral sequences using the Cohen–Taylor spectral sequence. Here we prove the convergence of the Čech and Sinha spectral sequences simultaneously by an independent method.

Theorem 5.17 *If M is simply connected and of dimension $d \geq 4$, both the Čech s.s. and Sinha s.s. for M converge to $H^*(\mathrm{Emb}(S^1, M))$.*

Proof We set a number s_d by $s_d = \min\{\frac{1}{3}d, 2\}$. If $d \geq 4$, clearly $s_d > 1$. Recall that $\{\check{\mathbb{E}}_r\}_r$ denotes the Čech s.s. By Lemma 5.7, we identify the Sinha s.s. with the spectral sequence $\overline{\mathbb{E}}_r$. We shall first show the claim that $\check{\mathbb{E}}_2^{-p,q} = 0$ if $q/p < s_d$. If a graph $G \in \mathsf{G}(n+1)$ has k discrete vertices, $H^*(D_G)$ is isomorphic to $H^*(SM)^{\otimes k} \otimes H^*(D_{G'}) \otimes \{\text{torsions}\}$, where $G' \in \mathsf{G}(n+1-k)$ is the graph made by removing discrete vertices. With this observation, and simple connectivity of M, we see that generators of the normalization $NA_n(M)$ are presented as $a_1 \cdots a_k b g_G$ where G is a graph in $\mathsf{G}(n+1)$ with r edges and k discrete vertices except for the vertex 0, a_t belongs to the t^{th} discrete tensor factor $H^{\geq 2}(SM)$, and $b \in H^*_{G'}$. We may ignore the torsion part in estimation of degree by the universal coefficient theorem. The bidegree $(-p,q)$ of this element satisfies $p = n + r$ and $q \geq 2k + rd$. Clearly we have $k + 2r \geq n + \epsilon$, with $\epsilon = 0$ or 1 according to whether the vertex 0 has valence 0 in G. With this, if $d \leq 5$, we have the following estimate:

$$\frac{q}{p} - \tfrac{1}{3}d \geq \frac{6k + (3r-p)d}{3(n+r)} \geq \frac{(6-d)k + d\epsilon}{3(n+r)} \geq 0.$$

If $d \geq 6$, we have the following estimate:

$$\frac{q}{p} - 2 = \frac{2\epsilon + (d-6)r}{n+r} \geq 0.$$

We have shown the claim. Since the filtration $\{F^{-p}\}$ of the Čech s.s. is exhaustive, and the total homology of each F^{-p} is of finite type, the Čech s.s. $\{\check{\mathbb{E}}_r\}_r$ converges to the total homology $H(NA_\bullet(M))$ of the normalized complex. By the same reasoning, $\{\overline{\mathbb{E}}_r\}$ also converges to $H(NA_\bullet(M))$. We shall show $\overline{\mathbb{E}}_r^{-p,q} = 0$ if $q/p < s_d$, for sufficiently large r. Suppose there exists a nonzero element $x \in \overline{\mathbb{E}}_\infty^{-p,q}$ with $q/p < s_d$. We consider x as an element of $(\overline{F}^{-p}/\overline{F}^{-p+1})H(NA_\bullet(M))$. Take a class x' in $\overline{F}^{-p}H(NA_\bullet(M))$ representing x. Take the smallest p' such that $F^{-p'}H(NA_\bullet(M))$ contains x'. Then $\check{\mathbb{E}}_\infty^{-p',q+p'-p}$ is not zero and $p' \geq p$ as $\overline{F}^{-p} \supset F^{-p}$. In the coordinate plane of bidegree, x' and x are on the same line of the form $-p + q = \text{constant}$. This and $p' \geq p$ imply that the "slope" of x' is smaller than s_d, which contradicts to the claim. This vanishing result on $\overline{\mathbb{E}}_r$ and (the cohomology version of) [4, Theorem 3.4] imply the convergence of $\overline{\mathbb{E}}_r$ and $\check{\mathbb{E}}_r$ to $H^*(\mathrm{Emb}(S^1, M))$. □

Remark 5.18 If the dimension of the target manifold M is 3, or if M is not simply connected, the Čech s.s. does not converge to the cohomology of the knot space but it does to the same target as the Sinha s.s. (see the proof of Theorem 5.17). The diagonal of the Sinha s.s. for long knots converges to the universal finite type invariants at least in the rational coefficient. So the Čech s.s. in dimension 3 may contain some information about knot invariants.

6 Algebraic presentations of the E_2–page of the Čech spectral sequence

In this section, we assume M is oriented and simply connected and $H^*(M)$ is a free k–module.

Definition 6.1 • A *Poincaré algebra of dimension d* is a graded commutative algebra $\mathcal{H}^*$ with a linear isomorphism $\epsilon\colon \mathcal{H}^d \to \mathsf{k}$ such that the bilinear form defined as the composition

$$\mathcal{H}^* \otimes \mathcal{H}^* \xrightarrow{\text{multiplication}} \mathcal{H}^* \xrightarrow{\text{projection}} \mathcal{H}^d \xrightarrow{\epsilon} \mathsf{k}$$

induces an isomorphism $\mathcal{H}^* \cong (\mathcal{H}^{d-*})^\vee$. We call ϵ the *orientation of* $\mathcal{H}$.

- For a Poincaré algebra $\mathcal{H}^*$, we denote by $\Delta_{\mathcal{H}}$ the *diagonal class for* $\mathcal{H}^*$ given by

$$\sum_i (-1)^{|a_i^*|} a_i \otimes a_i^* \in (\mathcal{H} \otimes \mathcal{H})^d,$$

where $\{a_i\}$ and $\{a_i^*\}$ are two bases of $\mathcal{H}^*$ such that $\epsilon(a_i \cdot a_j^*) = \delta_{ij}$, the Kronecker delta. This definition does not depend on a choice of a basis $\{a_i\}$.

- Let $\mathcal{H}$ be a Poincaré algebra $\mathcal{H}$ of dimension d with $\mathcal{H}^1 = 0$. We set $\mathcal{H}^{\leq d-2} = \bigoplus_{p \leq d-2} \mathcal{H}^p$ and $\mathcal{H}^{\geq 2} = \bigoplus_{p \geq 2} \mathcal{H}^p$, and define a graded k–module $\mathcal{H}^{\geq 2}[d-1]$ by $(\mathcal{H}^{\geq 2}[d-1])^p = X^{p-d+1}$ with $X^* = \mathcal{H}^{\geq 2}$. We denote by $\bar{a}$ the element in $(\mathcal{H}^{\geq 2}[d-1])^p$ corresponding to $a \in \mathcal{H}^{p-d+1}$. We define a Poincaré algebra $S\mathcal{H}$ of dimension $2d-1$ as follows. As a graded k–module, we set

$$S\mathcal{H}^* = \mathcal{H}^{\leq d-2} \oplus \mathcal{H}^{\geq 2}[d-1].$$

For $a, b \in \mathcal{H}^{\leq d-2}$, the multiplication $a \cdot b$ in $S\mathcal{H}$ is the one in $\mathcal{H}$ except for the case $|a| + |b| = d$, in which we set $a \cdot b = 0$. We set $a \cdot \bar{b} = \overline{ab}$ for $a \in \mathcal{H}^{\leq d-2}$ and $b \in \mathcal{H}^{\geq 2}$, and $\bar{a} \cdot \bar{b} = 0$ for $a, b \in \mathcal{H}^{\geq 2}$. We give the same orientation on $S\mathcal{H}$ as the one on $\mathcal{H}$ via the identity $S\mathcal{H}^{2d-1} = \mathcal{H}^d$.

- We regard $\mathcal{H} = H^*(M)$ as a Poincaré algebra with the orientation

$$H^d(M) \xrightarrow{w_M \cap} H_0(M) \cong \mathsf{k},$$

where w_M is the fundamental class of M determined by the orientation on M, and the isomorphism sends the class represented by a point to 1.

The following lemma is obvious:

Lemma 6.2 *With the notation of Definition 6.1, let $(b_{ij})_{ij}$ denote the inverse of the matrix $(\epsilon(a_i \cdot a_j))_{ij}$. Then*

$$\Delta_{\mathcal{H}} = \sum_{i,j} (-1)^{|a_j|} b_{ji} a_i \otimes a_j. \qquad \square$$

Under some assumptions, $S\mathcal{H}$ is isomorphic to $H^*(SM)$ (see the proof of Lemma 6.6), and the algebras $A^*_{\mathcal{H},G}$ and $B^*_{\mathcal{H},G}$ defined as follows are isomorphic to $H^*(D_G)$.

Definition 6.3 For a Poincaré algebra $\mathcal{H}$ of dimension d and graph $G \in \mathsf{G}(n)$, define a graded commutative algebra $A_{\mathcal{H},G}$ by

$$A_{\mathcal{H},G} = \mathcal{H}^{\otimes \pi_0(G)} \otimes \bigwedge\{y_1, \dots, y_n\}, \quad \deg y_i = d-1.$$

Here we regard $\pi_0(G)$ as an ordered set by the minimum in each component, and the tensor product is taken in this order. Furthermore, we also define a graded commutative algebra $B_{\mathcal{H},G}$ by

$$B_{\mathcal{H},G} = S\mathcal{H}^{\otimes n} \otimes \bigwedge\{y_{ij} \mid 1 \leq i, j \leq n \text{ and } i \sim_G j\}/J_G, \quad \deg y_{ij} = d-1.$$

Here $i \sim_G j$ means that the vertices i and j belong to the same connected component of G, and J_G is the ideal generated by the following relation:

$$\{e_i(a)-e_j(a), e_i(\bar{a})-e_j(\bar{a})-ay_{ij}, e_i(\bar{b})-e_j(\bar{b}), y_{ii}, y_{ij}+y_{jk}-y_{ik} \\ \mid a \in \mathcal{H}^{\le d-2}, b \in \mathcal{H}^d, 1 \le i,j,k \le n, i \sim_G j \sim_G k\}.$$

Here $e_j(\bar{a})$ is regarded as 0 if $a \in \mathcal{H}^0$.

For $i < j$, let $f_{ij}\colon \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes n}$ denote the map given by

$$f_{ij}(a \otimes b) = 1 \otimes \cdots \otimes a \otimes \cdots \otimes b \otimes \cdots \otimes 1$$

(a is the i^{th} factor, b is the j^{th} factor and the other factors are 1). We set

$$\Delta_{\mathcal{H}}^{ij} = f_{ij}(\Delta_{\mathcal{H}}) \in \mathcal{H}^{\otimes n}.$$

We sometimes regard $\Delta_{\mathcal{H}}^{ij}$ as an element of $(S\mathcal{H})^{\otimes n}$ via the projection and inclusion $\mathcal{H} \to \mathcal{H}^{\le d-1} \subset S\mathcal{H}$. We also regard it as an element of $A_{\mathcal{H},G}$ for a graph G via the map $\mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes \pi_0(G)}$ given by multiplication of factors in the same components with the standard commuting signs. We also set

$$\Delta_{S\mathcal{H}}^{ij} = f'_{ij}(\Delta_{S\mathcal{H}}) \in S\mathcal{H}^{\otimes n},$$

where $\Delta_{S\mathcal{H}}$ is the diagonal class for the Poincaré algebra $S\mathcal{H}$ and $f'_{ij}\colon S\mathcal{H}^{\otimes 2} \to S\mathcal{H}^{\otimes n}$ is the map defined by the same formula as f_{ij}. We regard $\Delta_{\mathcal{H}}^{ij}$ and $\Delta_{S\mathcal{H}}^{ij}$ as elements of $B_{\mathcal{H},G}$, similarly to the case of $A_{\mathcal{H},G}$.

As a graded algebra, $B_{\mathcal{H},G}^*$ is isomorphic to $(S\mathcal{H})^{\otimes \pi_0(G)} \bigwedge\{y_{ij} \mid i \sim_G j\}/(y_{ii}, y_{ij}+y_{jk}-y_{ik})$, but we need the presentation to describe maps induced by identifying vertices and removing edges.

The proof of the following lemma is easy and omitted.

Lemma 6.4 *Consider the Serre spectral sequence for a fibration*

$$F \to E \to B$$

with the base simply connected and the cohomology groups of the fiber and base finitely generated in each degree. If for each k there is at most a single p such that $E_\infty^{p,k-p} \ne 0$, the quotient map $F^p \to F^p/F^{p+1}$ has a unique section which preserves cohomological degree. Gathering these sections for all p, one can define an isomorphism of graded algebra $E_\infty \to H^*(E)$, which we call the ***canonical isomorphism****. The canonical isomorphisms are natural for maps between fibrations satisfying the above assumption.* □

Henceforth we regard the Euler number $\chi(M)$ as an element of the base ring k via the ring map $\mathbb{Z} \to \mathsf{k}$, and $\mathsf{k}^\times \subset \mathsf{k}$ denotes the subsets of the invertible elements.

Lemma 6.5 *We use the notation d_i, Δ_i, s_i and $\Delta_{ij}^!$ given in Definitions 5.8 and 5.14. Suppose $\chi(M) = 0 \in \mathsf{k}$. Set $\mathcal{H}^* = H^*(M)$. There exists a family of isomorphisms of graded algebras*

$$\{\varphi_G\colon A_{\mathcal{H},G} \xrightarrow{\cong} H^*(D_G) \mid n \ge 1, G \in \mathsf{G}(n)\}$$

which satisfies the following conditions:

(1) *Let $G \in \mathsf{G}(n)$ be a tree with i and $i+1$ disconnected. Set $H = d_i(G)$. The following diagram is commutative*:

$$\begin{array}{ccc} A^*_{\mathcal{H},G} & \xrightarrow{\varphi_G} & H^*(D_G) \\ \big\downarrow{\scriptstyle \bar{\Delta}_i} & & \big\downarrow{\scriptstyle \Delta_i^*} \\ A^*_{\mathcal{H},H} & \xrightarrow{\varphi_H} & H^*(D_H) \end{array}$$

Here the algebra map $\bar{\Delta}_i$ is defined as follows. For $a_1 \otimes \cdots \otimes a_p \in \mathcal{H}^{\otimes \pi_0(G)}$, we set

$$\bar{\Delta}_i(a_1 \otimes \cdots \otimes a_p) = \pm a_1 \otimes \cdots \otimes a_s \cdot a_t \otimes \cdots \otimes a_p \qquad \text{and} \qquad \bar{\Delta}_i(y_j) = y_{j'} \quad \text{with } j' = d_i(j).$$

Here $s, t \in \pi_0(G)$ are the connected components containing i and $i+1$, respectively, and $\pm$ is the standard sign in transposing graded elements.

(2) *For a graph $G \in \mathsf{G}(n)$, set $S = s_i(G)$. The following diagram is commutative*:

$$\begin{array}{ccc} A^*_{\mathcal{H},G} & \xrightarrow{\varphi_G} & H^*(D_G) \\ \big\downarrow{\scriptstyle \bar{s}_i} & & \big\downarrow{\scriptstyle s_i} \\ A^*_{\mathcal{H},S} & \xrightarrow{\varphi_S} & H^*(D_S) \end{array}$$

Here $\bar{s}_i$ is given by inserting the unit 1 as the factor of $H^{\otimes \pi_0(G)}$ which corresponds to the component containing $i+1$, and by skipping the subscript $i+1$, ie by the equality $\bar{s}_i(y_j) = y_{s_i(j)}$.

(3) *For a graph $G \in \mathsf{G}(n)$ and a permutation $\sigma \in \Sigma_n$, the following diagram is commutative*:

$$\begin{array}{ccc} A^*_{\mathcal{H},G} & \xrightarrow{\varphi_G} & H^*(D_G) \\ \big\downarrow{\scriptstyle \bar{\sigma}} & & \big\downarrow{\scriptstyle \sigma} \\ A^*_{\mathcal{H},\tau(G)} & \xrightarrow{\varphi_{\tau(G)}} & H^*(D_{\tau(G)}) \end{array}$$

Here $\tau = \sigma^{-1}$, the right vertical arrow is induced by the natural permutation of factors of the product $D_{\tau(G)} \to D_G$ and the left vertical arrow $\bar{\sigma}$ is the algebra map given by the permutation of tensor factors and subscripts.

(4) *For an edge (i, j) of a tree $G \in \mathsf{G}(n)$ with $i < j$, we define $K \in \mathsf{G}(n)$ by $E(K) = E(G) - \{(i, j)\}$. The following diagram is commutative*:

$$\begin{array}{ccc} A^*_{\mathcal{H},G} & \xrightarrow{\varphi_G} & H^*(D_G) \\ \big\downarrow{\scriptstyle \bar{\Delta}_{ij}} & & \big\downarrow{\scriptstyle \Delta^!_{ij}} \\ A^{*+d}_{\mathcal{H},K} & \xrightarrow{\varphi_K} & H^{*+d}(D_K) \end{array}$$

Here $\bar{\Delta}_{ij}$ is the right $A^*_{\mathcal{H},K}$–module homomorphism determined by $\bar{\Delta}_{ij}(1) = \Delta^{ij}_{\mathcal{H}}$, and $A^*_{\mathcal{H},G}$ is considered as an $A^*_{\mathcal{H},K}$–module via the natural algebra map $A^*_{\mathcal{H},K} \to A^*_{\mathcal{H},G}$.

Proof In this proof we fix a generator y of $H^{d-1}(S^{d-1})$, and we denote by y_i (or $\bar{y}_i$) the image of y by the inclusion to the i^{th} factor, $H^{d-1}(S^{d-1}) \to H^{d-1}(S^{d-1})^{\otimes n}$. We consider Serre spectral sequence for the fibration

$$(S^{d-1})^n \to D_G \to M^{\pi_0(G)},$$

where the projection is the restriction of that of the tangent sphere bundle. The first possibly nontrivial differential is $d_d\colon H^{d-1}((S^{d-1})^n) = E_d^{0,d-1} \to E_d^{d,0} = H^d(M)$, where d in the super- and subscripts is the dimension of M. This differential takes y_i to the generator of $H^d(M)$ multiplied by $\chi(M)$. As $\chi(M) = 0$, we have $d_d = 0$. Since the this differential on y_i is zero for degree reasons, y_i survives eternally, which implies $E_2 \cong E_\infty$. Clearly E_∞ satisfies the assumption of Lemma 6.4. We define φ_G as the composition

$$A_{\mathcal{H},G} \to E_2 = E_\infty \to H^*(D_G),$$

where the left map is the isomorphism given by identifying y_i in both of the sides and $\mathcal{H}^{\otimes \pi_0(G)}$ with $H^*(M^{\times \pi_0(G)})$ by the Künneth isomorphism, and the right map is the canonical isomorphism defined in Lemma 6.4. Parts (1), (2) and (3) obviously follow from naturality of the canonical isomorphisms. For (4), $H^*(D_G)$ is regarded as a $H^*(D_K)$–module via the pullback $\Delta_{ij}^*\colon H^*(D_K) \to H^*(D_G)$ by the inclusion $D_G \to \Delta_K$. This module structure is compatible with the $A^*_{\mathcal{H},K}$–module structure on $A^*_{\mathcal{H},G}$ via φ_G and φ_K by naturality of the canonical isomorphism. By a general property of a shriek map, the map $\Delta^!_{ij}$ is a $H^*(D_K)$–module homomorphism. So to prove the compatibility, we have only to check the image of 1. For simplicity, we may assume $n = 2$ and $(i, j) = (1, 2)$. We may write D_G as $SM \times_M SM$. The diagram

$$\begin{tikzcd}
H_{d-*}(M) \arrow[r, "\mathrm{PD}"] \arrow[d, "\Delta_*"] & H^*(M) \arrow[r, "\mathrm{proj}^*"] \arrow[d, "\Delta^!"] & H^*(SM \times_M SM) \arrow[d, "\Delta^!_{12}"] \\
H_{d-*}(M \times M) \arrow[r, "\mathrm{PD}"] & H^{*+d}(M \times M) \arrow[r, "\mathrm{proj}^*"] & H^{*+d}(SM \times SM)
\end{tikzcd}$$

is commutative, where PD denotes the cap product with the fundamental class. By the commutativity of the left square, we see that $\Delta^!(1)$ is the Poincaré dual class in $H^*(M \times M)$ of the diagonal $\Delta(M)$, which corresponds to $\Delta_{\mathcal{H}}$ by the Künneth isomorphism. By the commutativity of the right square, we see that $\Delta^!_{12}(1)$ corresponds to $f_{ij}\Delta_{\mathcal{H}}$. This completes the proof. □

Lemma 6.6 *We use the notation d_i, Δ_i, s_i and $\Delta^!_{ij}$ given in Definitions 5.8 and 5.14. Suppose $\chi(M) \in \mathsf{k}^\times$. Set $\mathcal{H} = H^*(M)$. There exists a family of isomorphisms of graded algebras*

$$\{\varphi_G\colon B_{\mathcal{H},G} \xrightarrow{\cong} H^*(D_G) \mid n \geq 1, G \in \mathsf{G}(n)\}$$

which satisfies the following conditions:

(1) *Let G and H be trees given in Lemma 5.12*(1). *The following diagram is commutative*:

$$\begin{tikzcd}
B_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\Delta}_i"] & H^*(D_G) \arrow[d, "\Delta_i^*"] \\
B_{\mathcal{H},H} \arrow[r, "\varphi_H"] & H^*(D_H)
\end{tikzcd}$$

Here $\bar{\Delta}_i$ *is defined by*

$$\bar{\Delta}_i(e_j(x)) = e_{j'}(x) \quad \text{for } x \in S\mathcal{H} \qquad \text{and} \qquad \bar{\Delta}_i(y_{jk}) = y_{j'k'},$$

where we set $j' = d_i(j)$ *and* $k' = d_i(k)$.

(2) *For a graph* $G \in \mathsf{G}(n)$, *set* $S = s_i(G)$. *The following diagram is commutative*:

$$\begin{tikzcd} B^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{s}_i"] & H^*(D_G) \arrow[d, "s_i"] \\ B^*_{\mathcal{H},S} \arrow[r, "\varphi_S"] & H^*(D_S) \end{tikzcd}$$

Here $\bar{s}_i$ *is given by inserting* 1 *in the* $(i+1)^{th}$ *factor of* $S\mathcal{H}^{\otimes n}$ *and skipping the subscript* $i+1$.

(3) *For a graph* $G \in \mathsf{G}(n)$ *and a permutation* $\sigma \in \Sigma_n$, *the following diagram is commutative*:

$$\begin{tikzcd} B^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\sigma}"] & H^*(D_G) \arrow[d, "\sigma"] \\ B^*_{\mathcal{H},\tau(G)} \arrow[r, "\varphi_{\tau(G)}"] & H^*(D_{\tau(G)}) \end{tikzcd}$$

Here τ *and the right vertical arrow are defined as in Lemma 6.5, and* $\bar{\sigma}$ *is the algebra homomorphism defined by the permutation of the tensors and subscripts.*

(4) *For an edge* $(i,j) \in E(G)$ *of a tree* $G \in \mathsf{G}(n)$ *with* $i < j$, *define* $K \in \mathsf{G}(n)$ *by* $E(K) = E(G) - \{(i,j)\}$. *The following square is commutative*:

$$\begin{tikzcd} B^*_{\mathcal{H},G} \arrow[r, "\varphi_G"] \arrow[d, "\bar{\Delta}_{ij}"] & H^*(D_G) \arrow[d, "\Delta^!_{ij}"] \\ B^{*+d}_{\mathcal{H},K} \arrow[r, "\varphi_K"] & H^{*+d}(D_K) \end{tikzcd}$$

Here $\bar{\Delta}_{ij}$ *is the right* $B^*_{\mathcal{H},K}$*–module homomorphism determined by* $\bar{\Delta}_{ij}(1) = \Delta^{ij}_{\mathcal{H}}$ *and* $\bar{\Delta}_{ij}(y_{ij}) = \Delta^{ij}_{S\mathcal{H}}$, *and* $B^*_{\mathcal{H},G}$ *is considered as a* $B^*_{\mathcal{H},K}$*–module via the algebra map* $f^G_K \colon B^*_{\mathcal{H},K} \to B^*_{\mathcal{H},G}$ *given by*

$$f^G_K(e_k(x)) = e_k(x) \quad \text{for } x \in S\mathcal{H} \qquad \text{and} \qquad f^G_K(y_{kl}) = \begin{cases} 0 & \text{if } (k,l) = (i,j), \\ y_{kl} & \text{if otherwise.} \end{cases}$$

Proof As in the proof of Lemma 6.5, we fix a generator $y \in H^{d-1}(S^{d-1})$. Note that d is even as $\chi(M) \neq 0$. We first show an isomorphism of algebras $S\mathcal{H}^* \cong H^*(SM)$. Consider the Serre spectral sequence for the tangent sphere fibration

$$S^{d-1} \to SM \to M.$$

The only nontrivial differential is $d_d \colon E^{0,d-1} = H^{d-1}(S^{d-1}) \to H^d(M)$. As $\chi(M)$ is invertible, d_d is an isomorphism. Since all other differentials vanish by degree reasons, $E_\infty \cong E_{d+1} \cong S\mathcal{H}$, where the second isomorphism is given by $E^{p,0}_{d+1} = H^p(M) \subset \mathcal{H}^{\leq d-2} \subset S\mathcal{H}$ for $p \leq d-2$ and $E^{p,d-1} = H^{d-1}(S^{d-1}) \otimes H^p(M) \ni y \otimes a \mapsto \bar{a} \in S\mathcal{H}$ for $p \geq 2$. Since $H^1(M) = 0$ and $H^*(M)$ is free, $H^{d-1}(M)=0$, which implies the fibration satisfies the conditions of Lemma 6.4. Composing this isomorphism with the canonical isomorphism $E_\infty \to H^*(SM)$, we have an isomorphism

$$S\mathcal{H}^* \cong H^*(SM). \tag{6-1}$$

If necessary, we modify y so that the composition $S\mathcal{H}^{2d-1} \to H^{2d-1}(SM) \to \mathsf{k}$ of (6-1) and the cap product with the fundamental class $\widehat{w}$ in Definition 5.8 coincides with the orientation given in Definition 6.1 by multiplying by a scalar.

We shall define the isomorphism φ_G. We may assume that $G \in \mathsf{G}(n)$ is connected, as in the disconnected case everything involved is a tensor product of the objects corresponding to connected subgraphs. Consider the Serre spectral sequence for the fibration

$$(S^{d-1})^{n-1} \to D_G \to SM$$

given by projection to the first component. As $E_2^{d,0}=S\mathcal{H}^d=0$, elements in $E_2^{0,d-1} \cong H^{d-1}(S^{d-1})^{\otimes n-1}$ survive eternally. As in the proof of Lemma 6.5, y_j denotes the copy of y living in the j^{th} factor of $H^*(S^{d-1})^{\otimes n-1}$, which is also regarded as a generator of $E_2^{0,d-1}$. We construct an isomorphism $\psi_G\colon S\mathcal{H}^* \otimes \bigwedge(y_1,\dots,y_{n-1}) \cong E_\infty \cong H^*(D_G)$ using (6-1) similarly to the construction of (6-1). Consider the Serre spectral sequence $\{\overline{E}_r^{p,q}\}$ for the fibration

$$(S^{d-1})^n \to D_G \to M$$

given by the projection of the sphere bundle. Let $\bar{y}_j$ be the copy of y in the j^{th} factor of $\overline{E}_2^{0,d-1} \cong (H^*(S^{d-1})^{\otimes n})^{*=d-1}$. For any i and j, since $d_d(\bar{y}_i)=d_d(\bar{y}_j)=(\text{a multiple of})\chi(M)w_M$, the element $\bar{y}_i-\bar{y}_j$ survives eternally by degree reasons. Clearly $\overline{E}_\infty$ satisfies the assumption of Lemma 6.4, so we can take the canonical isomorphism $\overline{E}_\infty^{*,*} \to H^*(D_G)$. We define an algebra map

$$\varphi'_G\colon (S\mathcal{H})^{\otimes n} \otimes \bigwedge\{y_{ij} \mid 1 \le i,j \le n\} \to \overline{E}_\infty^{*,*}$$

by $e_i(a) \mapsto a \in E_\infty^{*,0}$ for $a \in \mathcal{H}^{\le d-2}$, $e_i(\bar{b}) \mapsto b\bar{y}_i \in E_\infty^{*,d-1}$ for $b \in \mathcal{H}^{\ge 2}$, and $y_{ij} \mapsto \bar{y}_i-\bar{y}_j$. We see $\varphi'_G(J_G)=0$, where J_G is the ideal in Definition 6.3. For example, since $d_d(\bar{y}_i\bar{y}_j)=\chi(M)(\bar{y}_j-\bar{y}_i)w_M$ (up to $\mathsf{k}^\times$) and $\chi(M)$ is invertible, $(\bar{y}_i-\bar{y}_j)w_M=0$ in $\overline{E}_{d+1}^{d,d-1}$, which implies $\varphi'_G(e_i(\bar{b})-e_j(\bar{b}))=0$ for $b \in \mathcal{H}^d$. Annihilation of other elements in J_G is obvious. We define φ_G to be the unique map which makes the following diagram commutative:

$$\begin{tikzcd}
(S\mathcal{H})^{\otimes n} \otimes \bigwedge\{y_{ij}\} \arrow[r] \arrow[d, "\varphi'_G"] & (S\mathcal{H})^{\otimes n} \otimes \bigwedge\{y_{ij}\}/J_G \arrow[r, equal] & B^*_{\mathcal{H},G} \arrow[d, "\varphi_G"] \\
\overline{E}_\infty^{*,*} \arrow[rr, "\text{canonical isomorphism}"] & & H^*(D_G)
\end{tikzcd}$$

Since G is connected, $e_1\colon S\mathcal{H} \to S\mathcal{H}^{\otimes n}$ induces an isomorphism $\alpha_G\colon S\mathcal{H} \otimes \bigwedge\{y_{12},\dots,y_{1n}\} \cong B^*_{\mathcal{H},G}$. It is easy to see that the composition

$$S\mathcal{H} \otimes \bigwedge\{y_{12},\dots,y_{1n}\} \overset{\alpha_G}{\cong} B^*_{\mathcal{H},G} \xrightarrow{\varphi_G} H^*(D_G) \overset{\psi_G^{-1}}{\cong} S\mathcal{H} \otimes \bigwedge\{y_1,\dots,y_n\}$$

identifies the subalgebra $S\mathcal{H}$ in both sides and the sub-k–module $\mathsf{k}\langle y_{12},\dots,y_{1n}\rangle$ with $\mathsf{k}\langle y_1,\dots,y_n\rangle$ (since these are both isomorphic to $H^{d-1}(D_G)$), which implies the composition is an isomorphism and we conclude that φ_G is an isomorphism.

Parts (1), (2) and (3) obviously follow from naturality of the canonical isomorphism. We shall show (4). Since φ_G is an isomorphism, we may define $\bar\Delta_{ij}$ to be the map which makes the square in (4) commute. As in the proof of Lemma 6.5, $\bar\Delta_{ij}$ is a $B^*_{\mathcal{H},K}$–module homomorphism and we have $\bar\Delta_{ij}(1)=f_{ij}(\Delta_{\mathcal{H}})$. We shall show the equality $\bar\Delta_{ij}(y_{ij})=f_{ij}(\Delta_{S\mathcal{H}})$. We may assume $n=2$ and $G=(1,2)$. In this case, clearly $D_G=SM\times_M SM$. We consider the commutative diagram

$$\begin{tikzcd}
H^0(S^{d-1}) \arrow[d, "\Delta_1^!"] & H^0(SM) \arrow[l] \arrow[r, "\mathrm{PD}"] \arrow[d, "\Delta_2^!"] & H_{2d-1}(SM) \arrow[d, "(\Delta_2)_*"] \\
H^{d-1}(S^{d-1}\times S^{d-1}) & H^{d-1}(SM\times_M SM) \arrow[l] \arrow[r, "\mathrm{PD}"] \arrow[d, "\Delta_{12}^!"] & H_{2d-1}(SM\times_M SM) \arrow[d, "(\Delta_{12})_*"] \\
 & H^{2d-1}(SM\times SM) \arrow[r, "\mathrm{PD}"] & H_{2d-1}(SM\times SM)
\end{tikzcd}$$

where the left horizontal arrows are induced by the fiber restriction, the right ones are capping with the fixed fundamental classes, and $\Delta_1^!$ and $\Delta_2^!$ are the shriek maps induced by the diagonals. As d is even, $\Delta_1^!(1)=\bar y_1-\bar y_2$. As $\bar y_1-\bar y_2$ coincides with the image of $\varphi_G(y_{12})$ by the fiber restriction which induces an isomorphism in degree $d-1$, we have $\Delta_2^!(1)=\varphi_G(y_{12})$. So $\Delta_{12}^!(\varphi_G(y_{12}))=(\Delta_{12}\circ\Delta_2)^!(1)$. By the commutativity of the right-hand square, $(\Delta_{12}\circ\Delta_2)^!(1)$ is the diagonal class for SM. Thanks to the modification of y after the definition of (6-1), the diagonal class corresponds to $\Delta_{S\mathcal{H}}$ by φ_G. This implies $\bar\Delta_{12}(y_{12})=\Delta_{S\mathcal{H}}$. □

Definition 6.7 Let $\mathcal{H}$ be a Poincaré algebra of dimension d.

- We define a CDBA $A^{\star *}_{\mathcal{H}}(n)$ by the equality

$$A^{\star *}_{\mathcal{H}}(n)=\mathcal{H}^{\otimes n}\otimes\bigwedge\{y_i,g_{ij}\mid 1\le i,j\le n\}/\mathcal{I}.$$

Here, for the bidegrees, we set $|a|=(0,l)$ for $a\in(\mathcal{H}^{\otimes n})^{*=l}$, $|y_i|=(0,d-1)$ and $|g_{ij}|=(-1,d)$. The ideal $\mathcal{I}$ is generated by the elements

$$g_{ij}-(-1)^d g_{ji},\ (g_{ij})^2,\ g_{ii},\ (e_i(a)-e_j(a))g_{ij},\ g_{ij}g_{jk}+g_{jk}g_{ki}+g_{ki}g_{ij}\quad\text{for } 1\le i,j,k\le n \text{ and } a\in\mathcal{H}.$$

We call the last relation the 3–*term relation for* g_{ij}. The differential is given by $\partial(a)=0$ for $a\in\mathcal{H}^{\otimes n}$ and $\partial(g_{ij})=\Delta^{ij}_{\mathcal{H}}$; see Definition 6.3.

- Suppose d is even. We define a CDBA $B^{\star *}_{\mathcal{H}}(n)$ by the equality

$$B^{\star *}_{\mathcal{H}}(n)=(S\mathcal{H})^{\otimes n}\otimes\bigwedge\{g_{ij},h_{ij}\mid 1\le i,j\le n\}/\mathcal{J}$$

Here, for the bidegrees, we set $|a| = (0, l)$ for $a \in (\mathcal{H}^{\otimes n})^{*=l}$, $|g_{ij}| = (-1, d)$ and $|h_{ij}| = (-1, 2d-1)$. The ideal $\mathcal{J}$ is generated by the elements

$$g_{ij} - g_{ji}, (g_{ij})^2, g_{ii}, h_{ij} + h_{ji}, (h_{ij})^2, h_{ii}, e_{ij}(a)g_{ij}, e_{ij}(a)h_{ij}, e_{ij}(\bar{b})g_{ij} - e_i(b)h_{ij}, e_{ij}(\bar{b})h_{ij},$$
$$g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij}, h_{ij}h_{jk} + h_{jk}h_{ki} + h_{ki}h_{ij}, (h_{ij} + h_{ki})g_{jk} - (h_{ij} + h_{jk})g_{ki}$$

for $1 \le i, j, k \le n$, $a \in \mathcal{H}^{\le d-2}$ and $b \in \mathcal{H}^{\ge 2}$, where we regard $e_i(b)$ as 0 for $b \in \mathcal{H}^d$, and $e_{ij} : S\mathcal{H} \to (S\mathcal{H})^{\otimes n}$ is the map given by $e_{ij} = e_i - e_j$. The differential is given by $\partial(x) = 0$ for $x \in S\mathcal{H}^{\otimes n}$, $\partial(g_{ij}) = \Delta^{ij}_{\mathcal{H}}$ and $\partial(h_{ij}) = \Delta^{ij}_{S\mathcal{H}}$; see Definition 6.3.

- We equip the sequences $A_{\mathcal{H}} = \{A_{\mathcal{H}}(n)\}_n$ and $B_{\mathcal{H}} = \{B_{\mathcal{H}}(n)\}_n$ with the structures of $\mathcal{A}$–comodules of CDBA as follows. For $B_{\mathcal{H}}$, we define a partial composition and an action of Σ_n by the equalities

$$\mu \circ_i e_j(x) = e_{j'}(x), \quad \mu \circ_i (h_{jk}) = h_{j'k'}, \quad \mu \circ_i (g_{jk}) = g_{j'k'}, \quad e_j(x)^{\sigma} = e_{\tau(j)}(x),$$
$$h^{\sigma}_{jk} = h_{\tau(j),\tau(k)}, \quad g^{\sigma}_{jk} = g_{\tau(j),\tau(k)}, \quad \text{for } x \in S\mathcal{H} \text{ and } \sigma \in \Sigma_n,$$

where j' and k' are the numbers given by $j' = d_i(j)$ and $k' = d_i(k)$, and we set $\tau = \sigma^{-1}$ (see Definition 5.8 for d_i and μ). The definition of $A_{\mathcal{H}}$ is similar.

- We define simplicial CDBAs $A^{\star\star}_{\bullet}(\mathcal{H})$ and $B^{\star\star}_{\bullet}(\mathcal{H})$ as follows. For $B^{\star\star}_{\bullet}(\mathcal{H})$, we set

$$B^{\star\star}_n(\mathcal{H}) = B^{\star\star}_{\mathcal{H}}(n+1).$$

As in Definition 5.14, we relabel the involved subscripts with $0, \dots, n$. The face map $d_i : B^{\star\star}_n(\mathcal{H}) \to B^{\star\star}_{n-1}(\mathcal{H})$ is given by $d_i = \mu \circ_i (-)$ for $i < n$ and $d_n = \mu \circ_0 (-)^{\sigma}$ where $\sigma = (n, 0, 1, \dots, n-1)$. The degeneracy map $s_i : B^{\star\star}_n(\mathcal{H}) \to B^{\star\star}_{n+1}(\mathcal{H})$ is given by inserting 1 as the $(i+1)^{\text{th}}$ factor of $S\mathcal{H}^{\otimes n+1}$ and skipping the subscript $i+1$. $A^{\star\star}_{\bullet}(\mathcal{H})$ is defined similarly using $A^{\star\star}_{\mathcal{H}}$.

Remark 6.8 An algebra similar to the algebras $A^{\star\star}_{\mathcal{H}}(n)$ and $B^{\star\star}_{\mathcal{H}}(n)$ has already appeared as the E_2–page of Totaro's spectral sequence defined in [39].

In the rest of this section, we prove that $A_{\mathcal{H}}$ and $B_{\mathcal{H}}$ are isomorphic to A_M as $\mathcal{A}$–comodules of CDBA under different assumptions, and also prove similar statements for the simplicial CDBAs. We mainly deal with the case of $B_{\mathcal{H}}$. The case of $A_{\mathcal{H}}$ is similar.

Lemma 6.9 *The map*

$$\bigoplus_{G \in \mathsf{G}(n)^{\mathrm{dis}}} H^*_G g_G \to A_M$$

defined by the composition of the inclusion and quotient map is an isomorphism of k*–modules* (see Definition 5.8 for $\mathsf{G}(n)^{\mathrm{dis}}$).

Proof Let Π be the set of partitions of $\underline{n}$. The ideal $J(n)$ in Definition 5.14 has a decomposition $J(n) = \bigoplus_{\pi \in \Pi} J(n)_{\pi}$ such that $J(n)_{\pi} \subset \bigoplus_{\pi_0(G)=\pi} H_G$, since generators of $J(n)$ are sums of monomials which have the same connected components. If $\pi_0(G) = \pi_0(H) = \pi$, clearly $H^*_G = H^*_H$. We denote this module by H^*_{π}. We have $\bigoplus_{\pi_0(G)=\pi} H_G g_G = H_{\pi} \otimes (\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G)$. Similarly $J(n)_{\pi} = H_{\pi} \otimes J(n)'_{\pi}$

where $J(n)'_\pi$ is the sub-k–module of $\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G$ generated by multiples of 3–term relations, g_{ij}^2 and $g_{ij} - (-1)^d g_{ji}$. We have

$$A_M^* = \bigoplus_{\pi\in\Pi} \biggl\{ \Bigl(\bigoplus_{\pi_0(G)=\pi} H_G g_G \Bigr) / J(n)_\pi \biggr\} = \bigoplus_{\pi\in\Pi} H_\pi \otimes \biggl\{ \Bigl(\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G \Bigr) / J(n)'_\pi \biggr\}.$$

Note that $\bigoplus_{\pi\in\Pi} \{ (\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G) / J(n)'_\pi \}$ is isomorphic to the cohomology group of the configuration space $H^*(C_n(\mathbb{R}^d))$, whose basis is $\{g_G \mid G \in \mathsf{G}(n)^{\mathrm{dis}}\}$. So then $(\bigoplus_{\pi_0(G)=\pi} \mathsf{k} g_G)/J(n)'_\pi$ has a basis $\{g_G \mid G \in \mathsf{G}(n)^{\mathrm{dis}}, \pi_0(G) = \pi\}$, which implies the lemma. □

Under the assumptions and notation of Lemma 6.6, we identify H_G^* with $B_{\mathcal{H},G}$ by the isomorphism φ_G, so $A_M^*(n)$ is regarded as a quotient of $\bigoplus_{G\in\mathsf{G}(n)} B^*_{\mathcal{H},G} g_G$. With this identification, we set $\bar{h}_{ij} = y_{ij} g_{ij} \in A_M(n)$. $A_M(n)$ contains $S\mathcal{H}^{\otimes n}$ as the subalgebra $H_\varnothing g_\varnothing$, the summand corresponding to the graph $\varnothing \in \mathsf{G}(n)$. We regard $A_M(n)$ as a left $S\mathcal{H}^{\otimes n}$–module via the multiplication by $H_\varnothing g_\varnothing$. In the following lemma and its proof, h_G, $\bar{h}_G$ and y_G are defined similarly to g_G. For example, $h_G = h_{i_1,j_1} \cdots h_{i_r,j_r}$ for $E(G) = \{(i_1, j_1) < \cdots < (i_r, j_r)\}$.

Lemma 6.10 *Under the assumptions of Lemma 6.6 and the above notation, as an $S\mathcal{H}^{\otimes n}$–module, $A_M(n)$ is generated by the set $S = \{g_G \bar{h}_H \mid G, H \in \mathsf{G}(n), E(G) \cap E(H) = \varnothing, GH \in \mathsf{G}(n)^{\mathrm{dis}}\}$, and $B_{\mathcal{H}}(n)$ is generated by the set $S' = \{g_G h_H \mid G, H \in \mathsf{G}(n), E(G) \cap E(H) = \varnothing, GH \in \mathsf{G}(n)^{\mathrm{dis}}\}$.*

Proof $A_M(n)$ is generated by the elements $y_H g_G$, for graphs G and H, such that each connected component of H is contained in some connected component of G. We can express g_G as a sum of monomials g_{G_1} with $G_1 \in \mathsf{G}(n)^{\mathrm{dis}}$ and $\pi_0(G) = \pi_0(G_1)$ using the 3–term relation and the relation $g_{ij} = g_{ji}$ (this is standard procedure in the computation of $H^*(C_n(\mathbb{R}^d))$. So we may assume G is distinguished. For a sequence of edges $(i, k_1), (k_1, k_2), \ldots, (k_s, j)$ in G, we have $y_{ij} = y_{i,k_1} + \cdots + y_{k_s,j}$. By successive application of this equality, y_H is expressed as a sum of monomials y_{H_1} with H_1 being a subgraph of G. Thus any element of $A_M(n)$ is expressed as a $S\mathcal{H}^{\otimes n}$–linear combination of monomials $y_H g_G$ with $G \in \mathsf{G}(n)^{\mathrm{dis}}$ and $E(H) \subset E(G)$. Clearly $y_H g_G = \pm g_{G-H} \bar{h}_H$. Thus the set S generates $A_M(n)$. A proof for the assertion for $B_{\mathcal{H}}(n)$ is similar when one use 3–term relations for g_{ij} and h_{ij}, and the last relation for g_{ij} and h_{ij} in the ideal $\mathcal{J}$ in Definition 6.7. □

To prove that $B_{\mathcal{H}}(n)$ and $A_M(n)$ are isomorphic, we define a structure of a $B_{\mathcal{H},G}$–module on $B_{\mathcal{H}}(n)$ as follows. We first define two graded algebras $\widetilde{B}_{\mathcal{H},G}$ and $\widetilde{B}_{\mathcal{H}}(n)$. For a graph $G \in \mathsf{G}(n)$, we set

$$\widetilde{B}_{\mathcal{H},G} = S\mathcal{H}^{\otimes n} \otimes T\{y_{ij} \mid i < j \text{ and } i \sim_G j\} \quad \text{and} \quad \widetilde{B}_{\mathcal{H}}(n) = S\mathcal{H}^{\otimes n} \otimes \bigwedge\{g_{ij}, h_{ij} \mid 1 \le i < j \le n\},$$

where $T\{y_{ij}\}$ denotes the tensor algebra generated by the y_{ij}. For convenience, we set $y_{ij} = -y_{ji}$, $g_{ij} = g_{ji}$ and $h_{ij} = -h_{ji}$ for $i > j$. The degrees are the same as the elements of the same symbols in $B_{\mathcal{H},G}$ and $B_{\mathcal{H}}(n)$. We shall define a map of graded k–modules

$$(- \cdot -)\colon \widetilde{B}_{\mathcal{H},G} \otimes_{\mathsf{k}} \widetilde{B}_{\mathcal{H}}(n) \to B_{\mathcal{H}}(n).$$

We define $y_{ij}\cdot xg_Gh_H$ for $x\in S\mathcal{H}^{\otimes n}$ and $G,H\in \mathsf{G}(n)$ as follows. If $E(G)\cap E(H)\neq\varnothing$, we set $y_{ij}\cdot xg_Gh_H=0$. Suppose $E(G)\cap E(H)=\varnothing$. If $(i,j)\in E(G)$ is the t^{th} edge (in the lexicographical order), we set $y_{ij}\cdot xg_Gh_H=(-1)^{t+1+|x|}h_{ij}xg_Kh_H$ with $E(K)=E(G)-\{(i,j)\}$. If $(i,j)\in E(H)$ is an edge, we set $y_{ij}\cdot xg_Gh_H=0$. If $i\sim_{GH} j$, we take a sequence of edges $(k_0,k_1),\dots,(k_s,k_{s+1})$ of GH with $k_0=i$ and $k_{s+1}=j$ and set $y_{ij}\cdot xg_Gh_H=\sum_{l=0}^{s} y_{k_l,k_{l+1}}\cdot xg_Gh_H$. This does not depend on the choice of the sequence, because $g_Gh_H=0$ if GH is not a tree, which is proved by using the last three relations in the definition of $\mathcal{J}$ in Definition 6.7. If i and j are disconnected in GH, we set $y_{ij}\cdot xg_Gh_H=0$. For $z\in S\mathcal{H}^{\otimes n}$, we set $z\cdot xg_Gh_H=zxg_Gh_H$, the multiplication in $B_{\mathcal{H}}(n)$. We shall show that the map $(-\cdot-)$ annihilates the elements of $\mathcal{J}$ (we regard $\mathcal{J}$ as an ideal in $\widetilde{B}_{\mathcal{H}}(n)$). Direct computation shows that the generators of $\mathcal{J}$ are annihilated by any elements of $\widetilde{B}_{\mathcal{H},G}$. For example, $y_{ij}\cdot(g_{ij}g_{jk}+g_{jk}g_{ki}+g_{ki}g_{ij})=(h_{ij}+h_{ik})g_{jk}-(h_{ij}+h_{jk})g_{ki}=0$ and $y_{jk}\cdot\{(h_{ij}+h_{ki})g_{jk}-(h_{ij}+h_{jk})g_{ki}\}=h_{ij}h_{jk}+h_{jk}h_{ki}+h_{ki}h_{ij}=0$. We also easily see $y_{ij}\cdot xg_Gh_H=\pm(y_{ij}\cdot xg_{G'}h_{H'})g_{G-G'}h_{H-H'}$ for subgraphs $G'\subset G$ and $H'\subset H$ such that $i\sim_{G'H'}j$. These observations imply the assertion, and we see that $(-\cdot-)$ factors through $\widetilde{B}_{\mathcal{H},G}\otimes_{\mathsf{k}}B_{\mathcal{H}}(n)\to B_{\mathcal{H}}(n)$, which is also denoted by $(-\cdot-)$. Clearly the map $(-\cdot-)$ annihilates J_G in the definition of $B_{\mathcal{H},G}$. It also annihilates the commutativity relation $y_{ij}y_{kl}+y_{kl}y_{ij}$. If two paths connecting i and j or k and l have a common edge, both of the actions of $y_{ij}y_{kl}$ and $y_{kl}y_{ij}$ are zero, and otherwise the commutativity in $B_{\mathcal{H}}(n)$ implies the annihilation. Annihilation of these relations implies that the map $(-\cdot-)$ factors through a map $(-\cdot-)\colon B_{\mathcal{H},G}\otimes B_{\mathcal{H}}(n)\to B_{\mathcal{H}}(n)$, which defines a structure of $B_{\mathcal{H},G}$–module on $B_{\mathcal{H}}(n)$.

Theorem 6.11 *Suppose M is simply connected and oriented, and $H^*(M)$ is a free k–module. Set $\mathcal{H}=H^*(M)$.*

(1) *Suppose $\chi(M)=0\in\mathsf{k}$. The two $\mathcal{A}$–comodules of CDBA $A_M^{\star *}$ and $A_{\mathcal{H}}^{\star *}$ are isomorphic, and the two simplicial CDBAs $A_\bullet^{\star *}(M)$ and $A_\bullet^{\star *}(\mathcal{H})$ are isomorphic. In particular, the E_2–page of the Čech s.s. is isomorphic to the total homology of the normalization $NA_\bullet^{\star *}(\mathcal{H})$ as a bigraded k–module. The bigrading is given by $(\star-\bullet,*)$.*

(2) *Suppose $\chi(M)\in\mathsf{k}^\times$. The two $\mathcal{A}$–comodules of CDBA $A_M^{\star *}$ and $B_{\mathcal{H}}^{\star *}$ are isomorphic, and the two simplicial CDBAs $A_\bullet^{\star *}(M)$ and $B_\bullet^{\star *}(\mathcal{H})$ are isomorphic. In particular, the E_2–page of the Čech s.s. is isomorphic to the total homology of the normalization $NB_\bullet^{\star *}(\mathcal{H})$ as a bigraded k–module. The bigrading is given by $(\star-\bullet,*)$.*

Proof Part (1) obviously follows from Theorem 5.16 and Lemma 6.5. We shall prove (2). We define a map $\Phi_n\colon B_{\mathcal{H}}(n)\to A_M(n)$ of algebras by identifying the subalgebra $S\mathcal{H}^{\otimes n}$ and elements g_{ij} in both sides, and taking h_{ij} to $\bar{h}_{ij}$ (see the paragraph above Lemma 6.10). We easily verify that Φ_n is well defined. Then Φ_n fits into the following commutative diagram:

$$\begin{tikzcd}
\bigoplus_{G\in\mathsf{G}(n)^{\mathrm{dis}}} H_G g_G \arrow[d] \arrow[dr] & \\
B_{\mathcal{H}}(n) \arrow[r, "\Phi_n"] & A_M(n)
\end{tikzcd}$$

Here the vertical arrow is induced by the inclusion of a submodule $H_G g_G = B_{\mathcal{H},G} g_G \subset B_{\mathcal{H}}(n)$ given by the isomorphism φ_G in Lemma 6.6 and the module structure defined above, and the slanting arrow is given in Lemma 6.9. The vertical arrow and Φ_n are epimorphisms by Lemma 6.10, and the slanting arrow is an isomorphism by Lemma 6.9, so Φ_n is an isomorphism. By the definition of Φ_n and Lemma 6.6, the collection $\{\Phi_n\}_n$ commutes with the structures of an $\mathcal{A}$–comodule and degeneracy maps. The assertion for the E_2–page immediately follows from the isomorphism of simplicial objects. □

Remark 6.12 The Euler number $\chi(M)$ can be recovered from the Poincaré algebra $\mathcal{H}^* = H^*(M)$. It is the image of $\Delta_{\mathcal{H}}$ by the composition

$$(\mathcal{H}^{\otimes 2})^{*=d} \xrightarrow{\text{multiplication}} \mathcal{H}^d \xrightarrow{\epsilon} \mathsf{k}.$$

So under the assumptions of Theorem 6.11, the E_2–page of the Čech s.s. is determined by the cohomology algebra $H^*(M)$. (Different orientations give apparently different presentations, but they are isomorphic.)

7 Examples

In this section, we compute some of the E_2–page of Čech s.s. for the spheres and products of two spheres $S^k \times S^l$ with $(k,l) = (\text{odd}, \text{even})$ or $(\text{even}, \text{even})$, and deduce some results on cohomology groups for the products of spheres. We also prove Corollary 1.3. Our computation is restricted to low degrees and consists of only elementary linear algebra on differentials and degree argument based on Theorem 6.11. We briefly state the results for the cases of spheres since, in these cases, the Čech s.s. only gives less information than the combination of Vassiliev's (or Sinha's) spectral sequence for long knots and the Serre spectral sequence for a fibration $\mathrm{Emb}(S^1, S^d) \to STS^d$ (see the proof of Proposition 7.2), at least in the degrees where we have computed. We give concrete descriptions of the differentials in the case of $M = S^k \times S^l$ with k odd and l even. In the rest of this section, we set $\mathcal{H} = H^*(M)$ for a fixed orientation.

7.1 The case of $M = S^d$ with d odd

In this case $A_\bullet^{\star *}(\mathcal{H})$ is described as

$$A_n^{\star *}(\mathcal{H}) = \bigwedge\{x_i, y_i, g_{ij} \mid 0 \le i, j \le n\}/\mathcal{I},$$

where $|x_i| = (0,d)$, $|y_i| = (0, d-1)$, $|g_{ij}| = (-1, d)$ and $\mathcal{I}$ is the ideal generated by

$$(x_i)^2, (y_i)^2, (g_{ij})^2, g_{ii}, g_{ij} + g_{ji}, (x_i - x_j)g_{ij} \text{ and the 3–term relation for } g_{ij}.$$

The diagonal class is given by $\Delta_{\mathcal{H}} = x_0 - x_1 \in \mathcal{H} \otimes \mathcal{H}$.

Proposition 7.1 *Consider the Čech s.s. $\check{\mathbb{E}}_r^{pq}$ for the sphere S^d with odd $d \ge 5$. We abbreviate $\check{\mathbb{E}}_2^{pq}$ as (p,q). The following equalities hold*:

$$(-3, d) = \mathsf{k}\langle g_{12}\rangle, \quad (-1, d-1) = \mathsf{k}\langle y_1\rangle, \quad (0, d-1) = \mathsf{k}\langle y_0\rangle, \quad (0, d) = \mathsf{k}\langle x_0\rangle,$$

$$(-6,2d)=\mathsf{k}\langle g_{13}g_{24}, -g_{12}g_{34}+g_{14}g_{23}\rangle, \quad (-4,2d-1)=\mathsf{k}\langle y_1g_{23}-y_2g_{13}+y_3g_{12}\rangle,$$
$$(-5,2d)=\mathsf{k}\langle g_{01}g_{23}+g_{02}g_{13}+g_{13}g_{23}\rangle, \quad (-3,2d-1)=\mathsf{k}\langle y_0g_{12}\rangle,$$
$$(-3,2d)=\mathsf{k}\langle x_0g_{12}\rangle, \quad (-1,2d-1)=\mathsf{k}\langle x_0y_1, x_1y_0, x_1y_1\rangle, \quad (0,2d-1)=\mathsf{k}\langle x_0y_0\rangle.$$

For other (p,q) with $p+q\le 2d-1$, we have $(p,q)=0$. □

Proposition 7.2 *Let d be an odd number with $d\ge 5$.*

1. $\mathrm{Emb}(S^1,S^d)$ *is $(d-2)$–connected.*
2. *The Čech s.s. for S^d does not collapse at the E_2–page in any coefficient ring.*

Proof For (1), consider the fiber sequence

$$\mathrm{Emb}_c(\mathbb{R},\mathbb{R}^d)\to \mathrm{Emb}(S^1,S^d)\to STS^d,$$

where STS^d is the tangent sphere bundle of S^d, the left map is given by taking the tangent vector at a fixed point, and the right space is the space of long knots. As is well known, STS^d is $(d-2)$–connected and $\mathrm{Emb}_c(\mathbb{R},\mathbb{R}^d)$ is $(2d-7)$–connected. As $d\ge 5$, we have the claim. Part (2) follows from (1) and Proposition 7.1. (There are nonzero elements in the total degrees $d-3$ and $d-2$.) □

Remark 7.3 The reader may find inconsistency between [8, Proposition 3.9(3)] and Proposition 7.2(1). This is just a typo; $n-j-2$ should be replaced with $n-j-1$ (and $n-j-1$ with $n-j$) in the former proposition (see its proof).

7.2 The case of $M=S^d$ with d even

In this subsection, we assume $2\in\mathsf{k}^\times$. $B^{\star *}_\bullet(\mathcal{H})$ is described as

$$B^{\star *}_n(\mathcal{H})=\bigwedge\{z_i, g_{ij}, h_{ij}\mid 0\le i,j\le n\}/\mathcal{J},$$

where $|z_i|=(0,2d-1)$, $|g_{ij}|=(-1,d)$, $|h_{ij}|=(-1,2d-1)$ and $\mathcal{J}$ is the ideal generated by

$$(z_i)^2, (g_{ij})^2, (h_{ij})^2, g_{ii}, h_{ii}, g_{ij}-g_{ji}, h_{ij}+h_{ji}, (z_i-z_j)g_{ij}, (z_i-z_j)h_{ij}, (h_{ij}+h_{ki})g_{jk}-(h_{ij}+h_{jk})g_{ki},$$

and the 3–term relation for g_{ij} and h_{ij}. The diagonal classes are given by $\Delta_{\mathcal{H}}=0\in S\mathcal{H}\otimes S\mathcal{H}$ and $\Delta_{S\mathcal{H}}=z_0-z_1\in S\mathcal{H}\otimes S\mathcal{H}$.

Proposition 7.4 *Suppose $2\in\mathsf{k}^\times$. Consider the Čech s.s. $\check{\mathbb{E}}_r^{pq}$ for S^d with even $d\ge 4$. We abbreviate $\check{\mathbb{E}}_2^{pq}$ as (p,q). The following equalities hold*:

$$(-6,2d)=\mathsf{k}\langle g_{13}g_{24}\rangle, \qquad (-5,2d)=\mathsf{k}\langle g_{01}g_{23}+3g_{02}g_{13}+g_{03}g_{12}\rangle,$$
$$(-3,2d-1)=\mathsf{k}\langle h_{12}\rangle, \qquad (0,2d-1)=\mathsf{k}\langle z_0\rangle.$$

For other (p,q) with $p+q\le 2d-1$, we have $(p,q)=0$. □

For the case of $\mathsf{k}=\mathbb{F}_2$, the same statement as in Proposition 7.1 holds, except that "odd $d\ge 5$" is replaced with "even $d\ge 4$".

7.3 The case of $M = S^k \times S^l$ with k odd and l even

We fix generators $a \in H^k(S^k)$ and $b \in H^l(S^l)$. $\mathcal{H}$ is presented as $\wedge\{a,b\}$. We fix an orientation ϵ on $\mathcal{H}$ by $\epsilon(ab)=1$. We write a_i for $e_i(a)$ and b_i for $e_i(b)$, and $A_n(\mathcal{H})$ is presented as

$$A_n(\mathcal{H}) = \bigwedge\{a_i, b_i, y_i, g_{ij} \mid 0 \le i, j \le n\}/\mathcal{I},$$

where $|y_i| = (0, k+l-1)$, $|g_{ij}| = (-1, k+l)$ and $\mathcal{I}$ is the ideal generated by

$$(a_i)^2, (b_i)^2, (y_i)^2, (g_{ij})^2, g_{ii}, g_{ij}+g_{ji}, (a_i-a_j)g_{ij}, (b_i-b_j)g_{ij} \text{ and the 3–term relation for } g_{ij}.$$

The diagonal class is given by $\Delta_{\mathcal{H}} = a_0b_0 - a_1b_0 + a_0b_1 - a_1b_1 \in \mathcal{H}\otimes\mathcal{H}$. The module $NA_n(\mathcal{H})$ is generated by the monomials of the form $a_{p_1}\cdots a_{p_s}b_{q_1}\cdots b_{q_t}g_{i_1j_1}\cdots g_{i_rj_r}$ such that the set of subscripts $\{p_1,\dots,p_s,q_1,\dots,q_t,i_1,\dots,i_r,j_1,\dots,j_r\}$ contains the set $\{1,\dots,n\}$.

We shall present the total differential $\tilde{d}$ on

$$\check{\mathbb{E}}_1^{pq} = \bigoplus_{\star-\bullet=p} NA_\bullet^{\star,q}(\mathcal{H})$$

up to $p+q \le \max\{2k+l, k+2l\}$. For $(p,q) = (-1,k)$, $(-1,l)$, $(-1,k+l-1)$, $(-1,k+l)$, $(-1,2k)$, $(-1,2l)$, $(-1,2k+l)$, $(-1,k+2l)$, $(-1,2k+l-1)$, $(-1,k+2l-1)$, $(-2,2k)$, $(-2,2l)$, $(-2,3k)$ or $(-2,3l)$, $\tilde{d}$ is zero.

For $(p,q) = (-3,k+l)$, $\tilde{d}$ is presented by the following matrix

	g_{12}
g_{01}	0
a_1b_2	1
a_2b_1	-1

This is read as $\tilde{d}(g_{12}) = a_1b_2 - a_2b_1$. For $(p,q) = (-2,k+l)$,

	g_{01}	a_1b_2	a_2b_1
a_0b_1	1	1	1
a_1b_0	-1	1	1
a_1b_1	-1	-1	-1

For $(p,q) = (-4, 2k+l)$,

	a_1g_{23}	a_2g_{13}	a_3g_{12}
a_0g_{12}	1	0	-1
a_1g_{02}	1	1	0
a_1g_{12}	-1	0	1
a_2g_{01}	0	1	1
$a_1a_2b_3$	-1	1	0
$a_1a_3b_2$	1	0	1
$a_2a_3b_1$	0	1	-1

For $(p,q)=(-3,2k+l)$,

	a_0g_{12}	a_1g_{02}	a_1g_{12}	a_2g_{01}	$a_1a_2b_3$	$a_1a_3b_2$	$a_2a_3b_1$
a_0g_{01}	0	0	0	0	0	0	0
$a_0a_1b_2$	−1	1	0	0	−1	−1	0
$a_0a_2b_1$	1	0	0	1	0	−1	−1
$a_1a_2b_0$	0	1	0	−1	1	0	−1
$a_1a_2b_1$	0	0	1	−1	0	1	1
$a_1a_2b_2$	0	1	1	0	−1	−1	0

For $(p,q)=(-2,2k+l)$,

	a_0g_{01}	$a_0a_1b_2$	$a_0a_2b_1$	$a_1a_2b_0$	$a_1a_2b_1$	$a_1a_2b_2$
$a_0a_1b_0$	1	−1	−1	0	−1	1
$a_0a_1b_1$	1	1	1	0	1	−1

For $(p,q)=(-2,2k+l-1)$,

	a_1y_2	a_2y_1
a_0y_1	1	1
a_1y_0	1	1
a_1y_1	−1	−1

For $(p,q)=(-4,k+2l)$,

	b_1g_{23}	b_2g_{13}	b_3g_{12}
b_0g_{12}	−1	0	1
b_1g_{02}	−1	−1	0
b_1g_{12}	1	0	−1
b_2g_{01}	0	−1	−1
$a_1b_2b_3$	0	1	1
$a_2b_1b_3$	1	0	−1
$a_3b_1b_2$	−1	−1	0

For $(p,q)=(-3,k+2l)$,

	b_0g_{12}	b_1g_{02}	b_1g_{12}	b_2g_{01}	$a_1b_2b_3$	$a_2b_1b_3$	$a_3b_1b_2$
b_0g_{01}	0	0	0	0	0	0	0
$a_0b_1b_2$	0	1	0	1	1	0	−1
$a_1b_0b_2$	1	0	0	−1	−1	1	0
$a_1b_1b_2$	0	0	1	−1	−1	−1	0
$a_2b_0b_1$	−1	−1	0	0	0	−1	1
$a_2b_1b_2$	0	−1	−1	0	0	1	1

For $(p,q)=(-2,k+2l)$,

	b_0g_{01}	$a_0b_1b_2$	$a_1b_0b_2$	$a_1b_1b_2$	$a_2b_0b_1$	$a_2b_1b_2$
$a_0b_0b_1$	1	2	1	1	1	1
$a_1b_0b_1$	−1	0	−1	1	−1	1

For $(p,q)=(-2,k+2l-1)$,

	b_1y_2	b_2y_1
b_0y_1	-1	-1
b_1y_0	1	1
b_1y_1	1	1

By direct computation based on the above presentation we obtain the following result. Let k_2 (resp. k^2) denote the module $\mathsf{k}/2\mathsf{k}$ (resp. $\mathsf{k}\oplus\mathsf{k}$).

Proposition 7.5 *Suppose k is either of $\mathbb{Z}$ or $\mathbb{F}_{\mathfrak{p}}$ where $\mathfrak{p}$ is prime. Let k be an odd number and l be an even numbers with $k+5\le l\le 2k-4$ and $|3k-2l|\ge 2$, or $l+7\le k\le 2l-7$ and $|3l-2k|\ge 2$. We abbreviate $\check{\mathbb{E}}_2^{pq}$ for $S^k\times S^l$ as (p,q). We have the following isomorphisms*:

$$(0,k)=(-1,k)=(0,l)=(-1,l)=(-1,2k)=(-2,2k)=(-1,2l)=(-2,2l)=\mathsf{k}$$

$$(-2,3k)=(-3,3k)=(-2,3l)=(-3,3l)=(0,k+l-1)=(-1,k+l-1)=\mathsf{k},$$

$$(0,k+l)=\mathsf{k},\qquad (-1,k+l)=\mathsf{k}\oplus\mathsf{k}_2 \text{ or } \mathsf{k}^2,\qquad (-2,k+l)=0 \text{ or } \mathsf{k},$$

$$(0,2k+l-1)=\mathsf{k},\qquad (-1,2k+l-1)=\mathsf{k}^2,\qquad (-2,2k+l-1)=\mathsf{k},$$

$$(-1,2k+l)=\mathsf{k}_2 \text{ or } \mathsf{k},\qquad (-2,2k+l)=\mathsf{k}_2 \text{ or } \mathsf{k}^2,\qquad (-3,2k+l)=\mathsf{k}_2 \text{ or } \mathsf{k}^2,$$

$$(-4,2k+l)=0 \text{ or } \mathsf{k},\qquad (0,k+2l-1)=\mathsf{k},\qquad (-1,k+2l-1)=\mathsf{k}^2,$$

$$(-2,k+2l-1)=\mathsf{k},\qquad (-1,k+2l)=\mathsf{k}_2 \text{ or } \mathsf{k},\qquad (-2,k+2l)=\mathsf{k} \text{ or } \mathsf{k}^2,$$

$$(-3,k+2l)=\mathsf{k}^2,\qquad (-4,k+2l)=\mathsf{k}.$$

Here "$(p,q)=A$ or B" means $(p,q)=A$ if $\mathsf{k}=\mathbb{Z}$ or $\mathbb{F}_{\mathfrak{p}}$ with $\mathfrak{p}\neq 2$ and $(p,q)=B$ if $\mathsf{k}=\mathbb{F}_2$. For other (p,q) with $p+q\le\max\{k+2l,2k+l\}$ we have $(p,q)=0$. □

The isomorphisms of Proposition 7.5 hold under milder conditions on k and l. It suffices to ensure the bidegrees presented above are pairwise distinct. By degree argument, we obtain the following corollary:

Corollary 7.6 *Suppose k is either $\mathbb{Z}$ or $\mathbb{F}_{\mathfrak{p}}$ where $\mathfrak{p}$ is a prime. Let k be an odd number and l be an even number with $k+5\le l\le 2k-4$ and $|3k-2l|\ge 2$, or $l+7\le k\le 2l-7$ and $|3l-2k|\ge 2$. We set $H^*=H^*(\mathrm{Emb}(S^1,S^k\times S^l))$.*

(1) *We have isomorphisms*

$$H^i=\mathsf{k}\quad\text{for } i=k-1,k,2k-2,2k-1,l-1,l,2l-2,2l-1,k+l.$$

(2) *If $\mathsf{k}=\mathbb{F}_{\mathfrak{p}}$ with $\mathfrak{p}\neq 2$, we have isomorphisms*

$$H^i=\begin{cases}\mathsf{k} & \text{if } i=k+l-2,2k+l-3,2k+l-1,\\ \mathsf{k}^2 & \text{if } i=k+l-1,2k+l-2,\\ 0 & \text{if } i=2k+l-4.\end{cases}$$

(3) *If* $\mathsf{k} = \mathbb{Z}$, *we have isomorphisms*

$$H^i = \begin{cases} \mathsf{k} & \text{if } i = k+l-2, \\ \mathsf{k}^2 \oplus \mathsf{k}_2 & \text{if } i = k+l-1, 2k+l-2, \\ \mathsf{k} \oplus \mathsf{k}_2 & \text{if } i = 2k+l-3, \\ 0 & \text{if } i = 2k+l-4. \end{cases}$$

(4) *We have* $H^i = 0$ *for an integer* i *that satisfies* $i \leq \max\{k+2l, 2k+l\}$ *and is different from any of the following integers*:

$$k-1, k, l-1, l, 2k-2, 2k-1, 2l-2, 2l-1, 3k-3, 3k-2, 3l-3, 3l-2, k+l-2, k+l-1,$$
$$k+l, 2k+l-4, 2k+l-3, 2k+l-2, 2k+l-1, k+2l-4, k+2l-3, k+2l-2, k+2l-1.$$

Proof By an argument similar to the proof of Theorem 5.17, $\check{\mathbb{E}}_2^{-p,q} = 0$ if $q/p < \frac{1}{3}(k+l)$. We shall show that any differential $d_r : \check{\mathbb{E}}_r^{(-p-r,q+r-1)} \to \check{\mathbb{E}}_r^{-p,q}$ going into the term contained in the cohomology of the claim is zero. It is enough to show this for the case of $(-p,q) = (0, 2k+l-1)$ and $q+r-1 \geq k+2l-1$ since other cases are obvious, or follow from this case. We see

$$\frac{q+r-1}{p+r} = \frac{q-1}{r} + 1 \leq \frac{2k+l-2}{l-k+1} + 1 = \frac{k+2l-1}{l-k+1} < \tfrac{1}{3}(k+l).$$

So $\mathbb{E}_r^{(-p-r,q+r-1)} = 0$ and $d_r = 0$. □

7.4 The case of $M = S^k \times S^l$ with k, l even

We fix generators $a \in H^k(S^k)$ and $b \in H^l(S^l)$. $\mathcal{H}$ is presented as $\wedge\{a,b\}$. We fix an orientation ϵ on $\mathcal{H}$ by $\epsilon(ab) = 1$. We set $c = \bar{a} \in S\mathcal{H}$ and $d = \bar{b} \in S\mathcal{H}$. We write a_i for $e_i(a)$, b_i for $e_i(b)$, etc, and $B_n(\mathcal{H})$ is presented as

$$B_n(\mathcal{H}) = \bigwedge\{a_i, b_i, c_i, d_i, g_{ij}, h_{ij} \mid 0 \leq i, j \leq n\}/\mathcal{J}$$

where $|g_{ij}| = (-1, k+l)$, $|h_{ij}| = (-1, 2(k+l)-1)$ and $\mathcal{J}$ is the ideal generated by

$$(a_i)^2, (b_i)^2, (c_i)^2, (d_i)^2, a_ib_i, a_ic_i, b_id_i, c_id_i, a_id_i - b_ic_i (g_{ij})^2, (h_{ij})^2, g_{ii}, h_{ii}, g_{ij} - g_{ji}, h_{ij} + h_{ji},$$
$$(a_i - a_j)g_{ij}, (b_i - b_j)g_{ij}, (c_i - c_j)g_{ij} - a_ih_{ij}, (d_i - d_j)g_{ij} - b_ih_{ij}, (a_i - a_j)h_{ij}, (b_i - b_j)h_{ij}, (c_i - c_j)h_{ij},$$
$$(d_i - d_j)h_{ij}, (h_{ij} + h_{ik})g_{jk} - (h_{ij} + h_{jk})g_{ki} \text{ and the 3–term relations for } g_{ij} \text{ and } h_{ij}.$$

The diagonal classes are given by

$$\Delta_{\mathcal{H}} = a_0b_1 + a_1b_0 \in S\mathcal{H} \otimes S\mathcal{H} \quad \text{and} \quad \Delta_{S\mathcal{H}} = a_0d_0 + a_1d_0 + b_1c_0 - b_0c_1 - a_0d_1 - a_1d_1.$$

By an argument similar to the proof of Corollary 7.6, we obtain the following corollary:

Corollary 7.7 *Suppose* $2 \in \mathsf{k}^\times$. *Let* k *and* l *be two even numbers with* $k+2 \leq l \leq 2k-2$ *and* $|3k-2l| \geq 2$. *We set* $H^* = H^*(\mathrm{Emb}(S^1, S^k \times S^l))$. *We have isomorphisms*

$$H^i = \mathsf{k} \quad \text{for } i = k-1, k, l-1, l, k+l-3, k+l-2, k+l-1, 3k.$$

For any other degree $i \leq 2k+l$, *we have* $H^i = 0$. □

7.5 The case of 4–dimensional manifolds

In this subsection, we prove Corollary 1.3. We assume that M is a simply connected 4–dimensional manifold. So, as is easily observed, $\mathcal{H}$ is a free k–module for any k.

Definition 7.8 Set $\chi = \chi(M)$. We define a map $\alpha\colon (\mathcal{H}^2)^{\otimes 2} \oplus \mathsf{k}g_{01} \to (\mathcal{H}^2)^{\otimes 2} \oplus \mathcal{H}^4/\chi\mathcal{H}^4$ by

$$\alpha(a \otimes b) = (-a \otimes b - b \otimes a) + ab, \quad \alpha(g_{01}) = \mathrm{pr}_1(\Delta_{\mathcal{H}}).$$

Here g_{01} is a formal free generator (which will correspond to the element of the same symbol in $\check{\mathbb{E}}_1^{-2,4}$) and pr_1 is the projection

$$(\mathcal{H}^{\otimes 2})^{*=4} \to (\mathcal{H}^2)^{\otimes 2} \oplus (1 \otimes \mathcal{H}^4) \to (\mathcal{H}^2)^{\otimes 2} \oplus \mathcal{H}^4/\chi\mathcal{H}^4.$$

The next proposition follows from direct computation and degree argument based on Theorem 6.11.

Lemma 7.9 *We use the notation in Definition 7.8. Suppose k is a field and $\mathcal{H}^2$ is not zero.*

(1) *When $p+q=1$, $\check{\mathbb{E}}_r^{p,q}$ is stationary after E_2. In particular, $\check{\mathbb{E}}_2^{p,q} \cong \check{\mathbb{E}}_\infty^{p,q}$. We have isomorphisms*

$$\check{\mathbb{E}}_2^{p,q} \cong \begin{cases} \mathcal{H}^2 & \text{if } (p,q) = (-1,2), \\ 0 & \text{otherwise.} \end{cases}$$

(2) *There exists an isomorphism*

$$\check{\mathbb{E}}_2^{-2,4} \cong \mathrm{Ker}(\alpha)/\mathsf{k}(\mathrm{pr}_2(\Delta_{\mathcal{H}}) + 2g_{01}).$$

Here pr_2 is the projection $(\mathcal{H}^{\otimes 2})^{*=4} \to (\mathcal{H}^2)^{\otimes 2}$. The differential d_r coming into this term is zero for $r \geq 2$. □

Remark 7.10 Actually, Lemma 7.9 holds even when k is a not a field since torsion in the Künneth theorem does not affect the range.

Proof of Corollary 1.3 In this proof, we suppose k is a field. Set $H_2^{\mathbb{Z}} = H_2(M;\mathbb{Z})$. As is well known, there is a weak homotopy equivalence between $\mathrm{Imm}(S^1, M)$ and the free loop space LSM, and there is an isomorphism $\pi_1(LSM) \cong \pi_1(SM) \oplus \pi_2(SM)$. As M is simply connected, we have $\pi_1\,\mathrm{Imm}(S^1, M) \cong \pi_2(SM) \cong \pi_2(M) \cong H_2^{\mathbb{Z}}$.

By the Goodwillie–Weiss convergence theorem, connectivity of the standard projection $\mathrm{holim}_\Delta\, \mathcal{C}^\bullet(M) \to \mathrm{holim}_{\Delta_n}\, \mathcal{C}^\bullet(M)$ increases as n increases. Since Δ_n is a compact category in the sense of [13] and $\mathcal{C}^n(M)$ is simply connected for any n, by [13, Theorem 2.2] we see that $\mathrm{Emb}(S^1, M)$ is $\mathbb{Z}$–complete. In particular, $\pi_1(\mathrm{Emb}(S^1, M))$ is a pro-nilpotent group. So, by a theorem of Stallings [38], we only have to prove that the composition

$$\mathrm{Emb}(S^1, M) \xrightarrow{i_M} \mathrm{Imm}(S^1, M) \xrightarrow{\simeq} LSM \xrightarrow{\mathrm{cl}_1} K(H_2^{\mathbb{Z}}, 1)$$

induces an isomorphism on $H_1(-;\mathbb{Z})$ and a surjection on $H_2(-;\mathbb{Z})$. Here the rightmost map cl_1 is the classifying map; see [15].

Consider the spectral sequence $E_r^{p,q}$ associated to the Hochschild complex of $C_*(\widetilde{\mathcal{T}}_M)$. This spectral sequence is isomorphic to the Bousfield–Kan type cohomology spectral sequence associated to the well-known cosimplicial model for LSM given by $[n] \mapsto SM^{n+1}$. The quotient map $\widetilde{\mathcal{T}}_M \to \mathcal{T}_M$ induces a map $f_r \colon E_r^{pq} \to \check{\mathbb{E}}_r^{pq}$ of spectral sequences. For $r = \infty$, this map is identified with the map on the associated graded induced by the inclusion i_M. For $p+q=1$, by Lemma 7.9 (and similar computation for E_r^{pq}), f_2 is an isomorphism for any field k. Since $\pi_1(\mathrm{Emb}(S^1, M))$ is the same as π_1 of a finite stage of Taylor tower which is the finite homotopy limit of a simply connected finite cell complex, it is finitely generated, and so is H_1. By the universal coefficient theorem, i_M induces an isomorphism on $H_1(-;\mathbb{Z})$. For the part of $p+q=2$, we see $E_2^{pq}=0$ for $p<-2$ and $E^{-2,4} \cong \mathrm{Ker}(\alpha) \cap (\mathcal{H}^2)^{\otimes 2}$. Consider the zigzag

$$LSM \xrightarrow{L(\mathrm{cl}_2)} LK(H_2^{\mathbb{Z}}, 2) \xleftarrow{i_K} \Omega K(H_2^{\mathbb{Z}}, 2),$$

where the left map is induced by the classifying map $\mathrm{cl}_2 \colon SM \to K(H_2^{\mathbb{Z}}, 2)$ and the right one is the inclusion from the based loop space. Clearly the composition $\mathrm{cl}_1 \circ i_K \colon \Omega K(H_2^{\mathbb{Z}}, 2) \to K(H_2^{\mathbb{Z}}, 1)$ is a weak homotopy equivalence. Observe spectral sequences associated to the standard cosimplicial models of the above three spaces. Since the maps $L(\mathrm{cl}_2)$ and i_K are induced by cosimplicial maps, they induce maps on spectral sequences. In the part of total degree 2, we see that the filtration level F^{-2} for each of the three spectral sequences is the entire cohomology group, and the filtration level F^{-1} for the one for $\Omega K(H_2^{\mathbb{Z}}, 2)$ is zero. With these observations, we see that the image of $H^2(K(H_2^{\mathbb{Z}}, 1))$ in $H^2(LSM)$ by the map cl_1 is sent to a subspace V of $F^{-2}/F^{-1} \cong E_\infty^{-2,4} \subset E_2^{-2,4}$ isomorphically, and a basis of V is given by $\{a_i \otimes a_j - a_j \otimes a_i \mid i<j\}$ as elements of $E_2^{-2,4}$, where $\{a_i\}_i$ denotes a basis of $\mathcal{H}^2$. (We also see that these elements must be stationary.) If $\mathsf{k} \neq \mathbb{F}_2$, or if $\mathsf{k} = \mathbb{F}_2$ and the inverse of the intersection matrix has at least one nonzero diagonal component, the restriction of f_2 to V is a monomorphism by Lemmas 6.2 and 7.9. (Otherwise, the elements of the basis of V have the relation $\mathrm{pr}_0(\Delta_{\mathcal{H}}) = 0$.) This implies i_M induces a surjection on H_2 for any field k under the assumption of the theorem. By the universal coefficient theorem, we obtain the desired assertion on $H_2(-;\mathbb{Z})$. $\square$

Remark 7.11 If all of the diagonal components of the inverse of the intersection matrix on $H_2(M;\mathbb{F}_2)$ are zero, the map $f_2 \colon V \to \check{\mathbb{E}}_2^{-2,4}$ in the proof is not a monomorphism for $\mathsf{k} = \mathbb{F}_2$, but this does not necessarily imply the original (nonassociated graded) map is not a monomorphism. So in this case, it is still unclear whether i_M is an isomorphism on π_1.

8 Precise statement and proof of Theorem 1.5

Definition 8.1 • Fix a coordinate plane with coordinates (x, y). A *planar rooted n–tree* $(T, \mathfrak{e})$ consists of a 1–dimensional finite cell complex T and a continuous monomorphism $\mathfrak{e}$ from its realization $|T|$ to the half plane $y \geq 0$ such that:

- T is connected and $\pi_1(T)$ is trivial.

– The intersection of the image of $\mathfrak{e}$ and the x–axis consists of the image of n univalent vertices called *leaves*. These vertices are labeled by $1, \dots, n$ in the manner consistent with the standard order on the axis.
– T has a unique distinguished vertex, called the *root*, which is at least bivalent.
– Any vertex except for the leaves and root is at least trivalent.

An *isotopy* between n–trees $(T_1, \mathfrak{e}_1) \to (T_2, \mathfrak{e}_2)$ is an isotopy of the half plane onto itself which maps $\mathfrak{e}_1(|T_1|)$ onto $\mathfrak{e}_2(|T_2|)$ and the root to the root. (So an isotopy preserves the leaves, including the labels.) We will denote an isotopy class of planar rooted n–trees simply by T. The root vertex of a tree is usually denoted by v_r. For a vertex v of a tree, $|v|$ denotes the number which is the valence minus 1 if $v \neq v_r$, and equal to the valence if $v = v_r$ ($|v|$ is the number of the "out-going edges").

• Let Ψ_n be a category defined as follows. An object of Ψ_n is an isotopy class of planar rooted n–trees. There is a unique morphism $T \to T'$ if T' is obtained from T by successive contractions of internal edges (ie edges not adjacent to leaves).

• Let *Cat* be the category of small categories and functors. Let $i_n \colon \Psi_n \to \Psi_{n+1}$ be a functor which sends T to the tree made from T by attaching two edges to the n^{th} leaf of T and labeling the new leaves with n and $n+1$. We define a category Ψ as the colimit of the sequence $\Psi_1 \xrightarrow{i_1} \Psi_2 \xrightarrow{i_2} \cdots$ taken in *Cat*. $\mathcal{F}_n \colon \Psi_{n+1} \to \mathrm{P}_n$ denotes the functor given in [37, Definition 4.14], which sends a tree $T \in \Psi_{n+1}$ to the set of numbers i such that the shortest paths from i and $i+1$ to the root in T intersect only at the root. For the functor $\mathcal{G}_n \colon \mathrm{P}_{n+1} \to \Delta_n$, see Section 2.1. The square

$$\begin{tikzcd} \Psi_{n+2} \arrow[r, "\mathcal{G}_{n+1}\circ\mathcal{F}_n"] \arrow[d, "i_n"] & \Delta_n \arrow[d, "i_n"] \\ \Psi_{n+3} \arrow[r, "\mathcal{G}_{n+2}\circ\mathcal{F}_{n+1}"] & \Delta_{n+1} \end{tikzcd}$$

is clearly commutative, where the right vertical arrow is the natural inclusion, so we have the induced functor $\mathcal{G} \circ \mathcal{F} \colon \Psi \to \Delta$.

• Henceforth, for a symmetric sequence X and a vertex v of a tree in Ψ, we denote $X(|v|)$, $X(|v|-1)$ and $\underline{|v|-1}$ by $X(v)$, $X(v-1)$ and $\underline{v-1}$, respectively.

• For a $\mathcal{K}$–comodule X in $\mathcal{SP}$, we shall define a functor $\mathsf{F}^n X \colon \Psi_{n+2}^{\mathrm{op}} \to \mathcal{SP}$. The definition is similar to (a dual of) the construction of $\mathcal{D}_n[M]$ in [37, Definition 5.6]. For a tree $T \in \Psi_{n+2}$, define a space $\mathcal{K}_T^{\mathrm{nr}}$ by

$$\mathcal{K}_T^{\mathrm{nr}} = \prod_v \mathcal{K}(v).$$

Here v runs through all the nonroot and nonleaf vertices of T. This is denoted by K_T^{nr} in [37]. We set

$$\mathsf{F}^n X(T) = \mathrm{Map}(\mathcal{K}_T^{\mathrm{nr}}, X(v_r - 1)).$$

For a morphism $T \to T'$ given by the contraction of a nonroot edge e (an edge not adjacent to the root), the map $e^* \colon \mathsf{F}^n X(T') \to \mathsf{F}^n X(T)$ is the pullback by the inclusion $\mathcal{K}_T^{\mathrm{nr}} \to \mathcal{K}_{T'}^{\mathrm{nr}}$ to a face corresponding to the edge contraction (see [37, Definition 4.26]). For the i^{th} root edge e, the corresponding map is given by the following composition:

$$\begin{aligned}\mathrm{Map}\Big(\prod_{v\in T'}\mathcal{K}(v), X(\underline{v'_r-1})\Big) &= \mathrm{Map}\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v), X(\underline{v'_r-1})\Big) \to \mathrm{Map}\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v), \mathrm{Map}(\mathcal{K}(v_t), X(\underline{v_r-1}))\Big)\\ &\cong \mathrm{Map}\Big(\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v)\Big)\times\mathcal{K}(v_t), X(\underline{v_r-1})\Big) = \mathrm{Map}\Big(\prod_{v\in T}\mathcal{K}(v), X(\underline{v_r-1})\Big).\end{aligned}$$

Here v_t is the vertex of e which is not the root. For $1\le i\le |v_r|-1$, the arrow is the pushforward by the adjoint of the partial composition $(-\circ_i -)\colon \mathcal{K}(v_t)\mathbin{\hat{\otimes}} X(v'_r-1)\to X(v_r-1)$, and for $i=|v_r|$ it is the pushforward by the adjoint of the composition

$$\mathcal{K}(v_t)\mathbin{\hat{\otimes}} X(v'_r-1)\xrightarrow{\mathrm{id}\otimes(-)^\sigma}\mathcal{K}(v_t)\mathbin{\hat{\otimes}} X(v'_r-1)\xrightarrow{(-\circ_1-)} X(v_r-1),$$

where σ is the transposition of the first $|v'_r|-|v_t|$ and last $|v_t|-1$ letters. The functors $\{\mathsf{F}^n\}_n$ are compatible with the inclusions $i_n\colon \Psi_{n+2}\to\Psi_{n+3}$. Precisely speaking, there exists an obviously defined natural isomorphism $j_n\colon \mathsf{F}^n X\cong \mathsf{F}^{n+1}X|_{\Psi_{n+2}}$ because the inclusion does not change $|v_r|$. We define a functor $\mathsf{F}X\colon\Psi\to\mathcal{SP}$ by $\mathsf{F}X(T)$ being the colimit of the sequence $\mathsf{F}^nX(T)\xrightarrow{\cong}\mathsf{F}^{n+1}X(T)\xrightarrow{\cong}\mathsf{F}^{n+2}X(T)\xrightarrow{\cong}\cdots$.

- We define a category $\mathsf{G}(n)^+$ for an integer $n\ge 1$ as follows. Its objects are a symbol $*$ and the graphs G with set of vertices $V(G)=\underline{n}$ and set of edges $E(G)\subset\{(i,j)\mid i,j\in\underline{n}\text{ with } i\le j\}$. There is a unique morphism $G\to H$ if and only if either both of G and H are graphs and $E(G)\subset E(H)$, or $G=*$ and $H\neq\varnothing$, where $\varnothing$ denotes the graph with no edges. As in the definition, we allow graphs in $\mathsf{G}(n)^+$ to have *loops*, ie edges of the form (i,i) for $i\in\underline{n}$.

- We define a functor $\omega\colon\Psi_{n+2}^{\mathrm{op}}\to \mathit{Cat}$ by $\omega(T)=\mathsf{G}(|v_r|-1)^+$. For the contraction $T\to T'$ of an edge e, we define a map $e^*\colon\underline{v'_r-1}\to\underline{v_r-1}$ as follows. If e is a nonroot edge, e^* is the identity. If e is the i^{th} root edge for $1\le i\le|v_r|-1$, e^* is the order-preserving surjection with $e^*(j)=i$ for $i\le j\le i+|v_t|-1$. For $i=|v_r|$, e^* is the composition

$$\underline{v'_r-1}\xrightarrow{(-)^\sigma}\underline{v'_r-1}\xrightarrow{(e')^*}\underline{v_r-1},\quad\text{where }(e')^*(j)=\begin{cases}1 & \text{if } 1\le j\le |v_t|,\\ j-|v_t|+1 & \text{if } |v_t|+1\le j\le |v'_r|-1,\end{cases}$$

and σ is the permutation given in the previous item. For $G\in\mathsf{G}(|v'_r|-1)^+$, we define an object $e^*(G)\in\mathsf{G}(|v_r|-1)^+$ by

$$e^*(G)=\begin{cases} * & \text{if } G=*,\\ \text{the graph with the edge set } \{(e^*(s),e^*(t))\mid (s,t)\in E(G)\} & \text{otherwise.}\end{cases}$$

- We define a category $\widetilde{\Psi}_{n+2}$ as the Grothendieck construction for the (nonlax) functor ω

$$\widetilde{\Psi}_{n+2}=\int_{\Psi_{n+2}}\omega.$$

An object of $\widetilde{\Psi}_{n+2}$ is a pair (T,G) with $T\in\Psi_{n+2}$ and $G\in\omega(T)$. A map $(T,G)\to(T',G')$ is a pair of maps $e\colon T\to T'\in\Psi_{n+2}$ and $G\to e^*(G')\in\omega(T)$. The functor $i_n\colon\Psi_{n+2}\to\Psi_{n+3}$ and the identity $\omega(T)=\omega(i_n(T))$ naturally induce a functor $i_n\colon\widetilde{\Psi}_{n+2}\to\widetilde{\Psi}_{n+3}$. We denote by $\widetilde{\Psi}$ the colimit of the sequence $\{\widetilde{\Psi}_{n+2};i_n\}$.

- We fix a map $\mathcal{K}\to\mathcal{D}$ of operads and regard $\widetilde{\mathcal{T}}_M$ as a $\mathcal{K}$–comodule via this map.
- We shall define a functor $\mathsf{T}^n_M\colon\widetilde{\Psi}^{\mathrm{op}}_{n+2}\to\mathcal{SP}$. We set

$$\mathsf{T}^n_M(T,G)=\begin{cases} * & \text{if G has at least one loop or } G=*,\\ \mathrm{Map}(\mathcal{K}^{\mathrm{nr}}_T,\mathcal{T}_G) & \text{otherwise.}\end{cases}$$

For a map $(T,G)\to(T',G')$, we set

$$\begin{aligned}\mathrm{Map}\Big(\prod_{v\in T'}\mathcal{K}(v),\mathcal{T}_{G'}\Big)\to\mathrm{Map}\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v),\mathrm{Map}(\mathcal{K}(v_t),\mathcal{T}_G)\Big)&\cong\mathrm{Map}\Big(\Big(\prod_{\substack{v\in T\\ v\neq v_t}}\mathcal{K}(v)\Big)\times\mathcal{K}(v_t),\mathcal{T}_G\Big)\\ &=\mathrm{Map}\Big(\prod_{v\in T}\mathcal{K}(v),\mathcal{T}_G\Big).\end{aligned}$$

Here the arrow is the adjoint of the map $\mathcal{K}(v_t)\mathbin{\hat{\otimes}}\mathcal{T}_{G'}\to\mathcal{T}_G$ which is the composition of the map $\mathcal{K}(v_t)\mathbin{\hat{\otimes}}\mathcal{T}_{G'}\to\mathcal{T}_{e^*(G')}$ defined in view of Lemma 3.11 and the inclusion $\mathcal{T}_{e^*G'}\subset\mathcal{T}_G$ coming from $G\subset e^*(G')$. The collection $\{\mathsf{T}^n_M\}_n$ naturally induces a functor $\mathsf{T}_M\colon\widetilde{\Psi}\to\mathcal{SP}$ with natural isomorphism $\mathsf{T}_M|_{\widetilde{\Psi}_{n+2}}\cong\mathsf{T}^n_M$.

- Let $\mathcal{M}$ be a model category. Let $\eta\colon\widetilde{\Psi}\to\Psi$ be the functor given by the projection $\eta(T,G)=T$. Let $\eta_!\colon\mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}},\mathcal{M})\to\mathcal{F}un(\Psi^{\mathrm{op}},\mathcal{M})$ be the left Kan extension along η, ie

$$(\eta_!X)(T)=\operatorname*{colim}_{\omega(T)}X_T$$

for $X\in\mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}},\mathcal{M})$. Here abusing notation, for $T\in\Psi$ we denote by $\omega(T)$ the full subcategory $\{(T,G)\mid G\in\omega(T)\}$ of $\widetilde{\Psi}$, and by X_T the restriction of X to $\omega(T)$. Let $\eta^*\colon\mathcal{F}un(\Psi^{\mathrm{op}},\mathcal{M})\to\mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}},\mathcal{M})$ be the pullback, ie $\eta^*(Y)=Y\circ\eta$.

Remark 8.2 The category Ψ_n is equivalent to the category Ψ^o_n given in [37, Definition 4.12].

Notation Henceforth we omit $(-)^{\mathrm{op}}$ under (ho)colim. For example, $\mathrm{hocolim}_\Psi$ denotes $\mathrm{hocolim}_{\Psi^{\mathrm{op}}}$.

In the rest of this section, as before, all functor categories are endowed with the projective model structure (see Section 2.1).

Lemma 8.3 *Let $\mathcal{M}$ be a cofibrantly generated model category.*

(1) *The pair $(\eta_!,\eta^*)$ is a Quillen adjoint pair.*

(2) *The restriction*

$$\mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}},\mathcal{M})\to\mathcal{F}un(\omega(T)^{\mathrm{op}},\mathcal{M}),\quad X\mapsto X_T,$$

preserves weak equivalences and cofibrations. In particular, the natural map $\operatorname{hocolim}_{\omega(T)} X_T \to \mathbb{L}\eta_! X(T) \in \mathbf{Ho}(\mathcal{M})$ *is an isomorphism.*

(3) *For any functor* $X \in \mathcal{F}un(\widetilde{\Psi}^{\mathrm{op}}, \mathcal{M})$, *there is a natural isomorphism in* $\mathbf{Ho}(\mathcal{M})$

$$\operatorname*{hocolim}_{\Psi} \mathbb{L}\eta_! X \cong \operatorname*{hocolim}_{\widetilde{\Psi}} X.$$

Proof Part (1) is straightforward. We shall prove (2). Let I be a set of generating cofibrations of $\mathcal{M}$. Let C be a category. For objects $a \in C$ and $A \in \mathcal{M}$, the functor sending $b \in C$ to the coproduct of copies of A labeled by morphisms from b to a is denoted by $\boldsymbol{F}^a_A \in \mathcal{F}un(C^{\mathrm{op}}, \mathcal{M})$. A set of generating cofibrations of $\mathcal{F}un(C, \mathcal{M})$ is given by

$$I_C = \{\boldsymbol{F}^a_f : \boldsymbol{F}^a_A \to \boldsymbol{F}^a_B \mid a \in C \text{ and } f : A \to B \in I\}.$$

See [20, Theorem 11.6.1] for details. Since $\omega(T)$ is a full subcategory of $\widetilde{\Psi}$, the restriction functor sends $I_{\widetilde{\Psi}}$ into $I_{\omega(T)}$. Since the restriction preserves colimits, it preserves relative cell objects with respect to these generating sets. As any cofibration is a retract of a relative cell object, we have proved (2). Part (3) follows from (2) and a standard property of colimits. □

Theorem 8.4 (1) *There exists an isomorphism in* $\mathbf{Ho}(\mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{SP}))$

$$(\mathcal{G} \circ \mathcal{F})^*(\mathcal{C}^\bullet(M)^\vee) \cong \mathbb{L}\eta_! \mathsf{T}_M.$$

(2) *If* M *is simply connected and of dimension* ≥ 4, *there exists an isomorphism in* $\mathbf{Ho}(\mathcal{CH}_{\mathsf{k}})$

$$C^*(\mathrm{Emb}(S^1, M)) \cong \operatorname*{hocolim}_{\widetilde{\Psi}} C_* \circ \mathsf{T}_M.$$

Proof Let $T \in \Psi$ be an object and set $m = |v_r| - 1$, where v_r is the root vertex of T. By definition $\mathcal{T}_M(m) = \operatorname{colim}_{G \in \omega(T)} \mathcal{T}_G$. We shall show that the natural map

$$\operatorname*{hocolim}_{G \in \omega(T)} \mathcal{T}_G \to \operatorname*{colim}_{G \in \omega(T)} \mathcal{T}_G = \mathcal{T}_M(m) \in \mathbf{Ho}(\mathcal{SP})$$

is an isomorphism. Put $N_1 = \#\{(i, j) \mid i, j \in \underline{m} \text{ with } i \leq j\}$. By abuse of notation, we denote by P_{N_1} the subcategory of $\omega(T)$ consisting of nonempty graphs, which is actually isomorphic to P_{N_1}. The functor $\mathrm{P}_{N_1}^{\mathrm{op}} \ni G \mapsto \mathcal{T}_G \in \mathcal{SP}$ satisfies the assumption of Lemma 2.2(2), so the natural map $\operatorname{hocolim}_{\mathrm{P}_{N_1}} \mathcal{T}_G \to \operatorname{colim}_{\mathrm{P}_{N_1}} \mathcal{T}_G$ is an isomorphism. More precisely, for each k, $\mathrm{P}_{N_1}^{\mathrm{op}} \ni G \mapsto (\mathcal{T}_G)_k \in \mathcal{CG}_*$ satisfies the assumption for $\mathcal{M} = \mathcal{CG}_*$. Since a trivial fibration in $\mathcal{SP}$ is a level equivalence and a finite homotopy colimit is obtained by successive applications of a homotopy pushout, the finite homotopy colimit of a diagram of semistable connective spectra is π_*–isomorphic to the levelwise homotopy colimit. As $\mathcal{T}_M(m)$ is a cofiber of the natural map $\operatorname{colim}_{\mathrm{P}_{N_1}} \mathcal{T}_G \to \widetilde{\mathcal{T}}_M$, which is also a (levelwise) homotopy cofiber, we have the assertion. We define a natural transformation $\mathsf{T}_M \to \eta^* \circ \mathsf{F}(\mathcal{T}_M)$ by the pushforward by the constant map $\mathcal{T}_G \to \{*\} \subset \mathcal{T}_M(m)$ for $G \neq \varnothing \in \omega(T)$, and by the quotient map $\mathcal{T}_\varnothing \to \mathcal{T}_M(m)$ for $G = \varnothing$. By the assertion and Lemma 8.3(2), the derived adjoint of the natural transformation $\mathbb{L}\eta_! \mathsf{T}_M \to \mathsf{F}\mathcal{T}_M$ is an isomorphism in $\mathbf{Ho}(\mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{SP}))$. It is clear that F preserves weak equivalences, so by Theorem 4.4 we have isomorphisms in $\mathbf{Ho}(\mathcal{F}un(\Psi^{\mathrm{op}}, \mathcal{SP}))$

$$\mathsf{F}(\mathcal{C}_M^\vee) \cong \mathsf{F}\mathcal{T}_M \cong \mathbb{L}\eta_! \mathsf{T}_M.$$

We define a natural transformation $(\mathcal{G} \circ \mathcal{F})^*(\mathcal{C}^\bullet(M)^\vee) \to \mathsf{F}(\mathcal{C}_M^\vee)$ by the inclusion $\mathcal{C}^{m-1}(M) = \mathcal{C}_M(m) \subset \mathrm{Map}(\mathcal{K}_T^{\mathrm{nr}}, \mathcal{C}_M(m))$ onto constant maps. This is clearly a weak equivalence, so we have proved (1).

For (2), since the functor $C_*\colon \mathcal{SP} \to \mathcal{CH}_{\mathsf{k}}$ preserves homotopy colimits (of semistable spectra), by (1), Lemma 8.3(3) and Lemma 5.3, we have isomorphisms in $\mathbf{Ho}(\mathcal{CH}_{\mathsf{k}})$

$$\operatorname*{hocolim}_{\Psi} (\mathcal{G} \circ \mathcal{F})^* C_*(\mathcal{C}^\bullet(M)^\vee) \cong \operatorname*{hocolim}_{\Psi} \mathbb{L}\eta_! C_* \circ \mathsf{T}_M \cong \operatorname*{hocolim}_{\widetilde{\Psi}} C_* \circ \mathsf{T}_M.$$

By Lemma 5.3, Theorem 5.17 and the fact that $\mathcal{G} \circ \mathcal{F}\colon \Psi^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is (homotopy) right cofinal (see Proposition 4.15 and Theorem 6.7 of [37]), we have isomorphisms in $\mathbf{Ho}(\mathcal{CH}_{\mathsf{k}})$

$$C^*(\mathrm{Emb}(S^1, M)) \cong \operatorname*{hocolim}_{\Delta} C^*(\mathcal{C}^\bullet(M)) \cong \operatorname*{hocolim}_{\Delta} C_*(\mathcal{C}^\bullet(M)^\vee) \cong \operatorname*{hocolim}_{\Psi} (\mathcal{G} \circ \mathcal{F})^* C_*(\mathcal{C}^\bullet(M)^\vee).$$

Thus, we have an isomorphism $C^*(\mathrm{Emb}(S^1, M)) \cong \operatorname{hocolim}_{\widetilde{\Psi}} C_* \circ \mathsf{T}_M$. □

References

[1] **G Arone**, **M Szymik**, *Spaces of knotted circles and exotic smooth structures*, Canad. J. Math. 74 (2022) 1–23 MR Zbl

[2] **G Arone**, **V Turchin**, *On the rational homology of high-dimensional analogues of spaces of long knots*, Geom. Topol. 18 (2014) 1261–1322 MR Zbl

[3] **M Bendersky**, **S Gitler**, *The cohomology of certain function spaces*, Trans. Amer. Math. Soc. 326 (1991) 423–440 MR Zbl

[4] **A K Bousfield**, *On the homology spectral sequence of a cosimplicial space*, Amer. J. Math. 109 (1987) 361–394 MR Zbl

[5] **P Boavida de Brito**, **G Horel**, *Galois symmetries of knot spaces*, Compos. Math. 157 (2021) 997–1021 MR Zbl

[6] **P Boavida de Brito**, **M Weiss**, *Manifold calculus and homotopy sheaves*, Homology Homotopy Appl. 15 (2013) 361–383 MR Zbl

[7] **W Browder**, *Surgery on simply-connected manifolds*, Ergebnisse der Math. 65, Springer (1972) MR Zbl

[8] **R Budney**, *A family of embedding spaces*, from "Groups, homotopy and configuration spaces" (N Iwase, T Kohno, R Levi, D Tamaki, J Wu, editors), Geom. Topol. Monogr. 13, Geom. Topol. Publ., Coventry (2008) 41–83 MR Zbl

[9] **R Budney**, **D Gabai**, *Knotted* 3*–balls in* S^4, preprint (2019) arXiv 1912.09029

[10] **R Campos**, **T Willwacher**, *A model for configuration spaces of points*, Algebr. Geom. Topol. 23 (2023) 2029–2106 MR Zbl

[11] **R L Cohen**, *Multiplicative properties of Atiyah duality*, Homology Homotopy Appl. 6 (2004) 269–281 MR Zbl

[12] **R L Cohen**, **J D S Jones**, *A homotopy theoretic realization of string topology*, Math. Ann. 324 (2002) 773–798 MR Zbl

[13] **E D Farjoun**, *Bousfield–Kan completion of homotopy limits*, Topology 42 (2003) 1083–1099 MR Zbl

[14] **Y Felix**, **J-C Thomas**, *Configuration spaces and Massey products*, Int. Math. Res. Not. 2004 (2004) 1685–1702 MR Zbl

[15] **S M Gersten**, *The Whitehead theorem for nilpotent spaces*, Proc. Amer. Math. Soc. 47 (1975) 259–260 MR Zbl

[16] **E Getzler**, **J D S Jones**, *A_∞–algebras and the cyclic bar complex*, Illinois J. Math. 34 (1990) 256–283 MR Zbl

[17] **T G Goodwillie**, **J R Klein**, *Multiple disjunction for spaces of smooth embeddings*, J. Topol. 8 (2015) 651–674 MR Zbl

[18] **T G Goodwillie**, **M Weiss**, *Embeddings from the point of view of immersion theory, II*, Geom. Topol. 3 (1999) 103–118 MR Zbl

[19] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[20] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[21] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[22] **N Idrissi**, *The Lambrechts–Stanley model of configuration spaces*, Invent. Math. 216 (2019) 1–68 MR Zbl

[23] **D Kosanović**, *On homotopy groups of spaces of embeddings of an arc or a circle: the Dax invariant*, preprint (2021) arXiv 2111.03041

[24] **I Kříž**, **J P May**, *Operads, algebras, modules and motives*, Astérisque 233, Soc. Math. France, Paris (1995) MR Zbl

[25] **P Lambrechts**, **V Turchin**, **I Volić**, *The rational homology of spaces of long knots in codimension* > 2, Geom. Topol. 14 (2010) 2151–2187 MR Zbl

[26] **J-L Loday**, **B Vallette**, *Algebraic operads*, Grundl. Math. Wissen. 346, Springer (2012) MR Zbl

[27] **I Madsen**, **J r Tornehave**, *From calculus to cohomology: de Rham cohomology and characteristic classes*, Cambridge Univ. Press (1997) MR Zbl

[28] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. London Math. Soc. 82 (2001) 441–512 MR Zbl

[29] **M Markl**, *Operads and PROPs*, from "Handbook of algebra, V" (M Hazewinkel, editor), Elsevier, Amsterdam (2008) 87–140 MR Zbl

[30] **S Moriya**, *On Cohen–Jones isomorphism in string topology*, preprint (2021) arXiv 2003.03704v2

[31] **F Muro**, *Homotopy theory of nonsymmetric operads*, Algebr. Geom. Topol. 11 (2011) 1541–1599 MR Zbl

[32] **L Positselski**, *Two kinds of derived categories, Koszul duality, and comodule–contramodule correspondence*, Mem. Amer. Math. Soc. 996, Amer. Math. Soc., Providence, RI (2011) MR Zbl

[33] **B Richter**, **B Shipley**, *An algebraic model for commutative $H\mathbb{Z}$–algebras*, Algebr. Geom. Topol. 17 (2017) 2013–2038 MR Zbl

[34] **S Schwede**, *On the homotopy groups of symmetric spectra*, Geom. Topol. 12 (2008) 1313–1344 MR Zbl

[35] **D P Sinha**, *Manifold-theoretic compactifications of configuration spaces*, Selecta Math. 10 (2004) 391–428 MR Zbl

[36] **D P Sinha**, *Operads and knot spaces*, J. Amer. Math. Soc. 19 (2006) 461–486 MR Zbl

[37] **D P Sinha**, *The topology of spaces of knots: cosimplicial models*, Amer. J. Math. 131 (2009) 945–980 MR Zbl

[38] **J Stallings**, *Homology and central series of groups*, J. Algebra 2 (1965) 170–181 MR Zbl

[39] **B Totaro**, *Configuration spaces of algebraic varieties*, Topology 35 (1996) 1057–1067 MR Zbl

[40] **V A Vassiliev**, *On invariants and homology of spaces of knots in arbitrary manifolds*, from "Topics in quantum groups and finite-type invariants" (B Feigin, V Vassiliev, editors), Amer. Math. Soc. Transl. Ser. 2 185, Amer. Math. Soc., Providence, RI (1998) 155–182 MR Zbl

[41] **M Weiss**, *Embeddings from the point of view of immersion theory, I*, Geom. Topol. 3 (1999) 67–101 MR Zbl

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