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On the decategorification of some higher actions in Heegaard Floer homology

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We decategorify the higher actions on bordered Heegaard Floer strands algebras from recent work of Rouquier and the author, and identify the decategorifications with certain actions on exterior powers of homology groups of surfaces. We also suggest an interpretation for these actions in the language of open-closed TQFT, and we prove a corresponding gluing formula.

57K16; 18N25, 57K18

1 Introduction

In [15], Raphaël Rouquier and the author define a tensor product operation for higher representations of the dg monoidal category of Khovanov [11], which we call $\mathcal{U}$, and use it to reformulate aspects of cornered Heegaard Floer homology; see Douglas, Lipshitz and Manolescu [3; 4]. Part of this work involves defining 2–actions of $\mathcal{U}$ on the dg algebras $\mathcal{A}(\mathcal{Z})$ that bordered Heegaard Floer homology assigns to combinatorial representations $\mathcal{Z}$ of surfaces.

Ignoring gradings and thus working with decategorifications over $\mathbb{F}_2$, one can view $\mathcal{U}$ as a categorification of the algebra $\mathbb{F}_2[E]/(E^2)$ (an $\mathbb{F}_2$ analogue of $U(\mathfrak{gl}(1|1)^+)$), while if $\mathcal{Z}$ is a representation of a surface F, then $\mathcal{A}(\mathcal{Z})$ categorifies the vector space $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ where S_+ is a distinguished subset of the boundary of F. Thus, the 2–actions from [15] should categorify actions of $\mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$; the goal of this paper is to identify these actions explicitly using certain topological operations and to give an interpretation of these actions in the setting of open-closed TQFT.

To make things more precise, we recall that following Zarev [23] (but generalizing his definition slightly), a sutured surface is (F, S_+, S_-, Λ) where F is a compact oriented surface and Λ is a finite set of points in ∂F dividing ∂F into alternating subsets S_+ and S_-. We impose no topological restrictions, but note that the sutured surfaces representable by arc diagrams $\mathcal{Z}$ are those such that in each connected component of F (not of ∂F), both S_+ and S_- are nonempty (unlike Zarev [23], we allow arc diagrams to have circle components as well as interval components, and we do not impose nondegeneracy). In particular, no closed surface can be represented by an arc diagram.

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For an arc diagram $\mathcal{Z}$ representing a sutured surface (F, S_+, S_-, Λ), and each interval component I of S_+, the constructions of [15] define a 2–action of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$. On the other hand, there is a map ϕ_I from $H_1(F, S_+; \mathbb{F}_2)$ to $\mathbb{F}_2$ taking an element of $H_1(F, S_+; \mathbb{F}_2)$ to its boundary in $H_0(S_+; \mathbb{F}_2)$ and then pairing with the cohomology class of I. By summing ϕ_I over tensor factors, for $k \geq 1$ we get a map from $T^k H_1(F, S_+; \mathbb{F}_2)$ to $T^{k-1} H_1(F, S_+; \mathbb{F}_2)$ which induces a map Φ_I from $\wedge^k H_1(F, S_+; \mathbb{F}_2)$ to $\wedge^{k-1} H_1(F, S_+; \mathbb{F}_2)$.

Theorem 1.1 *The 2–action of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$ corresponding to I categorifies the action of $\mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ in which E acts by Φ_I.*

See Theorem 3.5 below for a more detailed statement of Theorem 1.1.

A TQFT interpretation

It is natural to ask whether the actions of $\mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ fit into a TQFT framework, with associated gluing results. Indeed, [15] reformulates and strengthens Douglas–Manolescu's gluing theorem for the algebras $\mathcal{A}(\mathcal{Z})$, which applies for certain decompositions of surfaces along 1–manifolds (given by certain decompositions of the arc diagram $\mathcal{Z}$). One could hope that such gluing theorems exist in even greater generality for the decategorified surface invariants $\wedge^* H_1(F, S_+; \mathbb{F}_2)$, yielding a TQFT-like construction for 1– and 2–manifolds.

Remark 1.2 Heegaard Floer homology is, in some nonaxiomatic sense, a 4–dimensional TQFT (spacetimes are 4–dimensional); accordingly, its decategorification should be a type of 3–dimensional TQFT involving the vector spaces $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ (and, for example, the Alexander polynomials of knots). The constructions under consideration for 1– and 2–manifolds should be part of a (loosely defined) extended-TQFT structure for decategorified Heegaard Floer homology.

A first observation is that a sutured surface (F, S_+, S_-, Λ) is nearly the same data as a morphism in the 2–dimensional open-closed cobordism category. As described by Lauda and Pfeiffer in [12], the objects of this category are finite disjoint unions of oriented intervals and circles. For two such objects X and Y, a morphism from X to Y is a compact oriented surface with its boundary decomposed into black regions (identified with $X \sqcup Y$) and colored regions. If (F, S_+, S_-, Λ) is a sutured surface and we label each component of S_+ as "incoming" or "outgoing", we get a morphism from S_+^{in} to S_+^{out} in this cobordism category. The black part of the boundary is S_+ and the colored part is S_-.

The actions of $\mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ suggest that one could try to assign the category of finite-dimensional $\mathbb{F}_2[E]/(E^2)$–modules to an interval. A sutured surface, with its S_+ boundary components labeled as incoming or outgoing, would be assigned a bimodule over tensor powers of $\mathbb{F}_2[E]/(E^2)$. For simplicity, we will restrict our attention here to sutured surfaces with no circular S_+ boundary components (all components of S_+ are intervals).

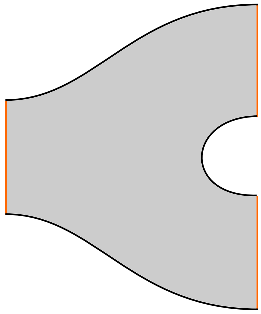

Figure 1: The open pair of pants; the S_+ boundary is shown in orange and the S_- boundary is shown in black (loosely following the visual conventions of [23]). Specifically, the input S_+ boundary is on the right while the output S_+ boundary is on the left.

For a surface F_1 with m intervals in its outgoing boundary and another surface F_2 with m intervals in its incoming boundary, let $F = F_2 \cup_{[0,1]^m} F_1$. We would want the bimodule of F to be a tensor product over $(\mathbb{F}_2[E]/(E^2))^{\otimes m}$ of the bimodules assigned to F_1 and F_2. The next theorem says this is true up to isomorphism; let $\mathbf{Alg}_{\mathbb{F}_2}$ denote the category whose objects are $\mathbb{F}_2$–algebras and whose morphisms are isomorphism classes of bimodules, with composition given by tensor product.

Theorem 1.3 *For F_1, F_2, and F as above, suppose that F_1 has m_{in} intervals in its incoming boundary and F_2 has m_{out} intervals in its outgoing boundary. We have a noncanonical isomorphism*

$$\wedge^* H_1(F, S_+; \mathbb{F}_2) \cong \wedge^* H_1(F_2, S_+; \mathbb{F}_2) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes m}} \wedge^* H_1(F_1, S_+; \mathbb{F}_2)$$

as bimodules over $((\mathbb{F}_2[E]/(E^2))^{\otimes m_{\mathrm{out}}}, (\mathbb{F}_2[E]/(E^2))^{\otimes m_{\mathrm{in}}})$. Thus, the exterior algebra vector spaces $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ give a functor from the "open sector" of the open-closed cobordism category into $\mathbf{Alg}_{\mathbb{F}_2}$.

In fact, a slightly more general version of Theorem 1.3 holds in which F_1 and F_2 can have S_+ circles in their boundaries as long as we are not gluing along them; see Theorem 4.2 below.

The tensor product case

As a special case of Theorem 1.3, we can glue interval S_+ components of two surfaces F' and F'' to the two input intervals of the "open pair of pants" cobordism shown in Figure 1. Let $P = F_1$ be the open pair of pants, let $F_2 = F' \sqcup F''$, and let F be the glued surface. We can identify $\wedge^* H_1(P, S_+; \mathbb{F}_2)$ with $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$, with right action of $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ given by multiplication and left action of $\mathbb{F}_2[E]/(E^2)$ given by the coproduct

$$\Delta(E) = E \otimes 1 + 1 \otimes E$$

(in fact, $\mathbb{F}_2[E]/(E^2)$ is a Hopf algebra with this coproduct together with counit $\varepsilon(E) = 0$ and antipode $S(E) = E$).

Corollary 1.4 *We have*

$$\wedge^* H_1(F, S_+; \mathbb{F}_2) \cong \wedge^* H_1(F', S_+; \mathbb{F}_2) \otimes \wedge^* H_1(F'', S_+; \mathbb{F}_2),$$

where the tensor product $\otimes$ is taken in the tensor category of finite-dimensional modules over the Hopf algebra $\mathbb{F}_2[E]/(E^2)$.

We can view Corollary 1.4 as a decategorification of the gluing result from [15] based on the higher tensor product operation $\circledast$. Thus, Theorem 1.3 suggests (at least at the decategorified level) a more general TQFT framework for the $\circledast$–based gluing results of [15].

Relationship to other work

Probably the closest analogue to the structures considered here can be found in Honda, Kazez and Matić's paper [7]. The data of a sutured surface (F, S_+, S_-, Λ) as discussed here is equivalent to the data (Σ, F) considered in [7, Section 7.1] (our F is the Σ of Honda, Kazez and Matić and our Λ is their F). The vector space $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ is isomorphic to an $\mathbb{F}_2$ version of Honda, Kazez and Matić's $V(\Sigma, F)$ which was subsequently studied by Mathews [16; 17; 18; 19] and Mathews and Schoenfeld [20]. In our notation, Honda, Kazez and Matić view this vector space as the sutured Floer homology of $F \times S^1$ with sutures given by $\Lambda \times S^1$, rather than as a Grothendieck group associated to $\mathscr{A}(\mathscr{Z})$. In other words, their surface invariants come from "trace decategorification" of 3–dimensional Heegaard Floer invariants rather than from Grothendieck-group-based decategorification of 2–dimensional Heegaard Floer invariants; these notions often agree, as they do here. See Cooper [1] for related work in the contact setting that discusses vector spaces similar to $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ in relation to Grothendieck groups of formal contact categories.

We can think of the gluings in Theorem 1.3 as successive self-gluings of two S_+ intervals in a sutured surface. These gluings can be interpreted as special cases of Honda, Kazez and Matić's gluings, where their gluing subsets γ and γ' cover our gluing S_+ intervals and extend a small bit past them on both sides. However, Honda, Kazez and Matić only assert the existence of a gluing map from the vector space of the original surface to the vector space of the glued surface (satisfying certain properties). Theorem 1.3 goes farther for the special gluings under consideration in that it shows how the vector space of the larger surface is recovered up to isomorphism as a tensor product.

Integral versions of the vector spaces $\wedge^*(F, S_+; \mathbb{F}_2)$, especially for closed F, or F with one boundary component (and implicitly $|\Lambda| = 2$), have also been studied in the context of TQFT invariants for 3–manifolds starting with Frohman and Nicas in [5]; see also Donaldson [2] and Kerler [10]. Building on work of Petkova [21], Hom, Lidman and Watson show in [6] that bordered Heegaard Floer homology (in the original formulation of Lipshitz, Ozsváth and Thurston [14] where F is closed) can be viewed as categorifying the $2+1$ TQFT described in [2] in which a surface F is assigned $\wedge^* H_1(F)$. Our perspective here differs in that we follow Zarev [23] rather than [14] and in that instead of $2+1$ TQFT structure we are (loosely) looking at the lower two levels of a $1+1+1$ TQFT.

Finally, the fact that the topological gluing considered in [15] can be viewed as the above open-pair-of-pants gluing was already noted in [15, Section 7.2.5], which also contains speculations about the connection to open-closed TQFT and extended TQFT.

Future directions

It would be desirable to treat 1–, 2–, and 3–manifolds at the same time, integrating the gluing results for surfaces here with the 3–manifold invariants mentioned above in something like a $1+1+1$ TQFT. One obstacle to doing this appears to be that while the isomorphism in the statement of Theorem 1.3 seems like something that could conceivably be proved using Mayer–Vietoris sequences, we were not able to find such a proof; the isomorphism we construct is not canonical and depends on suitable choices of bases. Geometrically, the issue seems to be that given arbitrary elements of $\wedge^* H_1(F_1, S_+; \mathbb{F}_2)$ and $\wedge^* H_1(F_2, S_+; \mathbb{F}_2)$, it is not clear how to pair them to get an element of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ in a canonical way (the endpoints of arcs don't necessarily match up in any nice way at the gluing interface).

It would also be desirable to categorify Theorem 1.3, such that the $\circledtimes$–based gluing results of [15] are recovered by gluing with an open pair of pants as in Corollary 1.4. Just as the proof of Theorem 1.3 depends on a choice of basis, it seems likely that a categorification of this theorem will depend on the arc diagrams $\mathcal{Z}$ chosen to represent the surfaces. For general arc diagrams $\mathcal{Z}_1$ and $\mathcal{Z}_2$ representing the surfaces F_1 and F_2 of Theorem 1.3, it is not even clear how one should glue these diagrams to get an arc diagram for the glued surface F (speculatively, something like [8, Figure 5(b)] followed by an "unzip" operation may be relevant).

Finally, preliminary computations indicate that close relatives of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ should arise in a TQFT with better structural properties than the "open" TQFT considered here, specifically one that is extended down to points and defined at least for all 0–, 1–, and 2–manifolds, with appropriate gluing theorems (including for gluing along circles). In work in progress, we study this extended TQFT as well as its relationship to the constructions of this paper.

Organization

In Sections 2.1 through 2.3 we review $\mathcal{U}$, the algebras $\mathcal{A}(\mathcal{Z})$, and the higher actions from [15]. Section 2.4 discusses decategorification for $\mathcal{U}$ and $\mathcal{A}(\mathcal{Z})$, showing that in the sense considered here, $\mathcal{A}(\mathcal{Z})$ categorifies $\wedge^* H_1(F, S_+; \mathbb{F}_2)$. Section 3 decategorifies the 2–actions of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$ from [15] and proves Theorem 1.1. Section 4 proves a generalized version of Theorem 1.3, and Section 5 discusses Corollary 1.4 in more generality.

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2 Decategorifying higher actions on strands algebras

2.1 The dg monoidal category $\mathcal{U}$

The following definition originated in [11] and was partly inspired by the strands dg algebras $\mathcal{A}(\mathcal{Z})$ in Heegaard Floer homology (we review these in Section 2.2). While Khovanov works over $\mathbb{Z}$, we work over $\mathbb{F}_2$ in order to interact properly with the $\mathbb{F}_2$–algebras $\mathcal{A}(\mathcal{Z})$.

Definition 2.1 Let $\mathcal{U}$ denote the strict $\mathbb{F}_2$–linear dg monoidal category freely generated (under $\otimes$ and composition) by an object e and an endomorphism τ of $e \otimes e$ modulo the relations $\tau^2 = 0$ and

$$(\mathrm{id}_e \otimes \tau) \circ (\tau \otimes \mathrm{id}_e) \circ (\mathrm{id}_e \otimes \tau) = (\tau \otimes \mathrm{id}_e) \circ (\mathrm{id}_e \otimes \tau) \circ (\tau \otimes \mathrm{id}_e).$$

We set $d(\tau) = 1$, and we let τ have degree -1 (we use the convention that differentials increase degree by 1).

The endomorphism algebra of $e^{\otimes n} \in \mathcal{U}$ is the dg algebra referred to as H_n^- in [11] (tensored with $\mathbb{F}_2$); in the language used in [15] it is a nil-Hecke algebra with a differential, and in the language used in [4] it is a nil-Coxeter algebra. We will use NC_n to denote the $\mathbb{F}_2$ version of this algebra. It has a graphical interpretation: $\mathbb{F}_2$–basis elements of NC_n are pictures like Figure 2, with n strands going from bottom to top (these pictures are in bijection with permutations on n letters). Multiplication is defined by vertical concatenation, with ab obtained by drawing a below b, except that if two strands cross and then uncross in the stacked picture (ie if the stacked picture has a double crossing) then the product is defined to be zero. The differential is defined by summing over all ways to resolve a crossing (see Figure 3), except that if a crossing resolution produces a double crossing between two strands then it contributes zero to the differential (see Figure 4). The endomorphism τ of $e \otimes e$ is represented by a single crossing between two strands.

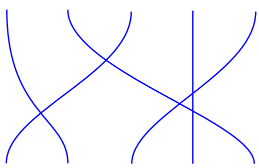

Figure 2: A basis element of NC_n for $n = 5$.

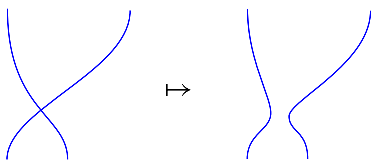

Figure 3: Resolving a crossing.

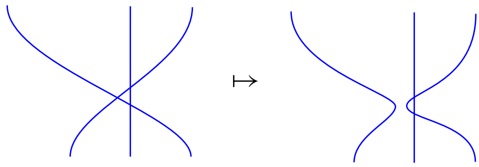

Figure 4: A resolution that produces a double crossing and thus does not contribute to the differential on NC_n.

2.2 Strands algebras

Let $\mathcal{Z}$ be an arc diagram as in [23, Definition 2.1.1], except that we allow (oriented) circles as well as intervals in $\boldsymbol{Z}$, and we do not impose any nondegeneracy condition. Thus, $\mathcal{Z}$ consists of:

- a finite collection $\boldsymbol{Z} = \{Z_1, \ldots, Z_l\}$ of oriented intervals and circles;
- a finite set of points $\boldsymbol{a}$ (with $|\boldsymbol{a}|$ even) in the interiors of the Z_i for $1 \leq i \leq l$;
- a two-to-one matching M of the points in $\boldsymbol{a}$.

An example is shown in Figure 5.

The definition of the dg strands algebra $\mathcal{A}(\mathcal{Z})$ over $\mathbb{F}_2$, from [23, Definition 2.2.2], generalizes in a straightforward way to this setting and is a special case of the general strands algebras treated in detail in [15]. One can view $\mathcal{A}(\mathcal{Z})$ as being defined by specifying an $\mathbb{F}_2$ basis consisting of certain pictures, along with rules for multiplying and differentiating basis elements.

Definition 2.2 A *strands picture* is a collection of strands drawn in $[0,1] \times \boldsymbol{Z}$, each with its left endpoint in $\{0\} \times \boldsymbol{a}$ and its right endpoint in $\{1\} \times \boldsymbol{a}$. The strands can be either solid or dotted and are considered only up to homotopy relative to the endpoints; by convention, strands are drawn "taut", sometimes with a bit of curvature for visual effect (see Figure 6). They must satisfy the following rules:

- Strands cannot move against the orientation of $\boldsymbol{Z}$ when moving from left to right (from 0 to 1 in $[0,1]$).
- No solid strands are horizontal, while all dotted strands are horizontal.
- If a solid strand has its left endpoint at $a \in \boldsymbol{a}$, and a is matched to $a' \in \boldsymbol{a}$ under M, then no strand can have its left endpoint at a', and similarly for right endpoints.

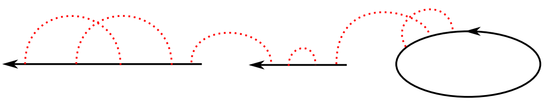

Figure 5: An arc diagram $\mathcal{Z} = (\boldsymbol{Z}, \boldsymbol{a}, M)$; $\boldsymbol{Z}$ consists of two intervals and a circle, $\boldsymbol{a}$ is the set of endpoints of the dotted (red) arcs, and M matches the two endpoints of each dotted arc.

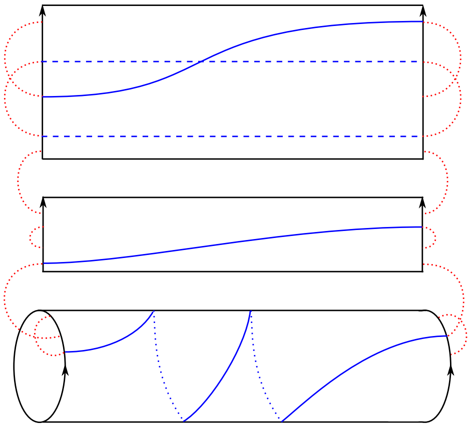

Figure 6: A strands picture (basis element for $\mathcal{A}(\mathcal{Z})$).

- If a dotted strand has its left (and thus right) endpoint at $a \in \boldsymbol{a}$, and a is matched to $a' \in \boldsymbol{a}$ under M, then there must be another dotted strand with its left (and thus right) endpoint at a' (we say this dotted strand is matched with the first one).

Definition 2.3 As an $\mathbb{F}_2$–vector space, $\mathcal{A}(\mathcal{Z})$ is defined to be the formal span of such strands pictures, so that strands pictures form an $\mathbb{F}_2$ basis for $\mathcal{A}(\mathcal{Z})$. The product of two basis elements of $\mathcal{A}(\mathcal{Z})$ is defined by concatenation (see Figure 7), with the following subtleties:

- If some solid strand has no strand to concatenate with, or if in some matched pair of dotted strands $\{s, s'\}$, neither s nor s' has a strand to concatenate with, the product is zero.
- When concatenating a solid strand with a dotted strand, one erases the dotted strand matched to the one involved in the concatenation, and makes the concatenated strand solid.
- If a double crossing is formed upon concatenation, the product of the basis elements is defined to be zero.

The differential of a basis element of $\mathcal{A}(\mathcal{Z})$ is the sum of all strands pictures formed by resolving a crossing in the original strands picture (in the sense of Figure 3 above), with the following subtleties:

- When resolving a crossing between a solid strand and a dotted strand, one erases the dotted strand matched to the one involved in the crossing resolution, and makes both the resolved strands solid.

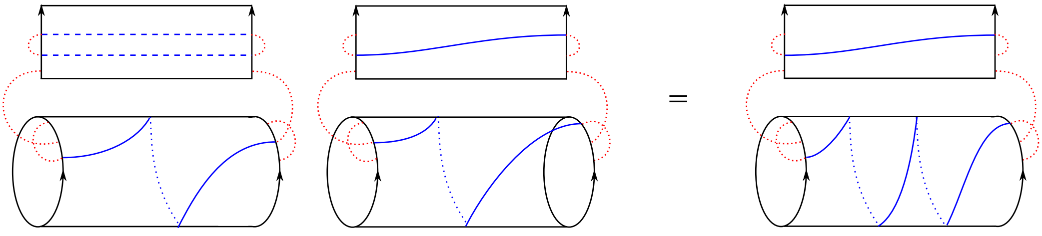

Figure 7: Example of a product in $\mathcal{A}(\mathcal{Z})$.

- If a double crossing is formed upon resolving a crossing (as in Figure 4 above), then this crossing resolution does not contribute a term to the differential.

Remark 2.4 Recall that a dg category over a field k is a category enriched in the symmetric monoidal category of chain complexes over k, ie graded $k[\partial]/(\partial^2)$–modules where ∂ has degree -1 or $+1$ depending on conventions, with the tensor product given as usual. Similarly, a differential category over k is a category enriched in the symmetric monoidal category of (ungraded) $k[\partial]/(\partial^2)$–modules (the symmetric monoidal structure is analogous to the graded case[1]).

While $\mathcal{U}$ is a dg category and not just a differential category, the grading on $\mathcal{A}(\mathcal{Z})$ is much more complicated: it is a grading by a nonabelian group $G(\mathcal{Z})$ rather than by $\mathcal{Z}$, and it depends on a choice of "grading refinement data". To avoid these complications, gradings were not fully treated in [15]; correspondingly, when decategorifying in this paper, we will work with Grothendieck groups defined over $\mathbb{F}_2$ rather than over $\mathbb{Z}$, and we will view $\mathcal{A}(\mathcal{Z})$ as a differential algebra.

Definition 2.5 We let $\mathcal{A}(\mathcal{Z}, k)$ be the $\mathbb{F}_2$–subspace of $\mathcal{A}(\mathcal{Z})$ spanned by strands pictures such that the number of solid strands plus half the number of dotted strands is k. In fact, $\mathcal{A}(\mathcal{Z}, k)$ is a dg subalgebra of $\mathcal{A}(\mathcal{Z})$ (ignoring unit), and if $|\boldsymbol{a}| = 2n$, we have $\mathcal{A}(\mathcal{Z}) = \bigoplus_{k=0}^{n} \mathcal{A}(\mathcal{Z}, k)$.

The basis elements of $\mathcal{A}(\mathcal{Z})$ with only dotted (horizontal) strands are idempotents of $\mathcal{A}(\mathcal{Z})$. Furthermore, for a general basis element a of $\mathcal{A}(\mathcal{Z})$, there is exactly one such idempotent (call it $\lambda(a)$) such that $\lambda(a)a = a$, and for all other such idempotents λ', we have $\lambda' a = 0$. We will refer to $\lambda(a)$ as the left idempotent of a; we can define a right idempotent $\rho(a)$ similarly.

Below we will identify $\mathcal{A}(\mathcal{Z})$ with the differential category whose objects are in bijection with the all-horizontal basis elements of $\mathcal{A}(\mathcal{Z})$, and whose morphism space from e to e' is $e'\mathcal{A}(\mathcal{Z})e$. Because each basis element of $\mathcal{A}(\mathcal{Z})$ has a unique left and right idempotent, we can view these elements as giving a basis for the morphism spaces of $\mathcal{A}(\mathcal{Z})$ as a category.

2.3 Higher actions on strands algebras

Let $\mathcal{Z} = (\boldsymbol{Z}, \boldsymbol{a}, M)$ be an arc diagram. As in [15, Section 7.2.4], we can view $\mathcal{Z}$ as a singular curve Z in the language of that paper, and $\mathcal{A}(\mathcal{Z})$ is the endomorphism algebra of a collection of objects in the strands category $\mathcal{S}(Z)$; see [15, Section 7.4.11]. For an interval I in $\boldsymbol{Z}$ (equivalently, a noncircular component of Z as in [15, Section 7.2.2]), the constructions of [15, Section 8.1.1] give us a differential bimodule E over $\mathcal{A}(\mathcal{Z})$.

Notation 2.6 We will call this bimodule $\mathcal{E}$ rather than E for notational clarity.

Closely related constructions appear in [4], although in that paper the relevant pictures were not explicitly organized into a bimodule over $\mathcal{A}(\mathcal{Z})$.

[1] And can be summarized by $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$, at least in characteristic 2, but our view is that in this paper "E" and "∂" are playing very different roles.

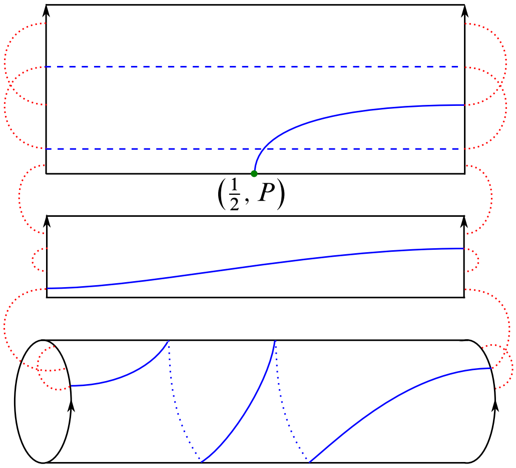

Figure 8: A strands picture for $\mathscr{E}$ (the distinguished interval I is the top interval).

As with the strands algebras, the bimodule $\mathscr{E}$ is defined by specifying an $\mathbb{F}_2$–basis of strands pictures, together with a differential and left and right actions of $\mathscr{A}(\mathscr{Z})$ in terms of basis elements. These strands pictures are almost the same as those described in Definition 2.2. To describe the difference, let P be the endpoint of the interval I such that in the orientation on $\boldsymbol{Z}$, I points from P to its other endpoint. Then, in a strands picture for $\mathscr{E}$, there should be one solid strand with its left endpoint at $\left(\frac{1}{2}, P\right) \in [0,1] \times \boldsymbol{Z}$ and with its right endpoint in $\{1\} \times \boldsymbol{a}$. See Figure 8; all other rules in Definition 2.2 are unchanged.

Definition 2.7 As an $\mathbb{F}_2$–vector space, $\mathscr{E}$ is defined to be the formal span of the strands pictures described above, which form an $\mathbb{F}_2$–basis for $\mathscr{E}$. The left and right actions of $\mathscr{A}(\mathscr{Z})$ on $\mathscr{E}$, and the differential on $\mathscr{E}$, are defined by concatenation and resolution of crossings as in Definition 2.3. We let $\mathscr{E}(k)$ be the $\mathbb{F}_2$–subspace of $\mathscr{E}$ spanned by strands pictures such that the number of solid strands plus half the number of dotted strands is k; then $\mathscr{E}(k)$ is a differential subbimodule of $\mathscr{E}$, and if $|\boldsymbol{a}| = 2n$, we have $\mathscr{E} = \bigoplus_{k=1}^{n} \mathscr{E}(k)$. Furthermore, $\mathscr{E}(k)$ is a bimodule over $(\mathscr{A}(\mathscr{Z}, k-1), \mathscr{A}(\mathscr{Z}, k))$ with all other summands of $\mathscr{A}(\mathscr{Z})$ acting as zero on $\mathscr{E}(k)$.

As with the basis elements of $\mathscr{A}(\mathscr{Z})$, to each basis element x of $\mathscr{E}$ we can associate a left idempotent $\lambda(x)$ and a right idempotent $\rho(x)$. We have $x = \lambda(x) x \rho(x)$, while for any other purely horizontal basis elements $\lambda' \neq \lambda(x)$ and $\rho' \neq \rho(x)$ of $\mathscr{A}(\mathscr{Z})$, we have $\lambda' x = 0$ and $x \rho' = 0$.

By [15, Lemma 8.1.2], the bimodule $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} \mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} \cdots \otimes_{\mathscr{A}(\mathscr{Z})} \mathscr{E}$ (with m factors) is isomorphic to the bimodule defined analogously to $\mathscr{E}$, but having solid strands with left endpoints at

$$\left\{\left(\frac{1}{m+1}, P\right), \left(\frac{2}{m+1}, P\right), \dots, \left(\frac{m}{m+1}, P\right)\right\}.$$

This bimodule (which we will call $\mathscr{E}^{\otimes m}$) also appears in [4], and as in that paper it admits a left action of NC_m defined diagrammatically by sticking strands pictures for NC_m on the bottom of strands pictures

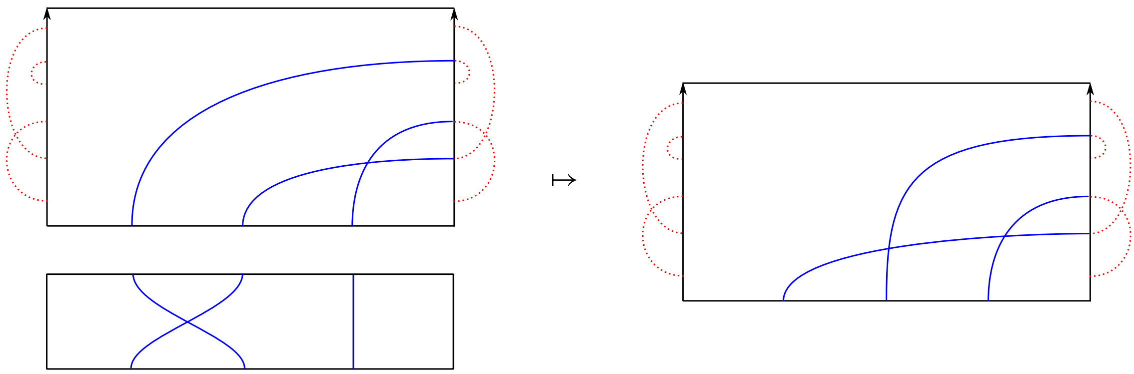

Figure 9: The action of an element of NC_3 on $\mathcal{E}^{\otimes 3}$.

for $\mathcal{E}^{\otimes m}$ (see Figure 9). These actions form a 2–action of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$ via differential bimodules and bimodule maps, which was defined in [15, Proposition 8.1.3]. In other words, they give a differential monoidal functor from $\mathcal{U}$ to the differential monoidal category of differential bimodules over $\mathcal{A}(\mathcal{Z})$ and chain complexes of bimodule maps between them.

2.4 Decategorification

2.4.1 Decategorifying $\mathcal{U}$

Definition 2.8 For a differential category A, we let $\bar{A}$ denote the smallest full differential subcategory of A–Mod (left differential modules over A) containing $\mathrm{Hom}(e, -)$ for all objects e of A and closed under mapping cones and isomorphisms. If A is a dg category, we let A–Mod be the category of left dg modules instead, and require that $\bar{A}$ be closed under degree shifts. We let $H(A)$ denote the homotopy category of A, and we let A^i denote the idempotent completion of A.

Remark 2.9 In the language of bordered Heegaard Floer homology [13; 14], $\bar{A}$ is essentially the same as the differential category of finitely generated bounded type D structures over A (in this setting it is typical to view A as a differential algebra with a distinguished set of idempotents rather than as a dg category).

It is a well-known result (see [9, Corollary 3.7]) that if A is a dg category, then $H(\bar{A})^i$ is equivalent to the full subcategory of the derived category $\mathcal{D}(A)$ (of left dg A–modules) on compact objects, ie the compact derived category of A.

We can view dg algebras such as NC_n as dg categories with one object. Khovanov shows in [11] that the Grothendieck group of the compact derived category of NC_n is zero for $n \geq 2$. For $n = 0$ and $n = 1$, NC_n is $\mathbb{F}_2$, so the Grothendieck group of its compact derived category is $\mathbb{Z}$ (Khovanov gets $\mathbb{Z}[q, q^{-1}]$ instead because he introduces an extra q–grading on NC_n which is identically zero, but we will not use this grading).

Corollary 2.10 *The Grothendieck group $K_0(H(\overline{NC}_n))$ is also $\mathbb{Z}$ for $n \in \{0,1\}$ and is zero for $n \geq 2$, where $H(\overline{NC}_n)$ is the homotopy category of $\overline{NC}_n$.*

Proof The inclusion of the triangulated category $H(\overline{NC}_n)$ into its idempotent completion is a monomorphism by [22, Corollary 2.3]. In fact, by [22, Theorem 2.1], $H(\overline{NC}_n)$ is already idempotent complete. □

Since we will primarily work with Grothendieck groups over $\mathbb{F}_2$ here, we introduce the following definition.

Definition 2.11 Let $\mathscr{C}$ be a category equipped with a collection of distinguished triangles $X \to Y \to Z \rightsquigarrow$ as in a triangulated category (but we do not require $\mathscr{C}$ to be triangulated or even to have a shift functor; we place no requirements on the collection of distinguished triangles). We let $K_0^{\mathbb{F}_2}(\mathscr{C})$ be the $\mathbb{F}_2$–vector space with basis given by isomorphism classes of objects of $\mathscr{C}$ modulo relations $[X]+[Y]+[Z]=0$ whenever there exists a distinguished triangle $X \to Y \to Z \rightsquigarrow$.

For a triangulated category $\mathscr{C}$, the above definition agrees with $K_0(\mathscr{C}) \otimes \mathbb{F}_2$. We see that $K_0^{\mathbb{F}_2}(H(\overline{NC}_n))$ is isomorphic to $\mathbb{F}_2$ for $n \in \{0,1\}$ and is zero otherwise.

Now, since $\mathscr{U}$ is a direct sum of NC_n (as a one-object dg category) over all $n \geq 0$, $K_0^{\mathbb{F}_2}(H(\overline{\mathscr{U}})) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. For notational convenience, we let

$$K_0^{\mathbb{F}_2}(\mathscr{U}) := K_0^{\mathbb{F}_2}(H(\overline{\mathscr{U}})).$$

Taking the monoidal structure on $\mathscr{U}$ into account, we see that as an $\mathbb{F}_2$–algebra,

$$K_0^{\mathbb{F}_2}(\mathscr{U}) \cong \mathbb{F}_2[E]/(E^2)$$

(this is Khovanov's identification $K_0(H^-) \cong \mathbb{Z}[q, q^{-1}, E_1]/(E_1^2)$ from [11], adapted to our setting).

2.4.2 Decategorifying the strands algebras As mentioned above, we will view the strands algebras $\mathscr{A}(\mathscr{Z})$ as differential categories with multiple (but finitely many) objects in bijection with the set of purely horizontal strands pictures for $\mathscr{Z}$. The homotopy category $H(\overline{\mathscr{A}(\mathscr{Z})})$ has a collection of distinguished triangles, namely those isomorphic to the image in the homotopy category of $X \xrightarrow{f} Y \to \mathrm{Cone}(f) \rightsquigarrow$ for some closed morphism $f \colon X \to Y$ in $\overline{\mathscr{A}(\mathscr{Z})}$.

Recall that the construction of a sutured surface (F, S_+, S_-, Λ) from an arc diagram $\mathscr{Z} = (\boldsymbol{Z}, \boldsymbol{a}, \Lambda)$ starts by taking $\boldsymbol{Z} \times [0,1]$, a collection of rectangles and annuli, and gluing on some 2–dimensional 1–handles. For each pair of points $\{p, q\}$ of $\boldsymbol{a}$ matched by M, one glues on a 1–handle with attaching zero-sphere $\{(p,1),(q,1)\}$ compatibly with the orientation on $\boldsymbol{Z}$. The result is F; one sets $S_+ := \boldsymbol{Z} \times \{0\}$ and $\Lambda := (\partial \boldsymbol{Z}) \times \{0\}$, with the rest of the boundary of F placed in S_-.

Proposition 2.12 [21] *For $\mathscr{Z} = (\boldsymbol{Z}, \boldsymbol{a}, M)$ with $\boldsymbol{Z}$ a single interval, $K_0(H(\overline{\mathscr{A}(\mathscr{Z})}))$ is isomorphic to $\wedge^* H_1(F;\mathbb{Z})$ where F is the surface represented by $\mathscr{Z}$. Specifically, for each k, $K_0(H(\overline{\mathscr{A}(\mathscr{Z},k)}))$ is isomorphic to $\wedge^k H_1(F;\mathbb{Z})$.*

It follows that $K_0^{\mathbb{F}_2}(H(\overline{\mathcal{A}(\mathcal{Z})}))$ is isomorphic to $\wedge^* H_1(F;\mathbb{F}_2)$, and in the $\mathbb{F}_2$ setting we do not need to consider Petkova's absolute $\mathbb{Z}/2\mathbb{Z}$ homological grading on $\mathcal{A}(\mathcal{Z})$.

Remark 2.13 Petkova views the surface F associated to a one-interval arc diagram $\mathcal{Z}$ as being closed, while we view it as having S^1 boundary with one S_+ interval and one S_- interval. Letting $\bar{F}$ denote the closed surface and F denote the surface with boundary, we have natural identifications

$$H_1(\bar{F}) \cong H_1(F) \cong H_1(F, S_+)$$

(with either $\mathbb{Z}$ or $\mathbb{F}_2$ coefficients).

Petkova's arguments readily generalize to show that for general $\mathcal{Z}$ as defined above, $K_0^{\mathbb{F}_2}(H(\overline{\mathcal{A}(\mathcal{Z})}))$ has an $\mathbb{F}_2$–basis given by the set of objects of $\mathcal{A}(\mathcal{Z})$ as a dg category, ie by the purely horizontal strands pictures for $\mathcal{Z}$.

Proposition 2.14 *If (F, S_+, S_-, Λ) is the sutured surface represented by a general arc diagram $\mathcal{Z}$, then the vector space $\wedge^* H_1(F, S_+;\mathbb{F}_2)$ has a basis in bijection with purely horizontal strands pictures for $\mathcal{Z}$.*

Proof It follows from the construction of (F, S_+, S_-, Λ) that F/S_+ is homotopy equivalent to a wedge product of circles, one for each pair of points of $\boldsymbol{a}$, and these circles form a basis for $H_1(F, S_+;\mathbb{F}_2)$. A basis for $\wedge^* H_1(F, S_+;\mathbb{F}_2)$ is then given by all subsets of the set of these circles. For each such subset X, there is a corresponding purely horizontal strands picture for $\mathcal{Z}$; if a circle (corresponding to $\{p, q\}$ matched by M) is in X, one draws a pair of dotted horizontal strands at p and q in the strands picture. This correspondence is a bijection, proving the proposition. □

Let $K_0^{\mathbb{F}_2}(\mathcal{A}(\mathcal{Z})) := K_0^{\mathbb{F}_2}(H(\overline{\mathcal{A}(\mathcal{Z})}))$ and $K_0^{\mathbb{F}_2}(\mathcal{A}(\mathcal{Z}, k)) := K_0^{\mathbb{F}_2}(H(\overline{\mathcal{A}(\mathcal{Z}, k)}))$.

Corollary 2.15 *We have natural identifications*

$$K_0^{\mathbb{F}_2}(\mathcal{A}(\mathcal{Z})) \cong \wedge^* H_1(F, S_+;\mathbb{F}_2) \quad \textit{and} \quad K_0^{\mathbb{F}_2}(\mathcal{A}(\mathcal{Z}, k)) \cong \wedge^k H_1(F, S_+;\mathbb{F}_2).$$

3 Actions on exterior powers of homology

Let $\mathcal{Z} = (\boldsymbol{Z}, \boldsymbol{a}, M)$ be an arc diagram representing a sutured surface (F, S_+, S_-, Λ) as in Figure 10, and let I be an interval component of S_+ (equivalently, let I be an interval component of $\boldsymbol{Z}$). The endomorphism Φ_I of $\wedge^* H_1(F, S_+;\mathbb{F}_2)$ defined in the introduction squares to zero and thus gives us an action of $\mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+;\mathbb{F}_2)$ in which E acts by Φ_I. In this section we identify this action with the action of $K_0^{\mathbb{F}_2}(\mathcal{U})$ on $K_0^{\mathbb{F}_2}(\mathcal{A}(\mathcal{Z}))$ coming from the 2–action of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$ described in Section 2.3.

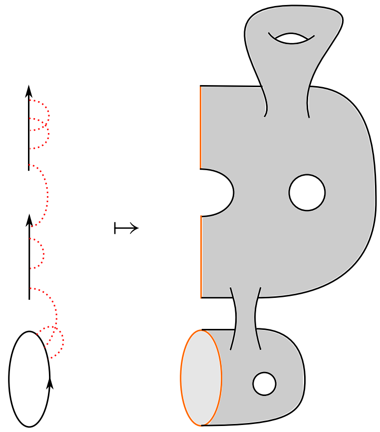

Figure 10: An arc diagram and the sutured surface it represents. The S_+ portion of the surface boundary is drawn in orange and the S_- portion is drawn in black.

Remark 3.1 For an element ω of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ that is a pure wedge product of arcs in F with boundary on S_+ and/or circles in F, we can depict ω by drawing all the arcs and circles of ω in a picture of F. See Figure 11 for an example. The element E of $\mathbb{F}_2[E]/(E^2)$ acts on this depiction of ω by summing over all ways of removing one arc incident with the component I of S_+; see Figure 12. An arc with both endpoints on I is "removed twice" which, in the sum with $\mathbb{F}_2$ coefficients, amounts to not being removed at all; indeed, such an arc represents the same homology class as a circle with no endpoints.

We first review an important structural property of the bimodule $\mathscr{E}$ from Section 2.3; the proposition below follows from [15, Section 8.1.4], but to keep this paper self-contained we include an independent proof.

Proposition 3.2 *As a left differential module over the differential category $\mathscr{A}(\mathscr{Z})$, $\mathscr{E}$ is an object of $\overline{\mathscr{A}(\mathscr{Z})}$.*

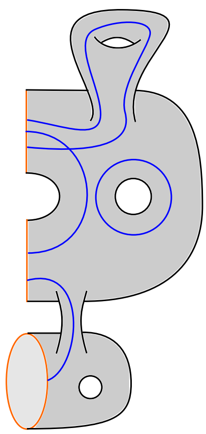

Figure 11: Depiction of a pure wedge-product element of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$.

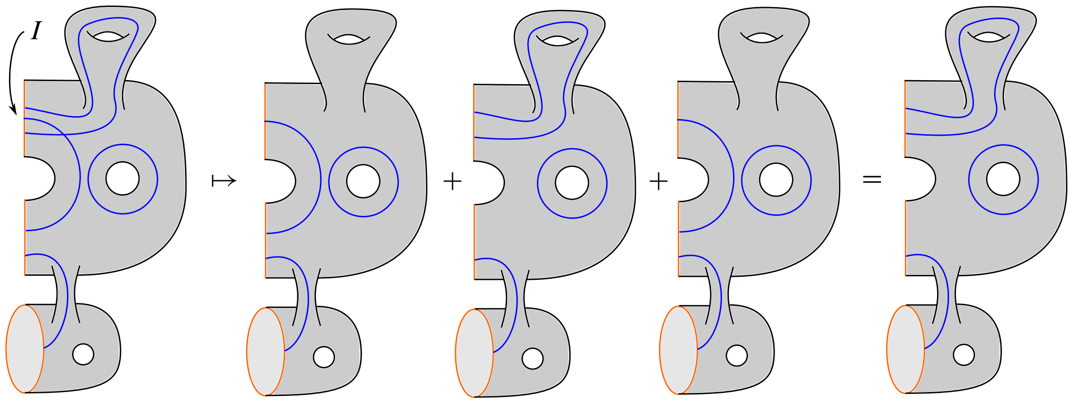

Figure 12: Action of $E \in \mathbb{F}_2[E]/(E^2)$ on $\omega \in \wedge^* H_1(F, S_+; \mathbb{F}_2)$ given a distinguished interval I of S_+.

Proof We first show that as a left module (disregarding the differential), $\mathscr{E}$ is isomorphic to a direct sum of modules of the form $\mathrm{Hom}(e, -)$ for objects e of $\mathscr{A}(\mathscr{Z})$. Indeed, consider the subset S of strands pictures for $\mathscr{E}$ (ie $\mathbb{F}_2$–basis elements of $\mathscr{E}$) such that the only moving strand is the one with left endpoint at $\left(\frac{1}{2}, P\right)$ in the language of Section 2.3. See Figure 13 for an example of an element of S. An arbitrary basis element x of $\mathscr{E}$ can be written as ay for unique basis elements $a \in \mathscr{A}(\mathscr{Z})$ and $y \in S$; indeed, after a homotopy relative to the endpoints, we can draw x such that all strands of x except the one with endpoint at $\left(\frac{1}{2}, P\right)$ only move on $\boldsymbol{Z} \times [0, \varepsilon]$ for some $\varepsilon < \frac{1}{2}$, and are horizontal on $\boldsymbol{Z} \times [\varepsilon, 1]$ (see Figure 14).

Cutting the diagram for x at $\boldsymbol{Z} \times \{\varepsilon\}$, we see a strands picture for a basis element $a \in \mathscr{A}(\mathscr{Z})$ on the left. On the right side of the cut, let y be the element of S obtained by making all the horizontal strands dotted and adding in their matching horizontal strands (according to the matching M). See Figure 15 for an example. We have $ay = x$; furthermore, for any $y \in S$ with left idempotent $\lambda(y)$, and any basis element

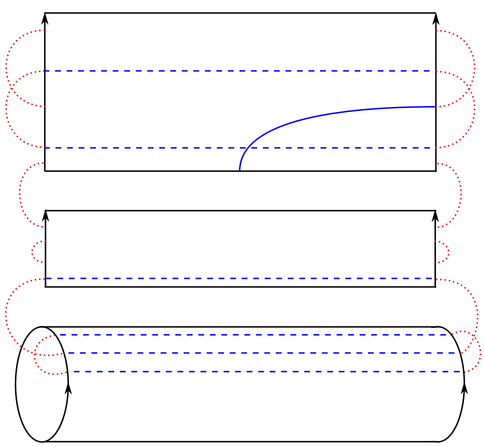

Figure 13: An element of the set S of special basis elements of $\mathscr{E}$.

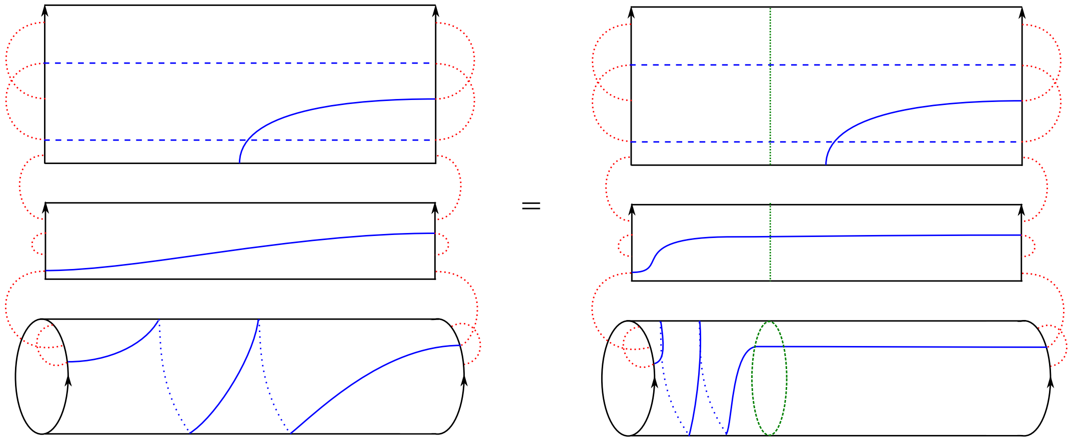

Figure 14: Stretching the basis element x of Figure 8 so that all "ordinary" moving strands only move on $\boldsymbol{Z} \times [0,\varepsilon]$; the green dashed lines on the right indicate where we will cut to factor x as ay with $a \in \mathcal{A}(\mathcal{Z})$ and $y \in S$.

a of $\mathrm{Hom}_{\mathcal{A}(\mathcal{Z})}(\lambda(y),-)$, we have that ay is a basis element for $\mathcal{E}$ and that a and y are recovered when splitting ay as above.

We have defined a bijection between our basis for $\mathcal{E}$ and the set of pairs (a, y) where y is an element of S with left idempotent $\lambda(y)$ and a is a basis element of $\mathrm{Hom}_{\mathcal{A}(\mathcal{Z})}(\lambda(y),-)$. Thus, we have an identification of $\mathcal{E}$ with $\bigoplus_{y\in S}\mathrm{Hom}_{\mathcal{A}(\mathcal{Z})}(\lambda(y),-)$ as vector spaces. This identification respects left multiplication by $\mathcal{A}(\mathcal{Z})$, so

$$\mathcal{E} \cong \bigoplus_{y\in S}\mathrm{Hom}_{\mathcal{A}(\mathcal{Z})}(\lambda(y),-)$$

as left modules over $\mathcal{A}(\mathcal{Z})$ (ignoring the differential).

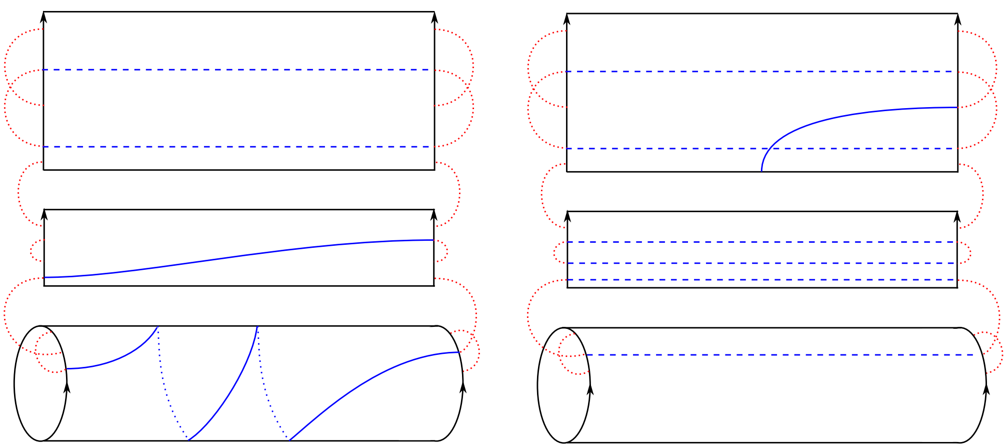

Figure 15: Factorizing the basis element x of Figure 8 as $a \in \mathcal{A}(\mathcal{Z})$ (left) times $y \in S$ (right).

Now, we can define a grading on the elements of S: say $y \in S$ has degree d if the moving strand σ of y with left endpoint $\left(\frac{1}{2}, P\right)$ encounters d points of $\boldsymbol{a}$ while traveling along a minimal path in $\boldsymbol{Z}$ from P to its right endpoint. Order the elements of S by increasing degree (choose any ordering of the elements of S in each given degree). Because the differential on $\mathscr{E}$, applied to $y \in S$, will only resolve crossings between the special strand σ of y and horizontal strands strictly below σ, the only nonzero terms of this differential will be of the form ay' for y' of degree strictly less than that of y (and thus y' that appear before y in the ordering on S). It follows that $\mathscr{E}$ is isomorphic to an iterated mapping cone built from $\mathrm{Hom}_{\mathscr{A}(\mathscr{Z})}(\lambda(y), -)$ for $y \in S$, so we have $\mathscr{E} \in \overline{\mathscr{A}(\mathscr{Z})}$. □

Remark 3.3 In the language of bordered Heegaard Floer homology, Proposition 3.2 says that $\mathscr{E}$ is the differential bimodule associated to a finitely generated left bounded type DA bimodule over $\mathscr{A}(\mathscr{Z})$ with δ^1_i zero for $i > 2$.

Proposition 3.2 gives us the following corollary.

Corollary 3.4 *We have a differential functor $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} -$ from $\overline{\mathscr{A}(\mathscr{Z})}$ to itself, and thus a functor $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} -$ from $H(\overline{\mathscr{A}(\mathscr{Z})})$ to itself.*

Proof Let $\mathscr{E} \cong \oplus_\alpha \mathscr{A}(\mathscr{Z}) \cdot e_\alpha$ (as a left module) and suppose we have $X \cong \oplus_\beta \mathscr{A}(\mathscr{Z}) \cdot x_\beta \in \overline{\mathscr{A}(\mathscr{Z})}$, where e_α and x_β are distinguished idempotents of $\mathscr{A}(\mathscr{Z})$, the sums over α and β are finite, for all (α, β) we have $e_\alpha \cdot' x_\beta \in \{e_\alpha, 0\}$ where $\cdot'$ denotes the right action of $\mathscr{A}(\mathscr{Z})$ on $\mathscr{E}$ (the proof of Proposition 3.2 implies this is possible), and there exist orderings of the α and β such that the differentials on $\mathscr{E}$ and X are strictly decreasing with respect to the order. Then

$$\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} X \cong \bigoplus_\beta \mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} (\mathscr{A}(\mathscr{Z}) \cdot x_\beta) \cong \bigoplus_\beta \mathscr{E} \cdot' x_\beta \cong \bigoplus_{\alpha,\beta} \mathscr{A}(\mathscr{Z}) \cdot (e_\alpha \cdot' x_\beta).$$

If we order the pairs (α, β) lexicographically such that the β coordinate dominates, then the differential on $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} X$ is strictly decreasing with respect to the order. It follows that $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} X \in \overline{\mathscr{A}(\mathscr{Z})}$; it is then a standard fact that $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} -$ gives a differential endofunctor of $\overline{\mathscr{A}(\mathscr{Z})}$. □

The differential functor $\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} -$ sends mapping cones to mapping cones, so the corresponding functor on homotopy categories sends distinguished triangles to distinguished triangles and thus induces an endomorphism $[\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} -]$ of $K_0^{\mathbb{F}_2}(\mathscr{A}(\mathscr{Z}))$.

Theorem 3.5 *Let $\mathscr{Z} = (\boldsymbol{Z}, \boldsymbol{a}, M)$ be an arc diagram and let (F, S_+, S_-, Λ) be the sutured surface represented by $\mathscr{Z}$. Let I be an interval component of S_+, or equivalently an interval component of $\boldsymbol{Z}$. Under the identification $K_0^{\mathbb{F}_2}(\mathscr{A}(\mathscr{Z})) \cong \wedge^* H_1(F, S_+; \mathbb{F}_2)$ from Corollary 2.15, the endomorphism $[\mathscr{E} \otimes_{\mathscr{A}(\mathscr{Z})} -]$ of $K_0^{\mathbb{F}_2}(\mathscr{A}(\mathscr{Z}))$ agrees with the endomorphism Φ_I of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ from the introduction. More specifically, the map $[\mathscr{E}(k) \otimes_{\mathscr{A}(\mathscr{Z},k)} -]$ from $K_0^{\mathbb{F}_2}(\mathscr{A}(\mathscr{Z}), k)$ to $K_0^{\mathbb{F}_2}(\mathscr{A}(\mathscr{Z}), k-1)$ agrees with Φ_I as a map from $\wedge^k H_1(F, S_+; \mathbb{F}_2)$ to $\wedge^{k-1} H_1(F, S_+; \mathbb{F}_2)$.*

Proof Let e be an object of $\mathcal{A}(\mathcal{Z})$ (viewed as a differential category); we have a corresponding basis element $[\mathrm{Hom}(e,-)]$ of $K_0^{\mathbb{F}_2}(\mathcal{A}(\mathcal{Z}))$. Applying $[\mathscr{E} \otimes_{\mathcal{A}(\mathcal{Z})} -]$ to $[\mathrm{Hom}(e,-)]$, we get

$$\sum_{y \in S,\, \rho(y)=e} [\mathrm{Hom}(\lambda(y),-)].$$

Viewing e as a purely horizontal strands picture and defining S as in the proof of Proposition 3.2, there is one element $y_s \in S$ with $\rho(y_s) = e$ for each strand s of e with endpoints in the interval I, and these are all the elements $y \in S$ with $\rho(y) = e$. For each such strand s (say with endpoints at $Q \in I$), the element y_s has a moving strand between $\left(\frac{1}{2}, P\right)$ and $(1, Q)$, and has the same horizontal strands as e except for s and its partner s' under the matching. Thus, $\lambda(y_s)$ is e with the strands s and s' removed.

It follows that $[\mathscr{E} \otimes_{\mathcal{A}(\mathcal{Z})} -]([\mathrm{Hom}(e,-)])$ is the sum of $[\mathrm{Hom}(e',-)]$ over all e' obtained from e by choosing one strand s in $[0,1] \times I$ and removing both s and its partner s'. In particular, for strands s in $[0,1] \times I$ such that s' is also in $[0,1] \times I$, the pair of strands (s, s') is removed from e twice, and since we are working over $\mathbb{F}_2$, removals of these strands contribute zero to $[\mathscr{E} \otimes_{\mathcal{A}(\mathcal{Z})} -]([\mathrm{Hom}(e,-)])$.

Now let ω be the element of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ corresponding to $[\mathrm{Hom}(e,-)]$ under the isomorphism of Corollary 2.15. Concretely, each pair of matched strands $\{s, s'\}$ of e gives a basis element of $H_1(F, S_+; \mathbb{F}_2)$, and ω is the wedge product of these elements over all such pairs $\{s, s'\}$. When we apply Φ_I to ω, we sum over all ways to remove a factor from this wedge product if the factor maps to $1 \in \mathbb{F}_2$ under the map ϕ_I from the introduction. Such factors are those corresponding to pairs of strands $\{s, s'\}$ of e in which one of $\{s, s'\}$, but not both, is in $[0,1] \times I$. It follows that $\Phi_I(\omega)$ corresponds to $[\mathscr{E} \otimes_{\mathcal{A}(\mathcal{Z})} -]([\mathrm{Hom}(e,-)])$, as desired. □

4 Gluing and TQFT

In this section, we prove (a slightly more general version of) Theorem 1.3 from the introduction. Let (F, S_+, S_-, Λ) be a sutured surface and suppose that $I_1 \neq I_2$ are interval components of S_+. Up to homeomorphism, there is a unique way to glue I_1 to I_2 and get an oriented surface $\overline{F}$. There are naturally defined subsets $\overline{S}_+$ and $\overline{S}_-$ of the boundary of $\overline{F}$, intersecting in a set of points $\overline{\Lambda}$ (which is Λ with the endpoints of I_1 and I_2 removed).

Lemma 4.1 *We have an isomorphism*

$$\wedge^* H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2) \cong (\wedge^* H_1(F, S_+; \mathbb{F}_2)) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \frac{\mathbb{F}_2[E]}{(E^2)},$$

where the action of $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ comes from the $\mathbb{F}_2[E]/(E^2)$ actions associated to I_1 and I_2, and the action of $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ on $\mathbb{F}_2[E]/(E^2)$ comes from multiplication. We can choose the isomorphism so that it intertwines the remaining actions of $\mathbb{F}_2[E]/(E^2)$ from S_+ intervals other than I_1 or I_2.

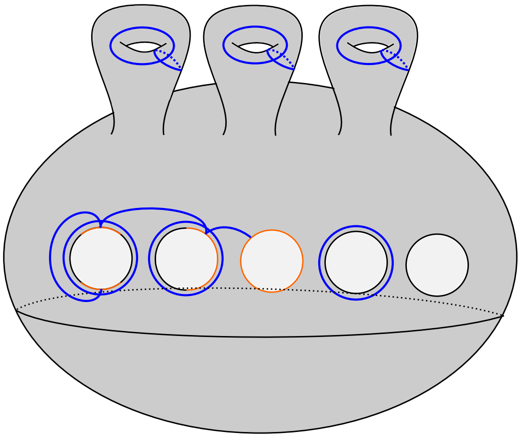

Figure 16: A standard model for a sutured surface, given by a sphere with some number of tori connect-summed on, as well as some number of disks removed and some even number of sutures on each boundary component. The S_+ boundary is drawn in orange and the S_- boundary is drawn in black. The set of blue arcs and circles gives a basis for $H_1(F, S_+; \mathbb{F}_2)$.

Proof Pick a homeomorphism between F and a finite disjoint union of standard sutured surfaces as shown in Figure 16 (spheres with some number of open disks removed and some even number of sutures on each boundary component, connect-summed with some number of tori). Figure 16 also indicates, with blue arcs and circles, a way to choose bases for $H_1(F, S_+; \mathbb{F}_2)$. One chooses

- for each torus that was connect-summed on, two circles giving a basis for the first homology of the torus;
- for all but one of the boundary components intersecting S_- nontrivially, a circle around the boundary component;
- a continuous map from a connected acyclic graph Γ_F to the surface F (an embedding on each edge of Γ_F) with one vertex on each component of S_+ — we will identify Γ_F with its image in F.

These circles, together with the edges of Γ_F, give a basis for $H_1(F, S_+; \mathbb{F}_2)$, so subsets of this set of arcs and circles give a basis for $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ consisting of wedge products of basis elements of $H_1(F, S_+; \mathbb{F}_2)$.

Now suppose I_1 and I_2 are intervals of S_+; we consider various cases.

Case 1 First, assume I_1 and I_2 live on distinct connected components of F. Choose Γ_F such that the vertices on I_1 and I_2 (say p_1 and p_2) are leaves of Γ_F, ie they have degree 1. When gluing F to get $\bar{F}$, we can ensure that p_1 and p_2 are glued to each other. If we let e_1 and e_2 denote the edges incident with p_1 and p_2, and modify Γ_F by removing p_1, p_2, e_1 and e_2 while adding the edge $e_1 \cup e_2$ as an embedded arc in $\bar{F}$, we get an acyclic graph $\Gamma_{\bar{F}}$ embedded in $\bar{F}$ with one vertex on each component of $\bar{S}_+$. See Figure 17.

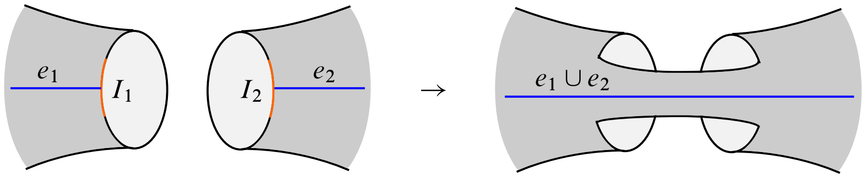

Figure 17: Left: arcs e_1 and e_2 in the surface F before gluing. Right: the arc $e_1 \cup e_2$ after gluing I_1 to I_2.

For an element ω of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$, obtained as a wedge product of basis elements of $H_1(F, S_+; \mathbb{F}_2)$, the I_1–action of $E \in \mathbb{F}_2[E]/(E^2)$ on ω is zero if e_1 is not a wedge factor of ω. Otherwise, write $\omega = e_1 \wedge \omega'$; we have $E \cdot \omega = \omega'$.

The I_2–action of $E \in \mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ is similar; informally, E acts by "removing e_2". It follows that $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ is a free module over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ with an $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$–basis given by elements $e_1 \wedge e_2 \wedge \omega'$ for all wedge products ω' in the other basis elements (not e_1 or e_2) of $H_1(F, S_+; \mathbb{F}_2)$. Thus, a basis for

$$(\wedge^* H_1(F, S_+; \mathbb{F}_2)) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \frac{\mathbb{F}_2[E]}{(E^2)}$$

is given by the set of elements $e_1 \wedge e_2 \wedge \omega'$, together with the elements $e_1 \wedge \omega' = e_2 \wedge \omega'$ (in each case ω' is a wedge product of basis elements of $H_1(F, S_+; \mathbb{F}_2)$ that are not e_1 or e_2). Meanwhile, a basis for $\wedge^* H_1(\bar{F}, \bar{S}_+; \mathbb{F}_2)$ is given by the set of elements $(e_1 \cup e_2) \wedge \omega'$ and ω' for the same set of ω'. We have a bijection between basis elements given by $e_1 \wedge e_2 \wedge \omega' \leftrightarrow (e_1 \cup e_2) \wedge \omega'$ and $(e_1 \wedge \omega' = e_2 \wedge \omega') \leftrightarrow \omega'$; this bijection is illustrated in Figure 18. Thus, we have an isomorphism of vector spaces as claimed in the statement of the theorem.

To see that this isomorphism intertwines the remaining actions of $\mathbb{F}_2[E]/(E^2)$ for S_+ intervals that are not I_1 or I_2, it suffices to consider the actions for the other two intervals (say I_1' and I_2') that intersect e_1 and e_2 respectively. We will consider the action for I_1'; the case of I_2' is similar. In the terminology used above, there are four types of basis elements of $\wedge^* H_1(F, S_+; \mathbb{F}_2)$: those of the forms $e_1 \wedge e_2 \wedge \omega'$,

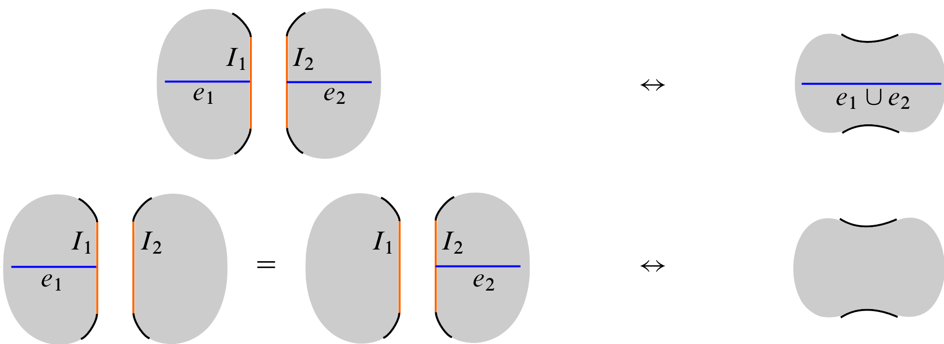

Figure 18: The bijection on basis elements in the first case of Lemma 4.1.

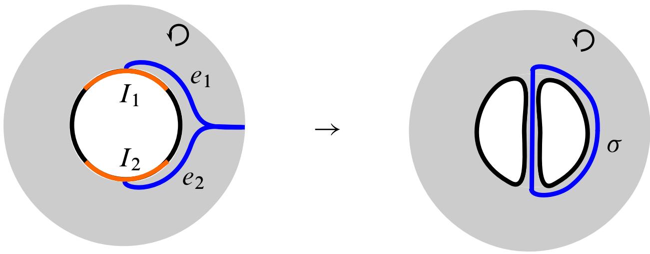

Figure 19: Left: local model near C for the arcs e_1 and e_2. Right: the circle σ after gluing I_1 to I_2. In both cases the curved arrow indicates the orientation on F; the induced boundary orientation on C is clockwise in this figure.

$e_1 \wedge \omega'$, $e_2 \wedge \omega'$, and ω'. The I_1'–action of $E \in \mathbb{F}_2[E]/(E^2)$ sums over all ways to remove one wedge factor corresponding to an arc with exactly one endpoint on I_1'; besides terms that modify ω', there is a "remove e_1" term that sends $e_1 \wedge e_2 \wedge \omega'$ to $e_2 \wedge \omega'$ and sends $e_1 \wedge \omega'$ to ω'. When we tensor over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ with the identity map on $\mathbb{F}_2[E]/(E^2)$, the "remove e_1" term of the action of E sends $e_1 \wedge e_2 \wedge \omega'$ to $e_2 \wedge \omega' = e_1 \wedge \omega'$ and sends $e_1 \wedge \omega' = e_2 \wedge \omega'$ to zero. On the other hand, as above there are two types of basis elements of $\wedge^* H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2)$: those of the form $(e_1 \cup e_2) \wedge \omega'$ and those of the form ω'. The I_1'–action of E has terms modifying ω' in the same way as above, and it also has "remove $e_1 \cup e_2$" terms sending $(e_1 \cup e_2) \wedge \omega'$ to ω' and sending ω' to zero. It follows that our choice of isomorphism intertwines the I_1' action of $\mathbb{F}_2[E]/(E^2)$.

Case 2 Next, assume I_1 and I_2 live on the same connected component F' of F; without loss of generality we can assume F is connected so that $F' = F$. We consider two further cases: either I_1 and I_2 live on the same connected component of ∂F, or they live on different connected components of ∂F.

Case 2-1 First assume I_1 and I_2 live on the same component C of ∂F, so that gluing I_1 to I_2 increases the number of boundary components of F by one while keeping the genus the same. When choosing a basis for $H_1(F, S_+; \mathbb{F}_2)$ as above, we can choose C for the unique not-fully S_+ boundary component of F that does not get a circle around it. We can also ensure that in the acyclic graph Γ_F, the vertices p_1 on I_1 and p_2 on I_2 are leaves of Γ_F.

Case 2-1a If there are any intervals of S_+ other than I_1 and I_2, or any fully S_+ circles, then p_1 and p_2 are incident with distinct edges $e_1 \neq e_2$ of Γ_F; we can furthermore choose Γ_F so that e_1 and e_2 share an endpoint q, and such that as embedded submanifolds of F, they look like the left side of Figure 19 in a small neighborhood of C and are identical outside this neighborhood (the picture should be appropriately modified if q lives on the circle C). As above, $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ is free over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ and has four types of basis elements, namely $e_1 \wedge e_2 \wedge \omega'$, $e_1 \wedge \omega'$, $e_2 \wedge \omega'$, and ω'. A basis for

$$(\wedge^* H_1(F, S_+; \mathbb{F}_2)) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \frac{\mathbb{F}_2[E]}{(E^2)}$$

is given by the elements $e_1 \wedge e_2 \wedge \omega'$ along with the elements $e_1 \wedge \omega' = e_2 \wedge \omega'$. Meanwhile, we can take $\Gamma_{\overline{F}}$ to be Γ_F with the edges e_1 and e_2 removed, and when choosing circles around boundary components

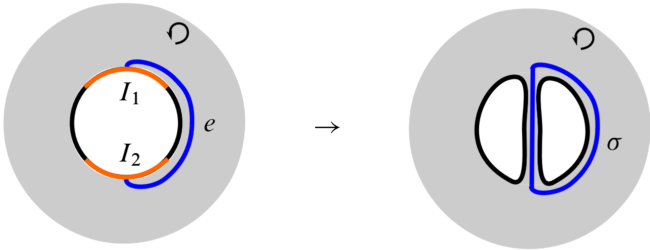

Figure 20: Left: local model near C for the arc e. Right: the circle σ after gluing I_1 to I_2.

to assemble a basis for $H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2)$, we can put a circle σ around the component of $\partial\overline{F}$ containing the segment of ∂F that goes from I_1 to I_2 when traversing the boundary in the oriented direction (see the right side of Figure 19). Then $\wedge^* H_1(\overline{F}, \overline{S}_+, \mathbb{F}_2)$ has basis elements of type $\sigma \wedge \omega'$ and ω'; we identify these with elements of type $e_1 \wedge e_2 \wedge \omega'$ and $e_1 \wedge \omega' = e_2 \wedge \omega'$ respectively. This bijection on basis elements gives us an isomorphism of vector spaces as in the statement of the theorem.

To see that this isomorphism intertwines the remaining actions of $\mathbb{F}_2[E]/(E^2)$ from S_+ intervals other than I_1 or I_2, it suffices to consider the interval I that contains the common endpoint q of e_1 and e_2. The I–action of $E \in \mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ has terms that modify ω' as well as "remove e_1" terms sending (for example) $e_1 \wedge e_2 \wedge \omega'$ to $e_2 \wedge \omega'$ and "remove e_2" terms sending (for example) $e_1 \wedge e_2 \wedge \omega'$ to $e_1 \wedge \omega'$. When we tensor over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ with the identity map on $\mathbb{F}_2[E]/(E^2)$, both the "remove e_1" and the "remove e_2" terms send $e_1 \wedge e_2 \wedge \omega'$ to $e_1 \wedge \omega' = e_2 \wedge \omega'$, and they send $e_1 \wedge \omega' = e_2 \wedge \omega'$ to zero. Since the "remove e_1" and "remove e_2" terms act in the same way, their contribution to the overall action of E is zero, and only the "modify ω'" terms remain. On the other hand, the I–action of E on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ only modifies ω' in terms of type $\sigma \wedge \omega'$ or ω', since σ is closed. It follows that our choice of isomorphism intertwines the I–action of $\mathbb{F}_2[E]/(E^2)$.

Case 2-1b Now assume that I_1 and I_2 are the only intervals of S_+ (but they still live on the same component C of ∂F) and that there are no fully S_+ circles; it follows that Γ_F has a unique edge e and it connects p_1 to p_2. We can assume e lives in a small neighborhood of C, and that in this neighborhood it looks like the left side of Figure 20. The I_1–action and I_2–action of $E \in \mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ agree; they both send $e \wedge \omega'$ to ω' and send ω' to zero. Thus

$$(\wedge^* H_1(F, S_+; \mathbb{F}_2)) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \frac{\mathbb{F}_2[E]}{(E^2)}$$

is canonically isomorphic to $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ where no tensor operation is performed. Meanwhile, we can take $\Gamma_{\overline{F}}$ to be empty, but in assembling a basis for $H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2)$, we again put a circle σ around the component of $\partial\overline{F}$ containing the segment of ∂F that goes from I_1 to I_2 when traversing the boundary in the oriented direction (see the right side of Figure 20). The correspondences $e \wedge \omega' \leftrightarrow \sigma \wedge \omega'$ and $\omega' \leftrightarrow \omega'$ give an isomorphism of vector spaces as in the statement of the theorem. There are no remaining S_+ intervals, so we do not need to check that this isomorphism intertwines any actions.

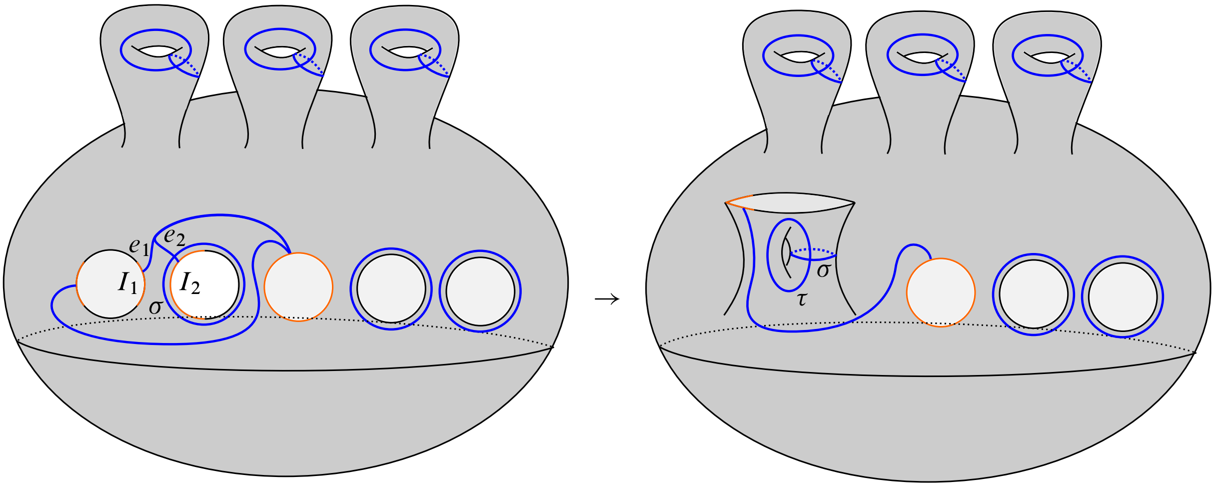

Figure 21: Left: F before gluing intervals I_1 and I_2 on the same component of F but different components of ∂F. Right: the glued surface $\overline{F}$.

Case 2-2 Next, assume that I_1 and I_2 live on different components C_1 and C_2 of ∂F; for visual simplicity, assume that in the model for F shown in Figure 16, C_1 and C_2 are next to each other. Gluing I_1 to I_2 decreases the number of boundary components of F by one and increases the genus of F by one.

Case 2-2a Also assume that there is either at least one S_+ interval that is not I_1 or I_2, or that there is at least one fully S_+ circle. As above, p_1 and p_2 are incident with distinct edges $e_1 \neq e_2$ of Γ_F, and we can choose Γ_F so that e_1 and e_2 share a vertex q and only diverge near C_1 and C_2. We also assume that C_1 is the unique not-fully S_+ boundary circle of F that does not get a circle around it as a basis element of $H_1(F, S_+; \mathbb{F}_2)$. Let σ be the circle around C_2; see the left side of Figure 21.

Basis elements for $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ can be of the form $e_1 \wedge e_2 \wedge \omega'$, $e_1 \wedge \omega'$, $e_2 \wedge \omega'$, or ω'; when we tensor with $\mathbb{F}_2[E]/(E^2)$ over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$, we have a basis whose elements are of type $e_1 \wedge e_2 \wedge \omega'$ or $e_1 \wedge \omega' = e_2 \wedge \omega'$. Meanwhile, we choose a basis for $H_1(\overline{F}, \overline{S}_+, \mathbb{F}_2)$ by choosing a homeomorphism with the standard surface shown on the right side of Figure 21. The graph $\Gamma_{\overline{F}}$ can be understood as Γ_F with e_1 and e_2 removed; we also have basis elements σ and τ of $H_1(F, S_+; \mathbb{F}_2)$ where $\sigma \subset \overline{F}$ comes from $\sigma \subset F$ and τ comes from e_1 and e_2. Basis elements of $\wedge^* H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2)$ are of the form $\tau \wedge \omega'$ or ω', where ω' is a wedge product of basis elements for $H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2)$ that are not τ. The correspondence $e_1 \wedge e_2 \wedge \omega' \leftrightarrow \tau \wedge \omega'$ and $(e_1 \wedge \omega' = e_2 \wedge \omega') \leftrightarrow \omega'$ gives an isomorphism of vector spaces as in the statement of the theorem. The proof that this isomorphism intertwines the remaining actions of $\mathbb{F}_2[E]/(E^2)$ proceeds as above.

Case 2-2b Finally, assume that I_1 and I_2 are the only S_+ intervals and that there are no fully S_+ circles (while I_1 and I_2 still live on different components of ∂F). Letting e be the arc of Γ_F connecting $p_1 \in I_1$ to $p_2 \in I_2$, basis elements for $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ are of the form $e \wedge \omega'$ or ω'. Meanwhile, defining τ as in Figure 21, basis elements for $\wedge^* H_1(\overline{F}, \overline{S}_+; \mathbb{F}_2)$ are of the form $\tau \wedge \omega'$ or ω'. The correspondence

$e \wedge \omega' \leftrightarrow \tau \wedge \omega'$ and $\omega' \wedge \omega'$ gives an isomorphism of vector spaces as in the statement of the theorem, and there are no remaining actions for this isomorphism to intertwine. □

Lemma 4.1 implies the following theorem.

Theorem 4.2 *Let (F, S_+, S_-, Λ) and $(F', S'_+, S'_-, \Lambda')$ be two sutured surfaces. For some $m \geq 0$, choose distinct intervals $I_1, \dots, I_m$ of S_+ and distinct intervals $I'_1, \dots, I'_m$ of S'_+. Use $I_1, \dots, I_m$ to define an action of $(\mathbb{F}_2[E]/(E^2))^{\otimes m}$ on $\wedge^* H_1(F, S_+; \mathbb{F}_2)$, and similarly for F'. Let $(\bar{F}, \bar{S}_+, \bar{S}_-, \bar{\Lambda})$ be the sutured surface obtained by gluing I_j to I'_j for $1 \leq j \leq m$ (in such a way that the result is oriented). Then we have an isomorphism*

$$\wedge^* H_1(\bar{F}, \bar{S}_+; \mathbb{F}_2) \cong \wedge^* H_1(F, S_+; \mathbb{F}_2) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes m}} \wedge^* H_1(F', S'_+; \mathbb{F}_2)$$

that intertwines the remaining actions of $\mathbb{F}_2[E]/(E^2)$ for intervals of S_+ and S'_+ that are not included in $\{I_1, \dots, I_m\}$ or $\{I'_1, \dots, I'_m\}$.

Proof We can write $\wedge^* H_1(F, S_+; \mathbb{F}_2) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes m}} \wedge^* H_1(F', S'_+; \mathbb{F}_2)$ as

$$\big((\wedge^* H_1(F \sqcup F', S_+ \sqcup S'_+; \mathbb{F}_2)) \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \mathbb{F}_2[E]/(E^2)\big) \cdots \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \mathbb{F}_2[E]/(E^2),$$

where there are m successive tensor products by $\mathbb{F}_2[E]/(E^2)$ over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ (one for each pair (I_j, I'_j)). The result now follows from Lemma 4.1. □

Corollary 4.3 *There is a functor from the full subcategory of the $(1+1)$–dimensional oriented open-closed cobordism category on objects with no closed circles (the "open sector" of the open-closed cobordism category) to $\mathbf{Alg}_{\mathbb{F}_2}$ sending an object with m intervals to the algebra $(\mathbb{F}_2[E]/(E^2))^{\otimes m}$ and sending a morphism (viewed as a sutured surface (F, S_+, S_-, Λ)) to $\wedge^* H_1(F, S_+; \mathbb{F}_2)$ (viewed as a bimodule over tensor products of $\mathbb{F}_2[E]/(E^2)$ for the input and output intervals of the morphism).*

5 The tensor product case

Figure 22 shows the open pair of pants surface P with a sutured structure (P, S_+, S_-, Λ). Let e_1 and e_2 be the arcs shown in the figure and let I_1, I_2 and I_3 be the S_+ intervals shown in the figure. Since $\{e_1, e_2\}$ is a basis for $H_1(P, S_+; \mathbb{F}_2)$, we have a basis $\{1, e_1, e_2, e_1 \wedge e_2\}$ for $\wedge^* H_1(P, S_+; \mathbb{F}_2)$. The three actions of $\mathbb{F}_2[E]/(E^2)$ on $\wedge^* H_1(P, S_+; \mathbb{F}_2)$ can be described as follows:

- For the I_1–action, E sends $1 \mapsto 0$, $e_1 \mapsto 1$, $e_2 \mapsto 0$, and $e_1 \wedge e_2 \mapsto e_2$.
- For the I_2–action, E sends $1 \mapsto 0$, $e_1 \mapsto 0$, $e_2 \mapsto 1$, and $e_1 \wedge e_2 \mapsto e_1$.
- For the I_3–action, E sends $1 \mapsto 0$, $e_1 \mapsto 1$, $e_2 \mapsto 1$, and $e_1 \wedge e_2 \mapsto e_1 + e_2$.

Using the I_1 and I_2 actions to define an action of $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ on $\wedge^* H_1(P, S_+; \mathbb{F}_2)$, we see that $\wedge^* H_1(P, S_+; \mathbb{F}_2)$ is a free module of rank 1 over $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$ with an $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$–basis given

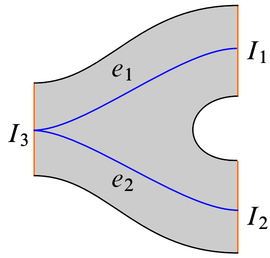

Figure 22: The open pair-of-pants surface P with sutured structure and basis $\{e_1, e_2\}$ for $H_1(P, S_+; \mathbb{F}_2)$.

by $\{e_1 \wedge e_2\}$. The I_3–action of $\mathbb{F}_2[E]/(E^2)$ is then given by applying the coproduct $\Delta(E) = E \otimes 1 + 1 \otimes E$, followed by multiplication in $(\mathbb{F}_2[E]/(E^2))^{\otimes 2}$.

Now, if we have sutured surfaces $(F', S'_+, S'_-, \Lambda')$ and $(F'', S''_+, S''_-, \Lambda'')$ with chosen intervals I' and I'' in S'_+ and S''_+ respectively, we can glue $F' \sqcup F''$ to P by gluing I' to I_1 and I'' to I_2. Applying Theorem 4.2 with $F_1 := F' \sqcup F''$ and $F_2 := P$, and letting $(\bar{F}, \bar{S}_+, \bar{S}_-, \bar{\Lambda})$ denote the glued surface,

$$\begin{aligned}\wedge^* H_1(\bar{F}, \bar{S}_+; \mathbb{F}_2) &\cong (\mathbb{F}_2[E]/(E^2))^{\otimes 2} \otimes_{(\mathbb{F}_2[E]/(E^2))^{\otimes 2}} \wedge^* H_1(F' \sqcup F'', S'_+ \sqcup S''_+; \mathbb{F}_2)\\ &\cong \wedge^* H_1(F' \sqcup F'', S'_+ \sqcup S''_+; \mathbb{F}_2)\\ &\cong \wedge^* H_1(F', S'_+; \mathbb{F}_2) \otimes \wedge^* H_1(F'', S''_+; \mathbb{F}_2)\end{aligned}$$

with I_3–action of E given by taking $\Delta(E) = E \otimes 1 + 1 \otimes E$ and then acting on the tensor product $\wedge^* H_1(F', S'_+; \mathbb{F}_2) \otimes \wedge^* H_1(F'', S''_+; \mathbb{F}_2)$. Corollary 1.4 follows from this computation.

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