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Towards a higher-dimensional construction of stable/unstable Lagrangian laminations

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We generalize some properties of surface automorphisms of pseudo-Anosov type. First, we generalize the Penner construction of a pseudo-Anosov homeomorphism, and show that if a symplectic automorphism is constructed by our generalized Penner construction, then it has an invariant Lagrangian branched submanifold and an invariant Lagrangian lamination. These invariants are higher-dimensional generalizations of a train track and a geodesic lamination in the surface case. As an application, we compute the Lagrangian Floer homology of some Lagrangians on plumbings of cotangent bundles of spheres.

53D05, 53D40, 57R17

1 Introduction

By the Nielsen–Thurston classification of surface diffeomorphisms, an automorphism $\psi\colon S \xrightarrow{\sim} S$ of a compact oriented surface S is of one of three types: periodic, reducible, or pseudo-Anosov. We recommend Casson and Bleiler [2] or Thurston [14]. Maher [7] shows that, for a suitable notion of randomness, a random element of the mapping class group is pseudo-Anosov.

Let us assume that ψ is of pseudo-Anosov type. For any closed curve $C \subset S$, it is known that there is a sequence $\{L_m\}_{m\in\mathbb{N}}$ of closed geodesics such that L_m is isotopic to $\psi^m(C)$ for all $m \in \mathbb{N}$, and $\{L_m\}_{m\in\mathbb{N}}$ converges to a closed subset $\mathscr{L}$ with respect to the Hausdorff metric on closed subsets. Moreover, $\mathscr{L}$ is a geodesic lamination. The definitions of a lamination, a geodesic lamination, and a Lagrangian lamination are the following:

Definition 1.1 (1) A *k–dimensional lamination* on an n–dimensional manifold M is a decomposition of a closed subset of M into k–dimensional submanifolds called *leaves* such that the closed subset is covered by charts of the form $I^k \times I^{n-k}$ where a leaf passing through a chart is a slice of the form $I^k \times \{\mathrm{pt}\}$.

(2) An 1–dimensional lamination $\mathscr{L}$ on a Riemannian 2–manifold (S, g) is a *geodesic lamination* if every leaf of $\mathscr{L}$ is geodesic.

(3) An n–dimensional lamination $\mathscr{L}$ on a symplectic manifold (M^{2n}, ω) is a *Lagrangian lamination* if every leaf of $\mathscr{L}$ is a Lagrangian submanifold.

For more details, we refer the reader to Farb and Margalit [5, Chapter 15].

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In [3], Dimitrov, Haiden, Katzarkov and Kontsevich defined the notion of a *pseudo-Anosov functor* of a triangulated category, and they gave examples of it on the Fukaya categories: a pseudo-Anosov map ψ on a compact oriented surface S induces a functor, also called ψ, on the derived Fukaya category $D^\pi \operatorname{Fuk}(S,\omega)$, where ω is an area form of S. In [3], the authors showed that ψ is a pseudo-Anosov functor.

In [3, Section 4], the authors listed a number of open questions. One of them is to find a symplectic automorphism ψ on a symplectic manifold M of dimension greater than 2 which has invariant transversal stable/unstable Lagrangian measured foliations. A slightly weaker version of the question is to define a symplectic automorphism ψ with invariant stable/unstable Lagrangian laminations.

The goal of the present paper is to answer the latter question. First, we define symplectic automorphisms of generalized Penner type.

Definition 1.2 Let M be a symplectic manifold. A symplectic automorphism $\psi\colon M \xrightarrow{\sim} M$ is of *generalized Penner type* if there are two collections $A=\{\alpha_1,\dots,\alpha_m\}$ and $B=\{\beta_1,\dots,\beta_l\}$ of Lagrangian spheres satisfying

- $\alpha_i \cap \alpha_j = \varnothing$, and $\beta_i \cap \beta_j = \varnothing$ for all $i \neq j$,
- $\alpha_i \pitchfork \beta_j$ for all i and j, and
- for each $\alpha_i \in A$ (resp. $\beta_j \in B$), there is at least one $\beta_j \in B$ (resp. $\alpha_i \in A$) such that $\alpha_i \cap \beta_j \neq \varnothing$,

so that ψ is a product of positive powers of Dehn twists τ_i along α_i and negative powers of Dehn twists σ_j along β_j, subject to the condition that every sphere appear in the product.

We will define a Dehn twist along a Lagrangian sphere in Section 2.2, partly to establish notation.

Then, we will define the notion of *Lagrangian branched submanifold* and *carried by*. These are higher-dimensional generalizations of the notion of train tracks and "carried by a train track" in surface theory. Roughly, in the surface theory, if a curve C is carried by a train track τ, then it is possible to encode C on τ with the extra data called "weights". We refer the reader to Farb and Margalit [5] for detail. Motivated by this, we will give the higher-dimensional generalizations of train tracks and the notion of "carried by" in Sections 3.1 and 3.3. Then, we prove Theorem 1.3 at the end of Section 3.

Theorem 1.3 *Let M be a symplectic manifold and let $\psi\colon M \xrightarrow{\sim} M$ be a symplectic automorphism of generalized Penner type. Then there exists a Lagrangian branched submanifold $\mathcal{B}_\psi$ such that if L is a Lagrangian submanifold which is carried* (*resp. weakly carried*) *by $\mathcal{B}_\psi$, then $\psi^m(L)$ is carried* (*resp. weakly carried*) *by $\mathcal{B}_\psi$ for all $m \in \mathbb{N}$.*

Remark 1.4 Theorem 1.3 cares about symplectic automorphisms of generalized Penner type. However, there should be a generalized version of Theorem 1.3 for arbitrary symplectic automorphisms, which we do not prove in the current paper.

In Section 4, we will prove that if a Lagrangian L is carried by a Lagrangian branched submanifold $\mathcal{B}$, one can encode L on $\mathcal{B}$ with extra data called *braids*. The definition of braids will appear in Section 4.3. In Sections 5 and 6, by using the notion of braids, we prove our main theorem, ie Theorem 1.5.

Theorem 1.5 *Let M be a symplectic manifold, and let $\psi: M \xrightarrow{\sim} M$ be a symplectic automorphism of generalized Penner type. Then there is a Lagrangian lamination $\mathcal{L}$ such that if L is a Lagrangian submanifold of M which is carried by $\mathcal{B}_\psi$, then there is a sequence of Lagrangian submanifolds L_m satisfying*

- *L_m is Hamiltonian isotopic to $\psi^m(L)$, and*
- *L_m converges to $\mathcal{L}$ as $m \to \infty$.*

Also, in Section 6.4, we will see how this generalizes to symplectic automorphisms which are not of generalized Penner type.

Finally, we will talk about Lagrangian Floer theory related to Theorems 1.3 and 1.5. The results will be written in Section 7.

Structure of the paper

This paper consists of 7 sections. In Section 2, we review plumbing spaces and generalized Dehn twists. We will prove Theorem 1.3 in Section 3 and Theorem 1.5 in Sections 4–6. In Section 7, we will discuss the relation of Theorems 1.3 and 1.5 to Lagrangian Floer theory and give a related calculation of Floer cohomology (Theorem 7.3).

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2 Preliminaries

In this section, we will review plumbings of cotangent bundles and generalized Dehn twists, partly to establish notation.

2.1 Plumbing spaces

Let α and β be oriented spheres S^n. We describe how to plumb $T^*\alpha$ and $T^*\beta$ at $p \in \alpha$ and $q \in \beta$. Let $U \subset \alpha$ and $V \subset \beta$ be small disk neighborhoods of p and q. Then, we identify T^*U and T^*V so that the base U (resp. V) of T^*U (resp. T^*V) is identified with a fiber of T^*V (resp. T^*U).

To do this rigorously, we fix coordinate charts $\psi_1\colon U \to \mathbb{R}^n$ and $\psi_2\colon V \to \mathbb{R}^n$. Then we obtain a compositions of symplectomorphisms

$$T^*U \xrightarrow{(\psi_1^*)^{-1}} T^*\mathbb{R}^n \simeq \mathbb{R}^{2n} \xrightarrow{f} \mathbb{R}^{2n} \simeq T^*\mathbb{R}^n \xrightarrow{\psi_2^*} T^*V,$$

where $f(x_1,\dots,x_n,y_1,\dots,y_n) = (y_1,\dots,y_n,-x_1,\dots,-x_n)$.

A plumbing space $P(\alpha,\beta)$ of $T^*\alpha$ and $T^*\beta$ is defined by $T^*\alpha \sqcup T^*\beta/\sim$, where $x \sim (\psi_2^* \circ f \circ \psi_1^{*-1})(x)$ for all $x \in T^*U$. Since $\psi_2^* \circ f \circ \psi_1^{*-1}$ is a symplectomorphism, $P(\alpha,\beta)$ has a natural symplectic structure induced by the standard symplectic structures of cotangent bundles.

Since the plumbing procedure is a local procedure, we can plumb a finite collection of cotangent bundles of the same dimension at finitely many points. For convenience, we plumb cotangent bundles of oriented manifolds.

Note that we can replace f by

$$g(x_1,\dots,x_n,y_1,\dots,y_n) = (-y_1,y_2,\dots,y_n,x_1,-x_2,\dots,-x_n).$$

If we plumb $T^*\alpha$ and $T^*\beta$ at one point using g, this plumbing space is symplectomorphic to the previous plumbing space $P(\alpha,\beta)$, which is plumbed using f. However, if we plumb at more than one point, then by replacing f with g at a plumbing point, the plumbing space will change.

Definition 2.1 Let $\alpha_1,\dots,\alpha_m$ be oriented manifolds of dimension n.

(1) A *plumbing datum* is a collection of pairs of nonnegative integers $(a_{i,j}, b_{i,j})$ for all $1 \le i \le j \le m$ and collections of distinct points

$$\{p_k^{i,j} \in \alpha_i \mid 1 \le i \le j \le m, 1 \le k \le a_{i,j} + b_{i,j}\},$$
$$\{q_k^{i,j} \in \alpha_j \mid 1 \le i \le j \le m, 1 \le k \le a_{i,j} + b_{i,j}\}.$$

(2) A *plumbing space* $P(\alpha_1,\dots,\alpha_m)$, with the given plumbing datum, is given by

$$P(\alpha_1,\dots,\alpha_m) = T^*\alpha_1 \sqcup \cdots \sqcup T^*\alpha_m/\sim,$$

where the equivalence relation $\sim$ is defined as follows: first, choose small disk neighborhoods $U_k^{i,j} \subset \alpha_i$ of $p_k^{i,j}$ and $V_k^{i,j} \subset \alpha_j$ of $q_k^{i,j}$ such that $U_{k_1}^{i_1,j_1} \cap U_{k_2}^{i_2,j_2} = \varnothing$ if $(i_1,j_1,k_1) \neq (i_2,j_2,k_2)$ and orientation-preserving coordinate charts $\psi_k^{i,j} : U_k^{i,j} \xrightarrow{\sim} \mathbb{R}^n$ and $\phi_k^{i,j} : V_k^{i,j} \xrightarrow{\sim} \mathbb{R}^n$; then for all $x \in T^*U_k^{i,j}$,

$$x \sim \begin{cases} (\phi_k^{i,j*} \circ f \circ (\psi_k^{i,j*})^{-1})(x) & \text{if } 1 \le k \le a_{i,j}, \\ (\phi_k^{i,j*} \circ g \circ (\psi_k^{i,j*})^{-1})(x) & \text{if } a_{i,j}+1 \le k \le a_{i,j}+b_{i,j}. \end{cases}$$

(3) A *plumbing point* is an identified point $p_k^{i,j} \sim q_k^{i,j} \in P(\alpha_1,\dots,\alpha_m)$.

Figure 1 shows some examples of plumbing spaces.

If α_i is of dimension $n \ge 2$, then specific choices of plumbing points do not change the symplectic topology of $P(\alpha_1,\dots,\alpha_m)$.

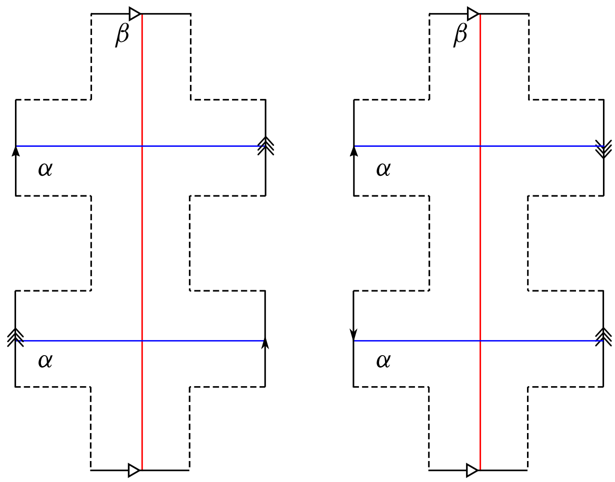

Figure 1: $P(\alpha \simeq S^1, \beta \simeq S^1)$ with plumbing datum $(2,0)$ (left) and $(1,1)$ (right).

2.2 Generalized Dehn twist

Let

$$T^*S^n = \{(u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|u\| = 1, \langle u, v\rangle = 0\},$$
$$S^n = \{(u, 0_{n+1}) \in T^*S^n\},$$

where $(u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $\langle u, v\rangle$ is the standard inner product of u and v in $\mathbb{R}^{n+1}$. Moreover, let 0_k be the origin in $\mathbb{R}^k$.

We fix a Hamiltonian function $\mu(u,v) = \|v\|$ on $T^*S^n \setminus S^n$. Then μ induces a circle action on $T^*S^n \setminus S^n$ given by

$$\sigma(e^{it})(u,v) = \Big(\Big(\cos(t)u + \sin(t)\frac{v}{\|v\|}\Big), (\cos(t)v - \sin(t)\|v\|u)\Big).$$

Let $r\colon [0,\infty) \to \mathbb{R}$ be a smooth decreasing function such that $r(0) = \pi$ and $r(t) = 0$ for all $t \geq \epsilon$ for a small positive number ϵ. If ω_0 is the standard symplectic form of T^*S^n, we define a symplectic automorphism $\tau\colon (T^*S^n, \omega_0) \xrightarrow{\sim} (T^*S^n, \omega_0)$ by

$$\tau(u,v) = \begin{cases} \sigma(e^{ir(\mu(u,v))})(u,v) & \text{if } v \neq 0_{n+1}, \\ (-u, 0_{n+1}) & \text{if } v = 0_{n+1}. \end{cases} \tag{2-1}$$

Let (M^{2n}, ω) be a symplectic manifold and let $L \simeq S^n$ be a Lagrangian sphere in M. By the Lagrangian neighborhood theorem — see Weinstein [16] — there is a neighborhood $N(L) \supset L$ and a symplectomorphism $\phi\colon T^*S^n \xrightarrow{\sim} N(L)$. We define a generalized Dehn twist τ_L along L by

$$\tau_L(x) = \begin{cases} (\phi \circ \tau \circ \phi^{-1})(x) & \text{if } x \in N(L), \\ x & \text{if } x \notin N(L). \end{cases} \tag{2-2}$$

Note that the support of τ_L is contained in $N(L)$. From now on, a generalized Dehn twist will just be called a Dehn twist.

Remark 2.2 We will use two specific Dehn twists $\tau, \tilde{\tau} : T^*S^n \xrightarrow{\simeq} T^*S^n$ which are defined by (2-1) and two functions $r, \tilde{r} : [0, \infty) \to \mathbb{R}$. The function r (resp. $\tilde{r}$) defining τ (resp. $\tilde{\tau}$) satisfies the above conditions in addition to $r(t) = \pi$ for all $t \le \frac{\epsilon}{2}$ (resp. $\tilde{r}'(0) < 0$). The two Dehn twists τ and $\tilde{\tau}$ are equivalent in the sense that $\tau \circ \tilde{\tau}^{-1}$ is a Hamiltonian isotopy.

Dehn twists have been studied extensively by Seidel. For example, Seidel [12] proved the following theorem.

Theorem 2.3 *Let α be a Lagrangian sphere and β be a Lagrangian submanifold of a symplectic manifold M. If α and β intersect transversally at only one point, $\beta \# \alpha$ is Lagrangian isotopic to $\tau_\alpha(\beta)$, where $\beta \# \alpha$ is a Lagrangian surgery of β and α.*

We prove Theorem 2.3 in the special case that β is also a sphere and $M = P(\alpha, \beta)$, as an illustration of the "spinning" procedure.

To define "spinning", we use the following notation. Let $y \in S^{n-1} \subset \mathbb{R}^n$. Then

$$\psi_y : T^*S^1 \simeq S^1 \times \mathbb{R} \to T^*S^n,$$
$$((\cos\theta, \sin\theta), t) \mapsto \big((\cos\theta(0_n, 1) + \sin\theta(y, 0)), (t\cos\theta(y, 0) - t\sin\theta(0_n, 1))\big)$$

is a symplectic embedding. Let W_y be the embedded symplectic surface $\psi_y(T^*S^1)$. We would like to note that $W_y = W_{-y}$.

Definition 2.4 Given a curve C in T^*S^1, its *spun image* $S(C)$ is $\bigcup_{y \in S^{n-1}} \psi_y(C)$.

Remark 2.5 A spun image $S(C)$ of a curve $C \subset T^*S^1$ is not an embedded submanifold of T^*S^n for all C. However, for some C, $S(C)$ is an embedded submanifold. For example, if C is invariant under the action $(\theta, t) \mapsto (-\theta, -t)$ on T^*S^1, then $S(C)$ is an embedded submanifold. Moreover, if $S(C)$ is a submanifold, then it is easy to prove that $S(C)$ is Lagrangian.

Proof of Theorem 2.3 We use $T^*\alpha$ and $T^*\beta$ to indicate neighborhoods of α and β inside $M = P(\alpha, \beta)$. Let p be the intersection point of α and β. Then, $T^*_p\alpha = \beta \cap T^*\alpha$. The closure of $T^*_p\alpha$ is denoted by D^-_p; we use D to indicate that this is a disk and the subscript p means that p is the center of D^-_p. The meaning of the negative sign in D^-_p will be explained in the next section. Since τ_α is supported on $T^*\alpha$,

$$\tau_\alpha(\beta) = \tau_\alpha(\beta \cap T^*\alpha) \cup \tau_\alpha(\beta \setminus T^*\alpha) = \tau_\alpha(D^-_p) \cup (\beta \setminus T^*\alpha).$$

There exists $\phi : T^*S^n \xrightarrow{\simeq} T^*\alpha$ such that $\tau_\alpha = \phi \circ \tau \circ \phi^{-1}$. Without loss of generality, $\phi((0_n, 1), 0_{n+1}) = p$ and

$$D^-_p = \phi\big(\{(0_n, 1, ty, 0) \mid t \in \mathbb{R}, y \in S^{n-1} \subset \mathbb{R}^n\}\big).$$

Then

$$(\phi \circ \tau_\alpha \circ \phi^{-1})(D^-_p) = (\phi \circ \tau)\big(\{(0_n, 1, ty, 0) \mid t \in \mathbb{R}, y \in S^{n-1} \subset \mathbb{R}^n\}\big)$$
$$= \bigcup_{y \in S^{n-1}} \phi\big(\{\tau(0_n, 1, ty, 0) \mid t \in \mathbb{R}\}\big).$$

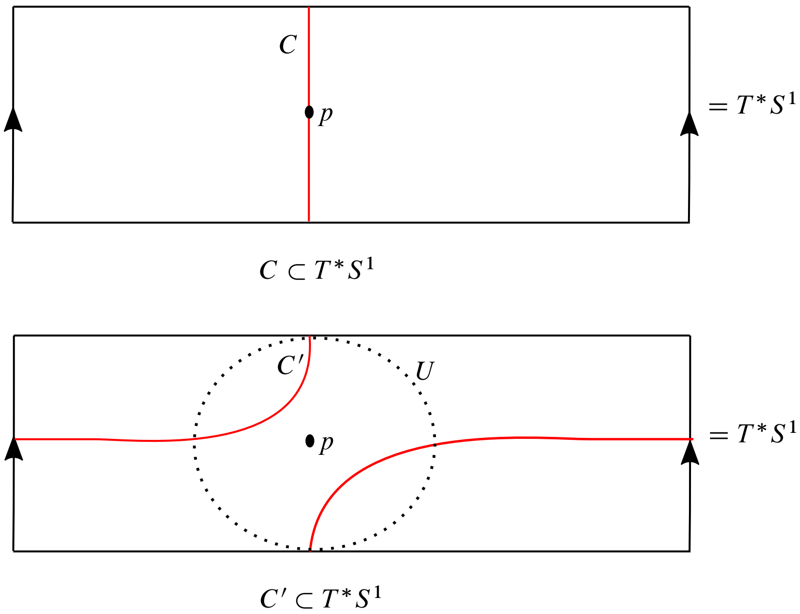

Figure 2: Curves C and C' in T^*S^1.

Since there is a curve C in T^*S^1 such that $\psi_y(C) = \{\tau(0_n, 1, ty, 0) \mid t \in \mathbb{R}\}$, $\tau_\alpha(D_p^-)$ is given by spinning with respect to p and ϕ.

Figure 2 represents $C \subset T^*S^1$. By Remark 2.5, it is easy to check that $\tau_\alpha(D_p^-)$ is Lagrangian.

One possible construction of $\beta \# \alpha$ is as follows. The Lagrangian surgery $\beta \# \alpha$ agrees with $\alpha \cup \beta$ outside of a small neighborhood U of p. On U, there is a Darboux chart ϕ satisfying

$$\phi(U) = \mathbb{C}^n, \quad \phi(\alpha) = \mathbb{R}^n, \quad \phi(\beta) = (i\mathbb{R})^n,$$
$$\phi(\beta \# \alpha) = \left\{ \left(x_1, \dots, x_n, -\frac{\epsilon x_1}{\sqrt{x_1^2 + \dots + x_n^2}}, \dots, -\frac{\epsilon x_n}{\sqrt{x_1^2 + \dots + x_n^2}} \right) \,\middle|\, x_i \in \mathbb{R} \right\}.$$

We refer the reader to Auroux [1]. Based on this construction, one could say that $\beta \# \alpha$ can be obtained by spinning a curve $C' \subset T^*S^1$ at p. Figure 2, bottom, represents $C' \subset T^*S^1$.

Similarly, we can construct a Lagrangian isotopy connecting $\tau_\alpha(\beta)$ and $\beta \# \alpha$ by spinning. □

3 Lagrangian branched submanifolds

In Section 3.1, we will define Lagrangian branched submanifolds. In Section 3.2, we will introduce a construction of a fibered neighborhood of a Lagrangian branched submanifold. In Section 3.3, we will defined the notion of "carried by" by using a fibered neighborhood. In Section 3.4, we will introduce the generalized Penner construction. Finally, we will give a proof of Theorem 1.3 in Section 3.5.

3.1 Lagrangian branched submanifolds

Thurston [15] used train tracks, which are 1–dimensional branched submanifolds of surfaces, and defined the notion of "carried by a train track". In this subsection, we generalize train tracks.

The generalization of a train track is an n–dimensional branched submanifold of a $2n$–dimensional manifold. We define the n–dimensional branched submanifolds with local models, as Floyd and Oertel defined a branched surface in a 3–dimensional manifold in [6; 9]. For our definition, we need a smooth function $s\colon \mathbb{R} \to \mathbb{R}$ such that $s(t) = 0$ if $t \le 0$ and $s(t) > 0$ if $t > 0$.

Definition 3.1 Let M^{2n} be a smooth manifold.

(1) A subset $\mathcal{B} \subset M$ is an *n–dimensional branched submanifold* if for every $p \in \mathcal{B}$, there exists a chart $\phi_p\colon U_p \xrightarrow{\sim} \mathbb{R}^{2n}$ about p such that $\phi_p(p) = 0$ and $\phi_p(\mathcal{B} \cap U_p)$ is a union of submanifolds $L_0, L_1, \dots, L_k$ for some $k \in \{0, \dots, n\}$, where

$$L_i := \{(x_1, \dots, x_n, s(x_1), s(x_2), \dots, s(x_i), 0, \dots, 0) \in \mathbb{R}^{2n} \mid x_j \in \mathbb{R}\}. \tag{3-1}$$

(2) A *sector* of $\mathcal{B}$ is a connected component of the set of all points in $\mathcal{B}$ that are locally modeled by L_0, ie $k = 0$.

(3) The *branch locus* $\mathrm{Locus}(\mathcal{B})$ of $\mathcal{B}$ is the complement of all the sectors.

(4) Let (M^{2n}, ω) be a symplectic manifold. A subset $\mathcal{B} \subset M$ is a *Lagrangian branched submanifold* if for every $p \in \mathcal{B}$, there exists a Darboux chart $\phi_p\colon (U_p, \omega|_{U_p}) \xrightarrow{\sim} (\mathbb{R}^{2n}, \omega_0)$ about p, satisfying that $\phi_p(\mathcal{B} \cap U_p)$ is a union of submanifolds $L_0, L_1, \dots, L_k$ for some $k \in \{0, \dots, n\}$ where L_i is defined in (3-1).

Remark 3.2 (1) At every point p of a branched submanifold $\mathcal{B}$, the tangent plane $T_p\mathcal{B}$ is well defined. Moreover, if $\mathcal{B}$ is Lagrangian, then $T_p\mathcal{B}$ is a Lagrangian subspace of T_pM.

(2) A point on the branch locus is (a smooth version of) an arboreal singularity in the sense of Nadler [8].

Example 3.3 (1) Every Lagrangian submanifold L is a Lagrangian branched submanifold. The branch locus $\mathrm{Locus}(L)$ is empty.

(2) Every train track of a surface equipped with an area form is a Lagrangian branched submanifold.

(3) Let (M, ω) be a symplectic manifold and let L_1 and L_2 be two Lagrangian submanifold of M such that

$$L_1 \pitchfork L_2 = L_1 \cap L_2 = \{p\}.$$

The Lagrangian surgery of L_1 and L_2 at p will be denoted by $L_2 \#_p L_1$. Then, $L_2 \#_p L_1 \cup L_1$ and $L_2 \#_p L_1 \cup L_2$ are examples of Lagrangian branched submanifolds.

In Section 3.3, we will define the notion of "carried by" which appears in Theorems 1.3 and 6.6. In order to define the notion of carried by, we will construct a fibered neighborhood first in Section 3.2.

3.2 Construction of fibered neighborhoods

Let $\mathscr{B}$ be a Lagrangian branched submanifold. A fibered neighborhood $N(\mathscr{B})$ of $\mathscr{B}$ is, roughly speaking, a codimension zero compact submanifold with boundary and corners of M, which is foliated by Lagrangian closed disks which are called *fibers*.

Definition 3.4 A *fibered neighborhood of* $\mathscr{B}$ is a union $\bigcup_{p\in\mathscr{B}} F_p$, where $\{F_p \mid p \in \mathscr{B}\}$ is a family of Lagrangian disks which are called *fibers* satisfying

(1) for any $p \in \mathscr{B}$, $F_p \pitchfork \mathscr{B}$,

(2) for any $p, q \in \mathscr{B}$, either $F_p = F_q$ or $F_p \cap F_q = \varnothing$,

(3) there exists a closed neighborhood $U \subset \mathscr{B}$ of $\mathrm{Locus}(\mathscr{B})$, such that $\{F_p \mid p \in U\}$ is a smooth family over each local sheet $L_i \cap U$,

(4) for each sector S of $\mathscr{B}$, $\{F_p \mid p \in S \setminus U\}$ is a smooth family,

(5) if $p \in S \cap \partial U$ where S is a sector of $\mathscr{B}$, then, for any sequence $\{q_n \in S \setminus U\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} F_{q_n}$ is a Lagrangian disk such that $\lim_{n\to\infty} F_{q_n} \subset F_p^{\circ} = F_p \setminus \partial F_p$.

Example 3.5 Let M be a symplectic manifold and let L be a Lagrangian submanifold of M. Then L is a Lagrangian branched submanifold of M. By the Lagrangian neighborhood theorem [16], for any Lagrangian submanifold L of M, there exists a small neighborhood $\mathcal{N}(L)$ of the zero section of T^*L such that a symplectic embedding $i_L : \mathcal{N}(L) \hookrightarrow M$ is defined on $\mathcal{N}(L)$. Without loss of generality, we assume that $\mathcal{N}(L)$ is a closed neighborhood. Then $\mathcal{N}(L)$ is foliated by closed Lagrangian disks $\mathcal{N}(L) \cap T_p^*L$. Thus, $\mathcal{N}(L)$ is a fibered neighborhood of L.

We will now give a specific construction of a fibered neighborhood $N(\mathscr{B})$. The rough sketch of the construction is as follows. If $p \in \mathscr{B}$ lies on a sector S of $\mathscr{B}$, by Example 3.5, there is a natural embedding $i_S : \mathcal{N}(S) \hookrightarrow M$. Then $i_s(\mathcal{N}(S) \cap T^*pS) \pitchfork \mathscr{B}$. Thus, it is natural to set $F_p := i_s(\mathcal{N}(S) \cap T^*pS) \pitchfork \mathscr{B}$. However, if one sets as above, the odds are that there are $p, q \in \mathscr{B} \setminus \mathrm{Locus}(\mathscr{B})$ near $\mathrm{Locus}(\mathscr{B})$ such that $F_p \cap F_q \neq \varnothing$, but $F_p \neq F_q$. See Figure 3 representing the case of $\dim M = 2$.

To handle this issue, we classify $p \in \mathscr{B}$ into three cases: "near the branch locus", "far from the branch locus", and "between the other two". Then, we construct a fiber F_p for p in each case.

Fibrations over near the branch locus First, we will construct fibers near the branch locus. For each connected component ℓ of $\mathrm{Locus}(\mathscr{B})$, we choose a small closed Lagrangian neighborhood L_ℓ of ℓ satisfying the following. Fix a Riemannian metric g or an almost complex structure J compatible with ω. Then, one can define a normal bundle for every Lagrangian submanifold. We choose any Lagrangian L_ℓ containing ℓ such that for any $x \in \ell$, $(T_x\phi_x^{-1}(L_i))^{\perp} \pitchfork T_x L_\ell$ for all i. Note that ϕ_x and L_i appeared in Definition 3.1.

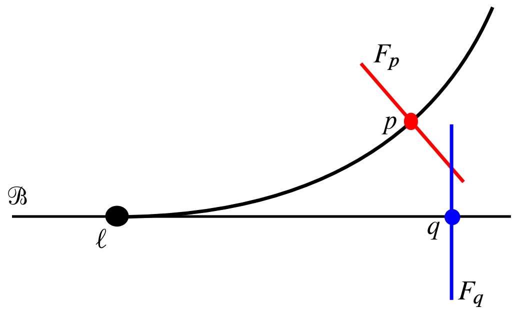

Figure 3: Black curves are part of a Lagrangian branched submanifold $\mathscr{B}$, the black point is a connected component ℓ of Locus($\mathscr{B}$), the red and blue points are $p, q \in \mathscr{B}$, and the red and blue lines are F_p and F_q.

Then, by Example 3.5, there exists a symplectic embedding

$$i_{L_\ell} : \mathcal{N}(L_\ell) \hookrightarrow M.$$

Let $U(\ell) = i_{L_\ell}(\mathcal{N}(\mathscr{L}_\ell))$.

Without loss of generality, we can choose a sufficiently small L_ℓ such that

$$\begin{aligned} i_{L_\ell}(\mathcal{N}(L_\ell) \cap T_x^* L_\ell) \cap \mathscr{B} &\neq \varnothing && \text{for all } x \in L_\ell, \\ i_{L_\ell}(\mathcal{N}(L_\ell) \cap T_x^* L_\ell) &\pitchfork \mathscr{B} && \text{for all } x \in L_\ell, \\ U(\ell) \cap U(\ell') &= \varnothing && \text{if } \ell \neq \ell'. \end{aligned}$$

If $p \in \mathscr{B}$ is "close" to the branch locus, ie there is a connected component ℓ of Locus($\mathscr{B}$) such that $p \in \mathscr{B} \cap U(\ell)$, then there exists $x \in L_\ell$ such that $p \in i_{L_\ell}(\mathcal{N}(L_\ell) \cap T_x^* L_\ell)$. Let $F_p := i_{L_\ell}(\mathcal{N}(L_\ell) \cap T_x^* L_\ell)$. Then F_p is a closed Lagrangian disk containing p.

By choosing a sufficiently small L_ℓ, for every $p \in \mathscr{B} \cap U(\ell)$,

$$F_p \pitchfork \mathscr{B} \quad \text{and} \quad \partial F_p \cap \mathscr{B} = \varnothing. \tag{3-2}$$

After possibly renaming $U(\ell)$, from now we assume that

$$U(\ell) = \bigcup_{p \in L_\ell} F_p.$$

If $p \in \mathscr{B} \cap U(\ell)$, then there is a unique $q \in L_\ell$ such that $p \in F_q$. We define $F_p := F_q$. Thus, for $p \in \mathscr{B}$ which is close to Locus($\mathscr{B}$), ie $p \in U(\ell)$ for some connected component ℓ of Locus($\mathscr{B}$), we can define a fiber F_p at p.

Fibrations far from the branch locus If $p \in \mathscr{B} \setminus \bigcup_\ell U(\ell)$, then there is a sector S of $\mathscr{B}$ containing p. Since S is Lagrangian, there is an embedding $i_S : \mathcal{N}(S) \hookrightarrow M$. We can assume $\mathcal{N}(S)$ is small enough that

$$\begin{aligned} F_q \cap i_S(\mathcal{N}(S)) \subset {F_q}^\circ = F_q \setminus \partial F_q && \text{for any } q \in \mathscr{B} \cap U(\ell), \\ \big(i_S(\mathcal{N}(S)) \setminus \cup U(\ell)\big) \cap \big(i_{S'}(\mathcal{N}(S')) \setminus \cup U(\ell)\big) = \varnothing && \text{if } S \neq S'. \end{aligned}$$

Figure 4, bottom right, represents examples of $\mathcal{N}(S)$.

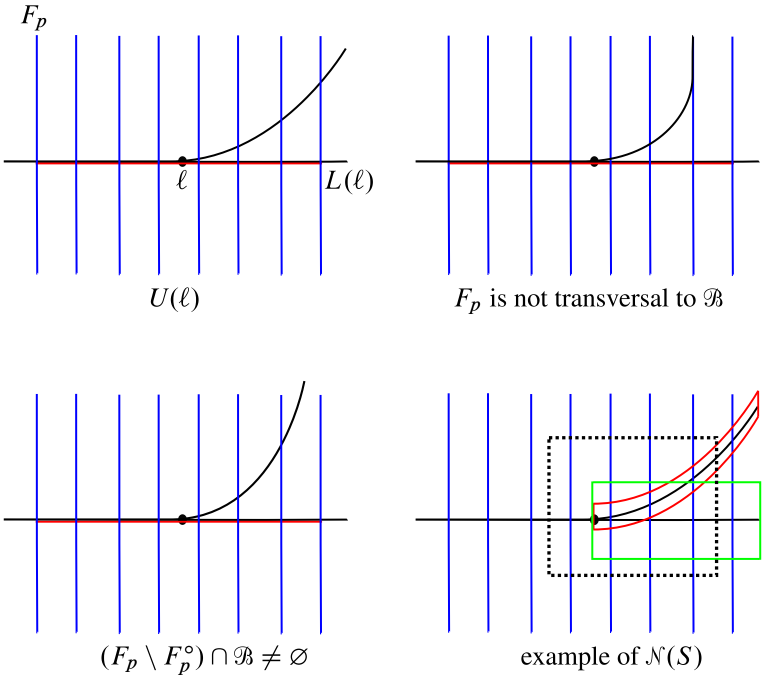

Figure 4: Black curves are part of a Lagrangian branched submanifold and the black marked points denote a connected component ℓ of Locus($\mathcal{B}$). In the top left, L_ℓ is in red, and the fibers F_p, for $p \in \mathcal{B} \cap U(\ell)$, are in blue; the top right and bottom left are not allowed by (3-2); and in the bottom right, the red and green boxes are examples of $\mathcal{N}(S)$ and the dotted box is an example of $U(\ell)$.

For any sector S, $S \setminus \bigcup_\ell \operatorname{Int} U(\ell)$ is a Lagrangian submanifold with boundary. The boundary of $S \setminus \bigcup_\ell \operatorname{Int} U(\ell)$ is a union of $S(\ell) := S \cap \partial(U(\ell))$. We fix a tubular neighborhood of $S(\ell)$, which is contained in $S \setminus \bigcup_\ell \operatorname{Int} U(\ell)$, and identify the tubular neighborhood with $S(\ell) \times [0, 1)$. For convenience, we will pretend that $S(\ell) \times [0, 1] \subset S$ and $S(\ell) \times \{0\} = S(\ell)$.

If $p \in S \setminus \bigcup_\ell \operatorname{Int} U(\ell)$ does not lie in any $S(\ell) \times (0, 1)$, then we set $F_p := i_S(\mathcal{N}(S) \cap T_p^* S)$. See Figure 5, top right.

Interpolation on $S(\ell) \times [0, 1]$ Let $p \in S(\ell)$. Then $F_{(p,0)}$ and $F_{(p,1)}$ are already constructed. We will construct $F_{(p,t)}$ from $F_{(p,0)}$ and $F_{(p,1)}$. The idea is to understand $F_{(p,0)}$ as a deformed $F_{(p,1)}$. In order to measure how much deformed $F_{(p,0)}$ is from $F_{(p,1)}$, we will construct a family of Lagrangian discs $B_{(p,t)}$ for all $t \in [0, 1]$, which are parallel to $F_{(p,1)}$. The family $B_{(p,t)}$ is defined by setting

$$B_{(p,t)} := i_S(\mathcal{N}(S) \cap T^*_{(p,t)} S).$$

We note that $B_{(p,t)}$ is parallel to $B_{(p,1)} = F_{(p,1)}$ so that there is a natural bijection map between $B_{(p,t)}$ and $B_{(p,1)}$.

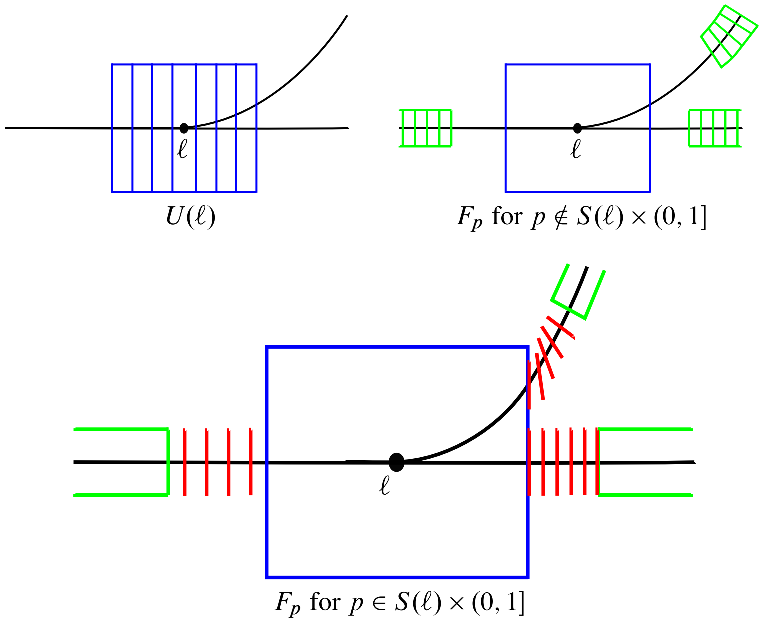

Figure 5: Black curves are part of a Lagrangian branched submanifold and marked points denote ℓ; in the top left, $U(\ell)$ is shaded blue, the vertical line segments are fibers. in the top right, the fiber F_p for $p \notin S(\ell) \times (0,1]$ is in green; and in the bottom, the fiber F_p for $p \in S(\ell) \times (0,1]$ is in red.

By applying the Lagrangian neighborhood theorem [16] to $B_{(p,0)}$,

$$F_{(p,0)} \cap i_S(\mathcal{N}(S)) = i_{B_{(p,0)}}(\text{the graph of a closed section in } T^*B_{(p,0)}).$$

Every closed section of $T^*B_{(p,0)}$ is an exact section because $B_{(p,0)}$ is a disk. Thus, there is a function $f_{(p,0)}\colon B_{(p,0)} \to \mathbb{R}$ such that

$$F_{(p,0)} \cap i_S(\mathcal{N}(S)) = i_{B_{(p,0)}}(\text{the graph of } df_{(p,0)}).$$

In other words, $F_{(p,0)}$ is obtained by deforming $B_{(p,0)}$. The deformation can be understood by using $f_{(p,0)}$.

Similarly, we will construct $F_{(p,t)}$ by deforming $B_{(p,t)}$. In order to deform, we define a function $f_{(p,t)}\colon B_{(p,t)} \to \mathbb{R}$ as

$$f_{(p,t)}\colon B_{(p,t)} \xrightarrow{\sim} B_{(p,0)} \xrightarrow{(1-t)f_{(p,0)}} \mathbb{R}.$$

The first arrow comes from the bijection between them. Then we set

$$F_{(p,t)} := i_{B_{(p,t)}}(\text{the graph of } df_{(p,t)}).$$

A fibered neighborhood $N(\mathcal{B})$ is given by the union of fibers, ie $N(\mathcal{B}) = \bigcup_{p \in \mathcal{B}} F_p$. Note that the construction of $N(\mathcal{B})$ is not unique.

3.3 Associated branched manifolds and the notion of "carried by"

We constructed a fibered neighborhood $N(\mathcal{B})$. In order to define what it means for a Lagrangian to be *carried by* $\mathcal{B}$, we introduce a projection map from $N(\mathcal{B})$ to an associated branched manifold $\mathcal{B}^*$.

Definition 3.6 Let $\mathcal{B}$ be a Lagrangian branched submanifold of M and let $N(\mathcal{B})$ be a fibered neighborhood of $\mathcal{B}$. Then, the *associated branched submanifold* $\mathcal{B}^*$ is defined by setting

$$\mathcal{B}^* := N(\mathcal{B})/\sim, \quad x \sim y \text{ if there exists an } F_p \text{ such that } x, y \in F_p.$$

Let $\pi: N(\mathcal{B}) \to \mathcal{B}^*$ denote the quotient map. We would like to remark that $\pi|_{\mathcal{B}}$ is not bijective, but $\mathcal{B}$ and $\mathcal{B}^*$ are equivalent as branched manifolds. We explain this with more detail in Remark 3.8.

We note that $\mathcal{B}^*$ is not contained in M. However, since $\mathcal{B}^*$ is a branched manifold, we can define the branch locus and sectors of $\mathcal{B}^*$ as follows:

Definition 3.7 (1) A *sector* of $\mathcal{B}^*$ is a connected component of

$$\{p \in \mathcal{B}^* \mid p \text{ has a neighborhood which is homeomorphic to } \mathbb{R}^n\}.$$

(2) A *branch locus* of $\mathcal{B}^*$ is the complement of all the sectors.

Remark 3.8 (1) Fibered neighborhoods $N(\mathcal{B})$ of $\mathcal{B}$ are not unique. However, if $N(\mathcal{B})$ is small enough, then $\mathcal{B}$ and $\mathcal{B}^*$ are equivalent as branched manifolds. For the equivalence between branched manifolds, we refer to Williams [17]. One can easily check their equivalence by using the Darboux chart that appeared in Definition 3.1. Thus, $\mathcal{B}^*$ is unique as a branched manifold under the assumption that $N(\mathcal{B})$ is small enough.

In the rest of this paper, when it comes to a Lagrangian branched submanifold $\mathcal{B}$, we will consider a triple $(\mathcal{B}, N(\mathcal{B}), \mathcal{B}^*)$ with an arbitrary choice of $N(\mathcal{B})$. Moreover, for any triple $(\mathcal{B}, N(\mathcal{B}), \mathcal{B}^*)$, the projection map is denoted by π for convenience.

(2) A fibered neighborhood $N(\mathcal{B})$ is a union of fibers, ie $N(\mathcal{B}) = \bigcup_{p \in \mathcal{B}} F_p$. In the equation, $\mathcal{B}$ is an index set. However, there is a possibility of having two distinct points $p, q \in \mathcal{B}$ such that $F_p = F_q$. From now on, we will use $\mathcal{B}^*$ as an index set and, by abuse of notation, F_x denotes $\pi^{-1}(x)$ for all $x \in \mathcal{B}^*$.

(3) Let x be a branch point of $\mathcal{B}^*$. Then there are sectors $S_0, S_1, \dots, S_l$ of $\mathcal{B}^*$ for some $l \geq 2$ such that

$$x \in \overline{S}_i \text{ for every } i = 0, 1, \dots, l,$$

$$F_x \cap \overline{\pi^{-1}(S_0)} = F_x \text{ and } F_x \cap \overline{\pi^{-1}(S_i)} \subset F_x^\circ = F_x \setminus \partial F_x \text{ for every } i = 1, 2, \dots, l.$$

Figure 6, right, represents this.

If a Lagrangian submanifold L (resp. Lagrangian branched submanifold $\mathcal{L}$) is contained in $N(\mathcal{B})$, there is a restriction of π to L (resp. $\mathcal{L}$). For convenience, we will simply use π instead of $\pi|_L: L \to \mathcal{B}^*$ (resp. $\pi|_{\mathcal{L}}: \mathcal{L} \to \mathcal{B}^*$).

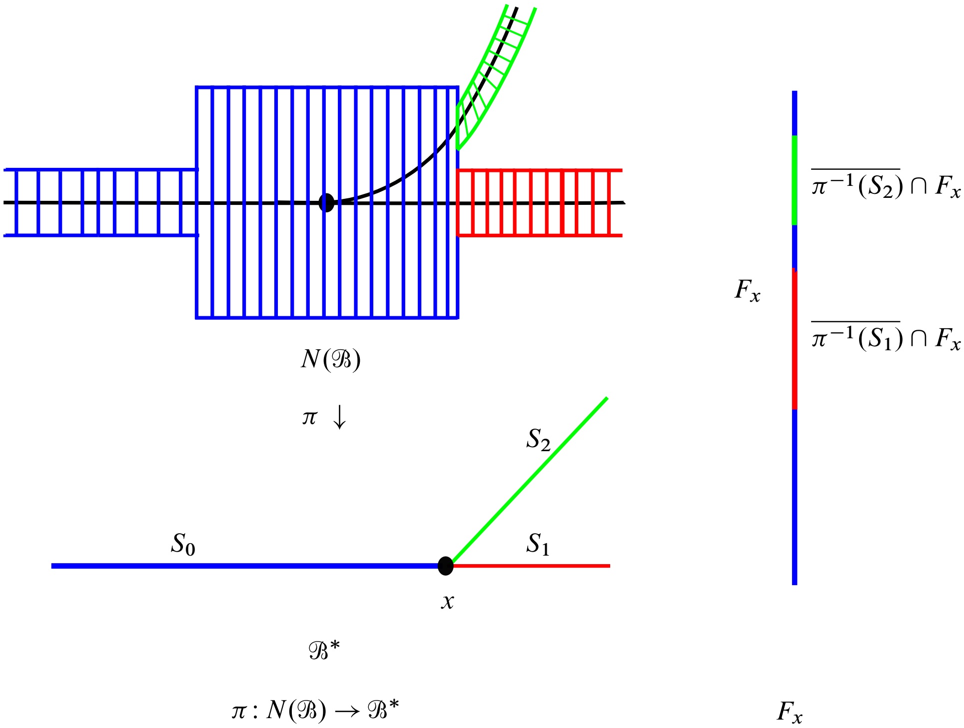

Figure 6: The left represents $\pi : N(\mathcal{B}) \to \mathcal{B}^*$. In $N(\mathcal{B})$, the blue, red, and green represent $\pi^{-1}(S_0)$, $\pi^{-1}(S_1)$, and $\pi^{-1}(S_2)$, where S_i is the corresponding sector of $\mathcal{B}^*$. The right represents F_x where x is in the branch locus of $\mathcal{B}^*$ to the left.

Definition 3.9 Let L be a Lagrangian submanifold (resp. $\mathcal{L}$ be a Lagrangian branched submanifold) of $N(\mathcal{B})$.

(1) A point x of L (resp. $\mathcal{L}$) is a *regular point* of π if $L \pitchfork F_{\pi(x)}$ (resp. $\mathcal{L} \pitchfork F_{\pi(x)}$) at x.

(2) A point x of L (resp. $\mathcal{L}$) is a *singular point* of π if x is not regular point of π. Moreover, $y \in \mathcal{B}^*$ is a *singular value* of π if there is a singular point x of π such that $\pi(x) = y$.

(3) L is *minimally singular with respect to* $\mathcal{B}$ if $\pi : L \to \mathcal{B}^*$ has no singular value on the branch locus of $\mathcal{B}^*$ and $|F_x \cap L| = |F_y \cap L|$, for any nonsingular value x and y which lie in the same sector of $\mathcal{B}^*$, where $|\cdot|$ means the cardinality of a set.

We recall that by definition, branched manifolds have tangent spaces even along the branch locus, so Definition 3.9 makes sense.

Definition 3.10 Let $\mathcal{B}$ and $\mathcal{L}$ be branched Lagrangian submanifolds.

(1) $\mathcal{L}$ is *strongly carried by* $\mathcal{B}$ if L (resp. $\mathcal{L}$) is Hamiltonian isotopic to $\mathcal{L}'$ such that $\mathcal{L}' \subset N(\mathcal{B})$ and $\pi : \mathcal{L}' \to \mathcal{B}^*$ has no singular value.

(2) $\mathscr{L}$ is *weakly carried by* $\mathscr{B}$ if $\mathscr{L}$ is Hamiltonian isotopic to $\mathscr{L}'$ such that $\mathscr{L}' \subset N(\mathscr{B})$, $\mathscr{L}'$ is minimally singular, and $\pi\colon \mathscr{L}' \to \mathscr{B}^*$ has countably many singular values.

We would like to remark that Lagrangian submanifolds are branched Lagrangian submanifold with empty branch locus. In the rest of this paper, if L is weakly carried by $\mathscr{B}$, then we will assume that $L \subset N(\mathscr{B})$ and L is minimally singular with respect to $\mathscr{B}$.

Note that the notion of "carried by" used by Thurston in [14] is our notion of "strongly carried by". For the case of surfaces, singularities of π can be easily resolved. However, for the case of higher-dimensional symplectic manifolds, there exists singularities which cannot be resolved. Thus, we defined the notion of "weakly carried by". We will give more detail in Section 3.4 with examples.

Thurston showed that for a pseudo-Anosov surface automorphism $\psi\colon S \xrightarrow{\sim} S$, there is a 1–dimensional branched submanifold τ which is called a train track such that $\psi(\tau)$ is strongly carried by τ. Our higher-dimensional generalization is slightly weaker, ie for some symplectic automorphism $\psi\colon (M,\omega) \xrightarrow{\sim} (M,\omega)$, we construct a Lagrangian branched submanifold $\mathscr{B}_\psi$ such that $\psi(\mathscr{B}_\psi)$ is weakly carried by $\mathscr{B}_\psi$. In other words, we allow nontransversality at countably many point $p \in \mathscr{B}_\psi$. However, we allow only one type of nontransversality. In the rest of the present subsection, we will describe the unique type of nontransversality.

Definition 3.11 Let L be weakly carried by $\mathscr{B}$. A *singular component* V of $\pi\colon L \to \mathscr{B}$ is a connected component of the set of all singular points of π.

Example 3.12 Let M be the symplectic manifold $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ equipped with the canonical symplectic form. The zero section $\mathscr{Z} := \mathbb{R}^n \times 0 \subset \mathbb{R}^{2n}$ is a Lagrangian branched submanifold. The fibered neighborhood $N(\mathscr{Z})$ is M with fibers $F_p := T_p^*\mathbb{R}^n$ for all $p \in \mathbb{R}^n = \mathscr{Z}$. Then, a Lagrangian submanifold

$$L_* := \{(tx, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid t \in \mathbb{R}, x \in S^{n-1} \subset \mathbb{R}^n\}$$

is weakly carried by $\mathscr{Z}$, and π_* has only one singular component

$$V_* := \{(0, x) \mid x \in S^{n-1}\},$$

where π_* is the projection map.

In order to understand the singularity, we would like to restrict π_* on L_*. By definition L_* is $\mathbb{R} \times S^{n-1}$, and the restriction is the map described as follows. First, the map collapses the center sphere $\{0\} \times S^{n-1}$ to a point and get two cones of S^{n-1} glued at the vertex. Then, second, the map projects each cone of S^{n-1} to a disk $\mathbb{D}^n$. Figure 7 describes the case of $n = 2$.

Definition 3.13 A singular component V of $\pi\colon L \to \mathscr{B}$ is of *real blow-up type* if there exists an open neighborhood U of V and a symplectomorphism $\phi\colon U \xrightarrow{\sim} \mathbb{R}^{2n}$ such that $\phi(U \cap \mathscr{B}) = \mathscr{Z}$, $\phi(V) = V_*$, and $\phi^{-1} \circ \pi_* \circ \phi = \pi$, where $\mathscr{Z}$, V_*, and π_* are defined in Example 3.12.

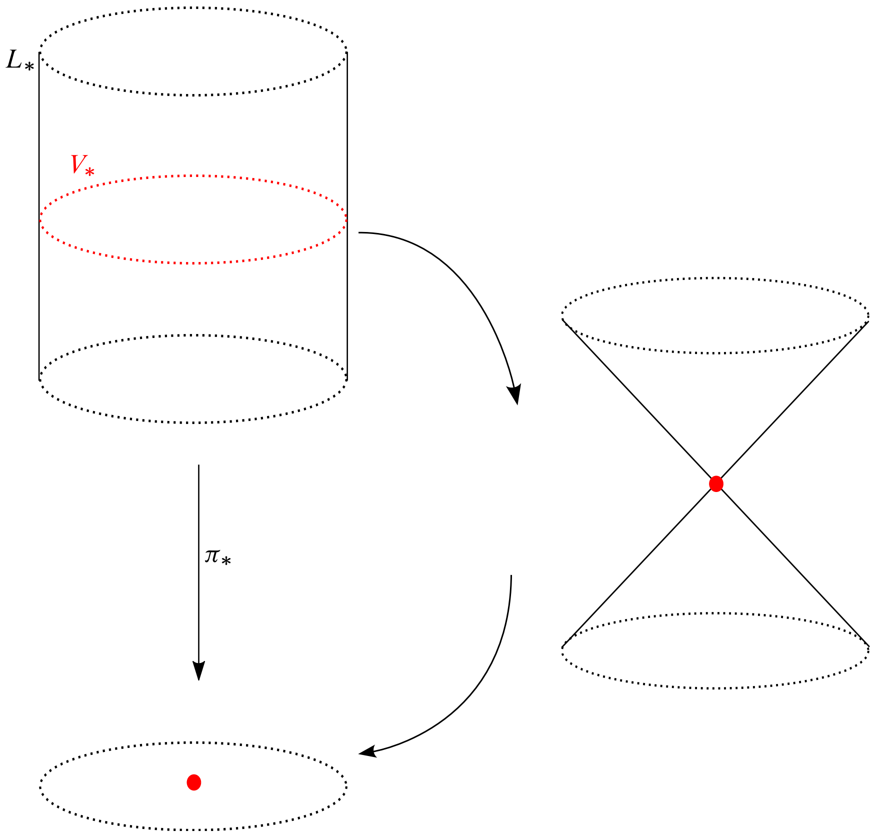

Figure 7: In the upper left, the Lagrangian L_* is shaded in black and the set of singular points V_* is shaded in red; the vertical arrow means the projection map π_*. In the lower left, the disk is a target of π_*; the red marked point is the singular value. In the middle right, the picture describes two cones glued at the vertex(red marked point), which is obtained by collapsing V_*.

Definition 3.14 A Lagrangian submanifold L (resp. Lagrangian branched submanifold $\mathscr{L}$) is *carried by* a Lagrangian branched submanifold $\mathscr{B}$ if L (resp. $\mathscr{L}$) is weakly carried by $\mathscr{B}$ and every singular component of π is a singular component of real blow-up type.

3.4 Examples of "weakly carried by"

In Section 3.4, we will give three examples of Lagrangians which are weakly carried by Lagrangian branched submanifolds. The first example is the lowest dimensional example, ie a 1–dimensional Lagrangian in a 2–dimensional symplectic manifold. The second example is a Lagrangian torus in T^*S^2. We will introduce these two examples in order to help the reader's understanding on the notion of "weakly carried by". The third example is a Lagrangian sphere in an A_3–surface singularity. With the example, we will explain why singular components occur naturally by iterating Dehn twists, which we will consider in the present paper.

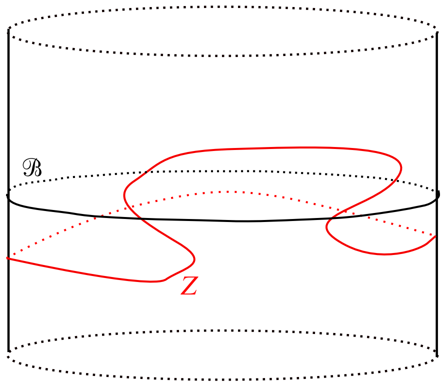

Figure 8: T^*S^1 together with the zero section $\mathscr{B}$ (black) and a Lagrangian Z (red) Hamiltonian isotopic to $\mathscr{B}$.

An example in T^*S^1 We consider the cotangent bundle of S^1. Let $\mathscr{B}$ denote the zero section of T^*S^1. Figure 8 describes T^*S^1 and $\mathscr{B}$. Let Z denote the red curve in Figure 8. Then Z is a Lagrangian which is Hamiltonian isotopic to $\mathscr{B}$.

By restricting a cotangent bundle map π on Z, Z is weakly carried by $\mathscr{B}$. However, by Hamiltonian isotoping Z, one obtains $\mathscr{B}$ and one can resolve the singularities of $\pi : Z \to \mathscr{B}$. In other words, Z is strongly carried by $\mathscr{B}$.

In [14], Thurston proved that on a surface, if a Lagrangian L is carried by a branched submanifold $\mathscr{B}$, then by isotoping L, one can resolve the singularities. Thus, Thurston used the notion of "carried by" without defining the notion of "weakly carried by" and his notion of "carried by" is the same to the notion of "strongly carried by".

Remark 3.15 Thurston resolved the singularities by isotoping, not Hamiltonian isotoping. Thus, for a 1–dimensional Lagrangian L which is weakly carried by a branched submanifold $\mathscr{B}$, it is possible that one cannot resolve the singularities of $\pi : L \to \mathscr{B}$, ie L is not strongly carried by. However, we do not discuss the existence of such examples in the current paper.

A torus in T^*S^2 We will introduce an example of a torus T in T^*S^2 such that T is weakly carried by, but not strongly carried by, the zero section $\mathscr{B}$. In order to describe the example, let assume that

$$T^*S^2 = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 \;\middle|\; \sum_{i=1}^{3} x_i^2 = 1, \sum_{i=1}^{3} x_i y_i = 0 \right\} \subset \mathbb{R}^6 \simeq T^*\mathbb{R}^3.$$

Then it is easy to check that $\omega|_{T^*S^2}$ is a symplectic form on T^*S^2, where $\omega = \sum_{i=1}^{3} dx_i \wedge dy_i$.

Let T be given by

$$T = \left\{ \big(\cos\theta(0,0,1) + \sin\theta(\cos\phi, \sin\phi, 0), -\sin\theta(0,0,1) + \cos\theta(\cos\phi, \sin\phi, 0)\big) \mid \theta, \phi \in \mathbb{R} \right\}.$$

Then it is easy to check that T is a Lagrangian submanifold of T^*S^2. By restricting the cotangent bundle map π on T, T is weakly carried by $\mathcal{B}$. However, L cannot be strongly carried by $\mathcal{B}$. If L is strongly carried by $\mathcal{B}$, then L should be a covering space of $\mathcal{B}$. However, since $\mathcal{B}$ is S^2, a torus T cannot be a covering space.

This example shows the reason why we need to define the notion of "weakly carried by" in a symplectic manifold of dimension greater than or equal to 4.

Singularity arising from iterating a Dehn twist We will give an exact Lagrangian sphere in A_3–surface singularity. By definition, A_3–surface singularity M is symplectically identified with

$$M := \{(x, y, z) \mid x^2 + y^2 + z^4 = 1\} \subset (\mathbb{C}^3, \omega_{\mathrm{std}}).$$

We will use well-known properties of M without proof. For details, we refer the reader to Wu [18].

The first property is that M is symplectically equivalent to the plumbing of two copies of T^*S^2 at one point, ie

$$M \simeq P(\alpha, \beta).$$

We defined $P(\alpha, \beta)$ in Section 2.1. The second property of M is that M is equipped with a Weinstein Lefschetz fibration $f(x, y, z) = z$. The Lefschetz fibration has three singular points. Fibers at regular points are T^*S^1.

The Lagrangian sphere which we will consider is $\tau^2(\beta)$, where τ is a Dehn twist along α. We will encode $\tau^2(\beta)$ on the base of the Lefschetz fibration. Figure 9 describes the base of the Lefschetz fibration $f : M \to \mathbb{C}$. Then α (resp. β) is a union of vanishing cycles over a curve connecting two singular points on the base, which is shaded red (resp. blue) in Figure 9, top. Similarly, $\tau^2(\beta)$ is a union of vanishing cycles over a curve shaded green in Figure 9, top.

Let $\mathcal{B}$ be the union of vanishing cycles over a curve shaded red in Figure 9, bottom. Then $\tau^2(\beta)$ is carried by $\mathcal{B}$. The projection map from $\tau^2(\beta)$ to $\mathcal{B}$ could be drawn as arrows on the base of f; see Figure 9, bottom.

One can observe that, in Figure 9, bottom, there is a arrow from a regular point x to a singular point y. On $\tau^2(\beta)$, the point x corresponds to the vanishing cycle on $f^{-1}(x)$. The vanishing cycle is projected to a point on $f^{-1}(y)$ by $\pi : \tau^2(\beta) \to \mathcal{B}$. Moreover, one can observe that the singular component is of real blow-up type. Thus, $\tau^2(\beta)$ is carried by $\mathcal{B}$.

Remark 3.16 The last example shows that a singular component could occur when we iterate a Dehn twist. We will consider the natural occurrence in later sections.

3.5 The generalized Penner construction

In this subsection, we give a higher-dimensional generalization of Penner construction [10] of pseudo-Anosov surface automorphisms. The generalization replaces Dehn twists by generalized Dehn twists along Lagrangian spheres.

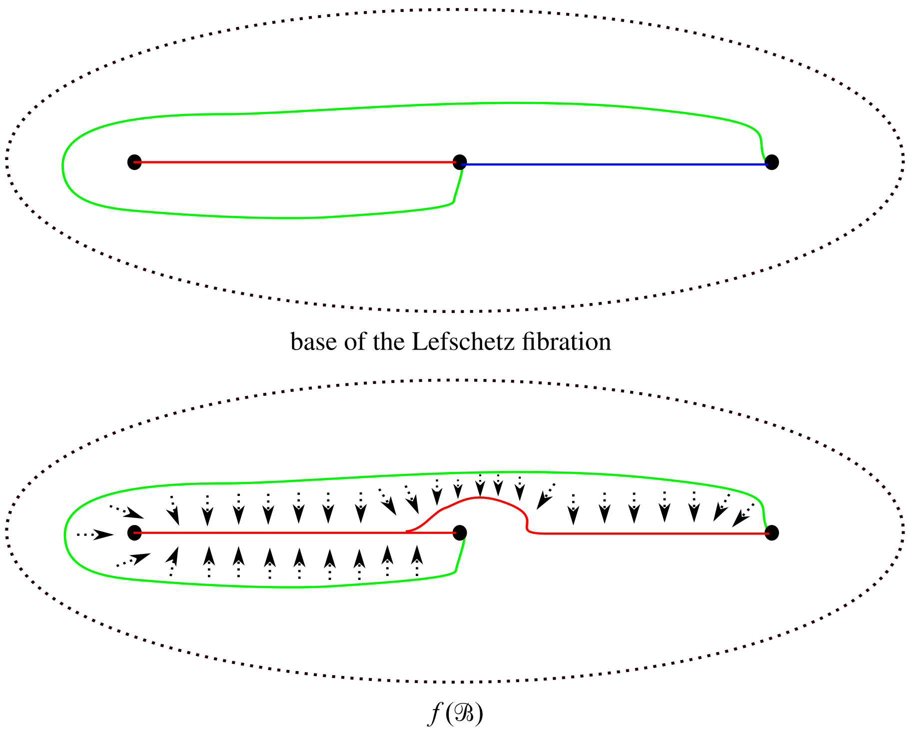

Figure 9: Top: base of the Lefschetz fibration f; the red, blue and green curves are images of α, β and $\tau^2(\beta)$. Bottom: the red curves are the image of $\mathcal{B}$ under f and the arrows are describing the projection maps from $\tau^2(\beta)$ to $\mathcal{B}$.

Generalized Penner construction Let M be a symplectic manifold. A symplectic automorphism $\psi : M \xrightarrow{\sim} M$ is of *generalized Penner type* if there are two collections,

$$A = \{\alpha_1, \dots, \alpha_m\}, \quad B = \{\beta_1, \dots, \beta_l\},$$

of Lagrangian spheres satisfying

$$\alpha_i \cap \alpha_j = \varnothing, \quad \beta_i \cap \beta_j = \varnothing \quad \text{for all } i \neq j,$$
$$\alpha_i \pitchfork \beta_j \quad \text{for all } i, j$$

such that ψ is a product of positive powers of Dehn twists τ_i along α_i and negative powers of Dehn twists σ_j along β_j, subject to the condition that every sphere appear in the product.

A Lagrangian sphere α_i (resp. β_j) is called a *positive* (resp. *negative*) sphere since only positive powers of τ_i (resp. negative powers of σ_j) appear in ψ.

Remark 3.17 (1) In Theorems 1.3 and 1.5, we can assume that the symplectic manifold M is a plumbing space. Every τ_i (resp. σ_j) is supported on a neighborhood of α_i (resp. β_j), which is denoted

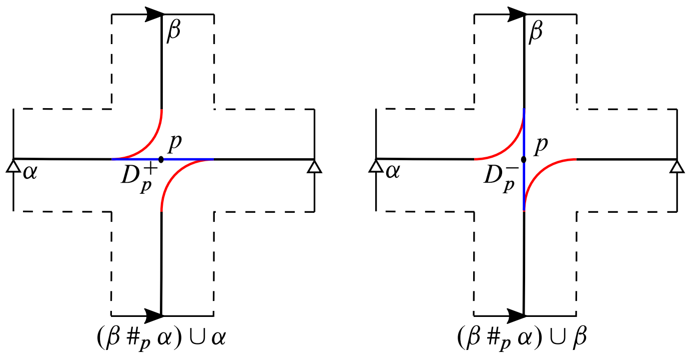

Figure 10: The blue curves represent D_p^+ in the left-hand picture and D_p^- in the right-hand picture; the red curves represent N_p in both.

by $T^*\alpha_i$ (resp. $T^*\beta_j$). Thus, ψ is supported on the union of $T^*\alpha_i$ and $T^*\beta_j$. By the transversality condition $\alpha_i \pitchfork \beta_j$, we can identify the union with a plumbing space

$$P = P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l).$$

Thus, it suffices to prove Theorems 1.3 and 1.5 on the plumbing space P, which we take to be connected.

(2) In [10], the Penner construction required that A and B fill the surface S; ie the complement of $A \cup B$ is a union of disks and annuli, one of whose boundary components is a component of ∂S. In the current paper, we do not require the analogue of the filling condition since we only construct an invariant Lagrangian branched submanifold and an invariant Lagrangian lamination, not an invariant singular foliation on all of M.

In the rest of this subsection, we define a set of Lagrangian branched submanifolds in a plumbing space $P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$. We start from the simplest plumbing space, having one positive and one negative sphere intersecting at only one point.

Example 3.18 Let α and β be n–dimensional spheres and let M be a plumbing $P(\alpha, \beta)$ which is plumbed at only one point p. Let $\beta \#_p \alpha$ be the Lagrangian surgery of α and β at p such that $\beta \#_p \alpha \simeq \tau_\alpha(\beta) \simeq \sigma_\beta^{-1}(\alpha)$. See Figure 10, which represents the case $n = 1$. The cross-shape is the plumbing space $P(\alpha, \beta)$, where α is the horizontal line and β is the vertical line.

The *neck N_p at p* connecting α and β is the closure of $(\beta \#_p \alpha) - (\alpha \cup \beta)$. In Figure 10, N_p is drawn in red. The *positive disk D_p^+ at p* is the closure of $\alpha - (\beta \#_p \alpha)$ and the *negative disk D_p^- at p* is the closure of $\beta - (\beta \#_p \alpha)$. The disks $D_p^\pm$ are drawn in blue in Figure 10. Then, by attaching D_p^+ or D_p^- to $\beta \#_p \alpha$, we obtain Lagrangian branched submanifolds $(\beta \#_p \alpha) \cup \alpha$ and $(\beta \#_p \alpha) \cup \beta$.

On a general plumbing space $M = P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$ with positive spheres α_i and negative spheres β_j, we similarly construct Lagrangian branched submanifolds. More precisely, given a plumbing point p, N_p, D_p^+ and D_p^- are the closures of $(\beta_j \#_p \alpha_i) - (\alpha_i \cup \beta_j)$, $\alpha_i - (\beta_j \#_p \alpha_i)$ and $\beta_j - (\beta_j \#_p \alpha_i)$, respectively. In order to construct a Lagrangian branched submanifold $\mathcal{B}$, let $D_p(\mathcal{B})$ be either D_p^+ or D_p^-. In other words, to construct $\mathcal{B}$, one choose either D_p^+ or D_p^- for each plumbing points p. Then we construct a Lagrangian branched submanifold $\mathcal{B}$ by setting

$$\mathcal{B} := \bigcup_i \Big(\alpha_i - \bigcup_{p\in\alpha_i} D_p^+\Big) \cup \bigcup_j \Big(\beta_j - \bigcup_{p\in\beta_j} D_p^-\Big) \cup \bigcup_p N_p \cup \bigcup_p D_p(\mathcal{B}). \tag{3-3}$$

There are 2^N possible choices of $\mathcal{B}$, where N is the number of plumbing points. Let $\mathbb{B}$ be the set of all 2^N Lagrangian branched submanifolds constructed above.

3.6 Proof of Theorem 1.3

In this subsection, let $M = P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$, let τ_i (resp. σ_j) be a Dehn twist along α_i (resp. β_j), and let ψ be of generalized Penner type.

In the rest of the paper, we assume that every Dehn twist, τ_i and σ_j, satisfies that

(1) τ_i (resp. σ_j) is supported on a small neighborhood $T^*\alpha_i$ (resp. $T^*\beta_j$) of α_i (resp. β_j);

(2) τ_i (resp. σ_j) agrees with the antipodal map on α_i (resp. β_j).

We define

$$\begin{gathered} A_p := \tau_i(D_p^+), \quad B_p := \sigma_j^{-1}(D_p^-) \quad \text{if } p \in \alpha_i \cap \beta_j, \\ \alpha_i' := \alpha_i - \bigcup_{p\in\alpha_i} (D_p^+ \cup A_p), \quad \beta_j' := \beta_j - \bigcup_{p\in\beta_j} (D_p^- \cup B_p). \end{gathered} \tag{3-4}$$

In words, A_p (resp. B_p) is a neighborhood of an antipodal point of p in α_i (resp. β_j). We are assuming that $D_p^\pm$, A_p and B_p are sufficiently small that they are disjoint to each other.

Recall that $\mathbb{B}$ is the set of Lagrangian branched submanifolds defined in Section 3.5; see the last sentence of that subsection.

Lemma 3.19 *For all k, there exists a function $F_{\tau_k} : \mathbb{B} \to \mathbb{B}$ such that $\tau_k(\mathcal{B})$ is carried by $F_{\tau_k}(\mathcal{B})$ for all $\mathcal{B} \in \mathbb{B}$. Similarly, there is a function $F_{\sigma_j^{-1}} : \mathbb{B} \to \mathbb{B}$ for all j such that $\sigma_j^{-1}(\mathcal{B})$ is carried by $F_{\sigma_j^{-1}}(\mathcal{B})$.*

Proof In this proof, τ_k is given by (2-2) and $\tilde{\tau} : T^*S^n \xrightarrow{\sim} T^*S^n$ defined in Section 2.2; ie $\tau_k = \phi \circ \tilde{\tau} \circ \phi^{-1}$ in a neighborhood of α_k, where ϕ is an identification of T^*S^n and a neighborhood of α_k.

Given $\mathcal{B} \in \mathbb{B}$, $\mathcal{B}$ admits the decomposition

$$\mathcal{B} = \bigcup_i \alpha_i' \cup \bigcup_j \beta_j' \cup \bigcup_p N_p \cup \bigcup_p A_p \cup \bigcup_p B_p \cup \bigcup_p D_p(\mathcal{B}), \tag{3-5}$$

where $D_p(\mathcal{B})$ is either D_p^+ or D_p^-. This follows from (3-3) and (3-4).

We prove the first statement for τ_k; the proof for σ_j^{-1} is analogous. Our strategy is to apply τ_k to α_i', β_j', N_p, A_p, B_p, and $D_p^\pm$. We claim:

(i) $\tau_k(\alpha_i') = \alpha_i'$ and $\tau_k(\beta_j') = \beta_j'$, and they are strongly carried by α_i' and β_j'.

(ii) If $p \notin \alpha_k$, then $\tau_k(N_p) = N_p$, $\tau_k(D_p^{\pm}) = D_p^{\pm}$, $\tau_k(A_p) = A_p$ and $\tau_k(B_p) = B_p$, and they are strongly carried by N_p, $D_p^{\pm}$, A_p and B_p.

(iii) If $p \in \alpha_k$, then $\tau_k(D_p^+) = A_p$, $\tau_k(A_p) = D_p^+$ and $\tau_k(B_p) = B_p$, and they are strongly carried by A_p, D_p^+ and B_p.

(iv) If $p \in \alpha_k$, then $\tau_k(D_p^-)$ and $\tau_k(N_p)$ are obtained by spinning with respect to p. Moreover, $\tau_k(D_p^-)$ is strongly carried by $N_p \cup (\alpha_k - D_p^+)$ and $\tau_k(N_p)$ is carried by $N_p \cup (\alpha_k - D_p^+)$.

By (3-5) and (i)–(iv), $\tau_k(\mathcal{B})$ is carried by $\mathcal{B}'$ such that

$$\mathcal{B}' = \bigcup_i \alpha_i' \cup \bigcup_j \beta_j' \cup \bigcup_p N_p \cup \bigcup_p A_p \cup \bigcup_p B_p \cup \bigcup_p D_p(\mathcal{B}'), \tag{3-6}$$

where $D_P(\mathcal{B}')$ is $D_p(\mathcal{B})$ if $p \notin \alpha_k$ and D_p^+ if $p \in \alpha_k$. Then $F_{\tau_k} : \mathbb{B} \to \mathbb{B}$ is defined by $F_{\tau_k}(\mathcal{B}) = \mathcal{B}'$.

For (i), since τ_k agrees with the antipodal map on α_k, $\tau_k(\alpha_k') = \alpha_k'$ and $\tau_k(\alpha_k')$ is strongly carried by α_k'. Moreover, since τ_k is supported on $T^*\alpha_k$, α_i' does not intersect the support of τ_k for all $i \neq k$. Thus, $\tau_k(\alpha_i')$ agrees with α_i' and $\tau_k(\alpha_i')$ is strongly carried by itself. The same proof applies to $\tau_k(\beta_j')$.

Statements (ii) and (iii) are proved in the same way.

For (iv), we compute $\tau_k(D_p^-)$ and $\tau_k(N_p)$ by spinning with respect to p and ϕ. We assume

$$\phi\big(((1, 0_n), 0_{n+1})\big) = p$$

without loss of generality. Using the notation from Section 2, D_p^- and N_p are contained in $\bigcup_{y \in S^{n-1}} \phi(W_y)$. Thus,

$$\begin{aligned} \tau_k(D_p^-) &= \bigcup_{y \in S^{n-1}} (\phi \circ \tilde{\tau} \circ \phi^{-1})(D_p^- \cap \phi(W_y)) \\ &= \bigcup_{y \in S^{n-1}} \big(\phi\big(\tilde{\tau}|_{W_y}(\phi^{-1}(D_p^-) \cap W_y)\big)\big) = \bigcup_{y \in S^{n-1}} \tau_k(D_p^-) \cap \phi(W_y), \end{aligned} \tag{3-7}$$

$$\begin{aligned} \tau_k(N_p) &= \bigcup_{y \in S^{n-1}} (\phi \circ \tilde{\tau} \circ \phi^{-1})(N_p \cap \phi(W_y)) \\ &= \bigcup_{y \in S^{n-1}} \phi\big(\tilde{\tau}|_{W_y}(\phi^{-1}(N_p) \cap W_y)\big) = \bigcup_{y \in S^{n-1}} \tau_k(N_p) \cap \phi(W_y). \end{aligned} \tag{3-8}$$

The restriction $\tilde{\tau}|_{W_y}$ is a Dehn twist on $W_y \simeq T^*S^1$ along the zero section. Thus, we obtain Figure 11 which represents intersections $\phi(W_y) \cap D_p^-$, $\phi(W_y) \cap N_p$, $\phi(W_y) \cap \tau_k(D_p^-)$, and $\phi(W_y) \cap \tau_k(N_p)$. Equation (3-8) and Figure 11 imply that $\tau_k(N_p)$ is carried by $N_p \cup (\alpha_k - D_p^+)$ This is because in each W_y, the vertical projection has no critical values. Thus, if there is a singular value, then the singular value is created when one takes the union in (3-8). One can easily check that $\tau_k(p)$ is the only singular value when one takes the union. Similarly, $\tau_k(D_p^-)$ is strongly carried by $N_p \cup (\alpha_k - D_p^+)$.

Then (i)–(iv) and (3-5) prove that $\tau_k(\mathcal{B})$ is carried by $F_{\tau_k}(\mathcal{B})$. □

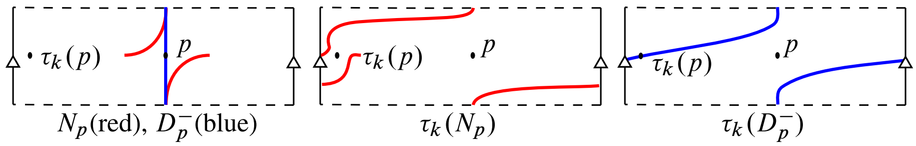

Figure 11: In the left picture, the blue curve represents D_p^- and the red curve represents N_p; in the middle picture, the red curve represents $\tau_k(N_p)$; and in the right picture, the blue curve represents $\tau_k(D_p^-)$.

Lemma 3.20 *If L is a Lagrangian submanifold which is carried* (*resp. weakly carried*) *by $\mathcal{B} \in \mathbb{B}$, then $\tau_k(L)$ is carried* (*resp. weakly carried*) *by $F_{\tau_k}(\mathcal{B})$. The case of σ_j^{-1} is analogous.*

Proof We can assume that L is contained in an arbitrary small neighborhood of $\mathcal{B}$. Then we apply a Dehn twist τ_k as we did in the proof of Lemma 3.19. The details are similar to the proof of Lemma 3.19. □

Proof of Theorem 1.3 Let $\psi : M \xrightarrow{\sim} M$ be a symplectic automorphism of generalized Penner type. Then we can write $\psi = \delta_1 \circ \cdots \circ \delta_l$, where δ_k is a Dehn twist τ_i or σ_j^{-1}. By Lemma 3.19, we have specific functions F_{τ_i} and $F_{\sigma_j^{-1}}$ acting on $\mathbb{B}$. We then define $F_\psi = F_{\delta_1} \circ \cdots \circ F_{\delta_l} : \mathbb{B} \to \mathbb{B}$.

We claim that F_ψ is a constant map, ie there is a unique $\mathcal{B}_\psi \in \mathbb{B}$ such that $F_\psi(\mathcal{B}) = \mathcal{B}_\psi$ for all $\mathcal{B} \in \mathbb{B}$, which we define as follows: in (3-3), for $p \in \alpha_i \cap \beta_j$, we set $D_p(\mathcal{B}_\psi) = D_p^+$ if the last τ_i in ψ appears later than the last σ_j^{-1}, and $D_p(\mathcal{B}_\psi) = D_p^-$ otherwise. Note that every Dehn twist τ_i and σ_j^{-1} appears in ψ; thus $\mathcal{B}_\psi$ is well defined. By (3-6), $F_\psi(\mathcal{B}) = \mathcal{B}_\psi$ for all $\mathcal{B} \in \mathbb{B}$. Lemma 3.20 completes the proof. □

Remark 3.21 (1) A singular value of $\pi : \psi^m(L) \to \mathcal{B}^*$ can be moved by isotoping $\psi^m(L)$.

(2) Every singular value of $\pi : \psi^m(\mathcal{B}_\psi) \to \mathcal{B}^*$ lies near $\pi(p)$, $\pi(\tau_i(p))$, or $\pi(\sigma_j^{-1}(p))$ by isotoping, where p is a plumbing point.

4 Encoding a Lagrangian on a Lagrangian branched submanifold

In the previous section, we generalized the notion of "carried by" for higher-dimensional symplectic manifolds. It is well known that on a surface, if a curve is carried by a train track, then one can encode the isotopy class of curve on the train track with an extra data. The extra data is called *weight*. We briefly review the notion of weight in Section 4.1, then generalize this for higher-dimensional case in Section 4.

4.1 Weights on a train track

We will briefly review the notion of weights on a train track with a simple example, and how one can construct a stable lamination of a surface automorphism of generalized Penner type from them in Section 4.1. We will introduce some well-known facts without proofs. For more detail, we refer the reader to Penner and Harer [11], or Farb and Margalit [5].

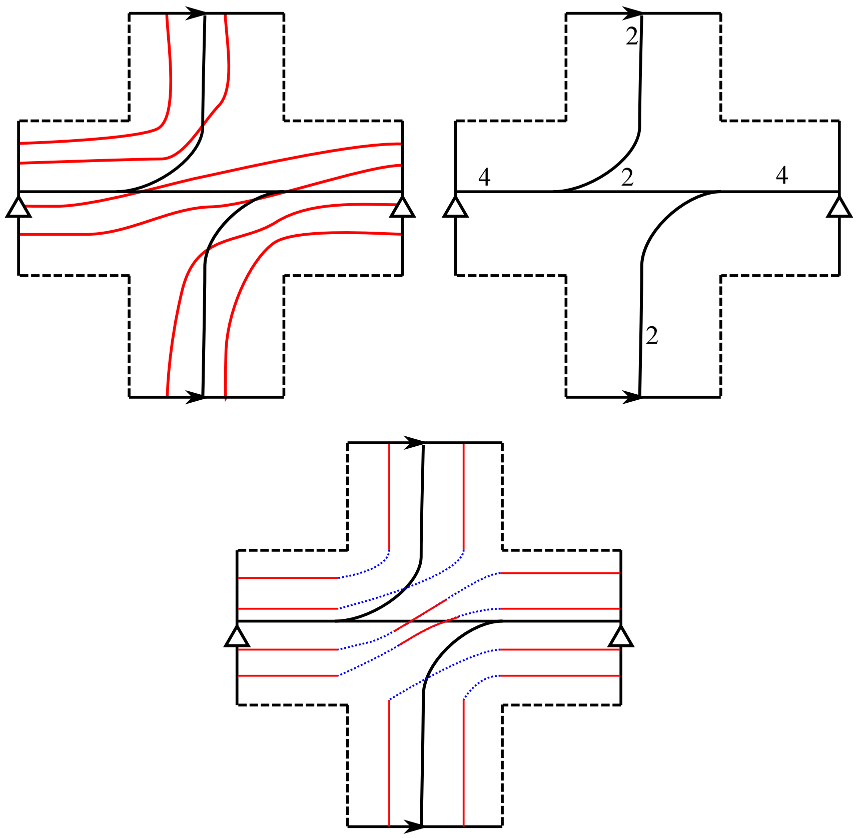

Figure 12: Top left: the cross shape is a surface S, the black graph is a Lagrangian branched submanifold $\mathcal{B}_\psi$, and the red curves are carried by $\mathcal{B}_\psi$. Top right: the numbers are the weights corresponding to the red curve in the diagram to the left. Bottom: the red curves are parallel copies of each edges and the blue dotted curves are the unique way to connecting the parallel copies.

At the end of Section 4.1, we will explain why the construction on surfaces does not work on the cases of a higher-dimensional symplectic manifold. Then, we will give a detailed organization of Section 4.

The notion of weights Let S be a surface obtained by plumbing two copies of T^*S^1 at one point. Two zero sections of each copies of T^*S^1 will be denoted by α and β as we did in previous sections. Similarly, let τ and σ denote Dehn twists along α and β respectively. We will fix a surface automorphism $\psi := \tau \circ \sigma^{-1}$. Then, by Section 3, there is a branched submanifold $\mathcal{B}_\psi$ such that if a curve $C \subset S$ is carried by $\mathcal{B}_\psi$, then $\psi(C)$ is also carried by $\mathcal{B}_\psi$. Moreover, as mentioned in Section 3.4, one can assume that there is no singular value of $\pi : C \to \mathcal{B}_\psi$ by isotoping. Figure 12, top left, describes the surface S and $\mathcal{B}_\psi$ together with an example of a curve C which is carried by $\mathcal{B}_\psi$.

Weights on a train track are collection of nonnegative numbers assigned on each edges of the train track. If a curve C is carried by a train track $\mathcal{B}$, then C gives weights on $\mathcal{B}$ by assigning the number of connected components of $\pi^{-1}(e)$ for each edge e of $\mathcal{B}_\psi$. Figure 12, top right, is an example of weights, which are induced from the curve C drawn in Figure 12, top left.

Conversely, one can recover the isotopy class of a curve C from a train track $\mathcal{B}$ which carries C and the weights induced from C. In order to recover, one can considers parallel copies of each edges. The numbers of copies are the weights on each edges. Then, it is known that there is a unique way to connect each copies to construct an isotopy class of the curve C. Figure 12, bottom, is the example of the recovering process.

Linear algebra on weights By Theorem 1.3, for a surface automorphism ψ of generalized Penner type, if a curve C is carried by a train track $\mathcal{B}_\psi$, then $\psi(C)$ is carried by $\mathcal{B}_\psi$. Since C and $\psi(C)$ are carried by $\mathcal{B}_\psi$, they induce weights on $\mathcal{B}_\psi$. Moreover, it is well known that the weights for $\psi(C)$ is obtained from the weights for C by doing linear algebra. We will review this with the example which we used above, ie S is the plumbing of $T^*\alpha$ and $T^*\beta$ and $\psi = \tau \circ \sigma^{-1}$.

Let C be a curve carried by $\mathcal{B}_\psi$ such that the induced weights on $\mathcal{B}_\psi$ are a, b and c, as drawn in Figure 13, top left. For simplicity, we write the weight for C in a vector

$$\vec{w}_C = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Figure 13, top right and bottom left, are $\tau(\mathcal{B}_\psi)$ and $\psi(\mathcal{B}_\psi)$. One can observe that $\psi(\mathcal{B}_\psi)$ is carried by $\mathcal{B}_\psi$ and induces weights $3b+2c$, $2b+c$ and a on $\mathcal{B}_\psi$. Thus, the weights for C and $\psi(C)$ satisfy

$$\vec{w}_{\psi(C)} = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \vec{w}_C. \tag{4-1}$$

Remark 4.1 In (4-1), a 3×3 matrix appears. One can replace this matrix with a 2×2 matrix. Since the weight assigned on the blue edge in Figure 13 should be the same to the sum of weights assigned on the red and black edges in Figure 13. This condition is called the *switch* condition. For the detail, see Farb and Margalit [5].

Stable lamination of ψ For a surface automorphism ψ of generalized Penner type, it is well known that the stable lamination of ψ is easily constructed from $\mathcal{B}_\psi$ and the linear algebra which we did above. For a rigorous treatment, we should define the notion of measured lamination and should explain how a measured lamination $\mathcal{L}$ can be encoded onto a pair $(\mathcal{B}, \vec{w}_\mathcal{L})$ of a train track $\mathcal{B}_\psi$ and weights $\vec{w}_\mathcal{L}$. However, for simplicity, we skip this excepts that the weight vector $\vec{w}_\mathcal{L}$ is an eigenvector of A_ψ corresponding to an eigenvalue $\lambda > 1$, where A_ψ is the matrix appearing in (4-1).

For more details including the notion of measured laminations and the existence of an eigenvalue $\lambda > 1$ of A_ψ, we refer the reader to Farb and Margalit [5].

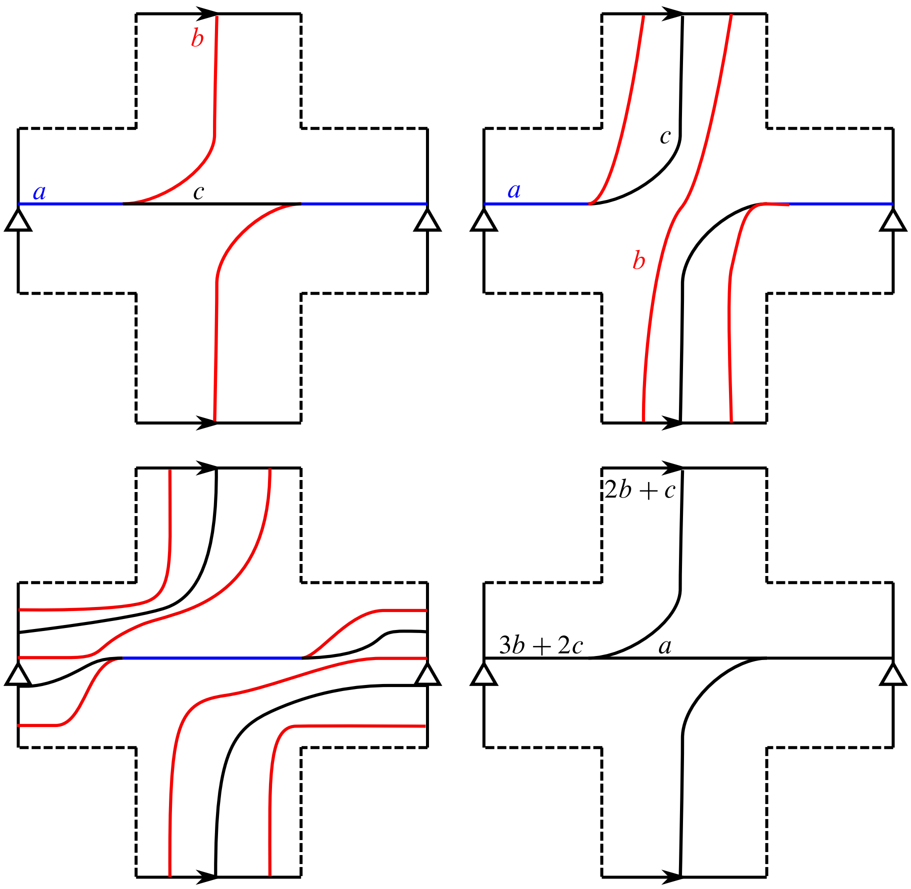

Figure 13: Top left: the Lagrangian branched submanifold $\mathcal{B}_\psi$ has three edges shaded blue, red and black, and a, b and c are weights on the edges, respectively. Top right describes $\sigma^{-1}(\mathcal{B}_\psi)$; blue, red and black edges are assigned the same weights a, b and c. Bottom left describes $\tau(\sigma^{-1}(\mathcal{B}_\psi)) = \psi(\mathcal{B}_\psi)$; blue, red and black edges are assigned the same weights a, b and c. Bottom right describes the projection of $\psi(\mathcal{B}_\psi)$ onto $\mathcal{B}_\psi$, and one finds new weights on each edge of $\mathcal{B}_\psi$.

A difficulty on higher-dimensional symplectic manifolds For a surface automorphism ψ of generalized Penner type, one can construct the stable lamination of ψ by doing some linear algebra on weights on a train track $\mathcal{B}_\psi$. This is because, in the case of a surface, the notion of carried by is the notion of strongly carried by, ie there is no singular component. However, in the case of a higher-dimensional symplectic manifold, the construction of laminations on surfaces does not work, because of singularities.

In Section 4.2, we will decompose $\mathcal{B}_\psi^*$ into a union of disks. The disks are of two types, one with singularities and one without singularities. Then, in Section 4.3, we will generalize the notion of weights. Since the generalization should have information on singularities, it will be defined by using the disks with singularities. In Section 5.1, we will generalize the linear algebra on weights. In Section 6.2, we will construct a stable Lagrangian lamination on a disc with singularities, and in Section 6.3, we will construct on a disc without singularities.

4.2 Singular and regular disks

As mentioned in the previous subsection, the construction of laminations on surfaces does not work because of singularities. In Section 4.2, for a symplectic automorphism ψ of generalized Penner type, we decompose $\mathcal{B}_\psi$ into a union of disks. The disks are classified into two types, with and without singularities.

Definition 4.2 Let assume that there is a pair $(\psi\colon M \xrightarrow{\sim} M, \mathcal{B}_\psi)$ of a symplectic automorphism ψ and a Lagrangian branched submanifold $\mathcal{B}$ such that $\psi^n(\mathcal{B}_\psi)$ is carried by $\mathcal{B}_\psi$ for all $n \in \mathbb{N}$. Let $\mathcal{B}_\psi^*$ be the associated branched manifold of $\mathcal{B}_\psi$. Then the triple $(\psi, \mathcal{B}_\psi, \mathcal{B}_\psi^*)$ *admits a decomposition into singular and regular discs* if $\mathcal{B}_\psi^*$ can be decomposed into the union of a finite number of disks $S_i \simeq \mathbb{D}^n$, which are called *singular disks*, and $R_j \simeq \mathbb{D}^n$, which are called *regular disks*, ie

$$\mathcal{B}_\psi^* = \bigcup_i S_i \cup \bigcup_j R_j \tag{4-2}$$

such that

1. each singular disk S_i is a closed disk contained in a closure of a sector of $\mathcal{B}^*$;
2. $S_i \cap S_j = \varnothing$ for any $i \neq j$;
3. every singular value of $\pi\colon \psi^m(\mathcal{B}_\psi) \to \mathcal{B}_\psi$ after weakly fibered isotopy lies in $\bigcup_i S_i^\circ$ for all $m \in \mathbb{N}$, where S_i° is the interior of S_i;
4. each regular disk R_j is a closed disk contained in a closure of a sector minus $\bigcup_i S_i^\circ$;
5. S_i and R_j (resp. R_i and R_j for $i \neq j$) meet only along their boundaries.

For convenience, we simply say that $\mathcal{B}_\psi^*$, instead of a triple $(\psi, \mathcal{B}_\psi, \mathcal{B}_\psi^*)$, admits a decomposition into singular and regular discs.

Definition 4.3 Let a triple $(\psi, \mathcal{B}_\psi, \mathcal{B}_\psi^*)$ admit a decomposition into singular and regular discs. A Lagrangian L which is carried by $\mathcal{B}_\psi$ is *compatible with the decomposition* if L is Hamiltonian isotopic to L' such that every singular value of $\pi\colon L' \to \mathcal{B}_\psi$ lies on a singular disc.

Remark 4.4 In Section 3, we used a decomposition of $\mathcal{B}$ with notation $D_p^\pm$, A_p, B_p and so on. However, the decomposition introduced in Definition 4.3 is a decomposition of the associated branched manifold $\mathcal{B}^*$, not $\mathcal{B}$.

In the rest of Section 4.2, we will introduce and use a specific decomposition of $\mathcal{B}_\psi^*$ for ψ of generalized Penner type. Since the specific decomposition of $\mathcal{B}_\psi^*$, together with the decomposition of $\mathcal{B}_\psi$ in (3-5), is likely to confuse the reader, we remark that here.

If $\mathcal{B}^*_\psi$ admits a decomposition into singular and regular discs, then one obtains a decomposition of $N(\mathcal{B}_\psi)$ as

$$N(\mathcal{B}_\psi) = \bigcup_i \pi^{-1}(S_i) \cup \bigcup_j \pi^{-1}(R_j).$$

Remark 4.5 In Section 4.3 (resp. Section 6.3), we will construct a Lagrangian lamination on $\overline{\pi^{-1}(S_i^\circ)}$ (resp. $\overline{\pi^{-1}(R_j^\circ)}$) which is the closure of $\pi^{-1}(S_i^\circ)$, not on $\pi^{-1}(S_i)$ (resp. $\subset \pi^{-1}(R_j)$). This is because $\pi^{-1}(S_i)$ (resp. $\pi^{-1}(R_j)$) is not a (closed) submanifold of M if S_i (resp. R_j) intersects the branch locus of $\mathcal{B}^*$.

Figure 6 is an example. If S_1 in Figure 6 is a singular disk, then $\pi^{-1}(S_1)$ is the union of the red box in Figure 6, left, and F_x.

Decomposition of $\mathcal{B}^*_\psi$ for ψ of generalized Penner type Let us assume that a symplectic automorphism $\psi\colon M \xrightarrow{\sim} M$ is of generalized Penner type. Then, in Section 3, we constructed a Lagrangian branched submanifold $\mathcal{B}_\psi$. We will now give a specific decomposition of $\mathcal{B}^*_\psi$ into singular and regular discs, which we will call *the standard decomposition of* $\mathcal{B}^*_\psi$.

By Remark 3.21, after weakly fiber isotoping, every singular value of $\pi\colon \psi^m(\mathcal{B}_\psi) \to \mathcal{B}^*_\psi$ lies in the interior of $S_p(\mathcal{B}_\psi)$ or $S_p^\pm$, where $S_p(\mathcal{B}_\psi) := \pi(D_p(\mathcal{B}_\psi))$, $S_p^+ := \pi(A_p)$ and $S_p^- := \pi(B_P)$. We note that as the notation suggests, $S_p(\mathcal{B})$ depends on $\mathcal{B}$, but $S_p^\pm$ does not. In the specific decomposition, $S_p(\mathcal{B}_\psi)$ and $S_p^\pm$ are singular disks of $\mathcal{B}^*_\psi$ and there is no other singular discs.

Remark 4.6 As mentioned in Remark 4.4, $S_p(\mathcal{B}_\psi)$ and $S_p^\pm$ are subsets of $\mathcal{B}^*_\psi$, not $\mathcal{B}_\psi$. However, in the rest of the current paper, if there is no chance of misunderstanding, we will abuse notation and will identify the singular disks with $D_p(\mathcal{B}_\psi)$, A_p and B_p. This is for notational convenience.

We will divide the complement of singular disks from $\mathcal{B}^*_\psi$, ie

$$\mathcal{B}^*_\psi \setminus \Big(\bigcup_p S_p(\mathcal{B}_\psi) \sqcup \bigcup_p S_p^+ \sqcup \bigcup_p S_p^-\Big), \tag{4-3}$$

into regular disks. In order to do this, we cut out a symplectic submanifold $W^{2n-2} \subset M^{2n}$, which is defined as follows: for each α_i (resp. β_j), there is an equator C_{α_i} (resp. C_{β_j}) $\simeq S^{n-1}$ such that

(1) for any plumbing point $p \in \alpha_i$ (resp. β_j), p lies on C_{α_i} (resp. C_{β_j});

(2) if $p \in \alpha_i \cap \beta_j$, then $T^*C_{\alpha_i} \equiv T^*C_{\beta_j}$ near p.

Note that the equators on Lagrangian spheres α_i and β_j are defined using identifications $\phi_{\alpha_i}\colon \alpha_i \xrightarrow{\sim} S^n$ and $\phi_{\beta_j}\colon \beta_j \xrightarrow{\sim} S^n$. Thus, by choosing proper identification ϕ_{α_i} and ϕ_{β_j}, we can assume the existence of C_{α_i} and C_{β_j}. Then

$$W := \bigcup_i T^*C_{\alpha_i} \cup \bigcup_j T^*C_{\beta_j}$$

is a $(2n-2)$–dimensional symplectic submanifold of M.

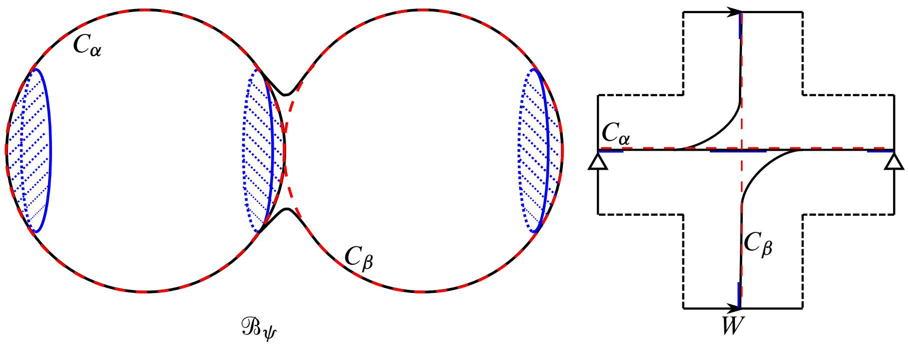

Figure 14: Left: the black curves represent $\mathcal{B}_\psi$ and the red dotted circles are C_α (left) and C_β (right). The blue shaded regions are singular disks. Right: W as a symplectic submanifold, black curves are $W \cap \mathcal{B}_\psi$, red dotted lines are C_α (horizontal) and C_β (vertical), blue shaded regions are intersections of W and singular disks.

We cut (4-3) along $\pi(W)$. The components of the complement of W are the regular discs R_j in the specific decomposition of $\mathcal{B}_\psi$. Each R_k is a manifold with corners, where the corners are at $R_k \cap \pi(W) \cap S_l$. Then the proof of Theorem 1.3 shows that this decomposition of $\mathcal{B}^*_\psi$ is a decomposition into singular and regular discs. More precisely, there are two types of singularities of $\pi \colon \psi^m(\mathcal{B}_\psi) \to \mathcal{B}_\psi$, one coming from a singularity of $\psi^{m-1}(\mathcal{B}_\psi)$ and the other occurring when one applies ψ. The proof of Theorem 1.3 shows two things; first, ψ sends a singular value of $\psi^{m-1}(\mathcal{B}_\psi)$ onto a singular disk, and second, a new born singular value lies on a singular disk.

Remark 4.7 (1) If $\mathcal{B}_{\psi_1} = \mathcal{B}_{\psi_2}$, then it is easy to check that the standard decomposition with respect to ψ_i are the same.

(2) In Section 3.5, we defined a set $\mathbb{B}$ of Lagrangian branched submanifolds in M. For all $\mathcal{B} \in \mathbb{B}$, one can find a symplectic automorphism ψ such that $\mathcal{B} = \mathcal{B}_\psi$. Together with the above argument, for all $\mathcal{B} \in \mathbb{B}$, $\mathcal{B}^*$ admits the standard decomposition.

Example 4.8 Let M be the plumbing of $T^*\alpha$ and $T^*\beta$ at one point p, where $\alpha, \beta \simeq S^2$. Let $\psi = \tau \circ \rho^{-1}$ where τ (resp. β) is a Dehn twist along α (resp. β). Then $\mathcal{B}_\psi = (\beta \#_p \alpha) \cup \alpha$ and $(\psi, \mathcal{B}_\psi, \mathcal{B}^*_\psi)$ admits the standard decomposition.

Figure 14, left, is a schematic picture of $\mathcal{B}_\psi$. The regions shaded blue are singular disks of the standard decomposition. The red dotted circles are C_α and C_β. Figure 14, right, is the symplectic submanifold W of codimension 2.

Remark 4.9 For a given symplectic manifold M and a given triple $(\psi, \mathcal{B}_\psi, \mathcal{B}^*_\psi)$, it is natural to ask which Lagrangians L are compatible with the standard decomposition of $\mathcal{B}^*_\psi$. One can easily check that if L is one of zero sections α_i or β_j, or if L is obtained by applying a series of Dehn twist to one of zero sections, then L is compatible with the standard decomposition. See Remark 3.17 for the notation α_i

and β_j and see the proof of Lemma 5.1. Also we note that by Wu [18], if M is an A_n–surface singularity, then every exact Lagrangian L is compatible with the standard decomposition of $\mathcal{B}^*_\psi$ if L is carried by $\mathcal{B}_\psi$. In the current paper, we simply assume that a Lagrangian M is compatible with the standard decomposition for convenience.

4.3 Braids

In Section 4.3, we will generalize the notion of weights on higher-dimensional symplectic manifolds. We will assume that a given triple $(\psi, \mathcal{B}_\psi, \mathcal{B}^*_\psi)$ admits a decomposition into singular and regular disks.

Let S be a singular disk in $\mathcal{B}^*_\psi$ and let S° be the interior of S. Then $\pi^{-1}(S^\circ) = \bigcup_{p \in S^\circ} F_p$ is symplectomorphic to $DT^*(\mathbb{D}^n)^\circ$. Thus, the closure $\overline{\pi^{-1}(S^\circ)}$ is symplectomorphic to $DT^*\mathbb{D}^n$ and there is a natural symplectomorphism between them. The boundary $\partial\overline{\pi^{-1}(S^\circ)}$ is a $\mathbb{D}^n$–bundle over $\partial S \simeq S^{n-1}$ and the natural symplectomorphism induces $\varphi\colon \partial\overline{\pi^{-1}(S^\circ)} \xrightarrow{\simeq} S^{n-1} \times \mathbb{D}^n$.

Definition 4.10 $D(S)$ (resp. $D(\partial S)$) is the $\mathbb{D}^n$–bundle $\overline{\pi^{-1}(S^\circ)}$ (resp. $\partial\overline{\pi^{-1}(S^\circ)}$) over S_i (resp. ∂S).

Definition 4.10 is for notational convenience.

Remark 4.11 Since $D(S)$ is symplectomorphic to a disk cotangent bundle of $\mathbb{D}^n$, coordinate charts on the base will induce a natural identification between $D(S)$ and $\mathbb{D}^n \times \mathbb{D}^n$. By restricting the identification on the boundary, $D(\partial S)$ is identified with $S^{n-1} \times \mathbb{D}^n$.

If L is a Lagrangian submanifold which is carried by $\mathcal{B}_\psi$ and if L is compatible with the decomposition of $\mathcal{B}^*_\psi$, then, for all $p \in \partial S$, $\varphi(L \cap F_p)$ is a finite collection of isolated points in $F_p \simeq \mathbb{D}^n$; recall that $\pi\colon L \to \mathcal{B}^*_\psi$ has no singular value on ∂S. Thus, $\varphi(L \cap D(\partial S))$ can be identified with a map from $\partial S \simeq S^{n-1}$ to the configuration space $\mathrm{Conf}_l(\mathbb{D}^n)$ of l points on $\mathbb{D}^n$ where $l = l(L, S)$, ie a braid. Since L is Lagrangian, $(\varphi^{-1})^*\omega$ vanishes on $\phi(L \cap D(\partial S))$.

From now on, we will define the braids on the boundary of a singular disk S. Let $f : S^{n-1} \to \mathrm{Conf}_l(\mathbb{D}^n)$ for some l. In other words, there are maps

$$f_1, \dots, f_l\colon S^{n-1} \to \mathbb{D}^n$$

such that $f(p) = \{f_1(p), \dots, f_l(p)\}$ with $f_i(p) \neq f_j(p)$ for all $i \neq j$. We define

$$(4\text{-}4)\qquad \begin{aligned} B(f) &:= \{(p, f_i(p)) \in S^{n-1} \times \mathbb{D}^n \mid p \in S^{n-1}, i \in \{1, \dots, \ell\}\}, \\ \widetilde{\mathrm{Br}}_{\partial S} &:= \{\varphi^{-1}(B(f)) \mid f : S^{n-1} \to \mathrm{Conf}_l(\mathbb{D}^n) \text{ such that } (\varphi^{-1})^*(\omega) \text{ is zero on } B(f) \text{ for some } l\}. \end{aligned}$$

Note that $\widetilde{\mathrm{Br}}_{\partial S}$ is a set of closed subsets of $D(\partial S)$ and independent of φ.

We define an equivalence relation on $\widetilde{\mathrm{Br}}_{\partial S}$ as follows: $b_0 \sim b_1$ for $b_i \in \widetilde{\mathrm{Br}}_{\partial S}$ if there exists a smooth 1–parameter family $b_t \in \widetilde{\mathrm{Br}}_{\partial S}$ connecting b_0 and b_1. Let $\mathrm{Br}_{\partial S} := \widetilde{\mathrm{Br}}_{\partial S}/\sim$.

Definition 4.12 Let $(\psi, \mathcal{B}_\psi, \mathcal{B}^*_\psi)$ admit a decomposition into singular and regular discs. If L is a Lagrangian submanifold which is carried by $\mathcal{B}$ and is compatible with the decomposition of $\mathcal{B}^*_\psi$, then the *braid* $b(L, S)$ of L on a singular disk S is the braid isotopy class of $\mathrm{Br}_{\partial S}$ which is given by

$$b(L, S) = [L \cap D(\partial S)] \in \mathrm{Br}_{\partial S}.$$

Remark 4.13 The word "braid" comes from the case of $\frac{1}{2}\dim M = n = 2$. If $n = 2$, then f in (4-4) is an element of $\pi_1(\mathrm{Conf}_l(\mathbb{D}^2))$, ie a braid. For a general n, we consider an element of $\pi_n(\mathrm{Conf}_l(\mathbb{D}^n))$.

The notion of braid is defined as an equivalence class in Definition 4.12. However, in the rest of the present paper, if it is not likely to be misunderstood, then we use the word "braid $b(L, S)$" to indicate a representative of the class. This is for the notational convenience. By considering a representative of a braid, we can consider $b(L, S)$ as a subset in $D(\partial S)$. For the case of $\frac{1}{2}\dim M = n = 2$ (resp. general n), a braid $b(L, S)$ is a union of circles (resp. S^{n-1}) embedded in $D(\partial S)$.

Definition 4.14 A *strand* of a braid $b(L, S)$ is a connected component of $b(L, S) \subset D(\partial S)$.

As similar to Remark 4.13, the word "strand" comes from the case of $\frac{1}{2}\dim M = n = 2$.

5 Action of a symplectomorphism

In Section 5.1, we briefly review how one can keep track of the action of a surface automorphism changing the isotopy classes of curves. The action can be written as a linear map acting on the set of weights. Also, we generalized the notion of weights in Section 5.

In Section 5, we generalize the "linear algebra on weights" for higher-dimensional cases.

5.1 Linear algebra on braids

We would like to generalize the linear algebra on weights, which we reviewed in Section 4.1. More precisely, we claim the following:

Claim ($\star$) *If L is carried by $\mathcal{B}_\psi$ and L is compatible with the standard decomposition of $\mathcal{B}^*_\psi$ for a symplectic automorphism ψ of generalized Penner type, then there is a systematic way to obtain*

$$\{b(\psi(L), S) \mid S \text{ is a singular disk of the standard decomposition of } \mathcal{B}^*_\psi\}$$

from

$$\{b(L, S) \mid S \text{ is a singular disk of the standard decomposition of } \mathcal{B}^*_\psi\}.$$

Moreover, the systematic way depends only on ψ, independent of L, as one has a matrix A_ψ for ψ of generalized Penner type as in Section 4.1.

Instead of proving ($\star$), we will prove Lemma 5.1, which considers Dehn twists τ_k and σ_j^{-1} instead of ψ. Recall Remark 3.17 saying that the symplectic manifold M is a plumbing space of copies of T^*S^n and τ_i (resp. σ_j) is a Dehn twist along one of the zero sections of T^*S^n. We also recall Lemma 3.20 saying the following: there exists a set $\mathbb{B}$ of Lagrangian branched submanifolds and functions $F_{\tau_k}: \mathbb{B} \to \mathbb{B}$ (resp. $F_{\sigma_j^{-1}}: \mathbb{B} \to \mathbb{B}$), such that if L is carried by $\mathcal{B} \in \mathbb{B}$, then $\tau_k(L)$ (resp. $\sigma_j^{-1}(L)$) is carried by $F_{\tau_k}(\mathcal{B})$ (resp. $F_{\sigma_j^{-1}}(\mathcal{B})$).

Lemma 5.1 *Let L be a Lagrangian submanifold of M such that L is carried by $\mathcal{B} \in \mathbb{B}$ and L is compatible with the standard decomposition of $\mathcal{B}^*$. Then $\tau_k(L)$ is compatible with the standard decomposition of $F_{\tau_k}(\mathcal{B})$. Moreover, there exists a systematic way to obtain*

$$\{b(\tau_k(L), S) \mid S \text{ is a singular disk of the standard decomposition of } F_{\tau_k}(\mathcal{B})^*\}$$

from

$$\{b(L, S) \mid S \text{ is a singular disk of the standard decomposition of } \mathcal{B}^*\}.$$

The case of σ_j^{-1} is analogous.

Remark 5.2 Since a symplectic automorphism ψ of generalized Penner type is a product of Dehn twists τ_k and σ_j^{-1}, Lemma 5.1 is enough to prove ($\star$).

We will prove Lemma 5.1 in Sections 5.2 and 5.3. The proof will be given for an example case. In the rest of Section 5.1, we will introduce the main idea of the proof. Also, we will introduce the example case which we will consider in Sections 5.2 and 5.3.

The main idea The main idea is to consider $\tau_k(N(\mathcal{B}))$ instead of $\tau_k(L)$. More precisely, for a given singular disk S' of $F_{\tau_k}(\mathcal{B}) = \mathcal{B}'$, we consider $\tau_k(N(\mathcal{B})) \cap D(\partial S')$. One can check that every connected component of $\tau_k(N(\mathcal{B})) \cap D(\partial S')$ is homeomorphic to $S^{n-1} \times \mathbb{D}^n$. For an arbitrary component, there is a map $f_{S \to S', i}: D(\partial S) \to D(\partial S')$, where S is a singular disk of $\mathcal{B}^*$ and a natural number i, such that the image of $f_{S \to S', i}$ is the connected component. In other words, every connected component of $\tau_k(N(\mathcal{B})) \cap D(\partial S')$ is given as the image of a function defined on $D(\partial S)$ where S is a singular disk of $\mathcal{B}$.

The subscription $(S \to S', i)$ of $f_{S \to S', i}$ means that it is a function explaining the contribution of $b(L, S)$ on $b(\tau_k(L), S')$. Since it is possible that there are multiple connected components of $\tau_k(N(\mathcal{B})) \cap D(\partial S')$, which are induced from the same singular disk S, one needs multiple functions, which are labeled by natural numbers i in the subscription.

Since L is carried by $\mathcal{B}$, $L \subset N(\mathcal{B})$. Thus, $\tau_k(L) \subset \tau_k(N(\mathcal{B}))$. By definition,

$$b(\tau_k(L), S') = \tau_k(L) \cap D(\partial S') \subset \tau_k(N(\mathcal{B})) \cap D(\partial S').$$

We consider the intersection of $b(\tau_k(L), S')$ with each connected components of $\tau_k(N(\mathcal{B})) \cap D(\partial S')$. In the connected component, which is the image of $F_{S\to S',i}$, $b(\tau_k(L), S')$ is given by

$$f_{S\to S',i}(b(L,S)).$$

Thus, the set $\{f_{S\to S',i}\}$ of functions gives the systematic way to construct

$$\{b(\tau_k(L),S) \mid S \text{ is a singular disk of the standard decomposition of } F_{\tau_k}(\mathcal{B})^*\}$$

from

$$\{b(L,S) \mid S \text{ is a singular disk of the standard decomposition of } \mathcal{B}^*\}.$$

The example case The symplectic manifold we consider is $M = P(\alpha,\beta_1,\beta_2)$, where α and β_j are spheres such that $\alpha\cap\beta_1 = \{p\}$ and $\alpha\cap\beta_2 = \{q\}$, ie M is a plumbing space of three copies of T^*S^n. Let τ_0 and σ_j be Dehn twists along α and β_j, and $\psi = \tau_0\circ\sigma_1^{-1}\circ\sigma_2^{-1}$. Then Theorem 1.3 gives a Lagrangian branched submanifold $\mathcal{B}_\psi$. For the case of $\dim M = 2n = 2$, Figure 15 describes the example symplectic manifold M. In the example, we will consider the effects of σ_2^{-1} on $\mathcal{B}_\psi$ in Section 5.2 and τ_0 in Section 5.3.

For convenience, we establish notation here. The standard decomposition of $\mathcal{B}_\psi$ has 6 singular disks which are centered at p, $\tau_0(p)$, $\sigma_1^{-1}(p)$, q, $\tau_0(q)$ and $\sigma_2^{-1}(q)$. As mentioned in Remark 4.6, we are abusing notation and pretending that the singular disks are in $\mathcal{B}_\psi$, not in $\mathcal{B}_\psi^*$. We also note that $\tau_0(p)$ and $\tau_0(q)$ are antipodal points of p and q on α. Similarly, $\sigma_1^{-1}(p)$ (resp. $\sigma_2^{-1}(q)$) is the antipodal point of p (resp. q) on β_1 (resp. β_2). Let $S_1,\dots,S_6$ denote the singular disks centered at p, $\tau_0(p)$, $\sigma_1^{-1}(p)$, q, $\tau_0(q)$ and $\sigma_2^{-1}(q)$ respectively. Moreover, let b_i denote $b(L,S_i)$ for $i = 1,\dots,6$.

Similarly, the Lagrangian branched submanifolds $\mathcal{B}' := F_{\sigma_2^{-1}}(\mathcal{B})$ and $\mathcal{B}'' := F_{\tau_0}(\mathcal{B})$ each have 6 singular disks. By definition of the standard decomposition, those singular disks are also centered at p, $\tau_0(p)$, $\sigma_1^{-1}(p)$, q, $\tau_0(q)$ and $\sigma_2^{-1}(q)$. As we did for $\mathcal{B}$, let $S_1',\dots,S_6'$ (resp. $S_1'',\dots,S_6''$) denote the singular disks of $\mathcal{B}'$ (resp. $\mathcal{B}''$) centered at p, $\tau_0(p)$, $\sigma_1^{-1}(p)$, q, $\tau_0(q)$ and $\sigma_2^{-1}(q)$. Moreover, we label

$$b_i = b(L,S_i),\quad b_i' = b(\sigma_2^{-1}(L),S_i'),\quad b_i'' = b(\tau_0(L),S_i'').$$

In the rest of this paper, we make specific choices of τ_0 and σ_j, given by (2-2) and $\tau\colon T^*S^n \xrightarrow{\sim} T^*S^n$, which is defined in Remark 2.2. In other words, $\tau_0 = \phi_\alpha\circ\tau\circ\phi_\alpha^{-1}$ and $\sigma_j = \phi_{\beta_j}\circ\tau\circ\phi_{\beta_j}^{-1}$, where ϕ_α (resp. ϕ_{β_j}) is a symplectomorphism from T^*S^n to a neighborhood of α (resp. β_j). The neighborhood of α (resp. β_j) will be denoted by $T^*\alpha$ (resp. $T^*\beta_j$).

Remark 5.3 Recall that τ is a Dehn twist on T^*S^n which agrees with the antipodal map

$$T^*S^n \xrightarrow{\sim} T^*S^n,\quad (u,v)\mapsto(-u,-v),$$

on a neighborhood of the zero section S^n.

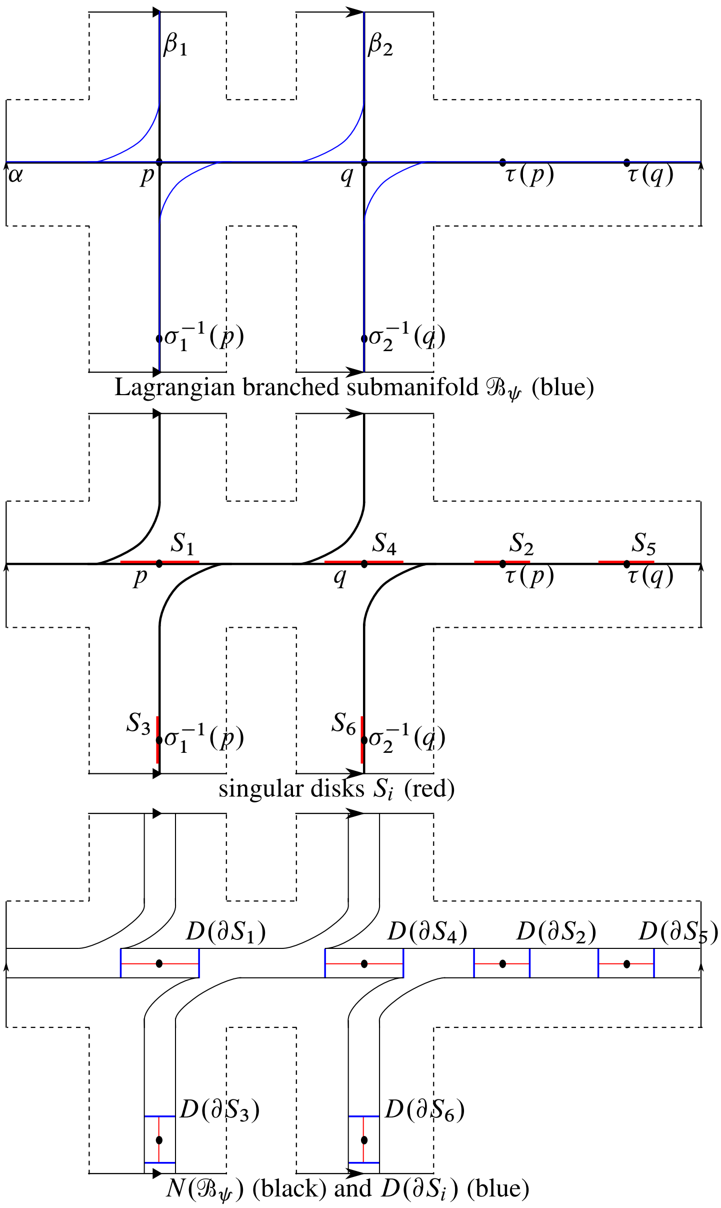

Figure 15: Top: the black curves represent α, β_1 and β_2 in $M = P(\alpha, \beta_1, \beta_2)$, and the blue curve is $\mathcal{B}_\psi$. Middle: the red curves are singular disks S_i. Bottom: the fibered neighborhood $N(\mathcal{B}_\psi)$ and a disk bundle $D(\partial S_i) \simeq \mathbb{D}^1 \times S^0$, ie two intervals attached at ∂S_i.

Remark 5.4 In the next sections, we will consider the example which we specified in the present subsection. Moreover, for convenience, we will assume that the dimension $2n$ of the symplectic manifold M is 4. For the case of $n = 2$, we specify identifications φ_i, φ_i' and φ_i'' from $D(\partial S_i)$, $D(\partial S_i')$ and $D(\partial S_i'')$ to $S^1 \times \mathbb{D}^2$. We would like to point out that there is no reason to choose these specific identifications, this is only for the notational convenience.

In order to construct $\varphi_1\colon D(\partial S_1) \xrightarrow{\simeq} S^1 \times \mathbb{D}^2$, we remark that

$$D(\partial S_1) = \partial\overline{\pi^{-1}(S_1^\circ)}, \quad D(S_1) = \overline{\pi^{-1}(S_1^\circ)}$$

by definition. Thus, in order to specify φ_1, it is enough to identify $D(S_1)$ and $\mathbb{D}^2 \times \mathbb{D}^2$. We remark that $D(S_1)$ is a disk bundle over $S_1 \simeq \mathbb{D}^2$.

By abuse of notation, let's assume that $S_1 \subset \mathbb{B}_\psi$, not $\mathbb{B}'_\psi$. Then S_1 is a Lagrangian disk in M. Thus, $D(S_1)$ is a small neighborhood of a Lagrangian disk S_1. By the Lagrangian neighborhood theorem [16], it is enough to choose coordinate charts on S_1. Similarly, it is enough to choose coordinate charts for S_i, S_i' and S_i''.

In order to choose specific coordinate charts, we use the symplectic submanifold $W \subset M$ defined in Section 4.2.

Let (x_1, x_2) be a coordinate chart on $S_1 \subset \alpha$ such that the x_1–axis agrees with $W \cap S_1$. There are two choices for the positive x_1–direction corresponding to the two orientations of $W \cap S_1$, or equivalently orientations of C_α. We can choose either of them. Then, let (y_1, y_2) be an oriented chart on S_2 such that the y_1–axis agrees with $W \cap \beta_1$ and $\omega(\partial_{x_1}, \partial_{y_1}) > 0$. The positive y_1–direction determines an orientation of C_{β_1}. On S_3, there exists an oriented chart (x_1, x_2) such that the positive x_1–direction agrees with the orientation of C_α. For the other singular disks, we obtain oriented coordinate charts from the orientations of C_α, C_{β_i}, α and β_i in the same way.

5.2 Effect of σ_2^{-1}

In Section 5.2, we discuss how $\{b_i' \mid i = 1, \dots, 6\}$ are obtained from $\{b_i \mid i = 1, \dots, 6\}$. Since σ_2^{-1} is supported on $T^*\beta_2$, a small neighborhood of β_2, b_i and b_i' are the same braid in $\mathrm{Br}_{\partial S_i}$ for $i = 1, 2, 3$ and 5. We will explain how b_6' is constructed.

We can obtain $\sigma_2^{-1}(\mathcal{B}_\psi)$ by spinning with respect to q in $T^*\beta_2$, ie $\sigma_2^{-1}(\mathcal{B}_\psi)$ is the union of curves in a 2–dimensional submanifold $\phi_{\beta_2}(W_y)$ over $y \in S^{n-1}$. Recall that the spinning and W_y are defined in Section 2.2.

Figure 16 represents a support of σ_2^{-1} in M, ie a small neighborhood of $\beta_2 \subset M$ where M is a symplectic manifold of dimension 2 given in Figure 15. Similarly, in Section 5.2, the rectangles in Figures 15–19 are the support of σ_2^{-1}.

By spinning blue, red, and green points in Figure 16, we obtain $\sigma_2^{-1}(\mathcal{B}_\psi) \cap D(\partial S_6')$. Let B, R and G be obtained by spinning constant curves drawn blue, red and green points in Figure 16, respectively.

Since $N(\mathcal{B}_\psi) \supset \mathcal{B}_\psi$, $\sigma_2^{-1}(N(\mathcal{B}_\psi)) \cap D(\partial S_6')$ is a neighborhood of $\sigma_2^{-1}(\mathcal{B}_\psi) \cap D(\partial S_6')$. By assuming that $N(\mathcal{B}_\psi)$ is a sufficiently small neighborhood of $\mathcal{B}_\psi$, $\sigma_2^{-1}(N(\mathcal{B}_\psi)) \cap D(\partial S_6')$ consists of three connected components, which are neighborhoods of B, R and G. Each connected component will be called $N(B)$, $N(R)$ and $N(G)$.

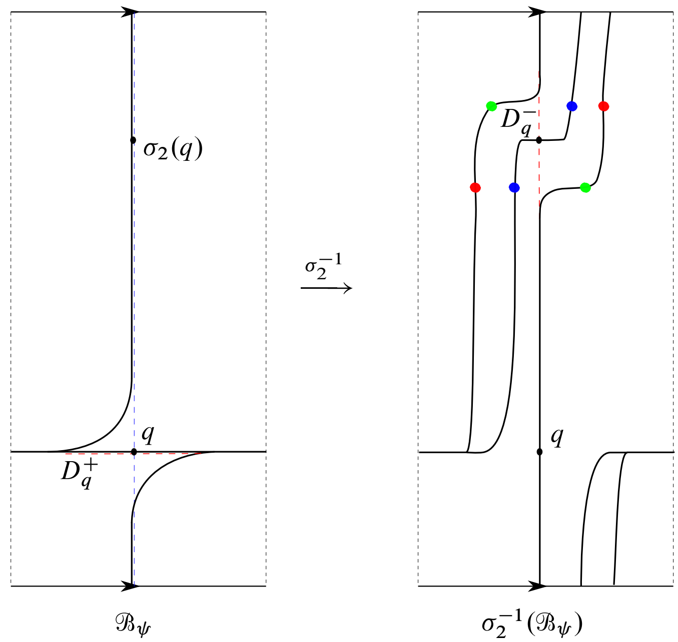

Figure 16: The left picture represents $\mathcal{B}_\psi \cap \phi_{\beta_2}(W_y)$ and the right picture represents $\sigma_2^{-1}(\mathcal{B}_\psi) \cap \phi_{\beta_2}(W_y)$.

Since $b_6' = \sigma_2^{-1}(L) \cap D(\partial S_6') \subset N(B) \sqcup N(R) \sqcup N(G)$, b_6' is divided into three groups, which are contained in $N(B)$, $N(R)$ and $N(G)$ respectively. We argue the group which is contained in $N(B)$ first.

Let assume that $\sigma_2^{-1}(S_4) = S_6'$. Then $\sigma_2^{-1}(\partial S_4) = \partial S_6'$. Moreover, if $\sigma_2^{-1}(D(\partial S_4))) \subset D(\partial S_6')$, then $N(B) = \sigma_2^{-1}(D(\partial S_4)) \subset D(\partial S_6')$. Also, one concludes that $\sigma_2^{-1}|_{D(\partial S_4)} \colon D(\partial S_4) \xrightarrow{\sim} N(B)$. If one can assume that b_4 is a subset of $D(\partial S_4)$ by definition of braids, the set of braids of b_6' inside $N(B)$ is $\sigma_4^{-1}(b_4)$.

However, $\sigma_2^{-1}(S_4)$ is not s_6'. Thus, we will construct a Hamiltonian isotopy Φ_t so that there exists a slightly smaller disk D_B is S_4 satisfying

$$(\Phi_1 \circ \sigma_2^{-1})(D_B) = S_6'.$$

Note that "slightly smaller" means that there is no singular value on $S_4 \setminus D_B$. Then

$$(\Phi_1 \circ \sigma_2^{-1})(D(\partial D_B)) = N(B),$$

where $D(\partial D_B)$ is defined as similar to Definition 4.10. The strands of b_6' in $N(B)$ will be given by $(\Phi_1 \circ \sigma_2^{-1})(D(\partial D_B) \cap L)$. Moreover, $D(\partial D_B)$ (resp. $D(\partial D_B) \cap L$) and $D(\partial S_4)$ (resp. $b_4 = D(\partial S_4) \cap L$) are naturally isotopic. Under the isotopic relation, there is a function $f_1 \colon D(\partial S_4) \to D(\partial S_6')$ such that the strands of b_6' in $N(B)$ are $f_1(b_4)$.

From now on, we will construct a specific Φ_t. For notational simplicity, we assume that $\dim(M) = 4$, but the construction of Φ_t is easily generalized for the case of higher dimensions.

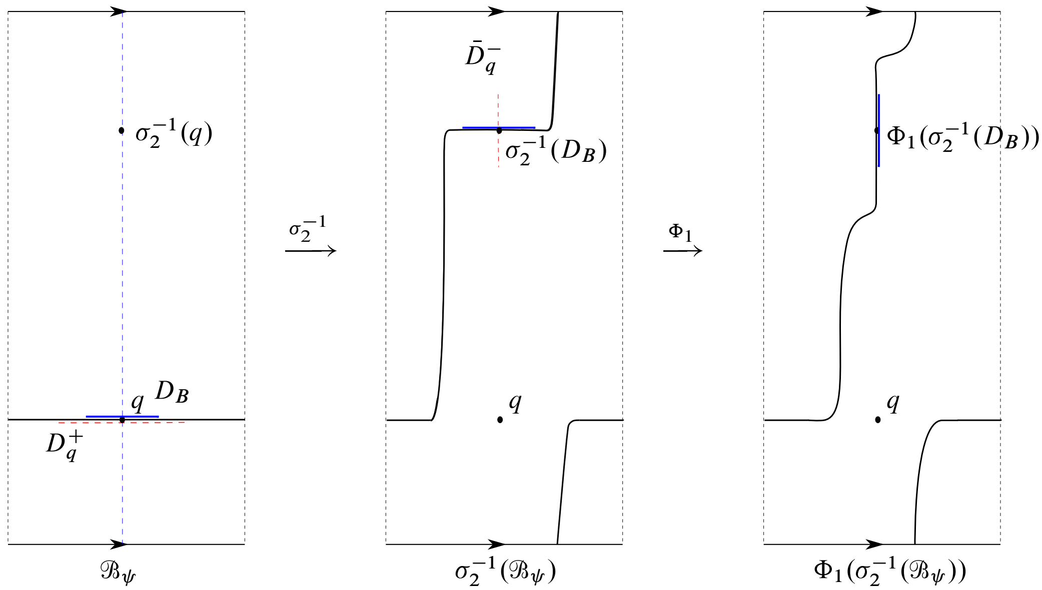

Figure 17: The blue curves represent $\widetilde{D}_B \cap \phi_{\beta_2}(W_y)$ in the left picture, $\sigma_2^{-1}(\widetilde{D}_B) \cap \phi_{\beta_2}(W_y)$ in the middle picture, and $\Phi_1(\sigma_2^{-1}(\widetilde{D}_B)) \cap \phi_{\beta_2}(W_y)$ in the right picture.

We choose a neighborhood $U \subset \beta_2$ of $\sigma_2^{-1}(q)$ and a Darboux chart $\phi_q : T^*U \xrightarrow{\sim} \mathbb{R}^4$ such that $\phi_q(\sigma_2^{-1}(q))$ is the origin. We remark that $T^*\beta_2$ denotes a neighborhood of β_2 in M, which is symplectomorphic to the cotangent bundle of β_2. Thus, for a subset U of β_2, one can assume that T^*U is a subset of M.

For convenience, let $\phi_q(x) = (x_1, x_2)$ where $x_i \in \mathbb{R}^2$. Then there is a Hamiltonian isotopy

$$\Phi_t(x) = \begin{cases} (\phi_q^{-1} \circ H_{t\delta(c_1\|x_1\|+c_2\|x_2\|)} \circ \phi_q)(x) & \text{if } x \in T^*U, \\ x & \text{if } x \notin T^*U, \end{cases} \tag{5-1}$$

where c_i is a positive constant, $\|\cdot\|$ is the standard norm on $\mathbb{R}^2$, and H_t and δ are defined as follows: let $H_t : \mathbb{R}^4 \to \mathbb{R}^4$ be a Hamiltonian isotopy given by

$$H_t = \begin{pmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & \cos t & 0 & -\sin t \\ \sin t & 0 & \cos t & 0 \\ 0 & \sin t & 0 & \cos t \end{pmatrix},$$

and let $\delta : [0, \infty) \to \mathbb{R}$ be a smooth decreasing function such that $\delta(x) = \frac{\pi}{2}$ for all $x < 1$ and $\delta(x) = 0$ for all $x > 2$.

Figure 17 represents the case of $\dim M = 2$. We note that the rectangles in Figure 17 represent a support of σ_2^{-1}. By choosing proper constants c_i, we obtain a small disk $D_B \subset S_4$ such that

$$(\Phi_1 \circ \sigma_2^{-1})(D(\partial D_B)) \subset D(\partial S_6').$$

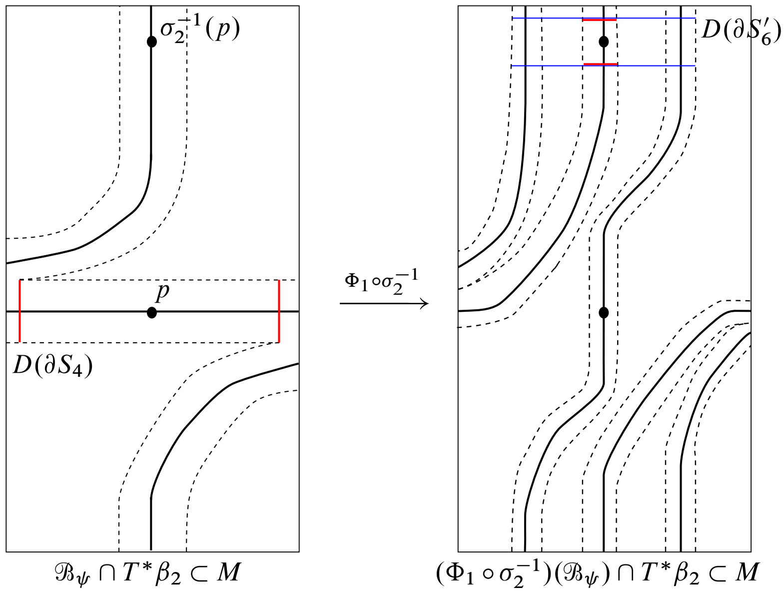

Figure 18: Left: the whole rectangle is a neighborhood of β_2 in M; thick black curves are parts of $\mathscr{B}_\psi$ and the dashed black curves are $N(\mathscr{B}_\psi)$; the red thick curves represent $D(\partial S_4)$. Right: thick black curves are $(\Phi_1 \circ \sigma_2^{-1})(\mathscr{B}_\psi)$, dashed black curves are $(\Phi_1 \circ \sigma_2^{-1})(N(\mathscr{B}_\psi))$, blue curves are $D(\partial S_6')$, and thick red curves represent the part of $D(\partial S_6')$ where $D(\partial S_4)$ contributes.

On a small neighborhood of D_B, σ_2^{-1} agrees with the antipodal map of $\phi_{\beta_2}(T^*\beta_2) \simeq T^*S^2$, as we mentioned in Remark 5.3. Then we obtain a map

$$f_1 \colon S^1 \times (\mathbb{D}^2)^\circ \simeq \pi^{-1}(\partial D_B) \xrightarrow{\Phi_1 \circ \sigma_2^{-1}} D(\partial S_6') \simeq S^1 \times \mathbb{D}^2, \quad (\theta, x, y) \mapsto (\theta + \pi, -r_1 x, -r_1 y).$$

The first and the last identifications are the natural identifications mentioned in Remark 4.11. The reason we consider the natural identification is for notational convenience, ie in order to write f_1 as a map on $(\theta, x, y) \in S^1 \times \mathbb{D}^2$. Then, the strands of b_6' in $N(B)$ is given by $f_1(b_4)$.

Figure 18 is a picture summarizing the whole process obtaining strands of b_6' in the first group, or equivalently, the picture explains how b_4 contributes on the construction of b_6', in the case $\dim M = 2n = 2$.

In order to study the construction of strands of b_6' in $N(R)$ and $N(G)$, one should consider

$$\widetilde{D}(\partial S_4) := \bigcup_{p \in \mathrm{Locus}(\mathscr{B}') \cap \partial S_4} F_p.$$

It is easy to check that $\widetilde{D}(\partial S_4)$ is a $\mathbb{D}^n$–bundle over ∂S_4 and $D(\partial S_4) \subset \widetilde{D}(\partial S_4)$.

Together with $\widetilde{D}(\partial S_4)$, we observe how b_6 contributes on the construction of b_6'. First, one can observe that b_6 and $L \cap (\widetilde{D}(\partial S_4) \setminus D(\partial S_4))$ are isotopic to each other. The isotopy connecting them is along the fibers on some regular disks such that the union of regular disks has ∂S_4 and ∂S_6 as their boundaries.

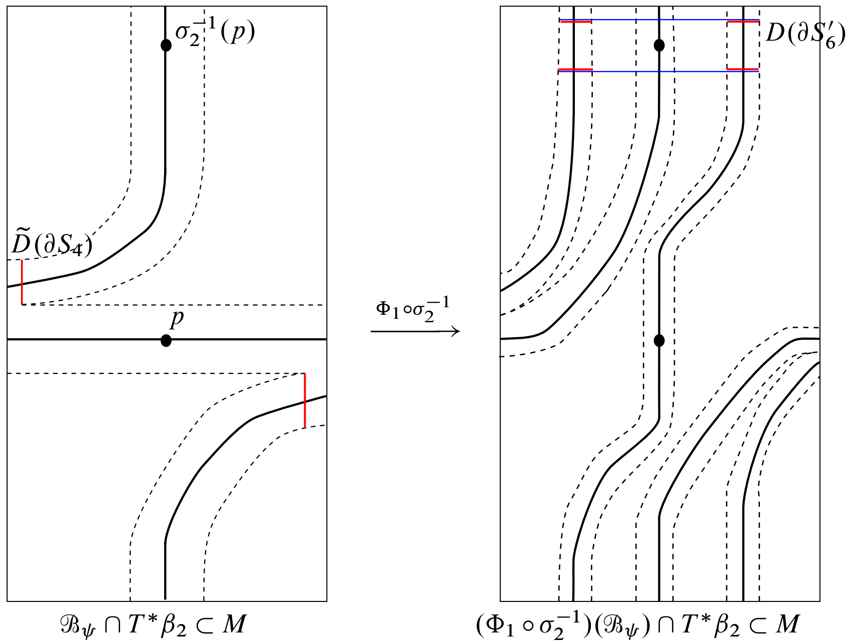

Figure 19: Left: the whole rectangle is a neighborhood of β_2 in M, thick black curves are parts of $\mathcal{B}_\psi$ and the dashed black curves are $N(\mathcal{B}_\psi)$, and the red thick curves represent $\widetilde{D}(\partial S_4)$. Right: thick black curves are $(\Phi_1 \circ \sigma_2^{-1})(\mathcal{B}_\psi)$, dashed black curves are $(\Phi_1 \circ \sigma_2^{-1})(N(\mathcal{B}_\psi))$, blue curves are $D(\partial S_6')$, and thick red curves represent the part of $D(\partial S_6')$ where $D(\partial S_4)$ contributes.

More precisely, the union of regular disks (resp. fibers on them) is homeomorphic to $S^{n-1} \times [0,1]$ (resp. a disk bundle over $S^{n-1} \times [0,1]$). The boundary of $S^{n-1} \times [0,1]$ corresponds to ∂S_4 and ∂S_6.

Similarly, one can observe that $L \cap (\widetilde{D}(\partial S_4) \setminus D(\partial S_4))$ and b_6 are isotopic to each other. The isotopy connecting them is the intersection of L and the fibers on the regular disks.

Second, one can describe the contribution of $L \cap (\widetilde{D}(\partial S_4) \setminus D(\partial S_4))$ on the contribution of b_6'. The contributions are given as two functions as the contribution of b_4 is described by the function f_1. For the case of $n = 2$ and under the identification defined in Remark 5.4, the two functions denoted by f_2 and f_3 are

$$f_2 \colon S^1 \times \mathbb{D}^2 \to S^1 \times \mathbb{D}^2, \quad (\theta, x, y) \mapsto (\theta, r_0 \cos\theta + r_2 x, r_0 \sin\theta + r_2 y),$$

and

$$f_3 \colon S^1 \times \mathbb{D}^2 \to S^1 \times \mathbb{D}^2,$$
$$(\theta, x, y) \mapsto (\theta, -r_0 \cos\theta + r_2(x \cos 2\theta - y \sin 2\theta), -r_0 \sin\theta + r_2(x \sin 2\theta + y \cos 2\theta)),$$

Similar to Figure 18, Figure 19 summarizes the whole process obtaining strands of b_6' in the second and third groups, or equivalently, the picture explains how $\widetilde{D}(\partial S_4)$ contributes on the construction of b_6', for the case of $\dim M = 2n = 2$.

Remark 5.5 (1) The constant r_1 is determined by the choice of an identification $\phi_{\beta_2}: T^*S^2 \xrightarrow{\sim} T^*\beta_2$, the fixed Dehn twist τ in Remark 2.2, and so on. However, r_1 has to be smaller than 1. This is because $\mathrm{Im}(f_1)$, $\mathrm{Im}(f_2)$ and $\mathrm{Im}(f_3)$ are mutually disjoint, since they correspond to $N(B)$, $N(R)$ and $N(G)$, respectively. Moreover, r_0 and r_2 are also positive numbers smaller than 1.

(2) Note that r_0 and r_2 are positive constants which are determined by specific choices. However, r_0 and r_2 have to satisfy $r_1 + r_2 < r_0$, since $\mathrm{Im}(f_1)$, $\mathrm{Im}(f_2)$ and $\mathrm{Im}(f_3)$ are mutually disjoint.

(3) To obtain f_1, we used a Hamiltonian isotopy Φ_t. Similarly, to obtain f_2 and f_3, we need a Hamiltonian isotopy.

The situation for b_4' is analogous. We obtain three maps g_1, g_2 and g_3 in the same way. At the end, b_4' is represented by $g_1(b_6) \sqcup g_2(b_6) \sqcup g_3(b_6)$. This proves Lemma 5.1 for the case of σ_2^{-1}.

Note that maps f_i and g_j are given by specific maps acting on $S^1 \times \mathbb{D}^2$, but we would like to consider them as maps on $\widetilde{\mathrm{Br}}_{\partial S_k}$ for some k. We summarize the effect of σ_2^{-1} as the matrix

$$\Sigma_{2,\mathcal{B}_\psi} = \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_1 + g_2 + g_3 \\ 0 & 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & f_1 & 0 & f_2 + f_3 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} b_1' \\ b_2' \\ b_3' \\ b_4' \\ b_5' \\ b_6' \end{pmatrix} = \Sigma_{2,\mathcal{B}_\psi} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ g_1(b_6) \sqcup g_2(b_6) \sqcup g_3(b_6) \\ b_5 \\ f_1(b_4) \sqcup f_2(b_6) \sqcup f_3(b_6) \end{pmatrix}.$$

Remark 5.6 In surface theory, we can do linear algebra on weights, but in a higher-dimensional case, we cannot do linear algebra with the matrix $\Sigma_{2,\mathcal{B}_\psi}$ because there is no module structure on $\widetilde{\mathrm{Br}}_{\partial S_i}$. In other words, the matrix $\Sigma_{2,\mathcal{B}_\psi}$ and sums of functions, for example $g_1 + g_2 + g_3$, are for notational convenience. Thus, the title of Section 5.1 is an abuse of terminologies.

5.3 Effect of τ_0

The situation for τ_0 is similar to that for σ_2^{-1}. For example, by observing how τ_0 acts on $D(\partial S_1)$, we obtain

$$h_1: S^1 \times \mathbb{D}^2 \to S^1 \times \mathbb{D}^2,$$

explaining the contribution of b_1 on the construction of b_3''. Then, h_1 is given by a translation on S^1 and a scaling on $\mathbb{D}^2$, as f_1 is. Similarly, we obtain h_2 and h_3, which explain the contributions of b_3 on the construction of b_3''. The maps h_2 and h_3 are of the same type as f_2 and f_3, respectively, ie

$$h_2(\theta,x,y) = (\theta \text{ or } \theta+\pi, \pm r_1\cos\theta + r_2 x, \pm r_1 \sin\theta + r_2 y),$$
$$h_3(\theta,x,y) = \big(\theta \text{ or } \theta+\pi, \pm r_1\cos\theta + r_2(x\cos 2\theta - y\sin 2\theta), \pm r_1\sin\theta + r_2(x\sin 2\theta + y\cos 2\theta)\big),$$

where r_1 and r_2 are constants.

We say that a map is of *scaling type* if a map is of the same type as f_1, in other words, if the map is given by a translation on S^1 and a scaling on $\mathbb{D}^2$. This is because the formula defining the map is given by a scaling on fibers. The maps of scaling type explain how the braids along the singular disk centered at p or antipodes of p, $b(L, S_p(\mathcal{B}_\psi))$ or $b(L, S_p^\pm)$, contribute on the braid along the singular points centered at the same points, $b\big(\delta(L), S_p(F_\delta(\mathcal{B}_\psi))\big)$ or $b(\delta(L), S_p^\pm)$, when one applies a Dehn twist δ.

We say that a map is of *the first* (*resp. second*) *singular type* if a map is of the same type as f_2 (resp. f_3). This is because they are related to a creation of new singular component. The maps of the first and second singular types explain how the braid $b(L, S_p^+)$ contributes on the construction of the braid $b\big(\delta(L), S_p(F_\delta(\mathcal{B}_\psi))\big)$.

To summarize, if b_i contributes the construction of b_j' and if the center of a singular disk corresponding to b_i is either the same point or the antipodal point of the center of the singular disk corresponding to b_j', maps of these three types explain the contribution of b_i on the construction of b_j'. Note that the center of a singular disk is defined in Remark 3.21.

The maps of these three types explain the effects of σ_2^{-1} on $\mathcal{B}$. However, to explain the effects of τ_0 on $\mathcal{B}_\psi$, we need maps of one more type. The reason is given in Figure 20, roughly. We note that the rectangles in Figure 20 are the support of τ_0 in M where M is given in Figure 15, ie a neighborhood of $\alpha \subset M$.

More precise reasoning is as follows. We note that α has two plumbing points, unlike β_i which has only one plumbing point. Thus, when we apply τ_0, b_i can contribute to b_j'' even if the centers of singular disks corresponding to b_i and b_j'' are neither the same nor antipodes of each other. For example, $L \cap \pi^{-1}(\pi(N_p))$ is stretched by τ_0. The stretched part $\tau_0\big(L\cap\pi^{-1}(\pi(N_p))\big)$ has intersection with $D(\partial S_4'')$ and $D(\partial S_5'')$ as one can see in Figure 20. Thus, b_4'' has some strands corresponding to $\tau_0\big(L\cap\pi^{-1}(\pi(N_p))\big)\cap D(\partial S_4)$. These strands are the contribution of b_3 on the construction of b_4''. Similarly, b_3 contributes to the construction of b_5'', and b_6 contributes to the constructions of b_1'' and b_2''.

To describe the contribution of b_3 on b_4'', without loss of generality, we assume that there is no singular value for

$$\tau_0\big(L\cap\pi^{-1}(\pi(N_p))\big)\cap D(S_4) \xrightarrow{\pi} S_4,$$

by Remark 3.21. Thus, $\tau_0\big(L\cap\pi^{-1}(\pi(N_p))\big)\cap D(S_4)$ is a union of disjoint Lagrangian disks on $D(S_4)$. We note that $D(S_4)$ is a disk bundle over $(S_4')^\circ$ which is an open disk. Thus, on the boundary $D(\partial S_4')$,

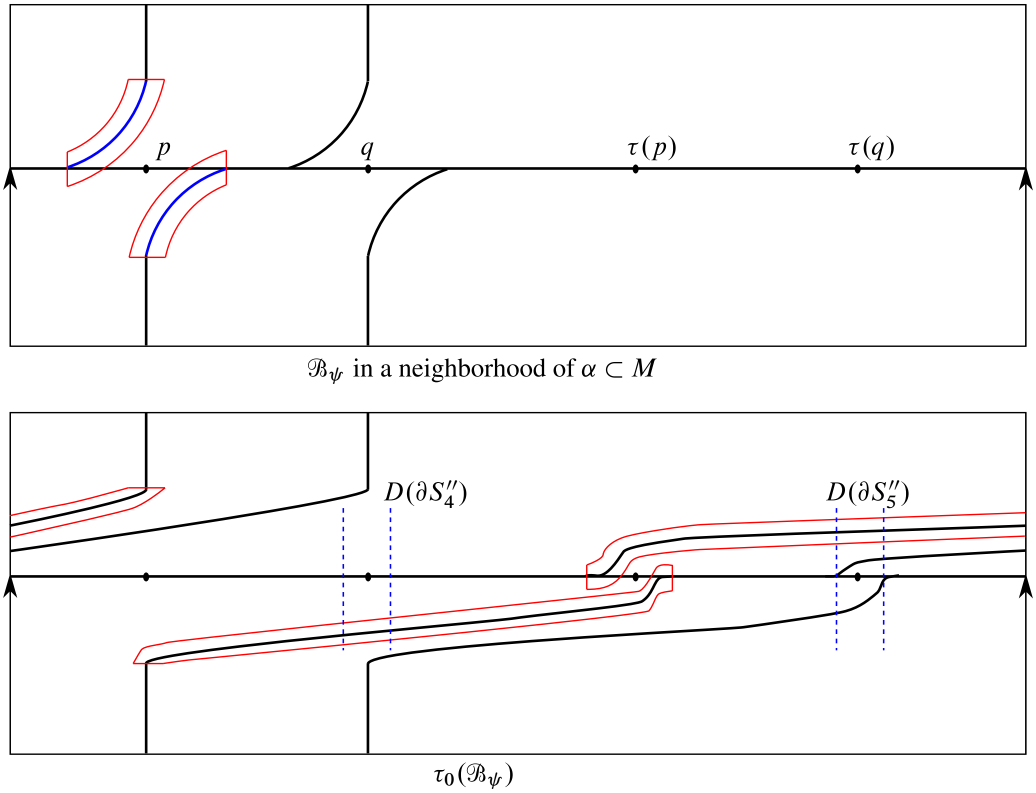

Figure 20: Top: the thick black and blue curves are $\mathcal{B}_\psi$ in a neighborhood of $\alpha \subset M$; in particular, the blue curves are N_p and the red parts are a fibered neighborhood of N_p, ie $\pi^{-1}(\pi(N_p))$. Bottom: the thick curves are $\tau_0(\mathcal{B}_\psi)$, the red parts are extended neighborhood of N_p by applying τ_0, and blue dashed lines are $D(\partial S''4)$ and $D(\partial S_5'')$.

b_3 contributes to b_4'' by adding strands near $\tau_0(N_p) \cap D(\partial S_4)$ which are not braided to each other. The number of the added strands is the same as the number of strands of b_3. In the same way, b_3 contributes to the construction of b_5''.

Remark 5.7 In the above argument, we said that the added strands are not braided to each other. To be more rigorous, we should specify the meaning of "not braided". We remark that $D(\partial S_4'')$ is identified with $S^{n-1} \times \mathbb{D}^n$ by the specific identification given in Remark 5.4. The added strands are not braided in $S^{n-1} \times \mathbb{D}^n$ after the identification.

As we did before, we would like to describe the added strands as an image of a function defined on $D(\partial S) = S^{n-1} \times \mathbb{D}^n$. In Section 5.3, we consider the case of $\dim(M) = 4$ as we did in Section 5.2 under the identifications given in Remark 5.4.

Let h_t be the function defined on $S^1 \times \mathbb{D}^2$. As we explained in Section 5.2, we expect that $h_t(b_3)$ can explain the contribution of b_3. However, for this case, $h_t(b_3)$ cannot do that. This is because the important factor is the number of strands of b_3, not that b_3 is braided. Thus, we define a trivial braid b_3^t such that b_3^t and b_3 have the same number of strands as

$$b_3^t := \varphi_3^{-1}(\{(\theta, x_0, y_0) \in S^1 \times \mathbb{D}^2 \mid (0, x_0, y_0) \in \varphi_3(b_3)\}) \subset D(\partial S_3).$$

Then, one obtains

$$h_t \colon S^1 \times \mathbb{D}^2 \xrightarrow{\varphi_1} \pi^{-1}(\partial S_1) \xrightarrow{\Phi_1 \circ \tau_0} \pi^{-1}(\partial S_4) \overset{\varphi_4'}{\simeq} S^1 \times \mathbb{D}^2, \quad (\theta, x, y) \mapsto (\theta, r_0 x + c_1, r_0 y + c_2),$$

where r_0 is a positive constant number less than 1 and Φ_1 is a Hamiltonian isotopy. We note that in Section 5.2, we needed a Hamiltonian isotopy. In a similar way, we can construct a Hamiltonian isotopy Φ_1. Then $h_t(\bar{b}_1^\circ)$ represents the added strands in b_4', which correspond to $\tau_0\big(L \cap \pi^{-1}(\pi(N_p))\big)$.

Similarly, if b_i contributes the construction of b_j'' and if the center of a singular disk corresponding to b_i is neither the same point nor the antipodal point of the center of the singular disk corresponding to b_j'', then the contribution of b_i on b_j'' can be described by a map like h_t. A map is of *trivial type* if a map is of the same type with h_t, because a map of trivial type adds strands which are not braided with each other.

Then, we can describe the effect of τ_0 on $\mathcal{B}_\psi$ as a matrix

$$\mathrm{T}_{0,\mathcal{B}_\psi} = \begin{pmatrix} 0 & i & 0 & 0 & 0 & h_t \\ h_1 & 0 & h_2 + h_3 & 0 & 0 & i_t \\ 0 & 0 & \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & h_t & 0 & i & 0 \\ 0 & 0 & i_t & h_1 & 0 & h_2 + h_3 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Among the entries, h_1, i and id are of scaling type, h_2 and h_3 are of the first and second singular types, and h_t and i_t are of trivial type.

Remark 5.8 A ψ of generalized Penner type is a product of Dehn twists. In the general case, when we apply ψ, each Dehn twist is followed by a Hamiltonian isotopy as σ_2^{-1} is followed by Φ_t in step two. Let $\psi_H = (\Phi_{1,1} \circ \delta_1) \circ \cdots \circ (\Phi_{l,1} \circ \delta_l)$, where $\psi = \delta_1 \circ \cdots \circ \delta_l$, δ_i is a Dehn twist, and $\Phi_{i,t}$ is a Hamiltonian isotopy which follows δ_i.

After applying the Hamiltonian isotopy, the effect of a Dehn twist τ_i (resp. σ_j^{-1}) on $\mathcal{B} \in \mathbb{B}$ is described by a matrix $\mathrm{T}_{i,\mathcal{B}}$ (resp. $\Sigma_{j,\mathcal{B}}$), whose entries are sums of maps of four types.

6 Proof of Theorem 1.5

In Sections 4 and 5, we generalized the notion of weights and linear algebra on weights. In this section, we prove our main theorem, ie Theorem 1.5, by using those generalizations.

6.1 Limit of a sequence of braids

By Lemma 5.1, one obtains braid sequences $\{b(\psi^m(L), S_i)\}_{m\in\mathbb{N}}$, where L is carried by $\mathcal{B}_\psi$, and S_i is a singular disk of $\mathcal{B}_\psi^*$. In the present subsection, we construct a limit of $\{b(\psi^m(L), S_i)\}_{m\in\mathbb{N}}$ as $m \to \infty$.

We argue with the above example, ie

$$M = P(\alpha, \beta_1, \beta_2), \quad \psi = \tau_0 \circ \sigma_1^{-1} \circ \sigma_2^{-1}, \quad \dim M = 4$$

For convenience, let

$$\mathcal{B} := \mathcal{B}_\psi, \quad \mathcal{B}' := F_{\sigma_2^{-1}}(\mathcal{B}), \quad \mathcal{B}'' := F_{\sigma_1^{-1}}(\mathcal{B}'),$$

and let S_i, S_i' and S_i'' denote singular disks of $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$. Using notation from Sections 5.2 and 5.3, we have matrices $\mathrm{T}_{0,\mathcal{B}''}$, $\Sigma_{1,\mathcal{B}'}$ and $\Sigma_{2,\mathcal{B}}$. Then we obtain $\Psi = \mathrm{T}_{0,\mathcal{B}''} \cdot \Sigma_{1,\mathcal{B}'} \cdot \Sigma_{2,\mathcal{B}}$ by defining a multiplication of maps as the composition of them. Note that a product of two arbitrary matrices is not defined since a composition of two arbitrary functions is not defined. For example, an input of $\Sigma_{2,\mathcal{B}}$ and an output of $\mathrm{T}_{0,\mathcal{B}''}$ are tuples of braids on singular disks of $\mathcal{B}^*$. Thus, $\Sigma_{2,\mathcal{B}} \cdot \mathrm{T}_{0,\mathcal{B}''}$ is defined. However, $\mathrm{T}_{0,\mathcal{B}''} \cdot \Sigma_{2,\mathcal{B}}$ is not defined since an input of $\mathrm{T}_{0,\mathcal{B}''}$ is a tuple of braids on singular disks of $\mathcal{B}^*$, but an output of $\Sigma_{2,\mathcal{B}}$ is a tuple of braids on singular disks of $\mathcal{B}'^*$.

Let $b_{i,m} = b(\psi^m(L), S_i)$. Then

$$\begin{pmatrix} b_{1,m} \\ b_{2,m} \\ b_{3,m} \\ b_{4,m} \\ b_{5,m} \\ b_{6,m} \end{pmatrix} := \Psi^m \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix}.$$

Thus, in order to keep track of braid sequences $\{b_{i,m}\}_{m\in\mathbb{N}}$, it is enough to keep track of Ψ^m.

Every entry of Ψ^m is a sum of compositions of $3m$ maps. The image of a composition of $3m$ maps is a solid torus. By Remark 5.5, the radius of each solid torus appearing in Ψ^m decreases exponentially and converges to zero as $m \to \infty$.

In order to be more precise, we consider ψ_H which is defined in Remark 5.8. One observes

$$b_{i,m} \subset \psi_H^m(N(\mathcal{B}_\psi)) \cap D(\partial S_i)$$

for all $m \in \mathbb{N}$ and $i = 1, \dots, 6$. Let

$$B_{i,m} := \psi_H^m(N(\mathcal{B}_\psi)) \cap D(\partial S_i).$$

Then $B_{i,m}$ is the disjoint union of solid tori. Each solid torus in $B_{i,m}$ is the image of a composition of $3m$ maps, appearing in Ψ^m. Conversely, for each composition of $3m$ maps appearing in Ψ^m, the image is a solid torus contained in $B_{i,m}$. The radii of solid tori in $B_{i,m}$ are decreasing exponentially and are converging to zero as $m \to \infty$.

Since $B_{i+1,m} \subset B_{i,m}$ for all $m \in \mathbb{N}$, there is a limit

$$B_{i,\infty} := \lim_{m\to\infty} B_{i,m} = \bigcap_{m\in\mathbb{N}} B_{i,m}.$$

Thus, $B_{i,\infty}$ is the union of infinite strands as a subset of $D(\partial S_i)$ and

$$\lim_{m\to\infty} b_{i,m} = B_{i,\infty}$$

as a sequence of closed sets in $D(\partial S_i)$.

Remark 6.1 (1) We have constructed a sequence of specific representatives $\{b_{i,m}\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} b_{i,m} = B_{i,\infty}.$$

For the purposes of extending the lamination to the singular and regular disks in Sections 6.2 and 6.3, we assume that the limit $B_{i,\infty}$ is a specific closed subset in $D(\partial S_i)$.

(2) Each strand of $B_{i,\infty}$ corresponds to an infinite sequence $\{f_m\}_{m\in\mathbb{N}}$ such that $f_1 \circ \cdots \circ f_{3m}$ appears in Φ^m for all $m \in \mathbb{N}$.

6.2 Lagrangian lamination on a singular disk

Let ψ be of generalized Penner type and let L be a Lagrangian submanifold which is carried by $\mathcal{B}_\psi$. In the previous sections, on each singular disk S_i, we gave an inductive description of a sequence $\{b_{i,m} := b(\psi^m(L), S_i)\}_{m\in\mathbb{N}}$. There is a limit $B_{i,\infty}$ of the sequence, which is independent of L. In this present subsection, we will construct a Lagrangian lamination $\mathcal{L}_i \subset \pi^{-1}(S_i)$ from $B_{i,\infty}$.

Lemma 6.2 *Let ψ be of generalized Penner type. For each singular disk S_i of $\mathcal{B}_\psi$, there is a Lagrangian lamination $\mathcal{L}_i \subset D(S_i)$, such that if L is a Lagrangian submanifold of M which is carried by $\mathcal{B}_\psi$, then for every $m \in \mathbb{N}$, there is a Lagrangian submanifold L_m which is Hamiltonian isotopic to $\psi^m(L)$ and $L_m \cap D(S_i)$ converges to $\mathcal{L}_i$ as a sequence of closed subsets.*

Proof Let ψ be of generalized Penner type, ie $\psi = \delta_1 \circ \cdots \circ \delta_l$, where δ_k is a Dehn twist τ_i or σ_j^{-1}. We will use similar notation as the previous subsections; for example, S_i denotes a singular disk of $\mathcal{B}_\psi$, Ψ denotes a matrix corresponding to ψ, $\varphi_i : D(\partial S_i) \xrightarrow{\sim} S^{n-1} \times \mathbb{D}^n$ denotes the identification induced from the fixed coordinate chart on S_i, and so on.

We will assume that L_m in Lemma 6.2 is $\psi_H^m(L)$ where ψ_H is defined in Remark 5.8. Then $\mathcal{L}_i$ is the limit of $\psi_H^m(L) \cap D(S_i)$ as $m \to \infty$. Thus, $\mathcal{L}_i \cap D(\partial S_i)$ is the limit of $\psi_H^m(L) \cap D(\partial S_i)$, ie $\mathcal{L}_i \cap D(\partial S_i) = D_{i,\infty}$. We will construct a Lagrangian lamination $\mathcal{L}_i$ when $B_{i,\infty}$ is given. Then we will prove that Lemma 6.2 holds with the constructed $\mathcal{L}_i$.

Construction of $\mathcal{L}_i$ As we mentioned in Remark 6.1, each strand of $B_{i,\infty}$ is identified with an infinite sequence $\{f_m\}_{m\in\mathbb{N}}$ such that $f_1 \circ \cdots \circ f_{lk}$ appears in Ψ^k for all $k \in \mathbb{N}$. For each strand $\{f_m\}_{m\in\mathbb{N}}$ of $B_{i,\infty}$, we will construct a Lagrangian submanifold of $D(S_i)$ whose boundary agrees with the strand $\{f_m\}_{m\in\mathbb{N}}$ in the construction part.

First, for a given strand $\{f_m\}_{m\in\mathbb{N}}$, let f_1 be of trivial type. Then the strand is identified with a sphere

$$\{(\theta, x_1, \dots, x_n) \mid \theta \in S^{n-1}\} \subset S^{n-1} \times \mathbb{D}^n \overset{\varphi_i}{\simeq} D(\partial S_i),$$

where x_i is a constant. A subsequence $\{f_m\}_{m\geq 2}$ determines constants x_i. Let

$$D := \{(p, x_1, \dots, x_n) \mid p \in S_i\} \subset \mathbb{D}^n \times \mathbb{D}^n \overset{\varphi_i}{\simeq} D(S_i).$$

Then $\varphi_i(D)$ is a Lagrangian disk in $D(S_i)$, whose boundary agrees with the strands $\{f_m\}_{m\in\mathbb{N}}$.

Second, let f_1 be not of trivial type, but there exists $m \in \mathbb{N}$ such that f_m is of trivial type. Let $k > 1$ be the smallest number such that f_k is of trivial type appearing in $\{f_m\}_{m\in\mathbb{N}}$. Then $\tilde{\psi} = \delta_{k_0} \circ \cdots \circ \delta_l \circ \delta_1 \circ \cdots \circ \delta_{k_0-1}$, where $k_0 \cong k \pmod l$, is of generalized Penner type satisfying the following: $\mathcal{B}_{\tilde{\psi}}$ has a singular disk $\widetilde{S}_j$ such that $\widetilde{B}_{j,\infty}$, the limit of the braid sequence corresponding to $\tilde{\psi}$ and $\widetilde{S}_j$, has a strand identified with $\{f_m\}_{m\geq k}$. Thus, there is a Lagrangian disk in $D(\widetilde{S}_j)$ whose boundary agrees with $\{f_m\}_{m\geq k}$. Let D denote the Lagrangian disk in $D(\widetilde{S}_j)$. Then there is a connected component of

$$\big((\Phi_{1,1} \circ \delta_1) \circ \cdots \circ (\Phi_{k_0,1} \circ \delta_k)\big)(D) \cap D(S_i)$$

whose boundary is $\{f_m\}_{m\in\mathbb{N}}$, where $\Phi_{i,t}$ is a Hamiltonian isotopy mentioned in Remark 5.8.

To summarize, if there is at least one map of trivial type in $\{f_m\}_{m\in\mathbb{N}}$, then we have a Lagrangian submanifold in $D(S_i)$, whose boundary agrees with $\{f_m\}_{m\in\mathbb{N}}$. Let $\mathcal{L}_{i,\infty}$ be the union of those Lagrangian submanifolds.

Finally, suppose that f_m is not of trivial type for any $m \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$, we will construct a sequence $\{f_m^k\}_{m\in\mathbb{N}}$ for each $k \in \mathbb{N}$, satisfying

(1) $\{f_m^k\}_{m\in\mathbb{N}}$ is a strand of $B_{i,\infty}$;

(2) if $m \leq kl$, then $f_m^k = f_m$;

(3) there exists a constant $N_k \in \mathbb{N}$ such that $f_{kl+N_k}^k$ is of trivial type.

To prove the existence of these sequences $\{f_m^k\}_{m\in\mathbb{N}}$ for all $k \in \mathbb{N}$, we use the fact that the limits $B_{i,\infty}$ depend only on ψ and are independent of L. Let k be a fixed positive integer. Then $f_1 \circ \cdots \circ f_{kl}$ explains an impact of $b_{i,0} = b(L, S_i)$ on $b_{j,k} = b(\psi^k(L), S_j)$ for some i and j.

Let consider $\tilde{b}_{i,m} = b\big(\psi^m(\psi^N(L)), S_i\big) = b_{i,m+N}$ for a sufficiently large integer N. Then $\tilde{b}_{i,0}$ is given by a union of images of $g_1 \circ \cdots \circ g_{Nl}$ which appears in the i^{th} row of Ψ^N. If we assume that there is at least one compact case having two or more plumbing points, then for a sufficiently large N, there exists a sequence of functions $g_1, \dots, g_{Nl}$ such that $g_1 \circ \cdots \circ g_{Nl}$ appears in the i^{th} row of Ψ^N and g_t is of trivial type for some $t \in [1, Nl]$. The reason is as follows: First, when we apply a Dehn twist along the compact core with two or more plumbing points, a function g_t of trivial type appears. The function g_t appears in a specific row. By applying ψ sufficiently many times, ie N times, one can guarantee that g_t appears in i^{th} row. This is because every Dehn twist along each compact core appears in ψ.

In $\tilde{b}_{i,kl} = b_{i,kl+Nl}$, there is a strand satisfying the last two conditions, and thus it guarantees the existence of $\{f^k_m\}_{m\in\mathbb{N}}$ for all $k \in \mathbb{N}$, assuming at least one compact core has two or more plumbing points. We note that the assumption excludes only one case, the plumbing of one positive and one negative sphere plumbed at only one point. The excluded case can be easily handled directly. For more detail, see Remark 6.3

Without loss of generality, there is a strand $\{f^k_m\}_{k\in\mathbb{N}}$ of $B_{i,\infty}$ for each $k \in \mathbb{N}$. These strands converge to $\{f_m\}_{m\in\mathbb{N}}$ as $k \to \infty$. Moreover, by definition of $\mathscr{L}_{i,\infty}$, the boundary of $\mathscr{L}_{i,\infty}$ contains strands $\{f^k_m\}_{m\in\mathbb{N}}$ for all $k \in \mathbb{N}$. Thus, the strand $\{f_m\}_{m\in\mathbb{N}}$ is contained in the boundary of $\mathscr{L}_i := \overline{\mathscr{L}}_{i,\infty}$, ie the closure of $\mathscr{L}_{i,\infty}$.

Remark 6.3 If there is no sphere with two or more plumbing points, then every sphere is plumbed at only one point. Thus, there is exactly one positive sphere and one negative sphere plumbed at one point. In this case, we can construct a Lagrangian lamination $\mathscr{L}$ on M by spinning. This is because only two spheres are plumbed, thus there is a plenty of symmetry, which comes from the symmetry of spheres. Then, $\mathscr{L}_i := \mathscr{L} \cap D(S_i)$ is a Lagrangian lamination which we want to construct in Lemma 6.2.

Convergence to $\mathscr{L}_i$ Let $L_m := \psi^m_H(L)$. We defined ψ_H in the fourth step of the proof of Lemma 5.1. We will prove that $L_m \cap D(S_i)$ converges to $\mathscr{L}_i$.

First, we will show that

$$\lim_{m\to\infty}(L_m \cap D(S_i)) = \lim_{m\to\infty}\big(\psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i)\big). \tag{6-1}$$

Since $\psi_H(N(\mathscr{B}_\psi)) \subset N(\mathscr{B}_\psi)$,

$$\psi^{m+1}_H(N(\mathscr{B}_\psi)) \cap D(S_i) \subset \psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i) \quad \text{for all } m \in \mathbb{N}.$$

Thus, we have the limit

$$\lim_{m\to\infty}\big(\psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i)\big) = \bigcap_m \big(\psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i)\big).$$

If we equip M with a Riemannian metric g, then $d_H\big(\psi^m_H(\mathscr{B}_\psi), \psi^m_H(N(\mathscr{B}_\psi))\big)$, where d_H is the Hausdorff metric induced by g, converges to zero as $m \to \infty$ for the same reason that $B_{i,m} := \psi^m_H(N(\mathscr{B}_\psi)) \cap D(\partial S_i)$ converges to an infinite braid $B_{i,\infty}$ in the last part of Section 6.1.

Since for a large integer N_0, L_{N_0} intersects $D(S_j)$ for any singular disk S_j, and $L_{m+N_0} \cap D(S_j)$ intersects every connected component of $\psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i)$. Thus,

$$\begin{aligned}
0 &\le \lim_{m\to\infty} d_H(L_{m+N_0} \cap D(S_i), \psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i)) \\
&\le \lim_{m\to\infty} [d_H(L_{m+N_0} \cap D(S_i), \psi^m_H(\mathscr{B}_\psi) \cap D(S_i)) + d_H(\psi^m_H(\mathscr{B}_\psi) \cap D(S_i), \psi^m(N(\mathscr{B}_\psi)) \cap D(S_i))] \\
&\le \lim_{m\to\infty} 2d_H(\psi^m_H(\mathscr{B}_\psi) \cap D(S_i), \psi^m_H(N(\mathscr{B}_\psi)) \cap D(S_i)) \\
&= 0.
\end{aligned}$$

This proves (6-1). Let $\mathbb{L}_i$ be the limit in (6-1).

Second, we show that $\mathbb{L}_i$ is $\mathscr{L}_i$. By the construction of $\mathscr{L}_i$, we know that

$$\mathscr{L}_i \subset \psi_H^m(N(\mathscr{B}_\psi)) \cap D(S_i) \quad \text{for every } m \in \mathbb{N}.$$

It implies that $\mathscr{L}_i \subset \mathbb{L}_i$. Moreover,

$$\mathscr{L}_i \cap \pi^{-1}(\partial S_i) = \mathbb{L}_i = B_{i,\infty} \cap D(S_i).$$

Because every connected component of $\mathbb{L}_i$ has a boundary on ∂S_i, this shows $\mathscr{L}_i = \mathbb{L}_i$. □

6.3 Lagrangian lamination on a regular disk

In the previous subsection, we constructed Lagrangian laminations on singular disks, when boundary data for singular disks were given. In the present subsection, first, we will define boundary data for a regular disk. Second, we will construct Lagrangian laminations on regular disks from the given data. Finally, we will prove Theorem 1.5 as a corollary of Lemmas 6.2 and 6.5.

Before defining the boundary data, we remark that, $\overline{\pi^{-1}(R_i^\circ)}$ is symplectomorphic to $DT^*\mathbb{D}^n$, where $\mathbb{D}^n$ is a disk, by Remark 4.5. Similar to Definition 4.10, let $D(R_j)$ (resp. $D(\partial R_j)$) denote the $\mathbb{D}^n$–bundle $\overline{\pi^{-1}(R_i^\circ)}$ (resp. $\partial\overline{\pi^{-1}(R_i^\circ)}$) over R_j (resp. ∂R_j).

We define a data $c_{j,m}$ on the boundary of a regular disk R_j for $\psi^m(L)$, by setting

$$c_{j,m} := L_m \cap D(\partial R_j).$$

We defined $L_m := \psi_H^m(L)$ in the proof of Lemma 6.2. Note that $c_{j,m}$ is a closed subset, not a class of a closed subset.

To obtain a limit of $c_{j,m}$, we consider

$$C_{j,m} := \psi_H^m(N(\mathscr{B}_\psi)) \cap D(\partial R_j),$$

as we did in Section 6.1. Since $\psi_H^m(N(\mathscr{B}_\psi)) \subset N(\mathscr{B}_\psi)$, $C_{j,m+1} \subset C_{j,m}$. Moreover, $C_{j,m}$ is the union of solid tori in $D(\partial R_j)$ when $n = 2$, or the union of $S^{n-1} \times \mathbb{D}^n$ for general n. If a symplectic manifold M is equipped with a Riemannian metric g, we can measure the radii of solid tori in $C_{j,m}$. The radii decrease exponentially and converge to zero as $m \to \infty$, for the same reason that radii of solid tori comprising $B_{i,m}$ decrease exponentially and converge to zero as $m \to \infty$ in Section 6.1. The limit of $c_{j,m}$ is given by

$$C_{j,\infty} = \lim_{m\to\infty} C_{j,m} = \bigcap_m C_{j,m}.$$

The next step is to smooth R_j. A regular disk R_j has corners. We will replace R_j with a smooth disk R_j'. This is because, at the end, a Lagrangian lamination will be given as graphs of closed sections. By smoothing R_j, it will be easier to handle closed sections.

To smooth R_j, we subtract a tubular neighborhood $N(\partial R_j) \subset R_j$ from R_j. Let $R_j' := R_j \setminus N(\partial R_j)$. Then R_j' is a smooth disk. We replace R_j with R_j'. To finish smoothing, we need to obtain boundary data for R_j' from $c_{j,m}$.

Each connected component of $c_{j,m}$ can be identified with a section of a bundle $D(\partial R_j)$ over ∂R_j. We can extend this section to a closed section of a bundle $\pi^{-1}(N(\partial R_j))$ over $N(\partial R_j)$ by computations. Then the graph of the extended section is a Lagrangian submanifold of $\pi^{-1}(N(\partial R_j))$. The boundary of the Lagrangian submanifold on $\partial R'_j$ makes up the boundary data for R'_j.

From now, we assume that a regular disk R_j is a smoothed disk. Lemma 6.4 claims that for a given data $c_{j,m}$ on a smoothed regular disk R_j, we can construct a Lagrangian submanifold $N_{j,m} \subset D(R_i)$ such that $\partial N_{j,m} = c_{j,m} \cap D(R_i)$.

Lemma 6.4 *Let Q be a closed subset of $\partial T^*\mathbb{D}^n$ such that there exists a disjoint union L of Lagrangian disks in $T^*\mathbb{D}^n$, which are transversal to fibers, such that $L \cap \partial T^*\mathbb{D}^n = Q$. Then we can construct a Lagrangian submanifold L uniquely up to Hamiltonian isotopy through Lagrangians transverse to the fibers.*

Proof To prove Lemma 6.4, we consider a identification $\varphi\colon \partial T^*\mathbb{D}^n \xrightarrow{\simeq} S^{n-1} \times \mathbb{D}^n$ which is defined as follows. If there is a global coordinate charts of the zero section $\mathbb{D}^n$ of $T^*\mathbb{D}^n$, then it induces an identification between $\mathbb{D}^n \times \mathbb{D}^n$ and $T^*\mathbb{D}^n$. By restricting the identification on $\partial T^*\mathbb{D}^n$, one obtains $\varphi\colon \partial T^*\mathbb{D}^n \xrightarrow{\simeq} S^{n-1} \times \mathbb{D}^n$. With the fixed identification φ, $\varphi(Q) = \varphi(\partial L)$ is isotopic to a union of spheres

$$\{S^{n-1} \times p_1, \dots, S^{n-1} \times p_m \mid p_i \in \mathbb{D}^n, m \text{ is the number of component of } L\}.$$

This is because $\varphi(L)$ is a union of Lagrangian disks in $\mathbb{D}^n \times \mathbb{D}^n \overset{\varphi}{\simeq} T^*\mathbb{D}^n$.

The proof of Lemma 6.4 consists of two parts: the construction of L and the uniqueness of L.

Construction We start the proof with the simplest case, ie Q consists of only one strand.

By fixing coordinate charts on $\mathbb{D}^n$, we can write down Q as a section of a disk bundle $\partial T^*\mathbb{D}^n$ over $\partial\mathbb{D}^n$, ie

$$Q := \{f_1(x_1, \dots, x_n)dx_1 + \cdots + f_n(x_1, \dots, x_n)dx_n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

Then, the simplest case is proved by determining a function $\phi\colon\mathbb{D}^n \to \mathbb{R}$ such that $d\phi = f_1dx_1 + \cdots + f_ndx_n$ on $\partial\mathbb{D}^n$. The graph of $d\phi$ is a Lagrangian submanifold which we would like to find. Note that there are infinitely many ϕ satisfying the conditions, but the Hamiltonian isotopy class of the graph of $d\phi$ is unique through Lagrangians transverse to the fibers.

If Q has two or more connected components l_i, then we can write l_i as a section over $\partial\mathbb{D}^n$. For each i, we need to determine functions $\phi_i\colon\mathbb{D}^n \to \mathbb{R}$ such that $d\phi_i$ agrees with l_i on $\partial\mathbb{D}^n$. Moreover, to avoid self-intersection, they should not be equal, ie $d\phi_i \neq d\phi_j$ for all $i \neq j$. Then, the union of graphs of $d\phi_i$ on $T^*\mathbb{D}^n$ is a Lagrangian submanifold L which we want to construct.

We discuss with the simplest nontrivial case, ie Q has two connected components l_0 and l_1, and the dimension $2n = 4$. Without loss of generality, we assume that l_0 is the zero section. Furthermore, we can assume that $\phi_0 \equiv 0$. We only need to determine ϕ_1 such that $d\phi_1$ does not vanish everywhere.

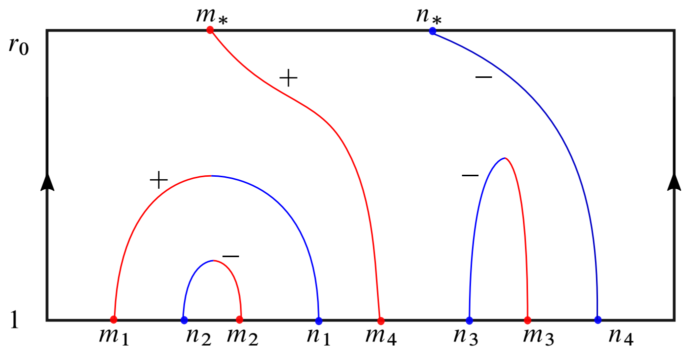

Figure 21: Example of a collection $\mathscr{C}$ on $[r_0, 1] \times S^1$.

We assume that there exists ϕ_1 satisfying the conditions. Then we will collect combinatorial data from ϕ_1, and we will construct a function $\tilde{\phi}_1$ satisfying conditions given by the combinatorial data. Through this, we will see what combinatorial data we need. We will end the construction by explaining how to obtain the combinatorial data from the given Q.

For convenience, we will use the polar coordinates instead of (x, y)–coordinates on $\mathbb{D}^2$. Let r_0 be a small positive number. We restrict the function ϕ_1 on $[r_0, 1] \times S^1$. On $\{1\} \times S^1 = \partial\mathbb{D}^2$, $d\phi_1$ agrees with l_1. On $\{r_0\} \times S^1$, $d\phi_1$ is approximately a constant section

$$a\,dx + b\,dy = a(\cos\theta\,dr - r_0\sin\theta\,d\theta) + b(\sin\theta\,dr + r_0\cos\theta\,d\theta),$$

where $d\phi_1(0,0) = a\,dx + b\,dy$ and (x, y) are the standard coordinate charts of $\mathbb{D}^2$.

We remark that on $\{r_0\} \times S^1$, the pair of graphs of $d\phi_i|_{\{r_0\}\times S^1}$ represents the trivial braid under the identification induced from the (x, y)–coordinates. Similarly, on $[r_0, 1] \times S^1$, the pair $(d\phi_0 \equiv 0, d\phi_1)$ implies an isotopy between two representatives of the trivial braid.

For every $r_* \in [r_0, 1]$, we can find all local maxima and minima of a function

$$\theta \mapsto \phi_1(r_*, \theta).$$

We mark (r_*, θ_*) as a red (resp. blue) point if the above function has a local maxima (resp. minima) at θ_*. If $r_* = 1$, there are same number of red/blue marked points on $\{1\} \times S^1$, and there is only one red/blue marked point on $\{r_0\} \times S^1$. On $[r_0, 1] \times S^1$, we have a collection $\mathscr{C}$ of curves shaded red and blue. If a curve in $\mathscr{C}$ is not a circle, then the curve has two end points on the boundary of $[r_0, 1] \times S^1$. There are exactly two curves connecting both boundary components of $[r_0, 1] \times S^1$, and those two curves have end points of the same color.

If we write $d\phi_1 = f\,d\theta + g\,dr$, then f is zero on curves in $\mathscr{C}$. Since $d\phi_1$ does not vanish, g cannot be zero on the curves. Thus, we can assign the sign of g for each curve. Figure 21 is an example of a collection $\mathscr{C}$.

Conversely, if we have a collection $\mathscr{C}$ of curves such that each curve is shaded red and blue and is equipped with a sign, then we can draw a graph of $\tilde{\phi}_1$ roughly. This is because the collection $\mathscr{C}$ determines the sign of horizontal directional derivative of $\tilde{\phi}_1$, ie $d\tilde{\phi}_1(\partial_\theta)$ on every point of $[r_0, 1] \times S^1$, and vertical directional derivative of $\tilde{\phi}_1$, ie $d\tilde{\phi}_1(\partial_r)$ on the curves. From these, one obtains a (rough) graph of $\tilde{\phi}_1$. Thus, in order to determine a function ϕ_1, it is enough to determine a collection $\mathscr{C}$ of curves in $[r_0, 1] \times S^1$ from the given Q.

From now on, we will construct a collection $\mathscr{C}$ from the given Q. For the given Q, we assume that a connected component l_0 of Q is the zero section, without loss of generality. For the other connected component l_1, one has $f_1, g_1 \colon S^1 \to \mathbb{R}$ such that l_1 is the graph of $f_1\, d\theta + g_1\, dr$ on $\{1\} \times S^1 = \partial\mathbb{D}^2$. We know that Q represents the trivial braid with respect to the standard (x, y)–coordinates of $\mathbb{D}^2$. Thus, there is an isotopy $\Gamma \colon [r_0, 1] \times S^1 \to \mathbb{D}^2$ such that

$$\Gamma(1, \theta) = (f(\theta), g(\theta)), \quad \Gamma(r_0, \theta) = (Ar_0 \cos\theta, A \sin\theta),$$
$$\Gamma(t, \theta) \neq (0, 0) \quad \text{for all } (t, \theta) \in [r_0, 1] \times S^1,$$

where A is a constant.

For every $r \in [r_0, 1]$, let $\gamma_r(\theta) = \Gamma(r, \theta)$. Then, γ_r is a closed curve in $\mathbb{D}^2$, for all r. Moreover, Γ is a path connecting γ_1 and γ_{r_0} in the loop space of $(\mathbb{D}^2)^\circ$ without touching the origin.

We mark (r, θ) on $[r_0, 1] \times S^1$ as a red (resp. blue) point if $\gamma_r(\theta)$ intersects dr–axis from right to left (resp. from left to right). These marked points comprise curves in $[r_0, 1] \times S^1$, and we have a collection $\mathscr{C}$ of curves, shaded red and blue, in $[r_0, 1] \times S^1$. We know that γ_1 has intersection points. The number of intersection points is an even number. When r decreases, there is a series of creations/removals of intersection points, which are given by finger moves along the dr–axis. Each finger move does not touch the origin. Thus, for a curve in $\mathscr{C}$, every intersection point composing the curve lies on either the positive dr–axis or the negative dr–axis. Then, we can assign a sign for each curve in $\mathscr{C}$.

Figure 22 is an example of Γ, corresponding to the case described by Figure 21.

The upper left of Figure 22 is γ_1 and the upper right is γ_{r_0}. Through the first arrow, we observe a finger move removing two intersection points. Those two intersection points correspond to m_2, a local maxima shaded red, and n_2, a local minima shaded blue. Thus, we obtain a curve connecting m_2 and n_2 in Figure 21. Moreover, the intersection points lie in the negative part of the dr–axis. Thus, we assign a negative sign to the curve. Similarly, we observe there are finger moves removing intersection points. We obtain curves connecting m_i and n_i for $i = 1, 2, 3$ in Figure 21. After the finger moves, there are only two intersection points corresponding to m_* and n_*, and we obtain curves connecting m_4 (resp. n_4) and m_* (resp. n_*).

We have constructed a collection $\mathscr{C}$ of curves on $[r_0, 1] \times S^1$ from an isotopy Γ. Thus, we can obtain a function $\phi_1 \colon [r_0, 1] \times S^1 \to \mathbb{R}$. In order to complete the proof, we need to extend ϕ_1 into a small disk with radius r_0. To extend ϕ_1, we assume that

$$\phi_1(x, y) = Ar \sin\theta = Ay$$

on the small disk.

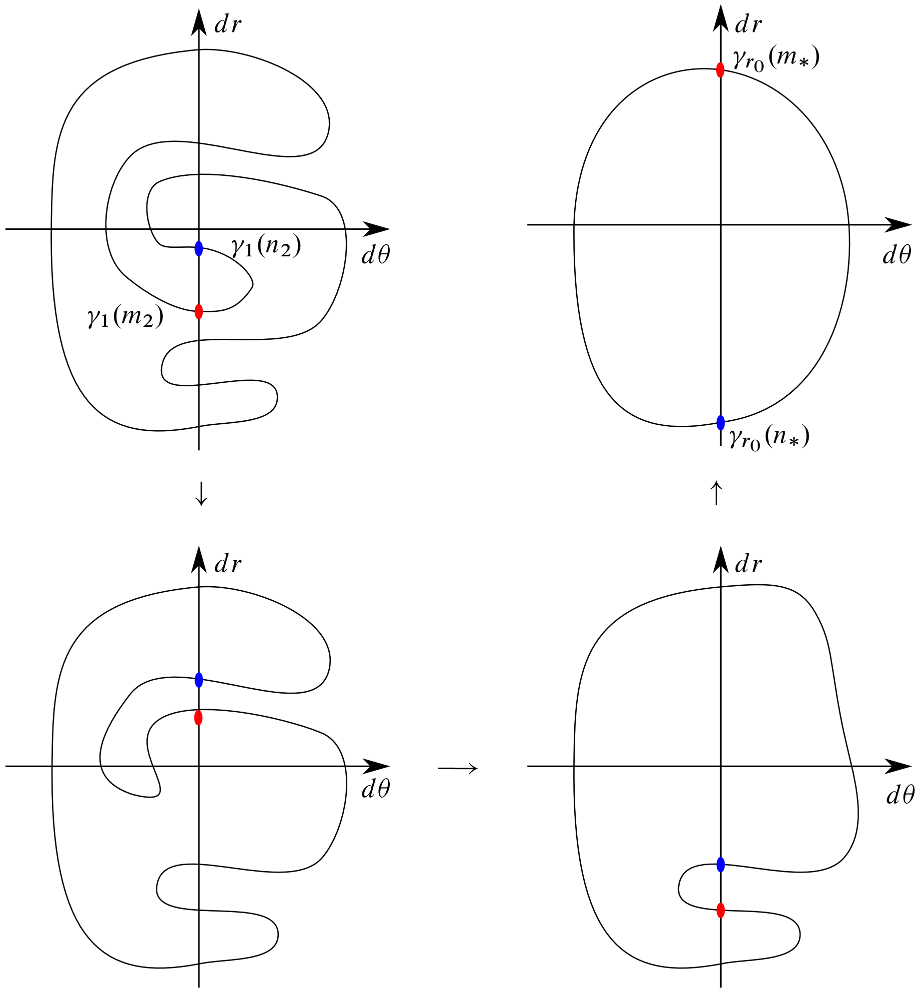

Figure 22: Creation of a collection $\mathscr{C}$.

The situation for the general case is analogous. If Q has more connected components l_i for $i = 0, \dots, k$, then we have to determine $\phi_i : \mathbb{D}^2 \to \mathbb{R}$ such that $d\phi_i = l_i$ on $\partial\mathbb{D}^2$, and $d\phi_i \neq d\phi_j$ for all $i \neq j$. We fix an isotopy Γ, and obtain a collection $\mathscr{C}$ of curves on $[r_0, 1] \times S^1$ from Γ. Each curve in $\mathscr{C}$ encodes restrictions on $d\phi_i - d\phi_j$ for some i and j. More precisely, $(\phi_i - \phi_j)$ has a local maxima (resp. minima) in the horizontal direction, only at a point of a curve shaded red (resp. blue), and $(d\phi_i - d\phi_j)(\partial_r)$ has the sign assigned on the curve. For the case of general dimension $2n$, we obtain combinatorial data from Q, ie a collection of curves on $[r_0, 1] \times S^{n-1}$ assigned a sign, and construct functions on $\mathbb{D}^n$ from the combinatorial data.

Uniqueness Recall that the construction consists of three steps. First, we choose an isotopy Γ connecting Q and the trivial representative of the trivial braid. Then, we obtained a collection $\mathscr{C}$ of curves from Γ, such that each curve encodes restrictions on $d\phi_i - d\phi_j$. The last step is to construct a set of functions $\{\phi_i : \mathbb{D}^n \to \mathbb{R}\}$.

The construction depends on choices in the first and last steps. More precisely, for the first step, the choice of isotopy Γ is not unique. If we choose an isotopy Γ, then there is a unique collection $\mathscr{C}$. However, a set $\{\phi_i\}$ of functions, which is constructed from the collection $\mathscr{C}$, is not unique. We will show that the Hamiltonian isotopy class of L, through Lagrangians transverse to the fibers, is independent of those choices.

First, we discuss the choice in the third step. Let us assume that we have a collection $\mathscr{C}$ of curves in $[r_0, 1] \times S^{n-1}$ and two sets of functions $\{\phi_i\}_i$ and $\{\zeta_i\}_i$ satisfying the restrictions encoded by $\mathscr{C}$. Then, by setting $\eta_{i,t} := (1-t)\phi_i + t\zeta_i$, we obtain a family of sets of functions such that every member of the family satisfies the restrictions encoded by $\mathscr{C}$.

Let L_t be the Lagrangian submanifold corresponding to $\{\eta_{i,t}\}$ for a fixed t. Then L_t is a Lagrangian isotopy connecting L_0, corresponding to $\{\phi_i\}$, and L_1, corresponding to $\{\zeta_i\}$. Since L_t is a disjoint union of Lagrangian disks in $T^*\mathbb{D}^n$, L_0 and L_1 are Hamiltonian isotopic. Thus, the Hamiltonian class of L through Lagrangians transverse to the fibers is independent of the choice of functions for the third step of the construction.

Before discussing the choice of the first step, note that a continuous change on a collection $\mathscr{C}$ does not make a change on the Hamiltonian isotopy class. More precisely, let $\mathscr{C}_0 = \{\gamma_1, \dots, \gamma_N\}$ be a collection of curves and let $\{\phi_i\}$ be a set of functions corresponding to $\mathscr{C}_0$. If $\{\gamma_{k,t}\}$ is a continuous family of curves with respect to t such that $\gamma_{k,0} = \gamma_k$ for all k, then we can obtain a continuous family $\{\phi_{1,t}, \dots, \phi_{N,t}\}$ such that $\phi_{i,0} = \phi_i$ and $\{\phi_{1,t}, \dots, \phi_{N,t}\}$ corresponds to $\mathscr{C}_t := \{\gamma_{1,t}, \dots, \gamma_{N,t}\}$. Then, it is easy to check that the Hamiltonian isotopy class of the union of graphs of $d\phi_{i,t}$ in $T^*\mathbb{D}^n$, through Lagrangians transverse to the fibers, is independent of t.

Finally, we will discuss the choice of Γ. Let Γ_0 and Γ_1 be two isotopies obtained from the given Q in the first step. Then we can understand Γ_0 and Γ_1 as paths on the loop space of the configuration space of $(\mathbb{D}^n)^\circ$. Since the loop space is simply connected, there is a continuous family $\{\Gamma_t\}_{t\in[0,1]}$ connecting γ_0 and γ_1.

Let $\mathscr{C}_t$ be the collection of curves obtained from Γ_t and let $\{\phi_i\}$ be a set of functions constructed from $\mathscr{C}_0$. There is $\{\phi_{i,t}\}$ corresponding to $\mathscr{C}_t$ such that $\phi_{i,0} = \phi_i$. Then, if L_t is the union of graphs of $d\phi_{i,t}$, then the Hamiltonian class of L_t is independent of t. This shows the uniqueness of L, up to Hamiltonian isotopy, through Lagrangians transverse to the fibers. □

For a smoothed regular disk R_j, there is a sequence of data $c_{j,m}$ for each $m \in \mathbb{N}$. Then, we can construct a sequence of Lagrangian submanifolds $N_{j,m} \subset D(R_j)$ such that $N_{j,m} \cap \partial D(R_j) = c_{j,m}$. The following lemma, Lemma 6.5, claims that we can construct $N_{j,m}$ wisely, so that $N_{j,m}$ converges to a Lagrangian lamination $\mathcal{N}_j$ as m goes to ∞.

Lemma 6.5 *It is possible to construct $N_{j,m} \subset D(R_j)$ so that the sequence $N_{j,m}$ converges to a Lagrangian lamination $\mathcal{N}_j \subset D(R_j)$ as $m \to \infty$.*

Proof Let the boundary condition $c_{j,m}$ be the set $\{l_{1,m}, \dots, l_{N_m,m}\}$, where $l_{i,m}$ is a connected component of $c_{j,m}$, or equivalently, $l_{i,m}$ is a strand of the braid represented by $c_{j,m}$. Note that $C_{j,m}$ is a disjoint union of solid tori in $D(\partial R_j)$, which is defined at the beginning of the present subsection. Then we can divide $c_{j,m}$ into a partition such that $l_{i,m}$ and $l_{j,m}$ are in the same subset if and only if $l_{i,m}$ and $l_{j,m}$ are in the same solid torus (or $S^{n-1} \times \mathbb{D}^n$, for a higher-dimensional case) in $C_{j,m}$. After that, we randomly choose a connected component $l_{s,m}$ from each subset of the partition.

By Lemma 6.4, there is $\phi_{s,m}\colon R_j \to \mathbb{R}$ such that $d\phi_{s,m} = l_{s,m}$ on ∂R_j. Then $\Gamma(d\phi_{s,m})$ is a Lagrangian disk in $D(R_j)$, where $\Gamma(d\phi_{s,m})$ is the graph of $d\phi_{s,m}$. We can choose a neighborhood $N(\Gamma(\phi_{s,m}))$ of $\Gamma(d\phi_{s,m})$ in $D(R_j)$, such that $N(\Gamma(d\phi_{s,m})) \simeq T^*\mathbb{D}^n$ and $N(\Gamma(d\phi_{s,m})) \cap D(\partial R_j)$ is the torus in $C_{j,m}$ containing $l_{s,m}$. Moreover, we can assume that

$$d_H(N(\phi_{s,m}), \Gamma(d\phi_{s,m})) < 2r^m,$$

where d_H is the Hausdorff metric induced by a fixed Riemannian metric and $r < 1$ is a small positive number.

We apply Lemma 6.4 to $\{l_{t,m+1} \in c_{j,m+1} \mid l_{t,m+1} \subset N(\Gamma(d\phi_{s,m}))\}$ in $N(\Gamma(d\phi_{s,m})) \simeq T^*\mathbb{D}^n$. Then we can construct $\phi_{t,m+1}\colon R_j \to \mathbb{R}$ such that $d\phi_{i,m+1} = l_{t,m+1}$ on ∂R_j and $\Gamma(d\phi_{s,m})$ is contained in $N(\phi_{s,m+1})$. We repeat this procedure inductively on $m \in \mathbb{N}$.

Let l be a strand of $C_{j,\infty}$. Then there is a sequence $l_{i_m,m} \in c_{j,m}$ such that $l_{i_m,m}$ converges to l. If we construct $\phi_{i,m}$ by repeating the above procedure, we know that

$$d_H(\Gamma(d\phi_{i_m,m}), \Gamma(d\phi_{i_n,n})) < 4r^{\max(m,n)}.$$

Thus, $d\phi_{i_m,m}$ converges. Moreover, by assuming that $\phi_{i,m}(p) = 0$ for every i and m, where p is a center of R_j, $\phi_{i_m,m}$ converges to a function ϕ. Then $\Gamma(d\phi)$ is a Lagrangian disk in $D(R_j)$ whose boundary is l, the stand of $C_{j,\infty}$. The union of $\Gamma(d\phi)$ is the Lagrangian lamination $\mathcal{N}_j$ which $N_{j,m}$ converges to. □

Proof of Theorem 1.5 By Lemma 6.2, there is a Lagrangian lamination $\mathcal{L}_i$ in $D(S_i)$, and by Lemma 6.5, there is a Lagrangian lamination $\mathcal{N}_j$ in $D(R_j)$. Moreover, every Lagrangian lamination agrees with each other along boundaries. Thus, we can glue them. Then we obtain a Lagrangian lamination $\mathcal{L}$ in M. □

6.4 A generalization

In the previous sections, we assumed that ψ is of generalized Penner type. In the present subsection, we discuss a symplectic automorphism $\psi\colon (M,\omega) \to (M,\omega)$, not necessarily to be of generalized Penner type, with some assumptions. In other words, we prove the following theorem.

Theorem 6.6 *Let $\psi\colon M \xrightarrow{\sim} M$ be a symplectic automorphism and let $\mathcal{B}_\psi$ be a Lagrangian branched submanifold such that $\psi(\mathcal{B}_\psi)$ is carried by $\mathcal{B}_\psi$. If the associated branched manifold $\mathcal{B}_\psi$ admits a* *decomposition into singular and regular disks, then there is a Lagrangian lamination $\mathscr{L}$ such that if L is a Lagrangian submanifold of M which is carried by $\mathscr{B}_\psi$ and if L is compatible with the decomposition of $\mathscr{B}^*_\psi$, then there is a Lagrangian submanifold L_m for all $m \in \mathbb{N}$, which is Hamiltonian isotopic to $\psi^m(L)$ and converges to $\mathscr{L}$ as closed sets as $m \to \infty$.*

First, we assume that there is a Lagrangian branched submanifold $\mathscr{B}_\psi$ such that $\psi(\mathscr{B}_\psi)$ is (weakly) carried by $\mathscr{B}_\psi$. Then if a Lagrangian submanifold L is (weakly) carried by $\mathscr{B}_\psi$, then $\psi(L)$ is carried by $\mathscr{B}_\psi$. This is because the proof of Lemma 3.19 carries over with no change.

As mentioned in Section 4.2, we assume that $\mathscr{B}^*_\psi$ admits a decomposition into a union of finite number of singular disks $S_i \simeq \mathbb{D}^n$ and regular disks $R_j \simeq \mathbb{D}^n$.

Proof of Theorem 6.6 First, we define data on the boundary of each singular and regular disk, in the same way we did for the case of ψ of generalized Penner type. Then, on a regular disk R_j, the proofs of Lemma 6.4 and Lemma 6.5 carry over with no change. Thus, we can construct a Lagrangian lamination on $D(R_j)$.

On a singular disk S_i, we define the boundary data in the same way. In other words, the boundary data is defined by the isotopy class of $\psi^m(L) \cap D(\partial S_i)$. We also can obtain a matrix Ψ, which explains how the sequences of braids are constructed inductively. However, the rest of the proof of Lemma 6.2 does not carry over. This is because in the proof of Lemma 6.2, functions of trivial type have a key role. To use the same proof, we need to show that there are enough functions of trivial type. However, the assumptions cannot imply the existence of enough functions of trivial type.

For a singular disk S_i, let $\{f_m\}_{m \in \mathbb{N}}$ be a strand of the limit braid on S_i. We note that each strand can be identified to an infinite sequence of functions. We forget specific functions f_m, but remember their types. Then, we obtain a sequence of types. The sequence of types determines the "shape" of strand, for example, how many times the strand is rotated.

We can construct a symplectomorphism ϕ which is of generalized Penner type such that $\mathscr{B}_\phi$ has a singular disk S such that the limit braid assigned on S has a strand of the same shape. In Section 4.3, we constructed a Lagrangian submanifold $L_0 \subset D(S)$ such that ∂L_0 is the strand. Since $D(S) \simeq D(S_i)$, we assume that L_0 is a Lagrangian submanifold in $D(S_i)$. By scaling and translating L_0 inside $D(S_i)$, we obtain a Lagrangian submanifold whose boundary agrees with the strand.

The rest of the proof is the same as the proof of Theorem 1.5. □

7 Application to Lagrangian Floer homology

One natural question following the construction of stable/unstable Lagrangian lamination is: how can we understand those constructed Lagrangian laminations in terms of Fukaya category? The purpose of Section 7 is to introduce one possible view-point of answering the question. More precisely, we expect that a symplectic automorphism of Penner type will induce a *pseudo-Anosov autoequivalence* in terms of Fan, Filip, Haiden, Katzarkov and Liu [4].

Remark 7.1 There are two different definitions of pseudo-Anosov autoequivalence. One is defined by Dimitrov, Haiden, Katzarkov and Kontsevich in [3] and the other is defined in [4].

Roughly, we expect that, for a given ϕ of Penner type, by counting intersection numbers of a Lagrangian submanifold L and the stable/unstable Lagrangian laminations, we can define a *mass function* for ϕ. Then, ϕ will induce a pseudo-Anosov autoequivalence with respect to that mass function.

We do not prove the above claim in the current paper. However, we prove Theorem 7.3 which relates the intersection numbers with Lagrangian Floer theory.

In Section 7.1, we state Theorem 7.3. In Section 7.2, we will give a proof of Theorem 7.3. Moreover, we will prove Lemmas 7.7 and 7.8, in order to weaken the difficulties of applying Theorem 7.3 together with Example 7.9.

Remark 7.2 (1) In order to do Lagrangian Floer theory, we should choose a suitable almost complex structure J. We will discuss our choice of almost complex structure in Section 7.1; see Remark 7.6.

(2) If M is a surface, ie a 2–dimensional symplectic manifold, then $\widetilde{L}_i = L_i$, and Theorem 7.3 is claiming that the rank of Lagrangian Floer homology of L_1 and L_2 is the same to the intersection number of L_1 and L_2. This is already proven in [3, Lemma 2.18].

7.1 Setting

First, we state Theorem 7.3. Then, we will define the terms in Theorem 7.3.

Theorem 7.3 *Let M be a plumbing space of Penner type, and let $\eta\colon M \xrightarrow{\sim} M$ be the involution associated to M. Assume that a transversal pair $L_1, L_2 \subset M$ of Lagrangian submanifolds satisfies*

(1) $\eta(L_i) = L_i$ *for* $i = 0, 1$;

(2) *if* $\widetilde{L}_i = L_i \cap M_i$, *then* $\widetilde{L}_i$ *is a Lagrangian submanifold of* $\widetilde{M}$ *such that* $\widetilde{L}_0$ *and* $\widetilde{L}_1$ *are not isotopic to each other*;

(3) $L_0 \cap L_1 = \widetilde{L}_0 \cap \widetilde{L}_1$;

(4) L_0 *and* L_1 *are not isotopic to each other.*

Then

$$\dim HF^0(L_1, L_2) + \dim HF^1(L_1, L_2) = i(\widetilde{L}_1, \widetilde{L}_2), \tag{7-1}$$

where $HF^k(L_1, L_2)$ denotes $\mathbb{Z}/2$–graded Lagrangian Floer homology over the Novikov ring of characteristic 2 and $i(\widetilde{L}_1, \widetilde{L}_2)$ denotes the geometric intersection number of $\widetilde{L}_1$ and $\widetilde{L}_2$ in the fixed surface $\widetilde{M}$.

In Section 7, we assume that our symplectic manifold M is a plumbing space

$$M = P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$$

of Penner type defined as follows.

Definition 7.4 A plumbing space $M = P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$ is of *Penner type* if α_i and β_j satisfy

(1) $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_l$ are n–dimensional spheres,

(2) $\alpha_i \cap \alpha_j = \varnothing$ and $\beta_i \cap \beta_j = \varnothing$ for all $i \neq j$.

Note that $P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$ is defined in Section 2.1.

From now on, we will define an involution $\eta\colon M \xrightarrow{\sim} M$, which is associated to M.

Involution η_0 on T^*S^n First, we will define an involution η_0 on T^*S^n. Let

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\},$$
$$T^*S^n = \{(x, y) \in S^n \times \mathbb{R}^{n+1} \mid x \in S^n, \langle x, y\rangle = 0\}.$$

Then we define $\eta_0\colon T^*S^n \xrightarrow{\sim} T^*S^n$ by

$$\eta_0(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = (x_1, x_2, -x_3, \dots, -x_{n+1}, y_1, y_2, -y_3, \dots, -y_{n+1}).$$

Let

$$W_0 = \{(\cos\theta, \sin\theta, 0, \dots, 0) \mid \theta \in [0, 2\pi]\} \subset S^n,$$
$$T^*S = \{(\cos\theta, \sin\theta, 0, \dots, 0, -r\sin\theta, r\cos\theta, 0, \dots, 0) \mid \theta \in [0, 2\pi], r \in \mathbb{R}\} \subset T^*S^n.$$

Then it is easy to check that T^*W_0 is the set of fixed points of η_0, ie $\eta_0^{\text{fixed}} = T^*W_0$.

Involution η associated to M First, we will construct an involution η_{α_i} and η_{β_j} on $T^*\alpha_i$ and $T^*\beta_j$ for every i and j. Note that $T^*\alpha_i, T^*\beta_j \subset M$.

For each α_i, we will choose a great circle $W_{\alpha_i} \subset \alpha_i$ such that W_{α_i} contains every plumbing point of α_i. Then there is a symplectic isomorphism $\phi_{\alpha_i}\colon T^*S^n \xrightarrow{\sim} T^*\alpha_i$ such that $\phi_{\alpha_i}(S^n) = \alpha_i$ and $\phi_{\alpha_i}(W_0) = W_{\alpha_i}$. One obtains an involution $\eta_{\alpha_i}\colon T^*\alpha_i \xrightarrow{\sim} T^*\alpha_i$ by setting

$$\eta_{\alpha_i} := \phi_{\alpha_i} \circ \eta_0 \circ (\phi_{\alpha_i})^{-1}.$$

Similarly, one obtains an involution $\eta_{\beta_j}\colon T^*\beta_j \xrightarrow{\sim} T^*\beta_j$.

Without loss of generality, one can assume that $\eta_{\alpha_i}(x) = \eta_{\beta_j}(x)$ for every $x \in T^*\alpha_i \cap T^*\beta_j$. Finally, the involution $\eta\colon M \xrightarrow{\sim} M$ is defined by

$$\eta(x) := \begin{cases} \eta_{\alpha_i}(x) & \text{if } x \in T^*\alpha_i, \\ \eta_{\beta_j}(x) & \text{if } x \in T^*\beta_j. \end{cases}$$

Let $\widetilde{M}$ be the set of fixed points of η, ie $\widetilde{M} = \{x \in M \mid \eta(x) = x\}$. It is easy to check that $\widetilde{M}$ is a 2–dimensional symplectic submanifold of M. Moreover, $\widetilde{M}$ is symplectomorphic to a plumbing space $P(S_{\alpha_1}, \dots, S_{\alpha_m}, S_{\beta_1}, \dots, S_{\beta_l})$ of Penner type. Note that S_{α_i} and S_{β_j} are embedded circles in α_i and β_j.

Definition 7.5 (1) The above η is called *the involution associated to M*.

(2) The above $\widetilde{M}$ is called *the fixed surface of M*.

Remark 7.6 It is easy to check that our setting is a special case of Seidel and Smith [13]. More precisely, [13] considers Lagrangian Floer cohomology on a symplectic manifold carrying a symplectic involution. Under various topological hypothesis, the authors proved a localization theorem, and the theorem implies a Smith-type inequality which is closely related to (7-1).

As a basic setup of Lagrangian Floer homology, [13] contains some analytic background; for example, the choice of almost complex structures. We follow their settings in order to do Lagrangian Floer homology. We refer the reader to [13, Section 3].

7.2 Proof of Theorem 7.3

Let M be a plumbing space of Penner type, η the associated involution of M, and L_0 and L_1 a transversal pair of Lagrangian submanifolds such that

(1) $\eta(L_i) = L_i$;

(2) $\widetilde{L}_i = L_i \cap \widetilde{M}$ is a Lagrangian submanifold of $\widetilde{M}$;

(3) $L_0 \cap L_1 = \widetilde{L}_0 \cap \widetilde{L}_1$;

(4) L_0 and L_1 are not isotopic to each other.

We will compute $\mathbb{Z}/2$–graded Lagrangian Floer homology $HF^*(L_0, L_1)$ over the Novikov field Λ of characteristic 2. To do this, we will prove that chain complexes $CF^*(L_0, L_1)$ and $CF^*(\widetilde{L}_0, \widetilde{L}_1)$ have the same generators and the same differential maps.

First, it is easy to show that $CF^*(L_0, L_1)$ and $CF^*(\widetilde{L}_0, \widetilde{L}_1)$ have the same generators since L_0 and L_1 satisfy that $L_0 \cap L_1 = \widetilde{L}_0 \cap \widetilde{L}_1$. Thus, $CF^*(L_0, L_1) = CF^*(\widetilde{L}_0, \widetilde{L}_1)$ as vector spaces.

Second, let ∂ (resp. $\tilde{\partial}$) denote the differential map on $CF^*(L_0, L_1)$ (resp. $CF^*(\widetilde{L}_0, \widetilde{L}_1)$). Then

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]:\mathrm{ind}([u])=1}} (\#\mathcal{M}(p, q; [u], J)) T^{\omega([u])} q,$$

where J is an almost complex structure on M, u is a holomorphic strip connecting p and q, and $\mathcal{M}(p, q; [u], J)$ is the moduli space of holomorphic strips. We skip the foundational details of the definition of ∂.

One can easily check that $\eta \circ u$ is also a holomorphic strip connecting p and q. Assume that for a holomorphic strip u, the image of u is not contained in $\widetilde{M}$. Then u and $\eta \circ u$ will be canceled together in $\partial(p)$, since the Novikov field Λ is of characteristic 2. Thus, in order to define the differential map ∂, it is enough to count holomorphic strips u such that the image of u is contained in $\widetilde{M}$.

On the other hand, in order to define $\tilde{\partial}\colon CF^*(\widetilde{L}_0, \widetilde{L}_1) \to CF^*(\widetilde{L}_0, \widetilde{L}_1)$, one needs to count the holomorphic strips on $\widetilde{M}$. Thus, $\partial(p) = \tilde{\partial}(p)$ for all $p \in L_0 \cap L_1 = \widetilde{L}_0 \cap \widetilde{L}_1$.

Under the assumptions, $HF^*(L_0, L_1) = HF^*(\widetilde{L}_0, \widetilde{L}_1)$. Note that the former is defined on M^{2n}, but the latter is defined on a surface $\widetilde{M}$. Thus, it is enough to check that

$$\dim HF^0(\widetilde{L}_1, \widetilde{L}_2) + \dim HF^1(\widetilde{L}_1, \widetilde{L}_2) = i(\widetilde{L}_1, \widetilde{L}_2).$$

By Remark 7.2, [3, Lemma 2.18] completes the proof. □

7.3 Example 7.9

In the present subsection, we will prove Lemmas 7.7 and 7.8 in order to slightly weaken the difficulty of applying Theorem 7.3. Then, we will give Example 7.9.

The difficulty of applying Theorem 7.3 is that there are too many conditions which L_0 and L_1 should satisfy. Lemmas 7.8 and 7.7 will give us plenty of Lagrangians satisfying the conditions after Hamiltonian isotopies.

Before giving the statement of Lemmas 7.7 and 7.8, we will establish notation. Since

$$M = P(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_l)$$

is a plumbing space of Penner type, we can construct a set $\mathbb{B}$ of Lagrangian branched submanifolds of M as we did in Section 3.4. Every Lagrangian branched submanifold $\mathcal{B} \in \mathbb{B}$ is a union of (parts of) α_i and β_j and Lagrangian connected sums α_i and β_j. However, there are two possible Lagrangian connected sums of α_i and β_j at each plumbing point $p \in \alpha_i \cap \beta_j$. They are $\alpha_i \#_p \beta_j$ and $\beta_j \#_p \alpha_i$. By assuming that α_i is a positive sphere and β_j is a negative sphere, one considers the Lagrangian connected sum $\beta_j \#_p \alpha_i$, not $\alpha_i \#_p \beta_j$. Similarly, by assuming that α_i is negative and β_j is positive, one can construct another set $\mathbb{B}^{\mathrm{op}}$ of Lagrangian branched submanifolds.

Lemma 7.7 *Let $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{B} \cup \mathbb{B}^{\mathrm{op}}$. Then there is a Hamiltonian isotopy $\Phi_t \colon M \to M$ such that*

1. $\Phi_t \circ \eta = \eta \circ \Phi_t$,
2. $\mathcal{B}_0 \pitchfork \Phi_1(\mathcal{B}_1)$,
3. *for every $q \in \mathcal{B}_0 \cap \Phi_1(\mathcal{B}_1)$, q is not a plumbing point or the antipodal point of a plumbing point.*

Proof Since $\mathcal{B}_1$ is a union of (parts of) compact cores and their Lagrangian connected sums, we will construct Hamiltonian isotopies perturbing each compact core α_i and β_j. Then, one obtains a perturbation of $\mathcal{B}_1$ as a union of (parts of) perturbations of α_i, β_j and Lagrangian connected sums of perturbed α_i and β_j.

First, we choose a smooth function $f_i \colon \alpha_i \to \mathbb{R}$ with isolated critical points such that

1. for every plumbing point $p \in \alpha_i$, $f_i(p) = f_i(-p) = 0$, where $-p$ is the antipodal point of p on α_i;
2. every critical point q of f_i lies on S_{α_i} and $q \neq p, -p$ for any plumbing point $p \in \alpha_i$;
3. $|df_i(x)| < \epsilon$ for all $x \in \alpha_i$ and for a sufficiently small fixed positive number ϵ;
4. $f_i \circ \eta_{\alpha_i} = f_i$, where η_{α_i} is the involution on $T^*\alpha_i$ defined in Section 7.1.

We remark that

$$T^*\alpha_i \overset{\phi_{\alpha_i}}{\simeq} T^*S^n = \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1, \langle x, y\rangle = 0\},$$

where $\phi_{\alpha_i} : T^*S^n \xrightarrow{\sim} T^*\alpha_i$ is the identification which we used in Section 7.1. Also, we remark that in (3), $|df_i(x)|$ is given by the standard metric on $\mathbb{R}^{2n+2}$.

Then, we can extend f_i to $\tilde{f_i} : T^*\alpha_i \to \mathbb{R}$ as follows. Let $\delta : [0,\infty) \to \mathbb{R}$ be a smooth decreasing function such that

$$\delta([0,\epsilon]) = 1, \delta([2\epsilon,\infty)) = 0.$$

We set

$$\tilde{f_i} : T^*\alpha_i \to \mathbb{R}, \tilde{f_i}(x,y) = \delta(|y|) f_i(x).$$

We get $\tilde{g}_j : T^*\beta_j \to \mathbb{R}$ in the same way.

These Hamiltonian functions $\tilde{f_i}$ and $\tilde{g}_j$ induce Hamiltonian isotopies on $T^*\alpha_i$ and $T^*\beta_j$. Moreover, these Hamiltonian isotopies could be extended on the plumbing space M since the Hamiltonian isotopies have compact supports on $T^*\alpha_i$ and $T^*\beta_j$.

Let $\Phi_{\alpha_i,t} : M \xrightarrow{\sim} M$ be the (extended) Hamiltonian isotopy associated to $\tilde{f_i}$. It is easy to check that

$$\begin{aligned} &\Phi_{\alpha_i,t} \circ \eta = \eta \circ \Phi_{\alpha_i,t}, \\ &\Phi_{\alpha_i,t}(\alpha_k) = \alpha_k \quad \text{if } k \neq i, \\ &\Phi_{\alpha_i,t}(\beta_j) = \beta_j \quad \text{for all } j, \\ &\Phi_{\alpha_i,1}(\alpha_i) = \Gamma(df_i), \end{aligned}$$

where $\Gamma(df_i)$ is the graph of df_i in $T^*\alpha_i \subset M$. Similarly, one can obtain a Hamiltonian isotopy $\Phi_{\beta_j,t} : M \xrightarrow{\sim} M$ for each β_j.

Let

$$\Phi_t = \prod_{\beta_j} \Phi_{\beta_j,t} \circ \prod_{\alpha_i} \Phi_{\alpha_i,t}.$$

It is easy to check that Φ_t satisfies the first condition of Lemma 7.7. Moreover, one can assume that $\Phi_1(\mathcal{B}_1)$ is constructed from $\Phi_1(\alpha_i)$ and $\Phi_1(\beta_j)$. Thus, it is easy to prove that $\mathcal{B}_0$ and $\Phi_1(\mathcal{B}_1)$ satisfy the second and the last conditions of Lemma 7.7. □

We will now explain how Lemma 7.7 weakens a difficulty of applying Theorem 7.3. The difficulty we will consider is the last condition of Theorem 7.3, ie $L_0 \cap L_1 = \widetilde{L}_0 \cap \widetilde{L}_1$. The other conditions can be weakened by a similar way.

Assume that L_0 (resp. L_1) is a Lagrangian submanifold which is carried by $\mathcal{B}_0$ (resp. $\mathcal{B}_1$) in $\mathbb{B} \cup \mathbb{B}^{\text{op}}$. Note that $\Phi_1(L_1)$ is carried by $\Phi_1(\mathcal{B}_1)$, where Φ_1 is the Hamiltonian isotopy constructed in Lemma 7.7.

We will count the numbers of intersections $L_0 \cap \Phi_1(L_1)$ and $\widetilde{L}_0 \cap \Phi_1(\widetilde{L}_1)$. If these numbers are the same, then $L_0 \cap \Phi_1(L_1) = \widetilde{L}_0 \cap \Phi_1(\widetilde{L}_1)$.

First, we remark that $\widetilde{L}_0$ (resp. $\Phi_1(\widetilde{L}_1)$) is a curve carried by a train track $\mathcal{B}_0 \cap \widetilde{M}$ (resp. $\Phi_1(\mathcal{B}_1) \cap \widetilde{M}$). Then, $\widetilde{L}_0$ (resp. $\Phi_1(\widetilde{L}_1)$) has weights on the train track $\mathcal{B}_0 \cap \widetilde{M}$ (resp. $\Phi_1(\mathcal{B}_1) \cap \widetilde{M}$). Moreover, the number of $\widetilde{L}_0 \cap \Phi_1(\widetilde{L}_1)$ is

$$\sum_{x \in \mathcal{B}_0 \cap \Phi_1(\mathcal{B}_1)} (\text{the weight of } \widetilde{L}_0 \text{ at } x) \cdot (\text{the weight of } \Phi_1(\widetilde{L}_1) \text{ at } x).$$

To count the number of $L_0 \cap \Phi_1(L_1)$, we can assume that $L_0 \cap \Phi_1(L_1)$ is contained in a small neighborhood of $\mathcal{B}_0 \cap \Phi_1(\mathcal{B}_1)$. Since L_0 is carried by $\mathcal{B}_0$, not strongly carried by, L_0 can have singular points. However, the singular points are "close" to one of plumbing points or the antipodes of plumbing points. Since the intersection points of $\mathcal{B}_0$ and $\Phi_1(\mathcal{B}_1)$ are not plumbing points or their antipodes, every $x \in L_0 \cap \Phi_1(L_1)$ is a regular point of L_0 (resp. $\Phi_1(L_1)$). It means that the number $|L_0 \cap \Phi_1(L_1)|$ is also given by

$$\sum_{x \in \mathcal{B}_0 \cap \Phi_1(\mathcal{B}_1)} (\text{the weight of } \widetilde{L}_0 \text{ at } x) \cdot (\text{the weight of } \Phi_1(\widetilde{L}_1) \text{ at } x).$$

Thus, $|L_0 \cap \Phi_1(L_1)| = |\widetilde{L}_0 \cap \Phi_1(\widetilde{L}_1)|$.

Lemma 7.8 *Let L_0 and L_1 be carried by $\mathcal{B}_0, \mathcal{B}_1 \in \mathbb{B} \cup \mathbb{B}^{\mathrm{op}}$. Then there is a Hamiltonian isotopy Φ_t such that*

$$L_0 \cap \Phi_1(L_1) = \widetilde{L}_0 \cap \Phi_1(\widetilde{L}_1).$$

Thus, if L_0 and L_1 are carried by $\mathcal{B}_0, \mathcal{B}_1 \in \mathbb{B} \cup \mathbb{B}^{\mathrm{op}}$, and if L_0 and L_1 satisfy conditions (1), (2) and (4) of Theorem 7.3, then one can apply Theorem 7.3 for L_0 and $\Phi_1(L_1)$.

Example 7.9 Let ψ_0 and ψ_1 be symplectomorphisms of Penner type, ie ψ_0 and ψ_1 are products of positive (resp. negative) powers of τ_i and negative (resp. positive) powers of σ_j, where τ_i and σ_j are Dehn twists along α_i and β_j respectively. Assume that L_0 (resp. L_1) is a Lagrangian submanifold of M, which is generated from one of compact cores by applying ψ_0 (resp. ψ_1), ie

$$L_0 = \psi_0(\alpha_k) \text{ or } \psi_0(\beta_j), \quad L_1 = \psi_1(\alpha_k) \text{ or } \psi_1(\beta_j).$$

Then $\eta(L_i) = L_i$ since

$$\eta(\alpha_i) = \alpha_i, \quad \eta(\beta_j) = \beta_j, \quad \eta \circ \tau_i = \tau_i \circ \eta, \quad \eta \circ \sigma_j = \sigma_j \circ \eta \quad \text{for all } i, j.$$

Moreover, $\widetilde{L}_i = \psi_i(\tilde{\alpha}_k)$ or $\psi_i(\tilde{\beta}_j)$. Thus, $\widetilde{L}_i$ is a Lagrangian submanifold of $\widetilde{M}$. Finally, L_i is carried by $\mathcal{B}_{\psi_i}$.

Thus, if L_0 and L_1 are not isotopic to each other, then one can apply Theorem 7.3.

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