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Models of G–spectra as presheaves of spectra

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Let G be a finite group. We give Quillen equivalent models for the category of G–spectra as categories of spectrally enriched functors from explicitly described domain categories to nonequivariant spectra. Our preferred model is based on equivariant infinite loop space theory applied to elementary categorical data. It recasts equivariant stable homotopy theory in terms of point–set-level categories of G–spans and nonequivariant spectra. We also give a more topologically grounded model based on equivariant Atiyah duality.

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Introduction

The equivariant stable homotopy category is of fundamental importance in algebraic topology. It is the natural home in which to study equivariant stable homotopy theory, a subject that has powerful and unexpected nonequivariant applications and is also of great intrinsic interest. The foundations were well established by the mid-1980s, and by then the importance of working with equivariant spectra had already become abundantly clear, especially with Carlsson's proof [1984] of the Segal conjecture. The following decade saw much further progress; Mackey functor and $RO(G)$–graded cohomology theories came of age, the Tate square and norm maps were introduced and given their first applications [Greenlees and May 1995b; 1997], and THH, TC and their applications to algebraic K–theory had made their appearance

[Bökstedt et al. 1993]. Summary accounts of where the subject stood in the mid-1990s are given in [Carlsson 1992; Greenlees and May 1995a; May 1996]. While there was continued work in the following decade, the subject really took hold in the mainstream of algebraic topology with its unexpected role in the 2009 solution of the Kervaire invariant problem by Hill, Hopkins and Ravenel [Hill et al. 2016]. For example, on a foundational level, understanding norms as maps of equivariant spectra plays a key role.

The first draft of this paper appeared in 2011, and the subject has truly blossomed in the decade since. Formally, just as the category of G–spaces is Quillen equivalent to the presheaf category of contravariant functors from the orbit category of G to spaces, the category of G–spectra is Quillen equivalent to the presheaf category of spectrally enriched contravariant functors from its full subcategory of suspension spectra of orbits to spectra. We shall say more about that shortly. The purpose of this paper is to replace the target presheaf category by one that is Quillen equivalent and yet is accessible to concrete constructions on the level of related presheaf categories of spaces and categories.

Setting up the equivariant stable homotopy category with its attendant model structures takes a fair amount of work. The first version was due to Lewis and May [Lewis et al. 1986b], and more modern versions that we shall start from are given in [Mandell and May 2002; Hill et al. 2016]. A result of Schwede and Shipley [2003] (reworked in [Guillou and May 2020] to give the starting point of this paper) asserts that any stable model category $\mathscr{M}$ is equivalent to a category $\mathbf{Pre}(\mathscr{D},\mathscr{S})$ of spectrally enriched presheaves with values in a chosen category $\mathscr{S}$ of spectra. However, the domain $\mathscr{S}$–category $\mathscr{D}$ is a *full* $\mathscr{S}$–subcategory of $\mathscr{M}$ and typically is as inexplicit and mysterious as $\mathscr{M}$ itself. From the point of view of applications and calculations, this is therefore only a starting point. One wants a more concrete understanding of the category $\mathscr{D}$. We shall give explicit equivalents to the domain category $\mathscr{D}$ in the case when $\mathscr{M} = G\mathscr{S}$ is the category of G–spectra for a finite group G, and we fix a finite group G throughout.

We shall define an $\mathscr{S}$–category (or spectral category) $G\mathscr{A}$ by applying a suitable infinite loop space machine to simply defined categories of finite G–sets. The spectral category $G\mathscr{A}$ is a spectrally enriched version of the Burnside category of G. We shall prove the following result:

Theorem 0.1 (main theorem) *There is a zigzag of Quillen equivalences*

$$G\mathscr{S} \simeq \mathbf{Pre}(G\mathscr{A},\mathscr{S})$$

relating the category of G–spectra to the category of spectrally enriched contravariant functors $G\mathscr{A} \to \mathscr{S}$.

Such functors are often called presheaves. We reemphasize the simplicity of our spectral category $G\mathscr{A}$: no prior knowledge of G–spectra is required to define it.

We give a precise description of the relevant categorical input and restate the main theorem more precisely in Section 1. The central point of the proof is to use equivariant infinite loop space theory to construct the spectral category $G\mathscr{A}$ from elementary categories of finite G–sets. We prove our main theorem in Section 2, using the equivariant Barratt–Priddy–Quillen (BPQ) theorem to compare $G\mathscr{A}$ to the spectral category $G\mathscr{D}_{\mathrm{All}}$ given by the suspension G–spectra $\Sigma^\infty_G(A_+)$ of based finite G–sets A_+, which is a standard choice for application of the theorem of Schwede and Shipley to $G\mathscr{S}$. The classical Burnside category of isomorphism classes of spans of finite G–sets leads to a calculation of the homotopy category $\mathrm{Ho}\, G\mathscr{D}_{\mathrm{All}}$ (see Theorem 1.12 below), and $G\mathscr{A}$ starts from the bicategory of such spans, in which isomorphisms of spans give the 2–cells.

Intuitively, (algebraic) Mackey functors can be viewed as functors from $\mathrm{Ho}\, G\mathscr{D}_{\mathrm{All}}$ to abelian groups, and the result of Schwede and Shipley says that G–spectra can be viewed as functors from $G\mathscr{D}_{\mathrm{All}}$ to spectra. We are lifting the standard purely algebraic understanding of Mackey functors to obtain an analogous algebraic understanding of G–spectra as functors from $G\mathscr{A}$ to spectra. Thus, the slogan is that G–spectra are spectral Mackey functors.

It is crucial to our work that the G–spectra $\Sigma^\infty_G(A_+)$ are self-dual. Our original proof took this as a special case of equivariant Atiyah duality (Section 4.2), thinking of A as a trivial example of a smooth closed G–manifold. We later found a direct categorical proof (Section 2.3) of this duality based on equivariant infinite loop space theory and the equivariant BPQ theorem. This allows us to give an illuminating new proof of the required self-duality as we go along. We give presheaf versions of a few standard constructions on G–spectra in Section 3. Switching gears, we give an alternative presheaf model for the category of G–spectra in terms of classical Atiyah duality in Section 4. Appendix A provides some background on the two model categories of G–spectra used here, equivariant orthogonal spectra and equivariant S–modules, and describes and compares the specialization of [Guillou and May 2020] to those categories that provides the starting point for our work.

We take what we need from equivariant infinite loop space theory as a black box in this paper. The additive and multiplicative space-level theories are worked out in [May et al. 2017] and [Guillou et al. 2019], respectively. The generalization from space-level to category-level input is based on general (and not necessarily equivariant) categorical coherence theory that is worked out in [Guillou et al. 2023]. What is needed for this paper is a small part of the full story there.

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In the interim, we teamed with Osorno and Mona Merling to fully work out the relevant infinite loop space theory, which turned out to be both surprisingly demanding and unexpectedly interesting. Also in the interim, Bohmann and Osorno [2015] introduced categorical Mackey functors and used these, together with our main result, to produce a functorial construction of equivariant Eilenberg–Mac Lane spectra for

Mackey functors. The prospect of applications like theirs was a major motivation for our variant of the Schwede and Shipley model for the homotopy category of G–spectra. A small error[1] in [Bohmann and Osorno 2015] is corrected in the short Appendix B, of this paper. Further applications to the concrete construction of genuine G–spectra are in development in their work and in work of Cary Malkiewich and Merling [2019; 2022]. During the delay, Jonathan Rubin combed through our draft and caught a great many errors of detail and infelicities. Needless to say, we are responsible for all that remain.

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Comparison with alternative approaches We also note that, since this article first appeared online in 2011, several alternative approaches have been given by other authors. First among these was the work of Barwick [2017]. A notable difference is that our spectral Burnside category $G\mathscr{A}$ is a group completion of Barwick's effective Burnside category. A second difference is that Barwick is working in the ∞–categorical setting, so that questions of strictness, such as those necessitating our Appendix B, do not arise. Moreover, Barwick's work provides a conceptual generalization that applies to handle the case of profinite groups, as well as other applications. Later, streamlined alternative approaches were given in [Nardin 2016; Clausen et al. 2020, Appendix A]. The version described in [Clausen et al. 2020] has the advantage of providing a monoidal equivalence (see also [Barwick et al. 2020, Section 11]). See Remark 3.9 for further discussion.

1 The bicategory $G\mathscr{E}$ and $\mathscr{S}$–category $G\mathscr{A}$

In this paper, $\mathscr{S}$ denotes the category of (nonequivariant) orthogonal spectra, and $G\mathscr{S}$ denotes the category of orthogonal G–spectra. For most of the paper, we index $G\mathscr{S}$ on a complete universe, but in Appendix A we allow a more general universe. See Appendix A for some discussion of the comparison between models of G–spectra. We first define the $\mathscr{S}$–category $G\mathscr{A}$ (Definition 1.13) and restate our main theorem. Conceptually $G\mathscr{A}$ can be viewed as obtained by applying a nonequivariant infinite loop space machine $\mathbb{K}$ to a category $G\mathscr{E}$ "enriched in permutative categories".[2] The term in quotes can be made categorically precise [Guillou 2010; Hyland and Power 2002; Schmitt 2007], but we shall use it just as an informal slogan since no real categorical background is necessary to our work here: we shall give direct elementary definitions of the examples we use, and they do satisfy the axioms specified in the cited sources. We then define (Definition 1.29) a G–category[3] $\mathscr{E}_G$ "enriched in permutative G–categories", from which $G\mathscr{E}$ is obtained by passage to G–fixed subcategories. Section 1.5 contains a discussion of duality that will be needed in Section 2 for the proof of our main theorem.

[1]We are grateful to Angélica Osorno for helping us discover and fix this error.

[2]A permutative category is a symmetric strict monoidal category.

[3]In general, we understand a G–category to be a category internal and not just enriched in G–sets, meaning that G can act on both objects and morphisms.

1.1 The bicategory $G\mathscr{E}$ of G–spans

In any category $\mathscr{C}$ with pullbacks, the bicategory of spans in $\mathscr{C}$ has 0–cells the objects of $\mathscr{C}$. The 1–cells from A to B are zigzags $B \leftarrow D \to A$ of morphisms in $\mathscr{C}$, and 2–cells between two such are diagrams

(1.1)
$$
\begin{tikzcd}
 & D \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\
B & & A \\
 & E \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Composites of 1–cells are given by (chosen) pullbacks

(1.2)
$$
\begin{tikzcd}
 & & F \arrow[dl] \arrow[dr] & & \\
 & E \arrow[dl] \arrow[dr] & & D \arrow[dl] \arrow[dr] & \\
C & & B & & A
\end{tikzcd}
$$

The identity 1–cells are the diagrams $A \xleftarrow{=} A \xrightarrow{=} A$. The associativity and unit constraints are determined by the universal property of pullbacks. Observe that the 1–cells $A \to B$ can just as well be viewed as objects over $B \times A$. Viewed this way, the identity 1–cells are given by the diagonal maps $\Delta\colon A \to A \times A$, and the composition can be displayed in the diagram

(1.3)
$$
\begin{tikzcd}
E \times D \arrow[d] & & F \arrow[ll] \arrow[d] \arrow[dr] & \\
C \times B \times B \times A & & C \times B \times A \arrow[ll, "\mathrm{id}\times\Delta\times\mathrm{id}"'] \arrow[r, "\pi"'] & C \times A
\end{tikzcd}
$$

where the square is a pullback and π is the projection. That is, composition is obtained from the obvious composition of maps to products by pulling back contravariantly along $\mathrm{id} \times \Delta \times \mathrm{id}$ and then pushing forward covariantly along π. See [Ponto and Shulman 2012, Theorem 5.2] for an illuminating discussion of bicategories of spans from this point of view.

Our starting point is the bicategory of spans of (unbased) finite G–sets. Here the disjoint union of G–sets over $B \times A$ gives us a symmetric monoidal structure on the category of 1–cells and 2–cells $A \to B$ for each pair (A, B). We can think of the bicategory of spans as a category "enriched in the category of symmetric monoidal categories". Again, the notion in quotes does not make obvious mathematical sense since there is no obvious monoidal structure on the category of symmetric monoidal categories, but category theory due to [Guillou 2010] (see also [Hyland and Power 2002; Schmitt 2007]) explains what these objects are and how to rigidify them to categories enriched in permutative categories.

We repeat that we have no need to go into such categorical detail. Rather than apply such category theory, we give a direct elementary construction of a strict structure that is equivalent to the intuitive notion of the category "enriched in symmetric monoidal categories" of spans of finite G–sets. We first define a bipermutative category $G\mathscr{E}(1)$ that is equivalent to the symmetric bimonoidal groupoid of finite G–sets.

Definition 1.4 Any finite G–set is isomorphic to one of the form $A = \underline{n}^{\alpha}$, where $\underline{n} = \{1, \dots, n\}$, α is a homomorphism $G \to \Sigma_n$, and G acts on $\underline{n}$ by $g \cdot i = \alpha(g)(i)$ for $1 \le i \le n$. *We understand finite G–sets to be of this restricted form from now on.* A G–map $f : \underline{m}^{\alpha} \to \underline{n}^{\beta}$ is a function $f : \underline{m} \to \underline{n}$ such that $f \circ \alpha(g) = \beta(g) \circ f$ for $g \in G$. The morphisms of $G\mathscr{E}(1)$ are the isomorphisms $\underline{n}^{\alpha} \to \underline{n}^{\beta}$ of G–sets.

The disjoint union $D \amalg E$ of finite G–sets $D = \underline{s}^{\sigma}$ and $E = \underline{t}^{\tau}$ is $\underline{s+t}^{\sigma \oplus \tau}$, with $\sigma \oplus \tau$ being the evident block sum $G \to \Sigma_{s+t}$. With the evident commutativity isomorphism, this gives the permutative groupoid[4] $G\mathscr{E}(1)$ of finite G–sets; the empty finite G–set is the unit for $\amalg$. To define the cartesian product, for each s and t, let $\lambda_{s,t} : \underline{st} \to \underline{s} \times \underline{t}$ denote the lexicographic ordering. Then $D \times E$ is $\underline{st}^{\sigma \otimes \tau}$, where $\sigma \otimes \tau$ is the permutation

$$\underline{st} \xrightarrow{\lambda_{s,t}} \underline{s} \times \underline{t} \xrightarrow{\sigma \times \tau} \underline{s} \times \underline{t} \xrightarrow{\lambda_{s,t}^{-1}} \underline{st}$$

as in [Guillou et al. 2023, (3.6)]. There is again an evident commutativity isomorphism, and $\amalg$ and $\times$ give $G\mathscr{E}(1)$ the structure of a bipermutative category in the sense of [May 1977]; the multiplicative unit is the trivial G–set $1 = (\underline{1}, \varepsilon)$, where $\varepsilon(g) = 1$ for $g \in G$.

As we will need it later, we also introduce the reordering permutation $\tau_{s,t} \in \Sigma_{st}$, defined as the composition

$$\underline{st} \xrightarrow{\lambda_{s,t}} \underline{s} \times \underline{t} \xrightarrow{\cong} \underline{t} \times \underline{s} \xrightarrow{\lambda_{t,s}^{-1}} \underline{ts} = \underline{st}$$

as in [Guillou et al. 2023, Definition 3.8].

We may view $G\mathscr{E}(1)$ as the groupoid of finite G–sets over the one-point G–set 1, and we generalize the definition as follows.

Definition 1.5 For a finite G–set A, we define a permutative groupoid $G\mathscr{E}(A)$ of finite G–sets over A. The objects of $G\mathscr{E}(A)$ are the G–maps $p\colon D \to A$. The morphisms $p \to q$, with $q\colon E \to A$, are the G–isomorphisms $f\colon D \to E$ such that $q \circ f = p$. Disjoint union of G–sets over A gives $G\mathscr{E}(A)$ the structure of a permutative category; its unit is the empty set over A. When $A = 1$, $G\mathscr{E}(A)$ is the ("additive") permutative category of the previous definition.

Remark 1.6 There is also a product $\times\colon G\mathscr{E}(A) \times G\mathscr{E}(B) \to G\mathscr{E}(A \times B)$. It takes (D, E) to $D \times E$, where D and E are finite G–sets over A and B, respectively. This product is also strictly associative and unital, with unit the unit of $G\mathscr{E}(1)$, and it has an evident commutativity isomorphism. Restriction to the object 1 gives the "multiplicative" permutative category of Definition 1.4. This product distributes over $\amalg$ and *almost* makes the enriched category $G\mathscr{E}$ of the next definition into a "category enriched in permutative categories", in the sense defined in [Guillou 2010]. The "almost" refers to the fact that the category we define does not have a strict unit, a problem that was encountered in [Bohmann and Osorno 2015] and is fixed in Appendix B below.

[4] Though the terminology "permutative category" is more prevalent than "permutative groupoid", we find it useful to remind the reader that we are only considering isomorphisms.

Definition 1.7 We define a bicategory $G\mathscr{E}$ with a permutative hom groupoid for each pair of objects as follows. The 0–cells of $G\mathscr{E}$ are the finite G–sets, which may be thought of as the categories $G\mathscr{E}(A)$. The permutative groupoid $G\mathscr{E}(A,B)$ of 1–cells and 2–cells $A \to B$ is $G\mathscr{E}(B \times A)$, as defined in Definition 1.5. The 1–cells are thought of as spans and the 2–cells as isomorphisms of spans. The composition

$$\circ\colon G\mathscr{E}(B,C) \times G\mathscr{E}(A,B) \to G\mathscr{E}(A,C)$$

is defined via pullbacks, as in the diagram (1.2). Precisely, following [Bohmann and Osorno 2015, Definition 7.2], we choose the pullback F in (1.2) to be the G–subset of $E \times D$, ordered lexicographically, consisting of the elements (e,d) such that d and e map to the same element of B. The diagonal map $\Delta_A\colon A \to A \times A$ serves as a unit 1–cell, and it is helpful to reinterpret composition in terms of the diagram (1.3).

Remark 1.8 This bicategory is almost a 2–category. The composition of spans is strictly associative, but if $|A| \geq 2$ then $\Delta_A\colon A \to A \times A$ acts as a strict unit only on the right and so should be called a pseudounit 1–cell. The point is that, with our chosen model for the pullback, the left map in the span composition

$$\begin{tikzcd}
& & \Delta_B \circ E \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
& B \arrow[dl, equal] \arrow[dr, equal] & & E \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B & & B & & A
\end{tikzcd}$$

must be order-preserving. Therefore, if f is not order-preserving, then $\Delta_B \circ E \neq E$. However, in view of the evident commutative diagram

$$\begin{tikzcd}
& \Delta_B \circ E \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "g \circ p_2"] & \\
B & E \arrow[l, "f"] \arrow[r, "g"'] & A
\end{tikzcd}$$

the function p_2 specifies a reordering isomorphism of spans

$$\Delta_B \circ E \xrightarrow{\ell_{B,E}} E. \tag{1.9}$$

In Appendix B, we show how to whisker the pseudounit 1–cells to obtain an equivalent construction $G\mathscr{E}'$ that still has a strictly associative composition but now has strict two-sided unit 1–cells. The construction is closely analogous to the usual whiskering of a degenerate basepoint in a space to obtain a nondegenerate basepoint.

Remark 1.10 We are suppressing some categorical details that are irrelevant to our work. The composition distributes over coproducts, and it should be defined on a "tensor product" rather than a cartesian product of permutative categories. Such a tensor product does in fact exist, in the sense that the 2–category of permutative categories has a pseudomonoidal structure [Hyland and Power 2002, Section 2.3]; however, we will not use this. Rather, we will use that composition is a pairing that gives rise to a pairing defined on the smash product of the spectra constructed from $G\mathscr{E}(B,C)$ and $G\mathscr{E}(A,B)$. This passage from pairings of permutative categories to pairings of spectra has a checkered history even nonequivariantly,[5] and it is here that a mistake occurred in earlier versions of this paper. As explained in [Guillou et al. 2023], categorical strictification and the full development of multiplicative equivariant infinite loop space theory resolve the relevant issues.

Before beginning work, we recall an old result that motivated this paper. The category $[G\mathscr{E}]$ of isomorphism classes of G–spans is obtained from the bicategory $G\mathscr{E}$ of G–spans by identifying spans from A to B if there is an isomorphism between them. Composition is again by pullbacks. We add spans from A to B by taking disjoint unions, which gives the morphism set $[G\mathscr{E}](A,B)$ the structure of an abelian monoid. We apply the Grothendieck construction to obtain an abelian group of morphisms $A \to B$. This gives an additive category $\mathscr{A}b[G\mathscr{E}]$.

Definition 1.11 Define $G\mathscr{D}_{\mathrm{All}}$ to be the full subcategory of $G\mathscr{S}$ whose objects are fibrant replacements of the G–spectra $\Sigma^\infty_G(A_+)$ in the stable model structure [Mandell and May 2002], where A runs over the finite G–sets, and let $\mathrm{Ho}\,G\mathscr{D}_{\mathrm{All}} \subset \mathrm{Ho}\,G\mathscr{S}$ denote its homotopy category.

Theorem 1.12 [Lewis et al. 1986a, Proposition 9.6][6] *The categories* $\mathrm{Ho}\,G\mathscr{D}_{\mathrm{All}}$ *and* $\mathscr{A}b[G\mathscr{E}]$ *are isomorphic.*

1.2 The precise statement of the main theorem

Infinite loop space theory associates a spectrum $\mathbb{K}\mathscr{A}$ to a permutative category $\mathscr{A}$. There are several machines available and all are equivalent [May 1978]. Since it is especially convenient for the equivariant generalization, we require $\mathbb{K}$ to take values in the category $\mathscr{S}$ of orthogonal spectra [Mandell et al. 2001], but symmetric spectra would also work. Slightly modifying the axiomatization of [May 1978], we require $\mathbb{K}$ to take values in positive[7] Ω–spectra and we require a natural map $\eta\colon B\mathscr{A} \to (\mathbb{K}\mathscr{A})_0$ whose composition with $(\mathbb{K}\mathscr{A})_0 \to \Omega(\mathbb{K}\mathscr{A})_1$ gives a group completion.

Since $\mathscr{S}$ is closed symmetric monoidal under the smash product, it makes sense to enrich categories in $\mathscr{S}$. Our preferred version of spectral categories is categories enriched in $\mathscr{S}$, abbreviated *$\mathscr{S}$–categories*. Model-theoretically, $\mathscr{S}$ is a particularly nice enriching category since its unit S is cofibrant in the stable model structure and $\mathscr{S}$ satisfies the monoid axiom [Mandell et al. 2001, Proposition 12.5].

When a spectral category $\mathscr{D}$ is used as the domain category of a presheaf category, the objects and maps of the underlying category are unimportant. The important data are the morphism spectra $\mathscr{D}(A,B)$, the unit maps $S \to \mathscr{D}(A,A)$ and the composition maps

$$\mathscr{D}(B,C) \wedge \mathscr{D}(A,B) \to \mathscr{D}(A,C).$$

[5] This starts from [May 1980], which is modernized, corrected and generalized in [Guillou et al. 2023], where pairings are subsumed as 2–ary morphisms in multicategories.

[6] All G–spectra in [Lewis et al. 1986b] are fibrant, but we are using orthogonal G–spectra here. The homotopy categories are equivalent.

[7] This means that $E_0 \to \Omega E_1$ need not be an equivalence.

The presheaves $\mathscr{D}^{\mathrm{op}} \to \mathscr{S}$ can be thought of as (right) $\mathscr{D}$–modules.

Recall that an object a in a permutative category $\mathscr{A}$ determines a point of $B\mathscr{A}$ and hence, via η, a point of $(\mathbb{K}\mathscr{A})_0$. Therefore, each $a \in \mathscr{A}$ determines a map $S \to \mathbb{K}\mathscr{A}$. We will use this to specify unit maps for spectral categories.

Definition 1.13 We define a spectral category $G\mathscr{A}$. Its objects are the finite G–sets A, which may be viewed as the spectra $\mathbb{K}G\mathscr{E}(A)$. Its morphism spectra are defined by $G\mathscr{A}(A,B) = \mathbb{K}G\mathscr{E}'(A,B)$, where $G\mathscr{E}'(A,B)$ is as defined in Definition B.2. Its unit maps $S \to G\mathscr{A}(A,A)$ are induced by the identity 1–cells in $G\mathscr{E}'(A,A)$, and its composition

$$G\mathscr{A}(B,C) \wedge G\mathscr{A}(A,B) \to G\mathscr{A}(A,C)$$

is induced by composition in $G\mathscr{E}'$.

As written, the definition makes little sense: to make the word "induced" meaningful requires a suitably behaved machine $\mathbb{K}$, as we will spell out in Section 2.2. For the purpose of Definition 1.13, the machine of [Elmendorf and Mandell 2009] would be sufficient, although it takes values in symmetric rather than orthogonal spectra. However, the proof of our main theorem, given in Section 2.4, will use the equivariant machine of [Guillou et al. 2023], and we will therefore use the same machine to make sense of Definition 1.13. Once this is done, we will have the presheaf category $\mathbf{Pre}(G\mathscr{A}, \mathscr{S})$ of $\mathscr{S}$–functors $(G\mathscr{A})^{\mathrm{op}} \to \mathscr{S}$ and $\mathscr{S}$–natural transformations. As shown for example in [Guillou and May 2020], it is a cofibrantly generated model category enriched in $\mathscr{S}$, or an *$\mathscr{S}$–model category* for short. As shown in [Mandell and May 2002], the category $G\mathscr{S}$ of (genuine) orthogonal G–spectra is also an $\mathscr{S}$–model category. Our main theorem can be restated as follows:

Theorem 1.14 (main theorem) *There is a zigzag of enriched Quillen equivalences connecting the $\mathscr{S}$–model categories $G\mathscr{S}$ and $\mathbf{Pre}(G\mathscr{A}, \mathscr{S})$.*

Therefore, G–spectra can be thought of as constructed from the very elementary category $G\mathscr{E}$ enriched in permutative categories, ordinary nonequivariant spectra and the black box of infinite loop space theory.

We have chosen to take all finite G–sets A as the objects of $G\mathscr{A}$. As we discuss in Theorem A.1, Theorem 1.14 holds just as well if we allow A to instead range only over the orbits G/H for subgroups $H \subset G$ (or even over one H in each conjugacy class). As discussed in Remark A.4, this can be viewed as a consequence of the fact that the spectral enrichment forces additivity. Intuitively, a G–spectrum is then described by its fixed-point spectra X^H, together with enriched restriction and transfer data. A bit more precisely, let $\mathscr{O}_G$ denote the category of orbits G/H and G–maps between them. For a G–spectrum X, passage to fixed-point spectra specifies a contravariant functor $X^{(-)}\colon \mathscr{O}_G \to \mathscr{S}$. The following reassuring result falls out of the proof of Theorem 1.14. We shall be more precise about this in Corollary 3.7.

Corollary 1.15 *The zigzag of equivalences induces a natural zigzag of equivalences between the fixed-point orbit functor,* $X \mapsto \{G/H \mapsto X^H\}$*, on* G*–spectra and the functor given by restricting presheaves* $G\mathscr{A} \to \mathscr{S}$ *to the (unenriched) orbit category.*

Thus, if X is a fibrant G–spectrum that corresponds to the presheaf Y, then X^H is equivalent to $Y(G/H)$.

Remark 1.16 For any n, the homotopy groups $\pi_n(X^H)$ define a Mackey functor, and so do the homotopy groups $\pi_n(Y(G/H))$. The corollary implies an isomorphism between these Mackey functors.

We view Theorem 1.14 as a G–spectrum analogue of the standard equivalence between G–spaces and space-valued presheaves on $\mathscr{O}_G$; see eg [Piacenza 1996]. As there, we do not in any sense view the theorem as giving a *replacement* for the category of G–spectra. We regard G–spectra as natural objects of intrinsic interest, and their presheaf descriptions as an illuminating perspective. We give some comparisons of functors to illustrate this in the brief Section 3.

1.3 The G–bicategory $\mathscr{E}_G$ of spans: intuitive definition

Everything we do depends on first working equivariantly and then passing to fixed points. We fix some generic notation. For a category $\mathscr{C}$, let $G\mathscr{C}$ be the category of G–objects in $\mathscr{C}$ and G–maps between them. Let $\mathscr{C}_G$ be the G–category of G–objects and nonequivariant maps, with G acting on morphisms by conjugation. The two categories are related conceptually by $G\mathscr{C} = (\mathscr{C}_G)^G$. The objects, being G–objects, are already G–fixed; we apply the G–fixed-point functor to hom sets. The reader may prefer to think of $\mathscr{C}_G$ as a category enriched in G–categories, with enriched hom objects the G–categories $\mathscr{C}_G(A, B)$ for G–objects A and B.

We apply this framework to the category of finite G–sets. We have already defined the G–fixed bicategory $G\mathscr{E}$, and we shall give two definitions of G–bicategories $\mathscr{E}_G$ with fixed-point bicategories equivalent to $G\mathscr{E}$. The first, given in this section, is more intuitive, but the second is more convenient for the proof of our main theorem.

Let U be a countable G–set that contains all orbit types G/H infinitely many times. Again let A, B and C denote finite G–sets, but now think of the D, E and F of (1.1) and (1.2) as finite subsets of the G–set U; these subsets need *not* be G–subsets. The action of G on U gives rise to an action of G on the finite subsets of U: for a finite subset D of U and $g \in G$, gD is another finite subset of U.

Definition 1.17 We define a G–groupoid $\mathscr{E}_G^U(A)$. The objects of $\mathscr{E}_G^U(A)$ are the nonequivariant maps $p\colon D \to A$, where A is a finite G–set and D is a finite subset of U. The morphisms $f : p \to q$, with $q\colon E \to A$, are the bijections $f : D \to E$ such that $q \circ f = p$. The group G acts on objects and morphisms by sending D to gD and sending a bijection $f : D \to E$ over A to the bijection $gf : gD \to gE$ over A given by $(gf)(gd) = g(f(d))$.

Definition 1.18 We define a bicategory $\mathscr{E}_G^U$ with objects the finite G–sets and with G–groupoids of morphisms between objects given by $\mathscr{E}_G^U(A,B) = \mathscr{E}_G^U(B \times A)$. Thinking of the objects of $\mathscr{E}_G^U(A,B)$ as nonequivariant spans $B \leftarrow D \rightarrow A$, composition and units are defined as in Definition 1.7.

Observe that taking disjoint unions of finite sets over A will not keep us in U and is thus not well defined. Therefore, the $\mathscr{E}_G^U(A)$ are not even symmetric monoidal (let alone permutative) G–categories in the naive sense of symmetric monoidal categories with G acting compatibly on all data.

1.4 The G–bicategory $\mathscr{E}_G$ of spans: working definition

We shall work with a less intuitive definition of $\mathscr{E}_G$, one that solves the problem of disjoint unions by avoiding any explicit use of them. It uses an especially convenient E_∞ operad of G–categories, denoted by $\mathscr{P}_G$. We recall it from [Guillou and May 2017], where we define a genuine permutative G–category to be an algebra over $\mathscr{P}_G$. More generally, in [Guillou et al. 2020] we define a genuine symmetric monoidal G–category to be a pseudoalgebra over $\mathscr{P}_G$, but we will not need that notion here. Such pseudoalgebras provide input for an equivariant infinite loop space machine.

To define $\mathscr{P}_G$, we apply our general point of view on equivariant categories to the category $\mathscr{C}at$ of small categories. Thus, for G–categories $\mathscr{A}$ and $\mathscr{B}$, let $\mathscr{C}at_G(\mathscr{A},\mathscr{B})$ be the G–category of functors $\mathscr{A} \to \mathscr{B}$ and natural transformations, with G acting by conjugation, and let $G\mathscr{C}at(\mathscr{A},\mathscr{B}) = \mathscr{C}at_G(\mathscr{A},\mathscr{B})^G$ be the category of G–functors and G–natural transformations.

Definition 1.19 Let $\mathscr{E}G$ be the groupoid[8] with object set G and a unique morphism, denoted by (h,k), from k to h for each pair of objects. Let G act from the right on $\mathscr{E}G$ by $h \cdot g = hg$ on objects and $(h,k) \cdot g = (hg,kg)$ on morphisms. Define $\mathscr{P}(j) = \mathscr{E}\Sigma_j$; this is the j^{th} category of an E_∞ operad of categories whose algebras are the permutative categories [Dunn 1994; May 1974]. Define $\mathscr{P}_G(j)$ to be the G–category

$$\mathscr{P}_G(j) = \mathscr{C}at_G(\mathscr{E}G, \mathscr{E}\Sigma_j) = \mathscr{C}at_G(\mathscr{E}G, \mathscr{P}(j)).$$

Here G acts trivially on $\mathscr{E}\Sigma_j$. The left action of G on $\mathscr{P}_G(j)$ is induced by the right action of G on $\mathscr{E}G$, and the right action of Σ_j is induced by the right action of Σ_j on $\mathscr{E}\Sigma_j$. The functor $\mathscr{C}at_G(\mathscr{E}G,-)$ is product-preserving and the operad structure maps are induced from those of $\mathscr{P}$. We interpret $\mathscr{P}(0)$ and $\mathscr{P}_G(0)$ to be trivial categories; $\mathscr{P}_G(1)$ is also trivial, with unique object denoted by $\mathbb{1}$.

Definition 1.20 Regard a finite G–set A as a discrete G–groupoid (identity morphisms only). Define the G–groupoid $\mathscr{E}_G(A)$ by

(1.21)
$$\mathscr{E}_G(A) = \coprod_{n \geq 0} \mathscr{P}_G(n) \times_{\Sigma_n} A^n = \Bigl(\coprod_{n \geq 1} \mathscr{P}_G(n) \times_{\Sigma_n} A^n \Bigr)_+ .$$

We interpret the term with $n = 0$ to be a trivial base category $*$, which explains the second equality, and we identify the $n = 1$ term with A.

[8] While $\mathscr{E}G$ is isomorphic as a G–category to the translation category of G, the action of G on that category is defined differently, as is explained in [Guillou et al. 2017, Proposition 1.8]. Our $\mathscr{E}G$ is the chaotic category of G, sometimes denoted by $\widetilde{G}$.

In the language of [Guillou and May 2017, Definition 4.5], $\mathscr{E}_G(A)$ is the free genuine permutative G–groupoid generated by the G–set A; its unit can be thought of as given by a disjoint trivial base category implicitly added to A. This is made precise by (1.24).

The following result is neither obvious nor difficult. It is proven in [Guillou and May 2017], where it is one ingredient in a categorical proof of the tom Dieck splitting theorem.

Theorem 1.22 [Guillou and May 2017, Theorem 5.9] *The G–fixed permutative groupoid $\mathscr{E}_G(A)^G$ is naturally isomorphic to the permutative groupoid $G\mathscr{E}(A)$ of Definition 1.5.*

The starting point of the proof is the observation that a functor $\mathscr{E}G \to \mathscr{E}\Sigma_n$ is uniquely determined by its object function $G \to \Sigma_n$. In particular, for a finite G–set $B = \underline{n}^\beta$, we may view the group homomorphism $\beta\colon G \to \Sigma_n$ as an object of the category $\mathscr{P}_G(n)$. With a little care, we see that a G–fixed object $(\beta; a_1, \dots, a_n)$ of $\mathscr{P}_G(n) \times_{\Sigma_n} A^n$ can be interpreted as a G–map $B \to A$ and that all finite G–sets over A are of this form.

Remark 1.23 Conceptually, Definition 1.20 hides an important identification and extension of functoriality that will be used crucially in Definition 1.28. A priori, $\mathscr{E}_G$ is a functor on *unbased* finite G–sets, but an alternative reformulation is

$$\mathscr{E}_G(A) = \mathbb{P}_G(A_+), \tag{1.24}$$

where $\mathbb{P}_G$ is the monad on the category of *based* G–categories, not just G–groupoids, whose algebras are the same as the $\mathscr{P}_G$–algebras. Thus, equation (1.24) exhibits $\mathscr{E}_G$ as a special case of the more general functor $\mathbb{P}_G$. With this reinterpretation, $\mathscr{E}_G(A)$ extends to a functor on all based finite G–sets and all based G–maps, not just those of the form f_+.

We need to be more precise about this identification and extended functoriality.

Definition 1.25 Let Λ be the category of finite based sets $\boldsymbol{n}$ and based injections.[9] For a finite based G–set $\mathscr{A}$, regarded as a discrete based G–category, insertion of basepoints makes the powers $\mathscr{A}^n$ into a covariant functor $\mathscr{A}^\bullet$ from Λ to based G–categories. Then $\mathbb{P}_G(\mathscr{A})$ is the categorical tensor product

$$\mathbb{P}_G(\mathscr{A}) = \mathscr{P}_G(\bullet) \otimes_\Lambda \mathscr{A}^\bullet.$$

Since any based injection $\sigma \in \Lambda(\boldsymbol{m}, \boldsymbol{n})$ can be written uniquely as the composition of a permutation of $\boldsymbol{m}$ followed by an order-preserving injection, the contravariant functoriality of $\mathscr{P}_G(\bullet)$ on based injections is given by combining the right Σ_m–action on $\mathscr{P}_G(m)$ with the contravariant functoriality with regards to ordered injections described in [May 1972, Notations 2.3]. Thus,

$$\mathbb{P}_G(\mathscr{A}) = \Big(\coprod_{n \geq 0} \mathscr{P}_G(n) \times \mathscr{A}^n \Big) \Big/ {\sim}, \tag{1.26}$$

[9] The category Λ is isomorphic to the category of finite (unbased) sets and injections. We use based here both for historical reasons and because it fits better into the machinery of infinite loop space theory.

where

$$(\sigma^*\mu; a) \sim (\mu; \sigma_* a) \quad \text{for } \mu \in \mathscr{P}_G(n),\ \sigma \in \Lambda(\boldsymbol{m}, \boldsymbol{n}) \text{ and } a \in \mathscr{A}^m.$$

As in [May 1972, Notations 2.3], we can first pass to orbits using the permutations in Λ and then use the equivalence relation induced by the proper injections to rewrite this as

(1.27)
$$\mathbb{P}_G(\mathscr{A}) = \Bigl(\coprod_{n\geq 0} \mathscr{P}_G(n) \times_{\Sigma_n} \mathscr{A}^n\Bigr)\Big/\sim,$$

thus highlighting the comparison with (1.21).

Definition 1.28 For a based G–map $f : A_+ \to B_+$, define a functor

$$f_! : \mathscr{E}_G(A) \to \mathscr{E}_G(B)$$

using the identification (1.24) and the functoriality of $\mathbb{P}_G$ on based maps.[10] In the case that $f^{-1}(*) = *$, so that f is in the image of the disjoint basepoint functor $X \mapsto X_+$, the functor $f_!$ is given by the disjoint union over n of the functors

$$\mathscr{P}_G(n) \times_{\Sigma_n} A^n \xrightarrow{\mathrm{id}\times_{\Sigma_n} f^n} \mathscr{P}_G(n) \times_{\Sigma_n} B^n.$$

If $i : A \to B$ is an injection of unbased finite G–sets, define an associated retraction $r : B_+ \to A_+$ of based finite G–sets by setting $ri(a) = a$ and $r(b) = *$ if $b \notin \mathrm{im}(i)$. Then define[11]

$$i^* = r_! : \mathscr{E}_G(B) \to \mathscr{E}_G(A).$$

By Remark 2.21 below, we may think of i^* as the dual of i.

The following definition gives the G–category analogue of Definition 1.7. It specifies a G–category (almost) "enriched in permutative G–categories".

Definition 1.29 We define a G–bicategory[12] $\mathscr{E}_G$ with a permutative G–groupoid hom object for each pair of objects as follows. The 0–cells of $\mathscr{E}_G$ are the finite G–sets A, which may be thought of as the G–categories $\mathscr{E}_G(A)$. The permutative G–groupoid $\mathscr{E}_G(A, B)$ of 1–cells and 2–cells $A \to B$ is $\mathscr{E}_G(B \times A)$, as defined in Definition 1.20. The composition

$$\circ : \mathscr{E}_G(B, C) \times \mathscr{E}_G(A, B) \to \mathscr{E}_G(A, C)$$

is given by the diagram

(1.30)
$$\begin{tikzcd}
\mathscr{E}_G(C \times B) \wedge \mathscr{E}_G(B \times A) \arrow[r, dashed, "\circ"] \arrow[d, "\omega"'] & \mathscr{E}_G(C \times A) \\
\mathscr{E}_G(C \times B \times B \times A) \arrow[r, "(\mathrm{id}\times\Delta\times\mathrm{id})^*"] & \mathscr{E}_G(C \times B \times A) \arrow[u, "\pi_!"']
\end{tikzcd}$$

[10] With the intuitive version of $\mathscr{E}_G$ described in Section 1.3, $f_! : \mathscr{E}_G(A) \to \mathscr{E}_G(B)$ is then just the pushforward functor obtained by composing maps over A with f.

[11] With the intuitive version of $\mathscr{E}_G$ described in Section 1.3, $i^* : \mathscr{E}_G(B) \to \mathscr{E}_G(A)$ is just the functor obtained by using i to pull back maps over B to maps over A.

[12] As in Remark 1.8, the bicategory $\mathscr{E}_G$ is almost a 2–category. It is just missing strict units, as we shall explain shortly.

Its first map ω is a pairing of free $\mathscr{P}_G$–algebras that will be made precise in Definition 1.35. Its second and third maps implement composition from the point of view of (1.3). They are specializations of the contravariant functoriality of $\mathscr{E}_G$ on injections and its covariant functoriality on surjections, as is made precise in Definition 1.28.

This composition is strictly associative, as we indicate in Remark 1.36. With $A = \underline{n}^\alpha$, $\mathscr{E}_G(A, A)$ has a pseudounit 1–cell

(1.31) $$\Delta_A = (\alpha; \Delta_A) \in \mathscr{E}_G(A \times A) = \mathscr{P}_G(n) \times_{\Sigma_n} (A \times A)^n,$$

where

$$\Delta_A = ((1,1), \dots, (n,n)) \in (A \times A)^n.$$

It is a strict right unit, but it is not a strict left unit (see Remark 1.36).

To rectify to obtain a strict unit, we need whiskered G–categories $\mathscr{E}'_G$ analogous to the whiskered categories $G\mathscr{E}'$, and we define them in Appendix B. They are defined in such a way that Theorem 1.22 has the following corollary by direct comparison of definitions:

Corollary 1.32 *The G–fixed category $(\mathscr{E}'_G)^G$ enriched in permutative categories is isomorphic to the category $G\mathscr{E}'$ enriched in permutative categories.*

In Definition 1.35 we will give an ad hoc definition of the pairing ω that is required in Definition 1.29. We place ω in a general multicategorical context in [Guillou et al. 2023, Definition 3.20]. We first comment on its domain; compare Remark 1.10.

Remark 1.33 We can define the smash product of based G–categories in the same way as the smash product of based G–spaces (see [Elmendorf and Mandell 2009, Lemma 4.20]). We are most interested in examples of the form $\mathscr{A}_+$ and $\mathscr{B}_+$ for unbased G–categories $\mathscr{A}$ and $\mathscr{B}$, and then $\mathscr{A}_+ \wedge \mathscr{B}_+$ can be identified with $(\mathscr{A} \times \mathscr{B})_+$. Therefore,

(1.34) $$\begin{aligned} \mathscr{E}_G(A) \wedge \mathscr{E}_G(B) &= \Big(\coprod_{m\geq 1} \mathscr{P}_G(m) \times_{\Sigma_m} A^m\Big)_+ \wedge \Big(\coprod_{n\geq 1} \mathscr{P}_G(n) \times_{\Sigma_n} B^n\Big)_+ \\ &\xrightarrow{\cong} \Big(\coprod_{\substack{m\geq 1\\ n\geq 1}} \mathscr{P}_G(m) \times_{\Sigma_m} A^m \times \mathscr{P}_G(n) \times_{\Sigma_n} B^n\Big)_+ \\ &\xrightarrow{\cong} \Big(\coprod_{\substack{m\geq 1\\ n\geq 1}} \mathscr{P}_G(m) \times \mathscr{P}_G(n) \times_{\Sigma_m \times \Sigma_n} A^m \times B^n\Big)_+. \end{aligned}$$

Note that this smash product does not have a $\mathscr{P}_G$–algebra structure.

Definition 1.35 The homomorphism $\otimes\colon \Sigma_m \times \Sigma_n \to \Sigma_{mn}$ defined using lexicographic ordering in Definition 1.4 is the object function of a functor

$$\mathscr{E}\Sigma_m \times \mathscr{E}\Sigma_n \to \mathscr{E}\Sigma_{mn}.$$

Applying the functor $\mathscr{C}at_G(\mathscr{E}G, -)$, we obtain pairings

$$\otimes\colon \mathscr{P}_G(m) \times \mathscr{P}_G(n) \to \mathscr{P}_G(mn);$$

on objects of $\mathscr{E}G$, $(\mu \otimes \nu)(g) = \mu(g) \otimes \nu(g)$. For G–sets A and B, we have the injection

$$\boxtimes\colon A^m \times B^n \to (A \times B)^{mn}$$

that sends $(a_1, \dots, a_m) \times (b_1, \dots, b_n)$ to the set of pairs (a_i, b_j), ordered lexicographically. Combining, there result functors

$$\omega_{m,n}\colon (\mathscr{P}_G(m) \times_{\Sigma_m} A^m) \times (\mathscr{P}_G(n) \times_{\Sigma_n} B^n) \to \mathscr{P}_G(mn) \times_{\Sigma_{mn}} (A \times B)^{mn},$$
$$\omega_{m,n}((\mu, a), (\nu, b)) = (\mu \otimes \nu, a \boxtimes b).$$

Using the description (1.34), the functors $\omega_{m,n}$ specify pairings of G–categories

$$\omega\colon \mathscr{E}_G(A) \wedge \mathscr{E}_G(B) \to \mathscr{E}_G(A \times B).$$

While $\mathscr{E}_G(A) \wedge \mathscr{E}_G(B)$ is not a $\mathscr{P}_G$–algebra, we show in [Guillou et al. 2023, Proposition 3.25] that ω is an example of a bilinear, or 2–ary, morphism in the multicategory of $\mathscr{P}_G$–algebras. The machine of [Guillou et al. 2023] then produces from this bilinear map a pairing of G–spectra, as we will discuss in Section 2.2 below.

Remark 1.36 The associativity of the composition $\circ$ defined in Definition 1.29 is an easy diagram chase, starting from the associativity of the pairing on $\mathscr{P}_G$. We illustrate how Definition 1.28 works by considering composites with the pseudounit objects Δ_A. Let E be a 1–cell in $\mathscr{E}_G(A, B)$ and choose an object

$$(\mu; (b_1, a_1), \dots, (b_m, a_m)) \quad \text{of} \;\; \mathscr{P}_G(m) \times (B \times A)^m$$

in the Σ_m–orbit E.

We first prove that $E \circ \Delta_A = E$. Suppose that $A = \underline{n}^\alpha$. Then the object

$$\big(\mu \otimes \alpha; ((b_i, a_i, j, j))\big) \quad \text{of} \;\; \mathscr{P}_G(mn) \times (B \times A \times A \times A)^{mn}$$

is in the Σ_{mn}–orbit $\omega(E, \Delta_A)$. The ordering of the four-tuples is lexicographic on i and j. The four-tuple (b_i, a_i, j, j) is in the image of $\mathrm{id} \times \Delta \times \mathrm{id}$ if and only if $a_i = j$. The retraction corresponding to this injection maps such a (b_i, a_i, j, j) to $(b_i, a_i, j) = (b_i, a_i, a_i)$ and all other (b_i, a_i, j, j) to the basepoint. Applying $\pi_!$, we arrive at

$$\sigma_*((b_1, a_1), \dots, (b_m, a_m)) \in (B \times A)^{mn},$$

where $\sigma\colon \boldsymbol{m} \to \boldsymbol{mn}$ is the ordered injection that sends i to $\lambda_{m,n}^{-1}(i, a_i)$. Therefore,

$$E \circ \Delta_A = \big(\mu \otimes \alpha; \sigma_*((b_1, a_1), \dots, (b_m, a_m))\big) = (\sigma^*(\mu \otimes \alpha); (b_1, a_1), \dots, (b_m, a_m)).$$

Since σ^* reverses the lexicographic ordering used to define $\mu \otimes \alpha$, we have the relation $\sigma^*(\mu \otimes \alpha) = \mu$.

Now let $B = \underline{p}^\beta$ and consider $\Delta_B \circ E$. Then the object

$$\big(\beta \otimes \mu; ((k, k, b_i, a_i))\big) \quad \text{of} \;\; \mathscr{P}_G(pm) \times (B \times B \times B \times A)^{pm}$$

is in the Σ_{pm}–orbit $\omega(\Delta_B, E)$. The ordering of the four-tuples is lexicographic on k and i. The four-tuple (k, k, b_i, a_i) is in the image of $\mathrm{id} \times \Delta \times \mathrm{id}$ if and only if $k = b_i$. The retraction corresponding to this injection maps all other (k, k, b_i, a_i) to the basepoint. Applying $\pi_!$, we arrive at

$$\tau_*((b_1, a_1), \ldots, (b_m, a_m)) \in (B \times A)^{pm},$$

where $\tau\colon \boldsymbol{m} \to \boldsymbol{pm}$ is the injection that sends i to $\lambda^{-1}_{p,m}(b_i, i)$. We have

$$\Delta_B \circ E = \bigl(\beta \otimes \mu, \tau_*((b_1, a_1), \ldots, (b_m, a_m))\bigr) = (\tau^*(\beta \otimes \mu); (b_1, a_1), \ldots, (b_m, a_m)),$$

but the injection τ is not ordered and $\tau^*(\beta \otimes \mu)$ is not equal to μ. We define

(1.37) $$\ell_{B,E}\colon \Delta_B \circ E \to E$$

to be the 2–cell induced by the (unique) morphism $\tau^*(\beta \otimes \mu) \to \mu$ in $\mathscr{P}_G(m)$. The structure $\mathscr{E}_G$ is only a bicategory, while $\mathscr{E}'_G$, defined in Appendix B, is a strict 2–category. The inclusion $\mathscr{E}_G \to \mathscr{E}'_G$ is a pseudofunctor with unit constraint given by ζ of Definition B.1. In [Guillou et al. 2023], the category of $\mathscr{P}_G$–algebras is given the structure of a multicategory. The composition functors in both $\mathscr{E}_G$ and $\mathscr{E}'_G$ are examples of bilinear maps in the multicategorical sense.

1.5 The categorical duality maps

Since various specializations are central to our work, we briefly recall how duality works categorically, following [Lewis and May 1986a, Section 1] for example. We then define maps of $\mathscr{P}_G$–algebras that will lead in Section 2.3 to the proof that the objects of $G\mathscr{A}$ are self-dual.

Let $\mathscr{V}$ be a closed symmetric monoidal category with product $\wedge$, unit S and hom objects $F(X, Y)$; write $DX = F(X, S)$. A pair of objects (X, Y) in $\mathscr{V}$ is a dual pair if there are maps $\eta\colon S \to X \wedge Y$ and $\varepsilon\colon Y \wedge X \to S$ such that the composites

$$X \cong S \wedge X \xrightarrow{\eta\wedge\mathrm{id}} X \wedge Y \wedge X \xrightarrow{\mathrm{id}\wedge\varepsilon} X \wedge S \cong X, \quad Y \cong Y \wedge S \xrightarrow{\mathrm{id}\wedge\eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon\wedge\mathrm{id}} S \wedge Y \cong Y$$

are identity maps. For any such pair, the adjoint $\tilde{\varepsilon}\colon Y \to DX$ of ε is an isomorphism. When (X, Y) and (X', Y') are dual pairs, the dual of a map $f\colon X \to X'$ is the composite

(1.38) $$Y' \cong Y' \wedge S_G \xrightarrow{\mathrm{id}\wedge\eta} Y' \wedge X \wedge Y \xrightarrow{\mathrm{id}\wedge f\wedge\mathrm{id}} Y' \wedge X' \wedge Y \xrightarrow{\varepsilon'\wedge\mathrm{id}} S_G \wedge Y \cong Y.$$

For any pair of objects X and Z, we have a natural map

(1.39) $$\zeta\colon Z \wedge DX = Z \wedge F(X, S) \to F(X, Z)$$

in $\mathscr{V}$, namely the adjoint of

$$\mathrm{id} \wedge \varepsilon\colon Z \wedge DX \wedge X \to Z \wedge S \cong Z,$$

where ε is the evident evaluation map. The map ζ is an isomorphism when either X or Z is dualizable [Lewis and May 1986a, Proposition 1.3]. When X is self-dual and Z is arbitrary, we have the composite isomorphism

(1.40) $$\delta = \zeta \circ (\mathrm{id} \wedge \tilde{\varepsilon})\colon Z \wedge X \to Z \wedge DX \to F(X, Z).$$

This map in various categories will play an important role in our work.

In Definitions 1.41 and 1.42, we will define two maps of $\mathscr{P}_G$–algebras that are central to duality and therefore to everything we do. Let $S^0 = \{*, 1\}$, where $*$ is the basepoint and 1 is not. We think of S^0 as 1_+, where 1 is the one-point G–set. In line with this convention, we also think of 1 as a trivial category with object 1. Remember that $\mathscr{E}_G(A) = \mathbb{P}_G(A_+)$ is the free $\mathscr{P}_G$–algebra generated by A_+, where we view finite G–sets as categories with only identity morphisms.

Definition 1.41 For a finite G–set $A = \underline{n}^\alpha$, define based G–maps

$$\varepsilon\colon (A\times A)_+ \to S^0, \quad r\colon (A\times A)_+ \to A_+ \quad \text{and} \quad \pi\colon A_+ \to S^0$$

by $r(a,b) = *$ if $a \neq b$ and $r(a,a) = a$, $\pi(a) = 1$ and $\varepsilon = \pi \circ r$, so that $\varepsilon(a,b) = *$ if $a \neq b$ and $\varepsilon(a,a) = 1$. Note that $r \circ \Delta_+ = \mathrm{id}_{A_+}$. We agree to again write ε for the induced map of $\mathscr{P}_G$–algebras

$$\varepsilon = \mathscr{E}_G\varepsilon\colon \mathscr{E}_G(A\times A) \to \mathscr{E}_G(1).$$

Definition 1.42 For a finite G–set $A = \underline{n}^\alpha$, regard the object $\Delta_A \in \mathscr{E}_G(A\times A)$ as the map of G–categories $i_A\colon 1 \to \mathscr{E}_G(A\times A)$ that sends the object 1 of the trivial category to the object Δ_A. By freeness, there results a map of $\mathscr{P}_G$–algebras

$$\eta\colon \mathscr{E}_G(1) \to \mathscr{E}_G(A\times A).$$

Explicitly,[13] η is the disjoint union over m of the maps

$$\mathscr{P}_G(m) \times_{\Sigma_m} 1^m \to \mathscr{P}_G(mn) \times_{\Sigma_{mn}} (A\times A)^{mn}$$

given by

$$\eta(\mu, 1^m) = (\mu \otimes \alpha; (\Delta_A)^m).$$

The following categorical observation will lead to our proof in Section 2.3 that the G–spectra $\Sigma_G^\infty(A_+)$ are self-dual. Since care of basepoints is crucial, we use the alternative notation $\mathbb{P}_G(A_+)$. Remember that $(A\times A)_+$ can be identified with $A_+ \wedge A_+$. We identify $1_+ \wedge A_+$ and $A_+ \wedge 1_+$ with A_+ at the bottom center of our diagrams.

Proposition 1.43 *In the diagrams below, square* (1) *commutes up to isomorphism, and the other three squares commute on the nose*:

$$\begin{tikzcd}
\mathbb{P}_G(A_+\wedge A_+)\wedge\mathbb{P}_G(A_+) \arrow[r, "\omega"] & \mathbb{P}_G(A_+\wedge A_+\wedge A_+) \arrow[d, "\mathbb{P}_G(\mathrm{id}\wedge\varepsilon)"] & \mathbb{P}_G(A_+)\wedge\mathbb{P}_G(A_+\wedge A_+) \arrow[l, "\omega"'] \arrow[d, "\mathrm{id}\wedge\varepsilon"] \\
\mathbb{P}_G(1_+)\wedge\mathbb{P}_G(A_+) \arrow[u, "\eta\wedge\mathrm{id}"] \arrow[r, "\omega"] \arrow[ur, phantom, "\Downarrow(1)"] & \mathbb{P}_G(A_+) & \mathbb{P}_G(A_+)\wedge\mathbb{P}_G(1_+) \arrow[l, "\omega"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}_G(A_+)\wedge\mathbb{P}_G(A_+\wedge A_+) \arrow[r, "\omega"] & \mathbb{P}_G(A_+\wedge A_+\wedge A_+) \arrow[d, "\mathbb{P}_G(\varepsilon\wedge\mathrm{id})"] & \mathbb{P}_G(A_+\wedge A_+)\wedge\mathbb{P}_G(A_+) \arrow[l, "\omega"'] \arrow[d, "\varepsilon\wedge\mathrm{id}"] \\
\mathbb{P}_G(A_+)\wedge\mathbb{P}_G(1_+) \arrow[u, "\mathrm{id}\wedge\eta"] \arrow[r, "\omega"] \arrow[ur, phantom, "(2)"] & \mathbb{P}_G(A_+) & \mathbb{P}_G(1_+)\wedge\mathbb{P}_G(A_+) \arrow[l, "\omega"']
\end{tikzcd}$$

[13] This uses that $\gamma(\mu;\alpha^n) = \mu\otimes\alpha$, where $\gamma\colon \mathscr{P}_G(m)\times\mathscr{P}_G(n)^m \to \mathscr{P}_G(mn)$, as explained in [Guillou et al. 2023, Section 3.1].

Proof In the right vertical arrows, ε means $\mathbb{P}_G(\varepsilon)$. Both right squares are naturality diagrams, so it remains to consider the squares on the left. The difference between squares (1) and (2) is closely analogous to the difference between left and right composition with Δ_A, as explained in Remark 1.36. Let $A = \underline{n}^\alpha$ and consider objects $(\mu, 1^m)$ of $\mathscr{P}(m) \times 1^m$ and (ν, a) of $\mathscr{P}(q) \times A^q$. We consider square (2) first, paying close attention to the order in which variables appear.

By Definitions 1.35 and 1.42,

$$\omega((\nu,a),(\mu,1^m)) = (\nu \otimes \mu, a \boxtimes 1^m) \qquad \text{in } \mathscr{P}(qm) \times A^{qm} \text{ and}$$

$$\omega \circ (\mathrm{id} \wedge \eta)((\nu,a),(\mu,1^m)) = (\nu \otimes \mu \otimes \alpha; a \boxtimes (\Delta_A)^m) \quad \text{in } \mathscr{P}_G(qmn) \times_{\Sigma_{qmn}} (A^3)^{qmn}.$$

Identifying $\underline{qm}$ with $\underline{q} \times \underline{m}$ lexicographically, the $(k,i)^{\text{th}}$ coordinate of $a \boxtimes 1^m$ is a_k. Identifying $\underline{qmn}$ with $\underline{q} \times \underline{m} \times \underline{n}$ lexicographically, the $(k,j,i)^{\text{th}}$ coordinate of $a \boxtimes (\Delta_A)^m$ is (a_k, i, i). By Definition 1.41, $\varepsilon \wedge \mathrm{id}$ sends this coordinate to the basepoint unless $a_k = i$, when it sends it to i. Noticing the agreement of lexicographic orderings, we see as in Remark 1.36 that the injection $\sigma\colon \boldsymbol{qm} \to \boldsymbol{qmn}$ such that

$$\sigma_*(a \boxtimes 1^m) = (\varepsilon \wedge \mathrm{id})_*(a \boxtimes (\Delta_A)^m)$$

is ordered and satisfies $\sigma^*(\nu \otimes \mu \otimes \alpha) = \nu \otimes \mu$.

Now consider square (1). By Definitions 1.35 and 1.42,

$$\omega((\mu,1^m),(\nu,a)) = (\mu \otimes \nu, 1^m \boxtimes a) \qquad \text{in } \mathscr{P}(mq) \times_{\Sigma_{mq}} A^{mq} \text{ and}$$

$$\omega \circ (\eta \wedge \mathrm{id})((\mu,1^m),(\nu,a)) = (\gamma(\mu;\alpha^n) \otimes \nu; (\Delta_A)^m \boxtimes a) \quad \text{in } \mathscr{P}_G(mnq) \times_{\Sigma_{mnq}} (A^3)^{mnq}.$$

Identifying $\underline{mq}$ with $\underline{m} \times \underline{q}$ lexicographically, the $(i,k)^{\text{th}}$ coordinate of $1^m \boxtimes a$ is a_k. Identifying $\underline{mnq}$ with $\underline{m} \times \underline{n} \times \underline{q}$ lexicographically, the $(i,j,k)^{\text{th}}$ coordinate of $(\Delta_A)^m \boxtimes a$ is (j,j,a_k). By Definition 1.41, $\mathrm{id} \wedge \varepsilon$ sends this coordinate to the basepoint unless $j = a_k$, when it sends it to j. Here the injection $\tau\colon \boldsymbol{mq} \to \boldsymbol{mnq}$ such that

$$\tau(1^m \boxtimes a) = (\mathrm{id} \wedge \varepsilon)_*((\Delta_A)^m \boxtimes a)$$

is not ordered and $\tau^*(\mu \otimes \alpha \otimes \nu)$ is not equal to $\mu \otimes \nu$ in $\mathscr{P}_G(mq)$. As in Remark 1.36, there is a unique 2–cell, necessarily an isomorphism,

$$\vartheta\colon (\mu \otimes \nu) \Rightarrow \tau^*(\mu \otimes \alpha \otimes \nu)$$

in $\mathscr{P}_G(mq)$. As the input varies, the 2–cells

$$(\vartheta, \mathrm{id})\colon (\mu \otimes \nu; 1^m \boxtimes a) \Rightarrow (\tau^*(\mu \otimes \alpha \otimes \nu), 1^m \boxtimes a)$$

specify the 2–natural isomorphism in the square (1). $\square$

2 The proof of the main theorem

2.1 The equivariant approach to Theorem 1.14

As we explain in [Guillou et al. 2023], following [Guillou and May 2017], equivariant infinite loop space theory associates an orthogonal G–spectrum $\mathbb{K}_G\mathscr{C}_G$ to a genuine permutative (or, more generally, genuine symmetric monoidal) G–category $\mathscr{C}_G$. The map $B\mathscr{C}_G = (\mathbb{K}_G\mathscr{C}_G)_0 \to \Omega(\mathbb{K}_G\mathscr{C}_G)_1$ is an equivariant group completion.[14]

Notation 2.1 We denote by $G\mathscr{S}$ the (closed symmetric monoidal) category of orthogonal G–spectra, indexed on a complete universe, and maps of such. A category enriched over $G\mathscr{S}$ will be referred to as a $G\mathscr{S}$–category.

The category $G\mathscr{S}$ has two further relevant enrichments. Using the closed structure yields a self-enrichment, which we write as $\mathscr{S}_G$. Thus, for G–spectra X and Y, the G–spectrum $\mathscr{S}_G(X,Y)$ is the mapping G–spectrum $F_G(X,Y)$. Applying fixed points to the mapping G–spectra gives an $\mathscr{S}$–enriched category, which we again write as $G\mathscr{S}$. This parallels the discussion at the start of Section 1.3.

Applying the functor $\mathbb{K}_G$ to $\mathscr{E}_G$ (Definition 1.29), we obtain the following equivariant analogue of Definition 1.13:

Definition 2.2 We define a G–spectral category, or $G\mathscr{S}$–category, $\mathscr{A}_G$. Its objects are the finite G–sets A, which may be viewed as the G–spectra $\mathbb{K}_G\mathscr{E}_G(A)$. Its morphism G–spectra $\mathscr{A}_G(A,B)$ are the $\mathbb{K}_G\mathscr{E}'_G(B\times A)$. Its unit G–maps $S_G \to \mathscr{A}_G(A,A)$ are induced by the points $I_A \in G\mathscr{E}'(A,A)$ (see Appendix B) and its composition G–maps

$$\mathscr{A}_G(B,C)\wedge \mathscr{A}_G(A,B)\to \mathscr{A}_G(A,C)$$

are induced by composition in $\mathscr{E}'_G$.

Again, as written, the definition makes little sense: to make the word "induced" meaningful requires properties of the equivariant infinite loop space machine $\mathbb{K}_G$ that we will spell out in Section 2.2. This depends on having a functor that takes pairings (alias bilinear maps) of free $\mathscr{P}_G$–algebras to pairings of G–spectra.

The equivariant and nonequivariant infinite loop space functors are related by the following result:

Theorem 2.3 [Guillou and May 2017] *There is a natural equivalence of spectra*

$$\iota\colon \mathbb{K}(G\mathscr{C})\to (\mathbb{K}_G\mathscr{C}_G)^G$$

for permutative G–categories $\mathscr{C}_G$ with G–fixed permutative categories $G\mathscr{C}$.

In view of Corollary 1.32, there results an equivalence of $\mathscr{S}$–categories

$$G\mathscr{A} \xrightarrow{\simeq} (\mathscr{A}_G)^G.$$

[14]The papers from around 1990, such as [Costenoble and Waner 1991; Shimakawa 1989], are not adequate, in part because genuine permutative G–categories were not explicitly defined and the group completion property was not worked out rigorously, but more substantially because a symmetric monoidal category of G–spectra had not yet been discovered. A key feature of the version of the Segal machine [Guillou et al. 2019] used in our proofs is that it is given by a symmetric monoidal functor, a claim that would not have made sense in 1990.

The proof of Theorem 1.14 goes as follows. We now write $G\mathscr{D}_{\mathrm{All}}$ for the spectral version of the category introduced in Definition 1.11. We start with the following Theorem 2.4, which is a specialization of [Guillou and May 2020, Lemma 1.35]; it is discussed in Section A.1. The essential point is that the collection $\{\Sigma_G^\infty A_+\}$ is a set of generators for $\mathrm{Ho}\, G\mathscr{S}$.

Theorem 2.4 *There is an $\mathscr{S}$–enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{S},$$

and it is a Quillen equivalence.

Remark 2.5 Instead of using $G\mathscr{D}_{\mathrm{All}}$, we can use its full subcategory $G\mathscr{D}_{\mathrm{Orb}}$ obtained by restricting the A to be orbits G/H, and then the result generalizes to compact Lie groups G; see Theorem A.1. We define $G\mathscr{D}_{\mathrm{Orb}}$ as we defined $G\mathscr{D}_{\mathrm{All}}$ in Definition 1.11, again using fibrant replacements. Then $G\mathscr{D}_{\mathrm{All}}$ and $G\mathscr{D}_{\mathrm{Orb}}$ are the G–fixed $\mathscr{S}$–categories obtained from full $G\mathscr{S}$–subcategories $\mathscr{D}_{\mathrm{All}}$ and $\mathscr{D}_{\mathrm{Orb}}$ of $\mathscr{S}_G$.

We will prove the following result in Section 2.4.

Theorem 2.6 (equivariant version of the main theorem) *There is a zigzag of weak equivalences connecting the $G\mathscr{S}$-categories $\mathscr{A}_G$ and $\mathscr{D}_{\mathrm{All}}$.*

A weak equivalence between $G\mathscr{S}$–categories with the same object sets is just an $G\mathscr{S}$–enriched functor that induces weak equivalences on morphism G–spectra.[15] On passage to G–fixed categories, this equivariant zigzag induces a zigzag of weak $\mathscr{S}$–equivalences connecting the $\mathscr{S}$–categories $G\mathscr{A}$ and $G\mathscr{D}_{\mathrm{All}}$. In turn, by [Guillou and May 2020, Proposition 2.4], this zigzag induces a zigzag of Quillen equivalences between $\mathbf{Pre}(G\mathscr{A}, \mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S})$. Since $\mathbf{Pre}(G\mathscr{D}_{\mathrm{All}}, \mathscr{S})$ is Quillen equivalent to $G\mathscr{S}$, it follows that Theorem 2.6 implies Theorem 1.14.

Remark 2.7 For a G–spectrum X, the functor $\mathbb{U}(X)$ (of Theorem 2.4) sends an orbit G/H to

$$F_G(\Sigma_G^\infty G/H_+, X)^G \cong X^H.$$

Keeping that fact in mind shows why Corollary 1.15 follows from the proof of Theorem 1.14.

To understand $G\mathscr{S}$ as an $\mathscr{S}$–category, we must first understand $\mathscr{S}_G$ as a $G\mathscr{S}$–category. That is, to understand the G–fixed spectra $F_G(X,Y)^G$, we must first understand the function G–spectra $F_G(X,Y)$. Using infinite loop space theory to model function spectra implicitly raises a conceptual issue: there is no known infinite loop space machine that knows about function spectra. That is, given input data X and Y (permutative G–categories, E_∞–G–spaces, Γ–G–spaces, etc) for an infinite loop space machine $\mathbb{K}_G$, we do not know what input data will have as output the function G–spectra $F_G(\mathbb{K}_G X, \mathbb{K}_G Y)$. The problem

[15] A more general definition is given in [Guillou and May 2020, Definition 2.3].

does not even make sense as just stated because the output G–spectra $\mathbb{K}_G X$ are always connective, whereas $F_G(\mathbb{K}_G X, \mathbb{K}_G Y)$ is generally not. The most that one could hope for in general is to detect the connective cover of $F(\mathbb{K}_G X, \mathbb{K}_G Y)$. In our case, the relevant function G–spectra are connective since the suspension G–spectra $\Sigma^\infty_G(A_+)$ are self-dual, as we shall reprove in Section 2.3.

2.2 Results from equivariant infinite loop space theory

The proof of Theorem 2.6 is the heart of this paper, and of course it depends on equivariant infinite loop space theory and in particular on the relationship between the G–spectra $\mathscr{A}_G(A) = \mathbb{K}_G \mathscr{E}_G(A)$ and the suspension G–spectra $\Sigma^\infty_G(A_+)$. We collect the results that we need from [Guillou et al. 2023] in this section. We warn the skeptical reader that the results of this paper depend fundamentally on Theorems 2.8 and 2.12. However, the proofs of those results require work far afield from the applications in this paper.

In fact, Theorem 2.6 is an application of a categorical version of the equivariant Barratt–Priddy–Quillen (BPQ) theorem for the identification of suspension G–spectra.[16] We state the theorem in full generality before restricting our attention to finite G–sets. We shall find use for the full generality in Section 2.5.

Recall from Remark 1.23 that $\mathscr{E}_G(A)$ can be identified with the category $\mathbb{P}_G(A_+)$, where $\mathbb{P}_G$ is the free $\mathscr{P}_G$–category functor on based G–categories. The functor $\mathbb{P}_G$ applies equally well to based topological G–categories.[17] We view a based G–space X as a topological G–category that is discrete in the categorical sense: its morphism and object G–spaces are both X, and its source, target, identity and composition maps are all its identity map. Thus, we have the topological $\mathscr{P}_G$–category $\mathbb{P}_G(X)$. The geometric realization of its nerve is the free E_∞–G–space generated by X.

Henceforward, we use the term *stable equivalence*, rather than weak equivalence, for the weak equivalences in our model categories of spectra and G–spectra. Guillou and May [2017, Theorem 6.2] established an equivariant version of the BPQ theorem, giving a natural equivalence between $\Sigma^\infty_G X_+$ and $\mathbb{K}_G \mathbb{P}_G(X)$. However, in order to produce our spectral category $\mathscr{A}_G$, we require a more structured version of that result.

First, it is essential that we have a machine with good multiplicative properties. The following result, which is proven in [Guillou et al. 2023], gives far more than we need. As explained in [Guillou et al. 2023, Section 3], we have a multicategory **Mult**($\mathscr{P}_G$) of (strict) $\mathscr{P}_G$–algebras and pseudomorphisms between them; it is a submulticategory of a multicategory **Mult**($\mathscr{P}_G$–**PsAlg**) of $\mathscr{P}_G$–pseudoalgebras. The multilinear maps of **Mult**($\mathscr{P}_G$) require $\mathscr{P}_G$–pseudomaps despite the restriction to strict $\mathscr{P}_G$–algebras as objects. We also have the multicategory **Mult**($G\mathscr{S}$) associated to the symmetric monoidal category of orthogonal G–spectra under the smash product.

[16] For $A = *$, Carlsson [1992, page 6] mentions a space-level version of the BPQ theorem. Shimakawa [1989, page 242] states and gives a sketch proof of a G–spectrum-level version.

[17] We understand a topological G–category to mean an internal category in the category of G–spaces.

Theorem 2.8 [Guillou et al. 2023] *$\mathbb{K}_G$ extends to a multifunctor*

$$\mathbb{K}_G\colon \mathbf{Mult}(\mathscr{P}_G) \to \mathbf{Mult}(G\mathscr{S}).$$

Remark 2.9 At one place in the duality proof of Section 2.3 below, we use from [Guillou et al. 2023, Proposition 9.24] that $\mathbb{K}_G$ converts 2–cells, such as ϑ in the proof of Proposition 1.43, to homotopies between maps of G–spectra.

Remark 2.10 In the proof of Theorem 2.6, we will use the fact that $\mathbb{K}_G$ takes values in positive Ω–G–spectra [Guillou et al. 2023].

Corollary 2.11 *The construction $\mathscr{A}_G$ given in Definition 2.2 defines a $G\mathscr{S}$–category.*

Proof It is shown in [Guillou et al. 2023, Section 3.5] that the pairing ω of Definition 1.35 is a bilinear morphism in $\mathbf{Mult}(\mathscr{P}_G)$. Moreover, the functors $(\mathrm{id}\times\Delta\times\mathrm{id})^*$ and $\pi_!$ of (1.30) are maps of $\mathscr{P}_G$–algebras. It follows that the composition $\mathscr{E}_G(B,C)\times\mathscr{E}_G(A,B)\xrightarrow{\circ}\mathscr{E}_G(A,C)$ is also bilinear. This remains true after applying the whiskering construction of Appendix B. Therefore, the multifunctor $\mathbb{K}_G$ produces a map of G–spectra $\mathscr{A}_G(B,C)\wedge\mathscr{A}_G(A,B)\to\mathscr{A}_G(A,C)$, as desired. The fact that the composition in $\mathscr{E}'_G$ is strictly associative and unital ensures that the same is true in $\mathscr{A}_G$. □

Theorem 2.8 yields another important consequence. Observe that the pairing ω of Definition 1.35 generalizes from G–sets A and B to G–spaces X and Y, giving a natural pairing

$$\omega\colon \mathbb{P}_G(X_+)\wedge\mathbb{P}_G(Y_+)\to\mathbb{P}_G(X_+\wedge Y_+).$$

Then Theorem 2.8 produces a map of G–spectra

$$\wedge\colon \mathbb{K}_G\mathbb{P}_G(X_+)\wedge\mathbb{K}_G\mathbb{P}_G(Y_+)\to\mathbb{K}_G\mathbb{P}_G(X_+\wedge Y_+).$$

This makes the assignment $X\mapsto\mathbb{K}_G\mathbb{P}_G(X_+)$ into a lax monoidal functor from (unbased) G–spaces to orthogonal G–spectra.

With this multiplicative machine in hand, it now makes sense to ask for a BPQ comparison that is also compatible with the multiplicative structure. That is another main result of [Guillou et al. 2023]. Recall that the assignment $X\mapsto\Sigma^\infty_G X_+$ is a strong monoidal functor from (unbased) G–spaces to orthogonal G–spectra.

Theorem 2.12 [Guillou et al. 2023] *There is a monoidal natural transformation*

$$\alpha\colon \Sigma^\infty_G(X_+)\to\mathbb{K}_G\mathbb{P}_G(X_+)$$

of functors from (unbased) G–spaces to orthogonal G–spectra, which restricts to a natural stable equivalence on the subcategory of G–CW complexes.

For the remainder of this section, we restrict our attention to the case when X is a finite G–set A, although we will return to the generality of G–spaces X in Section 2.5. We therefore use the identification (1.24) to rewrite $\mathbb{P}_G(A_+)$ as $\mathscr{E}_G(A)$.

That the transformation of Theorem 2.12 is monoidal means that we have a commutative diagram

(2.13)
$$\begin{tikzcd}
\Sigma^\infty_G(A_+) \wedge \Sigma^\infty_G(B_+) \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\wedge"', "\cong"] & \mathbb{K}_G\mathscr{E}_G(A) \wedge \mathbb{K}_G\mathscr{E}_G(B) \arrow[d, "\wedge"] \\
\Sigma^\infty_G(A\times B)_+ \arrow[r, "\alpha"] & \mathbb{K}_G\mathscr{E}_G(A\times B)
\end{tikzcd}$$

We restate the naturality of α with respect to G–maps $f: A \to B$ in the diagram

(2.14)
$$\begin{tikzcd}
\Sigma^\infty_G(A_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G f_+"'] & \mathbb{K}_G\mathscr{E}_G(A) \arrow[d, "\mathbb{K}_G f_!"] \\
\Sigma^\infty_G(B_+) \arrow[r, "\alpha"] & \mathbb{K}_G\mathscr{E}_G(B)
\end{tikzcd}$$

If $i: A \to B$ is an injection with retraction $r: B_+ \to A_+$, we have the induced map of G–spectra

$$\mathbb{K}_G i^* = \mathbb{K}_G r_!: \mathbb{K}_G\mathscr{E}_G(B) \to \mathbb{K}_G\mathscr{E}_G(A),$$

and (2.14) specializes to

(2.15)
$$\begin{tikzcd}
\Sigma^\infty_G(B_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G r"'] & \mathbb{K}_G\mathscr{E}_G(B) \arrow[d, "\mathbb{K}_G i^*"] \\
\Sigma^\infty_G(A_+) \arrow[r, "\alpha"] & \mathbb{K}_G\mathscr{E}_G(A)
\end{tikzcd}$$

By Remark 2.21 below, we may identify $\mathbb{K}_G i^*$ as the dual of $\mathbb{K}_G i$ and thus $\Sigma^\infty_G r$ as the dual of $\Sigma^\infty_G i_+$. We combine these diagrams to construct those that we need to prove Theorem 2.6. Let A, B and C be finite G–sets and recall Definition 1.29.

Proposition 2.16 *The following diagram of G–spectra commutes*:

(2.17)
$$\begin{tikzcd}
\Sigma^\infty_G(C\times B)_+ \wedge \Sigma^\infty_G(B\times A)_+ \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\wedge"', "\cong"] & \mathbb{K}_G\mathscr{E}_G(C\times B) \wedge \mathbb{K}_G\mathscr{E}_G(B\times A) \arrow[d, "\wedge"] \\
\Sigma^\infty(C\times B\times B\times A)_+ \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G r"'] & \mathbb{K}_G\mathscr{E}_G(C\times B\times B\times A) \arrow[d, "\mathbb{K}_G(\mathrm{id}\times\Delta\times\mathrm{id})^*"] \\
\Sigma^\infty(C\times B\times A)_+ \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty\pi"'] & \mathbb{K}_G\mathscr{E}_G(C\times B\times A) \arrow[d, "\mathbb{K}_G\pi_!"] \\
\Sigma^\infty_G(C\times A)_+ \arrow[r, "\alpha"] & \mathbb{K}_G\mathscr{E}_G(C\times A)
\end{tikzcd}$$

Here r is the retraction which sends the complement of the image of $\mathrm{id}\times\Delta\times\mathrm{id}$ *to the basepoint.*

The diagram (2.17) relates the composition pairing of the $G\mathscr{S}$-category $\mathscr{A}_G$ to remarkably simple and explicit maps between suspension G–spectra. In fact, recalling Definition 1.41 and again writing $\varepsilon = \Sigma^\infty_G\varepsilon$, we see that the left vertical composite in (2.17) can be identified with $\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}$. We have proven the following result, where we abuse notation by writing α for the composite

$$\Sigma^\infty_G(B\times A)_+ \to \mathbb{K}_G\mathscr{E}_G(B\times A) \to \mathbb{K}_G\mathscr{E}'_G(B\times A).$$

Theorem 2.18 *The following diagram of G–spectra commutes in* $\mathrm{Ho}\,G\mathscr{S}$:

$$\begin{tikzcd}
\Sigma^\infty_G(C\times B)_+\wedge\Sigma^\infty_G(B\times A)_+ \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\cong"'] & \mathscr{A}_G(B,C)\wedge\mathscr{A}_G(A,B) \arrow[ddd, "\circ"] \\
\Sigma^\infty_G(C_+)\wedge\Sigma^\infty_G(B\times B)_+\wedge\Sigma^\infty_G(A_+) \arrow[d, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"'] & \\
\Sigma^\infty_G(C_+)\wedge S_G\wedge\Sigma^\infty_G(A_+) \arrow[d, "\cong"'] & \\
\Sigma^\infty_G(C\times A)_+ \arrow[r, "\alpha"] & \mathscr{A}_G(A,C)
\end{tikzcd}$$

2.3 The self-duality of $\Sigma^\infty_G(A_+)$

Let A be a finite G–set and write $\mathbb{A}=\Sigma^\infty_G(A_+)$ for brevity of notation. As recalled in Section 1.5, in order to show that $\mathbb{A}$ is self-dual in $\mathrm{Ho}\,G\mathscr{S}$, we must define maps $\eta\colon S_G\to\mathbb{A}\wedge\mathbb{A}$ and $\varepsilon\colon\mathbb{A}\wedge\mathbb{A}\to S_G$ in the stable homotopy category $\mathrm{Ho}\,G\mathscr{S}$ such that the composites

(2.19)
$$\mathbb{A}\xrightarrow{\eta\wedge\mathrm{id}}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\xrightarrow{\mathrm{id}\wedge\varepsilon}\mathbb{A}\quad\text{and}\quad\mathbb{A}\xrightarrow{\mathrm{id}\wedge\eta}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\xrightarrow{\varepsilon\wedge\mathrm{id}}\mathbb{A}$$

are the identity map in $\mathrm{Ho}\,G\mathscr{S}$. Using the stable equivalence α and the definitions of η and ε from Definitions 1.41 and 1.42, we let η and ε be the composites

$$S_G\xrightarrow{\alpha}\mathbb{K}_G\mathscr{E}_G(1)\xrightarrow{\mathbb{K}_G\eta}\mathbb{K}_G\mathscr{E}_G(A\times A)\xrightarrow{\alpha^{-1}}\Sigma^\infty_G(A\times A)_+\cong\mathbb{A}\wedge\mathbb{A}$$

and

$$\mathbb{A}\wedge\mathbb{A}\cong\Sigma^\infty_G(A\times A)_+\xrightarrow{\alpha}\mathbb{K}_G\mathscr{E}_G(A\times A)\xrightarrow{\mathbb{K}_G\varepsilon}\mathbb{K}_G\mathscr{E}_G(1)\xrightarrow{\alpha^{-1}}S_G.$$

Abbreviate notation by setting $\mathscr{A}_G=\mathbb{K}_G\mathscr{E}_G$. The commutative diagram

$$\begin{tikzcd}
\mathscr{A}_G(A^2)\wedge\mathbb{A} \arrow[dr, "\mathrm{id}\wedge\alpha"] & & (\mathbb{A}^2)\wedge\mathbb{A}\cong\mathbb{A}^3\cong\mathbb{A}\wedge(\mathbb{A}^2) \arrow[ll, "\alpha\wedge\mathrm{id}"'] \arrow[rr, "\mathrm{id}\wedge\alpha"] \arrow[d, "\alpha"] & & \mathbb{A}\wedge\mathscr{A}_G(A^2) \arrow[dl, "\alpha\wedge\mathrm{id}"'] \arrow[dd, "\mathrm{id}\wedge\mathbb{K}_G\epsilon"] \\
& \mathscr{A}_G(A^2)\wedge\mathscr{A}_GA \arrow[r, "\wedge"] & \mathscr{A}_G(A^3) \arrow[dd, "\mathbb{K}_G(\mathrm{id}\times\epsilon)"] & \mathscr{A}_GA\wedge\mathscr{A}_G(A^2) \arrow[l, "\wedge"'] \arrow[dd, "\mathrm{id}\wedge\mathbb{K}_G\epsilon"'] & \\
\mathscr{A}_G1\wedge\mathbb{A} \arrow[uu, "\mathbb{K}_G\eta\wedge\mathrm{id}"] \arrow[ur, "\mathbb{K}_G\eta\wedge\alpha"'] & & & & \mathbb{A}\wedge\mathscr{A}_G1 \arrow[dl, "\alpha\wedge\mathrm{id}"'] \\
& \mathscr{A}_G1\wedge\mathscr{A}_GA \arrow[uu, "\mathbb{K}_G\eta\wedge\mathrm{id}"'] \arrow[r, "\wedge"] & \mathscr{A}_GA & \mathscr{A}_GA\wedge\mathscr{A}_G1 \arrow[l, "\wedge"'] & \\
S_G\wedge\mathbb{A} \arrow[uu, "\alpha\wedge\mathrm{id}"] \arrow[ur, "\alpha\wedge\alpha"'] \arrow[rr, "\cong"'] & & \mathbb{A} \arrow[u, "\alpha"'] & & \mathbb{A}\wedge S_G \arrow[ll, "\cong"] \arrow[ul, "\alpha\wedge\alpha"] \arrow[uu, "\mathrm{id}\wedge\alpha"']
\end{tikzcd}$$

proves that the first composite in (2.19) is the identity map in $\mathrm{Ho}\, G\mathscr{S}$; the second is dealt with similarly. Remembering that $\mathscr{E}_G(A) = \mathbb{P}_G(A_+)$, the center two squares are derived by use of the diagrams in Proposition 1.43.

Given Theorem 2.12, it is trivial that the outer parts of the diagram commute. The right central diagram is just a naturality diagram, as in Proposition 1.43. The left central diagram commutes up to homotopy by that result and Remark 2.9.

Specializing general observations about duality recalled in Section 1.5, we have the following corollary. This homotopical input is the crux of the proof of Theorem 2.6.

Corollary 2.20 *For finite G–sets A and B, the canonical map*

$$\delta = \zeta \circ (\mathrm{id} \wedge \tilde{\varepsilon}) \colon \mathbb{B} \wedge \mathbb{A} \to \mathbb{B} \wedge D\mathbb{A} \to F_G(\mathbb{A}, \mathbb{B})$$

of (1.40) *is a stable equivalence.*

We insert a mild digression concerning the identification of some of our maps.

Remark 2.21 For an injection $i \colon A \to B$ of finite G–sets, the composite (1.38) and the precise constructions of η and ε starting from Definitions 1.41 and 1.42 imply that the dual of i is the map $\mathbb{B} \to \mathbb{A}$ induced by the evident retraction $r \colon B_+ \to A_+$. A G–map $\pi \colon G/H \to G/K$ is a bundle, and the dual of $\Sigma^\infty \pi_+$ is the associated transfer map (see eg [Lewis and May 1986c, pages 182 and 192]). It can be identified explicitly by a similar (but not especially illuminating) inspection of definitions.

2.4 The proof that $\mathscr{A}_G$ is equivalent to $\mathscr{D}_{\mathrm{All}}$

We will have to chase large diagrams, and we again abbreviate notation by writing

$$\mathbb{A} = \Sigma^\infty_G(A_+), \quad \mathbb{B} = \Sigma^\infty_G(B_+) \quad \text{and} \quad \mathbb{C} = \Sigma^\infty_G(C_+)$$

for finite G–sets A, B and C. We also abbreviate notation by writing

$$\mathscr{A}_G(A) = \mathscr{A}_G(*, A).$$

This is the G–spectrum $\mathscr{A}_G(A) = \mathbb{K}_G \mathscr{E}_G(A)$, which is equivalent to $\mathbb{A}$ by Theorem 2.12. Remember that we are free to choose any bifibrant equivalents of the G–spectra $\mathbb{A}$ as the objects of $\mathscr{D}_{\mathrm{All}}$.

Proof of Theorem 2.6 We use model-categorical arguments, and we work with the stable model structure on $G\mathscr{S}$. We use [Guillou and May 2020, Section 2.4] to obtain a model structure on the category $G\mathscr{S}\mathbb{O}$–$\mathscr{C}at$ of $G\mathscr{S}$–categories with the same object set $\mathbb{O}$ as $G\mathscr{E}$. We emphasize that this is a model structure on a category of categories. Maps are weak equivalences or fibrations if they induce weak equivalences or fibrations on hom objects in $G\mathscr{S}$. Here the nature of the objects is irrelevant; we are concerned with $G\mathscr{S}$–categories with one object for each finite G–set A.

Let $\lambda \colon Q\mathscr{A}_G \to \mathscr{A}_G$ be a cofibrant approximation of $\mathscr{A}_G$. By [Guillou and May 2020, Theorem 2.16], since S_G is cofibrant in the stable model structure each morphism G–spectrum $Q\mathscr{A}_G(A, B)$ is cofibrant in $G\mathscr{S}$. The maps $\lambda\colon Q\mathscr{A}_G(A,B)\to\mathscr{A}_G(A,B)$ are stable acyclic fibrations. Digressively, since the $\mathscr{A}_G(A,B)$ are fibrant in the positive stable model structure (see Remark 2.10), that is also true of the $Q\mathscr{A}_G(A,B)$; we will use this fact in Section 2.5.

Let $\rho\colon Q\mathscr{A}_G\to RQ\mathscr{A}_G$ be a fibrant approximation of $Q\mathscr{A}_G$. The morphism G–spectra $RQ\mathscr{A}_G(A,B)$ are then bifibrant in the stable model structure. Therefore, $RQ\mathscr{A}_G(A)$ is bifibrant for each A, and it is stably equivalent to $\mathbb{A}$. We take the $RQ\mathscr{A}_G(A)$ as the bifibrant approximations of the $\mathbb{A}$ that we use to define the full $G\mathscr{S}$–subcategory $\mathscr{D}_{\mathrm{All}}$ of $G\mathscr{S}$.

We now have a zigzag

$$\mathscr{A}_G \xleftarrow[\sim]{\lambda} Q\mathscr{A}_G \xrightarrow[\sim]{\rho} RQ\mathscr{A}_G$$

of stable equivalences of $G\mathscr{S}$–categories. It remains to find a stable equivalence $RQ\mathscr{A}_G\to\mathscr{D}_{\mathrm{All}}$. To abbreviate notation, let us write $RQ\mathscr{A}_G(*,A)=RQ\mathscr{A}_G A$, and let

$$\gamma\colon RQ\mathscr{A}_G(A,B)\to\mathscr{D}_{\mathrm{All}}(A,B)=F_G(RQ\mathscr{A}_G A,RQ\mathscr{A}_G B)$$

be the adjoint of the composition map

$$\circ\colon RQ\mathscr{A}_G(A,B)\wedge RQ\mathscr{A}_G A\to RQ\mathscr{A}_G B.$$

By [Guillou and May 2020, Construction 5.6], this defines a $G\mathscr{S}$–functor

$$\gamma\colon RQ\mathscr{A}_G\to\mathscr{D}_{\mathrm{All}}.$$

It suffices to prove that each of the maps γ is a stable equivalence.

We define $\mathfrak{Q}_G$ to be the full $G\mathscr{S}$–subcategory of $\mathscr{S}_G$ with objects the $Q\mathscr{A}_G(A)$. It will play a role in our proof that γ is a stable equivalence. To abbreviate notation, we agree to write $Q\mathscr{A}_G(*,A)=Q\mathscr{A}_G A$. For finite G–sets A and B, let

$$\beta\colon Q\mathscr{A}_G(A,B)\to\mathfrak{Q}_G(A,B)=F_G(Q\mathscr{A}_G A,Q\mathscr{A}_G B)$$

be the adjoint of the composition map

$$\circ\colon Q\mathscr{A}_G(A,B)\wedge Q\mathscr{A}_G A\to Q\mathscr{A}_G B.$$

This defines a $G\mathscr{S}$–functor

$$\beta\colon Q\mathscr{A}_G\to\mathfrak{Q}_G.$$

For each finite G–set A, $\mathbb{A}$ is cofibrant and $\lambda\colon Q\mathscr{A}_G A\to\mathscr{A}_G A$ is an acyclic fibration in the stable model structure on $G\mathscr{S}$. Therefore, there is a map $\mu\colon\mathbb{A}\to Q\mathscr{A}_G A$ such that the diagram

$$\begin{tikzcd} & Q\mathscr{A}_G A \arrow[d, "\lambda"] \\ \mathbb{A} \arrow[ur, "\mu"] \arrow[r, "\alpha"'] & \mathscr{A}_G A \end{tikzcd}$$

commutes. Since α and λ are stable equivalences, so is μ. In the same way, we get a stable equivalence $\mu\colon \mathbb{B}\wedge\mathbb{A}\to Q\mathscr{A}_G(A,B)$.

For the remainder of the proof, we work in the homotopy category $\mathrm{Ho}\,G\mathscr{S}$. In particular, the distinction between $\mathbb{K}_G\mathscr{E}_G$ and $\mathbb{K}_G\mathscr{E}'_G$ vanishes. We claim that the following diagram of G–spectra commutes in $\mathrm{Ho}\,G\mathscr{S}$:

$$
\begin{tikzcd}
RQ\mathscr{A}_G(A,B) \arrow[r, "\gamma"] & F_G(RQ\mathscr{A}_G A, RQ\mathscr{A}_G B) \arrow[r, "F_G(\rho,\mathrm{id})", "\simeq"'] & F_G(Q\mathscr{A}_G A, RQ\mathscr{A}_G B) \arrow[d, "F_G(\mu,\mathrm{id})", "\simeq"'] \\
Q\mathscr{A}_G(A,B) \arrow[u, "\rho", "\simeq"'] \arrow[r, "\beta"] & F_G(Q\mathscr{A}_G A, Q\mathscr{A}_G B) \arrow[ur, "F_G(\mathrm{id},\rho)"'] \arrow[dr, "F_G(\mu,\mathrm{id})"] & F_G(\mathbb{A}, RQ\mathscr{A}_G B) \\
\mathbb{B}\wedge\mathbb{A} \arrow[u, "\mu", "\simeq"'] \arrow[r, "\delta", "\simeq"'] & F_G(\mathbb{A},\mathbb{B}) \arrow[r, "F_G(\mathrm{id},\mu)", "\simeq"'] & F_G(\mathbb{A}, Q\mathscr{A}_G B) \arrow[u, "F_G(\mathrm{id},\rho)"', "\simeq"]
\end{tikzcd}
$$

Indeed, modulo inversion of maps which are stable equivalences, it commutes on the nose. As before, we identify $\mathbb{B}\wedge\mathbb{A}=\Sigma^\infty_G B_+\wedge\Sigma^\infty_G A_+$ with $\Sigma^\infty_G(B\times A)_+$. The map δ is the stable equivalence of Corollary 2.20. The maps μ and ρ are also stable equivalences. The maps $F_G(\rho,\mathrm{id})$ and $F_G(\mu,\mathrm{id})$ that are labeled $\simeq$ are stable equivalences by [Guillou and May 2020, Lemma 1.22] since ρ and μ are maps between cofibrant objects and $RQ\mathscr{A}_G B$ is fibrant. The maps $F_G(\mathrm{id},\mu)$ and $F_G(\mathrm{id},\rho)$ that are labeled $\simeq$ are stable equivalences by [Mandell and May 2002, Proposition III.3.9], which shows that the functor $F_G(\mathbb{A},-)$ preserves stable equivalences. Provided that the diagram commutes, it follows that γ is a stable equivalence since all of the other outer arrows of the diagram are stable equivalences.

The top pentagon commutes since ρ is a map of $G\mathscr{S}$-categories, and both composites on the right give $F_G(\mu,\rho)$. It therefore remains to consider the lower pentagon. To prove that the diagram commutes in $\mathrm{Ho}\,G\mathscr{S}$, we consider its adjoint, which is displayed as the outer rectangle of the diagram

$$
\begin{tikzcd}
Q\mathscr{A}_G(A,B)\wedge Q\mathscr{A}_G A \arrow[rrr, "\circ"] \arrow[dr, "\lambda\wedge\lambda"] & & & Q\mathscr{A}_G B \arrow[dl, "\lambda"'] \\
& \mathscr{A}_G(A,B)\wedge\mathscr{A}_G A \arrow[r, "\circ"] & \mathscr{A}_G B & \\
\mathbb{B}\wedge\mathbb{A}\wedge\mathbb{A} \arrow[uu, "\mu\wedge\mu"] \arrow[ur, "\alpha\wedge\alpha"] \arrow[rrr, "\mathrm{id}\wedge\Sigma^\infty_G\varepsilon"] & & & \mathbb{B} \arrow[ul, "\alpha"] \arrow[uu, "\mu"']
\end{tikzcd}
$$

Here we have inserted the map $\circ\colon \mathscr{A}_G(A,B)\wedge\mathscr{A}_G A\to\mathscr{A}_G B$ and arrows λ into its source and target for purposes of the proof.

Since λ is a map of $G\mathscr{S}$–categories, it is apparent that all parts of the diagram commute except for the bottom trapezoid. Taking $(A,B,C)=(*,A,B)$ in Theorem 2.18, we see that the trapezoid commutes.

Since the wrong-way map λ is a stable equivalence and can be inverted upon passage to the homotopy category, this diagram and its adjoint commute there. $\square$

2.5 The identification of suspension G–spectra

We expand the adjoint $\mathscr{S}$–equivalences in Theorem 1.14 more explicitly as follows, using [Guillou and May 2020, Proposition 2.4]:

(2.22)
$$\begin{tikzcd} & G\mathscr{S} \arrow[r, shift left, "\mathbb{T}"] & \mathbf{Pre}(G\mathscr{D}_{\mathrm{All}},\mathscr{S}) \arrow[l, shift left, "\mathbb{U}"] \arrow[r, leftarrow, shift left, "\gamma_!"] \arrow[r, shift right, "\gamma^*"'] & \mathbf{Pre}((RQ\mathscr{A}_G)^G,\mathscr{S}) \arrow[d, shift right, "\rho^*"'] \\ \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \arrow[r, shift left, "\iota_!"] & \mathbf{Pre}((\mathscr{A}_G)^G,\mathscr{S}) \arrow[l, shift left, "\iota^*"] \arrow[r, leftarrow, shift left, "\lambda_!"] \arrow[r, shift right, "\lambda^*"'] & & \mathbf{Pre}((Q\mathscr{A}_G)^G,\mathscr{S}) \arrow[u, shift right, "\rho_!"'] \end{tikzcd}$$

The map $\iota\colon G\mathscr{A} \to (\mathscr{A}_G)^G$ is the equivalence of Theorem 2.3. Before passage to G–fixed points, the proof in Section 2.4 gives stable equivalences of $G\mathscr{S}$–categories

$$\rho\colon Q\mathscr{A}_G \to RQ\mathscr{A}_G, \quad \gamma\colon RQ\mathscr{A}_G \to \mathscr{D}_{\mathrm{All}} \quad \text{and} \quad \lambda\colon Q\mathscr{A}_G \to \mathscr{A}_G.$$

These maps give stable equivalences of $\mathscr{S}$–categories after passage to fixed points. Seeing this uses that the hom G–spectra in $RQ\mathscr{A}_G$ and $\mathscr{D}_{\mathrm{All}}$ are fibrant, while those in $Q\mathscr{A}_G$ and $\mathscr{A}_G$ are positive fibrant, as discussed in the proof of Theorem 2.6.

For a finite G–set B, $\Sigma^\infty_G B_+$ corresponds under this zigzag to the presheaf $\boldsymbol{B}$ that sends A to $G\mathscr{A}(A,B)$. This is almost a tautology since, for $E \in G\mathscr{S}$, $\mathbb{U}(E)$ is the presheaf represented by E, while $G\mathscr{E}(-,B)$ is the functor represented by B. In the proof of Theorem 2.6, we chose the bifibrant approximation of $\Sigma^\infty_G B_+$ in $\mathscr{D}_{\mathrm{All}}$ to be $RQ\mathscr{A}_G(B)$. With B fixed, that proof shows that γ gives an equivalence of presheaves

$$RQ\mathscr{A}_G(-,B) \to \gamma^*\mathbb{U}\,RQ\mathscr{A}_G(B)$$

(before passage to G–fixed points). The functors ρ^* and $\lambda_!$ and the isomorphism ι^* preserve representable functors, and therefore $\iota^*\lambda_!\rho^* RQ\mathscr{A}_G(-,B) \simeq \mathbb{K}_G\mathscr{E}_G(-,B)$.

This observation can be generalized from finite based G–sets B_+ to arbitrary based G–spaces X. To see this, we mix general based G–spaces X with finite based G–sets A_+ to obtain a functorial construction of a presheaf $\mathbf{Pr}_G(X)$.

Definition 2.23 For a based G–space X, define a presheaf $\mathbf{Pr}_G(X)\colon (\mathscr{A}_G)^{\mathrm{op}} \to \mathscr{S}_G$ by letting

$$\mathbf{Pr}_G(X)(A) = \mathbb{K}_G\mathbb{P}_G(X \wedge A_+).$$

The contravariant functoriality map

$$\mathbf{Pr}_G(X)\colon \mathscr{A}_G(A,B) \to F_G(\mathbf{Pr}_G(X)(B), \mathbf{Pr}_G(X)(A))$$

is the composite of the retraction $\mathscr{A}_G(A,B) = \mathbb{K}_G\mathscr{E}'_G(A,B) \to \mathbb{K}_G(\mathscr{E}_G(B\times A))$ (see Definition B.2) with the adjoint of the right vertical composite in the commutative diagram

$$\text{(2.24)}\qquad \begin{tikzcd}
\Sigma^\infty_G(X\wedge B_+)\wedge\Sigma^\infty_G(B_+\wedge A_+) \arrow[r, "\alpha\wedge\alpha"] \arrow[d, "\wedge"', "\cong"] & \mathbb{K}_G\mathbb{P}_G(X\wedge B_+)\wedge\mathbb{K}_G\mathbb{P}_G(B_+\wedge A_+) \arrow[d, "\wedge"] \\
\Sigma^\infty(X\wedge B_+\wedge B_+\wedge A_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty_G r"'] & \mathbb{K}_G\mathbb{P}_G(X\wedge B_+\wedge B_+\wedge A_+) \arrow[d, "\mathbb{K}_G\mathbb{P}_G(r)"] \\
\Sigma^\infty(X\wedge B_+\wedge A_+) \arrow[r, "\alpha"] \arrow[d, "\Sigma^\infty\pi"'] & \mathbb{K}_G\mathbb{P}_G(X\wedge B_+\wedge A_+) \arrow[d, "\mathbb{K}_G\mathbb{P}_G\pi"] \\
\Sigma^\infty_G(X\wedge A_+) \arrow[r, "\alpha"] & \mathbb{K}_G\mathbb{P}_G(X\wedge A_+)
\end{tikzcd}$$

Here r is the retraction of based G–sets associated to the diagonal inclusion and π is the projection. The diagram commutes by the same concatenation of commutative diagrams as in Proposition 2.16. Note that there is no need to whisker the G–categories $\mathbb{P}_G(X\wedge A_+)$ in order to get a strict functor. The spans in $\mathbb{P}_G(X\wedge A_+)$ are only composed on the right with spans in $\mathscr{A}_G$ in this construction, and the Δ_B were already strict units on the right. Therefore, use of the retraction does not destroy functoriality.

Theorem 2.25 *Let X be a based G–space. Under our zigzag of equivalences, $\Sigma^\infty_G X$ corresponds naturally to the presheaf $(\mathbf{Pr}_G(X))^G$ that sends A to $\mathbb{K}(\mathbb{P}_G(X\wedge A_+)^G)$.*

Proof Note that $\mathbb{K}_G\mathbb{P}_G(X\wedge -_+)$ is no longer a representable presheaf. We again work with G–spectra and obtain the conclusion after passage to G–fixed spectra. According to Theorem 2.12, we may replace $\Sigma^\infty_G X$ by the positive fibrant G–spectrum $\mathbb{K}_G\mathbb{P}_G(X)$, which we abbreviate to $\mathscr{A}_G(X)$ by a slight abuse of notation. After this replacement, the presheaf $\mathbb{U}(\Sigma^\infty_G X)$ may be computed as

$$\mathbb{U}(\Sigma^\infty_G X)(A) = F_G(RQ\mathscr{A}_G(A), \mathscr{A}_G(X)).$$

Therefore, following the chain of (2.22), we may compute $\rho^*\gamma^*\mathbb{U}(\Sigma^\infty_G X)$ as

$$\rho^*\gamma^*\mathbb{U}(\Sigma^\infty_G X) \simeq F_G(Q\mathscr{A}_G(-), \mathscr{A}_G(X)).$$

Replacing (B,A) by $(A,1)$ in (2.24) and recalling that $1_+ = S^0$, the right column gives the second map in the composite

$$\text{(2.26)}\qquad \mathbf{Pr}_G(X)(A)\wedge Q\mathscr{A}_G(A) \xrightarrow{\mathrm{id}\wedge\lambda} \mathbf{Pr}_G(X)(A)\wedge\mathscr{A}_G(A) \xrightarrow{\circ} \mathbf{Pr}_G(X)(1).$$

Its target is the G–spectrum $\mathscr{A}_G(X)$, and its adjoint gives a map of presheaves

$$\text{(2.27)}\qquad \lambda^*\mathbf{Pr}_G(X) \to F_G(Q\mathscr{A}_G(-), \mathscr{A}_G(X))$$

with domain $Q\mathscr{A}_G$. It remains to show that this map is an equivalence. To compute the adjoint (2.27), observe that (2.26) is the top horizontal composite in the diagram

$$\begin{tikzcd}
\mathbf{Pr}_G(X)(A) \wedge Q\mathscr{A}_G(A) \arrow[r, "\mathrm{id}\wedge\lambda"] & \mathbf{Pr}_G(X)(A) \wedge \mathscr{A}_G(A) \arrow[r, "\circ"] & \mathbf{Pr}_G(X)(1) \\
\Sigma^\infty_G(X \wedge A_+) \wedge Q\mathscr{A}_G(A) \arrow[u, "\alpha\wedge\mathrm{id}"] & \mathbf{Pr}_G(X)(A) \wedge \Sigma^\infty_G A_+ \arrow[u, "\mathrm{id}\wedge\alpha"'] & \\
\Sigma^\infty_G(X \wedge A_+) \wedge \Sigma^\infty_G A_+ \arrow[u, "\mathrm{id}\wedge\mu"] \arrow[ur, "\alpha\wedge\mathrm{id}"'] \arrow[r, "\cong"'] & \Sigma^\infty_G X \wedge \Sigma^\infty_G(A_+ \wedge A_+) \arrow[r, "\mathrm{id}\wedge\varepsilon"'] & \Sigma^\infty_G X \arrow[uu, "\alpha"']
\end{tikzcd}$$

The left pentagon commutes since $\lambda \circ \mu = \alpha$ and the right pentagon is a special case of (2.24). Therefore, the map (2.27) is the top horizontal composite in the diagram

$$\begin{tikzcd}
\mathbf{Pr}_G(X)(A) \arrow[r] & F_G(\mathscr{A}_G(A), \mathscr{A}_G(X)) \arrow[r, "F_G(\lambda,\mathrm{id})"] & F_G(Q\mathscr{A}_G(A), \mathscr{A}_G(X)) \arrow[d, "F_G(\mu,\mathrm{id})"] \\
\Sigma^\infty_G(X \wedge A_+) \arrow[u, "\alpha"] \arrow[r, "\delta"'] & F_G(\Sigma^\infty_G A_+, \Sigma^\infty_G X) \arrow[r, "F_G(\mathrm{id},\alpha)"'] & F_G(\Sigma^\infty_G A_+, \mathscr{A}_G(X))
\end{tikzcd}$$

The map α is a stable equivalence by Theorem 2.12. The map δ is the stable equivalence of (1.40). The map $F_G(\mathrm{id}, \alpha)$ is a stable equivalence by [Mandell and May 2002, Proposition III.3.9]. Finally, the map $F_G(\mu, \mathrm{id})$ is a stable equivalence by [Guillou and May 2020, Lemma 1.22]. □

3 Some comparisons of functors

3.1 Change-of-groups and fixed-point functors

We discuss several constructions on G–spectra from the point of view of Theorem 1.14. Categorical fixed points are already built into the setup: for any subgroup $H \subset G$, the functor of H–fixed points is given by evaluating presheaves at the orbit G/H. We will return to this in Construction 3.5.

Construction 3.1 (restriction to subgroups) Let $H \subset G$ be a subgroup. Then induction of G–sets provides a strong monoidal (in other words, coproduct-preserving) bifunctor $G \times_H (-) \colon H\mathscr{E} \to G\mathscr{E}$. Using our models for $H\mathscr{E}$ and $G\mathscr{E}$, we must declare a preferred ordering for an induced G–set $G \times_H A$, given an ordering of the H–set A. For this, we choose an ordering of G/H as well as a set of coset representatives for H in G. The choice of coset representatives gives a bijection of sets $G \times_H A \cong G/H \times A$, and we use the lexicographic ordering of $G/H \times A$ to order the induced G–set $G \times_H A$.

This extends to a (strict) 2–functor $G \times_H - \colon H\mathscr{E}' \to G\mathscr{E}'$ if, recalling that the 1–cell $I_A \in H\mathscr{E}'(A, A)$ is the identity of A as in Definition B.1, we then define $G \times_H I_A = I_{G\times_H A}$ for all H–sets A. For finite H–sets A and B, there is a unique G–equivariant isomorphism $G \times_H (A \amalg B) \cong (G \times_H A) \amalg (G \times_H B)$, though it is not order-preserving in general. It follows that the induction functor gives rise to a spectral functor $\mathbb{K}(G \times_H -)\colon H\mathscr{A} \to G\mathscr{A}$. Then

$$\mathbb{K}(G \times_H -)^*\colon \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(H\mathscr{A},\mathscr{S})$$

gives a model for the restriction $G\mathscr{S} \to H\mathscr{S}$.

Construction 3.2 (induction) Let $H \subset G$ be a subgroup. The spectrum-level induction functor $G_+ \wedge_H -\colon H\mathscr{S} \to G\mathscr{S}$ is left adjoint to restriction. Given the description of restriction provided in Construction 3.1, it follows that induction can be described as the enriched Kan extension (as in [Guillou and May 2020, Lemma 2.2])

$$\mathbb{K}(G \times_H -)_!\colon \mathbf{Pre}(H\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(G\mathscr{A},\mathscr{S})$$

along the spectral functor $\mathbb{K}(G \times_H -)\colon H\mathscr{A} \to G\mathscr{A}$.

Construction 3.3 (geometric inflation along a quotient) Let $N \trianglelefteq G$ be a normal subgroup. Then passage to N–fixed points defines a functor $\mathrm{Fix}^N\colon G\mathscr{E} \to G/N\mathscr{E}$. Note that since $\mathrm{Fix}^N(A)$ is a subset of A, the G/N–set $\mathrm{Fix}^N(A)$ inherits an ordering from that of A. Moreover, Fix^N preserves pullbacks and coproducts. It follows that Fix^N gives rise to a spectral functor $\mathbb{K}(\mathrm{Fix}^N)\colon G\mathscr{A} \to G/N\mathscr{A}$. Then

$$\mathbb{K}(\mathrm{Fix}^N)^*\colon \mathbf{Pre}(G/N\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(G\mathscr{A},\mathscr{S})$$

gives a model for the geometric inflation functor, whose image consists of G–spectra "concentrated over N". In the language of [Mandell and May 2002, Section VI.5], this is the functor $X \mapsto \widetilde{E\mathscr{F}}[N] \wedge \varepsilon^\# X$, where $\varepsilon\colon G \to G/N$ is the quotient homomorphism and $\varepsilon^\#$ is left adjoint to the N–fixed-point functor from G–spectra to G/N–spectra.

Construction 3.4 (geometric fixed points) Let $N \trianglelefteq G$ be a normal subgroup. Then the geometric N–fixed-point functor is left adjoint to geometric inflation. Given the description of geometric inflation provided in Construction 3.3, the enriched Kan extension (as in [Guillou and May 2020, Lemma 2.2])

$$\mathbb{K}(\mathrm{Fix}^N)_!\colon \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(G/N\mathscr{A},\mathscr{S})$$

gives a model for the geometric N–fixed-point functor $\Phi^N\colon G\mathscr{S} \to G/N\mathscr{S}$.

This construction extends to arbitrary subgroups as follows. For a subgroup $H \subset G$, the H–fixed-point functor $\mathrm{Fix}^H\colon G\mathscr{E} \to \mathscr{E}$ gives rise to a spectral functor $\mathbb{K}(\mathrm{Fix}^H)\colon G\mathscr{A} \to \mathscr{A}$, and the enriched Kan extension

$$\mathbb{K}(\mathrm{Fix}^H)_!\colon \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(\mathscr{A},\mathscr{S})$$

gives a model for the geometric H–fixed-point functor $\Phi^H\colon G\mathscr{S} \to \mathscr{S}$. We leave it to the reader to verify that, in the case of a normal subgroup, the two versions agree after restricting from G/N–spectra to underlying spectra.

Construction 3.5 (categorical fixed points) There is an inclusion $\iota\colon \mathscr{E} \hookrightarrow G\mathscr{E}$ of the finite sets as the G–trivial finite G–sets. This functor preserves pullbacks and coproducts and therefore induces a spectral functor $\mathbb{K}(\iota)\colon \mathscr{A} \hookrightarrow G\mathscr{A}$. As generalized equivariantly in Remark A.4, spectrally enriched presheaves on finite sets are determined by their value at a one-point set, and

$$\mathbb{K}(\iota)^*\colon \mathbf{Pre}(G\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(\mathscr{A},\mathscr{S}) \simeq \mathscr{S}$$

gives a model for the (categorical) G–fixed-point functor $(-)^G\colon G\mathscr{S} \to \mathscr{S}$. For a subgroup $H \subset G$, the H–fixed-point functor is given by first using the restriction functor of Construction 3.1 and then passing to fixed points.

Construction 3.6 (G–trivial G–spectra) Left adjoint to the G–fixed-point functor is the trivial G–action functor. Given the description of G–fixed points provided in Construction 3.5, the enriched Kan extension (as in [Guillou and May 2020, Lemma 2.2])

$$\mathbb{K}(\iota)_!\colon \mathscr{S} \simeq \mathbf{Pre}(\mathscr{A},\mathscr{S}) \to \mathbf{Pre}(G\mathscr{A},\mathscr{S})$$

gives a model for the trivial G–spectrum functor $\varepsilon^\#\colon \mathscr{S} \to G\mathscr{S}$ (using the notation of [Mandell and May 2002, Section VI.3]). This functor describes the tensoring of G–spectra over nonequivariant spectra. We return to this in Section 3.3.

3.2 Fixed-point orbit functors

We return to Corollary 1.15 and give a more precise formulation. We know from Construction 3.5 how to pass to H–fixed points for each H, but a more functorial perspective may be illuminating. Again let $\mathscr{O}_G$ denote the orbit category of G. For a G–spectrum X, passage to H–fixed-point spectra for $H \subset G$ gives a functor $X^\bullet\colon \mathscr{O}_G^{\mathrm{op}} \to \mathscr{S}$. Recall Remark 2.5. By definition, $G\mathscr{D}_{\mathrm{Orb}}$ is the image of the composition j of $\Sigma^\infty_{G,+}\colon \mathscr{O}_G \to G\mathscr{S}$ with our bifibrant replacement functor. Pulling back along j defines a functor

$$G\mathscr{S} \xrightarrow{\mathbb{U}} \mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}},\mathscr{S}) \xrightarrow{j^*} \mathrm{Pre}(\mathscr{O}_G,\mathscr{S}),$$

where the target denotes ordinary (ie unenriched) presheaves. On the other hand, we have the functor $k\colon \mathscr{O}_G \to G\mathscr{E}$ that associates to a map of finite G–sets its graph, considered as a span. This gives rise to a functor $\mathscr{O}_G \to G\mathscr{A}$, which we also denote by k. Now pullback along k gives a functor

$$\mathbf{Pre}(G\mathscr{A},\mathscr{S}) \xrightarrow{k^*} \mathrm{Pre}(\mathscr{O}_G,\mathscr{S}).$$

Corollary 3.7 *The zigzag of equivalences of Theorem 1.14 identifies the composition $j^* \circ \mathbb{U}$ with k^* up to equivalence.*

3.3 Tensors with spectra and smash products

There is another visible identification. The category $G\mathscr{S}$ and our presheaf categories are $\mathscr{S}$–complete, so they have tensors and cotensors over $\mathscr{S}$ (see [Guillou and May 2020, Section 5.1]). It is formal that the left adjoint of an $\mathscr{S}$–adjunction preserves tensors and the right adjoint preserves cotensors. A quick chase of our zigzag of Quillen $\mathscr{S}$–equivalences gives the following conclusion:

Proposition 3.8 *For a G–spectrum Y and a spectrum X, if Y corresponds to a presheaf $\mathscr{P}Y$ under our zigzag of weak equivalences, then the tensor $Y \odot X$ corresponds to the tensor $\mathscr{P}Y \odot X$.*

Remark 3.9 (smash products) We have not described the behavior of smash products under the equivalences of Theorem 1.14. On the presheaf side, one would expect to use Day convolution to describe the smash product, starting from the cartesian product of finite G–sets. Indeed, this is the approach taken in [Clausen et al. 2020], where a symmetric monoidal version of Theorem 1.14 is given. We warn the reader, however, of two notable differences in their approach. First, in the approach of [Clausen et al. 2020], the functor from G–spectra to presheaves is a *left* adjoint, so their right adjoint plays the role of our $\mathbb{T}$ in Theorem 2.4. Secondly, they produce a monoidal functor on the category of G–spectra by using that the category of G–spectra can be obtained as a monoidal category from the category of based G–spaces by inverting smash products with representation spheres [Clausen et al. 2020, Theorem A.2].

Remark 3.10 We here give a sketch of an approach to a monoidal version of Theorem 1.14. Starting from an enriched symmetric monoidal structure on $G\mathscr{D}_{\mathrm{All}}$, Day convolution provides a symmetric monoidal structure on our category of spectral presheaves, and Theorem 2.3 can be promoted to a monoidal Quillen equivalence, as in [Arone et al. 2022, Theorem 4.3]. It then remains to equip the spectral category $G\mathscr{A}$ with an enriched monoidal structure and promote Theorem 2.6 to a zigzag of *monoidal* weak equivalences.

However, there are several difficulties with this approach. First, starting with the enriched monoidal structure on $G\mathscr{D}_{\mathrm{All}}$, it is clear what to do on objects, since they are in bijective correspondence with finite G–sets. Namely, again employing the notation of Section 2.4, the objects are of the form $R\mathbb{A} = R\Sigma^\infty_G A_+$, and we define a product $\otimes$ on $G\mathscr{D}_{\mathrm{All}}$ by letting $R\mathbb{A} \otimes R\mathbb{B}$ be $R(\mathbb{A} \wedge \mathbb{B}) \cong R\Sigma^\infty_G (A \times B)_+$.

We next require a map of spectra

(3.11) $$F(R\mathbb{A}, R\mathbb{B}) \wedge F(R\mathbb{C}, R\mathbb{D}) \to F(R\mathbb{A} \otimes R\mathbb{C}, R\mathbb{B} \otimes R\mathbb{D}).$$

If we had a strong monoidal fibrant replacement functor R, this would provide isomorphisms $R\mathbb{A} \wedge R\mathbb{B} \cong R(\mathbb{A} \wedge \mathbb{B}) = R\mathbb{A} \otimes R\mathbb{B}$. These could then be combined with the map

$$F(R\mathbb{A}, R\mathbb{B}) \wedge F(R\mathbb{C}, R\mathbb{D}) \to F(R\mathbb{A} \wedge R\mathbb{C}, R\mathbb{B} \wedge R\mathbb{D})$$

to obtain the map (3.11). However, absent such a strong monoidal functor R, we do not see a way to define (3.11). We shall say a bit more fibrant replacement in Section A.3. One way around this problem would be to rework the entire theory with orthogonal G–spectra replaced by the S_G–modules of the equivariant version [Mandell and May 2002] of Elmendorf, Kriz, Mandell and May [Elmendorf et al. 1997]. Since all S_G–modules are fibrant, that would get around this problem; some relevant details are discussed in Sections 4.1 and A.4.

Another problem is that it is not straightforward to equip $G\mathscr{A}$ with an enriched monoidal structure. Again, it is clear what to do on objects. The machine developed in [Guillou et al. 2023] does convert the product functors

$$G\mathscr{E}(B\times A)\times G\mathscr{E}(D\times C)\xrightarrow{\times} G\mathscr{E}(B\times A\times D\times C)\xrightarrow{\cong} G\mathscr{E}(B\times D\times A\times C) \tag{3.12}$$

of Remark 1.6 to morphisms of spectra

$$\mathbb{K}G\mathscr{E}(B\times A)\wedge\mathbb{K}G\mathscr{E}(D\times C)\to\mathbb{K}G\mathscr{E}(B\times D\times A\times C).$$

However, recall from Definition 1.13 that the morphism spectra of $G\mathscr{A}$ are defined using $G\mathscr{E}'$ rather than $G\mathscr{E}$, so some care is required to handle that change. A little more seriously, even if we ignore the difference between $G\mathscr{E}$ and $G\mathscr{E}'$, the functors (3.12) do not give a strict 2–functor $G\mathscr{E}'\times G\mathscr{E}'\xrightarrow{\times} G\mathscr{E}'$ since the evident diagram relating products to composition (of 1–cells) only commutes up to isomorphism. We have not pursued this idea further, but we do not believe that the difficulties to this approach are insurmountable.

4 Atiyah duality for finite G–sets

It is illuminating to see that we can come very close to constructing an alternative model for the spectrally enriched category $G\mathscr{D}_{\mathrm{All}}$ just by applying the suspension G–spectrum functor Σ^∞_G to the category of based finite G–sets and G–maps and then passing to G–fixed points. This is based on a close inspection of classical Atiyah duality specialized to finite G–sets. However, it depends on working in the alternative category $G\mathscr{Z}$ of S_G–modules [Elmendorf et al. 1997; Mandell and May 2002] rather than in the category $G\mathscr{S}$ of orthogonal G–spectra. Because every object of $G\mathscr{Z}$ is fibrant and its suspension G–spectra are easily understood, it is considerably more convenient than $G\mathscr{S}$ for comparison with space-level constructions. This leads us to a variant, Theorem 4.19, of Theorem 0.1 that does not invoke infinite loop space theory. It is more topological and less categorical, and it best captures the geometric intuition behind our results. It is also more elementary.

4.1 The categories $G\mathscr{Z}$, $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ and $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$

Relevant background about $G\mathscr{Z}$ appears in Section A.4, and we just give a minimum of notation here. We alert the reader to one nonstandard notation. We indicate the tensor of a based G–space X and a G–spectrum E by $X\odot E=\Sigma^\infty_G X\wedge E$. Similarly, we later denote the tensor of a nonequivariant spectrum D and a G–spectrum E by $D\odot E$.

In analogy with Theorem 2.4, we have the following specialization of the same general result [Guillou and May 2020, Theorem 1.36] about stable model categories. It is discussed in Section A.1.

Theorem 4.1 *Let $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ be the full $\mathscr{Z}$–subcategory of $G\mathscr{Z}$ whose objects are cofibrant approximations of the suspension G–spectra $\Sigma^\infty_G(A_+)$, where A runs through the finite G–sets. Then there is an enriched*

Quillen adjunction

$$\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}, \mathscr{Z}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{Z},$$

and it is a Quillen equivalence.

We must be explicit about cofibrant approximation here. The construction of the category $G\mathscr{Z}$ of S_G–modules starts from the Lewis–May category $G\mathscr{S}p$ of G–spectra, and S_G–modules are G–spectra with additional structure. We have an elementary suspension G–spectrum functor $\Sigma^\infty_G\colon G\mathscr{T} \to G\mathscr{S}p$. As we recall in Section A.4, a suspension G–spectrum has a canonical S_G–module structure, so we may view Σ^∞_G as a functor $G\mathscr{T} \to G\mathscr{Z}$. Moreover, with codomain $G\mathscr{Z}$, this becomes a strong symmetric monoidal functor. However, the $\Sigma^\infty_G X$ are not cofibrant. As explained in Section A.4 below, there is a left Quillen equivalence $\mathbb{F}\colon G\mathscr{S}p \to G\mathscr{Z}$ such that the composite $\boldsymbol{\Sigma}^\infty_G = \mathbb{F} \circ \Sigma^\infty_G$ takes based G–CW complexes X, such as A_+ for a finite G–set A, to cofibrant S_G–modules. Therefore, $\boldsymbol{\Sigma}^\infty_G$ may be viewed as a cofibrant replacement functor for Σ^∞_G. In particular, we write $\boldsymbol{S}_G = \boldsymbol{\Sigma}^\infty_G S^0$ and have a cofibrant approximation $\gamma\colon \boldsymbol{S}_G \to S_G$ of the unit object S_G. Moreover, the cofibrant approximation $\boldsymbol{\Sigma}^\infty_G(A_+)$ is isomorphic over $\Sigma^\infty_G(A_+)$ to $\boldsymbol{S}_G \wedge \Sigma^\infty_G(A_+)$.

As before, we consider finite G–sets A, B and C, but we now agree to write

$$\mathbb{A} = \boldsymbol{\Sigma}^\infty_G A_+, \quad \mathbb{B} = \boldsymbol{\Sigma}^\infty_G B_+ \quad \text{and} \quad \mathbb{C} = \boldsymbol{\Sigma}^\infty_G C_+.$$

These are bifibrant objects of $G\mathscr{Z}$ and we let $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ and $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ be the full subcategories of $G\mathscr{Z}$ and $\mathscr{Z}_G$ whose objects are the S_G–modules $\mathbb{A}$, where A runs over the finite G–sets. Then $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ is enriched in $G\mathscr{Z}$ and $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}} = (\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}})^G$ is enriched in the category $\mathscr{Z}$ of S–modules. The functor $\boldsymbol{\Sigma}^\infty_G$ is almost strong symmetric monoidal. Precisely, by Proposition A.10 below, there is a natural isomorphism

$$\mathbb{A} \wedge \mathbb{B} \cong \boldsymbol{S}_G \wedge \boldsymbol{\Sigma}^\infty_G (A \times B)_+ \tag{4.2}$$

with appropriate coherence properties with respect to associativity and commutativity. Since S_G is the unit for the smash product, we can compose with

$$\gamma \wedge \mathrm{id}\colon \boldsymbol{S}_G \wedge \boldsymbol{\Sigma}^\infty_G (A \times B)_+ \to \boldsymbol{\Sigma}^\infty_G (A \wedge B)_+$$

to give a pairing as if $\boldsymbol{\Sigma}^\infty_G$ were a lax symmetric monoidal functor. However, the map $\gamma\colon \boldsymbol{S}_G \to S_G$ points the wrong way for the unit map of such a functor.

4.2 Space-level Atiyah duality for finite G–sets

To lift the self-duality of $\mathrm{Ho}\,\mathscr{D}_{\mathrm{All}}$ to obtain a new model for $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$, we need representatives in $G\mathscr{Z}$ for the maps

$$\eta\colon S_G \to \mathbb{A} \wedge \mathbb{A} \quad \text{and} \quad \varepsilon\colon \mathbb{A} \wedge \mathbb{A} \to S_G$$

in $\mathrm{Ho}\,G\mathscr{Z}$ that express the duality there. The map ε is induced from the elementary map ε of Definition 1.41. The observation that it plays a key role in Atiyah duality seems to be new. The definition of η requires desuspension by representation spheres.

Let A be a finite G–set and let $V = \mathbb{R}[A]$ be the real representation generated by A, with its standard inner product, so that $|a| = 1$ for $a \in A$. Since we are working on the space level, we may view $A_+ \wedge S^V$ as the wedge over $a \in A$ of the spaces (not G–spaces) $\{a\}_+ \wedge S^V$, with G acting by $g(a, v) = (ga, gv)$. There is no such wedge decomposition after passage to G–spectra.

Definition 4.3 Recall that $\varepsilon\colon (A \times A)_+ \to S^0$ is the G–map defined by $\varepsilon(a,b) = *$ if $a \neq b$ and $\varepsilon(a,a) = 1$. Recall too that $(A \times B)_+$ can be identified with $A_+ \wedge B_+$ and that the functor $\mathbf{\Sigma}^\infty_G$ is almost strong symmetric monoidal. We shall also write ε for the composite map of S_G–modules

(4.4)
$$\mathbb{A} \wedge \mathbb{A} \cong \boldsymbol{S}_G \wedge \mathbf{\Sigma}^\infty_G (A \times A)_+ \xrightarrow{\mathrm{id} \wedge \mathbf{\Sigma}^\infty_G \varepsilon} \boldsymbol{S}_G \wedge \boldsymbol{S}_G \xrightarrow{\gamma \wedge \gamma} S_G \wedge S_G \cong S_G,$$

where the first unlabeled isomorphism is an instance of (4.2).

Definition 4.5 Embed A as the basis of the real representation $V = \mathbb{R}[A]$. The normal bundle of the embedding is just $A \times V$, and its Thom complex is $A_+ \wedge S^V$. We obtain an explicit tubular embedding $\nu\colon A \times V \to V$ by setting

$$\nu(a, v) = a + \frac{\rho(|v|)}{|v|} v,$$

where $\rho\colon [0, \infty) \to [0, d)$ is a homeomorphism for some $d < \frac{1}{2}$; ν is a G–map since $|gv| = |v|$ for all g and v. Applying the Pontryagin–Thom construction, we obtain a G–map $t\colon S^V \to A_+ \wedge S^V$, which is an equivariant pinch map

$$S^V \to \textstyle\bigvee_{a \in A} S^V \cong A_+ \wedge S^V.$$

To be more precise, after collapsing the complement of the tubular embedding to a point, we use ν^{-1} to expand each small homeomorphic copy of S^V to the canonical full-sized one; explicitly, if $|w| < d$, then

$$\nu^{-1}(a + w) = \left(a, \frac{\rho^{-1}(|w|)}{|w|} w\right).$$

The diagonal map on A induces the Thom diagonal $\Delta\colon A_+ \wedge S^V \to A_+ \wedge A_+ \wedge S^V$, and we let

(4.6)
$$\eta = \eta_A\colon S^V \to A_+ \wedge A_+ \wedge S^V$$

be the composite $\Delta \circ t$. Explicitly,

(4.7)
$$\eta(v) = \begin{cases} \big(a, a, (\rho^{-1}(|w|)/|w|) w\big) & \text{if } v = a + w, \text{ where } a \in A \text{ and } |w| < d, \\ * & \text{otherwise.} \end{cases}$$

The negative sphere G–spectrum S^{-V} in $G\mathscr{S}p$ is obtained by applying the left adjoint of the V^{th} space functor to S^0, and S_G is isomorphic (on the point–set level) to $S^V \odot S^{-V}$ as is noted nonequivariantly in [Lewis and May 1986b, Proposition 4.2].[18] Taking the tensor of η with S^{-V}, we obtain a map of G–spectra

$$S_G \cong S^V \odot S^{-V} \to (A_+ \wedge A_+ \wedge S^V) \odot S^{-V} \cong (A_+ \wedge A_+) \odot S_G \cong \Sigma^\infty_G (A_+ \wedge A_+).$$

[18] The relevant display there has a typo, Ω^∞ for Σ^∞.

Applying the functor $\mathbb{F}$ to this map and smashing with $\boldsymbol{S_G}$ on the left, we obtain the map denoted by $\hat{\eta}_A$ in the diagram

$$S_G \cong S_G \wedge S_G \xleftarrow{\gamma\wedge\gamma} \boldsymbol{S_G} \wedge \boldsymbol{S_G} \xrightarrow{\hat{\eta}_A} \boldsymbol{S_G} \wedge \boldsymbol{\Sigma}_G^\infty (A \times A)_+ \cong \mathbb{A} \wedge \mathbb{A}. \tag{4.8}$$

The following result is a reminder about space-level Atiyah duality. The notion of a V–duality was defined and explained for smooth G–manifolds in [Lewis and May 1986a, Section 5]. Essentially, this states that the space-level maps η and ε make A_+ into a self-dual G–space, modulo inverting the G–space S^V. While our maps are specified precisely on the point–set level, we now pass to the homotopy category.

Proposition 4.9 *The maps*

$$\eta\colon S^V \to A_+ \wedge A_+ \wedge S^V \quad \text{and} \quad \varepsilon \wedge \mathrm{id}\colon A_+ \wedge A_+ \wedge S^V \to S^V$$

specify a V–duality between A_+ and itself.

Proof This could be proven from scratch by proving the required triangle identities, but in fact it is a special case of equivariant Atiyah duality for smooth G–manifolds, A being a 0–dimensional example. Our specification of η is a precise point–set-level specialization of the description of η for a general smooth G–manifold M given in [Lewis and May 1986a, page 152]. Similarly, we claim that our $\varepsilon \wedge \mathrm{id}$ is a precise point–set-level specialization of the definition of ε for a general smooth G–manifold given there. Indeed, letting s be the zero section of the normal bundle ν of the embedding $A \subset \mathbb{R}[A] = V$, we have the composite embedding

$$A \xrightarrow{\Delta} A \times A \xrightarrow{s\times\mathrm{id}} (A \times V) \times A \cong A \times A \times V.$$

The normal bundle of this embedding is $A \times V$, and we may view

$$\Delta \times \mathrm{id}\colon A \times V \to A \times A \times V$$

as giving a big tubular neighborhood. The Pontryagin–Thom map here is obtained by smashing the map $r\colon (A \times A)_+ \to A_+$ that sends (a,b) to a if $a = b$ and to $*$ if $a \neq b$ with the identity map of S^V. Composing with the map induced by the projection $\pi\colon A_+ \to S^0$ that sends a to 1, this gives $\varepsilon \wedge \mathrm{id}$. We observed this factorization of ε in Definition 1.41 and we have used it before, in the proof of Theorem 2.18. □

We obtain the spectrum-level duality maps displayed in (4.4) and (4.8) by tensoring with S^{-V}, applying the functor $\boldsymbol{S_G} \wedge \mathbb{F}$, and composing with γ.

4.3 The weakly unital categories $G\mathscr{B}$ and $\mathscr{B}_G$

Since the G–spectra $\mathbb{A}$ are self-dual, $F_G(\mathbb{A}, \mathbb{B})$ is naturally isomorphic to $\mathbb{B} \wedge \mathbb{A}$ in $\mathrm{Ho}\, G\mathscr{L}$, and the composition and unit

$$F_G(\mathbb{B}, \mathbb{C}) \wedge F_G(\mathbb{A}, \mathbb{B}) \to F_G(\mathbb{A}, \mathbb{C}) \quad \text{and} \quad S_G \to F_G(\mathbb{B}, \mathbb{B}) \tag{4.10}$$

can be expressed as maps

$$\mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \to \mathbb{C} \wedge \mathbb{A} \quad \text{and} \quad S_G \to \mathbb{A} \wedge \mathbb{A} \tag{4.11}$$

in $\mathrm{Ho}\, G\mathscr{Z}$. We want to understand these maps in terms of duality in $G\mathscr{Z}$, without use of infinite loop space theory. However, since we are working in $G\mathscr{Z}$, we must take the isomorphisms (4.2) and the cofibrant approximation $\gamma\colon \boldsymbol{S_G} \to S_G$ into account, and we cannot expect to have strict units. The notion of a weakly unital enriched category was introduced in [Guillou and May 2020, Section 3.5] to formalize what we see here.

Thus, we shall construct a weakly unital $G\mathscr{Z}$–category $\mathscr{B}_G$, analogous to $\mathscr{A}_G$, and compare it with $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$. The G–fixed category $G\mathscr{B}$ will be a weakly unital $\mathscr{Z}$–category. The objects of $\mathscr{B}_G$ and $G\mathscr{B}$ are the S_G–modules $\mathbb{A}$ for finite G–sets A, as in Section 4.1. The morphism S_G–modules of $\mathscr{B}_G$ are $\mathscr{B}_G(\mathbb{A},\mathbb{B}) = \mathbb{B} \wedge \mathbb{A}$. Composition is given by the maps

$$\mathrm{id} \wedge \varepsilon \wedge \mathrm{id}\colon \mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \to \mathbb{C} \wedge \mathbb{A}, \tag{4.12}$$

where ε is the map of (4.4); compare Theorem 2.18.

As recalled in Section 1.5, the adjoint $\tilde{\varepsilon}\colon \mathbb{A} \to D\mathbb{A} = F_G(\mathbb{A}, S_G)$ of ε is a stable equivalence, and we have the composite stable equivalence

$$\delta = \zeta \circ (\mathrm{id} \wedge \tilde{\varepsilon})\colon \mathbb{B} \wedge \mathbb{A} \to \mathbb{B} \wedge D\mathbb{A} \to F_G(\mathbb{A},\mathbb{B}). \tag{4.13}$$

Formal properties of the adjunction $(\wedge, F_G)$ give the following commutative diagram in $G\mathscr{Z}$, which uses δ to compare composition in $\mathscr{B}_G$ with composition in $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$:

$$\begin{tikzcd}
\mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \arrow[r, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\tilde{\varepsilon}\wedge\mathrm{id}\wedge\tilde{\varepsilon}"'] & \mathbb{C} \wedge \mathbb{A} \arrow[d, "\mathrm{id}\wedge\tilde{\varepsilon}"] \\
\mathbb{C} \wedge D\mathbb{B} \wedge \mathbb{B} \wedge D\mathbb{A} \arrow[r, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \arrow[d, "\zeta\wedge\zeta"'] & \mathbb{C} \wedge D\mathbb{A} \arrow[d, "\zeta"] \\
F_G(\mathbb{B},\mathbb{C}) \wedge F_G(\mathbb{A},\mathbb{B}) \arrow[r, "\circ"] & F_G(\mathbb{A},\mathbb{C})
\end{tikzcd} \tag{4.14}$$

At the bottom, we do not know that the function S_G–modules or their smash product are cofibrant, but all objects at the top are cofibrant and thus bifibrant. In general, to compute the smash product of G–spectra X and Y in the homotopy category, we should take the smash product of cofibrant approximations $\mathbb{Q}X$ and $\mathbb{Q}Y$ of X and Y. Since all objects of $G\mathscr{Z}$ are fibrant, to compute a map $X \wedge Y \to Z$ in the homotopy category, we should represent it by a map $\mathbb{Q}X \wedge \mathbb{Q}Y \to \mathbb{Q}Z$ and take its homotopy class. The diagram displays such a cofibrant approximation of the composition in $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$.

Specialized to our context of a category with self-dual objects, the definition [Guillou and May 2020, Definition 3.25] of a weakly unital $G\mathscr{Z}$–category requires, for each object $\mathbb{A}$, a "weak unit map" $\hat{\eta}_A\colon \mathbb{Q}S_G \to \mathbb{A} \wedge \mathbb{A}$ for some chosen cofibrant approximation $\gamma\colon \mathbb{Q}S_G \to S_G$, together with a weak equivalence $\hat{\xi}_{\mathbb{A}}\colon \mathbb{A} \xrightarrow{\simeq} \mathbb{A}$ such that certain unit diagrams relating $\hat{\eta}_A$, $\hat{\xi}_A$ and composition commute. We are led by (4.8) to choose our cofibrant approximation γ to be $\gamma\wedge\gamma\colon \boldsymbol{S_G}\wedge\boldsymbol{S_G}\to S_G\wedge S_G\cong S_G$, and to take $\hat{\eta}_A\colon \boldsymbol{S_G}\wedge\boldsymbol{S_G}\to\mathbb{A}\wedge\mathbb{A}$ to be the map displayed in (4.8). After composing with $\delta\colon\mathbb{A}\wedge\mathbb{A}\to F_G(\mathbb{A},\mathbb{A})$, $\hat{\eta}_A$ is a representative in $G\mathscr{Z}$ for the unit map $S_G\to F_G(\mathbb{A},\mathbb{A})$ that exists in $\operatorname{Ho} G\mathscr{Z}$. Finally, we specify the required equivalence $\hat{\xi}_A\colon\mathbb{A}\xrightarrow{\simeq}\mathbb{A}$.

Definition 4.15 Let $V=\mathbb{R}[A]$. For $a\in A$, define $\xi_a\colon\{a\}_+\wedge S^V\to\{a\}_+\wedge S^V$ by

(4.16) $$\xi_a(a,v)=\begin{cases}\bigl(a,(\rho^{-1}(|w|)/|w|)w\bigr) & \text{if } v=a+w \text{ and } |w|<d,\\ * & \text{otherwise,}\end{cases}$$

where ρ is as in Definition 4.5. Then the wedge of the ξ_a is a G–map

(4.17) $$\xi_A\colon A_+\wedge S^V\to A_+\wedge S^V;$$

ξ_A is G–homotopic to the identity map of $A_+\wedge S^V$ via the explicit G–homotopy

$$h(a,v,t)=\begin{cases}(a,v) & \text{if } t=0 \text{ or } v=a,\\ \bigl(a,(1-t)v+t(\rho^{-1}(t|w|)/|w|)w\bigr) & \text{if } v=a+w \text{ and } t|w|<d,\\ * & \text{otherwise.}\end{cases}$$

Tensoring with S^{-V} and using the natural isomorphisms

$$(X\wedge S^V)\odot S^{-V}\cong X\odot S_G\cong\Sigma^\infty_G X$$

for based G–spaces X, we see that the space-level G–equivalence ξ_A induces a spectrum-level G–equivalence $\hat{\xi}_A\colon\mathbb{A}\to\mathbb{A}$.

It is a bit tedious to verify that our definitions make $\mathscr{B}_G$ into a weakly unital $G\mathscr{Z}$–category, in the sense specified in [Guillou and May 2020, Definition 3.25]. Here are the details.

With η_A as specified in (4.6), easy and perhaps illuminating inspections show that the following unit diagrams already commute in $G\mathscr{T}$, before passage to homotopy:

$$\begin{tikzcd} B_+\wedge A_+\wedge S^V \arrow[r, "\mathrm{id}\wedge\eta_A"] \arrow[d, "\mathrm{id}\wedge\xi_A"'] & B_+\wedge A_+^3\wedge S^V \arrow[dl, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \\ B_+\wedge A_+\wedge S^V & \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} S^W\wedge B_+\wedge A_+ \arrow[r, "\eta_B\wedge\mathrm{id}"] \arrow[d, "\xi_B\wedge\mathrm{id}_A"'] & S^W\wedge B_+^3\wedge A_+ \arrow[dl, "\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}"] \\ S^W\wedge B_+\wedge A_+ & \end{tikzcd}$$

In both, A and B are finite G–sets. In the first, $V=\mathbb{R}[A]$. In the second, $W=\mathbb{R}[B]$ and we have moved S^W from the right to the left for clarity.

Tensoring with S^{-V} and S^{-W} and using (4.2) to pass to smash products of S_G–modules, a little diagram chase shows that the previous pair of diagrams in $G\mathscr{T}$ gives rise to the following pair of commutative diagrams in $G\mathscr{Z}$. These express the unit laws for a weakly unital $G\mathscr{Z}$–category $\mathscr{B}_G$ [Guillou and May 2020, Definition 3.25] with objects the $\mathbb{A}$ and composition as specified in (4.12). Again, the cited unit laws allow us to start with any chosen cofibrant approximation $\gamma\colon \mathbb{Q}S_G \to S_G$ of the unit S_G, and we were led by (4.8) to choose our cofibrant approximation γ to be $\gamma \wedge \gamma\colon \boldsymbol{S_G} \wedge \boldsymbol{S_G} \to S_G \wedge S_G \cong S_G$. The space-level diagrams above induce the required spectrum-level diagrams

$$\begin{tikzcd} \mathbb{B}\wedge\mathbb{A}\wedge\mathbb{Q}S_G \arrow[r, "\mathrm{id}\wedge\hat{\eta}_A"] \arrow[d, "\mathrm{id}\wedge\hat{\xi}_A\wedge\gamma"'] & \mathbb{B}\wedge\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} \arrow[d, "\circ"] \\ \mathbb{B}\wedge\mathbb{A}\wedge S_G \arrow[r, "\cong"'] & \mathbb{B}\wedge\mathbb{A} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathbb{Q}S_G\wedge\mathbb{B}\wedge\mathbb{A} \arrow[r, "\hat{\eta}_B\wedge\mathrm{id}"] \arrow[d, "\gamma\wedge\hat{\xi}_B\wedge\mathrm{id}"'] & \mathbb{B}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A} \arrow[d, "\circ"] \\ S_G\wedge\mathbb{B}\wedge\mathbb{A} \arrow[r, "\cong"'] & \mathbb{B}\wedge\mathbb{A} \end{tikzcd}$$

Taking $A = S^0$ in our second space-level diagram and changing B to A, we also obtain the following commutative diagrams in $G\mathscr{Z}$, where the second diagram is adjoint to the first:

$$\begin{tikzcd} \mathbb{Q}S_G\wedge\mathbb{A} \arrow[r, "\hat{\eta}_A\wedge\mathrm{id}"] \arrow[d, "\gamma\wedge\hat{\xi}_A"'] & \mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} \arrow[d, "\mathrm{id}\wedge\varepsilon"] \\ S_G\wedge\mathbb{A} \arrow[r, "\cong"'] & \mathbb{A} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathbb{Q}S_G \arrow[r, "\hat{\eta}_A"] \arrow[d, "\gamma"'] & \mathbb{A}\wedge\mathbb{A} \arrow[r, "\mathrm{id}\wedge\tilde{\varepsilon}"] & \mathbb{A}\wedge D\mathbb{A} \arrow[d, "\zeta"] \\ S_G \arrow[r, "\eta"'] & F_G(\mathbb{A},\mathbb{A}) \arrow[r, "F_G(\hat{\xi}_A{,}\mathrm{id})"'] & F_G(\mathbb{A},\mathbb{A}) \end{tikzcd} \tag{4.18}$$

Here η at the bottom left of the right diagram is adjoint to the identity map of $\mathbb{A}$. In effect, this uses $\delta = \zeta\circ(\mathrm{id}\wedge\tilde{\varepsilon})$ to compare the unit $S_G \xrightarrow{\eta} F_G(\mathbb{A},\mathbb{A})$ in $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ with the "weak unit" $S_G \leftarrow \mathbb{Q}S_G \to \mathbb{A}\wedge\mathbb{A}$ in $\mathscr{B}_G$.

4.4 The category of presheaves with domain $G\mathscr{B}$

The diagrams (4.14) and (4.18) show that the maps $\delta\colon \mathbb{A}\wedge\mathbb{B} \to F_G(\mathbb{A},\mathbb{B})$ specify a map of weakly unital $G\mathscr{Z}$–categories from the weakly unital $G\mathscr{Z}$–category $\mathscr{B}_G$ to the (unital) $G\mathscr{Z}$–category $\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$. Passing to G–fixed points, we obtain a weakly unital $\mathscr{Z}$–category $G\mathscr{B}$ and a map $\delta\colon G\mathscr{B} \to G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}$ of weakly unital $\mathscr{Z}$–categories. Weakly unital presheaves and presheaf categories are defined in [Guillou and May 2020, Definition 3.25]. By [Guillou and May 2020, Remark 3.26], we obtain the same category of presheaves $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}, \mathscr{Z})$ using unital or weakly unital presheaves. Since δ is an equivalence, we can adapt the methodology of [Guillou and May 2020, Section 2] to complete the proof of the following theorem, using the details relating the functor $\boldsymbol{\Sigma}^{\infty}_{G}$ to smash products from Section A.4. Since we find the use of weakly unital categories unpleasant and our main result Theorem 1.14 more satisfactory, we shall leave the details to the interested reader.

Theorem 4.19 *The categories* $\mathbf{Pre}(G\mathscr{B}, \mathscr{Z})$ *and* $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{All}}, \mathscr{Z})$ *are Quillen equivalent.*

Appendix A Enriched model categories of G–spectra

The results in this section show how to model categories of G–spectra as categories of presheaves of spectra, where G is any compact Lie group. We specialize results of [Guillou and May 2020] to provide and compare two such models. More precisely, in Section A.1, we establish Theorems 2.4 and 4.1, which state that G–spectra can be modeled as presheaves of spectra in both the orthogonal and S–module contexts. In Section A.2, we compare these two presheaf models. In Sections A.3 and A.4, we discuss suspension spectra for orthogonal spectra and S–modules, respectively, in order to be precise about the domain categories for our presheaves. We shall rely on [Elmendorf et al. 1997; Lewis et al. 1986b; Mandell and May 2002; Mandell et al. 2001] for definitions of the relevant categories.

A.1 Presheaf models for categories of G–spectra

We focus on two categories of G–spectra treated in detail in [Mandell and May 2002]. We have the closed symmetric monoidal category $\mathscr{S}$ of nonequivariant orthogonal spectra [Mandell et al. 2001]. Its function spectra are denoted by $F(X,Y)$. We also have the closed symmetric monoidal category $G\mathscr{S}$ of orthogonal G–spectra for a fixed G–universe U [Mandell and May 2002]. Its function G–spectra are denoted by $F_G(X,Y)$. In contrast to the previous sections, in this subsection and the next we allow G–spectra to be indexed over any G–universe. The homotopy type of $F_G(X,Y)$ very much depends on the choice of universe. Then $G\mathscr{S}$ is enriched over $\mathscr{S}$ via the G–fixed-point spectra $F_G(X,Y)^G$. In terms of the general context of [Guillou and May 2020], we are taking $\mathscr{V}=\mathscr{S}$ and $\mathscr{M}=G\mathscr{S}$. We have stable model structures on $\mathscr{S}$ and $G\mathscr{S}$ [Mandell and May 2002; Mandell et al. 2001].

Then [Guillou and May 2020, Theorem 1.36] specializes to give Theorem 2.4. It also gives the following more general result, in which G can be a compact Lie group and G–spectra can be indexed on any universe. (See also [Schwede and Shipley 2003, Example 3.4(i)]).

Theorem A.1 *Let $G\mathscr{D}_S$ be the full $\mathscr{S}$–subcategory of $G\mathscr{S}$ whose objects are fibrant approximations of the suspension G–spectra $\Sigma^\infty X_+$ for all X in any set S of compact G–spaces that contains G/H for at least one H in each conjugacy class of closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}_S,\mathscr{S}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{S},$$

and it is a Quillen equivalence. If $S\subset T$ are as prescribed and

$$\mathbb{R}\colon \mathbf{Pre}(G\mathscr{D}_T,\mathscr{S}) \to \mathbf{Pre}(G\mathscr{D}_S,\mathscr{S})$$

is the restriction along the inclusion $G\mathscr{D}_S\to G\mathscr{D}_T$, then $\mathbb{R}\circ\mathbb{U}_T=\mathbb{U}_S$ and therefore R induces an equivalence of presheaf homotopy categories.

Remark A.2 Adapting our work for finite groups to incomplete universes would require us to use incomplete Mackey functors and to reconcile the conflict between needing to use all orbits G/H to obtain generators for $\operatorname{Ho} G\mathscr{S}$ and needing to use only those orbits G/H that embed in the given universe to have self-duality of orbits, which is vital to our theory but irrelevant to Theorem A.1.

We have a second specialization of [Guillou and May 2020, Theorem 1.36]. We have the closed symmetric monoidal category $\mathscr{Z}$ of nonequivariant S–modules [Elmendorf et al. 1997].[19] Its function spectra are again denoted by $F(X,Y)$. We also have the closed symmetric monoidal category $G\mathscr{Z}$ of S_G–modules (for a fixed G–universe U as above) [Mandell and May 2002]. Its function G–spectra are denoted by $F_G(X,Y)$. Then $G\mathscr{Z}$ is enriched over $\mathscr{Z}$ via the G–fixed-point spectra $F_G(X,Y)^G$. We are taking $\mathscr{V} = \mathscr{Z}$ and $\mathscr{M} = G\mathscr{Z}$. We have stable model structures on $\mathscr{Z}$ and $G\mathscr{Z}$ [Elmendorf et al. 1997; Mandell and May 2002]. Again, Theorem 1.36 of [Guillou and May 2020] specializes to give Theorem 4.1. It also gives the following more general result, in which G can be a compact Lie group and G–spectra can be indexed on any universe:

Theorem A.3 *Let $G\mathscr{D}_S^{\mathscr{Z}}$ be the full $\mathscr{S}$–subcategory of $G\mathscr{Z}$ whose objects are cofibrant approximations of the suspension G–spectra $\Sigma^\infty X_+$ for all X in any set S of compact G–spaces that contains G/H for at least one H in each conjugacy class of closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D}_S^{\mathscr{Z}}, \mathscr{Z}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{Z},$$

and it is a Quillen equivalence. If $S \subset T$ are as prescribed and

$$\mathbb{R}\colon \mathbf{Pre}(G\mathscr{D}_T, \mathscr{Z}) \to \mathbf{Pre}(G\mathscr{D}_S, \mathscr{Z})$$

is the restriction along the inclusion $G\mathscr{D}_S^{\mathscr{Z}} \to G\mathscr{D}_T^{\mathscr{Z}}$, then $\mathbb{R} \circ \mathbb{U}_T = \mathbb{U}_S$ and therefore R induces an equivalence of presheaf homotopy categories.

Remark A.4 When G is finite, we focus on the set $S = \mathrm{Orb}$ of all orbit G–sets G/H and the set $T = \mathrm{All}$ of all finite G–sets. Here we can obtain an inverse equivalence to $\mathbb{R}$ by sending a presheaf defined on S to an *additive* presheaf defined on T, where additivity requires a presheaf that sends disjoint unions in T to finite products in $G\mathscr{S}$ or in $G\mathscr{Z}$. Thus, an interpretation of the equivalence of presheaves on $G\mathscr{D}_{\mathrm{Orb}}$ with presheaves on $G\mathscr{D}_{\mathrm{All}}$ is that presheaves on $G\mathscr{D}_{\mathrm{All}}$ are equivalent to additive presheaves. The intuition is that the spectral enrichment builds in additivity, just as functors enriched over abelian groups automatically preserve coproducts.

Homotopically, Theorems A.1 and A.3 are essentially the same result since $G\mathscr{S}$ and $G\mathscr{Z}$ are Quillen equivalent. On the point–set level they are quite different, and they have different virtues and defects.

We say just a bit about the proofs of these theorems. By [Guillou and May 2020, Theorem 4.32], the presheaf categories used in them are well-behaved model categories. The acyclicity condition there holds in Theorem A.1 because $\mathscr{S}$ satisfies the monoid axiom, by [Mandell et al. 2001, Proposition 12.5]. It holds in Theorem A.3 by use of the "cofibration hypothesis" of [Elmendorf et al. 1997, page 146], which

[19] The notation $\mathscr{S}$ is short for $\mathscr{I}\mathscr{S}$ and the notation $\mathscr{Z}$ is short for $\mathscr{M}_S$ in the original sources; as a silly mnemonic device, $\mathscr{Z}$ stands for the Z in the middle of [Elmendorf et al. 1997].

also holds equivariantly. The orbit G–spectra give compact generating sets in both $\mathrm{Ho}(G\mathscr{S})$ and $\mathrm{Ho}(G\mathscr{Z})$. We require bifibrant representatives. In Theorem A.1, the orbit G–spectra are cofibrant, and fibrant approximation makes them bifibrant.

By contrast, in Theorem A.3, all S_G–modules are fibrant, and cofibrant approximation makes them bifibrant. Here cofibrant approximation is given by a well-understood left adjoint that very nearly preserves smash products, as we shall explain in Section A.4.

Technically, Theorem 1.36 of [Guillou and May 2020] requires *either* that the unit object of the enriching category $\mathscr{V}$ be cofibrant *or* that every object in $\mathscr{V}$ be fibrant. The first hypothesis holds in $\mathscr{S}$ and the second holds in $\mathscr{Z}$. It is impossible to have both of these conditions in the same symmetric monoidal model category for the stable homotopy category [Lewis 1991; May 2009]. That is a key reason that both of these results are of interest.

A.2 Comparison of presheaf models of G–spectra

Theorems A.1 and A.3 are related by the following result, which is [Mandell and May 2002, Theorem IV.1.1]; the nonequivariant special case is [Mandell and May 2002, Theorem I.1.1]. In this result, $G\mathscr{S}$ is given its positive stable model structure from [Mandell and May 2002] and is denoted by $G\mathscr{S}_{\mathrm{pos}}$ to indicate the distinction; in that model structure, the sphere G–spectrum in $G\mathscr{S}$, like the sphere G–spectrum in $G\mathscr{Z}$, is not cofibrant. In [Mandell and May 2002], the result is proven for genuine G–spectra for compact Lie groups G. For arbitrary topological groups G, the same proof applies to classical G–spectra, that is G–spectra indexed on a universe with trivial G–action.

Theorem A.5 *There is a Quillen equivalence*

$$G\mathscr{S}_{\mathrm{pos}} \underset{\mathbb{N}^\#}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{Z}.$$

The functor $\mathbb{N}$ is strong symmetric monoidal, hence $\mathbb{N}^\#$ is lax symmetric monoidal.

The identity functor is a left Quillen equivalence $G\mathscr{S}_{\mathrm{pos}} \to G\mathscr{S}$. Therefore, Theorems A.1, A.3 and A.5, have the following immediate consequence:

Corollary A.6 *The categories $\mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}}, \mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}, \mathscr{Z})$ are Quillen equivalent. More precisely, there are left Quillen equivalences*

$$\mathbf{Pre}(G\mathscr{D}_{\mathrm{Orb}}, \mathscr{S}) \to G\mathscr{S} \leftarrow G\mathscr{S}_{\mathrm{pos}} \to G\mathscr{Z} \leftarrow \mathbf{Pre}(G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}, \mathscr{Z}).$$

In fact, we can compare the $\mathscr{S}$–category $G\mathscr{D}_{\mathrm{Orb}}$ with the $\mathscr{Z}$–category $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ via the right adjoint $\mathbb{N}^\#$. The adjunction

$$G\mathscr{S}_{\mathrm{pos}} \underset{\mathbb{N}^\#}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{Z}$$

is tensored over the adjunction

$$\mathscr{S}_{\mathrm{pos}} \underset{\mathbb{N}^\#}{\overset{\mathbb{N}}{\rightleftarrows}} \mathscr{Z}$$

in the sense of [Guillou and May 2020, Definition 3.20]. Indeed, since $G\mathscr{S}$ is a bicomplete $\mathscr{S}$–category, it is tensored over $\mathscr{S}$. While a more explicit definition is easy enough, for a spectrum X and G–spectrum Y, we can define the G–spectrum $Y \odot X$ to be $Y \wedge i_*\varepsilon^* X$, where $i_*\varepsilon^*\colon \mathscr{S} \to G\mathscr{S}$ is the change-of-groups and universe functor associated to $\varepsilon\colon G \to e$ that assigns a genuine G–spectrum to a nonequivariant spectrum. The same is true with $\mathscr{S}$ replaced by $\mathscr{Z}$. These functors are discussed in both contexts and compared in [Mandell and May 2002]. Results there (see [Mandell and May 2002, Theorem IV.1.1]) imply that

$$\mathbb{N}Y \odot \mathbb{N}X \cong \mathbb{N}(Y \odot X),$$

which is the defining condition for a tensored adjunction. Now Corollary 3.24 of [Guillou and May 2020] gives that the $\mathscr{S}$–category $\mathbb{N}^{\#}G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ is quasiequivalent to $G\mathscr{D}_{\mathrm{Orb}}$. Using [Guillou and May 2020, Remark 2.15 and Theorem 3.17], this implies a direct proof of the Quillen equivalence of Corollary A.6. Therefore, Theorems A.1 and A.3 are equivalent: each implies the other.

We reiterate the generality: the results above do *not* require G to be finite. In that generality, we do not know how to simplify the description of the domain category $G\mathscr{D}_{\mathrm{Orb}}$ to transform it into a weakly equivalent $\mathscr{S}$–category or $\mathscr{Z}$–category that is intuitive and perhaps even familiar, something accessible to study independent of knowledge of the category of G–spectra that we seek to understand. Our main theorem shows how to do just that when G is finite.

A.3 Suspension spectra and fibrant replacement functors in $G\mathscr{S}$

We here give some observations relevant to understanding the category $G\mathscr{D}_{\mathrm{Orb}}$ of Theorem A.1. From now on, the group G is finite and the universe is complete unless otherwise specified.

For an inner product space V and a based G–space X, the V^{th} space of the orthogonal G–spectrum $\Sigma^\infty_G X$ is $X \wedge S^V$. The functor Σ^∞_G, also denoted by F_0, is left adjoint to the zeroth space functor $(-)_0\colon G\mathscr{S} \to G\mathscr{T}$. Nonequivariantly, it is part of [Mandell et al. 2001, Lemma 1.8] that, for based spaces X and Y, $F_0X \wedge F_0Y$ is naturally isomorphic to $F_0(X \wedge Y)$. The categorical proof of that result in [Mandell et al. 2001, Section 21] applies equally well equivariantly to give the following result:

Proposition A.7 *The functor $\Sigma^\infty_G\colon G\mathscr{T} \to G\mathscr{S}$ is strong symmetric monoidal.*

Therefore, the zeroth space functor is lax symmetric monoidal, but of course that functor is not homotopically meaningful except on objects that are fibrant in the stable model structure. There is no known fibrant replacement functor in that model structure that is well behaved with respect to smash products. Recall from Remark 3.10 that the existence of a monoidal fibrant replacement functor is relevant to a monoidal version of our main result.

Although it is less useful for our purposes, we point out two different constructions of monoidal fibrant replacement functors in the *positive* stable model structure. The first is immediate from Theorem A.5 but does not appear in the literature.

Proposition A.8 *The unit $\eta\colon E \to \mathbb{N}^{\#}\mathbb{N}E$ of the adjunction between $G\mathscr{S}$ and $G\mathscr{Z}$ specifies a lax monoidal fibrant replacement functor on cofibrant objects for the positive stable model structure $G\mathscr{S}_{\mathrm{pos}}$.*

Remark A.9 Nonequivariantly, Kro [2007] has given a different lax monoidal positive fibrant replacement functor for orthogonal spectra. His construction does not require restriction to cofibrant objects. Parenthetically, as he notes, it does not apply to symmetric spectra. However, by [Mandell et al. 2001, Proposition 3.3], the unit $E \to \mathbb{N}^{\sharp}\mathbb{U}\mathbb{P}\mathbb{N}E$ of the composite of the adjunction $(\mathbb{P}, \mathbb{U})$ between symmetric and orthogonal spectra and the adjunction $(\mathbb{N}, \mathbb{N}^{\sharp})$ gives a lax monoidal positive fibrant replacement functor for symmetric spectra.

Unfortunately the restriction to the positive model structure in Proposition A.8 is necessary, and the only fibrant approximation functor we know of for use with the stable model structure employed in Theorem A.1 is that given by the small-object argument. The point is that the suspension G–spectra $\Sigma^{\infty}_{G}(G/H_{+})$ are cofibrant but not positive cofibrant.

Nonequivariantly, a homotopically meaningful version of the adjunction $(\Sigma^{\infty}, \Omega^{\infty})$ has been worked out for symmetric spectra by Sagave and Schlichtkrull [2012] and for symmetric and orthogonal spectra by Lind [2013], who compares his constructions with the adjunction $(\Sigma^{\infty}, \Omega^{\infty})$ in $\mathscr{S}p$ (see below) and with its analogue for $\mathscr{Z}$. This generalizes to the equivariant context, although details have not been written down.

A.4 Suspension spectra and smash products in $G\mathscr{Z}$

We here give some observations relevant to understanding the category $G\mathscr{D}^{\mathscr{Z}}_{\mathrm{Orb}}$ of Theorem A.3. In particular, we give properties of cofibrant approximations of suspension spectra that are used in Section 4. For more information, see [Elmendorf et al. 1996; Mandell and May 2002, Section IV.2] and the nonequivariant precursor [Elmendorf et al. 1997].

We have a category $G\mathscr{P}$ of (coordinate-free)-prespectra. Its objects Y are based G–spaces $Y(V)$ and based G–maps $Y(V) \wedge S^{W} \to Y(W-V)$ for $V \subset W$. Here V and W are sub-inner product spaces of a G–universe U. A G–spectrum is a G–prespectrum Y whose adjoint G–maps $Y(V) \to \Omega^{W-V}Y(W)$ are homeomorphisms. The (Lewis–May) category $G\mathscr{S}p$ of G–spectra is the full subcategory of G–spectra in $G\mathscr{P}$. The suspension G–prespectrum functor Π sends a based G–space X to $\{X \wedge S^{V}\}$. There is a left adjoint spectrification functor $L\colon G\mathscr{P} \to G\mathscr{S}p$, and the suspension G–spectrum functor $\Sigma^{\infty}_{G}\colon G\mathscr{T} \to G\mathscr{S}p$ is $L \circ \Pi$. Explicitly, let

$$Q_{G}X = \operatorname{colim} \Omega^{V}\Sigma^{V}X,$$

where V runs over the finite-dimensional subspaces of a complete G–universe U. Then the V^{th} G–space of $\Sigma^{\infty}_{G}X$ is $Q_{G}\Sigma^{V}X$.

All objects of $G\mathscr{S}p$ are fibrant, and the zeroth space functor $\Omega^{\infty}_{G}\colon G\mathscr{S}p \to G\mathscr{T}$ is now homotopically meaningful. For a based G–CW complex X (with based attaching maps), $\Sigma^{\infty}_{G}X$ is cofibrant in $G\mathscr{S}p$. In particular, the sphere G–spectrum $S_{G} = \Sigma^{\infty}_{G}S^{0}$ is cofibrant. Since G is a compact Lie group, the orbits G/H are G–CW complexes, hence the $\Sigma^{\infty}_{G}(G/H_{+})$ are cofibrant. However, $G\mathscr{S}p$ is not symmetric monoidal under the smash product. The implicit trade off here is intrinsic to the mathematics, as was explained by Lewis [1991]; see [May 2009] for a more recent discussion.

We summarize some constructions in [Elmendorf et al. 1997] that work in exactly the same fashion equivariantly as nonequivariantly. We have the G–space $\mathscr{L}(j)$ of linear isometries $U^j \to U$, with G acting by conjugation. These spaces form an E_∞ G–operad when U is complete. The G–monoid $\mathscr{L}(1)$ gives rise to a monad $\mathbb{L}$ on $G\mathscr{S}p$. Its algebras are called $\mathbb{L}$–spectra, and we have the category $G\mathscr{S}p[\mathbb{L}]$ of $\mathbb{L}$–spectra. It has a smash product $\wedge_{\mathscr{L}}$ which is associative and commutative but not unital. The action map $\xi\colon \mathbb{L}Y \to Y$ of an $\mathbb{L}$–spectrum Y is a stable equivalence.

Suspension G–spectra are naturally $\mathbb{L}$–spectra. In particular, the sphere G–spectrum S_G is an $\mathbb{L}$–spectrum. There is a natural stable equivalence

$$\lambda\colon S_G \wedge_{\mathscr{L}} Y \to Y$$

for $\mathbb{L}$–spectra Y. The S_G–modules are those Y for which λ is an isomorphism, and they are the objects of $G\mathscr{Z}$. All suspension G–spectra are S_G–modules, and so are all $\mathbb{L}$–spectra of the form $S_G \wedge_{\mathscr{L}} Y$. The smash product $\wedge$ on S_G–modules is just the restriction of the smash product $\wedge_{\mathscr{L}}$, and it gives $G\mathscr{Z}$ its symmetric monoidal structure.

We have a sequence of Quillen left adjoints

$$G\mathscr{T} \xrightarrow{\Sigma_G^\infty} G\mathscr{S}p \xrightarrow{\mathbb{L}} G\mathscr{S}p[\mathbb{L}] \xrightarrow{\mathbb{J}} G\mathscr{Z},$$

where $\mathbb{L}X$ is the free $\mathbb{L}$–spectrum generated by a G–spectrum X and $\mathbb{J}Y = S_G \wedge_{\mathscr{L}} Y$ is the S_G–module generated by an $\mathbb{L}$–spectrum Y. We let $\mathbb{F} = \mathbb{J}\mathbb{L}$; then $\mathbb{L}$, $\mathbb{J}$ and $\mathbb{F}$ are Quillen equivalences. The composite $\gamma = \xi \circ \lambda\colon \mathbb{F}Y \to Y$ is a stable equivalence for any $\mathbb{L}$–spectrum Y. We have defined $\boldsymbol{\Sigma}_G^\infty$ to be the composite functor $\mathbb{F}\Sigma_G^\infty$, and we have the natural stable equivalence of S_G–modules $\gamma\colon \boldsymbol{\Sigma}_G^\infty X \to \Sigma_G^\infty X$.

The tensor $Y \odot X$ of a G–prespectrum and a based G–space X has V^{th} G–space $Y(V) \wedge X$. When Y is a G–spectrum, the G–spectrum $Y \odot X$ is $L(\ell Y \odot X)$, where ℓY is the underlying G–prespectrum of Y [Lewis and May 1986b, Definition 3.1]. Tensors in $G\mathscr{S}p[\mathbb{L}]$ and $G\mathscr{Z}$ are inherited from those in $G\mathscr{S}p$. All of our left adjoints are enriched in $\mathscr{T}$ and preserve tensors. This leads to the following relationship between $\wedge$ and $\boldsymbol{\Sigma}_G^\infty$:

Proposition A.10 *For based G–spaces X and Y, there are natural isomorphisms*

$$\boldsymbol{\Sigma}_G^\infty X \wedge \boldsymbol{\Sigma}_G^\infty Y \cong (\boldsymbol{S}_G \wedge \boldsymbol{S}_G) \odot (X \wedge Y) \cong \boldsymbol{S}_G \wedge \boldsymbol{\Sigma}_G^\infty (X \wedge Y).$$

Proof We have $\Sigma_G^\infty X \cong S_G \odot X$ and therefore

$$\boldsymbol{\Sigma}_G^\infty X = \mathbb{F}\Sigma_G^\infty X \cong \mathbb{F}(S_G \odot X) \cong (\mathbb{F}S_G) \odot X = \boldsymbol{S}_G \odot X.$$

We also have

$$(\boldsymbol{S}_G \odot X) \wedge (\boldsymbol{S}_G \odot Y) \cong (\boldsymbol{S}_G \wedge \boldsymbol{S}_G) \odot (X \wedge Y),$$

and the conclusion follows. □

Appendix B Whiskering $G\mathscr{E}$ to obtain strict unit 1–cells

The bicategory $G\mathscr{E}$ of Definition 1.7 narrowly misses being a strict 2–category, and we whisker the unit 1–cells to obtain a strict 2–category $G\mathscr{E}'$.[20] Before focusing on specifics, we give an elementary general definition.

Definition B.1 For a category $\mathscr{D}$ with a privileged object Δ, define the whiskering $\mathscr{D}'$ of $\mathscr{D}$ at Δ by adjoining a new object I and an isomorphism $\zeta\colon I \to \Delta$. We have the inclusion $i\colon \mathscr{D} \to \mathscr{D}'$, and we define a retraction functor $r\colon \mathscr{D}' \to \mathscr{D}$ by $r(I) = \Delta$ and $r(\zeta) = \mathrm{id}_\Delta$. Thus, $r \circ i = \mathrm{Id}_\mathscr{D}$ and the isomorphism ζ on the object I together with the identity map on all other objects of $\mathscr{D}'$ defines a natural isomorphism $\mathrm{Id}_{\mathscr{D}'} \to i \circ r$. If $\mathscr{D}$ is a G–category and Δ is G–fixed, then $\mathscr{D}'$ is a G–category with I and ζ fixed by G, and then $\mathscr{D}$ and $\mathscr{D}'$ are G–equivalent.

The whiskered category $G\mathscr{E}'$ "enriched in permutative categories" and the whiskered G–category $\mathscr{E}'_G$ "enriched in permutative G–categories" are defined to have the same objects, or 0–cells, as $G\mathscr{E}$ and $\mathscr{E}_G$, namely the finite G–sets A in both cases.

Definition B.2 If $A \neq B$ or if $|A| \leq 1$ and $A = B$, we define $G\mathscr{E}'(A, B)$ to be the permutative category $G\mathscr{E}(A, B)$. For each A of cardinality at least 2, we define

$$G\mathscr{E}'(A, A) = G\mathscr{E}(A, A)',$$

where the whiskering is performed at the 1–cell Δ_A. We denote the adjoined 1–cell by I_A and the adjoined isomorphism 2–cell by $\zeta_A\colon I_A \to \Delta_A$. We specify a permutative structure on $G\mathscr{E}'(A, A)$ by setting

$$E \amalg F = \begin{cases} I_A & \text{if } (E, F) = (I_A, \varnothing) \text{ or } (\varnothing, I_A), \\ i(r(E) \amalg r(F)) & \text{otherwise.} \end{cases}$$

We have denoted the monoidal product by $\amalg$ since the product in $G\mathscr{E}(A \times A)$ is given by the disjoint union of spans. As the only 2–cell in $G\mathscr{E}'(A, A)$ with source or target $\varnothing$ is $\mathrm{id}_\varnothing$, this product extends uniquely to a functor. Since the retraction

$$r\colon G\mathscr{E}'(A, A) \to G\mathscr{E}(A, A)$$

is strict monoidal and an equivalence of categories, the symmetry isomorphism $\gamma\colon \amalg \cong \amalg\tau$ on $G\mathscr{E}(A, A)$ lifts uniquely to a symmetry isomorphism $\gamma\colon \amalg \cong \amalg\tau$ on $G\mathscr{E}'(A, A)$. Observe that the inclusion $i\colon G\mathscr{E}(A, A) \to G\mathscr{E}'(A, A)$ is strict monoidal.

To extend composition to functors

$$G\mathscr{E}'(B, C) \times G\mathscr{E}'(A, B) \xrightarrow{\circ} G\mathscr{E}'(A, C),$$

we declare I_A to be a strict 2–sided unit. It remains to define composition with a 2–cell with source or target I_A. Since every such 2–cell factors through ζ_A and composition with Δ_A is already defined,

[20] We thank Angélica Osorno for help with the material in this section.

it suffices to define composition with ζ_A. Since Δ_A is a strict right unit, for a span $B \leftarrow E \to A$, abbreviated E, we may define $E \circ \zeta_A \colon E \circ I_A \to E \circ \Delta_A$ to be the identity 2–cell id_E. We define $\zeta_B \circ E \colon I_B \circ E \to \Delta_B \circ E$ to be $\ell_{B,E}^{-1}$, where $\ell_{B,E}$ is the 2–cell defined in (1.9).

Remark B.3 In [Bohmann and Osorno 2015], and also in a previous version of this article, a different strictification of $G\mathscr{E}$ was proposed, namely just redefining composition with Δ_A to force this to be a unit 1–cell. Unfortunately, this breaks associativity, since the 1–cell Δ_A is decomposable under composition if $|A| \geq 2$.

We have a precisely analogous definition on the level of G–categories, obtaining a strict 2–category $\mathscr{E}'_G$ from $\mathscr{E}_G$.

Definition B.4 If $A \neq B$ or if $|A| \leq 1$ and $A = B$, we define $\mathscr{E}'_G(A, B)$ to be the permutative G–category $\mathscr{E}_G(A, B)$. For each A of cardinality at least 2, we define

$$\mathscr{E}'_G(A, A) = \mathscr{E}_G(A, A)'.$$

We denote the adjoined 1–cell by I_A and the adjoined isomorphism 2–cell by ζ_A. We specify a G–permutative structure on $\mathscr{E}'_G(A, A)$ by setting

$$\theta(\mu; E_1, \dots, E_n) = \begin{cases} I_A & \text{if } E_i = I_A \text{ and } E_j = \varnothing \text{ for all } j \neq i, \\ \theta(\mu; r(E_1), \dots, r(E_n)) & \text{otherwise.} \end{cases}$$

Observe that the inclusion $i \colon \mathscr{E}_G(A, A) \to \mathscr{E}'_G(A, A)$ is a map of $\mathscr{P}_G$–algebras.

To extend composition to a functor

$$\mathscr{E}'_G(B, C) \times \mathscr{E}'_G(A, B) \xrightarrow{\circ} \mathscr{E}'_G(A, C),$$

we declare the object $I_A \in \mathscr{E}'_G(A, A)$ to be a strict 2–sided unit. We define composition with a 2–cell whose source or target is of the form I_A exactly as in Definition B.2, except that to define $\zeta_B \circ E$ we now use the $\ell_{B,E}$ defined in (1.37).

References

[Arone et al. 2022] **G Arone**, **I Barnea**, **T M Schlank**, *Noncommutative CW–spectra as enriched presheaves on matrix algebras*, J. Noncommut. Geom. 16 (2022) 1411–1443 MR Zbl

[Barwick 2017] **C Barwick**, *Spectral Mackey functors and equivariant algebraic K–theory, I*, Adv. Math. 304 (2017) 646–727 MR Zbl

[Barwick et al. 2020] **C Barwick**, **S Glasman**, **J Shah**, *Spectral Mackey functors and equivariant algebraic K–theory, II*, Tunis. J. Math. 2 (2020) 97–146 MR Zbl

[Bohmann and Osorno 2015] **A M Bohmann**, **A Osorno**, *Constructing equivariant spectra via categorical Mackey functors*, Algebr. Geom. Topol. 15 (2015) 537–563 MR Zbl

[Bökstedt et al. 1993] **M Bökstedt**, **W C Hsiang**, **I Madsen**, *The cyclotomic trace and algebraic K–theory of spaces*, Invent. Math. 111 (1993) 465–539 MR Zbl

[Carlsson 1984] **G Carlsson**, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. 120 (1984) 189–224 MR Zbl

[Carlsson 1992] **G Carlsson**, *A survey of equivariant stable homotopy theory*, Topology 31 (1992) 1–27 MR Zbl

[Clausen et al. 2020] **D Clausen**, **A Mathew**, **N Naumann**, **J Noel**, *Descent and vanishing in chromatic algebraic K–theory via group actions* (2020) arXiv 2011.08233 To appear in Ann. Sci. École Norm. Sup.

[Costenoble and Waner 1991] **S R Costenoble**, **S Waner**, *Fixed set systems of equivariant infinite loop spaces*, Trans. Amer. Math. Soc. 326 (1991) 485–505 MR Zbl

[Dunn 1994] **G Dunn**, *E_n–monoidal categories and their group completions*, J. Pure Appl. Algebra 95 (1994) 27–39 MR Zbl

[Elmendorf and Mandell 2009] **A D Elmendorf**, **M A Mandell**, *Permutative categories, multicategories and algebraic K–theory*, Algebr. Geom. Topol. 9 (2009) 2391–2441 MR Zbl

[Elmendorf et al. 1996] **A D Elmendorf**, **L G Lewis, Jr**, **J P May**, *Brave new equivariant foundations*, from "Equivariant homotopy and cohomology theory" (J P May, editor), CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) 283–297 MR Zbl

[Elmendorf et al. 1997] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Math. Surv. Monogr. 47, Amer. Math. Soc., Providence, RI (1997) MR Zbl

[Greenlees and May 1995a] **J P C Greenlees**, **J P May**, *Equivariant stable homotopy theory*, from "Handbook of algebraic topology" (I M James, editor), North-Holland, Amsterdam (1995) 277–323 MR Zbl

[Greenlees and May 1995b] **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 543, Amer. Math. Soc., Providence, RI (1995) MR Zbl

[Greenlees and May 1997] **J P C Greenlees**, **J P May**, *Localization and completion theorems for MU–module spectra*, Ann. of Math. 146 (1997) 509–544 MR Zbl

[Guillou 2010] **B J Guillou**, *Strictification of categories weakly enriched in symmetric monoidal categories*, Theory Appl. Categ. 24 (2010) 564–579 MR Zbl

[Guillou and May 2017] **B J Guillou**, **J P May**, *Equivariant iterated loop space theory and permutative G–categories*, Algebr. Geom. Topol. 17 (2017) 3259–3339 MR Zbl

[Guillou and May 2020] **B J Guillou**, **J P May**, *Enriched model categories and presheaf categories*, New York J. Math. 26 (2020) 37–91 MR Zbl

[Guillou et al. 2017] **B J Guillou**, **J P May**, **M Merling**, *Categorical models for equivariant classifying spaces*, Algebr. Geom. Topol. 17 (2017) 2565–2602 MR Zbl

[Guillou et al. 2019] **B Guillou**, **J P May**, **M Merling**, **A M Osorno**, *A symmetric monoidal and equivariant Segal infinite loop space machine*, J. Pure Appl. Algebra 223 (2019) 2425–2454 MR Zbl

[Guillou et al. 2020] **B J Guillou**, **J P May**, **M Merling**, **A M Osorno**, *Symmetric monoidal G–categories and their strictification*, Q. J. Math. 71 (2020) 207–246 MR Zbl

[Guillou et al. 2023] **B J Guillou**, **J P May**, **M Merling**, **A M Osorno**, *Multiplicative equivariant K–theory and the Barratt–Priddy–Quillen theorem*, Adv. Math. 414 (2023) art. id. 108865 MR Zbl

[Hill et al. 2016] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. 184 (2016) 1–262 MR Zbl

[Hyland and Power 2002] **M Hyland**, **J Power**, *Pseudo-commutative monads and pseudo-closed* 2*–categories*, J. Pure Appl. Algebra 175 (2002) 141–185 MR Zbl

[Kro 2007] **T A Kro**, *Model structure on operads in orthogonal spectra*, Homology Homotopy Appl. 9 (2007) 397–412 MR Zbl

[Lewis 1991] **L G Lewis, Jr**, *Is there a convenient category of spectra?*, J. Pure Appl. Algebra 73 (1991) 233–246 MR Zbl

[Lewis and May 1986a] **L G Lewis, Jr**, **J P May**, *Equivariant duality theory*, from "Equivariant stable homotopy theory" (A Dold, B Eckmann, editors), Lecture Notes in Math. 1213, Springer (1986) 117–174 MR Zbl

[Lewis and May 1986b] **L G Lewis, Jr**, **J P May**, *The equivariant stable category*, from "Equivariant stable homotopy theory" (A Dold, B Eckmann, editors), Lecture Notes in Math. 1213, Springer (1986) 6–53 MR Zbl

[Lewis and May 1986c] **L G Lewis, Jr**, **J P May**, *Equivariant transfer*, from "Equivariant stable homotopy theory" (A Dold, B Eckmann, editors), Lecture Notes in Math. 1213, Springer (1986) 175–235 MR Zbl

[Lewis et al. 1986a] **L G Lewis, Jr**, **J P May**, **J E McClure**, *The Burnside ring and splittings in equivariant homology theory*, from "Equivariant stable homotopy theory" (A Dold, B Eckmann, editors), Lecture Notes in Math. 1213, Springer (1986) 236–298 MR Zbl

[Lewis et al. 1986b] **L G Lewis, Jr**, **J P May**, **M Steinberger**, **J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer (1986) MR Zbl

[Lind 2013] **J A Lind**, *Diagram spaces, diagram spectra and spectra of units*, Algebr. Geom. Topol. 13 (2013) 1857–1935 MR Zbl

[Malkiewich and Merling 2019] **C Malkiewich**, **M Merling**, *Equivariant A–theory*, Doc. Math. 24 (2019) 815–855 MR Zbl

[Malkiewich and Merling 2022] **C Malkiewich**, **M Merling**, *The equivariant parametrized h–cobordism theorem, the non-manifold part*, Adv. Math. 399 (2022) art. id. 108242 MR Zbl

[Mandell and May 2002] **M A Mandell**, **J P May**, *Equivariant orthogonal spectra and S–modules*, Mem. Amer. Math. Soc. 755, Amer. Math. Soc., Providence, RI (2002) MR Zbl

[Mandell et al. 2001] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. Lond. Math. Soc. 82 (2001) 441–512 MR Zbl

[May 1972] **J P May**, *The geometry of iterated loop spaces*, Lecture Notes in Math. 271, Springer (1972) MR Zbl

[May 1974] **J P May**, *E_∞ spaces, group completions, and permutative categories*, from "New developments in topology" (G Segal, editor), Lond. Math. Soc. Lect. Note Ser. 11, Cambridge Univ. Press (1974) 61–93 MR Zbl

[May 1977] **J P May**, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math. 577, Springer (1977) MR Zbl

[May 1978] **J P May**, *The spectra associated to permutative categories*, Topology 17 (1978) 225–228 MR Zbl

[May 1980] **J P May**, *Pairings of categories and spectra*, J. Pure Appl. Algebra 19 (1980) 299–346 MR Zbl

[May 1996] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) MR Zbl

[May 2009] **J P May**, *What precisely are E_∞ ring spaces and E_∞ ring spectra?*, from "New topological contexts for Galois theory and algebraic geometry" (A Baker, B Richter, editors), Geom. Topol. Monogr. 16, Geom. Topol. Publ., Coventry (2009) 215–282 MR Zbl

[May et al. 2017] **J P May**, **M Merling**, **A M Osorno**, *Equivariant infinite loop space theory*, *the space level story* (2017) arXiv 1704.03413 To appear in Mem. Amer. Math. Soc.

[Nardin 2016] **D Nardin**, *Parametrized higher category theory and higher algebra*, *IV*: *Stability with respect to an orbital ∞–category*, preprint (2016) arXiv 1608.07704

[Piacenza 1996] **R J Piacenza**, *The homotopy theory of diagrams*, from "Equivariant homotopy and cohomology theory" (J P May, editor), CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) 47–57 MR Zbl

[Ponto and Shulman 2012] **K Ponto**, **M Shulman**, *Duality and traces for indexed monoidal categories*, Theory Appl. Categ. 26 (2012) 582–659 MR Zbl

[Sagave and Schlichtkrull 2012] **S Sagave**, **C Schlichtkrull**, *Diagram spaces and symmetric spectra*, Adv. Math. 231 (2012) 2116–2193 MR Zbl

[Schmitt 2007] **V Schmitt**, *Tensor product for symmetric monoidal categories*, preprint (2007) arXiv 0711.0324

[Schwede and Shipley 2003] **S Schwede**, **B Shipley**, *Stable model categories are categories of modules*, Topology 42 (2003) 103–153 MR Zbl

[Shimakawa 1989] **K Shimakawa**, *Infinite loop G–spaces associated to monoidal G–graded categories*, Publ. Res. Inst. Math. Sci. 25 (1989) 239–262 MR Zbl

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Milnor invariants of braids and welded braids up to homotopy

JACQUES DARNÉ

We consider the group of pure welded braids (also known as loop braids) up to (link-)homotopy. The pure welded braid group classically identifies, via the Artin action, with the group of basis-conjugating automorphisms of the free group, also known as the McCool group $P\Sigma_n$. It has been shown recently that its quotient by the homotopy relation identifies with the group $hP\Sigma_n$ of basis-conjugating automorphisms of the reduced free group. We describe a decomposition of this quotient as an iterated semidirect product which allows us to solve the Andreadakis problem for this group, and to give a presentation by generators and relations. The Andreadakis equality can be understood, in this context, as a statement about Milnor invariants; a discussion of this question for classical braids up to homotopy is also included.

20F12, 20F14, 20F18, 20F28, 20F36; 16W25, 16W70, 20F05

Introduction

The present paper is a contribution to the theory of loop braids (also called *welded braids*), via the study of their finite-type invariants. Finite-type invariants were defined by Vassiliev [1990] and were much studied during the 1990s (see for instance [Gusarov 1994; Kontsevich 1993]), giving birth to a whole field of research, which is still very active nowadays. Finite-type invariants of string-links and braids have been the focus of several papers in the late 1990s, by Stanford [1996; 1998], Mostovoy and Willerton [2002], and Habegger and Masbaum [2000]. By then, finite-type invariants of braids were fairly well understood. Meanwhile, a generalization of finite-type invariants to virtual knotted objects was introduced in [Gusarov et al. 2000]. However, it was only much later that this definition was used and studied for welded knotted objects [Bar-Natan and Dancso 2016; 2017]. In the meantime, the interest for welded knotted objects had grown, as the link between welded diagrams, four-dimensional topology and automorphisms of the free group had become more apparent [Baez et al. 2007; Fenn et al. 1997; Satoh 2000]; see [Damiani 2017] for a survey of welded braids. In recent years, the study of these objects has been flourishing; see for instance [Audoux 2016; Bardakov and Bellingeri 2014; Damiani 2019; Kamada 2007; Meilhan and Yasuhara 2019; Nakamura et al. 2018]. In particular, link-homotopy for these objects (corresponding to self-virtualization moves in welded diagrams) has been the focus of several recent papers [Audoux et al. 2017a; 2017b; Audoux and Meilhan 2019].

The invariants under scrutiny in this paper appear naturally as filtrations on groups. Precisely, suppose G is a group whose elements are the objects one is interested in. For example, these could be mapping classes of a manifold, automorphisms of a group, (welded) braids up to isotopy, (welded) braids up to homotopy, etc. Suppose we are also given a filtration of G by subgroups: $G = G_1 \supseteq G_2 \supseteq \cdots$. Then one can consider the class $[g]_d$ of an element $g \in G$ inside G/G_{d+1} and hope to understand g through its approximations $[g]_d$, which become finer and finer as d grows to infinity. These approximation are often easier to understand than g. For instance, $[g]_d$ could be described by a finite family of integers (or other simple mathematical objects), that we would call *invariants of degree at most d*.

With this point of view, the question of comparing different filtrations on the same group (such as the Andreadakis problem — see Section 0.1) can be interpreted as a problem of comparison between different kinds of invariants. Conversely, comparing different notions of invariants on elements of a group can often be interpreted as a problem of comparison between different filtrations on the group, provided that these invariants are indexed by some kind of degree measuring their accuracy, and that they possess some compatibility with the group structure. It is mainly the latter point of view that we adopt below, working with filtrations on groups, with a rather algebraic point of view, getting back to the language of invariants only to interpret our results. This is motivated by the fact that the invariants we consider are strongly compatible with the group structures: not only do they come from filtrations by subgroups, as described above, but these filtrations are *strongly central*, a very nice property allowing us to study them using Lie algebras. Moreover, all the filtrations we consider do have a natural algebraic definition.

We consider mainly three kinds of filtrations (or invariants):

- **Minor invariants** correspond to *Andreadakis-like filtrations* (or the Johnson filtration for the mapping class group). These are defined for automorphism groups of groups, and their subgroups.
- **Finite-type (or Vassiliev) invariants** with coefficients in a fixed commutative ring $\Bbbk$ correspond to the *dimension filtration* $D_*^{\Bbbk} G = G \cap (1 + I^*)$, where I is the augmentation ideal of the group ring $\Bbbk G$.
- **The lower central series** on G is the minimal strongly central filtration on G.

The minimality of the lower central series means that the corresponding invariants of degree d contain as much information as possible for invariants possessing this compatibility with the group structure. Since the two other filtrations are also strongly central, and the Milnor invariants are of finite type, the above list goes from the coarsest invariants to the finest ones. Thus, although we will not always emphasize this in the sequel, the reader should keep in mind that a statement of the form "Milnor invariants of degree at most d distinguish classes of elements $g \in G$ modulo $\Gamma_{d+1}G$" *implies* that Milnor invariants of degree at most d are universal finite-type invariants of degree at most d, and that finite-type invariants of degree at most d distinguish classes of elements $g \in G$ modulo $\Gamma_{d+1}G$.

Main results

We are interested in the group of pure welded braids (or pure welded string-links) up to homotopy. This group identifies, through a version of the Artin action up to homotopy, with the group $hP\Sigma_n$ of (pure) basis-conjugating automorphisms of the *reduced free group* RF_n (see Definition 1.2). The key result of this paper is the decomposition theorem:

Theorem 3.1 *There is a decomposition of $hP\Sigma_n$ into a semidirect product*

$$hP\Sigma_n \cong \left[\left(\prod_{i<n} \mathcal{N}(x_n)/x_i\right) \rtimes (\mathrm{RF}_n/x_n)\right] \rtimes hP\Sigma_{n-1},$$

where $\mathcal{N}(x_n)/x_i$ is the normal closure of x_n inside RF_n/x_i, and the action of $\mathrm{RF}_n/x_n \cong \mathrm{RF}_{n-1}$ on the product is the diagonal one. Moreover, the semidirect product on the right is an almost direct one.

The reduced free group is studied in Section 1. In particular, using the version of the Magnus expansion for the reduced free groups introduced by Milnor, which takes values in the *reduced free algebra*, we are able to show an analogue of Magnus's theorem:

Theorem 1.12 *The Lie ring of the reduced free group identifies with the* ***reduced free Lie algebra*** *on the same set of generators.*

The restriction $hP\Sigma_n \cap \mathcal{A}_*(\mathrm{RF}_n)$ of the Andreadakis filtration $\mathcal{A}_*(\mathrm{RF}_n)$ of RF_n encodes Milnor invariants of pure welded braids. We are able to determine the structure of the associated graded Lie algebra in Section 2.1:

Theorem 2.9 *The Lie algebra $\mathcal{L}(hP\Sigma_n \cap \mathcal{A}_*(\mathrm{RF}_n))$ identifies, via the Johnson morphism, to the algebra of* ***tangential derivations*** *of the reduced free algebra.*

On the other hand, the decomposition of $hP\Sigma_n$ (Theorem 3.1) induces a decomposition of its lower central series, which in turn gives a decomposition of the associated Lie algebra (Theorem 3.8). We are thus able to compare the lower central series and the Andreadakis filtrations via a comparison of their associated graded Lie algebras, getting the promised comparison result, which we also show for the group hP_n of classical pure braids up to homotopy, embedded into $\mathrm{Aut}(\mathrm{RF}_n)$ via the Artin action:

Theorem 3.9 *The Andreadakis equality holds for $G = hP_n$ and $G = hP\Sigma_n$; that is,*

$$G \cap \mathcal{A}_*(\mathrm{RF}_n) = \Gamma_* G.$$

In other words, Milnor invariants of degree at most d classify braids up to homotopy (resp. welded braids up to homotopy) up to elements of $\Gamma_{d+1}(hP_n)$ (resp. $\Gamma_{d+1}(hP\Sigma_n)$).

Notice that there is no obvious link between this theorem and its analogue up to isotopy. On the one hand, for classical braids up to isotopy, the fact that Milnor invariants can detect the lower central series of P_n has been known for a long time [Habegger and Masbaum 2000; Mostovoy and Willerton 2002], but the result up to homotopy is new, and cannot be deduced from the former (as far as I know). On the other hand, for (pure) welded braids (that is, for basis-conjugating automorphisms of the free group), the result up to isotopy is still open. In fact, although [Bardakov 2003, Theorem 1] gives a decomposition of $P\Sigma_n$ similar to our decomposition theorem (see also Remarks 3.4 and 3.6), the pieces of this decomposition are poorly understood, far from the fairly complete description in our setting. Besides, one feature of hP_n and $hP\Sigma_n$ which makes them very different from P_n and $P\Sigma_n$ (and in fact, much easier to handle) is their *nilpotence*, which is used throughout the paper.

Finally, we use our methods to give a presentation of the group $hP\Sigma_n$. A classical result of McCool [1986] asserts that the group $P\Sigma_n$ of (pure) basis-conjugating automorphisms of the free group F_n is the group generated by generators χ_{ij} $(i \neq j)$ submitted to the *McCool relations*:

$$\begin{aligned} [\chi_{ik}\chi_{jk}, \chi_{ij}] &= 1 \quad \text{for } i, j, k \text{ pairwise distinct,} \\ [\chi_{ik}, \chi_{jk}] &= 1 \quad \text{for } i, j, k \text{ pairwise distinct,} \\ [\chi_{ij}, \chi_{kl}] &= 1 \quad \text{if } \{i, j\} \cap \{k, l\} = \varnothing. \end{aligned}$$

We show that we need to add three families of relation to get its quotient $hP\Sigma_n$:

Theorem 5.8 *The pure loop braid group up to homotopy $hP\Sigma_n$ is the group generated by generators χ_{ij} $(i \neq j)$ submitted to the McCool relations on the χ_{ij}, and the three families of relations,*

$$[\chi_{mi}, w, \chi_{mi}] = [\chi_{im}, w, \chi_{jm}] = [\chi_{im}, w, \chi_{mi}] = 1,$$

for $i, j < m$, $i \neq j$, and $w \in \langle \chi_{mk} \rangle_{k<m}$.

The method used for the group can be adapted to the Lie algebra associated to the lower central series of $hP\Sigma_n$. We show in Section 5.3 that it admits a similar presentation. We also give a presentation of the Lie algebra of hP_n in Corollary 3.12.

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0 Reminders: strongly central series and Lie rings

We give here a short introduction to the theory of strongly central filtrations and their associated Lie rings, whose foundations were laid by M Lazard [1954]. Details may be found in [Darné 2019; 2021].

0.1 A very short introduction to the Andreadakis problem

Let G be an arbitrary group. The left and right action of G on itself by conjugation are denoted respectively by $x^y = y^{-1}xy$ and ${}^yx = yxy^{-1}$. The *commutator* of two elements x and y in G is $[x,y] := xyx^{-1}y^{-1}$. If A and B are subsets of G, we denote by $[A,B]$ the subgroup generated by all commutators $[a,b]$ with $(a,b) \in A \times B$. We denote the *abelianization* of G by $G^{\mathrm{ab}} := G/[G,G]$ and its lower central series by $\Gamma_*(G)$; that is,

$$G = \Gamma_1(G) \supseteq [G,G] = \Gamma_2(G) \supseteq [G,\Gamma_2(G)] = \Gamma_3(G) \supseteq \cdots.$$

The lower central series is a fundamental example of a *strongly central filtration* (or *N–series*) on a group G:

Definition 0.1 A *strongly central filtration* G_* on a group G is a nested sequence of subgroups

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$$

such that $[G_i, G_j] \subseteq G_{i+j}$ for all $i, j \geqslant 1$.

In fact, the lower central series is the minimal such filtration on a given group G, as is easily shown by induction.

Recall that when G_* is a strongly central filtration, the quotients $\mathscr{L}_i(G_*) := G_i/G_{i+1}$ are abelian groups, and the whole graded abelian group $\mathscr{L}(G_*) := \bigoplus G_i/G_{i+1}$ is a Lie ring (ie a Lie algebra over $\mathbb{Z}$), where Lie brackets are induced by group commutators. The lower central series of a group is usually difficult to understand, but we are often helped by the fact that its associated Lie algebra is always generated in degree one.

Convention 0.2 If g is an element of a group G endowed with a (strongly central) filtration G_*, the *degree of g with respect to G_** is the minimal integer d such that $g \in G_d - G_{d+1}$. Since most of the filtrations we consider satisfy $\bigcap G_i = \{1\}$, this is well defined (if not, we could just say that $d = \infty$ for elements of $\bigcap G_i$). We often speak of *the class $\bar{g}$ of g in the Lie algebra $\mathscr{L}(G_*)$*, by which we mean the only nontrivial one, in $\mathscr{L}_d(G_*) = G_d/G_{d+1}$, where d is the degree of g with respect to G_*, unless a fixed degree is specified.

When G_* is a strongly central filtration on $G = G_1$, there is a universal way of defining a strongly central filtration on a group of automorphisms of G. Precisely, we get a strongly central filtration on a subgroup of $\mathrm{Aut}(G_*)$, the latter being the group of automorphisms of G preserving the filtration G_*:

$$\mathscr{A}_j(G_*) := \{\sigma \in \mathrm{Aut}(G_*) \mid \forall i \geqslant 1,\ [\sigma, G_i] \subseteq G_{i+j}\}. \tag{0-1}$$

The commutator is computed in $G \rtimes \mathrm{Aut}(G)$, which means that for $\sigma \in \mathrm{Aut}(G)$ and $g \in G$, $[\sigma, g] = \sigma(g)g^{-1}$. Thus, $\mathscr{A}_j(G_*)$ is the group of automorphisms of G_* acting trivially on the quotients G_i/G_{i+j} $(i \geqslant 1)$. For instance, $\mathscr{A}_1(G_*)$ is the group of automorphisms of G_* acting trivially on $\mathscr{L}(G_*)$. When G_* is the lower central series of a group G, then $\mathscr{L}(G) := \mathscr{L}(\Gamma_*(G))$ is generated (as a Lie ring) by $\mathscr{L}_1(G) = G^{\mathrm{ab}}$, so $\mathscr{A}_1(G)$ identifies with the group IA_G of automorphisms of G acting trivially on its abelianization G^{ab}. Thus $\mathscr{A}_*(G) := \mathscr{A}_*(\Gamma_*(G))$ is a strongly central filtration on IA_G, and we can try to understand how it compares to the minimal such filtration on IA_G, which is its lower central series:

Problem 1 (Andreadakis) *For a given group G, how close is the inclusion of $\Gamma_*(IA_G)$ into $\mathscr{A}_*(G)$ to being an equality*?

One way to attack this problem is to restrict to subgroups of IA_G. Precisely, if $K \subseteq IA_G$ is a subgroup, we can consider the following three strongly central filtrations on K:

$$\Gamma_*(K) \subseteq K \cap \Gamma_*(IA_G) \subseteq K \cap \mathscr{A}_*(G).$$

Definition 0.3 We say that the *Andreadakis equality* holds for a subgroup K of IA_G when

$$\Gamma_*(K) = K \cap \mathscr{A}_*(G).$$

Our three main tools in calculating Lie algebras are the following:

Lazard's theorem [1954, Theorem 3.1] (see also [Darné 2019, Theorem 1.36]) If A is a filtered ring (that is, A is filtered by ideals $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ such that $A_i A_j \subseteq A_{i+j}$), the subgroup $A^\times \cap (1 + A_1)$ of $A^\times$ inherits a strongly central filtration $A_*^\times := A^\times \cap (1 + A_*)$ whose Lie ring embeds into the graded ring $\mathrm{gr}(A_*)$, via

$$\mathscr{L}(A_*^\times) \hookrightarrow \mathrm{gr}(A_*), \quad \bar{x} \mapsto \overline{x - 1}.$$

If G is any group endowed with a morphism $\alpha\colon G \to A^\times$, then we can pull the filtration $A_*^\times$ back to G, and $\mathscr{L}(\alpha^{-1}(A_*^\times))$ embeds into $\mathscr{L}(A_*^\times)$, thus into $\mathrm{gr}(A_*)$.

Semidirect product decompositions [Darné 2021, Section 3.1] If G_* is a strongly central filtration, $G_* = H_* \rtimes K_*$ is a *semidirect product of strongly central filtrations* if $G_i = H_i \rtimes K_i$ is a semidirect product of groups for all i, and $[K_i, H_j] \subseteq H_{i+j}$ for all i and j. Then the strong centrality of G_* implies that H_* and K_* must be strongly central. This kind of decomposition is useful because it induces a decomposition of Lie algebras

$$\mathscr{L}(G_*) = \mathscr{L}(H_*) \rtimes \mathscr{L}(K_*).$$

Now, if $G = H \rtimes K$ is any semidirect product of groups, then its lower central series decomposes into a semidirect product $\Gamma_*(G) = \Gamma_*^K(H) \rtimes \Gamma_*(K)$ of strongly central filtrations, where $\Gamma_*^K(H)$ is defined by

$$H = \Gamma_1^K(H) \supseteq [G, H] = \Gamma_2^K(H) \supseteq [G, \Gamma_2^K(H)] = \Gamma_3^K(H) \supseteq \cdots.$$

When the semidirect product is an *almost-direct* one, which means that K acts trivially on H^{ab}, then $\Gamma_*^K(H) = \Gamma_*(H)$, so in this case

$$\mathscr{L}(H \rtimes K) = \mathscr{L}(H) \rtimes \mathscr{L}(K).$$

The Johnson morphism [Darné 2019, Section 1.4] A very useful tool to study a filtration of the form $\mathscr{A}_*(G_*)$ is the Johnson morphism, which encodes the fact that the associated graded Lie algebra $\mathscr{L}(\mathscr{A}_j(G_*))$ acts faithfully on the graded Lie algebra $\mathscr{L}(G_*)$. It is defined by

$$\tau\colon \mathscr{L}(\mathscr{A}_*(G_*)) \hookrightarrow \mathrm{Der}(\mathscr{L}(G_*)), \quad \bar{\sigma} \mapsto \overline{[\sigma, -]},$$

which means that it is induced by $\sigma \mapsto (x \mapsto \sigma(x)x^{-1})$. Its injectivity comes from the universality of the filtration $\mathscr{A}_*(G_*)$.

If we want to compare the filtration $\mathscr{A}_*(G_*)$ with another one, we can do so using comparison morphisms. For example, if K is a subgroup of $\mathrm{Aut}(G_*)$, the inclusion of $\Gamma_* K$ into $K \cap \mathscr{A}_*(G_*)$ induces a morphism $i_*\colon \mathscr{L}(K) \to \mathscr{L}(K \cap \mathscr{A}_*(G_*))$ which is injective if and only if $\Gamma_* K = K \cap \mathscr{A}_*(G_*)$. Thus we can show the Andreadakis equality by showing the injectivity of the morphism $\tau' := \tau \circ i_*$ (τ' is also sometimes called the Johnson morphism).

0.2 The case of the free group

Before beginning our study of the Andreadakis problem for the reduced free group, it may be useful to recall some basic facts about the free group case. Here F_n denotes the free group on n generators $x_1, \dots, x_n$.

Magnus expansions The assignment $x_i \mapsto 1 + X_i$ defines an embedding of F_n into the group of invertible power series on n noncommuting indeterminates $X_1, \dots, X_n$ with integral coefficients. In fact, it is easy to see that it defines a morphism to $1 + (X_1, \dots, X_n)$, and that this induces (using universal properties) an isomorphism of completed rings,

$$\widehat{\mathbb{Z}F_n} \cong \widehat{T[n]},$$

where the group ring $\mathbb{Z}F_n$ is completed with respect to the filtration by the powers of its augmentation ideal, and the tensor algebra $T[n]$ on n generators $X_1, \dots, X_n$ is completed with respect to the usual valuation. One shows that the above morphism from F_n to this ring is injective by showing directly that the image of a reduced nontrivial word must be nontrivial.

Magnus's theorem Using Lazard's theorem, we can get a surjection of $\mathscr{L}(F_n)$ onto the Lie ring generated in degree one inside $\mathrm{gr}(\widehat{T[n]}) \cong T[n]$, which is the free Lie ring $\mathfrak{L}[n]$ on n generators. Using freeness, one shows that this surjection has to be injective as well;

$$\mathscr{L}(F_n) \cong \mathfrak{L}[n].$$

The Andreadakis problem and the Johnson morphism In the case of the free group, the Johnson morphism defines an embedding of $\mathscr{L}(\mathscr{A}_*(F_n))$ into the Lie ring of derivations of the free Lie ring.

The Andreadakis problem for automorphisms of free groups is a difficult problem. The two filtrations were first conjectured to be equal [Andreadakis 1965, page 253]. This was disproved very recently [Bartholdi 2016], but the methods used do not give a good understanding of what is going on. The Andreadakis equality is known to hold for certain well-behaved subgroups, such as the pure braid group P_n [Darné 2021; Satoh 2017]. However, the problem stays largely open in general. In particular, it is open for the group $P\Sigma_n$ of basis-conjugating automorphisms (that is, for the group of pure welded braids), of which our group $hP\Sigma_n$ is a simpler version.

1 The reduced free group and its Lie algebra

In this first section, we introduce and study the *reduced free group*, which was first introduced by Milnor [1954] as the link group of the trivial link with n components. Using the Magnus expansion defined in [Milnor 1954], we determine its Lie ring.

Notation 1.1 Several of our constructions are functors on the category of sets. For such a functor Φ, we denote by $\Phi[X]$ its value at a set X. When X is finite with n elements, we will often denote $\Phi[X]$ by $\Phi[n]$ or by Φ_n.

1.1 The reduced free group

Definition 1.2 The *reduced free group* on a set X is the group defined by the presentation

$$\mathrm{RF}[X] := \langle X \mid \forall x \in X,\ \forall w \in F[X],\ [x, x^w] = 1\rangle.$$

This means that it is the largest group generated by X such that each element of X commutes with all its conjugates.

Since any x commutes with itself, the relations $[x, x^w]$ of Definition 1.2 can also be written $[x, [x, w]]$. The next result and its proof are taken from [Habegger and Lin 1990, Lemma 1.3]:

Proposition 1.3 *For any integer n, the group RF_n is n–nilpotent. For any set X, the group $\mathrm{RF}[X]$ is residually nilpotent.*

Proof We use the fact that $\mathrm{RF}[-]$ is a functor on pointed sets. First, for a finite set X, we show by induction on $n = |X|$ that $\mathrm{RF}_n = \mathrm{RF}[X]$ is n–nilpotent. This is obvious for $n = 1$, because $\mathrm{RF}_1 \cong \mathbb{Z}$. Suppose that RF_{n-1} is $(n-1)$–nilpotent. If $x \in X$, the normal subgroup $\mathcal{N}(x)$ of $\mathrm{RF}[X]$ generated by x is the kernel of the projection p_x from $\mathrm{RF}[X]$ to $\mathrm{RF}[X - \{x\}]$ sending x to 1. We have an exact sequence

$$1 \to \bigcap_{x\in X} \mathcal{N}(x) \to \mathrm{RF}[X] \xrightarrow{p=(p_x)} \prod_{x\in X} \mathrm{RF}[X - \{x\}].$$

Since the group on the right is $(n-1)$–nilpotent by the induction hypothesis, the morphism p must send $\Gamma_n(\mathrm{RF}[X])$ to 1, so that $\Gamma_n(\mathrm{RF}[X])$ is inside the kernel $\bigcap \mathcal{N}(x)$. Moreover, by definition of the reduced free group, for every $x \in X$, all elements of $\mathcal{N}(x)$ commute with x. Thus, an element of $\bigcap \mathcal{N}(x)$ commutes with all $x \in X$, so it is in the center $\mathcal{Z}(\mathrm{RF}[X])$. As a conclusion, $\Gamma_n(\mathrm{RF}[X]) \subseteq \mathcal{Z}(\mathrm{RF}[X])$, which means exactly that $\mathrm{RF}[X]$ is n–nilpotent.

Suppose now X infinite. Let w be an element of $\mathrm{RF}[X]$. It can be written as a product of a finite number of elements of X and their inverses. Denote be W such a finite subset of X. Then w is inside the image of the canonical injection $\mathrm{RF}[W] \hookrightarrow \mathrm{RF}[X]$, which is split by the projection from $\mathrm{RF}[X]$ to $\mathrm{RF}[W]$ sending $X - W$ to 1. Since $\mathrm{RF}[W]$ is $|W|$–nilpotent, this construction provides a nilpotent quotient of $\mathrm{RF}[X]$ in which the image of w is nontrivial, whence the residual nilpotence of $\mathrm{RF}[X]$. □

1.2 The reduced free algebra

Definition 1.4 Let Y be a set. If $s \geqslant 2$ is an integer, let us define $\Delta_s(Y)$ by

$$\Delta_s(Y) := \{(y_i) \in Y^s \mid \exists i \neq j,\ y_i = y_j\}.$$

The *reduced free algebra* on Y is the unitary associative ring defined by the presentation

$$A[Y] := \langle Y \mid \forall s,\ \forall (y_i) \in \Delta_s(Y),\ y_1 \cdots y_s = 0 \rangle.$$

For short, we often forget the mention of Y when it is clear from the context, and write only A for $A[Y]$.

Fact 1.5 *The algebra $A[Y]$ is graded by the degree of monomials. As a $\mathbb{Z}$–module, $A[Y]$ is a direct factor of the tensor algebra $T[Y]$; a (finite) basis of $A[Y]$ is given by **monomials without repetition** on the generators $y \in Y$, which are monomials of the form $y_1 \cdots y_s$ with $(y_i) \notin \Delta_s(Y)$.*

Proof Let R be the (free) $\mathbb{Z}$–submodule of $T[Y]$ generated by the $y_1 \cdots y_s$ such that $(y_i) \in \Delta_s(Y)$ (monomials *with repetition*). This module is clearly a homogeneous ideal of $T[Y]$. As a consequence, $A = T/R$. Moreover, if we denote by S the (free) $\mathbb{Z}$–submodule of T generated by monomials without repetition, then $T = S \oplus R$ as a $\mathbb{Z}$–module, so $A \cong S$. □

Definition 1.6 Let Y be a set. The *reduced free Lie algebra* on Y is the Lie algebra defined by the presentation

$$R\mathfrak{L}[Y] := \langle Y \mid \forall s,\ \forall (y_i) \in \Delta_s(Y),\ [y_1, \dots, y_s] = 0 \rangle,$$

where $[y_1, \dots, y_s]$ denotes $[y_1, [y_2, [\cdots [y_{s-1}, y_s] \cdots]]]$.

The following result uses some of the combinatorics of the free Lie ring recalled in the appendix:

Proposition 1.7 *The Lie subalgebra of $A[Y]$ generated by Y identifies with $R\mathfrak{L}[Y]$.*

Proof We need to prove that the intersection of the ideal R of relations defining $A[Y]$ with the free Lie algebra $\mathfrak{L}[Y] \subset T[Y]$ is exactly the module S of relations defining $R\mathfrak{L}[Y]$. The inclusion of S into R is clear: when we decompose a relation in S on the basis of TV, only monomials with exactly the same letters appear, counting repetitions. For the converse, let us first remark that thanks to Lemma A.14, S is the submodule of $\mathfrak{L}[Y]$ generated by all Lie monomials with repetition. Let $p \neq 0$ be an element of $R \cap \mathfrak{L}[Y]$, and let us consider its decomposition $p = \sum \lambda_w P_w$ on the Lyndon basis of $\mathfrak{L}[Y]$. Let w be the smallest Lyndon word such that $\lambda_w \neq 0$. It follows from Lemma A.7 that λ_w must be the coefficient of w in the decomposition of p into a linear combination of monomials of TV. Since $p \in R$, the word w must be with repetition, so $P_w \in S$. Then $p - \lambda_w P_w \in R \cap \mathfrak{L}[Y]$ has less terms than p in its decomposition on the Lyndon basis, giving us the result by induction. □

Remark 1.8 When Y is a finite set with n elements, we can extract finite presentations from the above presentations. Indeed, the ideal R and the Lie ideal S are both generated in degrees at most $n+1$, since $R_{n+1} = T[n]_{n+1}$ and $S_{n+1} = \mathfrak{L}[n]_{n+1}$ (a word of length $n+1$ must possess at least a repetition). As a consequence, the relations of degree at most $n+1$ are enough do describe $A[n]$ (resp. $R\mathfrak{L}[n]$), and there are finitely many of them.

Proposition 1.9 *Lyndon monomials* ***without*** *repetition on the y_i are a basis of $R\mathfrak{L}[Y]$. The rank of the degree-k part $R\mathfrak{L}[n]_k$ of $R\mathfrak{L}[n]$ is $(k-1)!\binom{n}{k}$.*

Proof Lemma A.14 implies that the module S in the proof of Proposition 1.7 is the submodule generated by all Lyndon monomials with repetition, which are thus a basis of S. As a consequence, Lyndon monomials *without* repetition give a basis of the quotient $R\mathfrak{L}[Y] = \mathfrak{L}[Y]/S$.

In order to determine the ranks, we need to count Lyndon words without repetition of length k in $y_1, \dots, y_n$. A word without repetition is Lyndon if and only if its first letter is the smallest one. Such a word is determined by the choice of k letters, and a choice of ordering of the $(k-1)$ letters left when the smallest one is removed. This gives $(k-1)!\binom{n}{k}$ such words, as announced. □

Proposition 1.10 *In $A[Y]^\times$, each element of $1+Y$ commutes with all its conjugates.*

Proof Let y be an element of Y. From the relation $y^2 = 0$, we deduce that $1+y$ is invertible, with $1-y$ as its inverse. Let $u \in A^\times$. Then $u(1+y)u^{-1} = 1 + uyu^{-1}$. Since $yAy = 0$, we can write

$$(1+y)(1+uyu^{-1}) = 1 + y + uyu^{-1} = (1+uyu^{-1})(1+y),$$

which is the desired conclusion. □

Notation 1.11 From now on, we denote by X and Y two sets endowed with a bijection $X \cong Y$ that we will denote by $x_i \mapsto y_i$ (we consider both X and Y indexed by a bijection from a set of indices I). This notation will allow us to distinguish between the group-theoretic world and its algebraic counterpart.

From Proposition 1.10, we get a well-defined morphism, which is an analogue of the Magnus expansion, and was introduced by Milnor [1954, Section 4],

$$\mu\colon \mathrm{RF}[X] \to A[Y]^\times, \quad x_i \mapsto 1 + y_i. \tag{1-1}$$

From Lazard's theorem [1954, Theorem 3.1] (see also [Darné 2019, Theorem 1.36]), we get an associated morphism between graded Lie algebras,

$$\bar{\mu}\colon \mathscr{L}(\mathrm{RF}[X]) \to \mathrm{gr}(A[Y]) \cong A[Y], \quad \bar{x}_i \mapsto y_i. \tag{1-2}$$

From this we deduce our first main theorem:

Theorem 1.12 *The above morphism* (1-2) *induces a canonical isomorphism between the Lie algebra of the reduced free group and the reduced free algebra*,

$$\mathscr{L}(\mathrm{RF}[X]) \cong R\mathfrak{L}[Y].$$

Proof Since $\mathscr{L}(\mathrm{RF}[X])$ is generated in degree 1 [Darné 2019, Proposition 1.19] (that is, generated by the $\bar{x}_i$), the morphism (1-2) defines a surjection from $\mathscr{L}(\mathrm{RF}[X])$ onto the Lie subalgebra of A generated by Y, which is $R\mathfrak{L}[Y]$ (Proposition 1.7). But $\mathscr{L}(\mathrm{RF}[X])$ is a reduced Lie algebra on X, by which we mean that the relations on the y_i defining $R\mathfrak{L}[Y]$ are true for the classes $\bar{x}_i$. Indeed, in $\mathrm{RF}[X]$, the normal closure $\mathscr{N}(x)$ of a generator $x \in X$ is commutative. As a consequence, if u is any element of $\mathscr{N}(x)$, then $[x, u] = 1$. Applying this to $u = [x_{r+1}, \dots, x_s, x, w] \in \mathscr{N}(x)$ (where our notation for iterated commutators is the same as above for iterated brackets in Lie algebras), we see that any $[x_1, \dots, x_r, x, x_{r+1}, \dots, x_s, x, w]$ is trivial in the group, hence so is its class in the Lie algebra. Thus $y_i \mapsto \bar{x}_i$ defines an inverse to our surjection, which has to be an isomorphism. □

Corollary 1.13 *The morphism* $\mu\colon x_i \mapsto 1 + y_i$ (1-1) *from* $\mathrm{RF}[X]$ *to* $A[Y]^\times$ *is injective.*

Proof Let w be an element of $\ker(\mu)$. If $w \neq 1$, then, by residual nilpotence of $\mathrm{RF}[X]$ (Proposition 1.3), there exists an integer k such that $w \in \Gamma_k - \Gamma_{k+1}$. Thus, $\bar{w}$ is a nontrivial element of $\mathscr{L}_k(\mathrm{RF}[X])$, sent to 0 by $\bar{\mu}$. But $\bar{\mu}$ is an isomorphism (Theorem 1.12), so this is not possible; our element w must be trivial. □

Remark 1.14 This statement also appears in [Bar-Natan 1995]; compare Proposition 5.2 therein.

Some remarks on finite presentations of nilpotent groups Every nilpotent group of finite type admits a finite presentation. This fact is easy to prove, by induction on the nilpotency class, using that finitely generated abelian groups are finitely presented, and that an extension of finitely presented groups is finitely presented. As a consequence, the reduced free group RF_n on $x_1, \dots, x_n$ must admit a finite presentation. Can we find a simple one? Considering that we have a finite presentation of the associated Lie algebra, the problem does not seem to be difficult, at first glance. Indeed, let G_n is the group admitting the same finite presentation as $R\mathscr{L}_n$ (see Remark 1.8), where brackets are replaced by commutators. These relations are true in RF_n (see the proof of Theorem 1.12), thus there is a map π from G_n onto RF_n, which must induce an isomorphism at the level of Lie rings. However, we can deduce that π is an isomorphism *only if we know that both these groups are nilpotent.* Which raises the question: do the relations defining G_n imply that it is nilpotent?

Thus we are led to ask ourselves: *what finite set of relation is needed to ensure that a group is nilpotent*? This question is strongly related to the following question: *can we give a simple finite presentation of the free nilpotent group of class c* (where “simple” is taken in some naive sense)?. This question is surprisingly difficult. The reader can convince himself that killing commutators of the form $[x_{i_0}, \dots, x_{i_c}]$ (or even $[x_{i_0}^{\pm}, \dots, x_{i_c}^{\pm}]$) does not seem to be enough, because the usual formulas of commutator calculus seem not to allow one to reduce to commutators of this particular form and length. Even killing all iterated commutators of length $c+1$ of the generators is only conjectured to be enough [Jackson 2008; Sims 1987].

To get a presentation known to work in general, we must take a much larger one. For instance, one can kill all iterated commutators of the generators of length between $c+1$ and $2c$. This can be improved slightly by killing only relations of the form $[x, y]$, where x and y are iterated commutators of the generators of length at most c, whose length add up to at least $c+1$. Indeed, all iterated commutators of length greater than c can be written as a product of conjugates of iterated commutators of the generators of length greater than c (by repeated use of the formulas $[a, bc] = [a, b] \cdot [a, c] \cdot [[c, a], b]$ and $[a, b^{-1}] = [b, a]^b$). And every such commutator has a subcommutator of the given form (to see that, it can help to think of commutator words as rooted planar binary trees).

In order to avoid these problems, and to keep our presentations simple, we will only give a presentation of RF_n *as a nilpotent group*, that is, we *assume* that the group G_n in the reasoning above is nilpotent, thus obtaining:

Proposition 1.15 *The reduced free group RF_n is the quotient of the free n–nilpotent group on $x_1, \dots, x_n$ by the finite set of relations*

$$\forall s \leqslant n \quad \forall (x_i) \in \Delta_s(X) \quad [x_1, \dots, x_s] = 1,$$

where $[x_1, \dots, x_s]$ denotes $[x_1, [x_2, [\cdots [x_{s-1}, x_s] \cdots]]]$.

The subtlety of this situation was not perceived in [Cohen 1995], where it was assumed that this presentation (with $n+1$ commutators included) would automatically define a nilpotent group. Note that several results of the present paper give some insight on the group-theoretic results of [Cohen 1995], which were stated only in terms of the underlying abelian groups, and become simpler when taking into account the Lie ring structure.

1.3 Centralizers

We will use Corollary 1.13 to compute the centralizers of generators in $\mathrm{RF}[X]$. First, we show a lemma about commutation relations in $A[Y]$:

Lemma 1.16 *Let $y \in Y$, and let λ be an integer. Define the λ–centralizer $C_\lambda(y)$ of y in $A[Y]$ to be*

$$C_\lambda(y) := \{u \in A[Y] \mid uy = \lambda yu\}.$$

If $\lambda \neq 1$, then $C_\lambda(y)$ is exactly $\langle y\rangle$. If $\lambda = 1$, then $C_\lambda(y) = \mathbb{Z}\cdot 1 \oplus \langle y\rangle$. As a consequence, $\mathbb{Z}\cdot 1 \oplus \langle y\rangle$ is the centralizer $C(y)$ of y. Also, $\langle y\rangle$ is the annihilator $\mathrm{Ann}(y)$ *of y, and it is also the set of elements u satisfying $uy = -yu$.*

Proof If u is an element of $\langle y\rangle$, then $uy = \lambda yu = 0$. Moreover, obviously, $1 \in C_1(y)$. This proves one inclusion. Let us prove the converse. Let u be an element of $C_\lambda(y)$. Let us decompose u as a sum of monomials without repetition $\sum \lambda_\alpha m_\alpha$ in A, and consider a monomial $m_\alpha \neq 1$ not containing y. Then λ_α is the coefficient of $m_\alpha y$ in $0 = uy - \lambda yu$, so it must be zero. Also, if μ is the coefficient of 1 in m, then the coefficient of y in $uy - \lambda yu$ is $(1-\lambda)\mu$, hence $\mu = 0$ if $\lambda \neq 1$. Thus all the monomials appearing in the decomposition of u (except possibly 1 if $\lambda = 1$) must contain y, so that u belongs to $\langle y\rangle$ (resp. to $\mathbb{Z} \oplus \langle y\rangle$ if $\lambda = 1$). □

The next lemma is [Habegger and Lin 1990, Lemma 1.10]:

Lemma 1.17 *Let $x \in X$. Let $C(x)$ be the centralizer of x in* $\mathrm{RF}[X]$. *Then $C(x)$ is exactly the normal closure $\mathcal{N}(x)$ of x.*

Proof The inclusion $\mathcal{N}(x) \subseteq C(x)$ follows from the definition of $\mathrm{RF}[X]$. Let us prove the converse. From Corollary 1.13, we know that $C(x) = C(1+y) \cap \mathrm{RF}[X] = (\mathbb{Z} \oplus \langle y\rangle) \cap \mathrm{RF}[X]$. Moreover, $\mathrm{RF}[X] \hookrightarrow A[Y]$ takes values in $1 + \bar{A}[Y]$ (where $\bar{A}$ is the augmentation ideal of A, that is, the set of polynomials with no constant term). As a consequence, this intersection is $(1 + \langle y\rangle) \cap \mathrm{RF}[X]$. But $1 + \langle y\rangle$ is exactly the set of elements sent to 1 by the projection $A[Y] \twoheadrightarrow A[Y - \{y\}]$. This projection induces the projection from $\mathrm{RF}[X]$ to $\mathrm{RF}[X - \{x\}]$, whose kernel is $\mathcal{N}(x)$, whence the result. □

Lemma 1.18 *Let $y \in Y$. Let $C_{\mathfrak{L}}(y)$ be the centralizer of y in $R\mathfrak{L}[Y]$. Then $C_{\mathfrak{L}}(y)$ is exactly the Lie ideal $\langle y\rangle$ generated by y.*

Proof If we now denote by $\langle y\rangle_A$ the ideal generated by y in A (denoted by $\langle y\rangle$ above), we have that $C_{\mathfrak{L}}(y) = C_1(y) \cap R\mathfrak{L}[Y] = \langle y\rangle_A \cap R\mathfrak{L}[Y]$ is the submodule of $R\mathfrak{L}[Y]$ generated by Lie monomials in which y appears, which is exactly $\langle y\rangle$. □

Proposition 1.19 *The center of* RF_n *is the intersection of the $\mathcal{N}(x_i)$, and also coincides with* $\Gamma_n(\mathrm{RF}_n)$; *it is free abelian of rank* $(n-1)!$

Proof The inclusions $\Gamma_n(\mathrm{RF}_n) \subseteq \bigcap \mathcal{N}(x_i) \subseteq \mathcal{Z}(\mathrm{RF}_n)$ were established in the proof of Proposition 1.3. Let w be a nontrivial element of $\mathcal{Z}(\mathrm{RF}_n)$. Since RF_n is nilpotent, $w \in \Gamma_k - \Gamma_{k+1}$ for some k, and $\bar{w}$ is a nontrivial element in the center of $\mathcal{L}(\mathrm{RF}_n) \cong R\mathfrak{L}_n$ (see Theorem 1.12). From Lemma 1.18, we deduce that $\bar{w}$ is in the Lie ideal $\langle y_1\rangle \cap \cdots \cap \langle y_n\rangle$. As a consequence, all y_i appear at least once in each Lie monomial of the decomposition of $\bar{w}$. Thus its degree must be at least n, which means that $w \in \Gamma_n(\mathrm{RF}_n)$.

Moreover, $\Gamma_n(\mathrm{RF}_n) = \Gamma_n(\mathrm{RF}_n)/\Gamma_{n+1}(\mathrm{RF}_n) = \mathcal{L}_n(\mathrm{RF}_n)$ identifies with the degree-n part $R\mathfrak{L}[n]_n$ of $R\mathfrak{L}[n]$, which is free abelian of rank $(n-1)!$ by Proposition 1.9. □

2 Derivations and the Johnson morphism

In order to tackle the Andreadakis problem for RF_n, we need to understand the associated Johnson morphism, whose target is the algebra of derivations of the reduced free Lie algebra.

2.1 Derivations

We begin our study of derivations by those of $A[Y]$, which are quite easy to handle.

Proposition 2.1 *Any derivation d of $A[Y]$ sends each element y of Y to an element of the ideal $\langle y\rangle$. Conversely, any application $d_Y : Y \to A[Y]$ sending each y into $\langle y\rangle$ extends uniquely to a derivation of $A[Y]$.*

Proof First, given a derivation d, we can apply it to the relation $y^2 = 0$. We get that $(dy)y + y(dy) = 0$. Thus $dy \in C_{-1}(y)$, which means that $dy \in \langle y\rangle$ by Lemma 1.16.

Suppose now that we are given a map $d_Y : Y \to A[Y]$ sending each y into $\langle y\rangle$. Then d_Y extends uniquely to a derivation d_T from $T[Y]$ to $A[Y]$ (the latter being a $T[Y]$–bimodule in the obvious sense) in the usual way,

$$d_T(y_{i_1}\cdots y_{i_l}) := \sum_{j=1}^{l} y_{i_1}\cdots y_{i_{j-1}} \cdot d_Y(y_{i_j}) \cdot y_{i_{j-1}} \cdots y_{i_l}.$$

From the hypothesis on d_Y, we deduce that d vanishes on the monomials with repetition (the sum on the left being a sum of monomials with repetition in this case), so it induces a well-defined derivation $d : A[Y] \to A[Y]$ extending d_Y. Unicity is obvious from the fact that Y generates the ring $A[Y]$. □

We now turn to the study of derivations of $R\mathfrak{L}[Y]$. We consider only derivations (strictly) increasing the degree, that is, sending Y into $R\mathfrak{L}[Y]_{\geqslant 2}$. In fact, we will mostly be concerned with homogeneous such derivations (which raise the degree by a fixed amount), but we will see that this distinction is not important for $R\mathfrak{L}[Y]$ (Corollary 2.3).

Proposition 2.2 *Let d be a derivation of $R\mathfrak{L}[Y]$. Then for any $y \in Y$,*

$$dy \in \langle y\rangle + \bigcap_{y' \neq y} \langle y'\rangle =: J_y,$$

where $\langle y\rangle$ is the Lie ideal generated by y. Conversely, any map from Y to $R\mathfrak{L}[Y]_{\geqslant 2}$ satisfying this condition can be extended uniquely to a derivation of $R\mathfrak{L}[Y]$.

Let us remark that the homogeneous ideal J_y differs from $\langle y\rangle$ only in degree $|Y| - 1$ (in particular, only when Y is finite), since the second term is generated by Lie monomials without repetition where all y' appear, save possibly y. Moreover, one easily sees that, for $|Y| = n$, the ideal J_y contains all of $R\mathfrak{L}[n]_{n-1}$.

Proof of Proposition 2.2 For $|Y|=2$, remark that $R\mathfrak{L}[Y]_2 \subset \langle y_1\rangle \cap \langle y_2\rangle$ and $R\mathfrak{L}[Y]_{\geqslant 3}=\{0\}$. As a consequence, any linear map raising the degree satisfies the condition and defines a derivation, so we have nothing to show.

Let us suppose that Y has at least three elements. Let d be a derivation of $R\mathfrak{L}[Y]$, and let $y \in Y$. Take $z \in Y-\{y\}$, and consider the relation $0=d([y,z,y])=[dy,z,y]+[y,z,dy]$. Let us decompose dy as a sum of monomials in $A[Y]$. Let m be a monomial which contains neither y nor z, and let λ be the coefficient of m in dy. Then the monomial mzy appears with coefficient 2λ in the decomposition of $[dy,z,y]+[y,z,dy]$, so λ must be trivial. Since this is true for any $z \neq y$, the only monomials without repetition not containing y that can appear in dy are the ones containing every element of Y save y, which are exactly the generators of J_y modulo $\langle y\rangle$. This shows that $dy \in J_y$.

To show the converse, we can restrict to homogeneous maps, since any map from Y to $R\mathfrak{L}[Y]_{\geqslant 2}$ is a sum of homogeneous ones, and a sum of derivations is a derivation. Suppose that we are given a homogeneous map $d_Y\colon Y \to R\mathfrak{L}[Y]_{\geqslant 2}$ sending each y into J_y. If d_Y is not of degree $|Y|-2$, this condition amounts to $d_Y(y) \in \langle y\rangle$. This Lie ideal stands inside the associative ideal $\langle y\rangle \subset A[Y]$. We can thus use Proposition 2.1 to extend this map to a derivation of $A[Y]$. This derivation sends Y into $R\mathfrak{L}[Y]$, hence it preserves $R\mathfrak{L}[Y] \subset A[Y]$. As a consequence, it restricts to a derivation of $R\mathfrak{L}[Y]$ extending d_Y.

We are left to study the case when Y has n elements and d_Y is of degree $n-2$. Then the conditions on the elements $d_Y(y)$ are empty. We can still extend d_Y to a derivation from $T[Y]$ to $A[Y]$, as in the proof of Proposition 2.1, but it does not vanish on the relations defining $A[Y]$. However, the induced Lie derivation from $\mathfrak{L}[Y]$ to $R\mathfrak{L}[Y]$ does vanish on the Lie monomials with repetition. Indeed, it vanishes on all elements of degree at least 3 (sent to $R\mathfrak{L}[Y]_{\geqslant n+1}=\{0\}$), and there are no such monomials in degree 2, since the elements $[y,y]$ are already trivial in $\mathfrak{L}[Y]$. As a consequence, it induces a well-defined derivation from $R\mathfrak{L}[Y]$ to itself. This derivation extends d_y and is the only one to do so, since $R\mathfrak{L}[Y]$ is generated by Y. □

Corollary 2.3 *Any derivation of $R\mathfrak{L}_n$ is the sum of homogeneous components*,

$$\mathrm{Der}(\mathfrak{L}_n) \cong \bigoplus_{k \geqslant 1} \mathrm{Der}_k(R\mathfrak{L}_n).$$

Proof If d is such a derivation, Proposition 2.2 shows that the homogeneous components of its restriction to Y extend uniquely to derivations of $R\mathfrak{L}_n$, whose sum coincides with d on Y, hence everywhere. Note that it makes sense to speak of this sum, because Y is finite, so that the number of nontrivial homogeneous components of $d|_Y$ is finite. □

The following theorem is an analogue of [Darné 2019, Proposition 2.41], replacing free nilpotent groups by reduced free groups.

Theorem 2.4 *Let $n \geqslant 2$ be an integer. The Johnson morphism is an isomorphism*:

$$\mathscr{L}(\mathscr{A}_*(\mathrm{RF}_n)) \cong \mathrm{Der}(R\mathfrak{L}_n).$$

Proof Take $|X|=|Y|=n$. Let d be a derivation of $R\mathfrak{L}[Y]$, of degree k. We need to lift it to an automorphism φ of RF$[X]$. We first suppose that $k \neq n-2$. Since $d(y_i) \in \langle y_i \rangle \cap R\mathfrak{L}_{k+1}[Y]$ (Proposition 2.2), we can write each $d(y_i)$ as a linear combination of Lie monomials of length $k+1$ containing y_i. The corresponding product of brackets in RF$[X]$ lifts $d(y_i)$ to an element w_i of $\Gamma_{k+1}(\mathrm{RF}[X]) \cap \mathcal{N}(x_i)$. The element $w_i x_i$ belongs to $\mathcal{N}(x_i)$, so it commutes with all its conjugates. As a consequence, $x_i \mapsto w_i x_i$ defines an endomorphism φ of RF$[X]$. Since φ acts trivially on the abelianization of RF$[X]$, which is nilpotent, it is an automorphism [Darné 2019, Lemma 2.38]. Moreover, by construction, $\tau(\bar{\varphi}) = d$.

Suppose now that $k = n-2$. Then $d(y_i)$ can be any element of $R\mathfrak{L}_{n-1}[Y]$. Choose any lift w_i in $\Gamma_{n-1}(\mathrm{RF}[X])$ of $d(y_i)$. Using the usual formulas of commutator calculus, we see that for any $w \in \mathrm{RF}[X]$, $[w_i x_i, w, w_i x_i] \equiv [x_i, w, x_i] \pmod{\Gamma_{n+1}}$. Since $[x_i, w, x_i] = 1$ and $\Gamma_{n+1}(\mathrm{RF}_n) = \{1\}$, we conclude that $[w_i x_i, w, w_i x_i] = 1$, which means exactly that $w_i x_i$ commutes with all its conjugate. The same construction as in the first case then gives an automorphism $\varphi \in \mathcal{A}_{n-2}$ such that $\tau(\bar{\varphi}) = d$. □

2.2 Tangential derivations

Definition 2.5 A *tangential derivation* of $R\mathfrak{L}[Y]$ is a derivation sending each $y \in Y$ to an element of the form $[y, w_y]$ (for some $w_y \in R\mathfrak{L}[Y]$).

Fact 2.6 *The subset* $\mathrm{Der}_\tau(R\mathfrak{L}[Y])$ *of tangential derivations is a Lie subalgebra of* $\mathrm{Der}(R\mathfrak{L}[Y])$.

Proof Let $d\colon y \mapsto [y, w_y]$ and $d'\colon y \mapsto [y, w'_y]$. Then an elementary calculation gives

$$[d, d'](y) = [y, [w_y, w'_y] + d(w'_y) - d'(w_y)], \tag{2-1}$$

whence the result. □

Proposition 2.7 *Let $n \geqslant 2$ be an integer. The Lie subalgebra of* $\mathrm{Der}(R\mathfrak{L}_n)$ *generated in degree* 1 *is the subalgebra* $\mathrm{Der}_\tau(R\mathfrak{L}_n)$ *of tangential derivations.*

Proof Consider the derivation d_{ij} sending y_i to $[y_i, y_j]$ and all the other y_k to 0. From Proposition 2.2, we know that these generate the module of derivations of degree 1. They are tangential derivations, so the Lie subalgebra they generate is inside $\mathrm{Der}_\tau(R\mathfrak{L}[Y])$. Let us show that it is all of $\mathrm{Der}_\tau(R\mathfrak{L}[Y])$. Consider the set D_i of tangential derivations sending all y_k to 0, save the i^{th} one. Such derivations vanish on all monomials which are not in $\langle y_i \rangle$, and preserve $\langle y_i \rangle$. Since elements of $\langle y_i \rangle$ commute with y_i, formula (2-1) implies that

$$c_i\colon \mathcal{R}\mathfrak{L}[Y] \to D_i, \quad t \mapsto (y_i \mapsto [y_i, t]),$$

is a morphism. It is obviously surjective, so D_i is a Lie subalgebra of $\mathrm{Der}_\tau(R\mathfrak{L}[Y])$. Moreover, its kernel is $\langle y_i \rangle$ (Lemma 1.18), so $D_i \cong R\mathfrak{L}[Y]/y_i$ is in fact the free reduced Lie algebra on the $c_i(y_j) = d_{ij}$ (for $j \neq i$). Since $\mathrm{Der}_\tau(R\mathfrak{L}[Y])$ is the (linear) finite direct sum of the D_i, it is indeed generated (as a Lie algebra) by the d_{ij}. □

Recall that the *McCool group* $P\Sigma_X$ is the group of automorphisms of the free group $F[X]$ on a set X fixing the conjugacy class of each generator $x \in X$.

Definition 2.8 The *reduced McCool group* $hP\Sigma_X$ is the subgroup of $\mathrm{Aut}(\mathrm{RF}[X])$ preserving the conjugacy class of each generator $x \in X$ of $\mathrm{RF}[X]$.

This group $hP\Sigma_X$ is also called $\mathrm{Aut}_C(\mathrm{RF}_X)$, but we prefer to think of it as version of $P\Sigma_X$ *up to homotopy* (this terminology will be explained in Section 4). When X is finite, we denote its elements by $x_1, \dots, x_n$ and $hP\Sigma_X$ by $hP\Sigma_n$.

Consider the filtration $\mathcal{A}_*(\mathrm{RF}_n)$ on $\mathrm{Aut}(\mathrm{RF}_n)$. It restricts to a filtration $hP\Sigma_n \cap \mathcal{A}_*(\mathrm{RF}_n)$ on $hP\Sigma_X$. Moreover, since $\mathcal{A}_*(\mathrm{RF}_n)$ is strongly central on the subgroup $\mathcal{A}_1(\mathrm{RF}_n)$ of automorphisms acting trivially on $\mathrm{RF}_n^{\mathrm{ab}}$, and $hP\Sigma_n \subset \mathcal{A}_1(\mathrm{RF}_n)$, this induced filtration is strongly central on $hP\Sigma_n$.

Theorem 2.9 *Let $n \geqslant 2$ be an integer. The Johnson morphism induces an isomorphism*

$$\mathcal{L}(hP\Sigma_n \cap \mathcal{A}_*(\mathrm{RF}_n)) \cong \mathrm{Der}_\tau(R\mathfrak{L}_n).$$

Proof Let $\varphi\colon x_i \mapsto x_i^{w_i}$ be a basis-conjugating automorphism belonging to $\mathcal{A}_k - \mathcal{A}_{k+1}$. Then

$$\tau(\varphi)(y_i) = [y_i, \overline{w}_i]$$

(where $\overline{w}_i$ is the class of w_i in Γ_k/Γ_{k+1}), so the Johnson morphism sends $\mathcal{L}(hP\Sigma_n \cap \mathcal{A}_*(\mathrm{RF}_n))$ into Der_τ. Moreover, it is injective by Theorem 2.4, and since $\tau(\chi_{ij}) = d_{ij}$, Proposition 2.7 implies that it is surjective. □

Theorem 2.9, together with Proposition 2.7, have an interesting consequence: *the group $hP\Sigma_n$ is maximal among subgroups of* $\mathrm{Aut}(\mathrm{RF}_n)$ *for which the Andreadakis equality can be true.* Indeed, let $hP\Sigma_n \subsetneq G \subseteq \mathrm{Aut}(\mathrm{RF}_n)$, and consider the comparison morphism $i_*\colon \mathcal{L}(G) \to \mathcal{L}(G \cap \mathcal{A}_*)$ obtained from the inclusion of $\Gamma_* G$ into $G \cap \mathcal{A}_*$. On the one hand, the Lie algebra $\mathcal{L}(G \cap \mathcal{A}_*)$ contains $\mathcal{L}(hP\Sigma_n \cap \mathcal{A}_*)$, and this inclusion must be strict, otherwise we could argue as in the proof of Lemma 5.3 to show that $G = hP\Sigma_n$. On the other hand, $\mathcal{L}(G)$ is generated in degree 1, so that $i_*(\mathcal{L}(G)) \subseteq \mathcal{L}(hP\Sigma_n \cap \mathcal{A}_*)$, the latter being the subalgebra of $\mathcal{L}(\mathcal{A}_*(\mathrm{RF}_n)) \cong \mathrm{Der}(R\mathfrak{L}_n)$ generated by its degree one. As a consequence, i_* cannot be surjective, whence the conclusion.

Here is another consequence of these theorems:

Corollary 2.10 *The group $hP\Sigma_n$ is generated by the χ_{ij} $(i \neq j)$, and $hP\Sigma_n^{\mathrm{ab}}$ identifies with the free abelian group generated by the $\bar{\chi}_{ij}$.*

In particular, the canonical morphism from $P\Sigma_n$ to $hP\Sigma_n$ is surjective. This means that when it comes to basis-conjugating automorphisms, all automorphisms of RF_n are *tame*. This is in striking contrast with the case of free nilpotent groups [Darné 2019, Section 2.6]. This fact is in fact obvious from the geometrical interpretation (recalled in Section 4), but we give an algebraic proof here, using much less machinery.

Proof of Corollary 2.10 Thanks to Proposition 2.7 and Theorem 2.9, we know that the classes of the χ_{ij} in $\mathscr{L}(\mathscr{A}_* \cap hP\Sigma_n)$ generate this Lie ring. By applying Lemma 5.3 to the finite filtration $\mathscr{A}_* \cap hP\Sigma_n$, we deduce that the χ_{ij} generate $hP\Sigma_n$.

As a consequence, the $\bar{\chi}_{ij}$ generate its abelianization. Moreover, the Johnson morphism from $hP\Sigma_n^{\mathrm{ab}}$ to $\mathrm{Der}_1(R\mathfrak{L}_n)$ sends the $\bar{\chi}_{ij}$ to the linearly independent elements d_{ij} of $\mathrm{Der}_1(R\mathfrak{L}_n)$. Thus the $\bar{\chi}_{ij}$ are a basis of $hP\Sigma_n^{\mathrm{ab}}$. □

We can also use the proof of Proposition 2.7 to compute the *Hirsch rank* of the nilpotent group $hP\Sigma_n$ (which is the rank of any associated Lie algebra). We recover the formula of [Audoux et al. 2017b, Remark 4.9]:

Corollary 2.11 *The Hirsch rank of the reduced McCool group is*

$$\mathrm{rk}(hP\Sigma_n) = \mathrm{rk}(\mathrm{Der}_\tau(R\mathfrak{L}_n)) = n \cdot \mathrm{rk}(R\mathfrak{L}_{n-1}) = \sum_{k=1}^{n-1} \frac{n!}{(n-k-1)! \cdot k}.$$

Proof The first equality is a direct consequence of Theorem 2.9. The second one stems from the proof of Proposition 2.7, where we have shown that $\mathrm{Der}_\tau(R\mathfrak{L}_n)$ is (linearly) a direct sum of n copies D_i of $R\mathfrak{L}_{n-1}$. The last one is a direct application of Proposition 1.9. □

3 The Andreadakis problem

The McCool group $P\Sigma_n \subset \mathrm{Aut}(F_n)$ is generated by the elements $\chi_{ij} : x_i \mapsto x_i^{x_j}$ (χ_{ij} fixes all the other x_t). The following relations, called the *McCool relations*, are known to define a presentation of the McCool group $P\Sigma_n$ [1986]. The reader can easily check that they are satisfied in $P\Sigma_n$:

$$\begin{aligned} [\chi_{ik}\chi_{jk}, \chi_{ij}] &= 1 \quad \text{for } i, j, k \text{ pairwise distinct,} \\ [\chi_{ik}, \chi_{jk}] &= 1 \quad \text{for } i, j, k \text{ pairwise distinct,} \\ [\chi_{ij}, \chi_{kl}] &= 1 \quad \text{if } \{i, j\} \cap \{k, l\} = \varnothing. \end{aligned}$$

Thanks to Corollary 2.10, we know that $hP\Sigma_n$ is naturally a quotient of $P\Sigma_n$. We will give in Section 5 three families of relations that need to be added to a presentation of $P\Sigma_n$ in order to get a presentation of $hP\Sigma_n$. This will rely on the semidirect product decomposition that we now describe.

3.1 A semidirect product decomposition

The following decomposition theorem is the central result of the present paper. From it we will deduce the Andreadakis equality for $hP\Sigma_n$ (Section 3.3) and a presentation of this group and of its Lie ring (Section 5):

Theorem 3.1 *There is a decomposition of $hP\Sigma_n$ as a semidirect product,*

$$hP\Sigma_n \cong \left[\left(\prod_{i<n} \mathcal{N}(x_n)/x_i\right) \rtimes (\mathrm{RF}_n/x_n)\right] \rtimes hP\Sigma_{n-1},$$

where $\mathcal{N}(x_n)/x_i$ is the normal closure of x_n inside RF_n/x_i, and the action of $\mathrm{RF}_n/x_n \cong \mathrm{RF}_{n-1}$ on the product is the diagonal one. Moreover, the semidirect product on the right is an almost direct one.

We prove this theorem in three steps. First, we show that $hP\Sigma_n$ decomposes into a semidirect product $\mathcal{K}_n \rtimes hP\Sigma_{n-1}$. Then we investigate the structure of $\mathcal{K}_n$, which decomposes as $\mathcal{K}'_n \rtimes \mathrm{RF}_{n-1}$. Finally, we investigate the structure of $\mathcal{K}'_n$, which is abelian and decomposes as the direct product of the $\mathcal{N}(x_n)/x_i$.

Step 1: decomposition of $hP\Sigma_n$ Elements of $hP\Sigma_n$ preserve the conjugacy class of x_n, so they preserve its normal closure $\mathcal{N}(x_n)$. As a consequence, any of these automorphisms induces a well-defined automorphism of $\mathrm{RF}_n/\mathcal{N}(x_n) \cong \mathrm{RF}_{n-1}$. In other words, the projection $x_n \mapsto 1$ from RF_n onto RF_{n-1} induces a well-defined morphism p_n from $hP\Sigma_n$ to $hP\Sigma_{n-1}$. Moreover, this morphism is a split projection, a splitting s_n being the map extending automorphisms by making them fix x_n. Let us denote by $\mathcal{K}_n$ the kernel of p_n. We thus get our first decomposition,

$$hP\Sigma_n \cong \mathcal{K}_n \rtimes hP\Sigma_{n-1}. \tag{3-1}$$

Moreover, it will follow from Lemma 3.3 below that this is indeed an almost direct product: $\mathcal{K}_n^{\mathrm{ab}}$ is generated by the classes of the χ_{in} and the χ_{ni}. From Corollary 2.10, we know that these are sent to linearly independent elements in $hP\Sigma_n^{\mathrm{ab}}$, so they freely generate $\mathcal{K}_n^{\mathrm{ab}}$. We thus get a direct product decomposition $hP\Sigma_n^{\mathrm{ab}} \cong \mathcal{K}_n^{\mathrm{ab}} \oplus hP\Sigma_{n-1}^{\mathrm{ab}}$, as announced.

Step 2: structure of $\mathcal{K}_n$ We first state an easy result on generators of factors in semidirect products.

Lemma 3.2 *Let $G = H \rtimes K$ be a semidirect product of groups. Suppose we are given a family (h_i) of elements of H, and a family (k_j) of elements of K such that their reunion generates G. Then K is generated by the k_j, and H is generated by the h_i^k, for $k \in K$.*

Proof Take an element $g \in G$ and write it as a product of $h_i^{\pm 1}$ and $k_j^{\pm 1}$. Then use the formula $kh = ({}^k h)k$ to push the k_j to the right. We obtain a decomposition $g = h'k$, where $h' \in H$ is a product of conjugates of the $h_i^{\pm 1}$ by elements of K, and $k \in K$ is a product of the $k_j^{\pm 1}$. This decomposition has to be the unique decomposition of g into a product of an element of H followed by and element of K. As a consequence, if $g \in H$, then $g = h'$, whereas if $g \in K$, then $g = k$, proving our claim. □

We can apply Lemma 3.2 to the χ_{ij} in $hP\Sigma_n \cong \mathcal{K}_n \rtimes hP\Sigma_{n-1}$. Indeed, the χ_{in} and the χ_{ni} are in $\mathcal{K}_n$, and the other χ_{ij} belong to $hP\Sigma_{n-1}$. Hence, $\mathcal{K}_n$ is generated by the conjugates of the χ_{in} and the χ_{ni} by products of the other χ_{ij} and their inverses. In fact, more is true:

Lemma 3.3 *The group $\mathcal{K}_n$ is generated by the χ_{in} and the χ_{ni}.*

Proof We use the above relations to show that the subgroup H of $\mathcal{K}_n$ generated by the χ_{in} and the χ_{ni} is normal in $hP\Sigma_n$, that is, $[hP\Sigma_n, H] \subseteq H$.

The commutator $[\chi_{in}, \chi_{\alpha\beta}]$ is obviously in H if $\alpha = n$ or $\beta = n$. Otherwise, it is trivial, except possibly when $\alpha = i$ or $\beta = i$. In the first case (since $\chi_{n\beta}$ and $\chi_{i\beta}$ commute),

$$1 = [\chi_{in}, \chi_{n\beta}\chi_{i\beta}] = [\chi_{in}, \chi_{n\beta}]({}^{\chi_{n\beta}}[\chi_{in}, \chi_{i\beta}]),$$

whence $[\chi_{in}, \chi_{i\beta}] \in H$. In the second case,

$$1 = [\chi_{in}\chi_{\alpha n}, \chi_{\alpha i}] = ({}^{\chi_{in}}[\chi_{\alpha n}, \chi_{\alpha i}])[\chi_{in}, \chi_{\alpha i}],$$

so, using the first case, $[\chi_{in}, \chi_{\alpha i}] \in H$.

In a similar fashion, the bracket $[\chi_{ni}, \chi_{\alpha\beta}]$ belongs to G if $\alpha = n$ or $\beta = n$. Otherwise, it is trivial, except when $\alpha = i$. But in this case,

$$1 = [\chi_{ni}, \chi_{i\beta}\chi_{n\beta}] = [\chi_{ni}, \chi_{i\beta}]({}^{\chi_{n\beta}}[\chi_{in}, \chi_{n\beta}]),$$

so $[\chi_{ni}, \chi_{i\beta}] \in H$. Thus, H is stable under conjugation by all generators of $hP\Sigma_n$, so it is normal in $hP\Sigma_n$. $\square$

Remark 3.4 We have used only the McCool relations here, so the analogue of Lemma 3.3 is also true in $P\Sigma_n$.

By looking at how elements of $\mathcal{K}_n$ act on x_n, we get a split projection q_n from $\mathcal{K}_n$ onto RF_{n-1}. Namely, if $\varphi \in \mathcal{K}_n$ is an automorphism sending each x_i to $x_i^{w_i}$, q_n sends φ onto the class $\bar{w}_n \in \mathrm{RF}_n/x_n \cong \mathrm{RF}_{n-1}$. This is well defined, because of Lemma 1.17,

$$x_n^v = x_n^w \iff x_n^{vw^{-1}} = 1 \iff vw^{-1} \in C(x_n) = \mathcal{N}(x_n) \iff \bar{v} = \bar{w}.$$

Moreover, this defines a morphism. Indeed, if φ and ψ send x_n respectively to $x_n^{w_n}$ and $x_n^{v_n}$, then

$$\psi\varphi(x_n) = \psi(x_n^{w_n}) = x_n^{v_n\psi(w_n)},$$

and since $\psi \in \mathcal{K}_n$, we have $\overline{\psi(w_n)} = \bar{w}_n$, whence

$$q_n(\psi\varphi) = \overline{v_n\psi(w_n)} = \bar{v}_n\bar{w}_n = q_n(\psi)q_n(\varphi).$$

This morphism q_n is a retraction of the inclusion t_n of $\mathrm{RF}_{n-1} \cong \mathrm{RF}_n/x_n$ into $\mathcal{K}_n$ sending $w \in \mathrm{RF}_n$ to the automorphism fixing all x_i save x_n, which is sent to x_n^w. If we call $\mathcal{K}'_n$ the kernel of q_n, we thus get a decomposition

$$\mathcal{K}_n = \mathcal{K}'_n \rtimes \mathrm{RF}_{n-1}. \tag{3-2}$$

Lemma 3.5 *The above decomposition is $hP\Sigma_{n-1}$–equivariant, with respect to the action of $hP\Sigma_{n-1}$ on $\mathcal{K}_n$ (and on $\mathcal{K}'_n \subset \mathcal{K}_n$) coming from conjugation in $hP\Sigma_n$, and to the canonical action of $hP\Sigma_{n-1}$ on RF_{n-1}. Precisely, q_n and t_n are $hP\Sigma_{n-1}$–equivariant morphisms.*

Proof If $\varphi \in \mathcal{K}_n$ sends x_i to $x_i^{w_i}$ as above, and $\chi \in hP\Sigma_{n-1}$, then $\chi\varphi\chi^{-1}$ sends x_n to $x_n^{\chi(w_n)}$, so

$$q_n(\chi\varphi\chi^{-1}) = \overline{\chi(w_n)} = \chi(\bar{w}_n) = \chi(q_n(\varphi)).$$

As for the equivariance of t_n, if $w \in \mathrm{RF}_{n-1}$, both $\chi \cdot t_n(w) \cdot \chi^{-1}$ and $t_n(\chi(w))$ fix all x_i save x_n, the latter being sent to $x_n^{\chi(w)}$, hence they are equal. □

Remark 3.6 A similar decomposition holds in $P\Sigma_n$, replacing RF_{n-1} by F_{n-1}. The same proof works, replacing the equality $C(x_n) = \mathcal{N}(x_n)$ (which is not true in this case) by the inclusion $C(x_n) \subset \mathcal{N}(x_n)$.

Step 3: structure of $\mathcal{K}'_n$ So far, we have not really used the fact that we consider welded braids *up to homotopy* (that is, automorphisms of RF_n, not of F_n). In fact, the analogues of the decomposition results above are true in the group $P\Sigma_n$ of welded braids (see Remarks 3.4 and 3.6). We now come to the part where the homotopy relation plays a crucial role. That is, we are going to use the relations defining RF_n in a crucial way. These relations, saying that each element x_i of the fixed basis commutes with its conjugates, can be rewritten as

$$\forall i \leqslant n \quad \forall s,t \in \mathrm{RF}_n \quad x_i^{s x_i t} = x_i^{st}.$$

In other words, for $w \in \mathrm{RF}_n$, x_i^w depends only on the class of w modulo x_i (that is, modulo the normal closure of x_i). These relations allow us to say more about the above decomposition of $\mathcal{K}_n$:

Lemma 3.7 *The kernel $\mathcal{K}'_n$ of the projection $q_n : \mathcal{K}_n \twoheadrightarrow \mathrm{RF}_{n-1}$ is an abelian group, isomorphic to the product of the $\mathcal{N}(x_n)/x_i$, where $\mathcal{N}(x_n)/x_i$ is the normal closure of x_n inside $\mathrm{RF}_n/x_i \cong \mathrm{RF}_{n-1}$. Precisely, the identification of $\mathcal{N}(x_n)/x_i$ with a factor of $\mathcal{K}'_n$ is induced by the map*

$$c_i : \mathcal{N}(x_n) \to \mathcal{K}'_n, \qquad u \mapsto \left(x_j \mapsto \begin{cases} x_i^u & \text{if } j = i \\ x_j & \text{otherwise} \end{cases} \right),$$

which is a well-defined group morphism. Furthermore, c_i is RF_{n-1}–equivariant, where $\mathrm{RF}_{n-1} \cong \langle \chi_{nj} \rangle_j$ acts via automorphisms on the source, and via conjugation on the target.

Proof We identify elements $w \in \mathrm{RF}_{n-1}$ with their image by $t_n : \mathrm{RF}_{n-1} \to \mathcal{K}_n$, that is, we denote by w the automorphism fixing all x_i save x_n, which is sent to x_n^w. Applying Lemma 3.2 to the semidirect product decomposition (3-2), we see that $\mathcal{K}'_n$ is generated by the elements χ_{in}^w, which we now compute. The automorphism χ_{in}^w fixes x_α if $\alpha \notin \{i,n\}$. On x_i and x_n, using that $\chi_{in}(w) \equiv w \pmod{x_n}$, we compute

$$\chi_{in}^w : x_i \mapsto x_i \mapsto x_i^{x_n} \mapsto x_i^{x_n^w}, \quad \chi_{in}^w : x_n \mapsto {}^w x_n \mapsto {}^{\chi_{in}(w)} x_n = {}^w x_n \mapsto x_n.$$

From this calculation, we see that all $\chi = \chi_{in}^w$ commute with every $\chi' = \chi_{jn}^v$, showing that $\mathcal{K}'_n$ is indeed abelian. If $j \neq i$, this is a consequence of the fact that these automorphisms act trivially modulo x_n,

$$\chi'(x_i^{x_n^w}) = x_i^{x_n^{\chi'(w)}} = x_i^{x_n^w}.$$

For $i = j$, it follows from the fact that the conjugates of x_n commute.

Consider now N_i the subgroup generated by the χ_{in}^w, for $w \in \mathrm{RF}_{n-1}$. The elements of N_i are automorphisms fixing all x_j save x_i, and sending x_i to an element x_i^u, for some $u \in \mathcal{N}(x_n)$. As a consequence, the map c_i is a surjection from $\mathcal{N}(x_n)$ onto N_i. Since, by definition of the reduced free group, $x_i^{s x_i t} = x_i^{st}$ for all $s, t \in \mathrm{RF}_n$, we see that $c_i(v)$ depends only on the class $\bar{v}$ of v in RF_{n-1}/x_i. We use this to show that c_i is a morphism,

$$c_i(u)c_i(v)\colon x_i \mapsto c_i(u)(x_i^{\bar{v}}) = (x_i^{\bar{u}})^{c_i(u)(\bar{v})} = x_i^{\overline{uv}} = c_i(uv)(x_i).$$

Now, the kernel of c_i is $C(x_i) \cap \mathcal{N}(x_n) = \mathcal{N}(x_i) \cap \mathcal{N}(x_n)$ (using Lemma 1.17). It thus induces an isomorphism between $\mathcal{N}(x_n)/(\mathcal{N}(x_i) \cap \mathcal{N}(x_n))$ and N_i. Moreover, since it is the image of $\mathcal{N}(x_n)$ in $\mathrm{RF}_n/\mathcal{N}(x_i)$, this group identifies with the normal closure of x_n inside $\mathrm{RF}_n/x_i \cong \mathrm{RF}_{n-1}$.

We are left to show that c_i is RF_{n-1}–equivariant. It is enough to show that it commutes with the actions of the generators. If $\varphi \in \langle \chi_{nj} \rangle_{j \neq i}$, then x_i does not appear in $\varphi(x_n)$, so

$$c_i(u)^{\varphi}\colon x_i \mapsto x_i \mapsto x_i^u \mapsto x_i^{\varphi(u)}, \quad c_i(u)^{\varphi}\colon x_n \mapsto \varphi(x_n) \mapsto \varphi(x_n) \mapsto x_n,$$

showing that $c_i(u)^{\varphi} = c_i(\varphi(u))$. It remains to check that $c_i(u)^{\chi_{ni}} = c_i(\chi_{ni}(u))$; $c_i(\chi_{ni}(u))$ identifies with $c_i(u)$, since χ_{ni} acts trivially modulo x_i. We thus need to check that χ_{ni} commutes with all the $c_i(u)$ (which are all elements in N_i). This comes from the two relations $x_n^{x_i} = x_n^{x_i^u}$ (because $u \in \mathcal{N}(x_n)$) and $x_i^{\chi_{ni}(u)} = x_i^u$ (because χ_{ni} acts trivially modulo x_i). This finishes the proof of the lemma, and of Theorem 3.1. □

3.2 The Lie algebra of the reduced McCool group

The decomposition of $hP\Sigma_n$ described in Theorem 3.1 induces a decomposition of its Lie algebra:

Theorem 3.8 *The Lie algebra $\mathcal{L}(hP\Sigma_n)$ decomposes into a semidirect product,*

$$\mathcal{L}(hP\Sigma_n) \cong \left[\left(\prod_{i<n} \langle y_i \rangle\right) \rtimes R\mathfrak{L}_{n-1}\right] \rtimes \mathcal{L}(hP\Sigma_{n-1}),$$

where $\langle y_i \rangle$ is the ideal generated by y_i inside $R\mathfrak{L}_{n-1}$, and the action of $R\mathfrak{L}_{n-1}$ on the product is the diagonal one.

Proof From the almost-direct product decomposition $hP\Sigma_n \cong \mathcal{K}_n \rtimes hP\Sigma_{n-1}$, comes a decomposition of the Lie algebra $\mathcal{L}(hP\Sigma_n) \cong \mathcal{L}(\mathcal{K}_n) \rtimes \mathcal{L}(hP\Sigma_{n-1})$. In the decomposition of $\mathcal{K}_n$ described in (3-2), we can replace the normal closure $\mathcal{N}(x_n)/x_i$ of x_n in RF_n/x_i by the normal closure $\mathcal{N}(x_i)/x_n$ of x_i in $\mathrm{RF}_n/x_n \cong \mathrm{RF}_{n-1}$. Indeed, the automorphism of RF_n exchanging x_i and x_n induces an isomorphism between these two, which is RF_{n-1}–equivariant, since x_i acts trivially on both of them. We thus have to compute

$$\mathcal{L}(\mathcal{K}_n) \cong \mathcal{L}\left[\left(\prod_{i<n} \mathcal{N}(x_i)\right) \rtimes \mathrm{RF}_{n-1}\right].$$

Since this is not a decomposition into an almost direct product, we have to use Section 3.1 of [Darné 2021]: we need to compute $\Gamma_*^{\mathrm{RF}_{n-1}}\big(\prod \mathcal{N}(x_i)\big)$, which is the product $\prod \Gamma_*^{\mathrm{RF}_{n-1}}(\mathcal{N}(x_i))$, since RF_{n-1} acts diagonally. In order to do this, consider the split short exact sequence of groups,

$$\mathcal{N}(x_i) \hookrightarrow \mathrm{RF}_{n-1} \twoheadrightarrow \mathrm{RF}_{n-1}/x_i \cong \mathrm{RF}_{n-2}.$$

From [Darné 2021, Proposition 3.4], this gives rise to a decomposition of $\Gamma_*(\mathrm{RF}_{n-1})$ into a semidirect product $\Gamma_*^{\mathrm{RF}_{n-2}}(\mathcal{N}(x_i)) \rtimes \Gamma_*(\mathrm{RF}_{n-2})$, where $\Gamma_*^{\mathrm{RF}_{n-2}}(\mathcal{N}(x_i))$ is defined by taking commutators with $\mathcal{N}(x_i) \rtimes \mathrm{RF}_{n-2} \cong \mathrm{RF}_{n-1}$ at each step, so is equal to $\Gamma_*^{\mathrm{RF}_{n-1}}(\mathcal{N}(x_i))$. As a consequence, $\mathcal{N}_*(x_i) := \Gamma_*^{\mathrm{RF}_{n-1}}(\mathcal{N}(x_i))$ is the intersection of $\Gamma_*(\mathrm{RF}_{n-1})$ with $\mathcal{N}(x_i)$. Its associated Lie algebra fits into the short exact sequence

$$\mathcal{L}(\mathcal{N}_*(x_i)) \hookrightarrow \mathcal{L}(\mathrm{RF}_{n-1}) \twoheadrightarrow \mathcal{L}(\mathrm{RF}_{n-2}).$$

Theorem 1.12 ensures that the projection on the right identifies with the projection of $R\mathfrak{L}_{n-1}$ onto $R\mathfrak{L}_{n-2}$ sending y_i to 0, whose kernel is $\langle y_i \rangle$. Thus $\mathcal{L}(\mathcal{N}_*(x_i)) \cong \langle y_i \rangle$, and

$$\mathcal{L}(\mathcal{N}(x_i) \rtimes \mathrm{RF}_{n-1}) \cong \mathcal{L}(\mathcal{N}_*(x_i)) \rtimes \mathcal{L}(\mathrm{RF}_{n-1}) \cong \langle y_i \rangle \rtimes R\mathfrak{L}_{n-1}. \qquad \square$$

3.3 The Andreadakis equality

Theorem 3.8 gives a complete description of the graded Lie ring associated to $\Gamma_*(hP\Sigma_n)$. On the other hand, Theorem 2.9 describes the Lie ring associated with the Andreadakis filtration $hP\Sigma_n \cap \mathcal{A}_*(\mathrm{RF}_n)$. Using these two results, we are now able to show:

Theorem 3.9 *The Andreadakis equality holds for $hP\Sigma_n$.*

Proof We want to show that the Johnson morphism $\tau' \colon \mathcal{L}(hP\Sigma_n) \to \mathrm{Der}(R\mathfrak{L}_n)$ is injective (see the end of Section 0.1). We make use of the commutative diagram

$$\begin{tikzcd}
\mathcal{L}(\mathcal{K}_n) \arrow[r, hook] \arrow[d, dashed, "\tau'"] & \mathcal{L}(hP\Sigma_n) \arrow[r, two heads] \arrow[d, "\tau'"] & \mathcal{L}(hP\Sigma_{n-1}) \arrow[d, "\tau'"] \\
\bullet \arrow[r, hook] & \mathrm{Der}_\tau(R\mathfrak{L}_n) \arrow[r, two heads] & \mathrm{Der}_\tau(R\mathfrak{L}_{n-1})
\end{tikzcd}$$

where the bottom projection is the one induced by $y_n \mapsto 0$. By induction (beginning at $n=2$), using the snake lemma, we only have to prove that the left map is injective, that is, that $\tau' \colon \mathcal{L}(\mathcal{K}_n) \to \mathrm{Der}(R\mathfrak{L}_n)$ is. Take an element

$$\varphi = ((w_i), w_n) \in \Gamma_j(\mathcal{K}_n) = \Bigg(\prod_{j<n} (\Gamma_j(\mathrm{RF}_n) \cap \mathcal{N}(x_n))/x_i\Bigg) \rtimes \Gamma_j(\mathrm{RF}_{n-1}),$$

meaning that φ is the automorphism conjugating x_n by $w_n \in \Gamma_j(\mathrm{RF}_{n-1})$ and x_i by $w_i \in \Gamma_j(\mathrm{RF}_n) \cap \mathcal{N}(x_n)$ for $i<n$, which depends only on the class of each w_i modulo $\mathcal{N}(x_i)$. Then $\tau'_j(\bar{\varphi})$ sends each y_i ($i \leqslant n$) to $[y_i, \bar{w}_i] \in \mathcal{L}_{j+1}(\mathrm{RF}_n)$. As a consequence, the equality $\tau'_j(\bar{\varphi}) = 0$ would mean that each $\bar{w}_i$ commutes

with y_i in $\mathscr{L}(\mathrm{RF}_n) \cong R\mathscr{L}_n$. By Lemma 1.18, this would imply that $\bar{w}_i \in \langle y_i \rangle$. However, in the course of the proof of Theorem 3.8, we have shown that $\langle y_i \rangle = \mathscr{L}(\Gamma_*(\mathrm{RF}_n) \cap \mathscr{N}(x_i))$. Thus there exists v_i in $\Gamma_j(\mathrm{RF}_n) \cap \mathscr{N}(x_i)$ such that $\bar{v}_i = \bar{w}_i$, that is, $w_i \equiv v_i \pmod{\Gamma_{j+1}(\mathrm{RF}_n)}$. But we can replace w_i by $w_i v_i^{-1}$ without changing φ, so all the w_i can be chosen to be in $\Gamma_{j+1}(\mathrm{RF}_n)$. This implies that $\varphi \in \Gamma_{j+1}(hP\Sigma_n)$, which means that $\bar{\varphi} = 0$ in $\mathscr{L}_j(hP\Sigma_n)$. This ends the proof that the kernel of τ' is trivial, and the proof of the theorem. □

3.4 Braids up to homotopy

Consider the (classical) pure braid group P_n. It can be embedded into the monoid of string-links on n strands. These string-links can be considered *up to (link-)homotopy*, which means that one adds to the isotopy relation the possibility for each strand to cross itself. This relation is obviously compatible with the monoid structure, and since every string-link is in fact homotopic to a braid, this quotient is a quotient of the pure braid group, called the group of braids up to homotopy, denoted by hP_n.

3.4.1 Decomposition and Lie algebra Goldsmith [1974] described hP_n as a quotient of P_n by a finite set of relations. These relations say exactly that for $j < k$, the generators A_{jk} commute with their conjugates by elements of $\langle A_{ik} \rangle_{i<k} \cong F_{k-1}$. This means exactly that the free factors in the decomposition of P_n are replaced by reduced free groups,

$$hP_{n+1} \cong \mathrm{RF}_n \rtimes hP_n.$$

This decomposition first appeared explicitly in [Habegger and Lin 1990], where a more topological proof is described.

Such a decomposition is compatible with the decomposition of the (classical) pure braid group, which means that the canonical projections give a morphism of semidirect products:

(3-3)
$$\begin{tikzcd}
F_n \arrow[r, hook] \arrow[d, two heads] & P_{n+1} \arrow[r, two heads] \arrow[d, two heads] & P_n \arrow[l, bend right] \arrow[d, two heads] \\
\mathrm{RF}_n \arrow[r, hook] & hP_{n+1} \arrow[r, two heads] & hP_n \arrow[l, bend right]
\end{tikzcd}$$

Since Goldsmith's relations are commutation relations, the projection from P_{n+1} onto hP_{n+1} induces an isomorphism between P_{n+1}^{ab} onto hP_{n+1}^{ab}. As a consequence, since the decomposition $P_{n+1} \cong F_n \rtimes P_n$ is an almost-direct product decomposition, the decomposition $hP_{n+1} \cong \mathrm{RF}_n \rtimes hP_n$ also is. It then induces a decomposition of the lower central series and of the corresponding Lie ring. Precisely, we get iterated semidirect product decompositions,

(3-4)
$$\Gamma_j(hP_{n+1}) = \Gamma_j(\mathrm{RF}_n) \rtimes \Gamma_j(hP_n),$$

inducing such decompositions of the associated graded Lie rings. Thus we get:

Proposition 3.10 *The group hP_{n+1} is n–nilpotent, and its Lie algebra decomposes as an iterated semidirect product of reduced free Lie algebras,*

$$\mathscr{L}(hP_{n+1}) \cong \mathscr{L}(\mathrm{RF}_n) \rtimes \mathscr{L}(hP_n) \cong R\mathfrak{L}_n \rtimes \mathscr{L}(hP_n).$$

From this, we can deduce the Hirsch rank of hP_n, recovering Milnor's formula, as quoted in [Habegger and Lin 1990, Section 3]:

Corollary 3.11 *The group hP_n has no torsion and its Hirsch rank is*

$$\mathrm{rk}(hP_n) = \sum_{k=1}^{n-1} (k-1)! \binom{n}{k+1}.$$

Proof That it has no torsion (even no torsion in its lower central series) comes from the fact that the $R\mathfrak{L}[m]$ do not, according to Proposition 1.9. The same proposition gives us the ranks of the $R\mathfrak{L}[m]_k$, allowing us to compute

$$\mathrm{rk}(\mathscr{L}_k(hP_n)) = \sum_{m=1}^{n-1} \mathrm{rk}(R\mathfrak{L}[m]_k) = (k-1)! \sum_{m=1}^{n-1} \binom{m}{k} = (k-1)! \binom{n}{k+1},$$

the last equality being obtained by iterating Pascal's formula, or by a combinatorial proof (replacing the choice of k elements $t_1, \dots, t_k$ among m elements, with m ranging from k to $n-1$, by the choice of $k+1$ elements $t_1, \dots, t_k, m+1$ among n elements). □

Let us also mention that we can deduce from the decomposition of $\mathscr{L}(hP_n)$ described in Proposition 3.10 and from the usual presentation of the pure braid group a presentation of this Lie ring, which is a quotient of the Drinfeld–Kohno Lie ring $\mathscr{L}(P_n)$ of infinitesimal braids (whose rational version was introduced in [Kohno 1985]).

Corollary 3.12 *The Lie ring of hP_n is generated by t_{ij} $(1 \leqslant i, j \leqslant n)$, under the Drinfeld–Kohno relations*

$$\begin{aligned} t_{ij} = t_{ji} \text{ and } t_{ii} &= 0 \quad \text{for all } i, j, \\ [t_{ij}, t_{ik} + t_{kj}] &= 0 \quad \text{for all } i, j, k, \\ [t_{ij}, t_{kl}] &= 0 \quad \text{if } \{i, j\} \cap \{k, l\} = \varnothing, \end{aligned}$$

to which are added, for each m, the vanishing of Lie monomials in the t_{im} $(i < m)$ with repetition.

Proof The proof in the classical case (see for instance the appendix of [Darné 2021]) adapts verbatim, by considering reduced free Lie rings instead of free Lie rings. □

Notice that as in the definition of the reduced free Lie ring (Definition 1.6 — see also Remark 1.8), one can give a simpler finite presentation by considering, for each m, only linear Lie monomials in the t_{im} $(i < m)$ of length at most m.

3.4.2 The Andreadakis problem The semidirect product $\mathrm{RF}_n \rtimes hP_n$ described above is the same thing as an action of hP_n on RF_n, also described by a morphism from hP_n to $\mathrm{Aut}(\mathrm{RF}_n)$. This is the *homotopy Artin action*, that we now study, using the fact that it is encoded by conjugation inside $hP_{n+1} = \mathrm{RF}_n \rtimes hP_n$.

First, we remark that this action is by basis-conjugating automorphisms. In fact, the compatibility diagram (3-3) gives rise to a commutative diagram

$$\begin{tikzcd} P_n \arrow[r, hook] \arrow[d, two heads] & \mathrm{Aut}_C(F_n) \arrow[d, two heads] \\ hP_n \arrow[r, "a"] & \mathrm{Aut}_C(\mathrm{RF}_n) \end{tikzcd}$$

where the morphism on the left is surjective by Corollary 2.10. The top map, which is the Artin action, is injective (the action is faithful) and its image is exactly the subgroup of basis-conjugating automorphisms fixing the *boundary element* $x_1 \cdots x_n$ [Birman 1974, Theorem 1.9]. Habegger and Lin [1990, Theorem 1.7] have shown that the analogous statements are true for hP_n: the homotopy Artin action induces an isomorphism between hP_n and the group $\mathrm{Aut}_C^{\partial}(\mathrm{RF}_n)$ of basis-conjugating automorphisms of RF_n preserving the product $x_1 \cdots x_n$. Precisely, they show that the latter admits the same decomposition as hP_n, and that the pieces of these decompositions are identified under the Artin morphism. We recover the faithfulness of the homotopy Artin action as part of our answer to the Andreadakis problem for $hP_n \subset \mathrm{Aut}_C(\mathrm{RF}_n)$ (see Corollary 3.14 below).

Theorem 3.13 *The Andreadakis equality holds for the image of the Artin action* $a\colon hP_n \to \mathrm{Aut}(\mathrm{RF}_n)$. *Namely,* $\Gamma_*(hP_n) = a^{-1}(\mathcal{A}_*(\mathrm{RF}_n))$.

Proof We adapt the proof for P_n given in [Darné 2021]. Let $w \in hP_n$, and suppose that w acts on RF_n as an element of $\mathcal{A}_j$. We want to show that it belongs to $\Gamma_j(hP_n)$. Our hypothesis can be written as

$$[w, \mathrm{RF}_n] \subseteq \Gamma_{j+1}(\mathrm{RF}_n),$$

where the bracket is computed in $\mathrm{RF}_n \rtimes hP_n$, which is exactly hP_{n+1}. Moreover, from the decomposition of the lower central series of hP_{n+1} described above (Section 3.4.1), we deduce that

$$\Gamma_j(hP_n) = hP_n \cap \Gamma_j(hP_{n+1}),$$

so the conclusion we seek is in fact $w \in \Gamma_j(hP_{n+1})$. Let us comb w: we write

$$w = \beta_n \cdots \beta_2 \in \mathrm{RF}_{n-1} \rtimes (\mathrm{RF}_{n-2} \rtimes (\cdots \rtimes \mathrm{RF}_1)) = hP_n.$$

Again, because of the decomposition of the lower central series of hP_n, we need to show that each β_i is in $\Gamma_j(P_{n+1})$. In the rest of the proof, we often write Γ_k for $\Gamma_k(hP_{n+1})$, its intersection with the subgroups under consideration being their own Γ_k, because of (3-4).

Let us suppose that $w \notin \Gamma_j(hP_{n+1})$. Then $w \in \Gamma_k - \Gamma_{k+1}$ for some $k < j$. Let i be maximal such that $\beta_i \notin \Gamma_{k+1}$. On the one hand, the generator $A_{i,n+1} \in \mathrm{RF}_n$ commutes with every β_k with $k < i$, so $[w, A_{i,n+1}] \equiv [\beta_i, A_{i,n+1}] \pmod{\Gamma_{k+2}}$. Moreover, by hypothesis, $[w, A_{i,n+1}] \in \Gamma_{j+1} \subseteq \Gamma_{k+2}$, so $[\beta_i, A_{i,n+1}] \in \Gamma_{k+2}$. Since β_i has degree k and $A_{i,n+1}$ has degree 1 in the lower central series, this means that $[\bar{\beta}_i, \bar{A}_{i,n+1}] = 0$ in the Lie algebra. On the other hand, β_i and $A_{i,n+1}$ belong to another copy of RF_n inside hP_{n+1}, namely $\langle A_{1,i}, \dots, A_{i-1,i}, A_{i,i+1}, \dots, A_{i,n+1}\rangle$. We denote this copy by $\widetilde{\mathrm{RF}}_n$. We remark that the equality $\Gamma_*(\widetilde{\mathrm{RF}}_n) = \widetilde{\mathrm{RF}}_n \cap \Gamma_*(hP_{n+1})$ is also true for this copy of RF_n, as one sees by switching the strands i and $n+1$ in the reasoning above. But then we can apply Lemma 1.18: since $\bar{\beta}_i$ commutes with the generator $\bar{A}_{i,n+1}$ of $\mathscr{L}(\widetilde{\mathrm{RF}}_n) \cong R\mathfrak{L}_n$, it must belong to the Lie ideal of $\mathscr{L}(\widetilde{\mathrm{RF}}_n)$ generated by $\bar{A}_{i,n+1}$. But this is impossible: by definition of β_i, the generator $\bar{A}_{i,n+1}$ cannot appear in $\bar{\beta}_i$. We thus get a contradiction, and our conclusion. □

From this, we can recover the injectivity part of the result of Habegger and Lin:

Corollary 3.14 [Habegger and Lin 1990, Theorem 1.7] *The homotopy Artin action is faithful.*

Proof If $w \in hP_n$ acts trivially on RF_n, then $a(w) \in \{1\} = \mathscr{A}_n(\mathrm{RF}_n)$, so $w \in \Gamma_n(hP_n)$ by Theorem 3.13. But $\Gamma_n(hP_n) = \{1\}$ (Proposition 3.10), whence $w = 1$. □

This injectivity of $a\colon hP_n \to hP\Sigma_n = \mathrm{Aut}_C(\mathrm{RF}_n)$ is weaker than our statement, which says that the lower central series are in fact compatible, since they both are the trace of the Andreadakis filtration $\mathscr{A}_*(\mathrm{RF}_n)$:

Corollary 3.15 *For all n, $hP_n \cap \Gamma_*(hP\Sigma_n) = \Gamma_*(hP_n)$.*

Proof Combine Theorems 3.13 and 3.9. □

Remark 3.16 The rationalization of the Lie ring $\mathscr{L}(P_n)$ is exactly $\mathscr{P}^{hsl}$ of [Bar-Natan 1995, Theorem 3], where different diagramatic descriptions for its enveloping algebra are discussed.

4 Topological interpretation

Consider the group P_n of pure braids. Via the decomposition $P_{n+1} \cong F_n \rtimes P_n$, we get an action of P_n on the free group F_n, which is the classical *Artin action*. Geometrically, it is best understood as the action of P_n, which is the *motion group* of n points in a plane, on the fundamental group of the plane with n points removed. As mentioned above (Section 3.4.2), this action is faithful, giving an embedding of P_n into $\mathrm{Aut}(F_n)$, whose image is exactly the subgroup $\mathrm{Aut}_C^{\partial}(F_n)$ of automorphisms fixing the conjugacy class of each generator x_i, and preserving the *boundary element* $x_1 \cdots x_n$ [Birman 1974, Theorem 1.9].

An analogous statement is true for the group $P\Sigma_n$ of pure *welded braids*. This group is a group of tube-shaped braids in $\mathbb{R}^4$, and can also be seen as the (pure) *motion group* of n unknotted circles in $\mathbb{R}^3$ (see [Damiani 2017] on the different definitions on this group). It acts on the fundamental group of $\mathbb{R}^3$

with n unknotted circles removed, which is again the free group F_n. This *Artin action* is again faithful, and its image is exactly the subgroup $\mathrm{Aut}_C(F_n)$ of automorphisms fixing the conjugacy class of each generator x_i [Goldsmith 1981].

The same statements are true *up to* (*link-*)*homotopy*. These have been recalled for braids in Section 3.4. For welded braids, link-homotopy of string links also makes sense (in $\mathbb{R}^4$), and for welded diagrams (which are another point of view on these objects), this relation corresponds to virtualization of self-crossings. It has been shown in [Audoux et al. 2017a, Theorem 2.34] that the group of welded braids up to homotopy is isomorphic to the group $\mathrm{Aut}_C(\mathrm{RF}_n) = hP\Sigma_n$ of automorphisms of RF_n fixing the conjugacy class of each generator x_i.

We sum up the situation with the following diagrams:

$$\begin{array}{cc} \text{up to isotopy} & \text{up to homotopy} \\ \begin{tikzcd} P_n \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Aut}_C^{\partial}(F_n) \arrow[d, hook] \\ P\Sigma_n \arrow[r, "\cong"] & \mathrm{Aut}_C(F_n) \end{tikzcd} & \begin{tikzcd} hP_n \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Aut}_C^{\partial}(\mathrm{RF}_n) \arrow[d, hook] \\ hP\Sigma_n \arrow[r, "\cong"] & \mathrm{Aut}_C(\mathrm{RF}_n) \end{tikzcd} \end{array} \tag{4-1}$$

4.1 Milnor invariants

Here we interpret our work in terms of *Milnor invariants* of welded braids up to homotopy. Milnor invariants were first defined in [Milnor 1957] for links, as integers with some indeterminacy. It appeared later that they were more naturally defined for string links, for which they are proper integers, the indeterminacy previously observed corresponding exactly to a choice of presentation of a link as the closure of a string-link. Here we focus on their definition for braids, which is not a restrictive choice when working up to homotopy.

If β is a pure braid, we can look at its image via the Artin action, which is a basis-conjugating automorphism $x_i \mapsto x_i^{w_i}$. The element w_i is well defined up to left multiplication by $x_i^{\pm 1}$, so it is well defined if we suppose that x_i does not appear in the class $\bar{w}_i \in F_n^{\mathrm{ab}}$. For each i, one can look at the image of the element $w_i \in F_n$ by the *Magnus expansion* $\mu\colon F_n \hookrightarrow \widehat{T[n]}$, getting an element of the completion of the free associative ring $\widehat{T[n]}$ on n generators $X_1, \dots, X_n$, which can be seen as the ring of noncommutative power series on these generators. Recall that the Magnus expansion is defined by $x_i \mapsto 1 + X_i$, and it is an injection of the free group F_n into $\widehat{T[n]}^{\times}$. Then the *Milnor invariants* are the coefficients of the $\mu(w_i)$. Precisely, if $i \leqslant n$ is an integer, and $I = (i_1, \dots, i_d)$ is any list of positive integers, then $\mu_{I,i}(\beta)$ is the coefficient of the monomial $X_{i_1} \cdots X_{i_d}$ in $\mu(w_i)$. Moreover, we call d the *degree* of the Milnor invariant $\mu_{I,i}$.

The first nontrivial Milnor invariants of β can also be obtained through the Johnson morphism. Namely, let d be the greatest integer such that $\beta \in \mathcal{A}_d(F_n)$ (we identify β with its image via the Artin action). By definition of w_i, x_i does not appear in the class $\bar{w}_i \in F_n^{\mathrm{ab}}$. Thus, we deduce from [Darné 2021, Lemma 6.3] that for all $j \geqslant 1$, $[x_i, w_i] \in \Gamma_{j+1}(F_n)$ if and only if $w_i \in \Gamma_d(F_n)$. This implies that d is maximal such that all w_i belong to $\Gamma_j(F_n)$. The image of $\bar{\beta} \in \mathscr{A}_d/\mathscr{A}_{d+1}$ by the Johnson morphism is the derivation of the free Lie algebra $\mathfrak{L}[n]$ given by $x_i \mapsto [x_i, \overline{w}_i]$, where $\overline{w}_i \in \Gamma_d/\Gamma_{d+1}(F_n) \cong \mathfrak{L}[n]_d$ is the class of w_i, possibly trivial (but nontrivial for at least one i).

Now, we can consider the element $\overline{w}_i$ as being inside $T[n]_d$, and the inclusion of $\mathfrak{L}[n]$ into $T[n]$ is exactly the graded map induced by the Magnus expansion μ. Precisely, if we call $\hat{T}_1^d$ the ideal of $T[n]$ defined by elements of valuation at least d (the valuation of a power series being the total degree of its least nontrivial monomial), then $\Gamma_d(F_n) = \mu^{-1}(1 + \hat{T}_1^d)$, and the induced map $\bar{\mu}\colon \Gamma_d/\Gamma_{d+1}(F_n) \hookrightarrow \hat{T}_1^d/\hat{T}_1^{d+1}$ identifies with the canonical inclusion of $\mathfrak{L}[n]_d$ into $T[n]_d$. As a consequence, the class $\overline{w}_i$ is the degree-d part of $\mu(w_i)$, which has valuation at least d. We sum this up in the following:

Proposition 4.1 *The group $\mathscr{A}_d(F_n) \cap P_n$ is the set of braids with vanishing Milnor invariants of degree at most $d-1$. Moreover, Milnor invariants of degree d of these braids can be recovered from their image by the Johnson morphism $\tau\colon \mathscr{A}_d/\mathscr{A}_{d+1} \hookrightarrow \mathrm{Der}_d(\mathfrak{L}[n])$.*

Obviously, since we have not used anywhere that the automorphism β preserves the boundary element, these constructions work for all welded braids (that is, for all basis-conjugating automorphisms of F_n).

Let us now explain how to define Milnor invariants for (welded) braids up to homotopy. First, we need to replace F_n by $\mathrm{R}F_n$. Then we can assume that x_i does not appear in w_i (since $x_i^{u x_i v} = x_i^{uv}$ in the reduced free group). The Magnus expansion must be replaced by the morphism (1-1), and we get only Milnor invariants without repetitions (that is, I must be without repetition in order to define a nontrivial $\mu_{I,i}$). Everything works as described above (using the work done in Section 1.2), so $\mathscr{A}_d(F_n)$ is exactly the subgroup where invariants of degree at most $d-1$ vanish. So we can reformulate Theorems 3.9 and 3.13 as:

Theorem 4.2 *Homotopy Milnor invariants of degree at most d classify braids (resp. welded braids) up to homotopy up to elements of $\Gamma_{d+1}(hP_n)$ (resp. $\Gamma_{d+1}(hP\Sigma_n)$).*

Remark 4.3 The group $\Gamma_{d+1}(hP_n)$ can also be seen as the set of braids which are homotopic to elements of $\Gamma_{d+1}(P_n)$.

4.2 Arrow calculus

We now explain briefly the precise link between our work and the work of Meilhan and Yasuhara [2019]. We will not give any definition here; the reader is referred to their paper for basic definitions and details.

Our understanding of the link between our work and theirs relies on the following remark: *calculus of arrows and w–trees is the same thing as commutator calculus in the welded braid group $P\Sigma_n$.* Precisely, when attaching a tree T to a diagram D, one has to select the points where the root and leaves of T are attached. If we consider a little arc around each of these points, we see that doing so consists of choosing n strands (which inherit their orientation from D). Then the data of T describes an element of the braid group on these strands, and doing the surgery along T is exactly the same as inserting the braid described by T at the chosen spot on D, to get the new diagram D_T. Namely, a single arrow from a strand j to a strand i describes the insertion of the braid χ_{ij}, and a tree with root at i describes the insertion of a commutator between the χ_{ij}, for varying j (note that any number of strands can be added).

In the light of this remark, we can see that many relations they describe correspond to algebraic relations written in the present paper. Also, two diagrams are w_k–equivalent if and only if they can be obtained from one another by inserting braids in $\Gamma_k(P_n)$ (for varying n). And we can in fact deduce our Andreadakis equality (Theorem 4.2) from their classification theorem of welded string links up to homotopy [Meilhan and Yasuhara 2019, Theorem 9.4]. They fell short of doing so, stating only their weaker Corollary 9.5. In fact, they did not look for the precise identification between trees and commutator calculus described here. They only knew that something of the sort should be true, but were interested in other matters at the time.

5 A presentation of the homotopy loop braid group

Goldsmith [1974] gave a presentation of the braid group up to homotopy (see also Section 3.4). She proved that, to a presentation of the pure braid group with generators A_{ij}, one has to add the family of relations making each $\langle A_{1k}, \dots, A_{k-1,k}\rangle$ into a reduced free group. The goal of the present section is to give a similar presentation of the loop braid group up to homotopy. The situation here it more intricate; to a presentation of $P\Sigma_n$ with generators χ_{ij}, we have to add three families of relations:

(R1) the relations saying that for all m, $\langle \chi_{mk}\rangle_{k<m}$ is reduced;

(R2) $[\chi_{im}, w, \chi_{jm}] = 1$, for $i, j < m$ and $w \in \langle \chi_{mk}\rangle_{k<m}$;

(R3) $[\chi_{im}, w, \chi_{mi}] = 1$, for $i < m$ and $w \in \langle \chi_{mk}\rangle_{k<m,k\neq i}$.

We remark that because of the symmetry with respect to the generators of RF_n, these relations are still true if we replace each symbol “$<$” by a symbol “$\neq$”, which would give a more symmetric (but bigger) set of relations.

Remark 5.1 These relations also describe the quotient of the group wB_n of all welded braids by the homotopy relation. Indeed, performing a homotopy cannot move endpoints of string links, so the subgroup of relations must be a subgroup of the *pure* welded braid group, like in the classical setting [Goldsmith 1974, Lemma 1].

5.1 Generators of nilpotent groups

One key argument in the determination of a presentation of $hP\Sigma_n$ consists in lifting generators from Lie rings to groups. Such generators will be obtained from combinatorics in the free Lie ring (see the appendix), and lifting them will use the nilpotence of the groups involved.

Convention 5.2 By a *finite* filtration, we always mean a separating one; a strongly central series G_* is *finite* if there exists a $i \geqslant 1$ such that $G_i = \{1\}$. In particular, if there exists a finite strongly central series on G, then G must be nilpotent (recall that $G_i \supseteq \Gamma_i G$).

Lemma 5.3 *Let G_* be a finite strongly central filtration on a (nilpotent) group G. Suppose that the x_α are elements of G such that their classes $\bar{x}_\alpha$ generate the Lie ring $\mathscr{L}(G_*)$. Then the x_α generate G.*

Proof Consider the subgroup K of G generated by the x_α. The canonical morphism from $\mathscr{L}(G_* \cap K)$ to $\mathscr{L}(G_*)$ comes from an injection between filtrations, so it is injective. By hypothesis, it is also surjective. By induction (using the five lemma), we deduce that $K/(G_j \cap K) = G/G_j$, for all j. Since there exists j such that $G_j = \{1\}$, this proves that $K = G$. □

The definition of the Lyndon monomials P_w (Section A.2) makes sense in any group, if we interpret letters as elements of the group, and brackets as commutators.

Proposition 5.4 *Let G be a nilpotent group generated by a set X, and $x \in X$. Then the normal closure $\mathcal{N}(x)$ of x in G is generated by Lyndon monomials P_w, for Lyndon words $w \in X^*$ containing x.*

Proof By taking images in G, it is enough to show this for the free nilpotent group $F_j[X] := F[X]/\Gamma_{j+1}$. In this case, $\mathcal{N}(x)$ is the kernel of the canonical projection from $F_j[X]$ to $F_j[X - \{x\}]$. Setting $\mathcal{N}_*(x) := \mathcal{N}(x) \cap \Gamma_*(F_j[X])$, we get a short exact sequence of filtrations translating into a short exact sequence of Lie rings

$$\mathscr{L}(\mathcal{N}_*(x)) \hookrightarrow \mathscr{L}(F_j[X]) \twoheadrightarrow \mathscr{L}(F_j[X - \{x\}]).$$

Since $\mathscr{L}(F_j[X])$ is the j^{th} truncation of the free Lie algebra on $Y = \overline{X}$, and the projection is the canonical one (sending $y = \bar{x}$ to 0), the subring $\mathscr{L}(\mathcal{N}_*(x))$ identifies with the j^{th} truncation of the ideal $\langle y \rangle$ of $\mathfrak{L}[Y]$. This ideal is linearly generated by Lyndon Lie monomials on Y containing y. Since these are the classes of the corresponding monomials in the group $F_j[X]$, Lemma 5.3 gives the desired conclusion. □

Corollary 5.5 *Let X be a set, and $x \in X$. The normal closure $\mathcal{N}(x)$ of x in* $\mathrm{RF}[X]$ *is free abelian on the Lyndon monomials P_w, for Lyndon words without repetition $w \in X^*$ containing x.*

Proof It is enough to show this for X finite. Then $\mathrm{RF}[X]$ is nilpotent, and we can apply Proposition 5.4 to show that Lyndon monomials without repetition containing x generate $\mathcal{N}(x)$. Indeed, in $\mathrm{RF}[X]$, the only nontrivial Lyndon monomials in elements of X are those without repetition. Moreover, $\mathcal{N}(x)$ is abelian, by definition of $\mathrm{RF}[X]$. We are thus left with proving that these elements are linearly independent. But any nontrivial linear relation between them would give a nontrivial linear relation between Lyndon monomials without repetition in $\mathscr{L}(\mathrm{RF}[X])$ (take l to be the minimal length of the monomials involved, and project the relation into Γ_l/Γ_{l+1}). Such a relation cannot hold (Proposition 1.9), so this proves the corollary. □

If $g, g_1, \dots, g_m$ are elements of a group, let us denote by $\mathrm{Lynd}(g; g_1, \dots, g_m)$ the family of Lyndon monomials (P_w), where w runs through Lyndon words without repetition on the letters $g, g_1, \dots, g_m$ which contain g. When considering these sets, we will choose an order on the letters making all g_i greater than g. In that case, elements of $\mathrm{Lynd}(g; g_1, \dots, g_m)$ are of the form $[[g, P_v], P_w]$, where neither v nor w contains g. As usual, we denote by $(g_1, \dots, \hat{g}_i, \dots, g_m)$ the $(m-1)$–tuple obtained from $(g_1, \dots, g_m)$ by removing the i^{th} component.

We now use Corollary 5.5 in order to get a basis of the group $\mathcal{K}'_n$ introduced in Section 3 from the decomposition obtained in Lemma 3.7.

Lemma 5.6 *A basis of the abelian group $\mathcal{K}'_n$ is given by*

$$\bigcup_i \mathrm{Lynd}(\chi_{in}; \chi_{n1}, \dots, \hat{\chi}_{ni}, \dots, \chi_{n,n-1}).$$

Proof We use notation from the proof of Lemma 3.7. Equivariance of the isomorphism c_i ensures that c_i^{-1} sends the set $\mathrm{Lynd}(\chi_{in}; \chi_{n1}, \dots, \hat{\chi}_{ni}, \dots, \chi_{n,n-1})$ to the set $\mathcal{B} := \mathrm{Lynd}(x_n; \chi_{n1}, \dots, \hat{\chi}_{ni}, \dots, \chi_{n,n-1})$, the latter brackets being computed in the semidirect product $(\mathcal{N}(x_n)/x_i) \rtimes \langle \chi_{nj} \rangle_j$. If $v \in \mathrm{RF}_{n-1}$, we denote by χ_v the automorphism of RF_n sending x_n to x_n^v and fixing all other generators (χ_v was denoted by $t_n(v)$ above). Elements of $\mathcal{B}$ are of the form $[[x_n, \chi_v], \chi_w]$, where χ_v and χ_w are Lyndon monomials in the χ_{nj} $(j \neq i)$, which means exactly that v and w are Lyndon monomials in the x_j $(j \neq i, n)$, since $t_n \colon v \mapsto \chi_v$ is a morphism. Recall that the class of χ_v in the Lie algebra $\mathcal{L}(\mathcal{A}_*(\mathrm{RF}_n))$ acts on the Lie algebra $R\mathfrak{L}_n$ via the tangential derivation $\tau(\bar{\chi}_v)$ induced by $[\chi_v, -]$, sending x_n to $[x_n, v]$ and all other x_i to 0. As a consequence, the class of $[[x_n, \chi_v], \chi_w]$ in the Lie algebra $\mathcal{L}(\mathcal{N}_*(x_n)/x_i) \subset R\mathfrak{L}_n$ is

$$\tau(\bar{\chi}_w)\tau(\bar{\chi}_v)(x_n) = \tau(\bar{\chi}_w)([v, x_n]) = [v, [w, x_n]] = [[x_n, w], v],$$

since the derivation $\tau(\bar{\chi}_w)$ vanishes on v. As a consequence, the family $\mathcal{B}$ is another lift of the basis of $\mathcal{L}(\mathcal{N}_*(x_n)/x_i)$ considered above, and the same proof as the proof of Corollary 5.5 (in $\mathrm{RF}_n/x_i \cong \mathrm{RF}_{n-1}$) shows that it is a basis of $\mathcal{N}(x_n)/x_i$, whence the result. □

Remark 5.7 In the semidirect product $(N(x_n)/x_i) \rtimes \langle \chi_{nj} \rangle_j$ which appears in the proof, the group $\langle \chi_{nj} \rangle_j$ is isomorphic to RF_{n-1} but its action is *not* the conjugation action.

5.2 The presentation

Let us recall the relations on the χ_{ij} that will give a presentation of $hP\Sigma_n$:

(R0) the McCool relations on the χ_{ij} (see the introduction);

(R1) $[\chi_{mi}, w, \chi_{mi}] = 1$, for $i < m$ and $w \in \langle \chi_{mk} \rangle_{k<m}$;

(R2) $[\chi_{im}, w, \chi_{jm}] = 1$, for $i, j < m$ and $w \in \langle \chi_{mk} \rangle_{k<m}$;

(R3) $[\chi_{im}, w, \chi_{mi}] = 1$, for $i < m$ and $w \in \langle \chi_{mk} \rangle_{k<m, k \neq i}$.

We now show that they indeed give the presentation that we were looking for:

Theorem 5.8 *The pure loop braid group up to homotopy $hP\Sigma_n$ is the quotient of $P\Sigma_n$ by relations* (R1), (R2) *and* (R3). *As a consequence, it admits the presentation*

$$hP\Sigma_n \cong \langle \chi_{ij}\ (i \neq j) \mid \text{(R0)}, \text{(R1)}, \text{(R2)}, \text{(R3)}\rangle.$$

Proof Let $\mathscr{G}_n$ be the group defined by the presentation of the theorem. The χ_{ij} in $hP\Sigma_n$ satisfy the above relations. As a consequence, there is and obvious morphism π from $\mathscr{G}_n$ to $hP\Sigma_n$. Since the χ_{ij} generate $hP\Sigma_n$ (Corollary 2.10), this morphism is surjective. We need to show that it is an isomorphism. We will do that by showing that $\mathscr{G}_n$ admits a decomposition similar to that of $hP\Sigma_n$, and that the pieces in the two decompositions are isomorphic via π. We do this in three steps, parallel to the proof of Theorem 3.1.

Step 1 We define a projection $\tilde{p}_n$ from $\mathscr{G}_n$ to $\mathscr{G}_{n-1}$ by sending χ_{ij} to χ_{ij} if $n \notin \{i, j\}$, and χ_{in} and χ_{nj} to 1. This morphism is well defined (from the presentations), and so is its obvious section $\tilde{s}_n : \mathscr{G}_{n-1} \hookrightarrow \mathscr{G}_n$. If we denote by $\widetilde{\mathscr{K}}_n$ the kernel of $\tilde{p}_n$, we get a semidirect product decomposition $\mathscr{G}_n = \widetilde{\mathscr{K}}_n \rtimes \mathscr{G}_{n-1}$ that fits in the following diagram:

$$\begin{tikzcd}
\widetilde{\mathscr{K}}_n \arrow[r, hook] \arrow[d, dashed] & \mathscr{G}_n \arrow[r, two heads, "\tilde{p}_n"'] \arrow[d, "\pi"] & \mathscr{G}_{n-1} \arrow[l, bend right, "\tilde{s}_n"'] \arrow[d, "\pi"] \\
\mathscr{K}_n \arrow[r, hook] & hP\Sigma_n \arrow[r, two heads, "p_n"'] & hP\Sigma_{n-1} \arrow[l, bend right, "s_n"']
\end{tikzcd}$$

By induction (using the five lemma), beginning with the isomorphism $\mathscr{G}_2 \cong hP\Sigma_2 \cong \mathbb{Z}^2$ (which is the group $\langle \chi_{12}, \chi_{21}\rangle$ of inner automorphisms of RF_2), we only need to show that the induced morphism between the kernels are isomorphisms.

Step 2 We can apply Lemma 3.2 to the above decomposition of $\mathscr{G}_n$; the proof of Lemma 3.3 only used the McCool relations, so it carries over without change to show that $\widetilde{\mathscr{K}}_n$ is generated by the χ_{in} together with the χ_{nj}. This shows directly that the map from $\widetilde{\mathscr{K}}_n$ to $\mathscr{K}_n$ is surjective (this fact also comes from the snake lemma and the induction hypothesis). Consider the map $\widetilde{\mathscr{K}}_n \to \mathscr{K}_n \twoheadrightarrow \mathrm{RF}_n$, where the second map is the projection q_n from $\mathscr{K}_n$ to RF_{n-1} defined in the proof of Theorem 3.1. This map sends the χ_{in} to 1 and the χ_{nj} to the x_j. From the relations (R1), we know that the assignment $x_j \mapsto \chi_{nj}$ defines a section $\tilde{t}_n$ from RF_{n-1} to $\widetilde{\mathscr{K}}_n$. This shows that the χ_{nj} generate a reduced free group inside $\widetilde{\mathscr{K}}_n$. If we denote by $\widetilde{\mathscr{K}}'_n$ the kernel of $\tilde{q}_n = q_n \circ \pi$, we get a semidirect product decomposition $\widetilde{\mathscr{K}}_n = \widetilde{\mathscr{K}}'_n \rtimes \mathrm{RF}_{n-1}$, similar to (3-2), that fits in the following diagram:

$$\begin{tikzcd}
\widetilde{\mathscr{K}}'_n \arrow[r, hook] \arrow[d, dashed] & \widetilde{\mathscr{K}}_n \arrow[r, two heads, "\tilde{q}_n"'] \arrow[d, "\pi"] & \mathrm{RF}_{n-1} \arrow[l, bend right, "\tilde{t}_n"'] \arrow[d, "\cong"] \\
\mathscr{K}'_n \arrow[r, hook] & \mathscr{K}_n \arrow[r, two heads, "q_n"'] & \mathrm{RF}_{n-1} \arrow[l, bend right, "t_n"']
\end{tikzcd}$$

Step 3 In order to show that the induced projection $\pi : \widetilde{\mathcal{K}}'_n \to \mathcal{K}'_n$ is an isomorphism, we need to investigate the structure of $\widetilde{\mathcal{K}}'_n$. By Lemma 3.2, it is generated by the χ_{in}^w for $w \in \langle \chi_{nj} \rangle \cong \mathrm{RF}_{n-1}$, and the relations (R2) say exactly that these commute with each other. Thus $\widetilde{\mathcal{K}}'_n$ is abelian. Let us fix i and denote by $\widetilde{N}_i$ the subgroup generated by the χ_{in}^w. It is the normal closure of χ_{in} in the subgroup $\widetilde{M}_i$ generated by χ_{in} and the χ_{nj}. Relations (R1) and (R2) imply that χ_{in} and the χ_{nj} commute with their conjugates in M_i, which is thus a quotient of RF_n. In particular, $\widetilde{M}_i$ is nilpotent, and we can apply Proposition 5.4 to get that $\widetilde{N}_i$ is generated by Lyndon monomials in χ_{in} and the χ_{nj} containing χ_{in}. We can even limit ourselves to the subset $\mathrm{Lynd}(\chi_{in}; (\chi_{nj})_j)$ of monomials without repetitions, the other ones being trivial by the argument above. Furthermore, the relations (R3) say exactly that among these, the ones containing χ_{ni} vanish. Thus, the abelian group $\widetilde{N}_i$ is generated by $\mathrm{Lynd}(\chi_{in}; \chi_{n1}, \dots, \hat{\chi}_{ni}, \dots, \chi_{n,n-1})$. Because of Lemma 5.6, we know that these monomials are sent to linearly independent elements in N_i (in fact, to a basis of this abelian group), so they must be a basis of $\widetilde{N}_i$, and the projection π induced a isomorphism between $\widetilde{N}_i$ and N_i. The projection $\pi : \widetilde{\mathcal{K}}'_n \to \mathcal{K}'_n$, being the direct product of these isomorphisms, is thus an isomorphism, which is the desired conclusion. □

Remark 5.9 The same remarks made at the end of Section 1.2 for RF_n hold true for $hP\Sigma_n$: it is finitely generated and nilpotent (of class $n-1$), so it has a finite presentation. However, in order to write down such a finite presentation, we need a presentation of the free $(n-1)$–nilpotent group on n^2 generators χ_{ij}. We can then add to such a presentation the relations similar to (R1), (R2) and (R3) that are iterated brackets of the generators (of any shape) of length at most $n-1$ to get an explicit finite presentation of $hP\Sigma_n$. In other words, the latter relations give a finite presentation of $hP\Sigma_n$ *as an $(n-1)$–nilpotent group*.

5.3 A presentation of the associated Lie ring

Using the above methods, one can also find a presentation of the Lie ring associated to $hP\Sigma_n$, similar to the presentation of $\mathcal{L}(hP_n)$ given in Corollary 3.12.

Proposition 5.10 *The Lie ring of $hP\Sigma_n$ is generated by x_{ij} $(1 \leqslant i \neq j \leqslant n)$, under the relations*

$$\begin{aligned} [x_{ik} + x_{jk}, x_{ij}] &= 0 \quad \text{for } i, j, k \text{ pairwise distinct}, \\ [x_{ik}, x_{jk}] &= 0 \quad \text{for } i, j, k \text{ pairwise distinct}, \\ [x_{ij}, x_{kl}] &= 0 \quad \text{if } \{i, j\} \cap \{k, l\} = \varnothing, \end{aligned}$$

to which are added, for each m, the families of relations

$$[x_{im}, [x_{mi}, t]] = 0, \quad [x_{im}, [x_{jm}, t]] = 0, \quad [x_{im}, [x_{mi}, t]] = 0,$$

where, in each case, t describes Lie monomials in the x_{mk} $(k < m)$.

Proof Since it is very similar to the proof of Theorem 5.8, we only outline the proof. Let $h\mathfrak{p}_n$ be the Lie ring defined by the presentation of the theorem. The relations are true for the classes of the χ_{ij} in $\mathcal{L}(hP\Sigma_n)$ (as direct consequences of the relations in the group $hP\Sigma_n$), so $x_{ij} \mapsto \bar{\chi}_{ij}$ defines a projection

π from $h\mathfrak{p}_n$ onto $\mathscr{L}(hP\Sigma_n)$. One shows that $h\mathfrak{p}_n$ admits a decomposition similar to the decomposition of $\mathscr{L}(hP\Sigma_n)$ described in Theorem 3.8. Indeed, the morphism from $h\mathfrak{p}_n$ to $h\mathfrak{p}_{n-1}$ sending x_{ij} on x_{ij} if $n \notin \{i, j\}$ and to 0 else is a well-defined projection p, which is split. From the relations, reasoning as in the proof of Lemma 3.3, one checks that the x_{in} together with the x_{ni} generate an ideal of $h\mathfrak{p}_n$, which has to be the kernel $\mathfrak{k}_n$ of p. They one argues exactly as in the proof of Theorem 5.8 to show (using the first family of relations) that $\mathfrak{k}_n$ decomposes as a semidirect product $\mathfrak{k}'_n \rtimes R\mathscr{L}_{n-1}$. Moreover, the projection π is compatible with the decompositions of $h\mathfrak{p}_n$ and $\mathscr{L}(hP\Sigma_n)$. Using the five lemma, we see that we only have to check that π induces an isomorphism between $\mathfrak{k}'_n$ and $\prod\langle y_i\rangle$. Since we know a basis of the target, whose elements are Lie monomials on the $\bar{\chi}_{in}$ and $\bar{\chi}_{ni}$, we are left with showing that the corresponding Lie monomials on the x_{in} and x_{ni} generate $\mathfrak{k}'_n$. Like in the proof of Theorem 5.8, the last two families of relations ensure exactly that, so π is indeed an isomorphism. $\square$

Remark 5.11 In the presentation, one can consider only the relations where t is a linear monomial of length at most m.

Remark 5.12 It is a difficult open question, very much related to the Andreadakis problem for $P\Sigma_n$, to decide whether the first three relations (the linearized McCool relations) define a presentation of the Lie ring of $P\Sigma_n$. It is only known to hold rationally [Berceanu and Papadima 2009].

Appendix Lyndon words and the free Lie algebra

For the comfort of the reader, we gather here some basic facts about Lyndon words. These describe a basis of the free Lie algebra, and we give a self-contained proof of this classical result involving as little machinery as possible. Our main sources for this appendix were Serre's lecture notes [1965] and Reutenauer's book [2003, 5.1].

A.1 Lyndon words

Let $\mathscr{A}$ be a set (called an *alphabet*) endowed with a fixed total order. We denote by $\mathscr{A}^*$ the free monoid generated by $\mathscr{A}$. Elements of $\mathscr{A}^*$ are *words* in $\mathscr{A}$, that is, finite sequence of elements of $\mathscr{A}$. The set $\mathscr{A}^*$ is endowed with the usual dictionary order induced by the order on $\mathscr{A}$.

The length of a word w is denoted by $|w|$. If v and w are words, v is a *suffix* (resp. a *prefix*) of w if there exists a word u such that $w = uv$ (resp. $w = vu$). It is called *proper* when it is nonempty and different from w.

Definition A.1 The *standard factorization* of a word w of length at least 2 is the factorization $w = uv$, where v is the smallest proper suffix of w.

Definition A.2 A *Lyndon word* is a nonempty word that is minimal among its nonempty suffixes.

Lemma A.3 *If $w = uv$ is a standard factorization, then v is a Lyndon word, and if w is Lyndon then so is u.*

Proof The fact that v is a Lyndon word is clear. Suppose that w is a Lyndon word. Let x be any proper suffix of u. Since $uv = w < xv$, if x is not a prefix of u, then $u < x$. Otherwise, $u = xy$ for some nonempty y, but then $xyv < xv$ implies $yv < v$, which contradicts the definition of v. □

The following proposition is the most basic result in the theory of Lyndon words:

Proposition A.4 *Every word $w \in \mathscr{A}^*$ factorizes uniquely as a product $l_1 \cdots l_n$ where n is an integer, the l_i are Lyndon words and $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_n$. We call this the* ***Lyndon factorization*** *of w.*

Proof We first prove unicity, by proving that in a factorization $w = l_1 \cdots l_n$ into a nonincreasing product of Lyndon words, l_n is the smallest nonempty suffix of w. Indeed, let v be a suffix of w. Decompose v as $yl_{k+1} \cdots l_n$, where y is a nonempty suffix of l_k (possibly equal to l_k). Then $v \geqslant y \geqslant l_k \geqslant l_n$.

We show existence by induction on the length of w. Take l_n to be the smallest nonempty suffix of w. Then $w = w'l_n$, and l_n is a Lyndon word. Moreover, a nonempty suffix of w' cannot be strictly smaller than l_n. Indeed, if y is a nonempty suffix of w' such that $y < l_n$, then either y is a (proper) prefix of l_n or $yl_n < l_n$. The second case contradicts the definition of l_n. In the first case, by definition of l_n, we get $yl_n > l_n = yu$, whence $l_n > u$. Thus both cases contradict the definition of l_n; we must have $y \geqslant l_n$. As a consequence, a factorization of w' satisfying the conditions of the proposition gives such a factorization for w, whence the conclusion. □

Proposition A.4 allows us to identify the abelian group $\mathbb{Z}\mathscr{A}^*$ with the symmetric algebra $S^*_{\mathbb{Z}}(\mathrm{Lynd})$. Note that this linear identification does not preserve the ring structure, since the Lyndon factorization of a product uv need not be the product of the Lyndon factorization of u with that of v.

A.2 The Lyndon basis of the free Lie algebra

In the sequel, $V = \mathbb{Z}\{\mathscr{A}\}$ is the free abelian group generated by the alphabet $\mathscr{A}$. We denote by $\mathfrak{L}V$ the free Lie algebra on V and by TV the free associative algebra on V. Recall that their universal properties imply that TV is the enveloping algebra of $\mathfrak{L}V$. We denote by $\iota\colon \mathfrak{L}V \to TV$ the canonical Lie morphism between them. Note that we do not know a priori that this map is injective (we do not assume the PBW theorem to be known).

Define an application $w \mapsto P_w$ from the set Lynd of Lyndon word on $\mathscr{A}$ to $\mathfrak{L}V$ as follows:

- Take $P_a := a \in V$ for any letter $a \in \mathscr{A}$.
- If w is a Lyndon word, consider its standard factorization $w = uv$ and define P_w to be $[P_u, P_v] \in \mathfrak{L}V$.

Lemma A.5 (standard factorization of a product of Lyndon words) *Let u and v be Lyndon words. Then uv is a Lyndon word if and only if $u < v$. Moreover, suppose that $u < v$, and denote by $u = xy$ the standard factorization of u, if u is not a letter. Then the standard factorization of uv is $u \cdot v$ if and only if u is a letter or $v \leqslant y$.*

Proof If uv is a Lyndon word, then $u < uv < v$. Conversely, suppose that $u < v$. Then either $uv < v$ or u is a prefix of v. But in this second case, $v = uw$, and $v < w$ implies that $uv < uw = v$, so in both cases $uv < v$. Now, take a proper suffix w of uv. If w is a suffix of v, then $w \geqslant v > uv$. If not, then $w = w'v$ with w' a proper suffix of u. Then $u < w'$ implies $uv < w'v = w$, finishing the proof that uv is a Lyndon word.

If u is a letter, then v is clearly the smallest proper suffix of uv. Let us assume that u is not a letter. Suppose that $v \leqslant y$. A proper suffix of uv is either a suffix of v, which is greater than v, or of the form wv, where w is a proper suffix of u. In the latter case, since y is the smallest proper suffix of u, we have $v \leqslant y \leqslant w < wv$. This shows that v is the smallest proper suffix of uv in this case. Conversely, if $v > y$, then yv is a Lyndon word by the first part of the proof. Hence $yv < v$, so v is not the smallest proper suffix of uv in this case. □

The following proposition and its proof are adapted from [Serre 1965, Theorem 5.3]. The proof is arguably the most technical one in the present appendix:

Proposition A.6 *The P_w for $w \in \mathrm{Lynd}$ linearly generate $\mathfrak{L}V$.*

Proof We only need to show that the $\mathbb{Z}$–module generated by the P_w is a Lie subalgebra. We show that if u and v are Lyndon words, then $[P_u, P_v]$ is a linear combination of P_w, with $|w| = |u| + |v|$ and $w < \max(u, v)$, by induction on $|u| + |v|$ and on $\max(u, v)$. To begin with, if u and v are letters, then we can suppose that $u < v$ (otherwise, use the antisymmetry relation). Then $[P_u, P_v] = P_{uv}$, and $uv < v$.

Now, take (u, v) such that $|u| + |v| > 2$, and suppose that our claim is proven for every (u', v') such that $|u'| + |v'| < |u| + |v|$, or $|u'| + |v'| = |u| + |v|$ and $\max(u', v') < \max(u, v)$. Using antisymmetry if needed, we can assume that $u < v$. We then use Lemma A.5. When u is not a letter, consider the standard factorization $u = xy$ of u. If u is a letter or $y \geqslant v$, then $u \cdot v$ is the standard factorization of uv, whence $[P_u, P_v] = P_{uv}$, and $uv < v$, proving our claim. Suppose that $y < v$. Then

$$[P_u, P_v] = [[P_x, P_y], P_v] = [[P_x, P_v], P_y] + [P_x, [P_y, P_v]].$$

Since $|x|, |y| < |u|$, we can use the induction hypothesis to write $[P_x, P_v]$ (resp. $[P_y, P_v]$) as a linear combination of P_w (resp. P_t) such that $|w| = |x| + |v|$ (resp. $|t| = |y| + |v|$), and $w < v$ (resp. $t < v$). Then, using that $x, y < v$ (since $x < xy = u < y < v$), we can apply the induction hypothesis to $[P_w, P_y]$ (resp. to $[P_x, P_t]$) to prove our claim, ending the proof of the proposition. □

The application $w \mapsto P_w$ extends to a map from $\mathscr{A}^*$ to TV defined as follows:

- Take $P_a := a \in V$ for any letter $a \in \mathscr{A}$.
- If w is a Lyndon word, consider its standard factorization $w = uv$ and define P_w to be $[P_u, P_v] \in \mathfrak{L}V$.
- If w is any word, consider its Lyndon factorization $w = l_1 \cdots l_n$. Define P_w to be $P_{l_1} \cdots P_{l_n} \in TV$.

The next lemma [Reutenauer 2003, Theorem 5.1], which says that the expression of the P_w in terms of associative monomials is governed by a triangular matrix, will play a key role in what follows.

Lemma A.7 *For any word w, the polynomial P_w is the sum of w and a linear combination of (strictly) greater words having the same length as w.*

Proof Note that if l is a Lyndon word and $l = uv$ with u and v nonempty, then $uv = l < v < vu$.

We use this to show the lemma for Lyndon words, by induction on their length. For letters, the result is obvious. Let l be a Lyndon word, and consider its standard factorization $l = uv$. Then u and v are Lyndon words, and $u < v$ (Lemmas A.3 and A.5). If the result is true for u and v, then $P_l = [P_u, P_v]$ is a linear combination of elements of the form $[s,t] = st - ts$, where $|s| = |u|$, $|t| = |v|$, $s \geqslant u$ and $t \geqslant v$. Then $ts \geqslant vu > uv$, and $st \geqslant uv$, with equality if and only if $s = u$ and $t = v$. Thus the word $l = uv$ appears with coefficient 1 in the decomposition of P_l, and $P_l - l$ is a linear combination of greater words, of the same length as l, which proves our claim.

Now, if w is any word, consider its Lyndon factorization $w = l_1 \cdots l_n$. Then $P_w := P_{l_1} \cdots P_{l_n}$ is a linear combination of $x_1 \cdots x_n$, where each x_i is a word satisfying $|x_i| = |l_i|$ and $x_i \geqslant l_i$. As a consequence, $|x_1 \cdots x_n| = |l_1 \cdots l_n|$, and $x_1 \cdots x_n \geqslant l_1 \cdots l_n$, with equality if and only if each x_i is equal to l_i. This last case only appears with coefficient 1, so the lemma is proved. □

The above application extends to a linear map $P : \mathbb{Z}\mathscr{A}^* \to TV$.

Proposition A.8 *The application $P : \mathbb{Z}\mathscr{A}^* \to TV$ defined above is injective.*

Proof Let m be a linear combination of words in the kernel of P. Suppose that w is such that no word smaller that w appears in m. Let λ be the coefficient of w in m. Then by Lemma A.7, λ is also the coefficient of w in $P_m = 0$, so it must be trivial. Thus, by induction, all coefficients of m have to be trivial, whence $m = 0$ and P is injective. □

We can now sum this up as the main result of this appendix:

Theorem A.9 *The map P induces a graded linear isomorphism*

$$\mathbb{Z}\{\mathrm{Lynd}\} \cong \mathfrak{L}V.$$

Otherwise said, the family $(P_w)_{w \in \mathrm{Lynd}}$ is a linear basis of $\mathfrak{L}V$.

Proof The P_w generate $\mathfrak{L}V$ (Proposition A.6) and, since their images in TV are linearly independent (Proposition A.8), they must be linearly independent. □

A.3 Primitive elements and the Milnor–Moore theorem

In proving the previous result, we have only used basic linear algebra, and the combinatorics of Lyndon words. In order to convince the reader of how powerful these techniques are, we will now recover the Milnor–Moore theorem for the algebra TV, using not much more machinery. The only additional tools we need are coalgebra structures and primitive elements.

The free commutative ring on the free abelian group V is denoted by $S^*(V)$. It is endowed with its usual Hopf algebra structure, whose coproduct is the only algebra morphism $\Delta\colon S^*(V) \to S^*(V) \otimes S^*(V)$ sending each element v of V to $v \otimes 1 + 1 \otimes v$. That is, it is the only bialgebra structure on $S^*(V)$ such that V consists of primitive elements. In fact, these are the only primitive elements in $S^*(V)$ [Serre 1965, Theorem 5.4]:

Proposition A.10 *The set of primitive elements of $S^*(V)$ is V.*

Proof By definition of the coproduct of $S^*(V)$, the subspace V is made of primitive elements. To show the converse, it is helpful to see $S^*(V)$ as the algebra $\mathbb{Z}[X_i]$ of polynomials in indeterminates X_i. Then $S^*(V) \otimes S^*(V)$ identifies with $\mathbb{Z}[X_i', X_i'']$, and the coproduct sends X_i to $X_i' + X_i''$. Since it is an algebra morphism, it sends a polynomial $f(X_i)$ to $f(X_i' + X_i'')$. Thus primitive elements are those f such that $f(X_i' + X_i'') = f(X_i') + f(X_i'')$, ie additive ones. But since we work over $\mathbb{Z}$, these are only the linear ones, which is the desired conclusion. □

The algebra TV is endowed with a Hopf structure defined exactly as the one for S^*V: it is the unique bialgebra structure such that elements of V are primitive ones. Since primitive elements are a Lie subalgebra, they contain the Lie subalgebra generated by V (which is the image $\iota(\mathfrak{L}V)$ of the canonical morphism $\iota\colon \mathfrak{L}V \to TV$).

Recall that Proposition A.4 allows us to identify $\mathbb{Z}\mathcal{A}^*$ with the symmetric algebra $S^*_{\mathbb{Z}}(\mathrm{Lynd})$. We will show the following:

Theorem A.11 (Milnor–Moore) *The application $P\colon S^*_{\mathbb{Z}}(\mathrm{Lynd}) \to TV$ defined in Section A.2 is an isomorphism of coalgebras.*

Proof Injectivity has already been shown (Proposition A.8). Let us first prove surjectivity. Let $p \neq 0$ be a homogeneous element of TV. Let w be the smallest monomial appearing in p, with coefficient λ. Then $p - \lambda P_w$ is homogeneous and contains only monomials greater than w (see Lemma A.7). By repeating this process, we can write p as a linear combination of P_w. Indeed, this process stops, since we consider only the finite set of words of fixed length (equal to the degree of p) whose letters appear in some monomial of p. This proves that P is surjective.

We are left to show that the application $P: w \mapsto P_w$ preserves the coproduct. We first remark that if l is a Lyndon word, then l is primitive in $S^*_{\mathbb{Z}}(\text{Lynd})$, and $P_l \in \iota(\mathfrak{L}V)$ is primitive in TV. For any word w, consider its Lyndon factorization $w = l_1 \cdots l_n$. Then we can write

$$\begin{aligned}\Delta(P_w) = \Delta(P_{l_1}) \cdots \Delta(P_{l_n}) &= (P_{l_1} \otimes 1 + 1 \otimes P_{l_1}) \cdots (P_{l_n} \otimes 1 + 1 \otimes P_{l_n}) \\ &= \sum_{\underline{n} = X \sqcup Y} P_{l_{x_1}} \cdots P_{l_{x_p}} \otimes P_{l_{y_1}} \cdots P_{l_{y_q}},\end{aligned}$$

where the sum is over all partitions of the set $\underline{n} = \{1, \dots, n\}$ into two subsets $X = \{x_1 < \cdots < x_p\}$ and $Y = \{y_1 < \cdots < y_q\}$. As a consequence,

$$\begin{aligned}\Delta(P_w) &= \sum_{\underline{n} = X \sqcup Y} P_{l_{x_1} \cdots l_{x_p}} \otimes P_{l_{y_1} \cdots l_{y_q}} \\ &= (P \otimes P)\Big(\sum_{\underline{n} = X \sqcup Y} l_{x_1} \cdots l_{x_p} \otimes l_{y_1} \cdots l_{y_q} \Big) \\ &= (P \otimes P)(\Delta(l_1) \cdots \Delta(l_n)) = (P \otimes P)(\Delta(w)). \qquad \square\end{aligned}$$

Corollary A.12 *The canonical map $\iota: \mathfrak{L}V \to TV$ identify $\mathfrak{L}V$ with the Lie algebra of primitive elements in TV.*

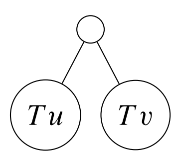
Proof Thanks to Theorems A.9 and A.11, this map identifies with $\mathbb{Z}\{\text{Lynd}\} \to S^*_{\mathbb{Z}}(\text{Lynd})$. But Proposition A.10 ensures that the set of primitive elements of the coalgebra $S^*_{\mathbb{Z}}(\text{Lynd})$ is exactly $\mathbb{Z}\{\text{Lynd}\}$, whence the result. □

Remark A.13 Neither our identification of the free abelian group $\mathbb{Z}\{\text{Lynd}\}$ with the primitives of TV nor our proof of Theorem A.11 requires the use of the fact that Lyndon words generate $\mathfrak{L}V$ (Proposition A.6); we only used that they are linearly independent (Proposition A.8) for that. The full strength of Theorem A.9 is only used to see that $P: \mathbb{Z}\mathscr{A}^* \hookrightarrow TV$ coincides with $\iota: \mathfrak{L}V \to TV$ (whence Corollary A.12).

A.4 Linear trees

The free Lie algebra can be seen as a quotient of the free abelian group $\mathbb{Z}\mathscr{M}(\mathscr{A})$ on the free magma $\mathscr{M}(\mathscr{A})$ on $\mathscr{A}$ by antisymmetry and the Jacobi identity. Elements of the free magma can be seen as parenthesized words in $\mathscr{A}$, or as finite rooted planar binary trees, whose leaves are indexed by elements of $\mathscr{A}$. The images of elements of the free magma in $\mathfrak{L}V$ are called *Lie monomials*.

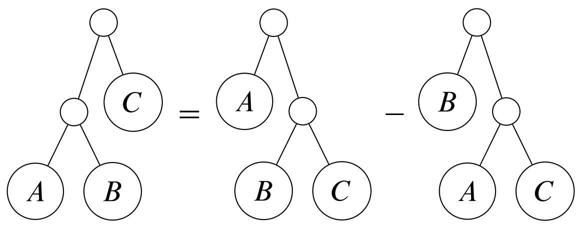
Lyndon words encode a family of rooted binary trees whose leaves are indexed by letters. Precisely, if w is a Lyndon word, the tree $T(w)$ is just one leaf indexed by w, if w is a letter. If not, take the standard factorization $w = uv$. Then $T(w)$ is given by a root, a left son $T(u)$ and a right son $T(v)$:

$T(uv) =$ 

The Lyndon basis of the free Lie algebras are the Lie monomials P_w obtained from such trees by interpreting nodes as Lie brackets. We call these *Lyndon monomials*

One can consider another family of Lie monomials, called *linear Lie monomials*, given by linear trees, that is, monomials which are letters or of the form $[y_1, \dots, y_n]$ $(= [y_1, [y_2, [\dots [y_{n-1}, y_n] \dots]]])$. It is easy to see, by induction, using the Jacobi identity, that these generate $\mathfrak{L}V$. In fact, the Jacobi identity can be written as

(A-1)



Using this as a rewriting rule (from left to right), one can write any tree (that is, any Lie monomial) as a linear combination of trees whose left son is a leaf. Applying the induction hypothesis to the right sons, one gets a linear combination of linear trees.

There are $n!$ linear Lie monomials in degree n, which is clearly strictly greater than the number of Lyndon words of length n, so they must be linearly dependent. It is the need to control this redundancy that leads us to consider Lyndon words (or, more generally, Hall sets).

Lemma A.14 *Any Lie monomial is a linear combination of linear Lie monomials with the same letters* (*counted with repetitions*). *Also*, *it is a linear combination of Lyndon monomials with the same letters.*

Proof The first part follows from the rewriting process that we have just described. The second one is a bit trickier: although we know that a decomposition into a linear combination of Lyndon monomials *exists* (Proposition A.6), we did not give an algorithm to compute it. However, we can use a homogeneity argument as follows: $\mathbb{Z}\mathcal{M}(\mathscr{A})$ is $\mathbb{N}\{\mathscr{A}\}$–graded, the degree of an element of the free magma $\mathcal{M}(\mathscr{A})$ being its image in the free commutative monoid $\mathbb{N}\{\mathscr{A}\}$ (which counts the number of appearance of each letter in a given nonassociative word). Moreover, the antisymmetry and the Jacobi relations are homogeneous with respect to this degree, so that the quotient $\mathfrak{L}[\mathscr{A}]$ is again a graded abelian group with respect to this degree. As a consequence, if we write a Lie monomial m of degree d as a linear combination of Lyndon monomials, taking the homogeneous component of degree d results in an expression of m as a linear combination of Lyndon monomials of degree $d \in \mathbb{N}\{\mathscr{A}\}$, as claimed. □

We remark that the expression of m obtained in the proof by taking the homogeneous component must in fact must be the same as the first one, because of Theorem A.9.

Linear trees can be used to define a basis of the reduced free Lie ring $R\mathfrak{L}[n]$ which could be used to replace the Lyndon basis in all our work (this is in fact the point of view used in [Meilhan and Yasuhara 2019]):

Lemma A.15 *For all integer $k \geqslant 2$, a basis of $R\mathfrak{L}[n]_k$ is given by Lie monomials which are letters or of the form $[y_{i_1}, \dots, y_{i_k}]$ where the $i_j \leqslant n$ are pairwise distinct and satisfy $i_k = \max_j(i_j)$.*

Proof Using antisymmetry, one sees that up to a sign, any Lie monomial without repetition is equal to a Lie monomial with the same letter where the right-most factor (the right-most leaf of the corresponding tree) bears the maximal index. Then we can use the rewriting rule (A-1) to get a linear combination of linear trees, and the right-most leaf stays the same throughout the process, as does the set of letters used. This shows that Lie monomials of the form described in the lemma generate the abelian group $R\mathfrak{L}[n]_k$. Moreover, there are $(k-1)!\binom{n}{k}$ such monomials, which is already known to be the rank of $R\mathfrak{L}[n]_k$ (Proposition 1.9); hence this family must be a basis of $R\mathfrak{L}[n]_k$. □

References

[Andreadakis 1965] **S Andreadakis**, *On the automorphisms of free groups and free nilpotent groups*, Proc. Lond. Math. Soc. 15 (1965) 239–268 MR Zbl

[Audoux 2016] **B Audoux**, *On the welded tube map*, from "Knot theory and its applications" (K Gongopadhyay, R Mishra, editors), Contemp. Math. 670, Amer. Math. Soc., Providence, RI (2016) 261–284 MR Zbl

[Audoux and Meilhan 2019] **B Audoux**, **J-B Meilhan**, *Characterization of the reduced peripheral system of links*, preprint (2019) arXiv 1904.04763

[Audoux et al. 2017a] **B Audoux**, **P Bellingeri**, **J-B Meilhan**, **E Wagner**, *Homotopy classification of ribbon tubes and welded string links*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 17 (2017) 713–761 MR Zbl

[Audoux et al. 2017b] **B Audoux**, **J-B Meilhan**, **E Wagner**, *On codimension two embeddings up to link-homotopy*, J. Topol. 10 (2017) 1107–1123 MR Zbl

[Baez et al. 2007] **J C Baez**, **D K Wise**, **A S Crans**, *Exotic statistics for strings in 4D BF theory*, Adv. Theor. Math. Phys. 11 (2007) 707–749 MR Zbl

[Bar-Natan 1995] **D Bar-Natan**, *Vassiliev homotopy string link invariants*, J. Knot Theory Ramifications 4 (1995) 13–32 MR Zbl

[Bar-Natan and Dancso 2016] **D Bar-Natan**, **Z Dancso**, *Finite-type invariants of w-knotted objects, I: w-knots and the Alexander polynomial*, Algebr. Geom. Topol. 16 (2016) 1063–1133 MR Zbl

[Bar-Natan and Dancso 2017] **D Bar-Natan**, **Z Dancso**, *Finite type invariants of w-knotted objects, II: Tangles, foams and the Kashiwara–Vergne problem*, Math. Ann. 367 (2017) 1517–1586 MR Zbl

[Bardakov 2003] **V G Bardakov**, *Structure of a conjugating automorphism group*, Algebra Logika 42 (2003) 515–541 MR Zbl In Russian; translated in Algebra Logic 42 (2003) 287–303

[Bardakov and Bellingeri 2014] **V G Bardakov**, **P Bellingeri**, *Groups of virtual and welded links*, J. Knot Theory Ramifications 23 (2014) art. id. 1450014 MR Zbl

[Bartholdi 2016] **L Bartholdi**, *Automorphisms of free groups, I: Erratum*, New York J. Math. 22 (2016) 1135–1137 MR Zbl

[Berceanu and Papadima 2009] **B Berceanu**, **Ş Papadima**, *Universal representations of braid and braid-permutation groups*, J. Knot Theory Ramifications 18 (2009) 999–1019 MR Zbl

[Birman 1974] **J S Birman**, *Braids, links, and mapping class groups*, Ann. of Math. Stud. 82, Princeton Univ. Press (1974) MR Zbl

[Cohen 1995] **F R Cohen**, *On combinatorial group theory in homotopy*, from "Homotopy theory and its applications" (A Adem, R J Milgram, D C Ravenel, editors), Contemp. Math. 188, Amer. Math. Soc., Providence, RI (1995) 57–63 MR Zbl

[Damiani 2017] **C Damiani**, *A journey through loop braid groups*, Expo. Math. 35 (2017) 252–285 MR Zbl

[Damiani 2019] **C Damiani**, *Towards a version of Markov's theorem for ribbon torus-links in* $\mathbb{R}^4$, from "Knots, low-dimensional topology and applications" (C C Adams, C M Gordon, V F R Jones, L H Kauffman, S Lambropoulou, K C Millett, J H Przytycki, R Ricca, R Sazdanovic, editors), Springer Proc. Math. Stat. 284, Springer (2019) 309–328 MR Zbl

[Darné 2019] **J Darné**, *On the stable Andreadakis problem*, J. Pure Appl. Algebra 223 (2019) 5484–5525 MR Zbl

[Darné 2021] **J Darné**, *On the Andreadakis problem for subgroups of* IA_n, Int. Math. Res. Not. 2021 (2021) 14720–14742 MR Zbl

[Fenn et al. 1997] **R Fenn**, **R Rimányi**, **C Rourke**, *The braid-permutation group*, Topology 36 (1997) 123–135 MR Zbl

[Goldsmith 1974] **D L Goldsmith**, *Homotopy of braids: in answer to a question of E Artin*, from "Topology conference" (R F Dickman, Jr, P Fletcher, editors), Lecture Notes in Math. 375, Springer (1974) 91–96 MR Zbl

[Goldsmith 1981] **D L Goldsmith**, *The theory of motion groups*, Michigan Math. J. 28 (1981) 3–17 MR Zbl

[Gusarov 1994] **M Gusarov**, *On n–equivalence of knots and invariants of finite degree*, from "Topology of manifolds and varieties" (O Viro, editor), Adv. Soviet Math. 18, Amer. Math. Soc., Providence, RI (1994) 173–192 MR Zbl

[Gusarov et al. 2000] **M Goussarov**, **M Polyak**, **O Viro**, *Finite-type invariants of classical and virtual knots*, Topology 39 (2000) 1045–1068 MR Zbl

[Habegger and Lin 1990] **N Habegger**, **X-S Lin**, *The classification of links up to link-homotopy*, J. Amer. Math. Soc. 3 (1990) 389–419 MR Zbl

[Habegger and Masbaum 2000] **N Habegger**, **G Masbaum**, *The Kontsevich integral and Milnor's invariants*, Topology 39 (2000) 1253–1289 MR Zbl

[Jackson 2008] **D A Jackson**, *Basic commutators in weights six and seven as relators*, Comm. Algebra 36 (2008) 2905–2909 MR Zbl

[Kamada 2007] **S Kamada**, *Braid presentation of virtual knots and welded knots*, Osaka J. Math. 44 (2007) 441–458 MR Zbl

[Kohno 1985] **T Kohno**, *Série de Poincaré–Koszul associée aux groupes de tresses pures*, Invent. Math. 82 (1985) 57–75 MR Zbl

[Kontsevich 1993] **M Kontsevich**, *Vassiliev's knot invariants*, from "I M Gelfand Seminar, II" (S Gelfand, S Gindikin, editors), Adv. Soviet Math. 16, Amer. Math. Soc., Providence, RI (1993) 137–150 MR Zbl

[Lazard 1954] **M Lazard**, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. École Norm. Sup. 71 (1954) 101–190 MR Zbl

[McCool 1986] **J McCool**, *On basis-conjugating automorphisms of free groups*, Canad. J. Math. 38 (1986) 1525–1529 MR Zbl

[Meilhan and Yasuhara 2019] **J-B Meilhan**, **A Yasuhara**, *Arrow calculus for welded and classical links*, Algebr. Geom. Topol. 19 (2019) 397–456 MR Zbl

[Milnor 1954] **J Milnor**, *Link groups*, Ann. of Math. 59 (1954) 177–195 MR Zbl

[Milnor 1957] **J Milnor**, *Isotopy of links*, from "Algebraic geometry and topology" (R H Fox, editor), Princeton Univ. Press (1957) 280–306 MR Zbl

[Mostovoy and Willerton 2002] **J Mostovoy**, **S Willerton**, *Free groups and finite-type invariants of pure braids*, Math. Proc. Cambridge Philos. Soc. 132 (2002) 117–130 MR Zbl

[Nakamura et al. 2018] **T Nakamura**, **Y Nakanishi**, **S Satoh**, **A Yasuhara**, *The pass move is an unknotting operation for welded knots*, Topology Appl. 247 (2018) 9–19 MR Zbl

[Reutenauer 2003] **C Reutenauer**, *Free Lie algebras*, from "Handbook of algebra, III" (M Hazewinkel, editor), Elsevier, Amsterdam (2003) 887–903 MR Zbl

[Satoh 2000] **S Satoh**, *Virtual knot presentation of ribbon torus-knots*, J. Knot Theory Ramifications 9 (2000) 531–542 MR Zbl

[Satoh 2017] **T Satoh**, *On the Andreadakis conjecture restricted to the "lower-triangular" automorphism groups of free groups*, J. Algebra Appl. 16 (2017) art. id. 1750099 MR Zbl

[Serre 1965] **J-P Serre**, *Lie algebras and Lie groups*, Benjamin, New York (1965) MR Zbl

[Sims 1987] **C C Sims**, *Verifying nilpotence*, J. Symbolic Comput. 3 (1987) 231–247 MR Zbl

[Stanford 1996] **T Stanford**, *Braid commutators and Vassiliev invariants*, Pacific J. Math. 174 (1996) 269–276 MR Zbl

[Stanford 1998] **T B Stanford**, *Vassiliev invariants and knots modulo pure braid subgroups*, preprint (1998) arXiv math/9805092

[Vassiliev 1990] **V A Vassiliev**, *Cohomology of knot spaces*, from "Theory of singularities and its applications" (V I Arnold, editor), Adv. Soviet Math. 1, Amer. Math. Soc., Providence, RI (1990) 23–69 MR Zbl

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Morse–Bott cohomology from homological perturbation theory

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We construct cochain complexes generated by the cohomology of critical manifolds in the abstract setup of flow categories for Morse–Bott theories under minimum transversality assumptions. We discuss the relations between different constructions of Morse–Bott theories. In particular, we explain how homological perturbation theory is used in Morse–Bott theories, and both our construction and the cascades construction can be interpreted as applications of homological perturbations. In the presence of group actions, we construct cochain complexes for the equivariant theory. Expected properties like the independence of approximations of classifying spaces and the existence of the action spectral sequence are proven. We carry out our construction for Morse–Bott functions on closed manifolds and prove it recovers the regular cohomology. We outline the project of combining our construction with polyfold theory.

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1 Introduction

Morse theory [59] enables one to analyze the topology of a manifold by studying Morse functions on that manifold, or more explicitly by studying critical points and gradient flow lines. Although Morse functions are generic among all differentiable functions, sometimes it is more convenient to work with more special functions. Morse–Bott functions were introduced by Bott in [8] as generalizations of Morse functions, and have proven to be extremely useful for studying spaces in the presence of symmetries; see Bott [9] and Bott and Samelson [10]. Inspired by ideas of Witten [76] and Gromov [37], Floer generalized Morse theory to various infinite-dimensional settings [27; 28; 29; 30]. Now there are many invariants in symplectic geometry, contact geometry and low-dimensional topology based on Floer's construction. Many of them have a "Morse theoretical" background, eg Dostoglou and Salamon [22], Kronheimer and Mrowka [50], Ozsváth and Szabó [62] and Seidel [70]. Many other invariants (see Eliashberg, Givental and Hofer [23], Fukaya, Oh, Ohta and Ono [34] and Seidel [71]) are closely related to Morse theory. Usually invariants are defined in the "Morse" case, ie critical points are isolated, and invariants or structural maps are defined by counting zero-dimensional moduli spaces. However, often it is more convenient to study the Morse–Bott case, where we need to "count" higher-dimensional moduli spaces, since there are several benefits:

(1) Morse–Bott functions usually reflect some extra symmetries of the problem, and computations in Morse–Bott theory are usually simpler because of the extra symmetries (see Bourgeois [12] and Diogo and Lisi [20]).
(2) Morse–Bott theories appear in equivariant theories; see Austin and Braam [3], Bourgeois and Oancea [14] and Lin [53].

There are two aspects of Morse–Bott theories in applications. First, we need to construct compactified moduli spaces of gradient flow lines/Floer trajectories from one critical manifold to another critical manifold. Moreover, we need the moduli spaces to be equipped with smooth structures so that the moduli spaces are manifolds or orbifolds. To achieve that, there are three main methods:

(1) geometric perturbations (see McDuff and Salamon [56]), where one perturbs geometric data like almost-compact structures or metrics (such methods were used in many classical treatments of Floer theories),
(2) the Kuranishi method (see [34], as well as Joyce [48] and McDuff and Wehrheim [57]),
(3) the polyfold method (see Hofer, Wysocki and Zehnder [44]).

There are many other methods for specific geometric settings (see Cieliebak and Mohnke [17], Ionel and Parker [46], Li and Tian [52] and Ruan and Tian [67]) and algebraic treatments; see Pardon [63]. Second, from critical manifolds and compactified moduli spaces of gradient flow lines/Floer trajectories we need to construct cochain complexes. We focus on the second part. In particular, we explain how to count when the dimension of moduli spaces is positive, assuming the moduli spaces are reasonably nice. However, we will discuss the transversality problem for the finite-dimensional Morse–Bott theory in Section 8 using geometric perturbations, and outline the polyfold method for the general case in Section 9.

1.1 Cohomology of flow categories

It turns out that all critical manifolds and *compactified* moduli spaces from a Morse–Bott setting determine a category, namely a flow category, which was first introduced by Cohen, Jones and Segal in [19] to organize all the moduli spaces of flow lines in Morse/Floer theories. Roughly speaking, the objects of a flow category come from critical points, and the morphisms are (broken) flow lines.

In the Morse case, the cochain complex is constructed by counting points in the zero-dimensional moduli spaces (the morphism space). However, in a general Morse–Bott case, higher-dimensional moduli spaces should contribute nontrivially to the construction. Given a general abstract Morse–Bott flow category, there are several methods to get a chain or cochain complex:

(1) **Austin and Braam's model** [3] The cochain complex is generated by differential forms of the critical manifolds, and the differential is defined by the pullback and pushforward of differential forms through the compactified moduli spaces.

(2) **Fukaya's model** [33] The chain complex is generated by a certain subcomplex of the singular chain complex of the critical manifolds, and the differential is defined by the pushforward and pullback of singular chains through the compactified moduli spaces.

(3) **The cascades model of Bourgeois** [12] **and Frauenfelder** [32] The cochain complex is generated by Morse cochain complexes of critical manifolds after we assign suitable Morse functions to each critical manifold. The differential is defined by counting "cascades".[1]

All of the methods above have to make some assumptions on the compactified moduli spaces of Morse/Floer trajectories. In the Morse–Bott setting, Morse/Floer trajectories can break into pieces with ends matched. Hence the boundary of a compactified moduli space consists of fiber products over critical manifolds. The minimal transversality requirement is that these fiber products are cut out transversely. Such a requirement is natural using any reasonable virtual technique. We work in the context of flow categories under such fiber products transversality assumptions.

Our first goal is to unify the three methods and provide a simple and clean construction, called the *minimal Morse–Bott construction*, to every Morse–Bott flow category. Moreover, we will explain the following guiding principle in Morse–Bott constructions:

Claim *Formal applications of the homological perturbation lemma tend to give well-defined constructions.*

It turns out that both cascades and the minimal construction fit into this principle, and the relations are described in the following diagram:

$$\begin{tikzcd}
 & \boxed{\begin{array}{c}\text{Austin and Braam's}\\ \text{model/Fukaya's model}\end{array}} \arrow[d] & \\
 & \text{homological pertubation lemma} \arrow[dl] \arrow[dr] & \\
\boxed{\text{cascades construction}} & & \boxed{\text{minimal construction}}
\end{tikzcd}$$

[1] Strictly speaking, the original cascades model [12; 32] was phrased using homological conventions; the abovementioned cochain complex is the linear dual of the homological cascades model.

In applications of the homological perturbation theory, one needs to choose some perturbation data (projections and homotopies). For the cascades model, the projections and homotopies are provided by Harvey and Lawson's work [39] on Morse theory. The minimal construction is based on a more direct construction of the projections and homotopies. For example, one can choose the projection to harmonic forms and the associated Green operator (as the homotopy) as the perturbation data. The principle above also works for structures more general than a "linear structure" like flow categories, as long as all the relevant moduli spaces satisfy the fiber products transversality assumption; see eg Cieliebak and Volkov [18]. However, this has gone beyond our scope here.

Our main theorem is that, with suitable orientations, one can associate a well-defined cochain complex generated by the cohomology of the object space (critical manifolds) to a flow category:

Theorem *To every oriented flow category we can assign a minimal Morse–Bott cochain complex* $(\mathrm{BC}, d_{\mathrm{BC}})$ *over* $\mathbb{R}$ *generated by the cohomology of the object space* (*with a suitable completion*) *in a functorial way.*

Of course, this theorem bears no meaning yet. We point out here that:

(1) When the flow category arises from a Morse–Bott function on a closed manifold, the cohomology of the minimal Morse–Bott cochain complex is the cohomology of the manifold.

(2) When the flow category arises from a Morse case (critical points are nondegenerate and hence isolated), the cochain complex is the usual cochain complex with differential defined by counting rigid points in the morphism space.

(3) There are analogous constructions for continuation maps and homotopies, which, in applications, will yield invariance with respect to various auxiliary geometric data (Hamiltonians, almost-complex structures, metrics etc).

The construction provides explicit formulae for how higher-dimensional moduli spaces contribute in the construction; in particular, there are error-correcting terms from moduli spaces related to the boundaries and corners. Like the cascades construction, to write down an explicit cochain complex we need to make some choices on each critical manifold. One of the advantages of the minimal construction is that the choices do not require any compatibility condition with the morphism space (moduli spaces). The cohomology theory on the level of flow categories in this paper simplifies many geometric constructions including products (Section 7.1.1), quotients (Section 7.2.1) and fibrations (Section 6.2.1), as such constructions are natural on the level of flow categories.

The theorem above is the simplest version. We also discuss several generalizations: the critical manifold C_i can be noncompact, the critical manifold C_i can be equipped with local systems and does not have to be orientable, and it is not necessary that the cochain complex is generated by the cohomology, any finite-dimensional subspace of differential forms satisfying a cohomological relation is sufficient. Such flexibility allows us to prove a Gysin exact sequence for sphere bundles over flow categories. In [79], we use the Gysin exact sequence to show that any exact filling of a simply connected flexibly fillable contact manifold has the same cohomology ring structure on even degrees.

1.2 Equivariant theories

Our second goal is developing an equivariant theory on the level of flow categories, which would serve as a model for defining equivariant Floer theory. When there is a group G symmetry on the Morse–Bott theory, the cohomology theory should be enriched to a G–equivariant theory. One typical method is approximating the homotopy quotient. Bourgeois and Oancea [16] used a construction inspired by the cascades method to define the S^1–equivariant symplectic homology in this spirit. In our case, the homotopy quotient construction is very natural on the level of flow categories. Hence we can combine the Borel construction and our minimal construction, and realize the equivariant cochain complex as a homotopy limit.

Theorem *Assume a compact Lie group G acts on an oriented flow category $\mathcal{C}$ and preserves the orientations. Then there is a cochain complex $(\mathrm{BC}^G, d^G_{\mathrm{BC}})$, whose homotopy type is unique, ie independent of all the choices in the construction, particularly the choice of finite-dimensional approximations of the classifying space $EG \to BG$.*

1.3 Constructions of flow categories

The remaining obstacle to using the minimal construction in applications is constructing a flow category. In Section 8, we construct flow categories for the finite-dimensional Morse–Bott theory using geometric methods. In general, geometric perturbations (perturbing metrics in Morse theory and perturbing almost-complex structures in Floer theory), may not be enough to guarantee the transversality assumption, and hence one needs to apply some abstract perturbations. In fact, our minimal construction is applicable to polyfold theory. We can enrich a flow category (a system of manifolds) to a system of polyfolds with sc–Fredholm sections, and the boundaries/corners of the polyfolds come from transverse fiber products of polyfolds. We will refer this system as a polyflow category. Then we can find a coherent perturbation scheme and apply the abstract perturbation theorem for polyfolds of Hofer, Wysocki and Zehnder [44] to get a flow category. In the presence of a group action, the theorem above on equivariant cohomology requires G–equivariant transversality. But we know that G–equivariant transversality is typically obstructed. In general, we need to apply the Borel construction using quotient theorems of Zhou [78] to the whole polyflow category instead of the flow category.

Organization

Section 2 discusses the motivation of the minimal construction from homological perturbation theory and interprets the cascades construction as an example of an application of the homological perturbation theory. Section 3 defines the minimal cochain complex, as well as continuation maps and homotopies explicitly, and proves that they satisfy the desired properties. Section 4 discusses the action spectral sequence.

Section 5 explains how the orientations used in Section 3 arise in Morse/Floer theories. Section 5 also generalizes the construction to the case with local systems and nonorientable manifolds. Section 6 generalizes the construction to flow categories with noncompact critical manifolds, and also provides a more general setup which allows us to prove statements like the Gysin exact sequence. Section 7 discusses the equivariant theory. Section 8 is devoted to the Morse–Bott theory on finite-dimensional manifolds (both open and closed) and proves that the minimal construction recovers the cohomology of the underlying manifold. Section 9 outlines the project of combining our construction with polyfold theory.

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2 Motivation from homological perturbation theory

2.1 Differential topology notation

We first set up some notation and transversality theory for manifolds with boundaries and corners, and orientation conventions.

2.1.1 Manifolds and submanifolds with boundaries and corners Unless stated otherwise, all manifolds we consider are manifolds possibly with boundaries and corners [58, Definition 1.6.1], ie for every point in the manifold there is an open neighborhood diffeomorphic to an open subset of $\mathbb{R}^n_+$, where $\mathbb{R}_+ := [0,\infty)$. A *closed* manifold is a *compact* manifold *without boundary*.

Definition 2.1 Let M be a manifold and $x \in M$ a point. Choosing a chart $\phi\colon \mathbb{R}^n_+ \supset U \to M$ near $x \in M$, the degeneracy index $d(x)$ of the point x is defined to be $\#\{v_i \mid v_i = 0\}$, where $(v_1,\dots,v_n) \in \mathbb{R}^n_+$ and $\phi(v_1,\dots,v_n) = x \in M$.

The degeneracy index d does not depend on the local chart ϕ [58, Corollary 1.5.1]. For $i \geq 0$, we define the *depth-i boundary* $\partial_i M$ to be

$$\partial_i M := \{x \in M \mid d(x) = i\}. \tag{2-1}$$

Then $\partial_0 M$ is the set of interior points of M. Note that all $\partial_i M$ are manifolds without boundary, and in most cases they are noncompact. Submanifolds of manifolds should be compatible with structures defined in (2-1):

Definition 2.2 A closed subset $N \subset M$ is a submanifold of M if and only if N is a manifold such that the inclusion $N \to M$ is a smooth embedding and, for all $i \geq 0$, we have $\partial_i N = N \cap \partial_i M$. In other words, (M, N) near x is locally modeled on $(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}, \mathbb{R}^k_+ \times \mathbb{R}^{n-m} \times \{0\}^{m-k})$ near 0 for every $x \in N$.

An instant corollary is that if N is submanifold of M and M is submanifold of K, then N is also a submanifold of K. Unless stated otherwise, we will only consider submanifolds defined as above. In particular, when M has no boundary, a submanifold does not have boundary either. Note that $\partial_i M$ is not a submanifold of M in the sense of Definition 2.2 unless $\dim M = 0$.

Remark 2.3 (1) Some authors require, in the definition of manifolds with boundaries and corners, the additional property that faces (the closure of connected components of $\partial_1 M$) are submanifolds (not in the sense of Definition 2.2 but a weaker sense, eg t–submanifolds in [58, Definition 1.7.3]); for example, [58, Definition 1.8.5]. Such a definition will rule out the "teardrop" shape. Although we do not use this definition, we note here that in Floer/Morse cohomology theories, which are the main applications of our abstract construction, the compactified moduli spaces of Floer/Morse trajectories are manifolds with boundaries and corners in this stronger sense. However, if we were to consider more general algebraic structures (more complicated than a cochain complex) arising from the compactified moduli spaces of pseudoholomorphic curves, a "teardrop" moduli space may appear; see for example [64, Figure 8].

(2) There are different notions of submanifolds in a manifold with boundaries and corners depending on the purpose. For example, there are notions of t–, d–, and p–submanifolds [58, Section 1.7] depending on the compatibility of tangent spaces at the boundary. However, our notion of submanifolds is stronger than any of that, as we require that $l = k$ in the definition of p–submanifolds [58, Definition 1.7.4]. This is equivalent to requiring that (M, N) near x is locally modeled on $(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}, \mathbb{R}^k_+ \times \mathbb{R}^{n-m} \times \{0\}^{m-k})$ near 0 for $x \in N$.

(3) Submanifolds in the sense of Definition 2.2 arise naturally as zero sets of sections $s \colon M \to E$ of a vector bundle E over a manifold M with boundaries and corners, if $s|_{\partial_i M}$ is transverse to 0 for all i. This can be viewed as a prototype of how compactified moduli spaces of Floer cylinders/holomorphic curves can be equipped with the structure of a manifold with boundaries and corners in the polyfold perspective. The transversality requirements above are equivalent to s being in general position [44, Definition 5.3.9].

Definition 2.4 Transversality is defined as follows, to accommodate the boundary and corner structures:

(1) Let C be a manifold *without boundary*, B a submanifold of C and M a manifold possibly with boundaries and corners. A smooth map $f \colon M \to C$ is *transverse to* B if and only if $f|_{\partial_i M} \pitchfork B$ for all i in the classical sense, ie $\mathrm{D} f_x(T\partial_i M) + T_{f(x)} B = T_{f(x)} C$ for all $x \in \partial_i M$ such that $f(x) \in B$.

(2) Let M be a manifold, and N_1 and N_2 two submanifolds. Then we say N_1 *is transverse to* N_2 if and only if, for all $i \geq 0$ and every $x \in \partial_i N_1 \cap \partial_i N_2$, we have that $\partial_i N_1$ is transverse to $\partial_i N_2$ in $\partial_i M$ in the classical sense, ie $T_x \partial_i N_1 + T_x \partial_i N_2 = T_x \partial_i M$.

Proposition 2.5 *We have the following implicit function theorems*:

(1) *Let C be a manifold* ***without boundary*** *and B be a submanifold. Given a manifold M along with a smooth map f, assume that $f: M \to C$ is transverse to B in the sense of Definition 2.4(1). Then $f^{-1}(B)$ is a submanifold of M (in the sense of Definition 2.2).*

(2) *Let N_1 and N_2 be two submanifolds of a manifold M such that N_1 is transverse to N_2 in the sense of Definition 2.4(2). Then $N_1 \cap N_2$ is a submanifold of M. The codimension of $N_1 \cap N_2$ is the sum of the codimensions of N_1 and N_2.*

Proof The first claim is standard. We sketch a proof of the second claim using the first claim (but not the "obvious" one, as we cannot assume $C = M$ and $B = N_2$ in the first claim since M and N_2 have nonempty boundaries). Let $x \in N_2$ with $d(x) = k$; we may assume the pair $(M, N_2, x) \cap U$, for an open set $U \subset M$, is modeled on $(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}, \mathbb{R}^k_+ \times \mathbb{R}^{m-k} \times \{0\}^{n-m}, 0)$, following Remark 2.3. We consider $f: N_1 \cap U \to \mathbb{R}^{n-m}$, the projection to the last $n-m$ coordinates. It is straightforward to check that transversality in Definition 2.4(2) implies (and is actually equivalent to) that 0 is a regular value of f. Since $f^{-1}(0) = N_1 \cap N_2 \cap U$, we endow $N_1 \cap N_2$ with the structure of submanifold with boundaries and corners in N_1 by the first claim, and hence the structure of submanifold with boundaries and corners in M. □

Since measure-zero sets on differentiable manifolds are well defined and our construction is based on integration, errors over a measure-zero set can be tolerated. In particular, we have the following useful notion:

Definition 2.6 Let M and N be two manifolds. A smooth map $f: M \to N$ is a *diffeomorphism up to zero-measure* if and only if there exist measure-zero closed sets $M_1 \subset M$ and $N_1 \subset N$ such that $f|_{M \backslash M_1}: M \backslash M_1 \to N \backslash N_1$ is a diffeomorphism.

2.1.2 Orientations Given an *oriented* vector bundle E over a manifold M, the determinant bundle $\det E$ is a trivial line bundle, which can be reduced further to a trivial $\mathbb{Z}/2$–bundle $\operatorname{sign} E$. Moreover, we can assign to $\operatorname{sign} E$ a $\mathbb{Z}/2$ grading $|\operatorname{sign} E| = \operatorname{rank} E$. The fiber of $\operatorname{sign} E$ over $x \in M$ is the set of equivalence classes of ordered bases $[(e_1, \dots, e_n)]$ of the fiber E_x, where $(e_1, \dots, e_n)$ is equivalent to $(e'_1, \dots, e'_n)$ if and only if the transformation matrix between them has positive determinant. Then the orientation of E induces a continuous section of $\operatorname{sign} E$, and we use $[E] \in \Gamma(\operatorname{sign} E)$ to denote the section induced by the orientation.

Given two vector bundles E and F over M, we fix a bundle isomorphism:

$$\begin{gathered} m_{E,F}: \operatorname{sign}(E) \otimes_{\mathbb{Z}/2} \operatorname{sign}(F) \to \operatorname{sign}(E \oplus F), \\ [(e_1, \dots, e_n)] \otimes [(f_1, \dots, f_m)] \mapsto [(e_1, \dots, e_n, f_1, \dots, f_m)]. \end{gathered}$$

Therefore orientations $[E]$ and $[F]$ determine an orientation of $E \oplus F$ through $m_{E,F}$, and hence we denote the induced orientation by

$$[E][F] := m_{E,F}([E], [F]). \tag{2-2}$$

Since $[(e_1, \dots, e_n, f_1, \dots, f_m)] = (-1)^{nm}[(f_1, \dots, f_m, e_1, \dots, e_n)]$, we have

$$[E][F] = (-1)^{|F||E|}[F][E].$$

Definition 2.7 For simplicity of notation, we introduce the following:

- A manifold M is oriented if and only if the tangent bundle TM is oriented, and we use $[M]$ to denote the orientation.
- $\partial[M]$ denotes the induced orientation (in the usual sense, so that Stokes' theorem holds without extra sign) on the depth-1 boundary $\partial_1 M$ for an oriented manifold M.
- Let $E \to M$ and $F \to N$ be two oriented vector bundles. We use $[E] + [F]$ to denote the induced orientation on $E \cup F \to M \cup N$, and $-[E]$ to denote the opposite orientation.
- Unless stated otherwise, the product $M \times N$ is oriented by the product orientation of M and N, and we use $[M \times N]$ to denote the product orientation. Then

$$\partial[M \times N] = \partial[M] \times [N] + (-1)^{\dim M}[M] \times \partial[N]. \tag{2-3}$$

- If $f : M \to N$ is a diffeomorphism, we use $f_*[M]$ as the orientation on N induced by $\mathrm{D}f : TM \to TN$ and $[M]$.
- Let $E \to N$ be an oriented vector bundle and $f : M \to N$ a smooth map. Then the bundle map $f^*E \to E$ induces a bundle map $\mathrm{sign}(f^*E) \to \mathrm{sign}(E)$. Through this map, the orientation $[E]$ induces an orientation on f^*E over M; the induced orientation is denoted by $f^*[E]$.

Example 2.8 Let C be a closed oriented manifold. We now explain our orientation convention for the normal bundle N of the diagonal $\Delta \subset C \times C = C_1 \times C_2$ using the notation introduced in Definition 2.7: Δ is oriented by the condition[2] $\pi_{1*}[\Delta] = [C_1]$, where $\pi_1 : C_1 \times C_2 \to C_1$ is the projection. Then there exists a unique orientation of N such that, when restricted to Δ, we have

$$[\Delta][N] = [TC_1][TC_2]|_\Delta.$$

For simplicity, we suppress the restrictions and the subscripts,[3] and the equation becomes

$$[\Delta][N] = [C][C] \quad \text{or equivalently} \quad [N][\Delta] = (-1)^{(\dim C)^2}[C][C]. \tag{2-4}$$

This determines our orientation convention for the normal bundle N.

2.2 Flow categories

Flow categories was introduced by Cohen, Jones and Segal [19] to organize the moduli spaces in Floer (co)homology, and were used to construct a stable homotopy type for Floer theories. Our construction will be based on the concept of flow categories, and hence we recall the definition first:

Definition 2.9 A *flow category* is a small category $\mathcal{C}$ with the following properties:

[2] This condition is equivalent to $\pi_{2*}[\Delta] = [C_2]$.

[3] We will never switch the order of the two copies of C.

(1) The object space $\mathrm{Obj}(\mathcal{C}) = \bigsqcup_{i\in\mathbb{Z}} C_i$ is a disjoint union of *closed* manifolds C_i, ie C_i is a compact manifold without boundary. The morphism space $\mathrm{Mor}(\mathcal{C}) = \mathcal{M}$ is a manifold. The source and target maps $s, t\colon \mathcal{M} \to C$ are smooth.

(2) Let $\mathcal{M}_{i,j}$ denote $(s\times t)^{-1}(C_i \times C_j)$. Then $\mathcal{M}_{i,i} = C_i$, corresponding to the identity morphisms, and s and t restricted to $\mathcal{M}_{i,i}$ are identities. $\mathcal{M}_{i,j} = \varnothing$ for $j < i$, and $\mathcal{M}_{i,j}$ is a *compact* manifold for $j > i$.

(3) Let $s_{i,j}$ and $t_{i,j}$ denote $s|_{\mathcal{M}_{i,j}}$ and $t|_{\mathcal{M}_{i,j}}$. For every strictly increasing sequence $i_0 < i_1 < \cdots < i_k$, $t_{i_0,i_1}\times s_{i_1,i_2}\times t_{i_1,i_2}\times\cdots\times s_{i_{k-1},i_k}\colon \mathcal{M}_{i_0,i_1}\times\mathcal{M}_{i_1,i_2}\times\cdots\times\mathcal{M}_{i_{k-1},i_k} \to C_{i_1}\times C_{i_1}\times C_{i_2}\times C_{i_2}\times\cdots\times C_{i_{k-1}}\times C_{i_{k-1}}$ is transverse to the submanifold $\Delta_{i_1}\times\cdots\times\Delta_{i_{k-1}}$ in the sense of Definition 2.4. Therefore the fiber product

$$\begin{aligned}&\mathcal{M}_{i_0,i_1}\times_{i_1}\mathcal{M}_{i_1,i_2}\times_{i_2}\cdots\times_{i_{k-1}}\mathcal{M}_{i_{k-1},i_k}\\ &\quad := (t_{i_0,i_1}\times s_{i_1,i_2}\times t_{i_1,i_2}\times\cdots\times s_{i_{k-1},i_k})^{-1}(\Delta_{i_1}\times\Delta_{i_2}\times\cdots\times\Delta_{i_{k-1}}) \subset \mathcal{M}_{i_0,i_1}\times\mathcal{M}_{i_1,i_2}\times\cdots\times\mathcal{M}_{i_{k-1},i_k}\end{aligned}$$

is a submanifold by Proposition 2.5.

(4) The composition $m\colon \mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k} \to \mathcal{M}_{i,k}$ is a smooth map such that

$$m\colon \bigsqcup_{i<j<k} \mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k} \to \partial\mathcal{M}_{i,k}$$

is a diffeomorphism up to zero-measure.

Example 2.10 Fix a Morse–Bott function f on a closed manifold M. Then there are finitely many critical values $v_1 < \cdots < v_n$. Let C_i denote the critical manifold corresponding to the critical value v_i, and $\mathcal{M}_{i,j}$ the *compactified* moduli space of *unparametrized gradient flow lines* from C_i to C_j. Since the function value increases along a gradient flow line, $\mathcal{M}_{i,j} = \varnothing$ when $i > j$. The source map $s\colon \mathcal{M}_{i,j} \to C_i$ and target map $t\colon \mathcal{M}_{i,j} \to C_j$ are defined to be the evaluation maps at the negative/positive ends of the flow line in $\mathcal{M}_{i,j}$. The composition map m is the concatenation of flow lines. It's a folklore theorem that the $\mathcal{M}_{i,j}$ are smooth manifolds with boundaries and corners if one chooses a suitable metric; see [3; 33] and Section 8. Therefore $\{C_i, \mathcal{M}_{i,j}\}$ forms a flow category. We emphasize here that the subscript i in C_i has nothing to do with Morse–Bott indices. Similar constructions also exist in Floer theories, as long as there is a background "Morse–Bott" functional and all the transversality conditions are met. For example, [19] gives an explicit construction of the flow category for the Hamiltonian Floer cohomology theory on $\mathbb{CP}^n$, where the background Morse–Bott functional is the symplectic action functional with the Hamiltonian[4] $H = 0$. There are also flow categories without obvious background Morse–Bott functionals, for example, the flow category for Khovanov homology [54].

We associate a natural cochain complex to each (oriented) flow category in a functorial way. The main application would be defining Hamiltonian–Floer cohomology or Morse cohomology under Morse–Bott nondegenerate conditions. Although we will be discussing the *abstract notion of flow categories*, it would be helpful to keep Example 2.10 in mind. In view of this, with a bit abuse of notation, we will refer to elements of $\mathcal{M}_{i,j}$ as Morse (or Floer) trajectories from C_i to C_j. Inspired by Example 2.10,

[4] [19] used homological convention, which gave the opposite category of a flow category in the sense of Definition 2.9.

Definition 2.9(2) is usually the consequence of the existence of some background functional, and the morphism space $\mathcal{M}_{i,j}$ is the *compactified* moduli space of "gradient flow lines",[5] that is, the space of possibly broken "gradient flow lines". Definition 2.9(3) is necessary for the smoothness of the composition map m. Roughly speaking, Definition 2.9(4) requires that the boundary of the morphism space is the space of nontrivial compositions of morphisms, although it is only about an essential portion of the correspondence. In applications, we can stratify $\mathcal{M}_{i,j}$ in a cell-like manner by a poset similar to the construction in [64] such that m respects the structure, but we will not need that level of precision here.

Remark 2.11 (1) A flow category is called *Morse* if C is a discrete set. Then the fiber product transversality becomes tautological, and it recovers the definition of a flow category in [19], up to taking the opposite category.

(2) In the context of Floer theories, the moduli spaces may not be manifolds in general, but instead some weighted objects with local symmetries, eg weighted branched orbifolds in [42]. All of our arguments hold for weighted branched orbifolds, since there is a well-behaved integration theory with Stokes' theorem [43].

(3) When the flow category comes from a Morse–Bott functional f, but f is not single valued,[6] we need to lift f to $\tilde{f}$ over the cyclic cover [19] to guarantee Definition 2.9(2). Such modification was already reflected in the usual construction by introducing the Novikov coefficient.

(4) In Definition 2.9, we require C_i to be compact and without boundary. However, the compactness assumption can be dropped: C_i could be a disjoint union of infinitely many closed manifolds or C_i could have noncompact components.[7] In such generalizations, compactness of $\mathcal{M}_{i,j}$ can be weakened to requiring that the target maps $t\colon M_{i,j} \to C_j$ are proper;[8] see Section 6.1 for details.

(5) For a background Morse–Bott function f, sometimes it is impossible to partition the critical manifolds by $\mathbb{Z}$ and in the order of increasing critical values; critical values may accumulate. For example, Hamiltonian Floer cohomology with Novikov coefficients will have this problem if the symplectic form is irrational. However, Gromov compactness for the Hamiltonian Floer equation implies that there is an action gap $\hbar$ such that there are no nonconstant flow lines when the action difference (energy) is smaller than $\hbar$. Therefore we can still divide all the critical manifolds into groups indexed by $\mathbb{Z}$ so that there are no nonconstant flow lines inside each group. Then the flow category can still be defined using the generalization in (4).

(6) We will mostly work with oriented C_i; see Definition 2.15. This assumption can be dropped at the price of working with local systems. We discuss this generalization in Section 5.

(7) The requirement of the partition of $\mathrm{Obj}(\mathcal{C})$ by $\mathbb{Z}$ is not necessary. We can certainly work with $\mathrm{Obj}(\mathcal{C})$ indexed by any set I, as long as we require that $\mathcal{M}_{i,j}$ has only finitely many degenerations for any

[5] It could be Floer flow lines, which, strictly speaking, are not gradient flow lines.

[6] For example, Hamiltonian Floer cohomology on (M,ω) with $\omega|_{\pi_2(M)} \neq 0$ has this property.

[7] But those noncompact manifolds should have finite topology; see Section 6.1 for details.

[8] One can instead ask $s\colon M_{i,j} \to C_i$ to be proper, but this will result in a theory analogous to the compactly supported cohomology.

$i, j \in I$, and the finite set of degeneration configurations is equipped with a partial order, whose minimum elements are built from $\mathcal{M}_{i,j}$ without boundary. This is precisely the setup in [63, Section 7], and is satisfied by more general constructions in [64]. When $\mathrm{Obj}(\mathcal{C})$ is indexed by $\mathbb{Z}$ with the properties in Definition 2.9, the set of degeneration configurations of $\mathcal{M}_{i,j}$ is precisely the set of strictly increasing sequences $S := \{i < \cdots < j\}$, where the partial order is given by $S_1 \leq S_2$ if and only if $S_2 \subset S_1$. Then the minimum element is $\{i < i+1 < \cdots < j-1 < j\}$, which corresponds to the fiber product of manifolds $\mathcal{M}_{*,*+1}$ without boundary. However, this level of generalization does not add much to the applications we have in mind, and hence we choose to work with the more down-to-earth version (Definition 2.9) to avoid more complication in notation.

Flow categories can be equipped with extra structures. For our construction, the most relevant structures are gradings and orientations. Given a flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$, for simplicity of notation, *we assume through out this paper that* $\dim \mathcal{M}_{i,j}$ *and* $\dim C_i$ *are well defined*. This requirement usually holds when each C_i has one component.

Remark 2.12 When $\dim C_i$ and $\dim \mathcal{M}_{i,j}$ are not well defined, then we need to work componentwise. For example, if a function f in Example 2.10 is Morse and C_i contains critical points of different Morse indices, then $\mathcal{M}_{i,i+1}$ has multiple connected components of different dimension. This generalization only results in complexity of notation; it is straightforward to see that our proofs still hold, and they can be viewed as formulae on one component.

Let $m_{i,j} := \dim \mathcal{M}_{i,j}$ for $i < j$ and $c_i := \dim C_i$. We formally define $m_{i,i} := c_i - 1$. By Definition 2.9(3) and (4) and Proposition 2.5, $t_{i,j} \times s_{j,k} : \mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \to C_j \times C_j$ is transverse to Δ_j and an open dense part of $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ can be identified with part of the boundary of $\mathcal{M}_{i,k}$. Then

$$m_{i,j} + m_{j,k} - c_j + 1 = m_{i,k} \quad \text{for all } i \leq j \leq k. \tag{2-5}$$

Definition 2.13 A flow category is *graded* if there is an integer d_i such that $d_i = d_j + c_j - m_{i,j} - 1$ for each $i \in \mathbb{Z}$ and all $i < j$. We will refer to $\{d_i\}$ as the grading structure.[9] Similarly, we define a $\mathbb{Z}/k$ grading structure if $d_i \in \mathbb{Z}/k$ and the relation holds in $\mathbb{Z}/k$.

Remark 2.14 The $\mathbb{Z}/k$ grading structure on a flow category is used to equip the Morse–Bott cochain complex with a $\mathbb{Z}/k$ grading. In the finite-dimensional Morse–Bott theory, a $\mathbb{Z}$ grading structure exists, ie d_i can be the dimension of the negative eigenspace of $\mathrm{Hess}(f)$ on C_i. For Hamiltonian Floer cohomology, a $\mathbb{Z}/2$ grading structure always exists and a $\mathbb{Z}$ grading structure exists if the first Chern class of the symplectic manifold vanishes; then d_i is related to the generalized Conley–Zehnder index [66].

Next, we define orientations on a flow category. Since $t_{i,j} \times s_{j,k} : \mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \to C_j \times C_j$ is transverse to the diagonal Δ_j, the pullback $(t_{i,j} \times s_{j,k})^* N_j$ of the normal bundle N_j of Δ_j by $t_{i,j} \times s_{j,k}$ is the normal

[9] When $\dim M_{i,j}$ or $\dim C_i$ are not well defined, a grading is an assignment of integers to each component of C_i satisfying similar relations.

bundle of $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} := (t_{i,j} \times s_{j,k})^{-1}(\Delta_j)$ in $\mathcal{M}_{i,j} \times \mathcal{M}_{j,k}$. If N_j is oriented, then we can pull back this orientation to orient the normal bundle of $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$. We define a coherent orientation on a flow category as follows:

Definition 2.15 A *coherent orientation* on a flow category is an assignment of orientations for each C_i, $\mathcal{M}_{i,j}$ and $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ such that:

(1) The normal bundle N_i of $\Delta_i \subset C_i \times C_i$ is oriented by $[N_i][\Delta_i] = (-1)^{c_i^2}[C_i][C_i]$, as in Example 2.8.

(2) $(t_{i,j} \times s_{j,k})^*[N_j][\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}] = (-1)^{c_j m_{i,j}}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}]$.

(3) $\partial[\mathcal{M}_{i,k}] = \sum_j (-1)^{m_{i,j}} m([\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}])$.

More precisely, (3) holds on where m is a diffeomorphism. One can combine (2) and (3) as

$$(t_{i,j} \times s_{j,k})^*[N_j] m^{-1}(\partial[\mathcal{M}_{i,k}]|_{m(\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k})}) = (-1)^{(c_j+1)m_{i,j}}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}].$$

Remark 2.16 Orientation conventions are by no means unique; however they typically differ by a global change. For example, in the context of Morse theory, Definition 3.3 differs from [65] by an opposite sign on the orientation of every $\mathcal{M}_{i,j}$. Although our orientation conventions for fiber products are different from [47], our conventions also enjoy the associativity property [47, Proposition 7.5(a)], and hence the uniqueness property in [47, Remark 7.6(iii)] holds.

We will discuss how coherent orientations arise in applications in Section 5.1. When the flow category is oriented as in Definition 2.15, we have the following form of Stokes' theorem:

$$\int_{\mathcal{M}_{i,k}} d\alpha = \sum_{i<j<k} (-1)^{m_{i,j}} \int_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} m^*\alpha.$$

Suppose that $\alpha \in \Omega^*(C_i)$, $\beta \in \Omega^*(C_k)$ and $i < j < k$. Because $s_{i,k} \circ m|_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} = s_{i,j} \circ \pi_1$ and $t_{i,k} \circ m|_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} = t_{j,k} \circ \pi_2$, where π_1 and π_2 are natural projections, we have

$$\int_{m(\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k})} s_{i,k}^*\alpha \wedge t_{i,k}^*\beta = \int_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} m^*s_{i,k}^*\alpha \wedge m^*t_{i,k}^*\beta = \int_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}} \pi_1^*s_{i,j}^*\alpha \wedge \pi_2^*t_{j,k}^*\beta. \tag{2-6}$$

Since we will only consider pullbacks of forms by source and target maps, it is convenient to think of $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ as contained in $\partial\mathcal{M}_{i,k}$, and suppress the composition map m.

2.2.1 Conventions for cochain complexes

In a typical homological algebra textbook, for example [75], a cochain complex is $\mathbb{Z}$ graded or $\mathbb{Z}/k$ graded for $k \geq 2$. As mentioned in Remark 2.14, the grading of the Morse–Bott cochain complex is a consequence of the grading structure in Definition 2.13, which is an extra piece of data on flow categories. Although the applications in our mind always have at least a $\mathbb{Z}/2$ grading structure, we will not assume this, and only work with Definition 2.9. As a result, our cochain complex is simply a vector space C with an operator $d : C \to C$ such that $d^2 = 0$. Then the cohomology $H(C, d)$ is defined as $\ker d / \operatorname{im} d$. The definitions of cochain maps and homotopies are similar and have the usual properties. It is clear that by forgetting the grading on a $\mathbb{Z}/k$ graded cochain complex we get a cochain complex in the above sense. Many basic properties in homological algebra survive for ungraded cochain complexes, eg the spectral sequence from a filtration, the exact triangle[10] from a short exact sequence, the mapping cone and mapping cylinder constructions.

2.3 Review of existing constructions

Throughout this subsection we fix a flow category $\mathcal{C} := \{C_i, \mathcal{M}_{i,j}\}$ such that there are finitely many nonempty C_i, for simplicity (for example, one can take the flow category from Example 2.10). Before giving our construction of the minimal Morse–Bott cochain complex in Section 3.2, we review the three constructions in the existing literature: Austin and Braam's pull–push construction, Fukaya's push–pull construction and the cascades construction. For simplicity, we completely neglect the issue of signs[11] and orientations.

2.3.1 Austin and Braam's Morse–Bott cochain complex ($\mathrm{BC}^{\mathrm{AB}}, d^{\mathrm{AB}}$) Austin and Braam [3] defined the Morse–Bott cochain complex of a flow category to be

$$\Big(\mathrm{BC}^{\mathrm{AB}} := \bigoplus_i \Omega^*(C_i), d^{\mathrm{AB}}\Big),$$

where $\Omega^*(C_i) = \bigoplus_{j=0}^{\dim C_i} \Omega^j(C_i)$ is the space of differential forms on C_i. The differential d^{AB} is defined as $\sum_{k\ge 0} d_k$, where d_k is defined by

$$\begin{aligned} d_0 &= \mathrm{d}\colon \Omega^*(C_i) \to \Omega^*(C_i), \\ d_k &: \Omega^*(C_i) \to \mathcal{D}^*(C_{i+k}) \quad \text{given by } \alpha \mapsto t_{i,i+k\,*} \circ s^*_{i,i+k}(\alpha) \text{ for } k \ge 1, \end{aligned} \tag{2-7}$$

where d is the usual exterior differential on differential forms. Here $\mathcal{D}^*(C)$ is the space of currents on C. The operator d_k taking values in $\mathcal{D}^*(C)$ instead of $\Omega^*(C)$ causes difficulties getting a well-defined *ungraded* cochain complex $(\mathrm{BC}^{\mathrm{AB}}, d^{\mathrm{AB}})$. Thus, to make it well-defined, the target maps $t_{i,j}$ are assumed to be fibrations in Austin and Braam's model. Under such assumptions, $t_{i,j\,*}$ is integration along the fiber, and hence d_k actually lands in $\Omega^*(C_{i+k})$. However, it was noticed in [51, Remark 2.4] that the fibration condition is obstructed for some Morse–Bott functions. That is, there exists a Morse Bott function f such that the fibration property fails for all metrics.

Remark 2.17 An equivalent form of the fibration condition was studied by Banyaga and Hurtubise under the name *the Morse–Bott–Smale condition* [4, Definition 3.4]. More precisely, let ϕ_t be the gradient flow of f. The Morse–Bott–Smale condition holds if and only if the unstable manifold $U(C_i) = \{x \mid x \in M \text{ and } \lim_{t\to-\infty}\phi_t(x) \in C_i\}$ and the stable manifold $S(p) = \{x \mid x \in M \text{ and } \lim_{t\to\infty}\phi_t(x) = p\}$ for $p \in C_j$ intersect transversely[12] for all C_i, C_j and $p \in C_j$. Note that $(U(C_i) \cap S(p))/\mathbb{R}$ is the intersection

[10] When we have a $\mathbb{Z}$ grading, the exact triangle is a long exact sequence.

[11] For curious readers who would like to verify those constructions, we point out that Austin and Braam [3] have incorrect orientations and signs. Although our construction is motivated by theirs, we will not appeal to any of their specific formulae in our proofs.

[12] Note that we use (un)stable manifolds of the *positive* gradient flow; this explains the discrepancy with [4, Definition 3.4].

of the preimage $t_{i,j}^{-1}(p)$ with the open stratum of $\mathcal{M}_{i,j}$ (the space of unbroken flow lines); it is easy to check that $U(C_i)$ is transverse to $S(p)$ if and only if p is a regular value of $t_{i,j}$ restricted to the open stratum. In particular, the fibration condition implies the Morse–Bott–Smale condition. On the other hand, the Morse–Bott–Smale condition implies the fibration condition by [4, Corollary 5.20] and Ehresmann's theorem. Latschev introduced another even stronger condition [51, Definition 2.3] to make sure the generalization of Harvey and Lawson's method [39] works in the context of Morse–Bott functions. The existence of a flow category only requires that $U(C_i)$ and $S(C_j)$ — the stable manifold of C_j — intersect transversely, and the iterated source and target maps from these transverse intersections are transverse for all i and j; see Section 8 (this holds automatically when the Morse–Bott–Smale condition holds). We refer to such a pair (f, g) of a function and a metric as a *Morse–Bott–Smale pair* in Section 8. It is important to note that the *Morse–Bott–Smale pair* condition is much weaker than the *Morse–Bott–Smale condition* (namely transversality vs pointwise transversality in a family). Moreover, Morse–Bott–Smale pairs always exist. In particular, there is a metric for Latschev's example that forms a Morse–Bott–Smale pair.

Remark 2.18 One way to get the fibration property is to fatten up all moduli spaces systematically; a construction in this spirit was carried out in [35] using CF–perturbations.

Remark 2.19 The Austin–Braam cochain complex $(\mathrm{BC}^{\mathrm{AB}}, d^{\mathrm{AB}})$ explained here is ungraded. However, we can grade $\alpha \in \Omega^j(C_i)$ by $j + d_i$, where d_i the dimension of the negative eigenspace of $\mathrm{Hess}(f)$ on C_i, (the grading structure in Remark 2.14). Then $(\mathrm{BC}^{\mathrm{AB}}, d^{\mathrm{AB}})$ is graded by $\mathbb{Z}$ and the degree of d^{AB} is 1. It is clear that $\mathrm{BC}^{\mathrm{AB}}$ is equipped with an (action) filtration $F_i := \bigoplus_{j=i}^{\infty} \Omega^*(C_j) \subset F_{i-1}$ compatible with the differential, which induces a spectral sequence. This structure does not depend on the grading and always exists for all flow categories; we will discuss the induced spectral sequence in Section 4. On the other hand, if there is a $\mathbb{Z}$ grading structure then the cochain complex has the structure of a multicomplex studied in [45], which can decompose the spectral sequence further by the grading.

2.3.2 Fukaya's Morse–Bott chain complex Fukaya [33] used "singular" chains of critical manifolds to model the *homology* of the manifold for the flow category in Example 2.10, and the Austin–Braam model can be viewed as the dual of Fukaya's model. The chain complex is defined to be

$$\Big(\mathrm{BC}^{\mathrm{F}} := \bigoplus_i C_*(C_i), \partial^{\mathrm{F}}\Big).$$

Here $C_*(C_i)$ is the space of singular chains on C_i and $\partial^{\mathrm{F}} := \sum_{k\geq 0} \partial_k$, with ∂_k defined by

$$\begin{aligned} \partial_0 = \partial &: C_*(C_i) \to C_*(C_i), \\ \partial_k &: C_*(C_{i+k}) \to C_*(C_i) \quad \text{given by } P \mapsto s_{i,i+k\,*} \circ t_{i,i+k}^*(P) \text{ for } k \geq 1, \end{aligned}$$

where ∂ is the usual boundary operator on singular chains. Now pushforward is well defined. Pullback is defined as follows. Let $P : \Delta \to C_{i+k}$ be a singular chain and assume the fiber product $\Delta \times_{C_{i+k}} \mathcal{M}_{i,i+k}$

is cut out transversely in the sense of Definition 2.4, and hence is a manifold with boundaries and corners. Then the projection to the second factor,[13]

$$\pi_{\mathcal{M}_{i,i+k}}\colon \Delta \times_{C_{i+k}} \mathcal{M}_{i,i+k} \to \mathcal{M}_{i,i+k},$$

is defined to be the pullback $t^*_{i,i+k}(P)$.

To guarantee this pullback is well defined for all singular chains in C_{i+k}, one also needs to assume the target map $t_{i,i+k}$ is a fibration. To drop this constraint, Fukaya constructed a quasi-isomorphic subset $C_{\mathrm{geo}}(C_i) \subset C_*(C_i)$ such that the fiber products in the definition of pullbacks are defined over $C_{\mathrm{geo}}(C_i)$ and the operators ∂_k are closed on $C_{\mathrm{geo}}(C_i)$. Then $\left(\bigoplus_i C_{\mathrm{geo}}(C_i), \sum_{k\geq 0} \partial_k\right)$ defines a chain complex. It is important to note that the construction of $C_{\mathrm{geo}}(C_i)$ depends on $\mathcal{M}_{i,j}$, $s_{i,j}$ and $t_{i,j}$.

2.3.3 The cascades model The cascades construction was first introduced by Bourgeois [12] and Frauenfelder [32]. In the following, we review their constructions, but in the cohomology context to align with Austin and Braam's construction. For each C_i, we choose a Morse–Smale pair (f_i, g_i).[14] Then the cascade cochain complex is defined to be

$$\left(\mathrm{BC}^{\mathrm{C}} := \bigoplus_i MC(f_i, g_i), d^{\mathrm{C}}\right),$$

where $MC(f_i, g_i)$ is the Morse *cochain* complex of C_i using the Morse–Smale pair (f_i, g_i). The differential d^{C} is defined to be $\sum_{k\geq 0} d_k^{\mathrm{C}}$, where d_k^{C} is defined by

$$d_0^{\mathrm{C}} = d_{\mathrm{M}}\colon MC(f_i, g_i) \to MC(f_i, g_i),$$

for d_{M} the usual Morse differential for (f_i, g_i), and

$$d_k^{\mathrm{C}}\colon MC(f_i, g_i) \to MC(f_{i+k}, g_{i+k}),$$

which is defined by the number of rigid cascades from C_i to C_{i+k} for all $k \geq 1$. A 0–cascade is an unparametrized gradient flow line for (f_i, g_i). For $k \geq 1$, a k–cascade from $a \in \mathrm{Crit}(f_i)$ to $b \in \mathrm{Crit}(f_j)$ for $i < j$ is a tuple for $i < r_1 < \cdots < r_k < j$,

$$(\gamma_i, m_{i,r_1}, \gamma_{r_1}, t_{r_1}, \ldots, m_{r_{k-1},r_k}, \gamma_{r_k}, t_{r_k}, m_{r_k,j}, \gamma_j),$$

where γ_* is a gradient flow line in C_*, $m_{*,*}$ is a point in $\mathcal{M}_{*,*}$, and the t_* are positive real numbers, satisfying $\gamma_i(-\infty) = a$, $\gamma_i(0) = s(m_{i,r_1})$, $\gamma_j(+\infty) = b$, $\gamma_j(0) = t(m_{r_k,j})$, $\gamma_{r_s}(t_{r_s}) = s(m_{r_s,r_{s+1}})$ and $\gamma_{r_s}(0) = t(m_{r_{s-1},r_s})$.

When appropriate transversality assumptions are met, the moduli space of all cascades from a to b form a manifold. Moreover, there is a natural compactification of the moduli space by including the "broken" cascades. Then the differential d^{C} for the cascades cochain complex comes from counting the zero-dimensional compactified moduli spaces of cascades.

[13] To be more precise, we need to choose a triangulation of $\Delta \times_{C_{i+k}} \mathcal{M}_{i,i+k}$.

[14] That is, stable manifolds and unstable manifolds of $\nabla_{g_i} f_i$ intersect transversely.

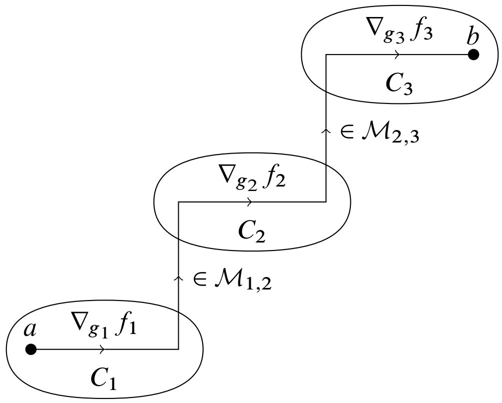


Figure 1: A 2–cascade.

Remark 2.20 The transversality for all compactified moduli spaces of cascades will become tautological if we assume $t_{i,j}$ is a fibration. In principle, we can obtain transversality for the cascades moduli spaces with generic choices of (f_i, g_i). However, the choice depends on $\mathcal{M}_{i,j}$, $s_{i,j}$ and $t_{i,j}$, just like Fukaya's model.

Remark 2.21 The cascades construction is very popular and has been deployed in many applications; see [7; 12; 21; 32; 68]. One advantage of the cascades model, besides being locally finite-dimensional, is the clear relation with the Morse model. More precisely, the additional Morse function f_i can be used to perturb the Morse–Bott function into a Morse function whose gradient flow lines can be identified with cascades. This identification was carried out by Banyaga and Hurtubise [5] in the context of finite-dimensional Morse–Bott theory, and Bourgeois and Oancea [15] in the context of symplectic homology with autonomous Hamiltonians.

2.4 Homological perturbation theory

The fibration condition in Austin and Braam's construction plays an important role in resolving the problem of the differential d_k taking values in the space of currents. Since fibration conditions are usually stronger than what one can get in any virtual techniques, we want to replace the fibration condition with a weaker transversality requirement, ie the fiber product transversality condition in Definition 2.9, which is generic in every reasonable virtual technique. Note that the operator d_k is defined using the pushforward of differential forms. Since pushforward is defined as the dual operator of pullback, the problem is rooted in the fact that the dual space of differential forms $\Omega^*(C_i)$ is the space of currents $\mathcal{D}^*(C_i)$ instead of itself. However, this problem never appears for finite-dimensional vector spaces; whenever a finite-dimensional space is equipped with a nondegenerate bilinear form, the dual space is identified with itself. To make use of this fact, we use the homological perturbation lemma, which is a method of constructing small cochain complexes from larger ones. The strategy is to formally apply the homological perturbation lemma to the almost-existing Austin–Braam cochain complex, and then directly verify that the formula suggested by the perturbation lemma is well defined and gives the desired algebraic relations. The theme of this paper can be summarized by the following slogan:

> Formal applications of the homological perturbation lemma can resolve the technical difficulty of infinite-dimensional cochain models.

2.4.1 A homological perturbation theorem Roughly speaking, the homological perturbation lemma takes in a cochain complex and perturbation data (in most cases projections and homotopies) and produces another cochain complex which is quasi-isomorphic to the input cochain complex. For simplicity, we consider a cochain complex $A = \bigoplus_{i=1}^{n} A_i$, where the A_i are $\mathbb{Z}/2$–linear spaces (ungraded as usual — i is *not* the grading!). Assume the differential d is in the form of $\sum_{k\geq 0} d_k$ with $d_k \colon A_i \to A_{i+k}$ for $k \geq 0$. Then $d^2 = 0$ implies that (A_i, d_0) is also a cochain complex for all i. The perturbation data consists of, for each $1 \leq i \leq n$, projections $p_i \colon A_i \to A_i$ and homotopies $H_i \colon A_i \to A_i$ between the identity and p_i:

$$\mathrm{id} - p_i = d_0 \circ H_i + H_i \circ d_0. \tag{2-8}$$

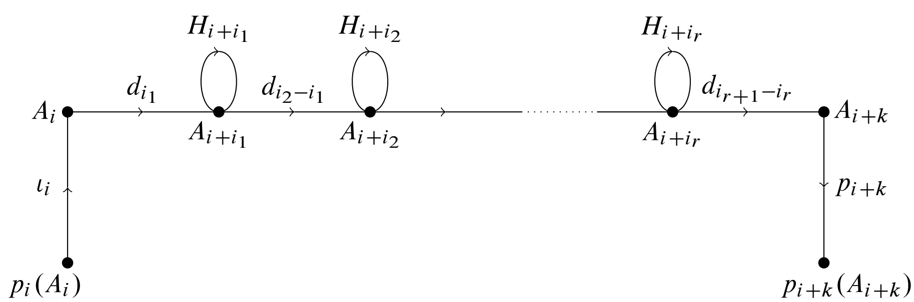
With this perturbation data, we have the following homological perturbation lemma:

Lemma 2.22 *There is a differential on $\bigoplus_i p_i(A_i)$ such that $\bigoplus_i p_i(A_i)$ is quasi-isomorphic to A.*

The lemma holds for general coefficient rings and graded complexes, once appropriate signs are assigned. Since we only use Lemma 2.22 to explain the motivation behind the formulae we give in Section 3, we will not go into the details of the signs nor the proof. What is more relevant to our purpose is the pattern of the formula for the differential on $\bigoplus p_i(A_i)$, which can be viewed as an analog of the perturbation theorem for A_∞ structures proved in [49]. For a strictly increasing sequence of integers $T = \{i_0 = 0, i_1, \dots, i_{r+1} = k\}$ for $r \geq 0$, we define the an operator $D_{k,T} \colon p_i(A_i) \to p_{i+k}(A_{i+k})$ for all integers i by

$$D_{k,T} = p_{i+k} \circ d_{i_{r+1}-i_r} \circ H_{i+i_r} \circ \cdots \circ H_{i+i_2} \circ d_{i_2-i_1} \circ H_{i+i_1} \circ d_{i_1-i_0} \circ \iota_i, \tag{2-9}$$

where $\iota_i \colon p_i(A_i) \to A_i$ denotes the inclusion. $D_{k,T}$ can be schematically explained as follows:



The new differential D on $\bigoplus_i p_i(A_i)$ is defined as

$$D = \sum_{k=0}^{\infty} D_k,$$

where $D_k = \sum_T D_{k,T}$ is the summation over all strictly increasing sequences T from 0 to k.

2.4.2 Cascades from homological perturbation In this section, we explain how to heuristically interpret the cascades *cochain* complex as a homological perturbation on the Austin–Braam cochain complex. The feature that the cascades construction does not require the fibration condition also reflects the theme of the paper.

We first explain the perturbation data used to get the cascades cochain complex, that is, a pair of projection and homotopy (p_i, H_i) on $\Omega^*(C_i)$ for every i. We require that the image $\operatorname{im} p_i$ is a finite-dimensional subspace of $\Omega^*(C_i)$. Given such perturbation data, we can formally write down operators $D_{k,T}$ from (2-9). Note that in the cascades construction we choose a Morse–Smale pair (f_i, g_i) on each critical manifold C_i. The perturbation data is then given by such a Morse–Smale pair using the construction in [39]. Before giving the construction, we first set up some notation. We will not be precise about signs and orientations.

Definition 2.23 Let C be an oriented closed manifold.

(1) $\mathcal{D}^*(C)$ denotes the space of currents[15] on C. There is a natural inclusion $\iota\colon \Omega^*(C) \to \mathcal{D}^*(C)$ given by

$$\iota(\alpha)(\beta) = \int_C \alpha \wedge \beta \quad \text{for all } \alpha \in \Omega^*(C).$$

(2) Let $\kappa \in \mathcal{D}^*(C \times C)$ be a current. Then the induced integral operator $I_\kappa\colon \Omega^*(C) \to \mathcal{D}^*(C)$ is defined as

$$I_\kappa(\alpha)(\beta) := (-1)^{\dim C} \kappa(\pi_1^*\alpha \wedge \pi_2^*\beta) \quad \text{for all } \alpha, \beta \in \Omega^*(C), \tag{2-10}$$

where π_1 and π_2 are projections of $C \times C$ to the first and second factors, respectively. We make the signs in (1) and (2) precise for the sake of Section 3.

(3) Let B be an oriented compact manifold and $i\colon B \to C$ a smooth inclusion. Then we can define a current $[B] \in \mathcal{D}^*(C)$ by

$$[B](\alpha) := \pm \int_B i^*\alpha \quad \text{for all } \alpha \in \Omega^*(C).$$

In general, one can define a current $[B]$ for any oriented singular chain B.

Let $\operatorname{Crit}(f_i)$ be the set of critical points of the Morse function f_i on C_i. We use $\phi_t^i\colon C_i \to C_i$ to denote the time-t flow of the gradient vector field $\nabla_{g_i} f_i$ on C_i. Then the pullback operator ${\phi_{-t}^i}^*\colon \Omega^*(C_i) \to \Omega^*(C_i)$ can be understood as the integral operator $I_{[\operatorname{graph} \phi_t^i]}$ of the current of $\operatorname{graph} \phi_t^i := \{(x, \phi_t^i(x))\} \subset C_i \times C_i$. The manifold $\bigcup_{0<t'<t} \operatorname{graph} \phi_{t'}^i \subset C_i \times C_i$ defines an integral operator $H_t^i := I_{[\bigcup_{0<t'<t} \operatorname{graph} \phi_{t'}^i]} = I_{[\bigcup_{0\le t'\le t} \operatorname{graph} \phi_{t'}^i]}$. Since $\partial\bigl(\bigcup_{0\le t'\le t} \operatorname{graph} \phi_{t'}^i\bigr) = \Delta_i \cup \operatorname{graph} \phi_t^i$, Stokes' theorem implies that

$$\mathrm{id} - {\phi_{-t}^i}^* = \mathrm{d} \circ H_t^i + H_t^i \circ \mathrm{d}. \tag{2-11}$$

It was proven in [39] that when $t \to \infty$, (2-11) converges to a projection–homotopy relation. To be more specific, let U_x and S_x denote the unstable and stable manifolds of the critical point $x \in \operatorname{Crit}(f_i)$:

$$U_x := \bigl\{y \in C_i \mid \lim_{t \to -\infty} \phi_t^i(y) = x\bigr\} \quad \text{and} \quad S_x := \bigl\{y \in C_i \mid \lim_{t \to \infty} \phi_t^i(y) = x\bigr\}. \tag{2-12}$$

[15] For basics of currents, we refer readers to [36].

In the sense of currents, we have the following:

$$\lim_{t\to\infty}[\operatorname{graph}\phi^i_t] = \sum_{x\in\operatorname{Crit}(f_i)}[S_x\times U_x], \quad\text{and}\quad \lim_{t\to\infty}\Big[\bigcup_{0<t'<t}\operatorname{graph}\phi^i_{t'}\Big] = \Big[\bigcup_{0<t'<\infty}\operatorname{graph}\phi^i_{t'}\Big]. \tag{2-13}$$

See [39, Theorems 2.3 and 3.3] for details.

Remark 2.24 Importantly, [39] studied $\lim_{t\to\infty}\phi^*_t$ (where ϕ^*_t is represented by $\{(\phi^i_t(x),x)\}\subset C_i\times C_i$) and [39, Theorem 3.3] stated that $\lim_{t\to\infty}\phi^*_t$ can be represented by $\sum_{x\in\operatorname{Crit}(f_i)}[U_x]\times[S_x]$. Then (2-11) projects $\Omega^*(C_i)$ to the Morse *chain* complex [39, Proposition 4.5], or equivalently the Morse *cochain* complex of $-f_i$. Since we need a projection to the Morse *cochain* complex of f_i to explain the cascades model, we need to work with $\lim_{t\to\infty}\phi^*_{-t}$ instead. This explains the discrepancy with [39].

Hence (2-13) defines two integral operators ${\phi^i_{-\infty}}^*, H^i_\infty\colon \Omega^*(C_i)\to\mathcal{D}^*(C_i)$ such that

$$\iota - {\phi^i_{-\infty}}^* = \mathrm{d}\circ H^i_\infty + H^i_\infty\circ\mathrm{d}, \tag{2-14}$$

where ι is the natural embedding $\Omega^*(C_i)\hookrightarrow\mathcal{D}^*(C_i)$; see [39, Theorems 2.3 and 3.3]. Note that

$${\phi^i_{-\infty}}^*(\alpha) = \sum_{x\in\operatorname{Crit}(f_i)}\Big(\int_{C_i}\alpha\wedge[S_x]\Big)\cdot[U_x] = \sum_{x\in\operatorname{Crit}(f_i)}\Big(\int_{S_x}\alpha|_{S_x}\Big)\cdot[U_x] \tag{2-15}$$

can be viewed as the projection from $\Omega^*(C_i)$ to the Morse cochain complex; see [39, Theorem 4.1]. By (2-14), H^i_∞ defines a homotopy between ι and the projection ${\phi^i_{-\infty}}^*$.

Remark 2.25 Strictly speaking, (2-14) is not a genuine projection–homotopy relation, since ${\phi^i_{-\infty}}^*$ lands in space of currents instead of differential forms. To get an honest projection–homotopy relation, we need to enlarge $\Omega^*(C)$ by adding some currents of singular chains. Roughly speaking, the enlargement is the minimal extension which contains $[U_x]$ and $[S_x]$ for $x\in\operatorname{Crit}(f_i)$ such that it is closed under ${\phi^i_{-\infty}}^*$, H^i_∞ and d. Such an enlargement depends on $\mathcal{M}_{i,j}$, $s_{i,j}$ and $t_{i,j}$, which leads to the choices in Remark 2.20.

From now on, we will neglect the issue in Remark 2.25 and show formally that the cascades construction can be understood as applying the construction in (2-9) to the Austin–Braam cochain complex using the perturbation data $({\phi^i_{-\infty}}^*, H^i_\infty)$. Before "proving" the claim, we first "define" the integration of pullbacks of currents from singular chains:

Definition 2.26 Let $\mathcal{M}$ be a compact manifold with two smooth maps $s,t\colon\mathcal{M}\to C_1,C_2$. Assume $B_1\subset C_1$ and $B_2\subset C_2$ are two submanifolds without boundary.[16] If s is transverse to B_1, t is transverse to B_2 and $s^{-1}(B_1)$ is transverse to $t^{-1}(B_2)$ with finite intersections, then we define

$$\int_{\mathcal{M}} s^*([B_1])\wedge t^*([B_2]) := \sum_{p\in s^{-1}(B_1)\cap t^{-1}(B_2)}\pm 1.$$

[16]The inclusion $B_*\subset C_*$ is not required to be proper, and hence B_* may not be closed. We only require that B_* is the interior of a compact manifold with boundaries and corners $\overline{B}_*$ so that the inclusion $B_*\hookrightarrow C_*$ is the restriction of a smooth map $\overline{B}_*\to C_*$. Therefore Definition 2.23(3) makes sense for B_1. In particular, the (un)stable manifolds satisfy the condition.

Definition 2.26 is natural in the sense that if we approximate the current $[B_1]$ by differential forms supported in a tubular neighborhood [36, Chapter 3, Section 1], then the limit of the integration of the pullbacks of the approximations is indeed the number of intersection points counted with sign.[17]

Now we apply (2-9). For $x \in \mathrm{Crit}(f_i)$, the first term D_0 in $D = \sum_{k \geq 0} D_k$ is defined by

$$D_0([U_x]) := {\phi^i_{-\infty}}^*(d_0([U_x])) = {\phi^i_{-\infty}}^*(\mathrm{d}([U_x])) = \sum_{y \in \mathrm{Crit}(f_i)} \Big(\int_{C_i} \mathrm{d}([U_x]) \wedge [S_y] \Big) \cdot [U_y].$$

It was proven in [39, Proposition 4.5] that when the Morse–Smale condition holds, $\int_{C_i} \mathrm{d}([U_x]) \wedge [S_y]$ equals the signed counts of rigid gradient flow lines from x to y. Therefore D_0 recovers the Morse differential on C_i. Next, we study the higher operators in D. Letting $x \in \mathrm{Crit}(f_i)$,

$$\begin{aligned} D_1([U_x]) = {\phi^{i+1}_{-\infty}}^* d_1[U_x] &= \sum_{y \in \mathrm{Crit}(f_{i+1})} \Big(\int_{C_{i+1}} d_1[U_x] \wedge [S_y] \Big) \cdot [U_y] && \text{(by (2-15))} \\ &= \sum_{y \in \mathrm{Crit}(f_{i+1})} \Big(\int_{\mathcal{M}_{i,i+1}} s^*_{i,i+1}[U_x] \wedge t^*_{i,i+1}[S_y] \Big) \cdot [U_y] && \text{(by (2-7))} \\ &= \sum_{y \in \mathrm{Crit}(f_{i+1})} \#(s^{-1}_{i,i+1}(U_x) \cap t^{-1}_{i,i+1}(S_y)) \cdot [U_y] && \text{(by Definition 2.26).} \end{aligned}$$

The last equality requires that $s^{-1}_{i,i+1}(U_x) \pitchfork t^{-1}_{i,i+1}(S_y)$. So D_1 counts points in $s^{-1}_{i,i+1}(U_x) \cap t^{-1}_{i,i+1}(S_y)$, which is exactly the 1–cascades in [12; 32]. By the same argument, $D_{2,\{0,2\}}$ counts rigid 1–cascades from C_i to C_{i+2}. Next we consider the operator $D_{2,\{0,1,2\}}$:

$$\begin{aligned} D_{2,\{0,1,2\}}([U_x]) &= {\phi^{i+2}_{-\infty}}^* \circ d_1 \circ H^{i+1}_\infty \circ d_1([U_x]) \\ &= \sum_{y \in \mathrm{Crit}(f_{i+2})} \Big(\int_{C_{i+2}} (d_1 \circ H^{i+1}_\infty \circ d_1[U_x]) \wedge [S_y] \Big) \cdot [U_y] && \text{(by (2-15))} \\ &= \sum_{y \in \mathrm{Crit}(f_{i+2})} \Big(\int_{\mathcal{M}_{i+1,i+2}} s^*_{i+1,i+2}(H^{i+1}_\infty \circ d_1[U_x]) \wedge t^*_{i+1,i+2}[S_y] \Big) \cdot [U_y] && \text{(by (2-7)).} \end{aligned}$$

Let us treat currents just like differential forms for simplicity. By definition,

$$\begin{aligned} \int_{C_{i+1}} H^{i+1}_\infty \circ d_1([U_x]) \wedge \alpha &= \int_{C_{i+1} \times C_{i+1}} \pi_1^*(d_1([U_x])) \wedge \Big[\bigcup_{0<t'<\infty} \mathrm{graph}\, \phi^{i+1}_{t'} \Big] \wedge \pi_2^* \alpha \\ &= \int_{\mathcal{M}_{i,i+1} \times C_{i+1}} s^*_{i,i+1}[U_x] \wedge (t_{i,i+1} \times \mathrm{id}_{C_{i+1}})^* \Big[\bigcup_{0<t'<\infty} \mathrm{graph}\, \phi^{i+1}_{t'} \Big] \wedge \pi_2^* \alpha. \end{aligned}$$

Then

$$H^{i+1}_\infty \circ d_1([U_x]) = \int_{\mathcal{M}_{i,i+1}} s^*_{i,i+1}[U_x] \wedge (t_{i,i+1} \times \mathrm{id}_{C_{i+1}})^* \Big[\bigcup_{0<t'<\infty} \mathrm{graph}\, \phi^{i+1}_{t'} \Big].$$

The right-hand side is the integration along the fiber $\mathcal{M}_{i,i+1}$ in the trivial fibration $\mathcal{M}_{i,i+1} \times C_{i+1}$. Therefore $D_{2,\{0,1,2\}}([U_x])$ equals

$$\sum_{y \in \mathrm{Crit}(f_{i+2})} \Big(\int_{\mathcal{M}_{i,i+1} \times \mathcal{M}_{i+1,i+2}} s^*_{i,i+1}[U_x] \wedge (t_{i,i+1} \times s_{i+1,i+2})^* \Big[\bigcup_{0<t'<\infty} \mathrm{graph}\, \phi^{i+1}_{t'} \Big] \wedge t^*_{i+1,i+2}[S_y] \Big) \cdot [U_y].$$

[17] The sign is determined by the orientations of B_1, B_2, C_1, C_2 and $\mathcal{M}$.

When transversality holds, by Definition 2.26 this equals

$$\sum_{y\in\mathrm{Crit}(f_{i+2})} \#\Big((s_{i,i+1}^{-1}(U_x)\times t_{i+1,i+2}^{-1}(S_y)) \pitchfork \Big((t_{i,i+1}\times s_{i+1,i+2})^{-1}\Big(\bigcup_{0<t'<\infty}\operatorname{graph}\phi_{t'}^{i+1}\Big)\Big)\Big)\cdot[U_y],$$

which can be interpreted as the counting of 2–cascades from C_i to C_{i+2} staying on C_{i+1} for finite time. Therefore $D_2 = D_{2,\{0,2\}} + D_{2,\{0,1,2\}}$ counts all rigid cascades from C_i to C_{i+2}. In general, assuming transversality for the cascade moduli spaces, we recover the whole cascades construction from (2-9). Hence the cascades construction fits into the homological perturbation philosophy.

3 The minimal Morse–Bott cochain complexes

In this section, we carry out the construction of the minimal Morse–Bott cochain complex for an abstract oriented flow category, which is applicable to both finite-dimensional Morse–Bott theory and Floer theories. The motivation of the construction comes from Lemma 2.22 and (2-9) with different perturbation data. We still need to make some choices (Definition 3.3) in the construction of the perturbation data. However, unlike the cascades construction, the choices in the minimal construction only depend on C_i, that is, there is no compatibility requirement with the morphism spaces $\mathcal{M}_{i,j}$.

This section is organized as follows: Section 3.1 constructs the perturbation data for the minimal Morse–Bott cochain complex. Section 3.2 constructs the Morse–Bott cochain complexes for every oriented flow category. Section 3.3 defines flow morphisms which can be viewed as the geometric analog of the continuation maps and shows that flow morphisms induce morphisms between Morse–Bott cochain complexes. Section 3.4 explains the compositions of flow morphisms. Section 3.5 defines flow homotopies and proves that flow homotopies induce homotopies between morphisms. Section 3.6 establishes that our construction is canonical on the cochain complex level, ie it is independent of all choices. Section 3.7 introduces flow subcategories and quotient categories, which are the geometric analogs of subcomplexes and quotient complexes, respectively. From now on, we will be very specific about the orientations and signs and provide rigorous arguments. Proofs in this section involve a lot of sign computations; we provide a detailed proof of $d_{\mathrm{BC}}^2 = 0$ for the coboundary map d_{BC} in Section 3.2. Proofs of other results in Sections 3.3, 3.4 and 3.5 will only be sketched.

3.1 Perturbation data for the minimal Morse–Bott cochain complex

In this subsection, we construct the perturbation data $\{(p_i, H_i)\}$ for the minimal Morse–Bott cochain complex of an oriented flow category $\mathcal{C} := \{C_i, \mathcal{M}_{i,j}\}$. Then (2-9) will motivate the definition of $D_{k,T}$ for the differential. We will show in the next subsection that they indeed define a cochain complex.

3.1.1 The projection p_i We start by defining a projection p_i on $\Omega^*(C_i) = \bigoplus_{j=1}^{\dim C_i} \Omega^j(C_i)$. First note that we have bilinear form on $\Omega^*(C_i)$ given by

$$\langle \alpha, \beta\rangle_i := (-1)^{\dim C_i\cdot|\beta|}\int_{C_i} \alpha\wedge\beta \quad \text{for all } \alpha,\beta\in\Omega^*(C_i). \tag{3-1}$$

We can pick representatives $\{\theta_{i,a}\}_{1\le a\le \dim H^*(C_i)} \subset \Omega^*(C_i)$ of a basis of $H^*(C_i)$, ie $\theta_{i,a}$ are closed forms such that the corresponding cohomology classes form a basis of $H^*(C_i)$. Such a choice gives us a quasi-isomorphic embedding $H^*(C_i) \to \Omega^*(C_i)$. Let $h(i)$ denote the image of the embedding above, so $h(i) := \langle \theta_{i,1}, \dots, \theta_{i,\dim H^*(C_i)}\rangle \subset \Omega^*(C_i)$. Note that (3-1) is nondegenerate on cohomology, and let $\{\theta^*_{i,a}\}_{1\le a\le \dim H^*(C_i)} \subset h(i)$ be the dual basis to the basis $\{\theta_{i,a}\}$ in the sense that

$$\langle \theta^*_{i,a}, \theta_{i,b}\rangle_i = \delta_{ab}. \tag{3-2}$$

Then we can define a projection $p_i : \Omega^*(C_i) \to h(i) \subset \Omega^*(C_i)$ by

$$p_i(\alpha) := \sum_{a=1}^{\dim H^*(C_i)} \langle \alpha, \theta_{i,a}\rangle_i \cdot \theta^*_{i,a}. \tag{3-3}$$

If we identify $H^*(C_i)$ with $h(i)$, then p_i can be thought of as a projection from $\Omega^*(C_i)$ to $H^*(C_i)$.

3.1.2 The homotopy H_i We now explain the related homotopy H_i. First note that the Poincaré dual of the diagonal $\Delta_i \subset C_i \times C_i$ can be represented by Thom classes. We can identify a tubular neighborhood of the diagonal Δ_i with the unit disk bundle of the normal bundle N_i of Δ_i. Then one way of writing Thom classes of the diagonal Δ_i is

$$\delta_i^n := \mathrm{d}(\rho_n \psi_i), \tag{3-4}$$

where ψ_i is the angular form of the sphere bundle $S(N_i)$ [11, Section 6] using the orientation in Example 2.8 and $\rho_n : \mathbb{R}^+ \to \mathbb{R}$ are smooth functions such that ρ_n is increasing, supported in $[0, 1/n]$ and is -1 near 0. For details of this construction, we refer readers to [11, Section 6]. We also include a brief discussion of this construction and its properties in Appendix A. The most important property of δ_i^n is that it converges to the Dirac current of Δ_i.

Lemma 3.1 *The Thom classes δ_i^n converge to the Dirac current δ_i of the diagonal Δ_i in the sense of currents: for all $\alpha \in \Omega^*(C_i \times C_i)$,*

$$\lim_{n\to\infty} \int_{C_i\times C_i} \alpha \wedge \delta_i^n = \int_{C_i\times C_i} \alpha \wedge \delta_i := \int_{\Delta_i} \alpha|_{\Delta_i}.$$

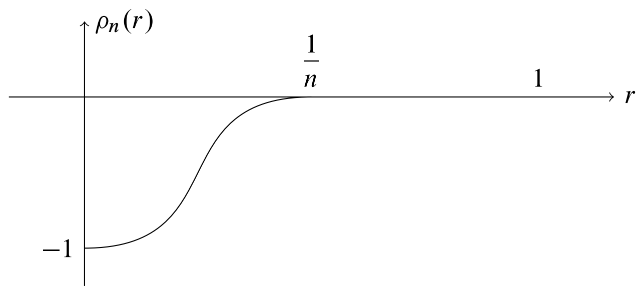


Figure 2: The graph of ρ_n.

We will prove Lemma 3.1 in Appendix A. By (2-10), for $\alpha,\beta\in\Omega^*(C_i)$, we have $\int_{C_i\times C_i}\pi_1^*\alpha\wedge\pi_2^*\beta\wedge\delta_i^n=(-1)^{(\dim C_i)^2}\int_{C_i\times C_i}\delta_i^n\wedge\pi_1^*\alpha\wedge\pi_2^*\beta=(-1)^{\dim C_i}\int_{C_i\times C_i}\delta_i^n\wedge\pi_1^*\alpha\wedge\pi_2^*\beta=I_{\delta_i^n}(\alpha)(\beta)$. Then Lemma 3.1 can be rewritten as

$$\lim_{n\to\infty} I_{\delta_i^n}=I_{\delta_i}=\mathrm{id}\colon \Omega^*(C_i)\to\Omega^*(C_i)$$

in the weak topology. On the other hand, under the orientation convention (2-4) we have another representative of the Poincaré dual of the diagonal by $\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*$, where π_1 and π_2 are the projections to the first and second factors of $C_i\times C_i$, respectively.

Proposition 3.2 *$\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*$ is cohomologous to δ_i^n for all n.*

Proof Since the pairing (3-1) is nondegenerate on $H^*(C_i\times C_i)$, it suffices to prove that

$$\int_{C_i\times C_i}\alpha\wedge\delta_i^n=\int_{C_i\times C_i}\alpha\wedge\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*$$

for any closed form α. Since all δ_i^n are cohomologous to each other for different n, Lemma 3.1 implies that if $\alpha\in\Omega^*(C_i\times C_i)$ is closed, then for all n

$$\int_{C_i\times C_i}\alpha\wedge\delta_i^n=\int_{\Delta_i}\alpha|_{\Delta_i}.$$

Therefore it suffices to show that, for all closed forms $\alpha\in\Omega^*(C_i\times C_i)$,

$$\int_{C_i\times C_i}\alpha\wedge\Big(\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*\Big)=\int_{\Delta_i}\alpha|_{\Delta_i}. \tag{3-5}$$

Since the cohomology of $C_i\times C_i$ is spanned by $\{\pi_1^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,d}\}_{1\le c,d\le\dim H^*(C_i)}$, it is enough to verify (3-5) for $\alpha=\pi_1^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,d}$. By definition $\langle\theta_{i,a}^*,\theta_{i,b}\rangle_i=\delta_{ab}$. Then if $c\neq d$,

$$\begin{aligned}\int_{C_i\times C_i}\pi_1^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,d}\wedge\Big(\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*\Big)&=\sum_a\pm\int_{C_i\times C_i}\pi_1^*\theta_{i,c}^*\wedge\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,d}\wedge\pi_2^*\theta_{i,a}^*\\&=\sum_a\pm\delta_{ca}\delta_{da}=0.\end{aligned}$$

Similarly, when $c=d$,

$$\begin{aligned}&\int_{C_i\times C_i}\pi_1^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,c}\wedge\Big(\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*\Big)\\&\qquad=\int_{C_i\times C_i}\pi_1^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,c}\wedge\pi_1^*\theta_{i,c}\wedge\pi_2^*\theta_{i,c}^*+\sum_{a\neq c}\pm\delta_{ca}\delta_{ca}\\&\qquad=(-1)^{|\theta_{i,c}|^2+|\theta_{i,c}|\cdot|\theta_{i,c}^*|}\int_{C_i\times C_i}\pi_1^*\theta_{i,c}^*\wedge\pi_1^*\theta_{i,c}\wedge\pi_2^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,c}\\&\qquad=(-1)^{|\theta_{i,c}|^2+|\theta_{i,c}|\cdot|\theta_{i,c}^*|+\dim C_i|\theta_{i,c}|}\Big(\int_{C_i}\theta_{i,c}^*\wedge\theta_{i,c}\Big)\langle\theta_{i,c}^*,\theta_{i,c}\rangle_i\\&\qquad=\int_{C_i}\theta_{i,c}^*\wedge\theta_{i,c}=\int_{\Delta_i}(\pi_1^*\theta_{i,c}^*\wedge\pi_2^*\theta_{i,c})|_{\Delta_i}.\end{aligned}$$

Thus (3-5) is proven. □

As a consequence of Proposition 3.2, there exist primitives $f_i^n \in \Omega^*(C_i \times C_i)$ such that

$$\mathrm{d} f_i^n = \delta_i^n - \sum_a \pi_1^* \theta_{i,a} \wedge \pi_2^* \theta_{i,a}^*, \tag{3-6}$$

$$f_i^n - f_i^m = (\rho_n - \rho_m)\psi_i. \tag{3-7}$$

Note that the integral operator I_{δ_i} of the Dirac current δ_i is the identity map from $\Omega^*(C_i)$ to itself. The integral operator $I_{\sum_a \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*}$ is the projection p_i in (3-3). Therefore, by (3-6), the integral operator $I_{f_i^n}$ of the primitive f_i^n satisfies

$$I_{\delta_i^n} - I_{\sum_a \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*} = I_{\mathrm{d} f_i^n} = \mathrm{d} \circ I_{f_i^n} + I_{f_i^n} \circ \mathrm{d}. \tag{3-8}$$

It is proven in Appendix A that f_i^n converges to a current $f_i \in \mathcal{D}^*(C_i \times C_i)$, and the corresponding integral operator I_{f_i} satisfies

$$\mathrm{id} - p_i = \mathrm{d} \circ I_{f_i} + I_{f_i} \circ \mathrm{d}, \tag{3-9}$$

which is the limit of (3-8). Therefore the integral operator $I_{f_i} = \lim I_{f_i^n}$ gives us the homotopy H_i for the projection p_i. This explains the perturbation data, which shall motivate the differential on the minimal Morse–Bott cochain complex. However, we will not use (3-9) to avoid working with currents (f_i is only a current), and always work with the approximation (3-8) and then take limits. More precisely, we will only use the "classical relation" (3-6).

From the discussion above, we have the following definition:

Definition 3.3 *Defining data* Θ for an oriented flow category $\mathcal{C}$ consists of

- quasi-isomorphic embeddings $H^*(C_i) \to \Omega^*(C_i)$, where the image is denoted by $h(\mathcal{C}, i)$ and we fix a basis $\{\theta_{i,a}\}$ of $h(\mathcal{C}, i)$ and a dual basis $\{\theta_{i,a}^*\}$ in the sense that $\langle \theta_{i,a}^*, \theta_{i,b} \rangle_i = \delta_{ab}$,
- a sequence of Thom classes with form $\delta_i^n = \mathrm{d}(\rho_n \psi_i)$ of the diagonal $\Delta_i \subset C_i \times C_i$ for all i,
- primitives f_i^n such that $\mathrm{d} f_i^n = \delta_i^n - \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*$ and $f_i^n - f_i^m = (\rho_n - \rho_m)\psi_i$ for all i.

Remark 3.4 The form $\sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta_{i,a}^*$ in Definition 3.3 does not depend on the basis $\{\theta_{i,a}\}$ for a fixed quasi-isomorphic embedding $H^*(C_i) \to \Omega^*(C_i)$.

3.1.3 The perturbed operator $D_{k,T,\Theta}$ Given defining data Θ, we are able to write down the operator $D_{k,T,\Theta}$ from (2-9) using the perturbation data introduced above. Those $D_{k,T,\Theta}$ will then be assembled to the differential on the minimal Morse–Bott cochain complex. To simplify the presentation, we first introduce the following notation:

(1) We use $[\alpha]$ to denote the cohomology class of a closed form $\alpha \in h(\mathcal{C}, i)$ and $|\alpha|$ to denote the degree of the differential form.

(2) We write $\mathcal{M}^{v,k}_{i_1,\ldots,i_r} := \mathcal{M}_{v,v+i_1} \times \cdots \times \mathcal{M}_{v+i_r,v+k}$ for $0 = i_0 < i_1 < i_2 < \cdots < i_r < i_{r+1} = k$ for $r \geq 0$, with the product orientation.

(3) For $\alpha \in \Omega^*(C_v)$, $\gamma \in \Omega^*(C_{v+k})$ and $f_{v+i_j} \in \Omega^*(C_{v+i_j} \times C_{v+i_j})$ for $1 \le j \le r$, we define the pairing $\mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f_{v+i_1}, \dots, f_{v+i_r}, \gamma]$ to be

$$\int_{\mathcal{M}^{v,k}_{i_1,\dots,i_r}} s^*_{v,v+i_1}\alpha \wedge (t_{v,v+i_1} \times s_{v+i_1,v+i_2})^* f_{v+i_1} \wedge \cdots \wedge (t_{v+i_{r-1},v+i_r} \times s_{v+i_r,v+k})^* f_{v+i_r} \wedge t^*_{v+i_r,v+k}\gamma. \tag{3-10}$$

Strictly speaking, before taking the wedge product we need to pullback $s^*_{v,v+i_1}\alpha$, $t^*_{v+i_r,v+k}\gamma$ and $(t_{v+i_{j-1},v+i_j} \times s_{v+i_j,v+i_{j+1}})^* f_{v+i_j}$ to $\mathcal{M}^{v,k}_{i_1,\dots,i_r}$ through the natural projections. This also applies to all similar formulae in this paper.

(4) For $\alpha \in h(\mathcal{C}, v)$ and $k \ge 1$, we define

$$\dagger(\mathcal{C}, \alpha, k) := (|\alpha| + m_{v,v+k})(c_{v+k} + 1), \tag{3-11}$$

$$\ddagger(\mathcal{C}, \alpha, k) := (|\alpha| + m_{v,v+k} + 1)(c_{v+k} + 1), \tag{3-12}$$

where $c_i := \dim C_i$, $m_{i,j} := \dim \mathcal{M}_{i,j}$ when $i < j$, and $m_{i,i} := c_i - 1$.

Then the perturbation data in Section 3.1 and (2-9) motivate the following definition:

Definition 3.5 Given defining data Θ and an increasing sequence $T := \{0 = i_0 < i_1 < \cdots < i_r < i_{r+1} = k\}$, we define a linear map $D_{k,T,\Theta} \colon H^*(C_v) \simeq h(\mathcal{C}, v) \to h(\mathcal{C}, v+k) \simeq H^*(C_{v+k})$ such that

$$\langle D_{k,T,\Theta}[\alpha], [\gamma]\rangle_{v+k} := (-1)^{\star} \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma] \tag{3-13}$$

for any $\gamma \in h(\mathcal{C}, v+k)$, where $\star := \sum_{j=0}^r \ddagger(\mathcal{C}, \alpha, i_j)$. In other words, by (3-2), we can write

$$D_{k,T,\Theta}([\alpha]) = \sum_a (-1)^{\star} \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \theta_{v+k,a}] \cdot [\theta^*_{v+k,a}]. \tag{3-14}$$

Remark 3.6 One way to understand the signs in (3-13) is to treat $D_{k,T,\Theta}$ as a composition of certain operators. Let $\alpha \in \Omega^*(C_i)$ and $f \in \Omega^*(C_j \times C_j)$. Then $\mathcal{M}_{i,j}$ defines an operator

$$\mathcal{M}_{i,j}(\alpha, f) := (-1)^{\ddagger(\mathcal{C},\alpha,0)} \int_{\mathcal{M}_{i,j}} s^*_{i,j}\alpha \wedge (t_{i,j} \times \mathrm{id}_j)^* f \in \Omega^*(C_j),$$

where $t_{i,j} \times \mathrm{id}_j \colon \mathcal{M}_{i,j} \times C_j \to C_j \times C_j$. Here, by omitting the pullback of projections for simplicity, $s^*_{i,j}\alpha \wedge (t_{i,j} \times \mathrm{id}_j)^* f$ is a differential form on $\mathcal{M}_{i,j} \times C_j$. Integrating along the $\mathcal{M}_{i,j}$ fiber in the trivial fibration $\mathcal{M}_{i,j} \times C_j$, we obtain a form on C_j. If $|f| = c_j - 1$, then $|\mathcal{M}_{i,j}(\alpha, f)| = |\alpha| + c_j - 1 - m_{i,j}$, so

$$\begin{aligned}\ddagger(\mathcal{C}, \mathcal{M}_{i,j}(\alpha, f), 0) &= (|\alpha| + c_j - 1 - m_{i,j} + m_{j,j} + 1)(c_j + 1) = (|\alpha| + c_j - 1 - m_{i,j} + c_j)(c_j + 1) \\ &\equiv \ddagger(\mathcal{C}, \alpha, j) \mod 2.\end{aligned}$$

Then for $g \in \Omega^*(C_k \times C_k)$,

$$\mathcal{M}_{j,k}(\mathcal{M}_{i,j}(\alpha, f), g) = (-1)^{\ddagger(\mathcal{C},\alpha,0)+\ddagger(\mathcal{C},\alpha,j)} \int_{\mathcal{M}_{i,j}\times\mathcal{M}_{j,k}} s^*_{i,j}\alpha \wedge (t_{i,j} \times s_{j,k})^* f \wedge (t_{j,k} \times \mathrm{id}_k)^* g.$$

In general, $(-1)^{\star}\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma]$ is the integral of the wedge product of compositions of such operators with $t^*_{s+i_r,s+k}\gamma$ on $\mathcal{M}_{s+i_r,s+k}$. When f is f^n_j for $n \gg 0$, $\mathcal{M}_{i,j}(\alpha, f)$ should be viewed as an approximation of $H_j \circ d_{j-i} \circ \iota_i(\alpha)$ in (2-9). In general, (3-14) can be viewed as (2-9) applied to the Austin–Braam complex using the perturbation data in this subsection.

The following lemma asserts that (3-13) is well defined and will be used in the proof of the main theorem; we prove it in Appendix A.

Lemma 3.7 *We have that* $\lim_{n\to\infty} \mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{s+i_1}, \dots, f^n_{s+i_r}, \gamma] \in \mathbb{R}$ *exists for every* $\alpha \in \Omega^*(C_s)$, $\gamma \in \Omega^*(C_{s+k})$ *and any defining data.*

3.2 The minimal Morse–Bott cochain complex

The main theorem of this subsection is that we can get a well-defined cochain complex out of an oriented flow category with any defining data. The cochain complex is generated by the cohomology $H^*(C_i)$ of the flow category, and hence it is called the minimal Morse–Bott cochain complex.

Definition 3.8 Given defining data Θ, the minimal Morse–Bott complex of an oriented flow category $\mathcal{C} := \{C_i, \mathcal{M}_{i,j}\}$ is defined by

$$\mathrm{BC}(\mathcal{C},\Theta) := \mathrm{BC} := \varinjlim_{q\to-\infty} \prod_{j=q}^{\infty} H^*(C_j),$$

ie the direct sum near the negative end and direct product near the positive end.[18] To be more precise, every element in BC is a function $A\colon \mathbb{Z} \to \prod_{i=-\infty}^{\infty} H^*(C_i)$ such that $A(i) \in H^*(C_i)$, and there exists $N_A \in \mathbb{Z}$ such that $A(i) = 0$ for all $i < N_A$. The differential $d_{\mathrm{BC},\Theta}\colon \mathrm{BC} \to \mathrm{BC}$ is defined as $\prod_{k\ge 1} d_{k,\Theta}$, where $d_{k,\Theta}\colon H^*(C_v) \to H^*(C_{v+k})$ is defined as

$$d_{k,\Theta} := \sum_T D_{k,T,\Theta}$$

for all increasing sequence $T = \{0 = i_0 < i_1 < \dots < i_r < i_{r+1} = k\}$ with $r \ge 0$. In other words,

$$\langle d_{k,\Theta}[\alpha], [\gamma]\rangle_{v+k} = \lim_{n\to\infty} \sum_T (-1)^{\star} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma] \tag{3-15}$$

for $\alpha \in h(\mathcal{C}, v)$, $\gamma \in h(\mathcal{C}, v+k)$ and $\star = \sum_{j=0}^{r} \ddagger(\mathcal{C}, \alpha, i_j)$. Defining $d_{i,\Theta} = 0$ for $i \le 0$, then for $A \in \mathrm{BC}$,

$$(d_{\mathrm{BC},\Theta} A)(i) := \sum_{j\in\mathbb{Z}} d_{i-j,\Theta} A(j).$$

Note that it is a finite sum. If moreover the flow category has a grading structure $\{d_i\}$, then BC is also graded. The grading of an element $\alpha \in H^*(C_i)$ is $|\alpha| + d_i$, which shall be viewed as in $\mathbb{Z}/k$ if $\{d_i\}$ is only a grading structure in $\mathbb{Z}/k$.

Remark 3.9 The degree of $d_{k,\Theta}[\alpha]$ in $H^*(C_{v+k})$ is $|\alpha| + c_{v+k} - m_{v,v+k}$ under the simplifying assumption after Remark 2.11 that c_i and $m_{i,j}$ are well defined. If the assumption is not satisfied, then

[18] Assume $\mathcal{C}$ arises from a Morse–Bott function f on a noncompact manifold (but $\mathcal{M}_{i,j}$ is still compact, so it cannot be any Morse–Bott function on any noncompact manifold). The differential in the cochain complex should increase the value of f, which forces the cochain complex to take the direct limit in the positive direction.

$d_{k,T,\Theta}$ can be decomposed with respect to the connected components of $\mathcal{M}^{v,k}_{i_1,\dots,i_r}$ so that each component has a well-defined degree in $H^*(C_{v+k})$. Then we need to keep track of the connected component in the proofs, which only results in complication of notation.

The main result of this section in the following:

Theorem 3.10 *Given an oriented flow category $\mathcal{C}$ and defining data Θ, $(\mathrm{BC}, d_{\mathrm{BC},\Theta})$ is a cochain complex. The cohomology $H(\mathrm{BC}, d_{\mathrm{BC},\Theta})$ is independent of the defining data Θ. If in addition the flow category is graded, then BC is also graded and the degree of $d_{\mathrm{BC},\Theta}$ is 1.*

Remark 3.11 (1) We prove in Section 8 that when the flow category comes from a Morse–Bott function f on a closed manifold M, the cohomology of the minimal Morse–Bott cochain complex is the regular cohomology $H^*(M,\mathbb{R})$. This follows from the definition if f is constant: since the flow category is $\{C_0 = M\}$ with only identities in the morphism space, $\mathrm{BC} = H^*(C_0,\mathbb{R}) = H^*(M;\mathbb{R})$ with $d_{\mathrm{BC}} = 0$. Therefore it suffices to show that the cohomology of the minimal Morse–Bott cochain complex is independent of the Morse–Bott function f.

(2) If all the critical manifolds C_i are discrete, then the defining data Θ is unique. Assume, for simplicity, that each C_i consists of one point. The minimal Morse–Bott cochain complex BC is generated by the critical points and equals the usual Morse cochain complex:

$$\mathrm{BC} = \varinjlim_{q\to-\infty} \prod_{j=q}^{\infty} H^*(C_j) = \varinjlim_{q\to-\infty} \prod_{j=q}^{\infty} \mathbb{R}. \tag{3-16}$$

Since $|f_i^n| = -1$, we have that $d_{k,\Theta}\colon H^*(C_v) \to H^*(C_{v+k})$ only has the leading term

$$\langle d_{k,\Theta}[1],[1]\rangle_{v+k} = \mathcal{M}^{v,k}[1,1] = \int_{\mathcal{M}_{v,v+k}} 1. \tag{3-17}$$

Therefore the differential $d_{\mathrm{BC},\Theta} := \sum_{k\ge 1} d_{k,\Theta_0}$ is just the signed counting of all zero-dimensional moduli spaces $\mathcal{M}_{v,v+k}$, which is the usual cochain differential in a nondegenerate Morse/Floer theory.

Remark 3.12 Theorem 3.10 is the simplest version. We generalize Theorem 3.10 in Sections 5 and 6 to the cases where C_i is not oriented, C_i is not compact, and the defining data is not minimal, ie the rank of the projection in the perturbation data is larger than $\dim H^*(C_i)$.

Corollary 3.13 *If the oriented flow category $\mathcal{C}$ has the property that $\dim C_i \le k$ for all i, then the minimal Morse–Bott cochain complex $\mathrm{BC}(\mathcal{C})$ only depends on $\mathcal{M}_{i,j}$ with $\dim \mathcal{M}_{i,j} \le 2k$.*

Proof Since $|f_i^n| = \dim C_i - 1 \le k-1$ and $|\alpha|, |\gamma| \le k$, if $\mathcal{M}_{i,j}$ appears in an integral in the definition of the differential with $\dim \mathcal{M}_{i,j} > 2k$, there is no way the pullbacks of those forms can contain a nontrivial component in $\bigwedge^{\dim \mathcal{M}_{i,j}} \mathcal{M}_{i,j}$. Therefore the integral must be zero. Note that when $k=0$, this amounts to saying that the cochain complex only depends on zero-dimensional moduli spaces (although the existence of 1–dimensional moduli spaces is needed to show that $d^2 = 0$). □

We first show that $(\mathrm{BC}, d_{\mathrm{BC},\Theta})$ is a cochain complex; the invariance is deferred to the next subsection. For simplicity, we first introduce notation:

(1) For $0 < i_1 < i_2 \cdots < i_r < k$, define

$$\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r} := \mathcal{M}_{v,v+i_1} \times \cdots \times (\mathcal{M}_{v+i_{p-1},v+i_p} \times_{v+i_p} \mathcal{M}_{v+i_p,v+i_{p+1}}) \times \cdots \times \mathcal{M}_{v+i_r,v+k} \tag{3-18}$$

with the product orientation.

(2) Define $\mathcal{M}^{v,k}_{i_1,\dots,i_r}[\mathrm{d}(\alpha, f_{v+i_1},\dots,f_{v+i_r},\gamma)]$ to be

$$\int_{\mathcal{M}^{v,k}_{i_1,\dots,i_r}} \mathrm{d}(s^*_{v,v+i_1}\alpha \wedge (t_{v,v+i_1}\times s_{v+i_1,v+i_2})^* f_{v+i_1} \wedge \cdots \wedge (t_{v+i_{r-1},v+i_r}\times s_{v+i_r,v+k})^* f_{v+i_r} \wedge t^*_{v+i_r,v+k}\gamma)$$

for $\alpha \in \Omega^*(C_v)$, $\gamma \in \Omega^*(C_{v+k})$ and $f_{v+i_j} \in \Omega^*(C_{v+i_j} \times C_{v+i_j})$.

(3) Define the pairing $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}[\alpha, f_{v+i_1},\dots,f_{v+i_{p-1}}, f_{v+i_{p+1}},\dots,f_{v+i_r},\gamma]$ over $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ to be

$$\begin{aligned}\int_{\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}} & s^*_{v,v+i_1}\alpha \wedge (t_{v,v+i_1}\times s_{v+i_1,v+i_2})^* f_{v+i_1} \wedge \cdots \wedge (t_{v+i_{p-2},v+i_{p-1}}\times s_{v+i_{p-1},v+i_{p+1}})^* f_{v+i_{p-1}} \\ & \wedge (t_{v+i_{p-1},v+i_{p+1}}\times s_{v+i_{p+1},v+i_{p+2}})^* f_{v+i_{p+1}} \wedge \cdots \wedge (t_{v+i_{r-1},v+i_r}\times s_{v+i_r,v+k})^* f_{v+i_r} \wedge t^*_{v+i_r,v+k}\gamma.\end{aligned}$$

(4) When we compose two operators, a trace term will appear. Therefore we introduce

$$\mathrm{Tr}^{v+i_p}\, \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f_{v+i_1},\dots,f_{v+i_{p-1}}, \theta\theta^*_{v+i_p}, f_{v+i_{p+1}},\dots,f_{v+i_r},\gamma] \tag{3-19}$$

to denote

$$\begin{aligned}\int_{\mathcal{M}^{v,k}_{i_1,\dots,i_r}} & s^*_{v,v+i_1}\alpha \wedge (t_{v,v+i_1}\times s_{v+i_1,v+i_2})^* f_{v+i_1} \wedge \cdots \\ & \wedge (t_{v+i_{p-1},v+i_p}\times s_{v+i_p,v+i_{p+1}})^* \Big(\sum_a \pi_1^*\theta_{v+i_p,a} \wedge \pi_2^*\theta^*_{v+i_p,a}\Big) \wedge \cdots \wedge (t_{v+i_r}\times s_{v+i_r})^* f_{v+i_r} \wedge t^*_{v+k}\gamma,\end{aligned}$$

where π_1 and π_2 are the projections of $C_{v+i_p}\times C_{v+i_p}$ to the first and second factors, respectively.

Heuristically speaking, the "Thom class" of $\mathcal{M}^{v,k}_{i_1,\dots,i_{p-1},\bar{i}_p,i_{p+1},\dots,i_r} \subset \mathcal{M}^{v,k}_{i_1,\dots,i_r}$ is given by the pullback of $(t_{v+i_{p-1},v+i_p}\times s_{v+i_p,v+i_{p+1}})^*\delta^n_{v+i_p} \in \Omega^*(\mathcal{M}_{v+i_{p-1},v+i_p}\times \mathcal{M}_{v+i_p,v+i_{p+1}})$ to $\mathcal{M}^{v,k}_{i_1,\dots,i_r}$ by the natural projection. Hence we have the following lemma, which is crucial to the proof that $d^2_{\mathrm{BC},\Theta} = 0$, and will be proven in Appendix A.

Lemma 3.14 *For an oriented flow category $\mathcal{C}$ and any defining data, we have*

$$\lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1},\dots,\delta^n_{v+i_p},\dots,f^n_{v+i_r},\gamma] = (-1)^* \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_{p-1},\bar{i}_p,i_{p+1},\dots,i_r}[\alpha, f^n_{v+i_1},\dots,f^n_{v+i_r},\gamma],$$

where $* = (|\alpha| + m_{v,v+i_p})c_{v+i_p}$.

Proposition 3.15 *We have that $(\mathrm{BC}, d_{\mathrm{BC},\Theta})$ is a cochain complex, that is, $d^2_{\mathrm{BC},\Theta} = 0$.*

Proof For simplicity, we will suppress the subscript Θ in the proof. It suffices to show that for all $\alpha \in h(\mathcal{C}, v)$ and $\gamma \in h(\mathcal{C}, v+k)$,

$$\Big\langle \sum_{i=1}^{k-1} d_{k-i} \circ d_i[\alpha], [\gamma] \Big\rangle_{v+k} = 0. \tag{3-20}$$

We first prove the following lemma:

Lemma 3.16 *For $r \geq 1$,*

$$\begin{aligned} 0 &= (-1)^{|\alpha| c_v} \int_{\partial \mathcal{M}_{v,v+k}} s^*_{v,v+k}\alpha \wedge t^*_{v,v+k}\gamma \\ &= \lim_{n\to\infty} \sum_{0<i_1<\cdots<i_r<k} (-1)^{\star_1} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\mathrm{d}(\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma)] \\ &\quad + \lim_{n\to\infty} \sum_{\substack{1\leq p\leq q\leq r \\ 0<i_1<\cdots<i_q<k}} (-1)^{\star_2} \operatorname{Tr}^{v+i_p} \mathcal{M}^{v,k}_{i_1,\dots,i_q} \\ &\qquad\qquad \cdot [\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_{p-1}}, \theta\theta^*_{v+i_p}, f^n_{v+i_{p+1}}, \dots, f^n_{v+i_q}, \gamma], \end{aligned} \tag{3-21}$$

where

$$\star_1 = |\alpha| c_v + \sum_{j=1}^{r} \dagger(\mathcal{C}, \alpha, i_j) \quad \text{and} \quad \star_2 = |\alpha|(c_v+1) + \sum_{j=1}^{p-1} \ddagger(\mathcal{C}, \alpha, i_j) + \sum_{j=p}^{q} \dagger(\mathcal{C}, \alpha, i_j). \tag{3-22}$$

Proof **Step 1** ($r = 1$) In this case, since $p = q = r = 1$ for the second term, we write $i = i_1$. Then $\star_2 = |\alpha|(c_v+1) + \dagger(\mathcal{C}, \alpha, i)$. Using the equation $\delta^n_* - \sum_a \pi_1^*\theta_{*,a} \wedge \pi_2^*\theta^*_{*,a} = \mathrm{d} f^n_*$ for any $n \in \mathbb{N}$,

$$\begin{aligned} (-1)^{\star_2} \operatorname{Tr}^{v+i} \mathcal{M}^{v,k}_i[\alpha, \theta\theta^*_{v+i}, \gamma] &= \sum_i (-1)^{\star_2} \mathcal{M}^{v,k}_i[\alpha, \delta^n_{v+i} - \mathrm{d} f^n_{v+i}, \gamma] \\ &= \lim_{n\to\infty} \sum_i (-1)^{\star_2} \mathcal{M}^{v,k}_i[\alpha, \delta^n_{v+i} - \mathrm{d} f^n_{v+i}, \gamma] \\ &= \lim_{n\to\infty} \sum_i (-1)^{\star_2} \mathcal{M}^{v,k}_i[\alpha, \delta^n_{v+i}, \gamma] + \lim_{n\to\infty} \sum_i (-1)^{\star_2+1} \mathcal{M}^{v,k}_i[\alpha, \mathrm{d} f^n_{v+i}, \gamma]. \end{aligned} \tag{3-23}$$

By Lemma 3.14,

$$\lim_{n\to\infty} \sum_i (-1)^{\star_2} \mathcal{M}^{v,k}_i[\alpha, \delta^n_{v+i}, \gamma] = \sum_i (-1)^{\star_2 + (|\alpha| + m_{v,v+i}) c_{v+i}} \mathcal{M}^{v,k}_{\bar{i}}[\alpha, \gamma]. \tag{3-24}$$

Since $(-1)^{\star_2 + (|\alpha| + m_{v,v+i}) c_{v+i}} = (-1)^{|\alpha| c_v + m_{v,v+i}}$ and $\partial[\mathcal{M}_{ik}] = \sum (-1)^{m_{i,j}} [\mathcal{M}_{ij}] \times_j [\mathcal{M}_{jk}]$, by Stokes' theorem this equals

$$\begin{aligned} &\sum_i (-1)^{|\alpha| c_v + m_{v,v+i}} \int_{\mathcal{M}_{v,v+i} \times_{v+i} \mathcal{M}_{v+i,v+k}} s^*_{v,v+i}\alpha \wedge t^*_{v+i,v+k}\gamma \\ &\qquad = (-1)^{|\alpha| c_v} \int_{\partial \mathcal{M}_{v,v+k}} s^*_{v,v+k}\alpha \wedge t^*_{v,v+k}\gamma = (-1)^{|\alpha| c_v} \int_{\mathcal{M}_{v,v+k}} \mathrm{d}(s^*_{v,v+k}\alpha \wedge t^*_{v,v+k}\gamma) = 0. \end{aligned}$$

Now, the second summand in (3-23) equals

$$\lim_{n\to\infty} \sum_i (-1)^{\star_2 + 1 + |\alpha|} \mathcal{M}^{v,k}_i[\mathrm{d}(\alpha, f^n_{v+i}, \gamma)].$$

Note that the difference between $\star_1$ and $\star_2$ in the $r = 1$ case is indeed $|\alpha|$. This proves the $r = 1$ case.

Step 2 (independence of r) We need to prove that the value of the right-hand side does not change from r to $r+1$. To do this, we apply Stokes' theorem to the exact term in (3-21) in the r case. The boundary $\partial(\mathcal{M}_{v,v+i_1} \times \cdots \times \mathcal{M}_{v+i_r,v+k})$ comes from fiber product at $v+w$ for all t and w such that $0 < i_1 < \cdots < i_t < w < i_{t+1} < \cdots < i_r < k$. Consider the boundary coming from the fiber product at $v+w$. After applying Stokes' theorem to the exact term in (3-21), the contribution from integration over the $\mathcal{M}^{v,k}_{i_1,\dots,i_t,\bar{w},\dots,i_r} \subset \mathcal{M}^{v,k}_{i_1,\dots,i_r}$ is

$$(-1)^{\star_3} \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_t,\bar{w},\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma], \tag{3-25}$$

where $\star_3 = |\alpha|c_v + \sum_{j=1}^r \dagger(\mathcal{C},\alpha,i_j) + m_{v,v+i_1} + \cdots + m_{v+i_t,v+w}$. By replacing the fiber product in $\mathcal{M}^{v,k}_{i_1,\dots,i_t,\bar{w},\dots,i_r}$ with the Cartesian product $\mathcal{M}^{v,k}_{i_1,\dots,i_t,w,\dots,i_r}$, Lemma 3.14 gives that (3-25) equals

$$(-1)^{\star_3+(|\alpha|+m_{v,v+w})c_{v+w}} \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_t,w,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, \delta^n_{v+w}, \dots, f^n_{v+i_r}, \gamma]. \tag{3-26}$$

We replace the Thom class δ^n_* by $\sum_a \pi_1^*\theta_{*,a} \wedge \pi_2^*\theta^*_{*,a} + \mathrm{d}f^n_*$ to get

$$(-1)^{\star_3+(|\alpha|+m_{v,v+w})c_{v+w}} \lim_{n\to\infty} \mathrm{Tr}^{v+w}\, \mathcal{M}^{v,k}_{i_1,\dots,i_t,w,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, \theta\theta^*_{v+w}, \dots, f^n_{v+i_r}, \gamma] \tag{3-27}$$

$$+ (-1)^{\star_3+(|\alpha|+m_{v,v+w})c_{v+w}} \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_t,w,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, \mathrm{d}f^n_{v+w}, \dots, f^n_{v+i_r}, \gamma]. \tag{3-28}$$

Let $\star_4$ denote $\star_3 + (|\alpha| + m_{v,v+w})c_{v+w}$. By (2-5),

$$\star_4 = |\alpha|(c_v+1) + \sum_{j=1}^{t} \ddagger(\mathcal{C},\alpha,i_j) + \dagger(\mathcal{C},\alpha,w) + \sum_{j=t+1}^{r} \dagger(\mathcal{C},\alpha,i_j) \mod 2.$$

Because $\star_5 := \star_4 + |\alpha| + \sum_{j=1}^t (c_{v+i_j}+1) \equiv |\alpha|c_v + \sum_{j=1}^r \dagger(\mathcal{C},\alpha,i_j) + \dagger(\mathcal{C},\alpha,w) \mod 2$ and $|f^n_{v+i_j}| \equiv c_{v+i_j}+1 \mod 2$, (3-28) equals

$$\lim_{n\to\infty} \sum_{0<i_1<\cdots<i_t<w<i_{t+1}<i_r<k} (-1)^{\star_5} \mathcal{M}^{v,k}_{i_1,\dots,i_t,w,i_{t+1},\dots,i_r}[\mathrm{d}(\alpha, f^n_{v+i_1}, \dots, f^n_{v+w}, \dots, f^n_{v+i_r}, \gamma)]. \tag{3-29}$$

Therefore, the right-hand side equals

$$\lim_{n\to\infty} \sum_{\substack{1\le p\le q\le r\\ 0<i_1<\cdots<i_q<k}} (-1)^{\star_2}\, \mathrm{Tr}^{v+i_p}\, \mathcal{M}^{s,k}_{i_1,\dots,i_q}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_{p-1}}, \theta\theta^*_{v+i_p}, f^n_{v+i_{p+1}}, \dots, f^n_{v+i_q}, \gamma]$$
$$+ \lim_{n\to\infty} \sum_{0<i_1<\cdots<i_t<w<i_{t+1}<i_r<k} (-1)^{\star_4}\, \mathrm{Tr}^{v+w}\, \mathcal{M}^{s,k}_{i_1,\dots,i_t,w,i_{t+1},\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, \theta\theta^*_{v+w}, \dots, f^n_{v+i_r}, \gamma]$$
$$+ \lim_{n\to\infty} \sum_{0<i_1<\cdots<i_t<w<i_{t+1}<i_r<k} (-1)^{\star_5} \mathcal{M}^{v,k}_{i_1,\dots,i_t,w,i_{t+1},\dots,i_r}[\mathrm{d}(\alpha, f^n_{v+i_1}, \dots, f^n_{v+w}, \dots, f^n_{v+i_r}, \gamma)].$$

This is the $r+1$ case, so we have proved the claim. ◻

Going back to the proof of Proposition 3.15, in the case of $r = k-1$ in Lemma 3.16, the terms

$$\lim_{n\to\infty} (-1)^{\star_1} \mathcal{M}^{v,k}_{1,\dots,k-1}[\mathrm{d}(\alpha, f^n_{v+1}, \dots, f^n_{v+k-1}, \gamma)], \tag{3-30}$$

and

$$\lim_{n\to\infty} \sum_{\substack{1\le p\le q\le k-1\\ 0<i_1<\cdots<i_q<k}} (-1)^{\star_2} \operatorname{Tr}^{v+i_p} \mathcal{M}^{v,k}_{i_1,\dots,i_q}[\alpha, f^n_{v+i_1},\dots, f^n_{v+i_{p-1}}, \theta\theta^*_{v+i_p}, f^n_{v+i_{p+1}},\dots, f^n_{v+i_q},\gamma] \tag{3-31}$$

sum to zero, where

$$\star_1 = |\alpha| c_v + \sum_{j=1}^{k-1} \dagger(\mathcal{C},\alpha,j) \quad \text{and} \quad \star_2 = |\alpha|(c_v+1) + \sum_{j=1}^{p-1} \ddagger(\mathcal{C},\alpha,i_j) + \sum_{j=p}^{q} \dagger(\mathcal{C},\alpha,i_j).$$

Since $\mathcal{M}^{v,k}_{1,\dots,k-1}$ is a closed manifold, (3-30) is 0 by Stokes' theorem. For the remaining term, we claim that (3-31) equals

$$\Big\langle \sum_{i=1}^{k-1} d_{k-i}\circ d_i[\alpha],[\gamma]\Big\rangle_{v+k}. \tag{3-32}$$

Since $|d_i\alpha| = |\alpha| + m_{v,v+i} + c_{v+i} \mod 2$,

$$\ddagger(\mathcal{C}, d_i\alpha, j) = \dagger(\mathcal{C},\alpha,i+j) \mod 2.$$

Then the claim simply follows from the definition of d_i. □

Remark 3.17 From the proof of Proposition 3.15, we see that there is no harm in suppressing the index n and $\lim_{n\to\infty}$ by Lemmas 3.7 and 3.14. If we write f_i as the limit of f^n_i in the space of currents such that

$$\delta_i = \pi_1^*\theta_{i,a}\wedge\pi_2^*\theta^*_{i,a} + \mathrm{d} f_i, \tag{3-33}$$

where δ_i is the Dirac current, then we can use (3-33) to do formal computations.

3.3 Flow morphisms induce cochain morphisms

Section 3.2 shows that a flow category carries enough geometric structure to define a cochain complex. In the following subsections, we study the analogous geometric data for cochain complex morphisms and homotopies. In this subsection, we introduce flow morphisms between flow categories, which is the underlying geometric data for defining continuation maps [2, Chapter 11]. We show that every flow category has an identity flow morphism from the flow category to itself. Using the identity flow morphism, we show that $H(\mathrm{BC}, d_{\mathrm{BC},\Theta})$ is independent of the defining data Θ, finishing the proof of Theorem 3.10.

3.3.1 Flow morphisms

Definition 3.18 An *oriented flow morphism* $\mathfrak{H}$ from an oriented flow category $\mathcal{C} := \{C_i, \mathcal{M}^C_{i,j}\}$ to another oriented flow category $\mathcal{D} := \{D_i, \mathcal{M}^D_{i,j}\}$ is a family of compact oriented manifolds $\{\mathcal{H}_{i,j}\}_{i,j\in\mathbb{Z}}$ such that:

(1) There are two smooth maps $s\colon \mathcal{H}_{i,j}\to C_i$ and $t\colon \mathcal{H}_{i,j}\to D_j$.

(2) There exists $N\in\mathbb{Z}$, such that $\mathcal{H}_{i,j}=\varnothing$ when $i-j>N$.

(3) For every $i_0<i_1<\cdots<i_k$, $j_0<\cdots<j_{m-1}<j_m$, the fiber product

$$\mathcal{M}^C_{i_0,i_1}\times_{i_1}\cdots\times_{i_k}\mathcal{H}_{i_k,j_0}\times_{j_0}\cdots\times_{j_{m-1}}\mathcal{M}^D_{j_{m-1},j_m}$$

is cut out transversely.

(4) There are smooth maps $m_L\colon \mathcal{M}^C_{i,j}\times_j \mathcal{H}_{j,k}\to\mathcal{H}_{i,k}$ and $m_R\colon \mathcal{H}_{i,j}\times_j \mathcal{M}^D_{j,k}\to\mathcal{H}_{i,k}$ such that

$$s\circ m_L(a,b)=s^C(a),\quad t\circ m_L(a,b)=t(b),\quad s\circ m_R(a,b)=s(a)\quad\text{and}\quad t\circ m_R(a,b)=t^D(b),$$

where s^C is the source map for the flow category $\mathcal{C}$ and t^D is the target map for the flow category $\mathcal{D}$.

(5) The map $m_L\cup m_R\colon \bigl(\bigcup_j \mathcal{M}^C_{i,j}\times_j\mathcal{H}_{j,k}\bigr)\cup\bigl(\bigcup_j \mathcal{H}_{i,j}\times_j\mathcal{M}^{\mathcal{D}}_{j,k}\bigr)\to\partial\mathcal{H}_{i,k}$ is a diffeomorphism up to zero-measure (Definition 2.6).

(6) The orientation $[\mathcal{H}_{i,j}]$ has the following properties:

$$\partial[\mathcal{H}_{i,j}]=\sum_{p>0}(-1)^{m^C_{i,i+p}}m_L([\mathcal{M}^C_{i,i+p}\times_{i+p}\mathcal{H}_{i+p,j}])+\sum_{p>0}(-1)^{h_{i,j}}m_R([\mathcal{H}_{i,j-p}\times_{j-p}\mathcal{M}^D_{j-p,j}]),$$

$$(t^C\times s)^*[N_j][\mathcal{M}^C_{i,j}\times_j\mathcal{H}_{j,k}]=(-1)^{c_jm^C_{i,j}}[\mathcal{M}^C_{i,j}][\mathcal{H}_{j,k}],$$

$$(t\times s^D)^*[N_j][\mathcal{H}_{i,j}\times_j\mathcal{M}^D_{j,k}]=(-1)^{d_jh_{i,j}}[\mathcal{H}_{i,j}][\mathcal{M}^D_{j,k}].$$

Here $c_i:=\dim C_i$, $m^C_{i,j}:=\dim\mathcal{M}^C_{i,j}$, $d_j:=\dim D_j$ and $h_{i,j}=\dim\mathcal{H}_{i,j}$.

By (4), we have a formula similar to (2-6). Thus it is convenient to use m_L and m_R to identify $\mathcal{M}^C_{i,j}\times_j\mathcal{H}_{j,k}$ and $\mathcal{H}_{i,j}\times_j\mathcal{M}^D_{j,k}$ with the corresponding parts of $\partial\mathcal{H}_{i,k}$. Hence in the following, we will suppress m_L and m_R, and treat $\mathcal{M}^C_{i,j}\times_j\mathcal{H}_{j,k}$ and $\mathcal{H}_{i,j}\times_j\mathcal{M}^D_{j,k}$ as though they are contained in $\partial\mathcal{H}_{i,k}$.

Remark 3.19 Condition (2) is important in obtaining a finite sum in the definition of the induced cochain morphism. In the context of Morse/Floer theories, the existence of N usually comes from some energy estimates. More precisely, $\mathcal{H}_{i,j}$ is typically the compactification of the space of solutions to parametrized Floer equations/gradient flow equations interpolating the geometric data for $\mathcal{C}$ and $\mathcal{D}$. Then there is usually some notion of energy $E(u)$ for a Floer cylinder/gradient flow u in the moduli space $\mathcal{H}_{i,j}$ such that $E(u)\geq 0$. Now we assume that the energy $E(u)$ satisfies inequality $E(u)\leq g(D_j)-f(C_i)+C$, where f and g are the background Morse–Bott functionals for $\mathcal{C}$ and $\mathcal{D}$, and C is a universal constant depending on the interpolating data we use to define the moduli space $\mathcal{H}_{i,j}$. Assuming the critical values do not accumulate for simplicity,[19] then if $j\ll i$ we have $E(u)<0$, ie there are no curves in $\mathcal{H}_{i,j}$.

Remark 3.20 Similar to Definition 2.13, we say $\mathfrak{H}$ is compatible with the grading structures on $\mathcal{C}$ and $\mathcal{D}$ if and only if $d(C_i)=d(D_j)+d_j-h_{i,j}$, where $\{d(C_i)\}$ and $\{d(D_j)\}$ are grading structures on $\mathcal{C}$ and $\mathcal{D}$, respectively. When this holds, the cochain morphism ϕ^H below will have degree 0.

The main result of this subsection is that oriented flow morphisms induce cochain morphisms between the minimal Morse–Bott cochain complexes. Let $\mathcal{C}:=\{C_i,\mathcal{M}^{\mathcal{C}}_{i,j}\}$ and $\mathcal{D}:=\{D_i,\mathcal{M}^{\mathcal{D}}_{i,j}\}$ be two oriented flow categories and assume $\mathfrak{H}=\{\mathcal{H}_{i,j}\}$ is an oriented flow morphism from $\mathcal{C}$ to $\mathcal{D}$. Then we introduce the following:

[19] When critical values accumulate see Remark 2.11.

(1) We write $c_i := \dim C_i$, $d_i := \dim D_i$, $m_{i,j}^C := \dim \mathcal{M}_{i,j}^C$, $m_{i,j}^D := \dim \mathcal{M}_{i,j}^D$ and $h_{i,j} := \dim \mathcal{H}_{i,j}$. We formally define $m_{i,i}^C = c_i - 1$ and $m_{i,i}^D = d_i - 1$ as before. We assume, as before, that those numbers are well defined. Then

$$h_{i,j} + m_{j,k}^D - d_j + 1 = h_{i,k} \quad \text{for } j \le k \qquad \text{and} \qquad m_{i,j}^C + h_{j,k} - c_i + 1 = h_{i,k} \quad \text{for } i \le j$$

by Definition 3.18.

(2) For $v, k \in \mathbb{Z}$, $0 < i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q < k$, we define

$$\mathcal{H}^{v,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q} := \mathcal{M}^C_{v,v+i_1} \times \cdots \times \mathcal{M}^C_{v+i_{p-1},v+i_p} \times \mathcal{H}_{v+i_p,v+j_1} \times \mathcal{M}^D_{v+j_1,v+j_2} \times \cdots \times \mathcal{M}^D_{v+j_q,v+k}$$

with the product orientation.

(3) $\mathcal{H}^{*,*}_{\ldots|\ldots}[\alpha, f_*, \ldots, f_*, \ldots, \gamma]$ is defined similarly to $\mathcal{M}^{*,*}_{\ldots}[\alpha, f_*, \ldots, \gamma]$ in (3-10).

(4) We define $\dagger(\mathfrak{H}, \alpha, k) = (|\alpha| + h_{v,v+k})(d_{v+k} + 1)$ and $\ddagger(\mathfrak{H}, \alpha, k) := (|\alpha| + h_{v,v+k} + 1)(d_{v+k} + 1)$ for $\alpha \in \Omega^*(C_v)$.

Let $\Theta_1 := \{h(\mathcal{C}, i), f_i^{C,n}\}$ and $\Theta_2 := \{h(\mathcal{D}, i), f_i^{D,n}\}$ be defining data for flow categories $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $\mathfrak{H} := \{\mathcal{H}_{i,j}\}$ be an oriented flow morphism from $\mathcal{C}$ to $\mathcal{D}$. The counterparts of Lemmas 3.7 and 3.14 hold for $\mathcal{H}$ by the same argument. Then define a linear operator $\phi^H_{k,\Theta_1,\Theta_2} : H^*(C_v) \to H^*(D_{v+k})$ for every $v, k \in \mathbb{Z}$ by

$$\begin{aligned}
\text{(3-34)}\quad & \langle \phi^H_{k,\Theta_1,\Theta_2}[\alpha], [\gamma]\rangle_{v+k} \\
&:= \sum_{\substack{p,q\ge 0\\ 0=i_0<i_1<\cdots<i_p\\ j_1<\cdots<j_q<j_{q+1}=k}} (-1)^* \mathcal{H}^{v,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q}[\alpha, f^C_{v+i_1}, \ldots, f^C_{v+i_p}, f^D_{v+j_1}, \ldots, f^D_{v+j_q}, \gamma] \\
&:= \lim_{n\to\infty} \sum_{\substack{p,q\ge 0\\ 0=i_0<i_1<\cdots<i_p\\ j_1<\cdots<j_q<j_{q+1}=k}} (-1)^* \mathcal{H}^{v,k}_{i_1,\ldots,i_p|j_1,\ldots,j_q}[\alpha, f^{C,n}_{v+i_1}, \ldots, f^{C,n}_{v+i_p}, f^{D,n}_{v+j_1}, \ldots, f^{D,n}_{v+j_q}, \gamma],
\end{aligned}$$

where

$$* := |\alpha| c_v + h_{v,v+j_1} + \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{q} \ddagger(\mathfrak{H}, \alpha, j_w).$$

The existence of N in Definition 3.18(2) implies that (3-34) is a finite sum and $\phi^H_{k,\Theta_1,\Theta_2} = 0$ for $k < -N$.

Theorem 3.21 *Let $\mathfrak{H} : \mathcal{C} \Rightarrow \mathcal{D}$ be an oriented flow morphism. If we fix defining data $\Theta_1 := \{h(\mathcal{C}, i), f_i^{C,n}\}$ and $\Theta_2 := \{h(\mathcal{D}, i), f_i^{D,n}\}$ for $\mathcal{C}$ and $\mathcal{D}$, respectively, then there is a linear map*

$$\phi^H_{\Theta_1,\Theta_2} = \prod_{k\in\mathbb{Z}} \phi^H_{k,\Theta_1,\Theta_2} : \mathrm{BC}(\mathcal{C}, \Theta_1) \to \mathrm{BC}(\mathcal{D}, \Theta_2)$$

given by (3-34) *such that*

$$\phi^H_{\Theta_1,\Theta_2} \circ d^C_{\mathrm{BC},\Theta_1} - d^D_{\mathrm{BC},\Theta_2} \circ \phi^H_{\Theta_1,\Theta_2} = 0.$$

In particular, $\phi^H_{\Theta_1,\Theta_2}$ induces a map $H(\mathrm{BC}(\mathcal{C}), d^C_{\mathrm{BC},\Theta_1}) \to H(\mathrm{BC}(\mathcal{D}), d^D_{\mathrm{BC},\Theta_2})$ on cohomology.

Proof Similar to the proof of Proposition 3.15, this theorem follows from the claim that, for $\alpha \in h(\mathcal{C}, v)$, $\gamma \in h(\mathcal{C}, v+k)$ with $k \in \mathbb{Z}$, and any $r \geq 1$, we have

$$\begin{aligned}
0 &= (-1)^{1+|\alpha|c_v+h_{v,v+k}} \int_{\partial H_{v,v+k}} s^*\alpha \wedge t^*\gamma \\
&= \sum_{\substack{0\leq p\leq r\\ 0<i_1<\cdots<i_p\\ j_1<\cdots<j_{r-p}<k}} (-1)^{*_1}\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{r-p}}[\mathrm{d}(\alpha, f^C_{v+i_1},\dots,f^C_{v+i_p}, f^D_{v+j_1},\dots,f^D_{v+j_{r-p}},\gamma)] \\
&\quad + \sum_{\substack{0\leq p\leq q\leq r, 1\leq t\leq p\\ 0<i_1<\cdots<i_p\\ j_1<\ldots<j_{q-p}<k}} (-1)^{*_2}\operatorname{Tr}^{v+i_t}\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{q-p}}[\alpha, f^C_{v+i_1},\dots,\theta^C\theta^C_{v+i_t}{}^*,\dots,f^D_{v+i_{q-p}},\gamma] \\
&\quad + \sum_{\substack{0\leq p\leq q\leq r, 1\leq t\leq q-p\\ 0<i_1<\ldots<i_p\\ j_1<\ldots<j_{q-p}<k}} (-1)^{*_3}\operatorname{Tr}^{v+j_t}\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{q-p}}[\alpha, f^C_{v+i_1},\dots,\theta^D\theta^D_{v+j_t}{}^*,\dots,f^D_{v+i_{q-p}},\gamma].
\end{aligned}$$

Here

$$\begin{aligned}
*_1 &= 1+|\alpha|(c_v+1)+h_{v,v+j_1}+\sum_{w=1}^{p}\dagger(\mathcal{C},\alpha,i_w)+\sum_{w=1}^{r-p}\dagger(\mathfrak{H},\alpha,j_w), \\
*_2 &= 1+|\alpha|c_v+h_{v,v+j_1}+\sum_{w=1}^{t-1}\ddagger(\mathcal{C},\alpha,i_w)+\sum_{w=t}^{p}\dagger(\mathcal{C},\alpha,i_w)+\sum_{w=1}^{q-p}\dagger(\mathfrak{H},\alpha,j_w), \\
*_3 &= 1+|\alpha|c_v+h_{v,v+j_1}+\sum_{w=1}^{p}\ddagger(\mathcal{C},\alpha,i_w)+\sum_{w=1}^{t-1}\ddagger(\mathfrak{H},\alpha,j_w)+\sum_{w=t}^{q-p}\dagger(\mathfrak{H},\alpha,j_w).
\end{aligned}$$

The proof is again by induction, and we omit it. Then for $r > k+N$, the first exact term is zero, as $\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{r-p}}$ is necessarily empty by Definition 3.18(2). We can directly check that the remaining terms are exactly $\langle(\phi^H\circ d^C - d^D\circ\phi^H)\alpha,\gamma\rangle_{v+k}$, and hence the theorem holds. □

Similar to Corollary 3.13, we have the following:

Corollary 3.22 *Assume that oriented flow categories $\mathcal{C}$ and $\mathcal{D}$ have the property that $\dim C_i, \dim D_i \leq k$ for all i. If $\mathfrak{H}\colon \mathcal{C} \Rightarrow \mathcal{D}$ is an oriented flow morphism, then $\phi^H : \mathrm{BC}(\mathcal{C},\Theta_1) \to \mathrm{BC}(\mathcal{D},\Theta_2)$ only depends on those $\mathcal{M}^C_{i,j}$, $\mathcal{H}_{i,j}$ and $\mathcal{M}^D_{i,j}$ of dimension $\leq 2k$.*

3.3.2 The identity flow morphism Next we show that, for every oriented flow category $\mathcal{C}$, there is an oriented flow morphism $\mathfrak{I}\colon \mathcal{C} \Rightarrow \mathcal{C}$, which is referred to as the identity flow morphism. Roughly speaking, when the flow category has a background Morse–Bott function, the identity flow morphism comes from the compactified moduli space of parametrized gradient flow lines, (flow lines not modulo the $\mathbb{R}$ translation action). Using the identity flow morphism, we show the Morse–Bott cohomology is independent of the defining data.

Definition/Lemma 3.23 *For an oriented flow category $\mathcal{C}$, there is a canonical oriented flow morphism $\mathfrak{I}\colon \mathcal{C} \Rightarrow \mathcal{C}$ given by $\mathcal{I}_{i,j} = \mathcal{M}_{i,j}\times[0, j-i]$ with the product orientation for $i \leq j$, and $\mathcal{I}_{i,j} = \varnothing$ for $i > j$. The source and target maps $s, t : \mathcal{I}_{i,j} \to C_i, C_j$ are defined as*

$$s = s^C \circ \pi_1 \quad \textit{and} \quad t = t^C \circ \pi_1,$$

where π_1 is the projection to the $\mathcal{M}$ component. The compositions m_L and m_R are defined by

$$m_L\colon \mathcal{M}_{i,k}\times_k \mathcal{I}_{k,j}\to \mathcal{I}_{i,j},\quad (a,b,t)\mapsto (m(a,b),t+k-i),$$
$$m_R\colon \mathcal{I}_{i,k}\times_k \mathcal{M}_{k,j}\to \mathcal{I}_{i,j},\quad (a,t,b)\mapsto (m(a,b),t),$$

where m is the composition in $\mathcal{C}$.

Before giving the proof, we will first use Definition/Lemma 3.23 to finish the proof of Theorem 3.10:

Proof of Theorem 3.10 Let Θ_1 and Θ_2 be defining data for the oriented flow category $\mathcal{C}$. We have shown in Proposition 3.15 that $(\mathrm{BC}, d_{\mathrm{BC},\Theta_1})$ and $(\mathrm{BC}, d_{\mathrm{BC},\Theta_2})$ are cochain complexes. By (3-34), the cochain morphism $\phi^I_{\Theta_1,\Theta_2}\colon (\mathrm{BC}, d_{\mathrm{BC},\Theta_1})\to(\mathrm{BC}, d_{\mathrm{BC},\Theta_2})$ induced by the identity flow morphism $\mathfrak{I}$ can be written as $\mathrm{id}+N$, where N is strictly upper triangular, ie N sends $H^*(C_s)$ to $\prod_{t=s+1}^{\infty} H^*(C_t)$. Note that $\sum_{n=0}^{\infty}(-N)^n$ is well defined on the cochain complex BC, and $\sum_{n=0}^{\infty}(-N)^n$ is the inverse to $\mathrm{id}+N$. Thus $\phi^I_{\Theta_1,\Theta_2}$ is an isomorphism, and hence induces an isomorphism on cohomology. □

Remark 3.24 When $\Theta_1=\Theta_2$, we show in Section 3.6 that $\phi^I_{\Theta_1,\Theta_2}$ is homotopic to the identity map. In particular, we will show that the construction, up to homotopy, is functorial with respect to the choice of defining data.

Proof of Definition/Lemma 3.23 Definition 3.18(2) follows from $\mathcal{I}_{i,j}=\varnothing$ for $i>j$. Condition (3) holds for $\mathfrak{I}$ due to the transversality property of the flow category $\mathcal{C}$. Since $m_L(\mathcal{M}_{i,k}\times_k \mathcal{I}_{k,j}) = \mathcal{M}_{i,k}\times_k \mathcal{M}_{k,j}\times[k-i,j-i]$ and $m_R(\mathcal{I}_{i,k}\times_k \mathcal{M}_{k.j}) = \mathcal{M}_{i,k}\times_k \mathcal{M}_{k,j}\times[0,k-i]$, the flow morphism conditions (4) and (5) are satisfied by $\mathfrak{I}$. Therefore we need only check (6), the orientation condition.

Unless stated otherwise, products of manifolds are always equipped with the product orientation. For $i<j$,

$$\begin{aligned}\partial[\mathcal{I}_{i,j}] &= \partial[\mathcal{M}_{i,j}\times[0,j-i]]\\ &= (-1)^{m_{i,j}+1}[\mathcal{M}_{i,j}\times\{0\}]+(-1)^{m_{i,j}}[\mathcal{M}_{i,j}\times\{j-i\}]+\sum_{i<k<j}(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[0,j-i]]\end{aligned}\tag{3-35}$$

$$= (-1)^{m_{i,j}+1}[\mathcal{M}_{i,j}\times\{0\}]+(-1)^{m_{i,j}}[\mathcal{M}_{i,j}\times\{j-i\}]\tag{3-36}$$

$$+\sum_{i<k<j}(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[0,k-i]]+\sum_{i<k<j}(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[k-i,j-i]].\tag{3-37}$$

Since the flow category $\mathcal{C}$ is oriented, for $i<k<j$

$$(t^C\times s^C)^*[N_k][\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}] = (-1)^{c_k m_{i,k}}[\mathcal{M}_{i,k}][\mathcal{M}_{k,j}].\tag{3-38}$$

Let π be the projection $\mathcal{I}_{i,j}\to\mathcal{M}_{i,j}$ for $i<j$. Then

$$(t\times s^C)^*N_k = \pi^*(t^C\times s^C)^*N_k|_{\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[0,k-i]},$$
$$(t^C\times s)^*N_k = \pi^*(t^C\times s^C)^*N_k|_{\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[k-i,j-i]}.$$

Therefore (3-38) implies

$$(t\times s^C)^*[N_k][\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[0,k-i]] = (-1)^{c_{i,k}m_{i,k}+m_{k,j}}[\mathcal{M}_{i,k}\times[0,k-i]][\mathcal{M}_{k,j}] \tag{3-39}$$
$$= (-1)^{c_{i,k}m_{i,k}+m_{k,j}}[\mathcal{I}_{i,k}][\mathcal{M}_{j,k}],$$

$$(t^C\times s)^*[N_k][\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[k-i,j-i]] = (-1)^{c_k m_{i,j}}[\mathcal{M}_{i,k}][\mathcal{M}_{k,j}\times[k-i,j-i]] \tag{3-40}$$
$$= (-1)^{c_k m_{i,j}}[\mathcal{M}_{i,k}][\mathcal{I}_{k,j}].$$

If we orient $\mathcal{I}_{i,k}\times_k\mathcal{M}_{k,j}$ by $(-1)^{m_{k,j}+c_k}[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}][[0,k-i]]$ and orient $[\mathcal{M}_{i,k}\times_k\mathcal{I}_{k,j}]$ by $[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}][[k-i,j-i]]$, then (3-39) implies that the first summand in (3-37) equals

$$(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[0,k-i]] = (-1)^{m_{i,j}+1}[\mathcal{I}_{i,k}\times_k\mathcal{M}_{k,j}] \tag{3-41}$$

and that

$$(t\times s^C)^*[N_k][\mathcal{M}_{i,k}\times_j\mathcal{I}_{k,j}] = (-1)^{c_k(m_{i,k}+1)}[\mathcal{I}_{i,k}][\mathcal{M}_{k,j}]. \tag{3-42}$$

And (3-40) implies that the second summand in (3-37) equals

$$(-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k\mathcal{M}_{k,j}\times[k-i,j-i]] = (-1)^{m_{i,k}}[\mathcal{M}_{i,k}\times_k\mathcal{I}_{k,j}] \tag{3-43}$$

and that

$$(t^C\times s)^*[N_k][\mathcal{M}_{i,k}\times_k\mathcal{I}_{k,j}] = (-1)^{c_k m_{i,k}}[\mathcal{M}_{i,k}][\mathcal{I}_{k,j}]. \tag{3-44}$$

We still have to consider the first two copies of $\mathcal{M}_{i,j}$ in (3-36). Since $m_L\colon \mathcal{I}_{i,i}\times_i\mathcal{M}_{i,j}\to\mathcal{M}_{i,j}$ and $m_R\colon \mathcal{M}_{i,j}\times_j\mathcal{I}_{j,j}\to\mathcal{M}_{i,j}$ are diffeomorphisms, we can orient $\mathcal{I}_{i,i}\times_i\mathcal{M}_{i,j}=C_i\times_i\mathcal{M}_{i,j}$ and $\mathcal{M}_{i,j}\times_j\mathcal{I}_{j,j}=\mathcal{M}_{i,j}\times_j C_j$ by $m_L^{-1}([\mathcal{M}_{i,j}])$ and $m_R^{-1}([\mathcal{M}_{i,j}])$. Then by Lemma 3.25 below and the discussion after,

$$(t\times s^C)^*[N_i][C_i\times\mathcal{M}_{i,j}] = (-1)^{c_i^2}[C_i][\mathcal{M}_{i,j}], \tag{3-45}$$

$$(t^C\times s)^*[N_j][\mathcal{M}_{i,j}\times_j C_j] = (-1)^{c_j m_{i,j}}[\mathcal{M}_{i,j}][C_j]. \tag{3-46}$$

Therefore

$$\begin{aligned}(-1)^{m_{i,j}+1}[\mathcal{M}_{i,j}\times\{0\}] &= (-1)^{m_{i,j}+1}m_R([\mathcal{I}_{i,i}\times_i\mathcal{M}_{i,j}]),\\ [(t\times s^C)^*N_j][\mathcal{I}_{i,i}\times_i\mathcal{M}_{i,j}] &= (-1)^{c_i^2}[\mathcal{I}_{i,i}][\mathcal{M}_{i,j}],\\ (-1)^{m_{i,j}}[\mathcal{M}_{i,j}\times\{j-i\}] &= (-1)^{m_{i,j}}m_L([\mathcal{M}_{i,j}\times_j\mathcal{I}_{j,j}]),\\ [(t^C\times s)^*N_i][\mathcal{M}_{i,j}\times_j\mathcal{I}_{j,j}] &= (-1)^{c_j m_{i,j}}[\mathcal{M}_{i,j}][\mathcal{I}_{j,j}].\end{aligned} \tag{3-47}$$

To sum up, (3-41), (3-42), (3-43), (3-44) and (3-47) prove the orientation condition, Definition 3.18(6). □

To state Lemma 3.25 we need some notation. Let E and F be two oriented finite-dimensional vector spaces and $l\colon E\to F$ be a linear map. We denote by Δ_F the diagonal subspace of $F\times F$. Suppose the ordered basis $(f_1,\dots,f_n)$ represents the orientation $[F]$ of F and the ordered basis $(e_1,\dots,e_m)$ represents the orientation of E. Then $((f_1,f_1),\dots,(f_n,f_n))$ determines an orientation $[\Delta_F]$ of Δ_F. Like (2-4), we orient the quotient bundle, ie the normal bundle, $(F\times F)/\Delta_F$ so that $[\Delta_F][(F\times F)/\Delta_F]=[F][F]$. The fiber product $E\times_l F$ is the graph of l in $E\times F$, so $\bigl((e_1,l(e_1)),\dots,(e_m,l(e_m))\bigr)$ determines an orientation $[E\times_l F]$ on $E\times_l F=\operatorname{graph} l$. The projection $\pi\colon E\times_l F\to E$ is an isomorphism and the orientation we put on $E\times_l F$ has the property that $\pi([E\times_l F])=[E]$. Since $(l,\mathrm{id})\colon (E\times F)/(E\times_l F)\to(F\times F)/\Delta_F$ is an isomorphism, we can orient $(E\times F)/(E\times_l F)$ by $(l,\mathrm{id})([(E\times F)/(E\times_l F)])=[(F\times F)/\Delta_F]$. What we describe here is the tangent picture of $\mathcal{M}_{i,j}\times_j C_j$: letting $(m,c)\in\mathcal{M}_{i,j}\times_j C_j$, the correspondences are $E=T_m\mathcal{M}_{i,j}$, $F=T_cC_j$ and $l=Ds|_m$, and the orientations match up.

Lemma 3.25 *Following the notation above, we have*

$$[(E\times F)/(E\times_l F)][E\times_l F]=(-1)^{\dim E\dim F}[E][F].$$

Proof The ordered basis $((0_F,f_1),\dots,(0_F,f_n))$ represents a basis for $(F\times F)/\Delta_F$ as well as the orientation $[(F\times F)/\Delta_F]$. Note that $((0_E,f_1),\dots,(0_E,f_n))$ represents a basis for $(E\times F)/(E\times_l F)$, and is mapped to $((0_F,f_1),\dots,(0_F,f_n))$ through the map (l,id); thus $((0_E,f_1),\dots,(0_E,f_n))$ represents the orientation on $(E\times F)/E\times_l F$. Since $\bigl((e_1,l(e_1)),\dots,(e_m,l(e_m)),(0_E,f_1),\dots,(0_E,f_n)\bigr)$ represents the orientation $[E][F]$,

$$[E\times_l F][(E\times F)/(E\times_l F)]=[E][F]\quad\text{or}\quad[(E\times F)/E\times_l F][(E\times_l F)]=(-1)^{\dim E\dim F}[E][F],$$

which yields (3-46). □

Similarly, consider $F\times_l E$ oriented by $\bigl((l(e_1),e_1),\dots,(l(e_m),e_m)\bigr)$. If we orient $(F\times E)/(F\times_l E)$ by $(\mathrm{id},l)([(F\times E)/(F\times_l E)])=[(F\times F)/\Delta_F]$, then

$$[(F\times E)/(F\times_l E)][F\times_l E]=(-1)^{(\dim F)^2}[F][E],$$

which yields (3-45).

3.4 Compositions of flow morphisms

Roughly speaking, the composition of flow morphisms is taking fiber products. Hence, in the Morse–Bott case, not every flow morphism can be composed, and we introduce the following concept:

Definition 3.26 Two flow morphisms $\mathfrak{H}\colon\mathcal{C}\to\mathcal{D}$ and $\mathfrak{F}\colon\mathcal{D}\to\mathcal{E}$ are *composable* if and only if the fiber products $\mathcal{M}^C_{i_1,i_2}\times_{i_2}\cdots\times_{i_{p-1}}\mathcal{M}^C_{i_{p-1},i_p}\times_{i_p}\mathcal{H}_{i_p,j_1}\times_{j_1}\mathcal{M}^D_{j_1,j_2}\times_{j_2}\cdots\times_{j_{q-1}}\mathcal{M}^D_{j_{q-1},j_q}\times_{j_q}\mathcal{F}_{j_q,k_1}\times_{k_1}\mathcal{M}^E_{k_1,k_2}\times_{k_2}\cdots\times_{k_{r-1}}\mathcal{M}^E_{k_{r-1},k_r}$ are cut out transversely.

Heuristically, one can define the composition $\mathfrak{F}\circ\mathfrak{H}$ of two composable morphisms $\mathfrak{F}$ and $\mathfrak{H}$ to be $(\mathcal{F}\circ\mathcal{H})_{i,k}=\bigcup_j\mathcal{H}_{i,j}\times_j\mathcal{F}_{j,k}$, where the orientation is determined by

$$(t^H\times s^F)^*[N_j][\mathcal{H}_{i,j}\times_j\mathcal{F}_{j,k}]=(-1)^{d_jh_{i,j}}[\mathcal{H}_{i,j}][\mathcal{F}_{j,k}]. \tag{3-48}$$

By Definition 3.18(2), $(\mathcal{F}\circ\mathcal{H})_{i,k}$ is a compact manifold. However, this is no longer a flow morphism, since the boundary can come from fiber products in the middle in addition to fiber products at the two ends,[20] violating Definition 3.18(5). Hence we introduce the following definition.

Definition 3.27 An *oriented flow premorphism* $\mathfrak{F}\colon \mathcal{C}\Rightarrow\mathcal{D}$ is a family of compact oriented manifolds $\mathcal{F}_{i,j}$ with smooth maps $s\colon \mathcal{F}_{i,j}\to C_i$ and $t\colon \mathcal{F}_{i,j}\to D_j$. Moreover, there exists N such that, for $i-j>N$, $\mathcal{F}_{i,j}=\varnothing$ and the fiber products $\mathcal{M}^C_{i_0,i_1}\times_{i_1}\cdots\times_{i_k}\mathcal{F}_{i_k,j_0}\times_{j_0}\cdots\times_{j_{l-1}}\mathcal{M}^D_{j_{l-1},j_l}$ are cut out transversely for all $i_0<\cdots<i_k$ and $j_0<\cdots<j_l$.

Given a flow premorphism $\mathfrak{F}$, one can still define ϕ^F by (3-34), which may not be a cochain morphism. Let $\mathfrak{H}$ and $\mathfrak{F}$ be two composable flow morphisms. Then $\mathfrak{F}\circ\mathfrak{H}$ is a flow premorphism by definition. We need to understand the relation between $\phi^{F\circ H}$ and $\phi^F\circ\phi^H$. The main result of this subsection is that they differ by a homotopy. Before stating the theorem, we first introduce some notation:

(1) $\mathcal{E}:=\{E_i,\mathcal{M}^E_{i,j}\}$ is an oriented flow category, $e_i:=\dim E_i$, $m^E_{i,j}:=\dim\mathcal{M}^E_{i,j}$ and $f_{i,j}:=\dim\mathcal{F}_{i,j}$. These are again assumed to be well defined for simplicity.

(2) For $k\in\mathbb{Z}$, $0<i_1<\cdots<i_p$, $j_1<\cdots<j_q$ and $k_1<\cdots<k_r<k$, we define $\mathcal{F}\times\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_r}$ to be

$$\mathcal{M}^C_{v,v+i_1}\times\cdots\times\mathcal{H}_{v+i_p,v+j_1}\times\mathcal{M}^D_{v+j_1,v+j_2}\times\cdots\times\mathcal{F}_{v+j_q,v+k_1}\times\cdots\times\mathcal{M}^E_{v+k_r,v+k}.$$

Note that we must have $q\geq 1$ for this to be defined.

(3) $(\mathcal{F}\times\mathcal{H})^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_r}[\alpha,f^C_{v+i_1},\dots,f^C_{v+i_p},f^D_{v+j_1},\dots,f^D_{v+j_q},f^E_{v+k_1},\dots,f^E_{v+k_r},\gamma]$ is defined similarly to (3-10).

To define the homotopy operator $P_{\Theta_1,\Theta_2,\Theta_3}$, or P for simplicity, for $k\in\mathbb{Z}$, $\alpha\in h(\mathcal{C},v)$ and $\gamma\in h(\mathcal{E},v+k)$, we define P by

$$\begin{aligned}\text{(3-49)}\quad &\langle P[\alpha],[\gamma]\rangle_{v+k}\\ &=\sum_{\substack{p,r\geq 0,q\geq 1\\ 0=i_0<i_1<\cdots<i_p,j_1<\cdots<j_q\\ k_1\leq\cdots<k_{r+1}=k}}(-1)^{\bigstar}F\\ &\qquad\times H^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_r}[\alpha,f^C_{v+i_1},\dots,f^C_{v+i_p},f^D_{v+j_1},\dots,f^D_{v+j_q},f^E_{v+k_1},\dots,f^E_{v+k_r},\gamma],\end{aligned}$$

where

$$\bigstar:=1+|\alpha|(c_v+1)+\dim(\mathcal{F}\circ\mathcal{H})_{v,v+k_1}+\sum_{w=1}^{p}\ddagger(\mathcal{C},\alpha,i_w)+h_{v,v+j_1}+\sum_{w=1}^{q}\ddagger(\mathfrak{H},\alpha,j_w)+\sum_{w=1}^{r}\dagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_w).$$

[20] Although, in this case, the breaking from fiber products in the middle should pair up and "cancel" with each other; this is morally why we have Theorem 3.28.

Theorem 3.28 *Let $\mathfrak{H}$ and $\mathfrak{F}$ be composable oriented flow morphisms from $\mathcal{C}$ to $\mathcal{D}$ and from $\mathcal{D}$ to $\mathcal{E}$, respectively. If we fix defining data Θ_1, Θ_2 and Θ_3 for $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, then there exists an operator $P_{\Theta_1,\Theta_2,\Theta_3}\colon \mathrm{BC}(\mathcal{C}) \to \mathrm{BC}(\mathcal{E})$ defined by (3-49) such that*

$$\phi^{F\circ H}_{\Theta_1,\Theta_3} - \phi^{F}_{\Theta_2,\Theta_3}\circ\phi^{H}_{\Theta_1,\Theta_2} + P_{\Theta_1,\Theta_2,\Theta_3}\circ d^{C}_{\mathrm{BC},\Theta_1} + d^{E}_{\mathrm{BC},\Theta_3}\circ P_{\Theta_1,\Theta_2,\Theta_3} = 0.$$

Proof For $\alpha \in h(\mathcal{C}, v)$, $\gamma \in h(\mathcal{E}, v+k)$ with $k \in \mathbb{Z}$, and any $l \geq 1$, we have

$$\begin{aligned}
0 &= \sum_{r\le l}\sum_{p+q=r-1}(-1)^{\bigstar_1}(\mathcal{F}\circ\mathcal{H})^{v,k}_{i_1,\dots,i_p|k_1,\dots,k_q}[\alpha,\dots,f^{C}_{v+i_*},\dots,f^{E}_{v+k_*},\dots,\gamma]\\
&\quad+\sum_{p+r+w=l}(-1)^{\bigstar_2}(\mathcal{F}\times\mathcal{H})^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_r|k_1,\dots,k_w}[\mathrm{d}(\alpha,\dots,f^{C}_{v+i_*},\dots,f^{D}_{v+j_*},\dots,f^{E}_{v+k_*},\dots,\gamma)]\\
&\quad+\sum_{\substack{p+q+w\le l,\\ u\ge 1}}(-1)^{\bigstar_3}\operatorname{Tr}^{v+i_u}(\mathcal{F}\times\mathcal{H})^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_w}[\alpha,\dots,f^{C}_{v+i_*},\dots,\theta\theta^{C}_{v+i_u}{}^{*},\dots,f^{D}_{v+j_*},\dots,f^{E}_{v+k_*},\dots,\gamma]\\
&\quad+\sum_{\substack{p+q+w\le l,\\ u\ge 1}}(-1)^{\bigstar_4}\operatorname{Tr}^{v+j_u}\mathcal{F}\times\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_w}[\alpha,\dots,f^{C}_{v+i_*},\dots,f^{D}_{v+j_*},\dots,\theta\theta^{D}_{v+j_u}{}^{*},\dots,f^{E}_{v+k_*},\dots,\gamma]\\
&\quad+\sum_{\substack{p+q+w\le l,\\ u\ge 1}}(-1)^{\bigstar_5}\operatorname{Tr}^{v+k_u}\mathcal{F}\times\mathcal{H}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q|k_1,\dots,k_w}[\alpha,\dots,f^{C}_{v+i_*},\dots,f^{D}_{v+j_*},\dots,f^{E}_{v+k_*},\dots,\theta\theta^{E}_{v+k_u}{}^{*},\dots,\gamma]
\end{aligned}$$

where we omit the obvious constraints $0 < i_1 < \dots < i_p$, $j_1 < \dots < j_q$ and $k_1 < \dots < k_w < k$. The indices for signs are

$$\begin{aligned}
\bigstar_1 &= 1+|\alpha|c_v+\dim(\mathcal{F}\circ\mathcal{H})_{v,v+k_1}+\sum_{s=1}^{p}\ddagger(\mathcal{C},\alpha,i_s)+\sum_{s=1}^{q}\ddagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_s),\\
\bigstar_2 &= |\alpha|c_v+\dim(\mathcal{F}\circ\mathcal{H})_{v,v+k_1}+\sum_{s=1}^{p}\dagger(\mathcal{C},\alpha,i_s)+h_{v,v+j_1}+\sum_{s=1}^{r}\dagger(\mathfrak{H},\alpha,j_s)+\sum_{s=1}^{w}\ddagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_s),\\
\bigstar_3 &= |\alpha|(c_v+1)+\dim(\mathcal{F}\circ\mathcal{H})_{v,v+k_1}+\sum_{s=1}^{u-1}\ddagger(\mathcal{C},\alpha,i_s)+\sum_{s=u}^{p}\dagger(\mathcal{C},\alpha,i_s)+h_{v,v+j_1}+\sum_{s=1}^{r}\dagger(\mathfrak{H},\alpha,j_s)+\sum_{s=1}^{w}\ddagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_s),\\
\bigstar_4 &= |\alpha|(c_v+1)+\dim(\mathcal{F}\circ\mathcal{H})_{v,v+k_1}+\sum_{s=1}^{p}\ddagger(\mathcal{C},\alpha,i_s)+h_{v,v+j_1}+\sum_{s=1}^{u-1}\ddagger(\mathfrak{H},\alpha,j_s)+\sum_{s=u}^{r}\dagger(\mathfrak{H},\alpha,j_s)+\sum_{s=1}^{w}\ddagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_s),\\
\bigstar_5 &= |\alpha|(c_v+1)+\dim(\mathcal{F}\circ\mathcal{H})_{v,v+k_1}+\sum_{s=1}^{p}\ddagger(\mathcal{C},\alpha,i_s)+h_{v,v+j_1}+\sum_{s=1}^{r}\ddagger(\mathfrak{H},\alpha,j_s)+\sum_{s=1}^{u-1}\dagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_s)\\
&\qquad+\sum_{s=u}^{w}\ddagger(\mathfrak{F}\circ\mathfrak{H},\alpha,k_s).
\end{aligned}$$

The proof is again by induction on l, which we omit. Then for $l \gg 0$, the exact term is zero. It is direct to check that the first term is $-\langle\phi^{F\circ H}\alpha,\gamma\rangle_{v+k}$, the third term is $-\langle P\circ d^{C}\alpha,\gamma\rangle_{v+k}$, the fourth term is $\langle\phi^{F}\circ\phi^{H}\alpha,\gamma\rangle_{v+k}$ and the last term is $-\langle d^{E}\circ P\alpha,\gamma\rangle_{v+k}$; hence the theorem follows. □

As a corollary, $\phi^{F\circ H}_{\Theta_1,\Theta_3}$ is a cochain map between $(\mathrm{BC}(\mathcal{C}), d^{C}_{\mathrm{BC},\Theta_1})$ and $(\mathrm{BC}(\mathcal{E}), d^{E}_{\mathrm{BC},\Theta_3})$, and is homotopic to $\phi^{F}_{\Theta_2,\Theta_3}\circ\phi^{H}_{\Theta_1,\Theta_2}$.

3.5 Flow homotopies induce cochain homotopies

In this subsection, we introduce the flow homotopies between flow premorphisms. Such structures can be viewed as the analog of the geometric data needed to define homotopies between continuation maps in Floer theories [2, Chapter 11].

Definition 3.29 An *oriented flow homotopy* $\mathscr{Y}$ between two flow *premorphisms* $\mathfrak{F} = \{\mathcal{F}_{i,j}\}$ and $\mathfrak{H} = \{\mathcal{H}_{i,j}\}$ from $\mathcal{C}$ to $\mathcal{D}$ is a family of oriented compact manifolds $\{\mathcal{Y}_{i,j}\}$ with smooth source and target maps $s\colon \mathcal{Y}_{i,j} \to C_i$ and $t\colon \mathcal{Y}_{i,j} \to D_j$ such that:

(1) There are smooth maps $\iota_F, \iota_H\colon \mathcal{F}_{i,j}, \mathcal{H}_{i,j} \to \mathcal{Y}_{i,j}$ such that $s \circ \iota_F = s^F$, $s \circ \iota_H = s^H$, $t \circ \iota_F = t^F$ and $t \circ \iota_H = t^H$ where s^F, s^H, t^F and t^H are the source and target maps for $\mathfrak{F}$ and $\mathfrak{H}$, respectively.

(2) There exists $N \in \mathbb{N}$ such that when $i - j > N$, we have $\mathcal{Y}_{i,j} = \varnothing$.

(3) For all $i_0 < \cdots < i_k$ and $j_0 < \cdots < j_l$, the fiber products $\mathcal{M}^C_{i_0,i_1} \times_{i_1} \cdots \times_{i_k} \mathcal{Y}_{i_k,j_0} \times_{j_0} \cdots \times_{j_{l-1}} \mathcal{M}^D_{j_{l-1},j_l}$ are cut out transversely.

(4) There are smooth maps $m_L\colon \mathcal{M}^C_{i,j} \times_j \mathcal{Y}_{j,k} \to \mathcal{Y}_{i,k}$ and $m_R\colon \mathcal{Y}_{i,j} \times_j \mathcal{M}^D_{j,k} \to \mathcal{Y}_{i,k}$ such that

$$s \circ m_L(a,b) = s^C(a), \quad t \circ m_L(a,b) = t(b), \quad s \circ m_R(a,b) = s(a) \quad \text{and} \quad t \circ m_R(a,b) = t^D(b).$$

Here s^C is the source map for $\mathcal{C}$ and t^D is the target map for $\mathcal{D}$.

(5) The map $\iota_F \cup \iota_H \cup m_L \cup m_R\colon \mathcal{F}_{i,k} \cup \mathcal{H}_{i,k} \cup \big(\bigcup_j \mathcal{M}^{\mathcal{C}}_{i,j} \times_j \mathcal{Y}_{j,k}\big) \cup \big(\bigcup_j \mathcal{Y}_{i,j} \times_j \mathcal{M}^{\mathcal{D}}_{j,k}\big) \to \partial \mathcal{Y}_{i,k}$ is a diffeomorphism up to measure-zero sets.

(6) The orientation $[\mathcal{Y}_{i,j}]$ has the following properties:

$$\partial[\mathcal{Y}_{i,j}] = \iota_F([\mathcal{F}_{i,j}]) - \iota_H([\mathcal{H}_{i,j}]) + \sum_{p>0} (-1)^{c_{i+p}+1} m_L([\mathcal{M}^C_{i,i+p} \times_{i+p} \mathcal{Y}_{i+p,j}]) \\ + \sum_{p>0} (-1)^{y_{i,j}} m_R([\mathcal{Y}_{i,j-p} \times_{j-p} \mathcal{M}^D_{j-p,j}]),$$

$$(t^C \times s)^*[N_j][\mathcal{M}^C_{i,j} \times_j \mathcal{Y}_{j,k}] = (-1)^{c_j m^C_{i,j}} [\mathcal{M}^C_{i,j}][\mathcal{Y}_{j,k}],$$
$$(t \times s^D)^*[N_j][\mathcal{H}_{i,j} \times_j \mathcal{M}^D_{j,k}] = (-1)^{d_j y_{i,j}} [\mathcal{Y}_{i,j}][\mathcal{M}^D_{j,k}],$$

where $y_{i,j} := \dim \mathcal{Y}_{i,j}$.

The main result of this subsection is that flow homotopies induce homotopies between the maps induced by the boundary flow premorphisms (which are not necessarily cochain morphisms). Before stating the theorem, we introduce the following notation:

(1) For $k \in \mathbb{Z}$, $0 < i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q < k$,

$$\mathcal{Y}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q} := \mathcal{M}^C_{v,v+i_1} \times \cdots \times \mathcal{M}^C_{v+i_{p-1},s+i_p} \times \mathcal{Y}_{v+i_p,v+j_1} \times \mathcal{M}^D_{v+j_1,v+j_2} \times \cdots \times \mathcal{M}^D_{v+j_q,v+k}.$$

(2) $\mathcal{Y}^{*,*}_{\dots}[\alpha, f^C_*, \dots, f^D_*, \dots, \gamma]$ is defined similarly to (3-10).

(3) For $\alpha \in h(\mathcal{C}, v)$, we define

$$\dagger(\mathscr{Y}, \alpha, k) := (|\alpha| + y_{v,v+k})(d_{v+k}+1) \quad \text{and} \quad \ddagger(\mathscr{Y}, \alpha, k) := (|\alpha| + y_{v,v+k}+1)(d_{v+k}+1).$$

To state the formula for the homotopy operator Λ^Y, we suppress the subscripts Θ_1 and Θ_2 for simplicity. Let $\alpha \in h(\mathcal{C}, v)$ and $\gamma \in h(\mathcal{D}, v+k)$. Then $\langle \Lambda^Y[\alpha], [\gamma]\rangle_{v+k}$ is defined to be

$$\sum_{\substack{p,q\geq 0\\ 0=i_0<\cdots<i_p\\ j_1<\cdots<j_{q+1}=k}} (-1)^{\clubsuit} \mathcal{Y}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q}[\alpha, f^C_{v+i_1}, \dots, f^D_{v+j_q}, \gamma], \tag{3-50}$$

where

$$\clubsuit := |\alpha|(c_v+1) + y_{v+i_p,v+j_1} + \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{q} \ddagger(\mathscr{Y}, \alpha, j_w).$$

Theorem 3.30 *Suppose $\mathscr{Y}$ is an oriented flow homotopy between two oriented flow premorphisms $\mathfrak{F}, \mathfrak{H}\colon \mathcal{C} \Rightarrow \mathcal{D}$. After fixing defining data Θ_1 and Θ_2 for $\mathcal{C}$ and $\mathcal{D}$, respectively, there exists an operator $\Lambda^Y_{\Theta_1,\Theta_2}\colon \mathrm{BC}(\mathcal{C}) \to \mathrm{BC}(\mathcal{D})$ defined by* (3-50) *such that*

$$d^D_{\mathrm{BC},\Theta_2} \circ \Lambda^Y_{\Theta_1,\Theta_2} + \Lambda^Y_{\Theta_1,\Theta_2} \circ d^C_{\mathrm{BC},\Theta_1} + \phi^F_{\Theta_1,\Theta_2} - \phi^H_{\Theta_1,\Theta_2} = 0.$$

Proof Similar to the proofs of Proposition 3.15 and Theorem 3.21, this theorem follows from the following claim, whose proof is again by induction and will be omitted.

For $\alpha \in h(\mathcal{C}, v)$, $\gamma \in h(\mathcal{D}, v+k)$ with $k \in \mathbb{Z}$, and any $r \geq 0$,

$$\begin{aligned} 0 = &\sum_{0\leq p\leq r} (-1)^{\clubsuit_1} \mathcal{Y}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{r-p}}[\mathrm{d}(\alpha, f^C_{v+i_1}, \dots, f^D_{v+j_{r-p}}, \gamma)] \\ &+ \sum_{\substack{0\leq p\leq q\leq r\\ 1\leq u\leq p}} (-1)^{\clubsuit_2} \mathrm{Tr}^{v+i_u}\, \mathcal{Y}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{q-p}}[\alpha, f^C_{v+i_1}, \dots, \theta\theta^C_{v+i_u}{}^*, \dots, f^D_{v+i_{q-p}}, \gamma] \\ &+ \sum_{\substack{0\leq p\leq q\leq r\\ 1\leq u\leq q-p}} (-1)^{\clubsuit_3} \mathrm{Tr}^{v+j_u}\, \mathcal{Y}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_{q-p}}[\alpha, f^C_{v+i_1}, \dots, \theta\theta^D_{v+j_u}{}^*, \dots, f^D_{v+i_{q-p}}, \gamma] \\ &+ \sum_{0\leq p\leq q<r} (-1)^{\clubsuit_4} (\mathcal{F}^{v,k}|_{i_1,\dots,i_p|j_1,\dots,j_{q-p}} - \mathcal{H}^{v,k}|_{i_1,\dots,i_p|j_1,\dots,j_{q-p}})[\alpha, f^C_{v+i_1}, \dots, f^D_{v+j_{q-p}}, \gamma]. \end{aligned}$$

Here

$$\clubsuit_1 = |\alpha|c_v + y_{v+i_p,v+j_1} + \sum_{w=1}^{p} \dagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{r-p} \dagger(\mathscr{Y}, \alpha, j_w),$$

$$\clubsuit_2 = |\alpha|(c_v+1) + y_{v+i_p,v+j_1} + \sum_{w=1}^{u-1} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=u}^{p} \dagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{q-p} \dagger(\mathscr{Y}, \alpha, j_w),$$

$$\clubsuit_3 = |\alpha|(c_v+1) + y_{v+i_p,v+j_1} + \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{u-1} \ddagger(\mathscr{Y}, \alpha, j_w) + \sum_{w=u}^{q-p} \dagger(\mathscr{Y}, \alpha, j_w),$$

$$\clubsuit_4 = |\alpha|c_v + y_{v,v+j_1} + \sum_{w=1}^{p} \ddagger(\mathcal{C}, \alpha, i_w) + \sum_{w=1}^{q-p} \dagger(\mathscr{Y}, \alpha, j_w). \qquad \square$$

Remark 3.31 Theorem 3.30 does not require that $\Phi^F_{\Theta_1,\Theta_2}$ or $\Phi^H_{\Theta_1,\Theta_2}$ is a cochain morphism. When they are (in fact, that one of them is a cochain morphism would imply the other is also by Theorem 3.30), Theorem 3.30 implies that they are homotopic to each other.

3.6 The minimal Morse–Bott cochain complex is canonical

Unlike the Morse case, where the defining data is unique, there is a lot of freedom in choosing the defining data for the minimal Morse–Bott cochain complex: choices of quasi-isomorphic embeddings, choices of Thom classes and choices of f_i^n. The cochain morphism $\phi^H_{\Theta,\Theta'}$ induced from the flow morphism $\mathfrak{H}$ by (3-34) also depends on Θ and Θ'. Although Theorem 3.10 asserts that the cohomology is independent of the defining data, it is important to have the isomorphism be canonical in a functorial way with respect to the choice of defining data. In this section, we prove that the construction of the minimal Morse–Bott cochain complex $(\mathrm{BC}, d_{\mathrm{BC},\Theta})$ is natural with respect to the defining data Θ. Moreover, we will show that the cochain morphism $\phi^H_{\Theta,\Theta'}$ from (3-34) is also canonical in a suitable sense. To explain the claim above in more detail, we introduce the following category of defining data of an oriented flow category:

Definition 3.32 Given an oriented flow category $\mathcal{C}$, $\mathcal{D}ata(\mathcal{C})$ is defined to be the category whose objects are defining data of $\mathcal{C}$, and there is exactly one morphism between any two objects.

For every object Θ in $\mathcal{D}ata(\mathcal{C})$, we can associate it with a cochain complex $(\mathrm{BC}, d_{\mathrm{BC},\Theta})$. The following theorem says that such an assignment can be completed to a functor $\mathcal{D}ata(\mathcal{C}) \to \mathcal{K}(\mathcal{C}h)$, where $\mathcal{K}(\mathcal{C}h)$ is the homotopy category of cochain complexes.

Theorem 3.33 *There is a functor $\mathcal{BC}(\mathcal{C})\colon \mathcal{D}ata(\mathcal{C}) \to \mathcal{K}(\mathcal{C}h)$ defined by*

$$\Theta \mapsto (\mathrm{BC}, d_{\mathrm{BC},\Theta}) \quad \textit{and} \quad (\Theta_1 \to \Theta_2) \mapsto (\phi^I_{\Theta_1,\Theta_2}\colon (\mathrm{BC}, d_{\mathrm{BC},\Theta_1}) \to (\mathrm{BC}, d_{\mathrm{BC},\Theta_2})),$$

where $\mathfrak{I}$ is the identity flow morphism used to define $\phi^I_{\Theta_1,\Theta_2}$ by (3-34).

Proof Step 1 ($\phi^I_{\Theta,\Theta}$ is homotopic to the identity) It is not hard to check that $\phi^{I\circ I}_{\Theta,\Theta}$ can be written as $\mathrm{id}+M$ with M strictly upper triangular. Note that for $i<j$, $I_{i,j}=\mathcal{M}_{i,j}\times[0,j-i]$ and $(I\circ I)_{i,j}=\bigcup_{k,i\le k\le j} I_{i,k}\times_k I_{k,j}$ have an interval direction. Since the pullback of differential forms by source and target maps cannot cover that interval direction, we have

$$\begin{aligned} I^{v,k}_{\dots,p|q,\dots}[\dots,f_{v+p},f_{v+q},\dots]=(I\circ I)^{v,k}_{\dots,p|q,\dots}[\dots,f_{v+p},f_{v+q},\dots]&=0 \quad \text{if } p\ne q,\\ I^{v,k}_{\dots,p|}=(I\circ I)^{v,k}_{\dots,p|}&=0 \quad \text{if } p\ne k,\\ I^{v,k}_{|q,\dots}=(I\circ I)^{v,k}_{|q,\dots}&=0 \quad \text{if } q\ne 0.\end{aligned}$$

Therefore, for $k \in \mathbb{N}^+$, $\alpha \in h(\mathcal{C}, v)$ and $\gamma \in h(\mathcal{C}, v+k)$, we have

$$\begin{aligned}\langle M[\alpha], [\gamma]\rangle_{v+k} = \sum_{\substack{1\le p\le q\le k\\ 0<i_1<\cdots<i_q<k}} &(-1)^{\spadesuit_1} I \circ I^{v,k}_{i_1,\dots,i_p|i_p,\dots,i_q}[\alpha, f_{v+i_1}, \dots, f_{v+i_p}, f_{v+i_p}, \dots, f_{v+i_q}, \gamma]\\ &+ \sum_{\substack{1\le p\\ 0<i_1<\cdots<i_p=k}} (-1)^{\spadesuit_2} I \circ I^{v,k}_{i_1,\dots,i_p|}[\alpha, f_{v+i_1}, \dots, f_{v+i_p}, f_{v+i_p}, \gamma]\\ &\qquad + \sum_{\substack{1\le p\\ 0=i_1<\cdots<i_p<k}} (-1)^{\spadesuit_3} I \circ I^{v,k}_{|i_1,\dots,i_p}[\alpha, f_{v+i_1}, f_{v+i_1}, \dots, f_{v+i_p}, \gamma],\end{aligned}$$

where $\spadesuit_1$, $\spadesuit_2$ and $\spadesuit_3$ are determined according to (3-34).

Similarly, we have a decomposition $\phi^I_{\Theta,\Theta} = \mathrm{id} + N$ with N strictly upper triangular. Note that $(I \circ I)_{v+i_p,v+i_p} = I_{v+i_p,v+i_p} = C_{v+i_p}$, and hence

$$\begin{aligned}(I \circ I)^{v,k}_{i_1,\dots,i_p|i_p,\dots,i_q}&[\alpha, f_{v+i_1}, \dots, f_{v+i_p}, f_{v+i_p}, \dots, f_{v+i_q}, \gamma]\\ &= I^{v,k}_{i_1,\dots,i_p|i_p,\dots,i_q}[\alpha, f_{v+i_1}, \dots, f_{v+i_p}, f_{v+i_p}, \dots, f_{v+i_q}, \gamma].\end{aligned}$$

Similarly for the remaining two terms of M and N. Thus we have $N = M$. Then by Theorem 3.28,

$$(\mathrm{id} + M) - (\mathrm{id} + M)^2 = P \circ d_{\mathrm{BC},\Theta} + d_{\mathrm{BC},\Theta} \circ P.$$

Since $\mathrm{id} + M$ is a cochain isomorphism,

$$\mathrm{id} - (\mathrm{id} + M) = (\mathrm{id} + M)^{-1} \circ P \circ d_{\mathrm{BC},\Theta} + d_{\mathrm{BC},\Theta} \circ (\mathrm{id} + M)^{-1} \circ P.$$

Thus $\mathrm{id} + M = \mathrm{id} + N = \phi^I_{\Theta,\Theta}$ is homotopic to the identity.

Step 2 (functoriality) Given three defining data Θ_1, Θ_2 and Θ_3, by the same argument as above we have, up to homotopy, that

$$\phi^I_{\Theta_1,\Theta_3} = \phi^{I\circ I}_{\Theta_1,\Theta_3}.$$

By Theorem 3.28,

$$\phi^{I\circ I}_{\Theta_1,\Theta_3} - \phi^I_{\Theta_2,\Theta_3} \circ \phi^I_{\Theta_1,\Theta_2} + P \circ d_{\mathrm{BC},\Theta_1} + d_{\mathrm{BC},\Theta_3} \circ P = 0.$$

Thus $\phi^I_{\Theta_1,\Theta_3}$ is homotopic to $\phi^I_{\Theta_2,\Theta_3} \circ \phi^I_{\Theta_1,\Theta_2}$. □

Remark 3.34 A similar mechanism of proof appeared in [63, Proposition 7.7.4], where the situation is Morse and the auxiliary data (which can be viewed as the analog of the defining data) are choices in the construction of virtual fundamental cycles.

To explain the functoriality for flow morphisms, we introduce the following category:

Definition 3.35 Letting $\mathcal{C}$ and $\mathcal{D}$ be oriented flow categories, $\mathcal{D}ata(\mathcal{C} \to \mathcal{D})$ is defined to be the category whose objects are defining data of $\mathcal{C}$ and $\mathcal{D}$. There is exactly one morphism from Θ_1 to Θ_2 if Θ_1 and Θ_2 are defining data for the same flow category or Θ_1 and Θ_2 are defining data for $\mathcal{C}$ and $\mathcal{D}$, respectively.

Then $\mathcal{D}ata(\mathcal{C})$ and $\mathcal{D}ata(\mathcal{D})$ are full subcategories of $\mathcal{D}ata(\mathcal{C} \to \mathcal{D})$. If there is an oriented flow morphism $\mathfrak{H}\colon \mathcal{C} \to \mathcal{D}$, then for any defining data Θ and Θ' of $\mathcal{C}$ and $\mathcal{D}$, respectively, we can assign a cochain morphism $\phi^H_{\Theta,\Theta'}\colon (\mathrm{BC}(\mathcal{C}), d^C_{\mathrm{BC},\Theta}) \to (\mathrm{BC}(\mathcal{D}), d^D_{\mathrm{BC},\Theta'})$. The next theorem states that such an assignment along with $\mathcal{BC}(\mathcal{C})$ and $\mathcal{BC}(\mathcal{D})$ is a functor.

Theorem 3.36 *For an oriented flow morphism $\mathfrak{H}$, there is a functor*

$$\Phi^H\colon \mathcal{D}ata(\mathcal{C} \to \mathcal{D}) \to \mathcal{K}(\mathcal{C}h)$$

which extends functors $\mathcal{BC}(\mathcal{C})$ and $\mathcal{BC}(\mathcal{D})$ by sending the morphism $\Theta^C \to \Theta^D$ to $\phi^H_{\Theta^C,\Theta^D}$. Here Θ^C and Θ^D are defining data for $\mathcal{C}$ and $\mathcal{D}$, respectively.

Proof We only need to prove the functoriality. We use Θ^C and Θ^D to denote defining data for $\mathcal{C}$ and $\mathcal{D}$, respectively. By Theorem 3.28, $\phi^{H\circ I}_{\Theta^C_1,\Theta^D}$ is homotopic to both

$$\phi^H_{\Theta^C_2,\Theta^D} \circ \phi^I_{\Theta^C_1,\Theta^C_2} \quad \text{and} \quad \phi^H_{\Theta^C_1,\Theta^D} \circ \phi^I_{\Theta^C_1,\Theta^C_1}.$$

Since, by Theorem 3.33, $\phi^I_{\Theta^C_1,\Theta^C_1}$ is homotopic to the identity, $\phi^H_{\Theta^C_2,\Theta^D} \circ \phi^I_{\Theta^C_1,\Theta^C_2}$ is homotopic to $\phi^H_{\Theta^C_1,\Theta^D}$. Similarly, $\phi^I_{\Theta^D_1,\Theta^D_2} \circ \phi^H_{\Theta^C,\Theta^D_1}$ is homotopic to $\phi^H_{\Theta^C,\Theta^D_2}$. □

3.7 Flow subcategories and flow quotient categories

In this section, we introduce subcategories and quotient categories in the setting of flow categories, which on the cochain complex level correspond to subcomplexes and quotient complexes.

Definition 3.37 Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category. A subset A of $\mathbb{Z}$ is called a $\mathcal{C}$–subset if $j \notin A$ implies $\mathcal{M}_{i,j} = \varnothing$ for all $i \in A$.

A basic example of a $\mathcal{C}$–subset is the set of integers bigger than a fixed number.

Proposition 3.38 *Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category and A be a $\mathcal{C}$–subset. Then $\mathcal{C}_A = \{C_i, \mathcal{M}_{i,j}, i, j \in A\}$ and $\mathcal{C}_{/A} = \{C_i, \mathcal{M}_{i,j}, i, j, \notin A\}$ are flow categories.*

Proof It is clear that both $\mathcal{C}_A$ and $\mathcal{C}_{/A}$ are subcategories. Then it is sufficient to prove that the boundary of morphism spaces comes from fiber products of the morphisms spaces for both $\mathcal{C}_A$ and $\mathcal{C}_{/A}$. Since the boundary $\partial\mathcal{M}_{i,k}$ comes from $\mathcal{M}_{i,j} \times_j \mathcal{M}_{i,k}$, if both $i, k \in A$, then $j \in A$, otherwise one of $\mathcal{M}_{i,j}$ and $\mathcal{M}_{j,k}$ is empty. Similarly for $\mathcal{C}_{/A}$. □

We will call $\mathcal{C}_A$ a *flow subcategory* and $\mathcal{C}_{/A}$ the associated *flow quotient category*.

Remark 3.39 A finer definition of subcategory is using a subset of components of $\mathrm{Obj}(\mathcal{C})$ such that a similar condition to Definition 3.37 holds.

From Definition 3.8, when the defining data of $\mathcal{C}_A$ and $\mathcal{C}_{/A}$ are restrictions of a defining data on $\mathcal{C}$, we have the tautological short exact sequence

$$0 \to \mathrm{BC}(\mathcal{C}_A) \to \mathrm{BC}(\mathcal{C}) \to \mathrm{BC}(\mathcal{C}_{/A}) \to 0 \tag{3-51}$$

by the obvious inclusion and projection. To make the structure more compatible with concepts introduced here and our future applications [79], we lift the short exact sequence to the flow morphism level. We first introduce the following:

Lemma 3.40 *Assume $(V_0 \oplus V_1, d)$ is a cochain complex with the property that $d(V_0) \subset V_0$, that is, d has a decomposition into $d_{00} + d_{10} + d_{11}$, where $d_{ab}\colon V_a \to V_b$. Suppose we have another cochain complex $(V_0' \oplus V_1', d')$ with the same property. Assume the following squares are commutative up to homotopies H_1 and H_2 with the property that $\operatorname{im} H_1 \subset V_0'$, $V_0 \subset \ker H_2$ and the middle morphism ϕ has the same decomposition $\phi_{00} + \phi_{10} + \phi_{11}$, ie $\phi(V_0) \subset V_0'$:*

$$\begin{tikzcd}
0 \arrow[r] & V_0 \arrow[r] \arrow[d, "\psi"] & V_0 \oplus V_1 \arrow[r] \arrow[d, "\phi"] & V_1 \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & V_0' \arrow[r] & V_0' \oplus V_1' \arrow[r] & V_1' \arrow[r] & 0
\end{tikzcd}$$

Then they induce a morphism between the long exact sequences of cohomology.

Proof We only need to prove the following square is commutative:

$$\begin{tikzcd}
H(V_1) \arrow[r, "d_{10}"] \arrow[d, "\eta"] & H(V_0) \arrow[d, "\psi"] \\
H(V_1') \arrow[r, "d_{10}'"] & H(V_0')
\end{tikzcd}$$

By $\operatorname{im} H_1 \subset V_0'$ and $V_0 \subset \ker H_2$, we have $\psi = \phi_{00}$ and $\eta = \phi_{11}$ on cohomology. Then the claim follows because the square below is commutative up to the homotopy[21] ϕ_{10}:

$$\begin{tikzcd}
(V_1, d_{11}) \arrow[r, "d_{10}"] \arrow[d, "\phi_{11}"] & (V_0, -d_{00}) \arrow[d, "\phi_{00}"] \\
(V_1', d_{11}') \arrow[r, "d_{10}'"] & (V_0', -d_{00}')
\end{tikzcd}$$

□

Proposition 3.41 *Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category and A a $\mathcal{C}$–subset. Then we have two flow morphisms $\mathfrak{I}_A\colon \mathcal{C}_A \Rightarrow \mathcal{C}$ and $\mathfrak{P}_A\colon \mathcal{C} \Rightarrow \mathcal{C}_{/A}$, which induces a short exact sequence $0 \to \mathrm{BC}(\mathcal{C}_A) \to \mathrm{BC}(\mathcal{C}) \to \mathrm{BC}(\mathcal{C}_{/A}) \to 0$. The induced long exact sequence is isomorphic to that of* (3-51) *if the defining data for $\mathcal{C}_A$ and $\mathcal{C}_{/A}$ are the restriction of defining data on $\mathcal{C}$.*

Proof $\mathfrak{I}_A$ is the identity flow morphism of $\mathcal{C}_A$ when the target lands in A, and the empty set otherwise. $\mathfrak{P}_A$ is the identity flow morphism of $\mathcal{C}_{/A}$ when the source lands outside A, and the empty set otherwise. Similar to the proof of Proposition 3.38, both $\mathfrak{I}_A$ and $\mathfrak{P}_A$ are oriented flow morphisms. Since the induced cochain morphism of $\mathfrak{I}_A$ maps $\mathrm{BC}(\mathcal{C}_A)$ isomorphically to the subspace of $\mathrm{BC}(\mathcal{C})$ generated by $H^*(C_i)$ for $i \in A$, and the induced cochain morphism of $\mathfrak{P}_A$ vanishes on the subspace of $\mathrm{BC}(\mathcal{C})$ generated by $H^*(C_i)$ for $i \in A$ and maps the subspace generated by $H^*(C_i)$ for $i \notin A$ isomorphically to $\mathrm{BC}(\mathcal{C}_{/A})$,

[21] See Remark 3.42 for the explanation of the sign, although it does not affect the map on cohomology.

then we have a short exact sequence as below. Moreover, we claim that we have the diagram of short exact sequences which is commutative up to homotopy

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{BC}(\mathcal{C}_A) \arrow[r, "\phi^{I_A}"] \arrow[d, "\mathrm{id}"] & \mathrm{BC}(\mathcal{C}) \arrow[r, "\phi^{P_A}"] \arrow[d, "\mathrm{id}"] & \mathrm{BC}(\mathcal{C}_{/A}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathrm{BC}(\mathcal{C}_A) \arrow[r, "i"] & \mathrm{BC}(\mathcal{C}) \arrow[r, "\pi"] & \mathrm{BC}(\mathcal{C}_{/A}) \arrow[r] & 0 \end{tikzcd}$$

where the second row is the tautological short sequence (3-51). This is equivalent to proving ϕ^{I_A} is homotopic to inclusion i, and ϕ^{P_A} is homotopic to the projection π. Note that $\phi^{I_A} = i + N$ with N a strict upper triangular matrix and $N = \phi^{I_A} - i = i \circ (\phi^{I_{\mathcal{C}_A}} - \mathrm{id})$. Similar to the proof of Theorem 3.33, we have that $\mathfrak{I}_A \circ \mathfrak{I}_{\mathcal{C}_A}$ and $\mathfrak{I}_A$ induce the same map. Hence $(i+N)\circ(\mathrm{id}+N)$ is homotopic to $i+N$ by Theorem 3.28, and so $i+N$ is homotopic to i if we multiply $(\mathrm{id}+N)^{-1}$ to the right of the homotopy relation. Similarly, ϕ^{P_A} is homotopic to the projection π. It is clear from Theorem 3.28 that those homotopies satisfy the conditions of Lemma 3.40, and hence the claim follows. □

Remark 3.42 The conclusion of Lemma 3.40 can be rephrased as saying that $V_0 \to V_0 \oplus V_1 \to V_1 \to V_0[1]$ and $V'_0 \to V'_0 \oplus V'_1 \to V'_1 \to V'_0[1]$ are equivalent distinguished triangles in $\mathcal{K}(\mathcal{C}h)$.[22] In view of Section 3.6, the minimal Morse–Bott cochain complex is only well defined in $\mathcal{K}(\mathcal{C}h)$. It is natural to expect that we only get well-defined distinguished triangles in $\mathcal{K}(\mathcal{C}h)$.

Definition 3.43 Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented flow categories, A a $\mathcal{C}$–subset and B a $\mathcal{D}$–subset. We say an oriented flow morphism $\mathfrak{H}$ maps A to B, if and only if $\mathcal{H}_{i,j} = \varnothing$ whenever $i \in A$ and $j \notin B$.

Proposition 3.44 *Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented flow categories, A a $\mathcal{C}$–subset and B a $\mathcal{D}$–subset. Assume an oriented flow morphism $\mathfrak{H}$ maps A to B. Then we have oriented flow morphisms $\mathfrak{H}_A : \mathcal{C}_A \Rightarrow \mathcal{D}_B$ and $\mathfrak{H}_{/A} : \mathcal{C}_{/A} \Rightarrow \mathcal{D}_{/B}$, and on the cochain level they induce a morphism between the long exact sequences.*

Proof The restriction of $\mathfrak{H}$ is $\mathfrak{H}_A$ when the source and target land in A and B, respectively. $\mathfrak{H}_{/A}$ is the restriction of $\mathfrak{H}$ when source and target land in complements of A and B respectively. Then $\mathfrak{H}_A$ and $\mathfrak{H}_{/A}$ are flow morphisms by a direct check similar to Proposition 3.38. We define $\mathfrak{F}$ to be the flow morphism from $\mathcal{C}_A$ to $\mathcal{D}$ which is the restriction of $\mathfrak{H}$ to $\mathcal{C}_A$. Since $\mathcal{H}_{i,j} = \varnothing$ whenever $i \in A$ and $j \notin B$, we have that $\mathfrak{H}$ must land in $\mathcal{D}_B$. Then by the same argument as in Theorem 3.33, $\mathfrak{H} \circ \mathfrak{I}_A$, $\mathfrak{I}_B \circ \mathfrak{H}_A$ and $\mathfrak{F}$ induce the same cochain morphism. Then Theorem 3.28 implies that both $\phi^H \circ \phi^{I_A}$ and $\phi^{I_B} \circ \phi^{H_A}$ are homotopic to $\phi^{\mathfrak{F}}$. Similarly, $\phi^{H_{/A}} \circ \phi^{P_A}$ and $\phi^{P_B} \circ \phi^H$ are homotopic. It is clear that the homotopies and ϕ^H satisfy the conditions in Lemma 3.40, and hence the claim follows. □

Remark 3.45 It is clear that the identity flow morphism maps A to A. Hence Proposition 3.44 implies that the long exact sequence from Proposition 3.41 is independent of the defining data and is isomorphic to the long exact sequence induced from (3-51).

[22] When (V, d) is ungraded, $V[1]$ simply means $(V, -d)$.

4 The action spectral sequence

Given a Morse–Bott function on a closed manifold M, there is a spectral sequence converging to $H^*(M)$ with the first page generated by the cohomology of critical manifolds (sometimes twisted by a local system). Such a spectral sequence is sometimes referred to as the Morse–Bott spectral sequence. For flow categories, Austin and Braam's construction [3] comes with a spectral sequence, which is induced by the an action filtration. Moreover, it was shown under the fibration condition that the spectral sequence from Austin and Braam's construction (from the first page) is isomorphic to the Morse–Bott spectral sequence. Similar spectral sequences from action filtration in Floer theory can be found in many places, eg [70]. Often the spectral sequence is an invariant of the Morse–Bott function, ie independent of other auxiliary structures. For example, in the finite-dimensional Morse–Bott theory, any reasonable construction should recover the Morse–Bott spectral sequence, which can be constructed using only the Morse–Bott function in a purely topological manner.

The goal of this section is to prove those results for the minimal Morse–Bott cochain complex. The existences of an "action" filtration is encoded in the definition of a flow category by requiring $\mathcal{M}_{i,j}=\varnothing$ for $i>j$, since we secretly order C_i by their critical values of the hypothetical Morse–Bott functional. For basics of spectral sequences arising from filtrations, we refer readers to [55; 75].

Letting $\mathcal{C}:=\{C_i,\mathcal{M}_{i,j}\}$ be an oriented flow category, we have the following "action" filtration on the minimal Morse–Bott cochain complex BC:

$$F_p\mathrm{BC}:=\prod_{i\geq p} H^*(C_i)\subset F_{p-1}\mathrm{BC}\subset \mathrm{BC}.$$

It is clear from definition that the differential $d_{\mathrm{BC},\Theta}$ is compatible with this filtration for any defining data Θ. The associated spectral sequence can be described explicitly as follows. We define Z^p_{k+1} to be the space of $\alpha_0\in H^*(C_p)$ such that there exist $\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\in H^*(C_*)$ with (we suppress the subscript Θ in $d_{i,\Theta}$ for simplicity)

$$\begin{aligned}
d_1\alpha_0&=0,\\
d_2\alpha_0+d_1\alpha_1&=0,\\
d_3\alpha_0+d_2\alpha_1+d_1\alpha 12&=0,\\
&\ \ \vdots\\
d_k\alpha_0+d_{k-1}\alpha_1+\cdots+d_1\alpha_{k-1}&=0.
\end{aligned}\tag{4-1}$$

We define B^p_{k+1} to be the space of $\alpha\in H^*(C_p)$ such that there exist $\alpha_0,\alpha_1,\ldots,\alpha_{k-1}\in H^*(C_*)$ with

$$\begin{aligned}
\alpha&=d_k\alpha_0+d_{k-1}\alpha_1+\cdots+d_1\alpha_{k-1},\\
0&=d_{k-1}\alpha_0+d_{k-2}\alpha_1+\cdots+d_1\alpha_{k-2},\\
&\ \ \vdots\\
0&=d_1\alpha_0.
\end{aligned}\tag{4-2}$$

On Z^p_{k+1}/B^p_{k+1}, there is a map $\partial_{k+1}\colon Z^p_{k+1}/B^p_{k+1} \to Z^{p+k+1}_{k+1}/B^{p+k+1}_{k+1}$ defined by $\partial_{k+1}\alpha_0 := d_{k+1}\alpha_0 + d_k\alpha_1 + \cdots + d_2\alpha_{k-1}$. Since the differential on the minimal Morse–Bott cochain complex has the special form $\prod d_i$, unwrapping Leray's theorem on the spectral sequence associated to a filtered complex, we have the following:

Proposition 4.1 [55] *Following the notation above,*

$$B^p_1 \subset B^p_2 \subset \cdots \subset B^p_k \subset \bigcup_k B^p_k = B^p_\infty \subset Z^p_\infty = \bigcap_k Z^p_k \subset \cdots \subset Z^p_k \subset \ldots \subset Z^p_2 \subset Z^p_1.$$

Additionally, ∂_k is a well-defined map from Z^p_k/B^p_k to Z^{p+k+1}_k/B^{p+k+1}_k such that $\partial_k^2 = 0$ and $Z^p_{k+1}/B^p_{k+1} \simeq H^p(Z_k/B_k, \partial_k)$. Here we view the superscript p as a grading and then ∂_k has grading $k+1$ on Z_k/B_k. Hence we have a spectral sequence $(E^p_k := Z^p_k/B^p_k, \partial_k)$ with

$$E^p_\infty := Z^p_\infty/B^p_\infty \simeq F_pH(\mathrm{BC}, d_{\mathrm{BC}})/F_{p+1}H(\mathrm{BC}, d_{\mathrm{BC}}),$$

where $F_pH(\mathrm{BC}, d_{\mathrm{BC}})$ is the associated filtration on the cohomology of $(\mathrm{BC}, d_{\mathrm{BC}})$. In other words, the spectral sequence (E^p_k, ∂_k) is the spectral sequence induced from the filtration $F_p\mathrm{BC}$.

Remark 4.2 Since we do not assume $\mathcal{C}$ carries a grading structure, we do not have a grading on BC (as well as its relation to the natural degree on $H^*(C_*)$) in general. In particular, we will not get a multicomplex in [4]. The cost is that we cannot further refine the spectral sequence in E^p_k using their degrees on $H^*(C_p)$.

The second page of the spectral sequence is computed by taking the cohomology with respect to $\partial_1 = d_1$ in (3-15). Since d_1 is computed using $\mathcal{M}_{*,*+1}$, which are manifolds without boundary, d_1 is simply the pullback and pushforward of cohomology. It is more accessible in good cases; works in this direction using cascades constructions can be found in [20; 21]. In general, even though d_i depends on defining data in general for $i \geq 2$, ∂_i does not for any i.

Proposition 4.3 *Every page of the spectral sequence is independent of the defining data.*

Proof The identity flow morphism $\mathfrak{I}$ induces a cochain map $\phi^I_{\Theta_1,\Theta_2}\colon (\mathrm{BC}, d_{\mathrm{BC},\Theta_1}) \to (\mathrm{BC}, d_{\mathrm{BC},\Theta_2})$. The cochain map ϕ^I preserves the filtrations, thus it induces a morphism between spectral sequences. Since the induced map on the zeroth page is the identity it induces isomorphisms on every page. □

Remark 4.4 Proposition 4.3 only asserts the invariance of the spectral sequence with respect to defining data for a fixed flow category. However, the spectral sequence is expected to be an invariant of the hypothetical Morse–Bott functional, ie independent of other choices (metrics, almost-complex structures, abstract perturbations) in the construction of the flow category. To prove this claim, one needs to study the underlying moduli problem and deploy some virtual techniques. We will touch on this aspect of the theory briefly in Section 9. The spectral sequence is also expected to be independent of the specific construction method. It is an interesting question to find applications of those invariants, particularly in the quantitative aspects of symplectic geometry like symplectic embedding problems.

The final page of the spectral sequence only recovers the associated graded of the cohomology with respect to the induced filtration. We define

$$E_\infty := \varprojlim_p \varinjlim_q \bigoplus_{i=q}^{p} E^i_\infty,$$

ie the direct sum at the negative end and the direct limit at the positive end of E^i_∞. Following [55, Proof of Lemma 3.10], we have the following exact sequence (note that we are using field coefficients):

$$0 \to \varprojlim_p F_p H(\mathrm{BC}, d_{\mathrm{BC}}) \to H(\mathrm{BC}, d_{\mathrm{BC}}) \to E_\infty \to \varprojlim_p{}^1 F_p H(\mathrm{BC}, d_{\mathrm{BC}}) \to 0.$$

In some good cases, like $F_p\mathrm{BC} = 0$ for $p \gg 0$, E_∞ is (noncanonically) isomorphic to the Morse–Bott cohomology. For example, the symplectic cohomology considered in [70] satisfies this condition, as the symplectic action is bounded from above.

5 Orientations and local systems

The aim of this section is explaining how orientation conventions in Definitions 2.15, 3.18 and 3.29 arise in applications. In applications like Morse or Floer theories, coherent orientations usually use extra structures from the moduli problem, namely the gluing theorem for the determinant line bundles of Fredholm sections; see [31]. Similar properties and constructions exist in Floer theories of different flavors beyond cohomology theory, eg [13; 34; 71]. In this section, we explain the structure which is necessary for the existence of coherent orientations on flow categories and how they arise in applications. Then we generalize the construction of the minimal Morse–Bott cochain complex to flow categories with local systems, where critical manifolds C_i can be nonorientable.

5.1 Orientations for flow categories

5.1.1 Orientations in the Morse case We first review how coherent orientations arise in the construction of Hamiltonian Floer cohomology in the nondegenerate (Morse) case following [1]. We will not just orient 0– and 1–dimensional moduli spaces but all of them, and show that they satisfy Definition 2.15. Assume a symplectic manifold (M, ω) is symplectically aspherical, that is, $\omega|_{\pi_2(M)} = 0$. Let $H_t : S^1 \times M \to \mathbb{R}$ be a Hamiltonian such that all contractible 1–periodic orbits of the Hamiltonian vector field X_{H_t} are nondegenerate. For simplicity, we assume that every moduli space of Floer cylinders is cut out transversely. We note here that the orientation problem is independent from many other aspects of the theory, and in particular, the transversality problem.[23] In other words, we have a flow category $\{x_i, \mathcal{M}_{i,j}\}$, where x_i is a nondegenerate contractible periodic orbit and $\mathcal{M}_{i,j}$ is the *compactified* moduli space of Floer cylinders from x_i to x_j, where the symplectic action of x_i is smaller than that of x_j if and only if $i < j$.

[23] In the nontransverse case, the discussion of the determinant line bundle below can be lifted to the underlying Banach manifolds/polyfolds. However, when transversality holds, there is a canonical isomorphism depending on the section/perturbation from the determinant bundle of the moduli space to $o_{i,j}$ that it is compatible with gluing, ie (4) and (5).

To orient $\mathcal{M}_{i,j}$ in a coherent way such that Definition 2.15 holds, we recall the following extra structures that can be associated to the moduli spaces $\mathcal{M}_{i,j}$ in the Hamiltonian Floer cohomology:

(1) For every periodic orbit x_i, we can assign an orientation line o_i with a $\mathbb{Z}/2$ grading. Such a line is constructed from the determinant line of a perturbed $\bar{\partial}$ operator over $\mathbb{C}$ with one positive end at infinity [1, (1.4.8)] and the grading is the index of the operator (modulo 2).

(2) For every point in $\mathcal{M}_{i,j}$, there is an orientation line with a $\mathbb{Z}/2$ grading coming from the determinant line bundle of the linearized Floer equation at that point. All these lines form a line bundle $o_{i,j}$ over $\mathcal{M}_{i,j}$. We refer readers to [80] for the topology on the determinant bundle.

(3) By the gluing theorem for linear Fredholm operators [1, Lemma 1.4.5], we have a grading-preserving isomorphism over $\mathcal{M}_{x,y}$:

$$\rho_{i,j}\colon s^*o_i \otimes o_{i,j} \to t^*o_j. \tag{5-1}$$

Over $\mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \subset \partial\mathcal{M}_{i,k}$, there is a grading-preserving isomorphism

$$\rho_{i,j,k}\colon \pi_1^*o_{i,j} \otimes \pi_2^*o_{j,k} \to o_{i,k},$$

where π_1 and π_2 are the two projections. Note that $\rho_{i,j}$ and $\rho_{i,j,k}$ are compatible in the sense that there is commutative diagram over $\mathcal{M}_{i,j} \times \mathcal{M}_{j,k}$ up to multiplying by a positive number:

$$\begin{tikzcd}
s^*o_i \otimes \pi_1^*o_{i,j} \otimes \pi_2^*o_{j,k} \arrow[r, "\rho_{i,j}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\rho_{i,j,k}"] & \pi_1^*t^*o_j \otimes \pi_2^*o_{j,k} \arrow[r, equal] & \pi_2^*s^*o_j \otimes \pi_2^*o_{j,k} \arrow[r, "\pi_2^*\rho_{j,k}"] & \pi_2^*t^*o_k \arrow[r, equal] & t^*o_k \arrow[d] \\
s^*o_i \otimes o_{i,k} \arrow[rrrr, "\rho_{i,k}"] & & & & t^*o_k
\end{tikzcd}$$

(4) Let $\bar{\partial}_{i,j}$ be the Floer operator cutting out $\mathcal{M}_{i,j}$. When transversality holds for every moduli space, $\ker \mathrm{D}\bar{\partial}_{i,j}$ is a vector bundle over $\mathcal{M}_{i,j}$. Then $\ker \mathrm{D}\bar{\partial}_{i,j}$ contains an oriented trivial line subbundle $\underline{\mathbb{R}}$ induced by the $\mathbb{R}$ translation action, and

$$\ker \mathrm{D}\bar{\partial}_{i,j} = T\mathcal{M}_{i,j} \oplus \underline{\mathbb{R}}. \tag{5-2}$$

Moreover, we have a grading-preserving isomorphism $\phi_{i,j}\colon o_{i,j} \to \det\ker \mathrm{D}\bar{\partial}_{i,j}$.

(5) On $\mathcal{M}_{i,j} \times \mathcal{M}_{j,k}$, we have an isomorphism $\ker \mathrm{D}\bar{\partial}_{i,j} \oplus \ker \mathrm{D}\bar{\partial}_{j,k} \xrightarrow{\phi} \ker \mathrm{D}\bar{\partial}_{i,k}$ and the following diagram commutes (we suppress the pullbacks):

$$\begin{tikzcd}
o_{i,j} \otimes o_{j,k} \arrow[r, "\rho_{i,j,k}"] \arrow[d, "\phi_{i,j}\otimes\phi_{j,k}"] & o_{i,k} \arrow[d, "\phi_{i,k}"] \\
\det\ker \mathrm{D}\bar{\partial}_{i,j} \otimes \det\ker \mathrm{D}\bar{\partial}_{j,k} \arrow[r, "\det\phi"] & \det\ker \mathrm{D}\bar{\partial}_{i,k}
\end{tikzcd}$$

(6) Let $\underline{\mathbb{R}}_r$, $\underline{\mathbb{R}}_s$ and $\underline{\mathbb{R}}_t$ be the trivial subbundles in $\ker \mathrm{D}\bar{\partial}_{i,j}$, $\ker \mathrm{D}\bar{\partial}_{j,k}$ and $\ker \mathrm{D}\bar{\partial}_{i,k}$, respectively. Then by [1, Lemma 1.5.7],

$$\phi(\langle r,s\rangle) = t \quad \text{and} \quad \phi(\langle -r,s\rangle) \text{ is pointing out along } \mathcal{M}_{i,j} \times \mathcal{M}_{j,k} \subset \partial\mathcal{M}_{i,k} \text{ in (5-2).} \tag{5-3}$$

Proposition 5.1 *If we fix an orientation for every o_i, then* (3) *and* (4) *determine an orientation of $\mathcal{M}_{i,j}$ and $[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}] = (-1)^{m_{i,j}+1}\partial[\mathcal{M}_{i,k}]|_{\mathcal{M}_{i,j}\times\mathcal{M}_{j,k}}$.*

Proof Given orientations of o_i, the isomorphism $\rho_{i,j}$ determines an orientation of $o_{i,j}$. Then by (4) and $\phi_{i,j}$, there is an induced orientation $[\mathcal{M}_{i,j}]$. We claim this orientation satisfies the claimed relation. By (3), $\rho_{i,j,k}$ preserves the orientations. Therefore $\phi\colon \ker \mathrm{D}\bar{\partial}_{i,j} \oplus \ker \mathrm{D}\bar{\partial}_{j,k} \to \ker \mathrm{D}\bar{\partial}_{i,k}$ preserves the orientations. That is, $[\mathcal{M}_{i,j}][\mathbb{R}_r][\mathcal{M}_{j,k}][\mathbb{R}_s] = [\mathcal{M}_{i,k}][\mathbb{R}_t]$. Then by (6), we have $[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}] = (-1)^{m_{i,j}+1}\partial[\mathcal{M}_{i,k}]|_{\mathcal{M}_{i,j}\times\mathcal{M}_{j,k}}$. □

Orientations from Proposition 5.1 can be used to prove $d^2 = 0$ for Hamiltonian Floer cohomology in the nondegenerate case. Moreover, orientations $-[\mathcal{M}_{i,j}]$ fit into the orientation convention in Definition 2.15.

5.1.2 Orientations in the Morse–Bott case We should expect similar structures and properties in Morse–Bott theories. We phrase the structures as a definition and explain how to get an oriented flow category from there. Before stating the definition, we introduce some notation:

(1) Let $E \to M$ be a vector bundle. Then $\det E := \bigwedge^{\max} E$ with $\mathbb{Z}/2$ grading $\operatorname{rank} E \pmod 2$. We write $\det C := \det TC$.

(2) For $\mathbb{Z}/2$ graded line bundles o_1 and o_2, unless stated otherwise the map $o_1 \otimes o_2 \to o_2 \otimes o_1$ is defined by

$$v_1 \otimes v_2 \to (-1)^{|o_1|\cdot|o_2|} v_2 \otimes v_1 \tag{5-4}$$

for vectors v_1 and v_2 in o_1 and o_2, respectively.

(3) Let Δ be the diagonal in $C \times C$ with normal bundle N. Unless stated otherwise, $\det \Delta \otimes \det N \to \det C \otimes \det C$ on Δ is the map induced by the isomorphism $T\Delta \oplus N \to TC \oplus TC$. In particular, if we orient N following Example 2.8, such a map preserves orientations.

Definition 5.2 An *orientation structure* on a flow category $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ consists of the following structures:

(1) There are topological line bundles o_i over C_i with $\mathbb{Z}/2$ gradings for every C_i, and topological line bundles $o_{i,j}$ over $\mathcal{M}_{i,j}$ with $\mathbb{Z}/2$ gradings for every $\mathcal{M}_{i,j}$.

(2) There is a grading-preserving bundle isomorphism over $\mathcal{M}_{i,j}$

$$\rho_{i,j}\colon s^* o_i \otimes s^* \det C_i \otimes o_{i,j} \to t^* o_j, \tag{5-5}$$

and a grading-preserving bundle isomorphism over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \subset \partial\mathcal{M}_{i,k}$

$$\rho_{i,j,k}\colon \pi_1^* o_{i,j} \otimes (t \times s)^* \det T\Delta_j \otimes \pi_2^* o_{j,k} \to o_{i,k}|_{\mathcal{M}_{i,j}\times_j\mathcal{M}_{j,k}}. \tag{5-6}$$

The bundle isomorphisms are compatible in the sense that the following diagram over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ is commutative up to multiplying by a positive number:

$$\begin{tikzcd}
s^* o_i \otimes s^* \det C_i \otimes \pi_1^* o_{i,j} \otimes (t \times s)^* \det \Delta_j \otimes \pi_2^* o_{j,k} \arrow[r, "\mathrm{id} \otimes \rho_{i,j,k}"] \arrow[d, "\rho_{i,j} \otimes \mathrm{id}"] & s^* o_i \otimes s^* \det C_i \otimes o_{i,k} \arrow[dd, "\rho_{i,k}"] \\
\pi_2^* s^* o_j \otimes \pi_2^* s^* \det C_j \otimes \pi_2^* o_{j,k} \arrow[d, "\pi_2^* \rho_{j,k}"] & \\
t^* o_k \arrow[r] & t^* o_k
\end{tikzcd} \tag{5-7}$$

The diagram makes sense because over the fiber product $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$, we have $\pi_1^* t^* o_j = \pi_2^* s^* o_j$ and $(t \times s)^* \det \Delta_j = \pi_2^* s^* \det C_j$.

(3) There are vector bundles $V_{i,j}$ over $\mathcal{M}_{i,j}$ with smooth bundle maps

$$S_{i,j} \colon V_{i,j} \to TC_i \quad \text{and} \quad T_{i,j} \colon V_{i,j} \to TC_j$$

covering $s_{i,j} \colon \mathcal{M}_{i,j} \to C_i$ and $t_{i,j} \colon \mathcal{M}_{i,j} \to C_j$, respectively. Moreover, there is an oriented trivial subbundle $\underline{\mathbb{R}}$ of $V_{i,j}$ such that $S_{i,j}(\underline{\mathbb{R}}) = T_{i,j}(\underline{\mathbb{R}}) = 0$,

$$V_{i,j} = T\mathcal{M}_{i,j} \oplus \underline{\mathbb{R}}, \tag{5-8}$$

$S_{i,j}|_{T\mathcal{M}_{i,j}} = \mathrm{d}s_{i,j}$ and $T_{i,j}|_{T\mathcal{M}_{i,j}} = \mathrm{d}t_{i,j}$. There is a grading-preserving isomorphism

$$\phi_{i,j} \colon s^* \det C_i \otimes o_{i,j} \otimes t^* \det C_j \to \det V_{i,j}. \tag{5-9}$$

(4) On $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ we have $V_{i,j} \times_{TC_j} V_{j,k} = V_{i,k}$, and the following diagram commutes, where the last map is induced by the isomorphism $V_{i,j} \oplus V_{j,k} = (t \times s)^* N_j \oplus V_{i,k}$:

$$\begin{tikzcd}
(t\times s)^* \det N_j \otimes s^* \det C_i \otimes o_{i,j} \otimes (t\times s)^* \det \Delta_j \otimes o_{j,k} \otimes t^* \det C_k \arrow[r, "\rho_{i,j,k}"] \arrow[d] & (t\times s)^* \det N_j \otimes s^* \det C_i \otimes o_{i,k} \otimes t^* \det C_k \arrow[ddd, "\phi_{i,k}"] \\
s^* \det C_i \otimes o_{i,j} \otimes (t\times s)^* (\det \Delta_j \otimes \det N_j) \otimes o_{j,k} \otimes t^* \det C_k \arrow[d] & \\
s^* \det C_i \otimes o_{i,j} \otimes t^* \det C_j \otimes s^* \det C_j \otimes o_{j,k} \otimes t^* \det C_k \arrow[d, "\phi_{i,j} \otimes \phi_{j,k}"] & \\
\det V_{i,j} \otimes \det V_{j,k} \arrow[r] & (t\times s)^* \det N_j \otimes \det V_{i,k}
\end{tikzcd}$$

(5) Let $\underline{\mathbb{R}}_r$, $\underline{\mathbb{R}}_s$ and $\underline{\mathbb{R}}_t$ be the trivial subbundles in $V_{i,j}$, $V_{j,k}$ and $V_{i,j}$, respectively. We have

$$\langle r, s \rangle = t \quad \text{and} \quad \langle -r, s \rangle \text{ is pointing out along } \mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \subset \partial \mathcal{M}_{i,k}. \tag{5-10}$$

In applications, the topological line bundle o_i is the determinant line bundle of a perturbed Floer equation with exponential decay at the end over a domain with one positive end. For details on exponential decay, we refer readers to [12; 32]. The topological line bundle $o_{i,j}$ usually comes from the determinant bundle of the Floer equation with exponential decay at both ends over a cylinder. The bundle isomorphism and its compatible diagram come from a version of the linear gluing theorem for Fredholm operators [1; 31]. $V_{i,j}$ is the kernel of the linearized Floer operator defining $\mathcal{M}_{i,j}$ and the trivial subbundle comes from the $\mathbb{R}$ translation. The last condition (5) comes from a similar argument as in [1, Lemma 1.5.7]. The bundle $o_{i,j}$ can be defined on the background Banach manifold or polyfolds [44, Chapter 6], however $V_{i,j}$ is defined only when transversality holds. Definition 5.2(3) states the relation between $V_{i,j}$, $o_{i,j}$ and $T\mathcal{M}_{i,j}$, and (4) states the compatibility with the gluing map $\rho_{i,j,k}$.

Remark 5.3 Similar to Definition 2.13, Definition 5.2 is a simplified version. In general, we should associate each component of C_i with a line bundle and each component of $\mathcal{M}_{i,j}$ with a bundle isomorphism satisfying similar compatibility conditions.

Remark 5.4 Definition 5.2 is modeled on the classical treatment of the Floer equation [12; 32]. That is, we mod out the $\mathbb{R}$ translation after solving the Floer equation. Hence we expect that bundles $V_{i,j}$ over $\mathcal{M}_{i,j}$ contain a trivial oriented $\mathbb{R}$ direction. If we use the polyfold setup, then the Floer operator is defined on polyfolds of cylinders with the $\mathbb{R}$ translation already quotiented out; see [26; 73]. One can adjust Definition 5.2 to be consistent with such a point of view.

Proposition 5.5 *Assume the flow category $\mathcal{C}$ has an orientation structure, all the line bundles o_i are oriented and all C_i are oriented. Then $\mathcal{C}$ can be coherently oriented.*

Proof By the map $\rho_{i,j}$ in (5-5), if the o_i and C_i are oriented, then there are induced orientations $[o_{i,j}]$ on $o_{i,j}$. By (5-7), over the fiber product $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ we have

$$\rho_{i,j,k}(\pi_1^*[o_{i,j}] \otimes (t \times s)^*[\Delta_j] \otimes \pi_2^*[o_{j,k}]) = [o_{i,k}]. \tag{5-11}$$

Using $\phi_{i,j}$ in Definition 5.2(4), we have an orientation $[V_{i,j}]$ on $V_{i,j}$. Then by (5-11), the commutative diagram in Definition 5.2(4) implies that the natural map $V_{i,j} \oplus V_{j,k} \to (t \times s)^* N_j \times V_{i,k}$ induces

$$[V_{i,j}] \otimes [V_{j,k}] \mapsto (-1)^{c_j(m_{i,j}+1)} (t \times s)^*[N_j] \otimes [V_{i,k}]$$

on the prescribed orientations. By Definition 5.2(3), the orientation $[V_{i,j}]$ induces an orientation $[\mathcal{M}_{i,j}]$. Hence on $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \subset \partial \mathcal{M}_{i,k}$,

$$[\mathcal{M}_{i,j}][\mathbb{R}_r][\mathcal{M}_{j,k}][\mathbb{R}_s] = (-1)^{c_j(m_{i,j}+1)} (t \times s)^*[N_j][\mathcal{M}_{i,k}][\mathbb{R}_t].$$

Then Definition 5.2(5) implies that

$$[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}] = (-1)^{c_j m_{i,j} + m_{i,j} + 1} (t \times s)^*[N_j] \partial [\mathcal{M}_{i,k}]|_{\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}}.$$

Then the orientations $-[\mathcal{M}_{i,j}]$ satisfy Definition 2.15.[24] □

[24]One can certainly modify the definition of coherent orientations of a flow category (Definition 2.15) so that $[\mathcal{M}_{i,j}]$ gives a coherent orientation. Then the signs in (3-15) do not factorize nicely.

When the o_i are not oriented or the C_i are not oriented, Definition 5.2 gives all the structures we need to work with the local system o_i. We discuss such generalization in Section 5.2.

5.1.3 Orientations for flow morphisms We explain how the orientation convention in Definition 3.18 arise in application.

Definition 5.6 Assume $\mathfrak{H} = \{\mathcal{H}_{i,j}\}$ is a flow morphism from flow category $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ have orientation structures. A compatible orientation structure on $\mathfrak{H}$ is the following:

(1) There are $\mathbb{Z}/2$ graded line bundles $o^{H}_{i,j}$ over $\mathcal{H}_{i,j}$. Over $\mathcal{H}_{i,j}$, we have a grading-preserving isomorphism

$$\rho^{H}_{i,j} : s^* o^{C}_i \otimes s^* \det C_i \otimes o^{H}_{i,j} \to t^* o^{D}_j . \tag{5-12}$$

(2) Over the fiber product $\mathcal{M}^{C}_{i,j} \times_j \mathcal{H}_{j,k} \subset \partial \mathcal{H}_{i,k}$, we have a grading-preserving isomorphism

$$\rho^{C,H}_{i,j,k} : \pi_1^* o^{C}_{i,j} \otimes (t \times s)^* \det \Delta^{C}_j \otimes \pi_2^* o^{H}_{j,k} \to o^{H}_{i,k}. \tag{5-13}$$

Over the fiber product $\mathcal{H}_{i,j} \times_j \mathcal{M}^{D}_{j,k} \subset \partial \mathcal{H}_{i,k}$, we have a grading-preserving isomorphism

$$\rho^{H,D}_{i,j,k} : \pi_1^* o^{H}_{i,j} \otimes (t \times s)^* \det \Delta^{D}_j \otimes \pi_2^* o^{D}_{j,k} \to o^{H}_{i,k}. \tag{5-14}$$

(3) The bundle isomorphisms in (1) and (2) are compatible in the sense that over $\mathcal{M}^{C}_{i,j} \times_j \mathcal{H}_{j,k}$ and $\mathcal{H}_{i,j} \times_j \mathcal{M}^{D}_{j,k}$, we have the commutative diagrams

$$\begin{tikzcd}
s^* o^{C}_i \otimes s^* \det C_i \otimes \pi_1^* o^{C}_{i,j} \otimes (t \times s)^* \det \Delta^{C}_j \otimes \pi_2^* o^{H}_{j,k} \arrow[r, "\mathrm{id} \otimes \rho^{C,H}_{i,j,k}"] \arrow[d, "\rho^{C}_{i,j} \otimes \mathrm{id}"] & s^* o^{C}_i \otimes s^* \det C_i \otimes o^{H}_{i,k} \arrow[dd, "\rho^{H}_{i,k}"] \\
\pi_2^* s^* o^{D}_j \otimes \pi_2^* s^* \det D_j \otimes \pi_2^* o^{D}_{j,k} \arrow[d, "\rho^{H}_{j,k}"] & \\
t^* o_k \arrow[r] & t^* o_k
\end{tikzcd} \tag{5-15}$$

and

$$\begin{tikzcd}
s^* o^{C}_i \otimes s^* \det C_i \otimes \pi_1^* o^{H}_{i,j} \otimes (t \times s)^* \det \Delta^{D}_j \otimes \pi_2^* o^{D}_{j,k} \arrow[r, "\mathrm{id} \otimes \rho^{H,D}_{i,j,k}"] \arrow[d, "\rho^{H}_{i,j} \otimes \mathrm{id}"] & s^* o^{C}_i \otimes s^* \det C_i \otimes o^{H}_{i,k} \arrow[dd, "\rho^{H}_{i,k}"] \\
\pi_2^* s^* o^{D}_j \otimes \pi_2^* s^* \det D_j \otimes \pi_2^* o^{D}_{j,k} \arrow[d, "\rho^{D}_{j,k}"] & \\
t^* o_k \arrow[r] & t^* o_k
\end{tikzcd} \tag{5-16}$$

respectively.

(4) There is a grading-preserving isomorphism $\phi^{H}_{i,j} : s^* \det C_i \otimes o^{H}_{i,j} \otimes t^* \det D_j \to \det T\mathcal{H}_{i,j}$.

(5) On $\mathcal{M}^{C}_{i,j} \times_j \mathcal{H}_{j,k} \subset \partial\mathcal{H}_{i,k}$ we have $V^{C}_{i,j} \times_{TC_j} T\mathcal{H}_{j,k} = T\mathcal{H}_{i,k}$, and the following diagram commutes, where the last row is induced by the isomorphism $V^{C}_{i,j} \oplus T\mathcal{H}_{j,k} \to (t\times s)^* N^{C}_j \oplus T\mathcal{H}_{i,k}$:

$$\begin{tikzcd}
(t\times s)\det N^{C}_j \otimes s^*\det C_i \otimes o^{C}_{i,j} \otimes (t\times s)^*\det\Delta^{C}_j \otimes o^{H}_{j,k} \otimes t^*\det D_k \arrow[r, "\rho^{C,H}_{i,j,k}"] \arrow[d] & (t\times s)^*\det N^{C}_j \otimes s^*\det C_i \otimes o^{H}_{i,k} \otimes t^*\det D_k \arrow[ddd, "\phi^{H}_{i,k}"] \\
s^*\det C_i \otimes o^{C}_{i,j} \otimes (t\times s)^*(\det\Delta^{C}_j \otimes N^{C}_j) \otimes o^{H}_{j,k} \otimes t^*\det D_k \arrow[d] & \\
s^*\det C_i \otimes o^{C}_{i,j} \otimes t^*\det C_j \otimes s^*\det C_j \otimes o^{H}_{j,k} \otimes t^*\det D_k \arrow[d, "\phi^{C}_{i,j}\otimes\phi^{H}_{j,k}"] & \\
\det V^{C}_{i,j} \otimes \det T\mathcal{H}_{j,k} \arrow[r] & (t\times s)^*\det N^{C}_j \otimes \det T\mathcal{H}_{i,k}
\end{tikzcd}$$

On $\mathcal{H}_{i,j} \times_j \mathcal{M}^{D}_{j,k} \subset \partial\mathcal{H}_{i,k}$, we have $T\mathcal{H}_{i,j} \times_{TD_j} V^{D}_{j,k} = T\mathcal{H}_{i,k}$, and the following diagram commutes, where the last row is induced by the isomorphism $T\mathcal{H}_{i,j} \oplus V^{D}_{j,k} \to (t\times s)^* N^{D}_j \oplus T\mathcal{H}_{i,k}$:

$$\begin{tikzcd}
(t\times s)\det N^{D}_j \otimes s^*\det C_i \otimes o^{H}_{i,j} \otimes (t\times s)^*\det\Delta^{D}_j \otimes o^{D}_{j,k} \otimes t^*\det D_k \arrow[r, "\rho^{H,D}_{i,j,k}"] \arrow[d] & (t\times s)^*\det N^{D}_j \otimes s^*\det C_i \otimes o^{H}_{i,k} \otimes t^*\det D_k \arrow[ddd, "\phi^{H}_{i,k}"] \\
s^*\det C_i \otimes o^{H}_{i,j} \otimes (t\times s)^*(\det\Delta^{D}_j \otimes N^{D}_j) \otimes o^{D}_{j,k} \otimes t^*\det D_k \arrow[d] & \\
s^*\det C_i \otimes o^{H}_{i,j} \otimes t^*\det D_j \otimes s^*\det D_j \otimes o^{D}_{j,k} \otimes t^*\det D_k \arrow[d, "\phi^{H}_{i,j}\otimes\phi^{D}_{j,k}"] & \\
\det T\mathcal{H}_{i,j} \otimes \det V^{D}_{j,k} \arrow[r] & (t\times s)^*\det N^{D}_j \otimes \det T\mathcal{H}_{i,k}
\end{tikzcd}$$

(6) Let $\underline{\mathbb{R}}_s$ and $\underline{\mathbb{R}}_t$ be the trivial lines in $V^{C}_{i,j}$ and $V^{D}_{j,k}$, respectively. Then s points in along $\mathcal{M}^{C}_{i,j} \times_j \mathcal{H}_{j,k} \subset \partial\mathcal{H}_{i,k}$ and t points out along $\mathcal{H}_{i,j} \times_j \mathcal{M}^{D}_{j,k} \subset \partial\mathcal{H}_{i,k}$.

In the example of Hamiltonian Floer cohomology for nondegenerate Hamiltonians, the bundle $o^{H}_{i,j}$ is the determinant line bundle of the time-dependent Floer equation [2, page 384]. In the Morse–Bott case, $o^{H}_{i,j}$ is the determinant line bundle of the time-dependent Floer equation with exponential decay at both ends. By the same argument as in Proposition 5.5, we have the following:

Proposition 5.7 *Let $\mathcal{C}$ and $\mathcal{D}$ be two flow categories with orientation structures and $\mathfrak{H}$ be a flow morphism from $\mathcal{C}$ to $\mathcal{D}$ with a compatible orientation structure. Assume o^{C}_i, o^{D}_i, C_i and D_i are oriented, and $\mathcal{C}$ and $\mathcal{D}$ are oriented using Proposition 5.5. Then Definition 5.6*(1) *and* (4) *determine orientations on $\mathcal{H}_{i,j}$ such that $\mathfrak{H}$ is an oriented flow morphism from $\mathcal{C}$ to $\mathcal{D}$.*

Remark 5.8 A compatible orientation structure on a flow premorphism is Definition 5.6(1) and (4), and hence we have enough structures to orient the spaces in a flow premorphism when o_i^C, o_j^D, C_i and D_i are oriented. The composition $\mathfrak{F}\circ\mathfrak{H}$ of two composable flow morphisms $\mathfrak{F}$ and $\mathfrak{H}$ with compatible orientation structures has a natural compatible orientation structure, where

$$o_{i,j}^{F\circ H}|_{\mathcal{H}_{i,j}\times_j\mathcal{F}_{j,k}} = \pi_1^* o_{i,j}^H \otimes (t_{i,j}^H \times s_{j,k}^F)^* \det \Delta_j^D \otimes \pi_2^* o_{j,k}^F.$$

5.1.4 Orientations for flow homotopies In applications, a flow homotopy from $\mathfrak{H}$ to $\mathfrak{F}$ usually comes from considering a time-dependent Floer equation with an extra $[0,1]_z$ parameter [2, page 414] such that when $z=0$ the equation defines the flow morphism $\mathfrak{H}$, and when $z=1$ the equation defines the flow morphism $\mathfrak{F}$. Hence we have the following definition:

Definition 5.9 Let $\mathfrak{H}$ and $\mathfrak{F}$ be two flow premorphisms with orientation structures from $\mathcal{C}$ to $\mathcal{D}$ whose orientation structures are compatible with those of $\mathcal{C}$ and $\mathcal{D}$. A flow homotopy $\mathscr{Y}$ between $\mathfrak{H}$ and $\mathfrak{F}$ is said to have a compatible orientation structure if:

(1) There are $\mathbb{Z}/2$ graded line bundles $o_{i,j}^Y$ over $\mathcal{Y}_{i,j}$. Over $\mathcal{Y}_{i,j}$ there is a grading-preserving isomorphism

(5-17) $$\rho_{i,j}^Y : s^* o_i^C \otimes s^* \det C_i \otimes o_{i,j}^Y \to t^* o_j^D.$$

(2) Over the fiber product $\mathcal{M}_{i,j}^C \times_j \mathcal{Y}_{j,k} \subset \mathcal{Y}_{i,k}$, we have a grading-preserving isomorphism

(5-18) $$\rho_{i,j,k}^{C,Y} : \pi_1^* o_{i,j}^C \otimes (t\times s)^* \det \Delta_j^C \otimes \pi_2^* o_{j,k}^Y \to o_{i,k}^Y.$$

Over the fiber product $\mathcal{Y}_{i,j}\times_j \mathcal{M}_{j,k}^D \subset \partial\mathcal{Y}_{i,k}$, we have a grading-preserving isomorphism

(5-19) $$\rho_{i,j,k}^{Y,D} : \pi_1^* o_{i,j}^Y \otimes (t\times s)^* \det \Delta_j^D \otimes \pi_2^* o_{j,k}^D \to o_{i,k}^Y.$$

(3) $\rho_{i,j}^Y$, $\rho_{i,j,k}^{C,Y}$ and $\rho_{i,j,k}^{Y,D}$ are compatible so that commutative diagrams similar to Definition 5.6(3) hold.

(4) On $\mathcal{H}_{i,j}\subset\partial\mathcal{Y}_{i,j}$ we have $o_{i,j}^Y|_{\mathcal{H}_{i,j}} = o_{i,j}^H$ and $\rho_{i,j}^Y|_{\mathcal{H}_{i,j}} = \rho_{i,j}^H$; similarly for $\mathcal{F}_{i,j}\subset\partial\mathcal{Y}_{i,j}$.

(5) $T\mathcal{Y}_{i,j}|_{\mathcal{H}_{i,j}} = \underline{\mathbb{R}}_z \oplus T\mathcal{H}_{i,j}$ with z pointing in along the boundary and $T\mathcal{Y}_{i,j}|_{\mathcal{F}_{i,j}} = \underline{\mathbb{R}}_z \oplus T\mathcal{F}_{i,j}$ with z pointing out along the boundary. And there is a $\mathbb{Z}/2$–bundle isomorphism

$$\phi_{i,j}^Y : \underline{\mathbb{R}}_z \otimes s^* \det C_i \otimes o_{i,j}^Y \otimes t^* \det D_j \to \det T\mathcal{Y}_{i,j}$$

such that $\phi_{i,j}^Y|_{\mathcal{H}_{i,j}} = \mathrm{id}_{\underline{\mathbb{R}}_z} \otimes \phi_{i,j}^H$ and $\phi_{i,j}^Y|_{\mathcal{F}_{i,j}} = \mathrm{id}_{\underline{\mathbb{R}}_z} \otimes \phi_{i,j}^F$.

(6) On $\mathcal{M}_{i,j}^C \times_j \mathcal{Y}_{j,k} \subset \partial\mathcal{Y}_{i,k}$ we have $V_{i,j}^C \times_{TC_j} T\mathcal{Y}_{j,k} = T\mathcal{Y}_{i,k}$, and the following diagram (we suppress the pullback notation) commutes, where the last row is induced by the isomorphism $V_{i,j}^C \oplus T\mathcal{Y}_{j,k} \to$

$(t\times s)^*N_j^C\oplus T\mathcal{Y}_{i,k}$:

$$\begin{tikzcd}
\underline{\mathbb{R}}_z\otimes\det N_j^C\otimes\det C_i\otimes o_{i,j}^C\otimes\det\Delta_j^C\otimes o_{j,k}^Y\otimes\det D_k \arrow[r, "\rho^{C,Y}_{i,j,k}"] \arrow[d] & \underline{\mathbb{R}}_z\otimes\det N_j^C\otimes\det C_i\otimes o_{i,k}^Y\otimes\det D_k \arrow[ddd, "\phi^Y_{i,k}"] \\
\underline{\mathbb{R}}_z\otimes\det C_i\otimes o_{i,j}^C\otimes\det\Delta_j^C\otimes N_j^C\otimes o_{j,k}^Y\otimes\det D_k \arrow[d] & \\
\det C_i\otimes o_{i,j}^C\otimes\det C_j\otimes\underline{\mathbb{R}}_z\otimes\det C_j\otimes o_{j,k}^Y\otimes\det D_k \arrow[d, "\phi^C_{i,j}\otimes\phi^Y_{j,k}"] & \\
\det V_{i,j}^C\otimes\det T\mathcal{Y}_{j,k} \arrow[r] & (t\times s)^*\det N_j^C\otimes\det T\mathcal{Y}_{i,k}
\end{tikzcd}$$

On $\mathcal{Y}_{i,j}\times_j\mathcal{M}_{j,k}^D\subset\partial\mathcal{Y}_{i,k}$ we have $T\mathcal{Y}_{i,j}\times_{TD_j}V_{j,k}^D=T\mathcal{Y}_{i,k}$, and the following diagram commutes, where the last row is induced by the isomorphism $T\mathcal{Y}_{i,j}\oplus V_{j,k}^D\to(t\times s)^*N_j^D\oplus T\mathcal{Y}_{i,k}$ twisted by $(-1)^{d_j}$ (because of the extra $\mathbb{R}_z$):

$$\begin{tikzcd}
\underline{\mathbb{R}}_z\otimes\det N_j^D\otimes\det C_i\otimes o_{i,j}^Y\otimes\det\Delta_j^D\otimes o_{j,k}^D\otimes\det D_k \arrow[r, "\rho^{Y,D}_{i,j,k}"] \arrow[d] & \det N_j^D\otimes\det C_i\otimes o_{i,k}^Y\otimes\det D_k \arrow[ddd, "\phi^Y_{i,k}"] \\
\underline{\mathbb{R}}_z\otimes\det C_i\otimes o_{i,j}^Y\otimes\det\Delta_j^D\otimes N_j^D\otimes o_{j,k}^D\otimes\det D_k \arrow[d] & \\
\det C_i\otimes o_{i,j}^Y\otimes\det D_j\otimes\underline{\mathbb{R}}_z\otimes\det D_j\otimes o_{j,k}^D\otimes\det D_k \arrow[d, "\phi^Y_{i,j}\otimes\phi^D_{j,k}"] & \\
\det T\mathcal{Y}_{i,j}\otimes\det V_{j,k}^D \arrow[r, "(-1)^{d_j}"] & \det N_j^D\otimes\det T\mathcal{Y}_{i,k}
\end{tikzcd}$$

(7) Let $\underline{\mathbb{R}}_s$ and $\underline{\mathbb{R}}_t$ be the trivial lines in $V_{i,j}^C$ and $V_{j,k}^D$, respectively. Then s points in along $\mathcal{M}_{i,j}^C\times\mathcal{Y}_{j,k}\subset\partial\mathcal{Y}_{i,k}$ and t points out along $\mathcal{Y}_{i,j}\times\mathcal{M}_{j,k}^D\subset\partial\mathcal{Y}_{i,k}$.

If we can fix orientations of o_i^C, o_i^D, C_i and D_i, then (1), (4) and (5) imply that the induced orientations of $\mathcal{Y}_{i,j}$, $\mathcal{H}_{i,j}$ and $\mathcal{F}_{i,j}$ satisfy

$$\partial[\mathcal{Y}_{i,j}|_{\mathcal{H}_{i,j}}]=-[\mathcal{H}_{i,j}]\quad\text{and}\quad\partial[\mathcal{Y}_{i,j}|_{\mathcal{F}_{i,j}}]=[\mathcal{F}_{i,j}].$$

In general, we have the analog of Proposition 5.5 and 5.7:

Proposition 5.10 *Let $\mathscr{Y}$ be a flow homotopy between two flow premorphisms $\mathfrak{H}$ and $\mathfrak{F}$ from $\mathcal{C}$ to $\mathcal{D}$. Assume everything is equipped with compatible orientation structures, and o_i^C, o_i^D, C_i and D_i are oriented. If $\mathcal{C}$, $\mathcal{D}$, $\mathfrak{H}$ and $\mathfrak{F}$ are oriented by Propositions 5.5 and 5.7, then $\mathcal{Y}_{i,j}$ can be oriented by Definition 5.9*(1) *and* (5) *so that $\mathscr{Y}$ is an oriented flow homotopy between $\mathfrak{H}$ and $\mathfrak{F}$.*

5.2 Local systems

From the discussion in Section 5.1, to orient a flow category, a flow morphism or a flow homotopy with orientation structures, we need to orient o_i and C_i. However, in the Morse–Bott case, it is possible that C_i is not orientable or o_i is not orientable. Hence we need to upgrade the minimal Morse–Bott cochain complex to a version with local systems. In fact, Definitions 5.2, 5.6 and 5.9 already provide all the structures needed to define a cochain complex without any orientable assumptions; the generator will be the cohomology of C_i twisted by o_i. In the case of finite-dimensional Morse–Bott theory, let C be a critical manifold with stable bundle S. Then in view of the Thom isomorphism, the contribution from a critical manifold C to the total cohomology should be the cohomology with local system $H^*(C, \det S)$. In the abstract setting, if a flow category has an orientation structure, then the line bundle o_i plays the role of $\det S$.

We will introduce a more compact definition, just like Definition 2.15. First we introduce some notation. Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be a flow category. Over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \subset \partial\mathcal{M}_{i,k}$, we have an induced isomorphism $T\mathcal{M}_{i,j} \oplus T\mathcal{M}_{j,k} \to (t\times s)^* N_j \oplus T\partial\mathcal{M}_{i,k}$. If we use the identification $t^*TC_j \to (t\times s)^*N_j$ given by $v \mapsto (-v, v)$, we have an isomorphism $T\mathcal{M}_{i,j} \oplus T\mathcal{M}_{j,k} \to t^*C_j \oplus T\partial\mathcal{M}_{i,k}$. Therefore we have an isomorphism over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$:

$$\det\mathcal{M}_{i,j} \otimes \det\mathcal{M}_{j,k} \to t^*\det C_j \otimes \det\partial\mathcal{M}_{ik}.$$

Using the isomorphism $\underline{\mathbb{R}}_{\mathrm{out}} \oplus T\partial\mathcal{M}_{i,k} = T\mathcal{M}_{i,k}$, there is a natural isomorphism $\det\partial\mathcal{M}_{i,k} \to \det\mathcal{M}_{i,k}$ preserving compatible orientations. Hence we have an isomorphism of degree 1

$$\det\mathcal{M}_{i,j} \otimes \det\mathcal{M}_{j,k} \to t^*\det C_j \otimes \det\mathcal{M}_{ik},$$

which induces an isomorphism

$$f\colon \det\mathcal{M}_{i,j} \otimes t^*\det^* C_j \otimes \det\mathcal{M}_{j,k} \to \det\mathcal{M}_{j,k}, \tag{5-20}$$

where $\det^* C_j = (\det C_j)^*$. Here f is induced by the natural isomorphism $t^*\det C_j \otimes t^*\det^* C_j = \mathbb{R}$ and the order-switch convention (5-4).

Definition 5.11 Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be a flow category. Then a *local system* on $\mathcal{C}$ consists of the following:

(1) There is a line bundle o_i on each C_i.

(2) Over the $\mathcal{M}_{i,j}$, there is a bundle isomorphism

$$\rho_{i,j}\colon s^*o_i \otimes \det\mathcal{M}_{i,j} \otimes t^*\det^* C_j \to t^*o_j$$

such that the following diagram over $\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k} \subset \partial\mathcal{M}_{i,k}$ commutes, where f is defined in (5-20):

$$\begin{tikzcd}
s^*o_i \otimes \det\mathcal{M}_{i,j} \otimes t^*\det^* C_j \otimes \det\mathcal{M}_{j,k} \otimes t^*\det^* C_k \arrow[r, "\rho_{i,j}"] \arrow[d, "f"] & s^*o_j \otimes \det\mathcal{M}_{j,k} \otimes t^*\det^* C_k \arrow[r, "\rho_{j,k}"] & t^*o_k \arrow[d] \\
s^*o_i \otimes \det\mathcal{M}_{i,k} \otimes t^*\det^* C_k \arrow[rr, "(-1)^{m_{i,j}+1}\rho_{i,k}"] & & t^*o_k
\end{tikzcd}$$

Proposition 5.12 *If $\mathcal{C}$ has an orientation structure, then o_i is a local system on $\mathcal{C}$.*

Proof Since $\mathcal{C}$ has an orientation structure, ie we have isomorphisms $\rho^C_{i,j} : s^*o_i \otimes s^* \det C_i \otimes o_{i,j} \to t^*o_j$, $V_{i,j} = T\mathcal{M}_{i,j} \oplus \underline{\mathbb{R}}$ and $\phi_{i,j} : s^*o_i \otimes o_{i,j} \otimes t^*o_j \to \det V_{i,j}$, using the natural orientation on $\underline{\mathbb{R}}$ and isomorphisms $\phi_{i,j}$ and $\rho^C_{i,j}$ we get an isomorphism $\rho_{i,j} : s^*o_i \otimes \det \mathcal{M}_{i,j} \otimes t^* \det^* C_j \to t^*o_j$. Similarly to Proposition 5.5, Definition 5.2(4) and (5) imply the commutative diagram in Definition 5.11. $\square$

Similarly, we have the following definitions of local systems on flow morphism and flow homotopies:

Definition 5.13 Let $\mathfrak{H} = \{\mathcal{H}_{i,j}\}$ be a flow morphism from the flow category $\mathcal{C}$ to the flow category $\mathcal{D}$. Both $\mathcal{C}$ and $\mathcal{D}$ are equipped with local systems. We say $\mathfrak{H}$ has a *compatible local system* if, on each $\mathcal{H}_{i,j}$, we have an isomorphism

$$\rho^H_{i,j} : s^*o^C_i \otimes \det \mathcal{H}_{i,j} \otimes t^* \det^* C_j \to t^*o^D_j$$

such that the two following diagrams over $\mathcal{M}^C_{i,j} \times_j \mathcal{H}_{j,k} \subset \partial\mathcal{H}_{i,k}$ and $\mathcal{H}_{i,j} \times_j \mathcal{M}^D_{j,k} \subset \partial\mathcal{H}_{i,k}$, respectively, commute, where the map f in the first columns of both diagrams is defined in a similar way to (5-20):

$$\begin{tikzcd}
s^*o^C_i \otimes \det \mathcal{M}^C_{i,j} \otimes t^* \det^* C_j \otimes \det \mathcal{H}_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^C_{i,j}"] \arrow[d, "f"] & s^*o^C_j \otimes \det \mathcal{H}_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^H_{j,k}"] & t^*o^D_k \arrow[d] \\
s^*o^C_i \otimes \mathcal{H}_{i,k} \otimes t^* \det^* D_k \arrow[rr, "(-1)^{m^C_{i,j}+1}\rho^H_{i,k}"] & & t^*o^D_k
\end{tikzcd}$$

$$\begin{tikzcd}
s^*o^C_i \otimes \det \mathcal{H}_{i,j} \otimes t^* \det^* D_j \otimes \det \mathcal{M}^D_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^H_{i,j}"] \arrow[d, "f"] & s^*o^D_j \otimes \det \mathcal{M}^D_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^D_{j,k}"] & t^*o^D_k \arrow[d] \\
s^*o^C_i \otimes \mathcal{H}_{i,k} \otimes t^* \det^* D_k \arrow[rr, "(-1)^{h_{i,k}+1}\rho^H_{i,k}"] & & t^*o^D_k
\end{tikzcd}$$

Definition 5.14 A compatible local system on a flow premorphism $\mathfrak{H}$ from $\mathcal{C}$ to $\mathcal{D}$ consists of bundle isomorphisms $\rho^H_{i,j} : s^*o^C_i \otimes \det \mathcal{H}_{i,j} \otimes t^* \det^* D_j \to t^*o^D_j$ on every $\mathcal{H}_{i,j}$.

Definition 5.15 Let $\mathscr{Y}$ be a flow morphism between flow premorphisms $\mathfrak{H}$ and $\mathfrak{F}$ from the flow category $\mathcal{C}$ to the flow category $\mathcal{D}$. Assume $\mathcal{C}$, $\mathcal{D}$, $\mathfrak{H}$ and $\mathfrak{F}$ are equipped with compatible local systems. We say $\mathscr{Y}$ has a *compatible local system* if on each $\mathcal{Y}_{i,j}$ we have an isomorphism $\rho^Y_{i,j} : s^*o^C_i \otimes \det \mathcal{Y}_{i,j} \otimes t^* \det^* D_j \to t^*o^D_j$ such that:

(1) Under the identification $\det \mathcal{Y}_{i,j}|_{\mathcal{F}_{i,j}} = \det \mathcal{F}_{i,j}$ induced by $\underline{\mathbb{R}}_{\text{out}} \oplus T\mathcal{F}_{i,j} = T\mathcal{Y}_{i,j}|_{\mathcal{F}_{i,j}}$, we have $\rho^Y_{i,j}|_{\mathcal{F}_{i,j}} = \rho^F_{i,j}$. Under the identification $\det \mathcal{Y}_{i,j}|_{\mathcal{H}_{i,j}} = \det \mathcal{H}_{i,j}$ induced by $\underline{\mathbb{R}}_{\text{in}} \oplus T\mathcal{H}_{i,j} = T\mathcal{Y}_{i,j}|_{\mathcal{H}_{i,j}}$, we have $\rho^Y_{i,j}|_{\mathcal{H}_{i,j}} = \rho^H_{i,j}$.

(2) The following two diagrams over $\mathcal{M}^C_{i,j} \times_j \mathcal{Y}_{j,k} \subset \partial\mathcal{Y}_{i,k}$ and $\mathcal{Y}_{i,j} \times_j \mathcal{M}^D_{j,k} \subset \partial\mathcal{Y}_{i,k}$, respectively, commute, where the map f in the first columns of both diagrams is defined in a similar way to (5-20):

$$\begin{tikzcd}
s^* o^C_i \otimes \det \mathcal{M}^C_{i,j} \otimes t^* \det^* C_j \otimes \det \mathcal{Y}_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^C_{i,j}"] \arrow[d, "f"] & s^* o^C_j \otimes \det \mathcal{Y}_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^Y_{j,k}"] & t^* o^D_k \arrow[d] \\
s^* o^C_i \otimes \mathcal{Y}_{i,k} \otimes t^* \det^* D_k \arrow[rr, "(-1)^{c_j} \rho^Y_{i,k}"] & & t^* o^D_k
\end{tikzcd}$$

$$\begin{tikzcd}
s^* o^C_i \otimes \det \mathcal{Y}_{i,j} \otimes t^* \det^* D_j \otimes \det \mathcal{M}^D_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^Y_{i,j}"] \arrow[d, "f"] & s^* o^D_j \otimes \det \mathcal{M}^D_{j,k} \otimes t^* \det^* D_k \arrow[r, "\rho^D_{j,k}"] & t^* o^D_k \arrow[d] \\
s^* o^C_i \otimes \mathcal{Y}_{i,k} \otimes t^* \det^* D_k \arrow[rr, "(-1)^{y_{i,k}+1} \rho^Y_{i,k}"] & & t^* o^D_k
\end{tikzcd}$$

The propositions below follow from arguments similar to the proof of Proposition 5.12.

Proposition 5.16 *Let $\mathcal{C}$ and $\mathcal{D}$ be two flow categories with orientation structures. Assume $\mathfrak{H}$ is a flow morphism with compatible orientation structures. If $\mathcal{C}$ and $\mathcal{D}$ are given local systems using Proposition 5.12, then $\mathfrak{H}$ has a compatible local system. If $\mathfrak{H}$ is only a flow premorphism from $\mathcal{C}$ to $\mathcal{D}$ with compatible orientation structure, then $\mathfrak{H}$ can be given a compatible local system.*

Proposition 5.17 *Let $\mathcal{C}$ and $\mathcal{D}$ be two flow categories with orientation structures, and $\mathfrak{H}$ and $\mathfrak{F}$ two flow premorphism with compatible orientation structures. Assume $\mathscr{Y}$ is a flow morphism with compatible orientation structures. If $\mathcal{C}$ and $\mathcal{D}$ are given local systems using Proposition 5.12 and $\mathfrak{H}$ and $\mathfrak{F}$ are given local systems using Proposition 5.16, then $\mathscr{Y}$ has a compatible local system.*

5.2.1 De Rham theory with local systems To generalize the construction of the minimal Morse–Bott cochain complex to flow categories with local systems, we first recall the de Rham theory with local systems [11, Section 7]. Let M be manifold and o a local system over M. The de Rham complex $\Omega^*(M, o)$ with local system o is defined as sections of $\bigwedge T^*M \otimes_{\mathbb{Z}/2} o$. The usual exterior differential lifts to a differential on $\Omega^*(M, o)$, which is still denoted by d. The associated cohomology is denoted by $H^*(M, o)$. The wedge product defines a bilinear map

$$\Omega^*(M, o) \times \Omega^*(M, o') \to \Omega^*(M, o \otimes o'),$$

which induces a map on cohomology. Using local systems, the integration is well defined for forms in $\Omega^*(M, \det M)$ without assuming M is oriented. Moreover, we have the form of Stokes's theorem

$$\int_M \mathrm{d}\alpha = \int_{\partial M} i^*\alpha,$$

where $i\colon \Omega^*(M,\det M)\to\Omega^*(\partial M,\det\partial M)$ is defined by the restriction map and the isomorphism $\det M|_{\partial M}\to\det\partial M$ induced by the isomorphism $\underline{\mathbb{R}}_{\mathrm{out}}\oplus T\partial M=TM$.

Let C be a closed manifold with a local system o. Since there is a canonical isomorphism from $o^*\otimes o$ to the trivial line bundle, we have a paring

$$H^*(C,o^*)\times H^*(C,o\otimes\det C)\to\mathbb{R} \tag{5-21}$$

by integrating over C. It is a nondegenerate pairing just like the usual case.

5.2.2 The minimal Morse–Bott cochain complex for flow categories with local systems Let $\mathcal{C}=\{C_i,\mathcal{M}_{i,j}\}$ be a flow category with a local system. Define $o_i^*\boxtimes(o_i\otimes\det C_i)$ to be $\pi_1^*o_i^*\otimes\pi_2^*(o_i\otimes\det C_i)$. Since $\pi_2^*\det C_i$ is canonically isomorphic to $\det\Delta_i$ and $(o_i^*\boxtimes o_i)|_{\Delta_i}=o_i^*\otimes o_i=\underline{\mathbb{R}}$, when $\omega\in\Omega^*(C_i\times C_i,o_i^*\boxtimes(o_i\otimes\det C_i))$ is restricted to the diagonal Δ_i, we have $\omega|_{\Delta_i}\in\Omega^*(\Delta_i,\det\Delta_i)$. Therefore $\int_{\Delta_i}\omega$ is well defined. In particular, $\int_{\Delta_i}$ descends to a well-defined map on $H^*(C_i\times C_i,o_i^*\boxtimes(o_i\otimes\det C_i))$. Since the pairing in (5-21) is nondegenerate, $\int_{\Delta_i}$ is represented by a class in

$$H^*(C_i\times C_i,(o_i\otimes\det C_i)\boxtimes o_i^*)=H^*(C_i,o_i\otimes\det C_i)\otimes H^*(C_i,o_i^*).$$

If we choose representatives $\{\theta_{i,a}\}\subset\Omega^*(C_i,o_i\otimes\det C_i)$ of a basis of $H^*(C_i,o_i\otimes\det C_i)$ and representatives $\{\theta_{i,a}^*\}\subset\Omega^*(C_i,o_i^*)$ of the dual basis in $H^*(C_i,o_i^*)$ in the sense that $\langle\theta_{i,a}^*,\theta_{i,b}\rangle=(-1)^{\dim C_i\cdot|\theta_{i,b}|}\int_C\theta_{i,a}^*\wedge\theta_{i,b}=\delta_{ab}$, then $\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*$ represents $\int_{\Delta_i}$ by the same proof as in Proposition 3.2. On the other hand, there is a natural isomorphism $\pi_1^*\det C_i\otimes\pi_2^*\det C_i\simeq\det\Delta_i\otimes\det N_i$ over the diagonal Δ_i, induced by the isomorphism $TC_i\oplus TC_i=T\Delta_i\oplus N_i$. Using the natural identification $\pi_2^*\det C_i\to\det\Delta_i$, there is an induced isomorphism $\pi_1^*\det C_i\to N_i$. A similar isomorphism was already used in the definition of (5-20). Using this isomorphism, if a form in $\Omega^*(C_i\times C_i,(o_i\otimes\det C_i)\boxtimes o_i^*)$ is supported in the tubular neighborhood of Δ_i, then it can be viewed as a form in $\Omega^*(N_i,\det N_i)$. Using the twisted Thom isomorphism in [72], we get another representative of $\int_{\Delta_i}$ by the Thom classes δ_i^n. As a consequence, we find primitives $f_i^n\in\Omega^*(C_i\times C_i,(o_i\otimes\det C_i)\boxtimes o_i^*)$ such that

$$\mathrm{d}f_i^n=\delta_i^n-\sum_a\pi_1^*\theta_{i,a}\wedge\pi_2^*\theta_{i,a}^*,$$

with a relation similar to (3-7). Similarly to Definition 3.3, such choices are referred to as *defining data*.

Given defining data on a flow category with a local system, we define the minimal Morse–Bott chain complex to be

$$\mathrm{BC}(\mathcal{C}):=\varinjlim_{q\to-\infty}\prod_{j=q}^{\infty}H^*(C_j,o_j^*)=\varinjlim_{q\to-\infty}\prod_{j=q}^{\infty}H^*(C_j,o_j) \tag{5-22}$$

(since $o_i\simeq o_i^*$, but not canonically). Next, we explain how (3-15) for d_k in the construction of the minimal Morse–Bott cochain complex still makes sense in the setting of local systems. Let $\alpha\in\Omega^*(C_v,o_v^*)$ and $\gamma\in\Omega^*(C_{v+k},o_{v+k}\otimes\det C_{v+k})$. Then $s^*\alpha\wedge t^*\gamma\in\Omega^*(\mathcal{M}_{v,v+k},s^*o_v^*\otimes t^*o_{v+k}\otimes t^*\det C_{v+k})$. By Definition 5.11, we have an isomorphism

$$\rho_{v,v+k}\colon s^*o_v\otimes\det\mathcal{M}_{v,v+k}\otimes t^*\det^*C_{v+k}\to t^*o_{v+k},$$

which induces an isomorphism

$$\det \mathcal{M}_{v,v+k} \to s^* o_v^* \otimes t^* o_{v+k} \otimes t^* \det C_{v+k}. \tag{5-23}$$

Let $\psi_{v,v+k}$ denote the inverse of (5-23) with an extra negative sign. The extra negative sign is to match the negative sign in the proof of Proposition 5.5. Using $\psi_{v,v+k}$, we can view $s^*\alpha \wedge t^*\gamma$ as in $\Omega^*(\mathcal{M}_{v,v+k}, \det \mathcal{M}_{v,v+k})$, and hence the integration $\int_{\mathcal{M}_{v,v+k}} s^*\alpha \wedge t^*\gamma$ is well defined.

Next, we consider $\mathcal{M}_i^{v,k}[\alpha, f_{v+i}^n, \gamma]$. Then $s^*\alpha \wedge (t \times s)^* f_{v+i}^n \wedge t^*\gamma$ is a form in

$$\Omega^*(\mathcal{M}_{v,v+i} \times \mathcal{M}_{v+i,v+k}, s^* o_v^* \otimes (t \times s)^*((o_{v+i} \otimes \det C_{v+i}) \boxtimes o_{v+i}^*) \otimes t^*(o_{v+k} \otimes \det C_{v+k})).$$

Since

$$\begin{aligned} s^* o_v^* \otimes (t \times s)^*((o_{v+i} \otimes \det C_{v+i}) \boxtimes o_{v+i}^*) \otimes t^*(o_{v+k} \otimes \det C_{v+k}) \\ = (s^* o_v^* \otimes t^*(o_{v+i} \otimes \det C_{v+i})) \boxtimes (s^* o_{v+i}^* \otimes t^*(o_{v+k} \otimes \det C_{v+k})), \end{aligned}$$

using $\psi_{v,v+i}$ and $\psi_{v+i,v+k}$, we get a bundle isomorphism

$$\begin{aligned} s^* o_v^* \otimes (t \times s)^*((o_{v+i} \otimes \det C_{v+i}) \boxtimes o_{v+i}^*) \otimes t^*(o_{v+k} \otimes \det C_{v+k}) \to \det \mathcal{M}_{v,v+i} \boxtimes \det \mathcal{M}_{v+i,v+k} \\ \to \det(\mathcal{M}_{v,v+i} \times \mathcal{M}_{v+i,v+k}). \end{aligned}$$

Thus $\mathcal{M}_i^{v,k}[\alpha, f_{v+i}^n, \gamma]$ is defined. Similarly, $\mathcal{M}_{i_1,\ldots,i_r}^{v,k}[\alpha, f_{v+i_1}^n, \ldots, f_{v+i_r}^n, \gamma]$ makes sense in the local system setting. Thus the differential $d_{\mathrm{BC}} = \prod d_k$ is well defined and $d_{\mathrm{BC}}^2 = 0$ by the same proof as in Theorem 3.10.

Theorem 5.18 *Let $\mathcal{C}$ be a flow category with a local system. Then $(\mathrm{BC}(\mathcal{C}), d_{\mathrm{BC}})$ is cochain complex for any defining data and the cohomology is independent of defining data.*

Similarly, we have analogs of Theorems 3.21, 3.28, 3.30, 3.33 and 3.36 in the setting of local systems by the same argument.

6 Generalizations

In this section, we give two generalizations of the minimal Morse–Bott cochain complex. The first is dropping the compactness assumption on the C_i in flow categories. The second extracts abstract properties used in the construction of the minimal Morse–Bott cochain complex and provides more flexibility in choosing the "homological perturbation" data. Such generalization leads to Gysin exact sequences for flow categories.

6.1 Proper flow categories

We first generalize to the case that C_i is not compact. However, we cannot work with every noncompact manifold. Hence we introduce the following:

Definition 6.1 A manifold C is called of *finite type* if and only if C is the interior of a compact manifold M with boundary.

In particular, any closed manifold is of finite type. An infinite-genus surface is not of finite type. For any manifold C of finite type, $H^*(C)$ is a finite-dimensional vector space.

Definition 6.2 A *proper flow category* is defined similarly to Definition 2.9, except for the following two differences:

(1) C_i is a manifold such that each connected component of C_i is of finite type.

(2) $\mathcal{M}_{i,j}$ is not assumed to be compact. However, the target map $t_{i,j}: \mathcal{M}_{i,j} \to C_j$ is proper.

To explain the generalization of the minimal Morse–Bott cochain complex to proper flow categories, we first explain the counterpart of the perturbation data. Although the following discussion does not require a coherent orientation as explained in Section 5, we assume $\{C_i, \mathcal{M}_{i,j}\}$ is equipped with a coherent orientation for simplicity. In particular, C_i is oriented. Let C be an oriented manifold of finite type and $\Omega^*_c(C)$ denote the space of compactly supported differential forms on C. Then we have a bilinear form

$$\Omega^*(C) \times \Omega^*_c(C) \to \mathbb{R} \quad \text{given by } (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle := (-1)^{\dim C \cdot |\beta|} \int_C \alpha \wedge \beta,$$

and Lefschetz duality asserts that the bilinear form is nondegenerate on cohomology.

Definition 6.3 Let C be an oriented manifold of finite type. We define $\Omega^*_{c,\cdot}(C \times C)$ to be

$$\{\alpha \in \Omega^*_{c,\cdot}(C \times C) \mid \operatorname{supp}(\alpha) \subset K \times C \text{ for some compact set } K\}.$$

Similarly, we define $\Omega^*_{\cdot,c}(C \times C)$ to be

$$\{\alpha \in \Omega^*_{\cdot,c}(C \times C) \mid \operatorname{supp}(\alpha) \subset C \times K \text{ for some compact set } K\}.$$

$\Omega^*_{c,\cdot}(C \times C)$ and $\Omega^*_{\cdot,c}(C \times C)$ are both cochain complexes using the usual exterior differential. Moreover, $H^*_{c,\cdot}(C \times C) := H^*(\Omega^*_{c,\cdot}(C \times C), \mathrm{d}) = H^*_c(C) \otimes H^*(C)$ and $H^*_{\cdot,c}(C \times C) := H^*(\Omega^*_{\cdot,c}(C \times C), \mathrm{d}) = H^*(C) \otimes H^*_c(C)$, where $H^*_c(C)$ is the cohomology of compactly supported differential forms. The following proposition is an analog of the Lefschetz duality with a similar proof to [11, Theorem 12.15]:

Proposition 6.4 *The bilinear form $\Omega^*_{c,\cdot}(C \times C) \times \Omega^*_{\cdot,c}(C \times C) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \int_{C \times C} \alpha \wedge \beta$ descends to cohomology. The induced bilinear form on cohomology is nondegenerate.*

To explain the perturbation data for proper flow categories, we need to interpret the diagonal $\Delta \subset C \times C$ as a cohomology class and represent the cohomology class two different ways: Thom classes which approximate the Dirac current of the diagonal, and a trace term. Let $\alpha \in \Omega^*_{\cdot,c}(C \times C)$. Then $\operatorname{supp}(\alpha) \cap \Delta$ is compact, and hence $\int_\Delta \alpha$ is well defined. Moreover, for $\alpha \in \Omega^*_{\cdot,c}(C \times C)$ we have $\int_\Delta \mathrm{d}\alpha = 0$ by Stokes' theorem. Therefore Δ determines a linear function $[\Delta]$ on $H^*_{\cdot,c}(C \times C)$. In particular, $[\Delta]$ can be represented by a cohomology class in $H^*_{c,\cdot}(C \times C)$ by Proposition 6.4. Since C is of finite type, both $H^*(C)$ and $H^*_c(C)$ are finite-dimensional. Let $\{\theta_a \in \Omega^*_c(C)\}_{1 \le a \le \dim H^*_c(C)}$ be representatives of a basis

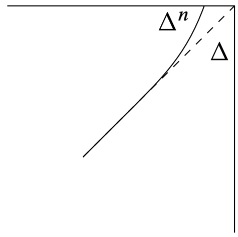


Figure 3: The graph of Δ^n near the boundary.

of $H_c^*(C)$ in $\Omega_c^*(C)$, and $\{\theta_a^* \in \Omega^*(C)\}_{1\le a\le \dim H^*(C)}$ be representatives of a basis of $H^*(C)$ in $\Omega^*(C)$, such that $\langle \theta_a^*, \theta_b\rangle = \delta_{ab}$. The following proposition is proven by the same argument as in Proposition 3.2:

Proposition 6.5 *$\sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta_a^* \in \Omega^*_{c,\cdot}(C\times C)$ represents $[\Delta]$, ie for any closed form $\alpha \in \Omega^*_{\cdot,c}(C\times C)$,*

$$\int_{C\times C} \alpha \wedge \Big(\sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta_a^*\Big) = \int_\Delta \alpha.$$

The Dirac current δ of the diagonal Δ and any of its approximations given in (3-4) are not in $\Omega^*_{c,\cdot}(C\times C)$. To overcome this problem, we need to perturb Δ to Δ^n so that $\Delta^n \subset K\times C$ for a compact set K and Δ^n approximates Δ in a suitable sense. In order to do this, we write C as $M\cup(0,1)\times\partial M$ for a manifold M with boundary ∂M. We fix a smooth nondecreasing function $f\colon \mathbb{R}\to\mathbb{R}_+$ such that $f(x)=0$ for $x\le 0$ and $f(x)<x$ for $x>0$. Then we define $\Delta^n \subset C\times C$ to be

$$\Delta^n := \Delta_M \cup \Delta_{(0,1-1/n)\times\partial M} \cup \tilde{\Delta}^n,$$

where

$$\Big[1-\frac{1}{n},1\Big)\times\partial M\times\Big[1-\frac{1}{n},1\Big)\times\partial M \supset \tilde{\Delta}^n := \Big\{\Big(1-\frac{1}{n}+f(r),x,1-\frac{1}{n}+r,x\Big) \;\Big|\; r\in\Big[0,\frac{1}{n}\Big), x\in\partial M\Big\}.$$

Proposition 6.6 *$\int_{\Delta^n}$ defines the same map on $H^*_{\cdot,c}(C\times C)$ for all $n\in\mathbb{N}$ and equals $\int_\Delta$.*

Proof The claim follows from the fact that any class in $H_c^*(C)$ has a representative supported in $M\subset C = M\cup(0,1)\times\partial M$ and $\Delta_n\cap(C\times M) = \Delta\cap(C\times M)$ for all n. □

The Thom class of Δ^n constructed from (3-4) gives a form $\delta^n \in \Omega^*_{c,\cdot}(C\times C)$ — in a sufficiently small tubular neighborhood of Δ^n — representing the map $\int_{\Delta^n} = \int_\Delta$ through the nondegenerate bilinear form in Proposition 6.4. As a consequence of Propositions 6.4 and 6.5, δ^n and $\sum_a \pi_1^*\theta_a\wedge\pi_2^*\theta_a^*$ are cohomologous in $\Omega^*_{c,\cdot}(C\times C)$, ie we can find primitives f^n such that

$$\mathrm{d}f^n = \delta^n - \sum_a \pi_1^*\theta_a\wedge\pi_2^*\theta_a^*.$$

The following proposition shows that we can choose δ^n and f^n carefully so that the relation (3-7) holds asymptotically. Such a result is crucial for setting up the convergence results and follows directly from the construction.

Proposition 6.7 *Fix a tubular neighborhood of $\Delta \subset C \times C$. Then there exist Thom classes δ^n of Δ^n and primitives f^n such that $f^n - f^m = (\rho_n - \rho_m)\psi$ on $C \times (M \cup (0, 1-2/\min(n,m)) \times \partial M)$.*

Following the same idea as in Definition 3.3, the bases $\{\theta_{i,a}\}$ and $\{\theta^*_{i,a}\}$, along with Thom classes δ^n_i and primitives f^n_i in Proposition 6.7 for each C_i, give defining data for a proper flow category. Next, we show the analog of Lemmas 3.7 and 3.14 hold for proper flow categories:

Lemma 6.8 *Let $\mathcal{C}$ be an oriented proper flow category. Given defining data as above, then for every $\alpha \in \Omega^*(C_v)$, $\gamma \in \Omega^*_c(C_{v+k})$:*

(1) $\lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma] \in \mathbb{R}$ *exists.*

(2) *For $* = (|\alpha| + m_{v,v+i_p})c_{v+i_p}$,*

$$\lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, \delta^n_{v+i_p}, \dots, f^n_{v+i_r}, \gamma] = (-1)^* \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_{p-1},\bar{i}_p,i_{p+1},\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma].$$

Proof Since the target map t is proper,

$$t^*\gamma \in \Omega^*_c(\mathcal{M}_{v+i_r,v+k}) \quad \text{and} \quad (t\times s)^* f^n_{v+i_j} \in \Omega^*_{c,\cdot}(\mathcal{M}_{v+i_{j-1},v+i_j} \times \mathcal{M}_{v+i_j,v+i_{j+1}}),$$

so $s^*\alpha \wedge (t\times s)^* f^n_{v+i_1} \wedge \cdots \wedge (t\times s)^* f^n_{v+i_r} \wedge t^*\gamma \in \Omega^*_c(\mathcal{M}^{v,k}_{i_1,\dots,i_r})$. Hence $\mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma]$ makes sense. For the convergence, take $\mathcal{M}^{v,k}_i[\alpha, f^n_{v+i}, \gamma]$ as an example. Let K denote the subset $s_{v+i,v+k}(t^{-1}_{v+i,v+k}(\mathrm{supp}(\gamma)))$ of C_{v+i}. Then we only need f^n_{v+i} for its value on $C_{v+i} \times K$ to determine $\mathcal{M}^{v,k}_i[\alpha, f^n_{v+i}, \gamma]$. By the properness, K is compact. We write $C_{v+i} = M \cup (0,1) \times \partial M$. Therefore, for n big enough, $K \subset M \cup (0, 1-2/n) \times \partial M$. Hence for n and m big enough, the difference $f^n_{v+i} - f^m_{v+i}$ on $C_{v+i} \times K$ is prescribed in Proposition 6.7. Therefore the argument in the proof of Lemma 3.7 can be applied to prove the convergence. Similarly, $\lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1}, \dots, f^n_{v+i_r}, \gamma]$ exists. The second claim follows from a similar argument and the proof of Lemma 3.14. □

Similarly to Definition 6.2, we have proper flow morphisms, proper flow premorphisms and proper flow homotopies if we require the target maps to be proper. With Lemma 6.8, all results in Section 3 can be generalized to proper flow categories with the same proof.

6.2 Flexible defining data

The following discussion works for proper flow categories with orientation structures. However, for simplicity of notation, we only work with oriented flow categories. Let $\mathcal{C}$ be an oriented flow category. From the discussion in Section 3, the essential property we need for the construction is the following relation:

$$\delta^n_i = \mathrm{d} f^n_i + \sum_a \pi_1^*\theta_{i,a} \wedge \pi_2^*\theta^*_{i,a}. \tag{6-1}$$

In fact, it is not necessary to construct our cochain complex from the cohomology of the critical manifolds. We only need to find differential forms $\{\theta_{i,a}\}$ and $\{\theta^*_{i,a}\}$ such that (6-1) holds and they are dual bases in the sense that $\langle \theta^*_{i,a}, \theta_{i,b}\rangle_i = \delta_{ab}$. Such generalization provides some flexibility in applications. For example, one can use the generalized construction to prove Gysin exact sequences for sphere bundles over flow categories.

Definition 6.9 For an oriented closed manifold C, a reduction of $\Omega^*(C)$ is a pair (A, A^*) such that:

(1) A and A^* are finite-dimensional subspaces of $\Omega^*(C)$ with $\dim A = \dim A^*$.

(2) There exists a basis $\{\theta_a\}$ of A and a basis $\{\theta^*_a\}$ of A^* such that $\langle \theta^*_a, \theta_b\rangle = \delta_{ab}$.

(3) $\sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta^*_a$ is cohomologous to the Thom class δ^n.

Example 6.10 In the construction of the minimal Morse–Bott cochain complex on an oriented flow category, we use that the reduction $A = A^*$ equals the image of chosen quasiembedding $H^*(C) \to \Omega^*(C)$.

Using Definition 6.9, we can generalize defining data to the following: a reduction (A_i, A^*_i) for C_i, a family of Thom classes δ^n_i converging to the Dirac current of the diagonal Δ_i, and primitives f^i_n such that (6-1) and (3-7) hold. We will call this generalization *defining data with reductions*.

Let $\mathcal{C}$ be an oriented flow category. Given defining data with reductions A, we define

$$\mathrm{BC}(\mathcal{C}, A) := \varinjlim_{j\to-\infty} \prod_{i=j}^{\infty} A^*_i. \tag{6-2}$$

The differential is defined as $d_A = \prod_{i=0}^{\infty} d_{A,i}$, where

$$d_{A,0}\alpha := (-1)^{|\alpha|(c_v+1)+c_v} \sum_a \biggl(\int_{C_v} \mathrm{d}\alpha \wedge \theta_{v,a}\biggr)\theta^*_{v,a} = (-1)^{c_v+|\alpha|} \sum_a \langle \mathrm{d}\alpha, \theta_{v,a}\rangle \theta^*_{v,a}, \tag{6-3}$$

with d the normal exterior differential and $\alpha \in A^*_v$. For $k \geq 1$ and $\gamma \in A_{v+k}$,

$$\langle d_{A,k}\alpha, \gamma\rangle_{v+k} = \lim_{n\to\infty} \sum_{\substack{r\geq 0\\ 0=i_0<i_1<\cdots<i_r<k}} (-1)^{\star} \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, f^n_{v+i_r}, \gamma], \tag{6-4}$$

where $\star = \sum_{j=0}^{r} \ddagger(\mathcal{C}, \alpha, i_j)$.

Remark 6.11 We can view (6-3) as the composition of d with the projection (3-3) twisted by a sign. The extra sign could be eliminated by using a suitable change of coordinates on A^*_v (ie conjugate by an automorphism of A^*_v). Then the sign in (6-4) would be modified accordingly. The upshot is that there is no canonical orientation and sign convention, but different conventions typically differ by a "change of coordinates".

One important feature of our construction is that the choices we make on the critical manifolds C_i (reductions, Thom classes and primitives f^n_i) are independent of the structures of the flow categories, flow morphisms or flow homotopies.

Example 6.12 Now we can (heuristically) rephrase the perturbation data for the cascades construction as a reduction. Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category. We neglect the difference between differential forms and currents, as well as orientations and signs for now. For a Morse–Smale pair (f_i, g_i) on a critical manifold C_i, let $A_i := \{[S_x]\}_{x\in\mathrm{Crit}(f_i)}$ and $A_i^* := \{[U_x]\}_{x\in\mathrm{Crit}(f_i)}$. Then, by [39],

$$[\Delta_i] - \sum_{x\in\mathrm{Crit}(f_i)} [S_x][U_x] = d\lim_{t\to\infty}\Bigl[\bigcup_{t'<t}\operatorname{graph}\phi^i_{t'}\Bigr],$$

and $[U_x]$ is the dual of $[S_x]$. Thus $\{A_i, A_i^*\}$ is a reduction.[25]

One should be able to modify our construction to make the argument above rigorous. In particular, we need an extension of the space of differential forms to include $[S_x]$ and $[U_x]$ as well as the homotopy operator. However, such an extension will depend on $\mathcal{M}_{i,j}$, which explains various transversality requirements of the gradient flows of f_i with $\mathcal{M}_{i,j}$ in the cascades construction.

In general, a reduction for manifolds of finite type with local systems is defined as follows:

Definition 6.13 For a manifold C of finite type with a local system o, a reduction is a pair (A, A^*) such that:

(1) A and A^* are finite-dimensional subspaces of $\Omega_c^*(C, o\otimes\det C)$ and $\Omega^*(C, o^*)$, respectively, such that $\dim A = \dim A^*$.

(2) There exists a basis $\{\theta_a\}$ of A and a basis $\{\theta_a^*\}$ of A^* such that $\langle\theta_a^*, \theta_b\rangle = \delta_{ab}$.

(3) $\sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta_a^*$ represents the same map as $\int_\Delta$ on $H^*(C, o^*)\otimes H_c^*(C, o\otimes\det C)$.

Constructions in Section 3 combined with results in Sections 5.2 and 6.1 yield the following results by identical proofs:

Theorem 6.14 (1) *Let $\mathcal{C}$ be a proper flow category with local systems and let A be defining data with reductions. Then* (6-2), (6-3) *and* (6-4) *define a cochain complex* $(\mathrm{BC}(\mathcal{C}, A), d_A)$, *and the homotopy type of* $(\mathrm{BC}(\mathcal{C}, A), d_A)$ *is independent of the defining data.*

(2) *Let $\mathcal{D}$ be another proper flow category with local systems, B defining data with reductions for $\mathcal{D}$ and $\mathfrak{H}$ a proper flow morphism from $\mathcal{C}$ to $\mathcal{D}$ with compatible local systems. Then* (3-34) *defines a cochain morphism* $\phi^H_{A,B}\colon(\mathrm{BC}(\mathcal{C}, A), d_A)\to(\mathrm{BC}(\mathcal{C}, B), d_B)$ *and the homotopy type of $\phi^H_{A,B}$ is independent of the defining data.*

(3) *Let $\mathcal{E}$ be another proper flow category with local systems, C defining data with reductions for $\mathcal{E}$ and $\mathfrak{F}$ a proper flow morphism from $\mathcal{D}$ to $\mathcal{E}$ with compatible local systems. Assume $\mathfrak{H}$ and $\mathfrak{F}$ are composable. Then $\phi^{F\circ H}_{A,C}$ and $\phi^F_{B,C}\circ\phi^H_{A,B}$ are homotopic.*

(4) *Let $\mathfrak{H}$ and $\mathfrak{F}$ be two proper flow premorphisms from $\mathcal{C}$ to $\mathcal{D}$ with compatible local systems. Assume there exists a proper flow homotopy $\mathscr{Y}$ from $\mathfrak{H}$ to $\mathfrak{F}$ with compatible local systems. Then $\phi^H_{A,B}$ is homotopic to $\phi^F_{A,B}$.*

[25] The "homotopy operator" f_i^n in our construction might be different from the "homotopy operator" $\bigl[\bigcup_{0<t<\infty}\operatorname{graph}\phi^i_t\bigr]$ in the cascades construction. However, the homotopy operator in our construction is irrelevant as long as the convergence results in Section 3 hold.

Remark 6.15 When $\mathcal{C}$ is a single manifold C, let (A, A^*) be a reduction. Then the independence result in Theorem 6.14 shows that the cohomology of $(A^*, d_{A,0})$ is $H^*(C, o^*)$. In particular, $\dim A = \dim A^* \geq \dim H^*(C, o^*)$.

We end this subsection with a general way of constructing a reduction (but not all reductions arise in this way).

Proposition 6.16 *Let C be a manifold of finite type with a local system o, and assume A and A^* are finite-dimensional subspaces of $\Omega^*_c(C, o \otimes \det C)$ and $\Omega^*(C, o^*)$, respectively. If d is closed on both A and A^*, the pairing $A^* \otimes A \to \mathbb{R}$ given by $(\alpha, \beta) \mapsto (-1)^{\dim C \cdot |\beta|} \int_C \alpha \wedge \beta$ is nondegenerate, and both $A \hookrightarrow \Omega^*(C, o \otimes \det C)$ and $A^* \hookrightarrow \Omega^*(C, o^*)$ induce surjections on cohomology, then (A, A^*) is a reduction.*

Proof Let $\{\theta_a\}$ be a basis of A, and $\{\theta^*_a\}$ the dual basis under the pairing. It remains to verify Definition 6.13(3). We first claim that $T := \sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta^*_a$ is closed. By our assumption that A and A^* are closed under d, we have $\mathrm{d}T \in \pi_1^* A \wedge \pi_2^* A^* \subset \Omega^*_{c,\cdot}(C \times C, (o \otimes \det C) \boxtimes o^*)$. Moreover, the pairing on $(\pi_1^* A \wedge \pi_2^* A^*) \otimes (\pi_1^* A^* \wedge \pi_2^* A)$ from integration is nondegenerate by the nondegeneracy of the paring on $A^* \otimes A$. Therefore to show $\mathrm{d}T = 0$, it is sufficient to prove that for any $\theta^*_p \in A^*$ and $\theta_q \in A$,

$$\int_{C\times C} \mathrm{d}T \wedge \pi_1^*\theta^*_p \wedge \pi_2^*\theta_q = 0.$$

Hence

$$\begin{aligned}\int_{C\times C} \mathrm{d}T \wedge \pi_1^*\theta^*_p \wedge \pi_2^*\theta_q &= \int_{C\times C}\Big(\sum_a \pi_1^*\mathrm{d}\theta_a \wedge \pi_2^*\theta^*_a + (-1)^{|\theta_a|}\pi_1^*\theta_a \wedge \pi_2^*\mathrm{d}\theta^*_a\Big) \wedge \pi_1^*\theta^*_p \wedge \pi_2^*\theta_q \\ &= (-1)^{|\theta^*_q|\cdot|\theta^*_p| + \dim C\cdot|\theta_q|} \int_C \mathrm{d}\theta_q \wedge \theta^*_p + \int_C \mathrm{d}\theta^*_p \wedge \theta_q.\end{aligned}$$

Since the only case where the above integration is nonzero is when $|\theta_q| + |\theta^*_p| = \dim C - 1$, the above integration is $\int_C \mathrm{d}(\theta^*_p \wedge \theta_q) = 0$. As a consequence, T is closed. Since $A \hookrightarrow \Omega^*(C, o \otimes \det C)$ and $A^* \hookrightarrow \Omega^*(C, o^*)$ induce surjections on cohomology, every class $H^*(C, o^*) \otimes H^*_c(C, o \otimes \det C)$ can be represented by an element in $\pi_1^* A \wedge \pi_2^* A$. Then by the same argument as in Proposition 3.2, T represents the diagonal. Hence (A, A^*) is a reduction. $\square$

6.2.1 Gysin sequences Let C be a manifold and $\pi : E \to C$ an oriented sphere bundle over C with fiber S^k. Then we have an exact sequence [11, Section 14]

$$\cdots \to H^*(C) \xrightarrow{\pi^*} H^*(E) \xrightarrow{\pi_*} H^{*-k}(C) \xrightarrow{\wedge e} H^{*+1}(C) \to \cdots,$$

where e is the Euler class of E. In this section, we generalize it to the setting of flow categories. This construction plays an important role in proving the uniqueness of the cohomology ring of exact symplectic fillings of a flexibly fillable contact manifold in [79].

Definition 6.17 Let $\mathcal{C}$ be an oriented flow category. A k–sphere bundle over $\mathcal{C}$ is a functor $\pi : \mathcal{E} \to \mathcal{C}$ such that π maps E_i to C_i and $\mathcal{M}^E_{i,j}$ to $\mathcal{M}^C_{i,j}$, both $\pi : E_i \to C_i$ and $\pi : \mathcal{M}^E_{i,j} \to \mathcal{M}^C_{i,j}$ are k–sphere bundles, and $s^E_{i,j}$ and $t^E_{i,j}$ are bundle maps covering $s_{i,j}$ and $t_{i,j}$. A k–sphere bundle $\pi : \mathcal{E} \to \mathcal{C}$ is said to be oriented if and only if $\pi : E_i \to C_i$ are oriented sphere bundles, and there is an orientation on each bundle $\pi : \mathcal{M}^E_{i,j} \to \mathcal{M}^C_{i,j}$ with both bundle maps $s^E_{i,j}$ and $t^E_{i,j}$ preserving the orientation.

Proposition 6.18 *Let $\pi : \mathcal{E} \to \mathcal{C}$ be an oriented k–sphere bundle. Then $\mathcal{E}$ is oriented using the convention*

$$[E_i] = [C_i][S^k], \quad [\mathcal{M}^E_{i,j}] = (-1)^k[\mathcal{M}^C_{i,j}][S^k].$$

Proof This is proven in Definition/Proposition 7.2. □

Theorem 6.19 *Let $\pi : \mathcal{E} \to \mathcal{C}$ be an oriented k–sphere bundle. There exist flow morphisms $\Pi^* : \mathcal{C} \Rightarrow \mathcal{E}$ and $\Pi_* : \mathcal{E} \Rightarrow \mathcal{C}$ and defining data Θ and Ξ for $\mathcal{C}$ and $\mathcal{E}$, respectively (where Θ is minimal but Ξ is defining data with reductions), such that we have a short exact sequence*

$$0 \to \mathrm{BC}(\mathcal{C}, \Theta) \xrightarrow{\phi^{\Pi^*}} \mathrm{BC}(\mathcal{E}, \Xi) \xrightarrow{\phi^{\Pi_*}} \mathrm{BC}(\mathcal{C}, \Theta) \to 0.$$

Assume $\mathcal{C}$ has a grading structure (Definition 2.13). Then we have a long exact sequence

$$\cdots \to H^*(\mathcal{C}) \xrightarrow{\pi^*} H^*(\mathcal{E}) \xrightarrow{\pi_*} H^{*-k}(\mathcal{C}) \to H^{*+1}(\mathcal{C}) \to \cdots. \tag{6-5}$$

Otherwise, we have an exact triangle (without grading).

Before giving the proof, we first explain the defining data Θ and Ξ. The defining data for $\mathcal{C}$ is any minimal defining data Θ. For the defining data of $\mathcal{E}$, we fix an angular form $\psi_i \in \Omega^k(E_i)$ such that $\mathrm{d}\psi_i = -\pi^* e_i$, where e_i is the Euler class (viewed in $\Omega^{k+1}(C_i)$) of the sphere bundle $E_i \to C_i$. When k is even, the cohomology class $[e_i]$ is zero. Hence when k is even, we can choose ψ such that $e_i = 0 \in \Omega^{k+1}(C_i)$. Assume $\{\theta_{i,a}\}$ is the chosen basis in the minimal defining data Θ, with $\{\theta^*_{i,a}\}$ the dual basis. Then for each $\theta_{i,a}$ there exists a unique $\eta \in \langle \theta_{i,a} \rangle = \langle \theta^*_{i,a} \rangle$ such that $[(-1)^{|\theta^*_{i,a}|+1}\theta^*_{i,a} \wedge e_i] = [\eta]$ in cohomology. In other words, we can find $\eta_{i,a}$ such that $(-1)^{|\theta^*_{i,a}|+1}\theta^*_{i,a} \wedge e_i - \mathrm{d}\eta_{i,a} \in \langle \theta_{i,a} \rangle$. If we write $m = \dim H^*(C_i)$, then we define

$$A_i = A_i^* := \langle \pi^*\theta_{i,1}, \ldots, \pi^*\theta_{i,m}, \pi^*\theta^*_{i,1} \wedge \psi_i - \pi^*\eta_{i,1}, \ldots, \pi^*\theta^*_{i,m} \wedge \psi_i - \pi^*\eta_{i,m} \rangle.$$

The above construction ensures that d is closed on $A_i = A_i^*$. Since $\int_{E_i} \pi^*\theta_{i,a} \wedge (\pi^*\theta^*_{i,b} \wedge \psi_i - \pi^*\eta_{i,b}) = \int_{C_i} \theta_{i,a} \wedge \theta^*_{i,b}$, for any nonzero vector v in $A = A^*$, there is a vector $u \in A = A^*$ with $\langle v, u \rangle \neq 0$. In particular, the pairing is nondegenerate on $A \otimes A^*$. That $A \to \Omega^*(E_i)$ induces a surjection on cohomology follows from the Gysin sequence associated to the sphere bundle $E_i \to C_i$. Therefore by Proposition 6.16, (A_i, A_i^*) is a reduction for E_i. Moreover, we can choose $\eta_{i,a}$ such that the following holds:

Lemma 6.20 *We write $\pi^*\theta^*_{i,a} \wedge \psi_i - \pi^*\eta_{i,a}$ as $\xi_{i,a}$. Then there exist $\{\eta_{i,a}\}$ from the construction above such that $\langle \pi^*\theta_{i,a}, \xi_{i,b} \rangle_i \neq 0$ if and only if $a = b$ and $\langle \xi_{i,a}, \xi_{i,b} \rangle_i = 0$ for any a and b.*

Proof We have some freedom in the choice of $\eta_{i,a}$, since we can modify it by an element in $\langle \theta_{i,a} \rangle$. The first claim is obvious by integrating the S^k fiber first. The only nontrivial part is proving $\langle \xi_{i,a}, \xi_{i,b} \rangle_i = 0$ for any a and b. We will proceed by induction. Assume for $a, b \le N < \dim H^*(C_i)$ that $\langle \xi_{i,a}, \xi_{i,b} \rangle_i = 0$.

Then we can find $\xi_{i,N+1}$ such that $\langle \xi_{i,a}, \xi_{i,N+1}\rangle_i = 0$ for any $a \le N+1$. We first take any $\bar{\xi}_{i,N+1}$ in the form $\pi^*\theta^*_{i,N+1} \wedge \psi_i - \pi^*\bar{\eta}_{i,N+1} \in A$ from the construction above. Then we define

$$\xi_{i,N+1} := \bar{\xi}_{i,N+1} - \sum_{a=1}^{N} \langle \xi_{i,a}, \bar{\xi}_{i,N+1}\rangle_i \pi^*\theta_{i,a}.$$

It is straightforward to check that $\langle \xi_{i,a}, \xi_{i,N+1}\rangle_i = 0$ for any $a \le N$. Now note that, by degree reasons, if $\langle \xi_{i,N+1}, \xi_{i,N+1}\rangle_i \neq 0$ we must have $|\xi_{i,N+1}| = \frac{1}{2}\dim E_i$. In this case,

$$\langle \xi_{i,N+1}, \xi_{i,N+1}\rangle_i = ((-1)^{(\dim E_i/2)+1} - 1) \int_{C_i} \eta_{i,N+1} \wedge \theta^*_{i,N+1}.$$

However, no matter what the parity of $\frac{1}{2}\dim E_i$ is, we can add a multiple of $\pi^*\theta_{i,N+1}$ to $\xi_{i,N+1}$ to make sure that $\langle \xi_{i,N+1}, \xi_{i,N+1}\rangle_i = 0$. Note that this modification does not affect the property that $\langle \xi_{i,a}, \xi_{i,N+1}\rangle_i = 0$ for any $a \le N$, as $\langle \xi_{i,a}, \pi^*\theta_{i,N+1}\rangle_i = 0$ for $a \le N$. The above argument also proves the induction foundation when $N = 1$. Hence the claim follows. □

In order to obtain the proof of Theorem 6.19, we need to use the following approximations of Dirac currents of diagonals and primitives f^n on the sphere bundle $E_i \to C_i$:

Proposition 6.21 *Let $\pi\colon E \to C$ be an oriented k–sphere bundle over an oriented closed manifold. Let $A = A^*$ be the reduction on $\Omega^*(E)$ built from the previous discussion (in particular, we choose ψ_i such that $d\psi_i = 0$ if k is even). Suppose T is the closed form in $\pi_1^*A \wedge \pi_2^*A$ representing the diagonal in the definition of reduction. Then there exist approximations $\delta^{E,n}$ of the Dirac current of the diagonal Δ_E such that:*

(1) *There exist forms $f^{E,n}$ on $E \times E$ such that*

$$df^{E,n} = \delta^{E,n} - T.$$

(2) *Lemmas 3.7 and 3.14 hold for $f^{E,n}$. In particular, the construction in Section 6.2 works for $f^{E,n}$.*

(3) *Let $\pi_{1,2}$ denote the projection $E \times E \to C \times C$. Then $f^{E,n}$ can be written as sums of differential forms in the form $(\pi_{1,2}^*\alpha) \wedge \beta$ with $\alpha \in \Omega^*(C \times C)$ and $\deg(\beta) \le k$, ie the fiber degree of $f^{E,n}$ is at most k. In other words, if $v_1, \dots, v_{k+1}$ are $k+1$ vertical vectors in $T_p(E \times E)$ for $p \in C \times C$, then $f^{E,n}(v_1 \wedge \dots \wedge v_{k+1} \wedge \cdots) = 0$.*

Proof See Appendix B. □

Proof of Theorem 6.19 The defining data Θ and Ξ are explained above. We now explain the flow morphisms Π_* and Π^*. On the space level, Π_* is the same as the identity flow morphism $\mathfrak{I}^E$ for $\mathcal{E}$. The only difference is that the source map on Π^* is the projection to C_i. Similarly, Π_* from $\mathcal{E}$ to $\mathcal{C}$ on the space level is the same as the identity flow morphism $\mathfrak{I}^E$, but the target map for Π_* is the projection to C_i. We point out here that if the flow category $\mathcal{C}$ is an actual space (concentrated in one level), then Π^* and Π_* induce π^* and π_* on cohomology by definition.

With the defining data Θ and Ξ, we get maps

$$0 \to \mathrm{BC}(\mathcal{C}, \Theta) \xrightarrow{\phi^{\Pi^*}} \mathrm{BC}(\mathcal{E}, \Xi) \xrightarrow{\phi^{\Pi_*}} \mathrm{BC}(\mathcal{C}, \Theta) \to 0. \tag{6-6}$$

We will show (6-6) is a short exact sequence. Using the reduction from Lemma 6.20, the dual basis of $\{\pi^*\theta_{i,a}\} \cup \{\xi_{i,a}\}$ is $\{\xi_{i,a}\} \cup \{\pi^*\theta_{i,a}\}$, up to sign. Then $\mathrm{BC}(\mathcal{E}, \Xi)$ can be decomposed into $V_0 \oplus V_1$ as a vector space, where V_0 is generated by $\langle \pi^*\theta_{i,a}\rangle$ and V_1 is generated by $\langle \xi_{i,a}\rangle$. Next we use approximations of the Dirac currents of the diagonal and primitives f_E^n from Proposition 6.21. By Proposition 6.21(3), if $\gamma \in \langle \pi^*\theta_{i,v+k}\rangle$, then ${\Pi^*}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q}[\alpha, f^{C,n}_{v+i_1}, \dots, f^{C,n}_{v+i_p}, f^{E,n}_{v+j_1}, \dots, f^{E,n}_{v+j_q}, \gamma]$ in the definition of Φ^{Π^*} is zero. Otherwise we cannot cover the fiber directions to get a nonzero integration, as the total fiber degree contributed by $f^{E,n}_{v+j_1}, \dots, f^{E,n}_{v+j_q}$ is at most kq, but the total fiber dimension in ${\Pi^*}^{v,k}_{i_1,\dots,i_p|j_1,\dots,j_q}$ is $k(q+1)$. Hence $\operatorname{im}\phi^{\Pi^*} \subset V_0$. Moreover, ϕ^{Π^*} is an isomorphism onto V_0, as it is the identity plus a strictly upper triangle matrix, similar to the proof of Theorem 3.10 using the identity flow morphism. Similarly, $V_0 \subset \ker \phi^{\Pi_*}$ and $\phi^{\Pi_*}|_{V_1} : V_1 \to \mathrm{BC}(\mathcal{C}, \Theta)$ is an isomorphism. Therefore (6-6) is a short exact sequence, and the induced long exact sequence is the Gysin exact sequence (6-5). □

Remark 6.22 There are two cases of the Gysin exact sequence for which we do not need to appeal to Proposition 6.21:

(1) When $\mathcal{C}$ is a single space C, the reduction of the sphere bundle E can be viewed as decomposed into two copies of $H^*(C)$, which corresponds to the classical Gysin exact sequence. This is explained in Proposition 6.24.

(2) When $\dim C_i \le 1$ for all i, $\deg f^{E,n_i} = \dim C_i + k - 1 \le k$, and Proposition 6.21(3) holds tautologically for any defining data.

These two cases are enough for the argument in [79].

By Corollaries 3.13 and 3.22, we have the following:

Corollary 6.23 *If $\mathcal{C}$ is a Morse flow category and $\mathcal{E}$ an oriented k–sphere bundle over $\mathcal{C}$, then the Gysin exact sequence only depends on $\mathcal{M}^E_{i,j}$ of dimension no greater than $2k$.*

The next proposition follows from direct computation:

Proposition 6.24 *If $\mathcal{C}$ is a single space C, then an oriented k–sphere bundle $\mathcal{E}$ over $\mathcal{C}$ is an oriented k–sphere bundle $\pi : E \to C$. Then the Gysin exact sequence in Theorem 6.19 is the classical Gysin exact sequence*

$$\cdots \to H^i(C) \xrightarrow{\pi^*} H^i(E) \xrightarrow{\pi_*} H^{i-k}(C) \xrightarrow{\wedge(-1)^{\dim C+1}e} H^{i+1}(C) \to \cdots,$$

where $e \in H^*(C)$ is the Euler class of $\pi : E \to C$ and π_* is the integration along the fiber following the convention in [11, Section 6].

Proof Let $\{\theta_1, \dots, \theta_k\}$ and $\{\theta_1^*, \dots, \theta_k^*\}$ be representatives of a basis and the dual basis of $H^*(C)$. Assume ψ is the Thom class of E such that $d\psi = -\pi^*e$, where e is a closed differential form representing the Euler class. $\mathrm{BC}(\mathcal{C})$ is $\langle \theta_1^*, \dots, \theta_k^* \rangle = \langle \theta_1, \dots, \theta_k \rangle$ with zero differential. On the other hand, by the proof of Theorem 6.19, $\mathrm{BC}(\mathcal{E})$ is the reduction $A^* = A$ in the form

$$\langle \pi^*\theta_1, \dots, \pi^*\theta_k, \xi_1 := \pi^*\theta_1^* \wedge \psi - \pi^*\eta_1, \dots, \xi_k := \pi^*\theta_k^* \wedge \psi - \pi^*\eta_k \rangle.$$

The differential d_A on $\pi^*\theta_i$ is zero. Since (6-3), in this case, can be equivalently expressed for $\gamma \in A$, we have

$$\langle d_{A,0}\xi_i, \gamma \rangle = (-1)^{|\xi_i|(\dim E+1)+\dim E} \int_E \pi^*((-1)^{|\theta_i^*|+1}\theta_i^* \wedge e - \mathrm{d}\eta_i) \wedge \gamma.$$

Since $\int \mathrm{d}\xi_i \wedge \pi^*\theta_j = 0$, it is sufficient to compute the case when $\gamma = \xi_j$:

$$\begin{aligned}\langle d_{A,0}\xi_i, \xi_j \rangle &= (-1)^{|\xi_i|(\dim E+1)+\dim E} \int_E \pi^*((-1)^{|\theta_i^*|+1}\theta_i^* \wedge e - \mathrm{d}\eta_i) \wedge (\pi^*\theta_j^* \wedge -\pi^*\eta_j) \\ &= (-1)^{|\xi_i|(\dim E+1)+\dim E} \int_E \pi^*((-1)^{|\theta_i^*|+1}\theta_i^* \wedge e - \mathrm{d}\eta_i) \wedge \pi^*\theta_j^* \wedge \psi.\end{aligned}$$

Note that

$$\int_E \pi^*(\mathrm{d}\eta_i \wedge \theta_j^*) \wedge \psi = \int_C \mathrm{d}\eta_i \wedge \theta_j^* = \int \mathrm{d}(\eta_i \wedge \theta_j^*) = 0.$$

Then we have

$$\begin{aligned}\langle d_{A,0}\xi_i, \xi_j \rangle &= (-1)^{|\xi_i|(\dim E+1)+\dim E} \int_E \pi^*((-1)^{|\theta_i^*|+1}\theta_i^* \wedge e \wedge \theta_j^*) \wedge \psi \\ &= (-1)^{|\xi_i|\dim E+\dim C+1} \int_C \theta_i^* \wedge e \wedge \theta_j^*.\end{aligned}$$

On the other hand,

$$\langle \pi^*\theta_j, \xi_j \rangle = (-1)^{|\theta_j|+|\xi_j|\dim E}.$$

As a consequence,

$$d_{A,0}\xi_i = \sum_j (-1)^{|\xi_i|\dim E+\dim C+1+|\theta_j|+|\xi_j|\dim E} \Big(\int_C \theta_i^* \wedge e \wedge \theta_j^*\Big)\pi^*\theta_j.$$

Note that to have a nonzero integration it is necessary to have $|\xi_i| + |\xi_j| + 1 = \dim E$, and hence

$$|\xi_i|\dim E + \dim C + 1 + |\theta_j| + |\xi_j|\dim E = \dim C + 1 + |\theta_j| = \dim C + |\xi_i| \mod 2$$

and

$$d_{A,0}\xi_i = (-1)^{\dim C+|\xi_i|}\pi^*\Big(\sum_j \Big(\int_C \theta_i^* \wedge e \wedge \theta_j^*\Big)\theta_j\Big).$$

Since

$$\Big\langle \theta_j^*, \Big(\int_C \theta_i^* \wedge e \wedge \theta_j^*\Big)\theta_j \Big\rangle = (-1)^{|\theta_j^*|\cdot|\theta_j|} \int_C \theta_j^* \wedge \theta_i^* \wedge e = (-1)^{|\theta_j|^2} \langle \theta_j^*, \theta_i^* \wedge e \rangle,$$

we know that

$$\Big[(-1)^{\dim C+|\xi_i|} \sum_j \Big(\int_C \theta_i^* \wedge e \wedge \theta_j^*\Big)\theta_j\Big] = [(-1)^{\dim C+1}\theta_i^* \wedge e] \in H^*(C). \tag{6-7}$$

Next, by Theorem 3.21 and similar computation as above, $\phi^{\Pi^*}(\theta_i) = \pi^*\theta_i$ and $\phi^{\Pi_*}(\xi_i) = \theta_i^*$. Then the connecting map $\delta\colon H^{*-k}(C) \to H^{*+1}(C)$ is given by $\delta(\theta_i^*) = (-1)^{\dim C+1}\theta_i^* \wedge e$ by (6-7). □

Remark 6.25 To explain the sign twist compared to [11, Section 14], recall from (6-3) that $d_A\xi_i$ is, roughly speaking, $(-1)^{\dim E+|\xi_i|}\mathrm{d}\xi_i$ (then project to A). Then $(-1)^{\dim E+|\xi_i|}\mathrm{d}\xi_i = (-1)^{\dim C+1}\pi^*\theta_i^* \wedge e$.

In other words, if we consider the Gysin exact sequence following [11, Section 14] but with the cochain complex $(\Omega^*(E), (-1)^{\dim E+*}\mathrm{d})$, then we get the long exact sequence with sign twist in Proposition 6.24.

Next, we consider the functoriality of Gysin exact sequences.

Definition 6.26 Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented flow categories, and $\pi_E : \mathcal{E} \to \mathcal{C}$ and $\pi_F : \mathcal{F} \to \mathcal{D}$ be two oriented k–sphere bundles. Assume $\mathfrak{H} : \mathcal{C} \Rightarrow \mathcal{D}$ is an oriented flow morphism. A compatible k–sphere bundle $\mathfrak{T}$ over $\mathfrak{H}$ is a flow morphism (not oriented a priori) from $\mathcal{E}$ to $\mathcal{F}$ such that $\mathcal{T}_{i,j}$ is an S^k–bundle over $\mathcal{H}_{i,j}$ and s^T, t^T are bundle maps covering s^H, t^H. It is oriented if the sphere bundles $\mathcal{T}_{i,j} \to \mathcal{H}_{i,j}$ are oriented and s^T, t^T preserve the orientation.

Similar to Proposition 6.18, we have that if the k–sphere bundle $\mathfrak{T}$ over $\mathfrak{H}$ is oriented, then $\mathcal{T}$ is an oriented flow morphism from $\mathcal{E}$ to $\mathcal{F}$ with orientation $[\mathcal{T}_{i,j}] = [\mathcal{H}_{i,j}][S^k]$.

Proposition 6.27 *Let $\mathcal{C}$ and $\mathcal{D}$ be two oriented flow categories, and $\pi_E : \mathcal{E} \to \mathcal{C}$ and $\pi_F : \mathcal{F} \to \mathcal{D}$ be two oriented k–sphere bundles. Assume $\mathfrak{H} : \mathcal{C} \Rightarrow \mathcal{D}$ is an oriented flow morphism and $\mathfrak{T}$ is a compatible oriented k–sphere bundle over $\mathfrak{H}$. Then we have a morphism between the Gysin exact sequences below, assuming they have grading structures. Otherwise it is a commutative diagram of exact triangles*:

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(\mathcal{C}) \arrow[r, "\pi^*"] \arrow[d, "\phi^H"] & H^*(\mathcal{E}) \arrow[r, "\pi_*"] \arrow[d, "\phi^T"] & H^{*-k}(\mathcal{C}) \arrow[r] \arrow[d, "\phi^H"] & H^{*+1}(\mathcal{C}) \arrow[r] \arrow[d, "\phi^H"] & \cdots \\
\cdots \arrow[r] & H^*(\mathcal{D}) \arrow[r, "\pi^*"] & H^*(\mathcal{F}) \arrow[r, "\pi_*"] & H^{*-k}(\mathcal{D}) \arrow[r] & H^{*+1}(\mathcal{D}) \arrow[r] & \cdots
\end{tikzcd}$$

Proof We define $\mathfrak{P}$ to be a flow morphism from $\mathcal{C}$ to $\mathcal{F}$ which on the space level is same as $\mathfrak{T}$, but the source map is $\pi \circ t^T_{i,j}$, where π is the projection $E_i \to C_i$. We claim that $\phi^P = \phi^{T\circ \Pi^*_E} = \phi^{\Pi^*_F \circ H}$. By the argument in Theorem 3.33, the contribution from $\mathfrak{T} \circ \Pi^*_E$ containing $(\Pi^*_E)_{i,j}$ for $i<j$ is zero due to the extra interval direction in $(\Pi^*_E)_{i,j}$. Then it is easy to identify $\phi^P = \phi^{T\circ\Pi^*_E}$ on the nose. On the other hand, the contribution from $\Pi^*_F \circ \mathfrak{H}$ containing $(\Pi^*_F)_{i,j}$ for $i<j$ is zero and $(\Pi^*_F)_{j,j} \times_{D_j} \mathcal{H}_{i,j} \simeq \mathcal{T}_{i,j}$ by Definition 6.26. Hence ϕ^P can also be identified with $\phi^{\Pi^*_F\circ H}$ on the nose. Then by Theorem 3.28, $\phi^T \circ \pi^*$ is homotopic to $\pi^* \circ \phi^H$. Similarly, $\phi^H \circ \pi_*$ is homotopic to $\pi_* \circ \phi^T$. By the same argument as in Theorem 6.19, using the special defining data in Proposition 6.21, the homotopies above and ϕ^T satisfy the conditions of Lemma 3.40, and hence the claim follows. □

7 Equivariant theory

The aim of this section is to construct an equivariant theory for a flow category with a smooth group action. Our method is based on the approximation of the homotopy quotient. In the context of Floer theory, a construction in this spirit can be found in [16]. All the results in this section, namely Theorems 7.1 and 7.14, can be generalized to proper flow categories with local systems. However, for simplicity, we only consider oriented flow categories.

7.1 Parametrized cohomology

Similar to the construction of parametrized symplectic homology in [16], we need the parametrized cohomology of an oriented flow category, ie we need to take the product of a flow category $\mathcal{C}$ with a closed oriented manifold B. Since taking a product with B automatically falls into the Morse–Bott case, using the theory developed in previous sections, we have a direct, also geometric construction. Then all that remains are some orientation checks.

Let $\mathcal{C} = \{C_i, \mathcal{M}_{i,j}\}$ be an oriented flow category and B an oriented compact manifold throughout this section. We construct the product flow category $\mathcal{C} \times B$ first. The parametrized cohomology is defined to be the cohomology of $\mathcal{C} \times B$. Each map $f\colon B_1 \to B_2$ induces an oriented flow morphism $\mathfrak{H}(f)\colon \mathcal{C} \times B_2 \Rightarrow \mathcal{C} \times B_1$. Similarly, a homotopy induces a flow homotopy. The main result of this subsection is that, after taking the minimal Morse–Bott cochain complex, we have a contravariant functor by this product construction.

Theorem 7.1 *Let $\mathcal{C}$ be an oriented flow category. Then we have a contravariant functor*

$$\mathcal{C}\times\colon \mathcal{K}(\mathcal{M}an) \to \mathcal{K}(\mathcal{C}h),$$

where $\mathcal{K}(\mathcal{M}an)$ is the category whose objects are closed oriented manifolds and morphisms are homotopy classes of smooth maps.

7.1.1 Product flow categories The first step towards the construction of the functor $\mathcal{C}\times$ is to construct the functor on objects, that is, the product flow categories.

Definition/Proposition 7.2 *If we orient $C_i \times B$ and $\mathcal{M}_{i,j} \times B$ by $[C_i \times B] = [C_i][B]$ and $[\mathcal{M}_{i,j} \times B] = (-1)^{\dim B}[\mathcal{M}_{i,j}][B]$, then $\mathcal{C} \times B = \{C_i \times B, \mathcal{M}_{i,j} \times B\}$ is an oriented flow category.*

Remark 7.3 The reason we oriented $\mathcal{M}_{i,j} \times B$ by $(-1)^{\dim B}[\mathcal{M}_{i,j}][B]$ is that in Definition 5.2 and Proposition 5.5 we mod out the $\mathbb{R}$ translation from the right in the construction of coherent orientations in applications which motivate those definitions.

Definition 7.4 Let $E_1 \to M_1$ and $E_2 \to M_2$ be two vector bundles. Then $E_1 \boxplus E_2$ is defined to $\pi_1^* E_1 \oplus \pi_2^* E_2$ over $M_1 \times M_2$, where $\pi_1, \pi_2\colon M_1 \times M_2 \to M_1, M_2$ are the projections.

Proof of Definition/Proposition 7.2 It is clear that we only need to verify that $\mathcal{C} \times B$ satisfies the orientation property in Definition 2.15. Note that

$$\partial[\mathcal{M}_{i,k} \times B] = \sum_j (-1)^{\dim B + m_{i,j}} [\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}][B].$$

Let N_B be the normal bundle of Δ_B in $B \times B$, and orient it by $[\Delta_B][N_B] = [B][B]$. Then the normal bundle of $\Delta_{C_j \times B}$ is $N_j \boxplus N_B$. If we orient $N_j \boxplus N_B$ by the product orientation, then $[\Delta_{C_j \times B}][N_j \boxplus N_B] = [C_j \times B][C_j \times B]$, ie $[N_j \boxplus N_B]$ satisfies our orientation convention (2-4) for $C_j \times B$.

Then

$$
\begin{aligned}
&[N_i \boxplus N_B]\partial[\mathcal{M}_{i,k} \times B|_{\mathcal{M}_{i,j}\times_j \mathcal{M}_{j,k}\times B}] \\
&\qquad = (-1)^{\dim B + m_{i,j}}[N_i \boxplus N_B][\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}][B] \\
&\qquad = (-1)^{\dim B + m_{i,j} + \dim B(m_{i,k}-1) + \dim B^2}[N_i][\mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}][\Delta_B][N_B] \\
&\qquad = (-1)^{\dim B + m_{i,j} + \dim B(m_{i,k}-1) + c_j m_{i,j} + \dim B^2}[\mathcal{M}_{i,j}][\mathcal{M}_{j,k}][B][B] \\
&\qquad = (-1)^{\dim B + m_{i,j} + \dim B(m_{i,k}-1) + c_j m_{i,j} + \dim B^2 + \dim B m_{j,k}}[\mathcal{M}_{i,j} \times B][\mathcal{M}_{j,k} \times B].
\end{aligned}
$$

Because

$$
\begin{aligned}
&\dim B + m_{i,j} + \dim B(m_{i,k}-1) + c_j m_{i,j} + \dim B^2 + \dim B m_{j,k} \\
&\qquad\qquad = \dim B + m_{i,j} + (m_{i,j} + \dim B)(c_j + \dim B) \mod 2,
\end{aligned}
$$

by Definition 2.15, $\mathcal{C} \times B$ is an oriented flow category. □

Remark 7.5 It is very natural to expect a Künneth formula for $\mathcal{C} \times B$. Indeed, $H(\mathcal{C} \times B) \simeq H(\mathcal{C}) \otimes H(B)$. Since we will not use it, we omit the proof.

7.1.2 Flow morphisms between product flow categories The second step is to construct the functor on morphisms: we want to associate every smooth map $f: B_1 \to B_2$ with a cochain map $\mathrm{BC}(\mathcal{C} \times B_2) \to \mathrm{BC}(\mathcal{C} \times B_1)$. To that end, we first construct a flow morphism $\mathfrak{H}(f)$ from $\mathcal{C} \times B_2$ to $\mathcal{C} \times B_1$, which is defined similarly to the identity flow morphism of $\mathcal{C} \times B_1$. Then the associated cochain map is $\phi^{H(f)}$, defined by Theorem 3.21.

Definition 7.6 Let $\mathcal{C}$ be an oriented flow category and $f: B_1 \to B_2$ a smooth map between two closed oriented manifolds. Then we define $\mathfrak{H}(f) = \{\mathcal{H}^f_{i,j}\}$ as follows:

(1) $\mathcal{H}^f_{i,j} = \mathcal{M}_{i,j} \times [0, j-i] \times B_1$ with the product orientation when $i \le j$, and $\mathcal{H}^f_{i,j} = \varnothing$ when $i > j$.

(2) The source and target maps s and t are defined by

$$
s: \mathcal{H}^f_{i,j} \to C_i \times B_2, \quad (m,t,b) \mapsto (s^C(m), f(b)) \quad \text{and} \quad t: \mathcal{H}^f_{i,j} \to C_j \times B_1, \quad (m,t,b) \mapsto (t^C(m), b)
$$

for $m \in \mathcal{M}_{i,j}$, $t \in [0, j-i]$ and $b \in B_1$, and where s^C and t^C are source and target maps of $\mathcal{C}$.

(3) For $m \in \mathcal{M}_{i,j}$, $n \in \mathcal{M}_{j,k}$, $t \in [0, k-j]$ and $b_1 \in B_1, b_2 \in B_2$ such that $(m,n) \in \mathcal{M}_{i,j} \times_j \mathcal{M}_{j,k}$ and $f(b_1) = b_2$, we define

$$
m_L: (\mathcal{M}_{i,j} \times B_2) \times_j \mathcal{H}^f_{j,k} \to \mathcal{H}^f_{i,k} \quad \text{by} \quad (m, b_2, n, t, b_1) \mapsto (m, n, t+j-i, b_1).
$$

(4) For $m \in \mathcal{M}_{i,j}$, $n \in \mathcal{M}_{j,k}$, $t \in [0, j-i]$ and $b_1 \in B_1$ such that $(m,n) \in \mathcal{M}_{i,j} \times_k \mathcal{M}_{j,k}$ and $f(b_1) = b_2$, we define

$$
m_R: \mathcal{H}^f_{i,j} \times_j (\mathcal{M}_{j,k} \times B_1) \to \mathcal{H}^f_{i,k} \quad \text{by} \quad (m, t, b_1, n, b_1) \mapsto (m, n, t, b_1).
$$

Proposition 7.7 *The flow morphism $\mathfrak{H}(f)$ in Definition 7.6 is an oriented flow morphism $\mathcal{C} \times B_2 \Rightarrow \mathcal{C} \times B_1$.*

Proof All we need to do is the orientation check. It is analogous to the proof of Definition/Lemma 3.23. □

Remark 7.8 In other words, $\mathfrak{H}(f)$ can be viewed as the identity flow morphism on $\mathcal{C} \times B_1$ with source maps twisted by f. In view of the Künneth formula, the morphism induced by $\mathfrak{H}(f)$ is given by $\mathrm{id} \otimes f^*$ twisted by an appropriate sign. We can similarly define another flow morphism from $\mathcal{C} \times B_1$ to $\mathcal{C} \times B_2$ as the identity flow morphism on $\mathcal{C} \times B_1$ with target maps twisted by f. Then the induced map on cohomology is $\mathrm{id} \otimes f_*$ twisted by an appropriate sign, where $f_*\colon H^*(B_1) \to H^{*+\dim B_2 - \dim B_1}(B_2)$ is the pushforward.

7.1.3 Flow homotopies between product flow categories

For an oriented flow category $\mathcal{C}$, we now have enough ingredients to define the functor $\mathcal{C}\times\colon \mathcal{K}(\mathcal{M}an) \to \mathcal{K}(\mathcal{C}h)$:

$$
\begin{aligned}
B &\mapsto \mathrm{BC}(\mathcal{C} \times B) && \text{on objects,} \\
(B_1 \xrightarrow{f} B_2) &\mapsto \big(\mathrm{BC}(\mathcal{C} \times B_2) \xrightarrow{\phi^{H(f)}} \mathrm{BC}(\mathcal{C} \times B_1)\big) && \text{on morphisms.}
\end{aligned}
$$

To finish the proof of Theorem 7.1, we still need to show that homotopic smooth maps induce homotopic cochain maps, and the functoriality of $\mathcal{C}\times$.

Let $f, g\colon B_1 \to B_2$ be two smooth maps and $D\colon [0,1] \times B_1 \to B_2$ a homotopy between them such that $D|_{\{0\}\times B_1} = f$ and $D_{\{1\}\times B_1} = g$. We claim there is a flow homotopy $\mathscr{Y}(D)$ between the $\mathfrak{H}(f)$ and $\mathfrak{H}(g)$.

Definition 7.9 We define $\mathscr{Y}(D) = \{\mathcal{Y}^D_{i,j}\}$ as follows:

(1) For $i \le j$, we define $\mathcal{Y}^D_{i,j} = [0,1] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1$ with the product orientation. For $i < j$, we define $\mathcal{Y}^D_{i,j} = \varnothing$.

(2) The source map s is defined as
$$s\colon [0,1] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1 \to C_i \times B_2, \quad (z, m, t, b) \mapsto (s^{C}(m), D_z(b)).$$

(3) The target map t is defined as
$$t\colon [0,1] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1 \to C_i \times B_1, \quad (z, m, t, b) \mapsto (t^{C}(m), b).$$

(4) We define $\iota_f\colon \mathcal{H}^f_{i,j} \xrightarrow{=} \{0\} \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1 \subset \mathcal{Y}^D_{i,j}$.

(5) We define $\iota_g\colon \mathcal{H}^g_{i,j} \xrightarrow{=} \{1\} \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1 \subset \mathcal{Y}^D_{i,j}$.

(6) We define
$$
\begin{aligned}
m_L\colon (\mathcal{M}_{i,j} \times B_2) \times_j ([0,1] \times \mathcal{M}_{j,k} \times [0, k-j] \times B_1) &\to [0,1] \times \mathcal{M}_{i,k} \times [0, k-i] \times B_1 = \mathcal{K}^D_{i,k}, \\
(m, b_2, z, n, t, b_1) &\mapsto (z, m, n, t + j - i, b_1).
\end{aligned}
$$

(7) We define
$$
\begin{aligned}
m_R\colon ([0,1] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1) \times_j (\mathcal{M}_{j,k} \times B_1) &\to [0,1] \times \mathcal{M}_{i,k} \times [0, k-i] \times B_1 = \mathcal{Y}^D_{i,k}, \\
(z, m, t, b_1, n, b_1) &\mapsto (z, m, n, t, b_1).
\end{aligned}
$$

Proposition 7.10 *$\mathscr{Y}(D)$ in Definition 7.9 is an oriented flow homotopy from $\mathfrak{H}(f)$ to $\mathfrak{H}(g)$.*

Proof We need only check the orientations, and it is analogous to the proof of Definition/Lemma 3.23. □

To complete the proof of Theorem 7.1, we still have to prove the functoriality. Let $g\colon B_1 \to B_2$ and $f\colon B_2 \to B_3$ be two smooth maps. It is not hard to see that $\mathfrak{H}(f)$ and $\mathfrak{H}(g)$ can be composed. We claim that there is a homotopy $\mathscr{Y}^c$ from $\mathfrak{H}(f) \circ \mathfrak{H}(g)$ to $\mathfrak{H}(f \circ g) \circ \mathfrak{I}$, where $\mathfrak{I}$ is the identity flow morphism on $\mathcal{C} \times B_3$.

Definition 7.11 $\mathscr{Y}^c = \{\mathcal{Y}^c_{i,j}\}$ is defined as follows:

- $\mathcal{Y}^c_{i,j} = [0,2] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1$ with product orientation for $i \le j$. We define $\mathcal{Y}^c_{i,j} = \varnothing$ for $i < j$.
- The source map s is defined as
$$s\colon [0,2] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1, \to C_i \times B_3, \quad (z,m,t,b) \mapsto (s^C(m), f \circ g(b)).$$
- The target map t is defined as
$$t\colon [0,2] \times \mathcal{M}_{i,j} \times [0, j-i] \times B_1 \to C_i \times B_1, \quad (s,m,t,b) \mapsto (t^C(m), b).$$
- Since $(\mathcal{H}^{f\circ g} \circ \mathcal{I})_{i,k} = \bigcup_{i \le j \le k} \mathcal{I}_{i,j} \times_j \mathcal{H}^{f\circ g}_{j,k}$, we define $\iota_{\mathfrak{H}(f\circ g)\circ\mathfrak{I}}$ in two cases:
  (1) When $j = i$, we define $\iota_{\mathfrak{H}(f\circ g)\circ\mathfrak{I}}$ as
$$\mathcal{I}_{i,i} \times_i \mathcal{H}^{f\circ g}_{i,k} = (C_i \times B_3) \times_i (\mathcal{M}_{i,k} \times [0,k-i] \times B_1) \to [0,2] \times \mathcal{M}_{i,k} \times [0,k-i] \times B_1,$$
$$(c, b_3, m, t, b_1) \mapsto (0, m, t, b_1).$$
  (2) When $j > i$, we define $\iota_{\mathfrak{H}(f\circ g)\circ\mathfrak{I}}$ on $\mathcal{I}_{i,j} \times_j \mathcal{H}^{f\circ g}_{j,k}$ as
$$(\mathcal{M}_{i,j} \times [0,j-i] \times B_3) \times_j (\mathcal{M}_{j,k} \times [0,j-i] \times B_1) \to [0,2] \times \mathcal{M}_{i,k} \times [0,k-i] \times B_1,$$
$$(m, t_1, b_3, n, t_2, b_1) \mapsto \left(\frac{t_1}{j-i}, m_L(m,n), t_2 + j - i, b_1\right).$$
- For $j < k$, we define $\iota_{\mathfrak{H}(f)\circ\mathfrak{H}(g)}$ on $\mathcal{H}^f_{i,j} \times_j \mathcal{H}^g_{j,k}$ as
$$(\mathcal{M}_{i,j} \times [0,j-i] \times B_2) \times_j (\mathcal{M}_{j,k} \times [0,k-j] \times B_1) \to [1,2] \times \mathcal{M}_{i,k} \times [0,k-i] \times B_1,$$
$$(m, t_1, b_2, n, t_2, b_1) \mapsto \left(\frac{t_2}{k-j} + 1, m, n, t_1 + k - j, b_1\right).$$

When $k = j$, we define $\iota_{\mathfrak{H}(f)\circ\mathfrak{H}(g)}$ as

$$(\mathcal{M}_{i,k} \times [0,k-i] \times B_2) \times_j (C_k \times B_1) \to [1,2] \times \mathcal{M}_{i,k} \times [0,k-i] \times B_1, \quad (m, t, b_2, c, b_1) \mapsto (2, m, t, b_1).$$

- We define
$$m_L\colon (\mathcal{M}_{i,j} \times B_3) \times_j ([0,2] \times \mathcal{M}_{j,k} \times [0,k-j] \times B_1) \to [1,2] \times \mathcal{M}_{i,k} \times [0,k-i] \times B_1 \subset \mathcal{Y}^c_{i,k},$$
$$(m, b_3, z, n, t, b_1) \mapsto \left(\tfrac{1}{2}z + 1, (m,n), t + j - i, b_1\right).$$

- We define

$$m_R\colon ([0,2]\times\mathcal{M}_{i,j}\times[0,j-i]\times B_1)\times_j(\mathcal{M}_{j,k}\times B_1)\to[0,1]\times\mathcal{M}_{i,k}\times[0,k-i]\times B_1\subset\mathcal{Y}^c_{i,k},$$
$$(z,m,t,b,n,b)\mapsto\left(\tfrac{1}{2}z,(m,n),t,b\right).$$

Proposition 7.12 *$\mathscr{Y}^c$ in Definition 7.11 is an oriented flow homotopy from $\mathfrak{H}(f)\circ\mathfrak{H}(g)$ to $\mathfrak{H}(f\circ g)\circ\mathfrak{I}$.*

Proof The proof is analogous to the proof of Definition/Lemma 3.23. □

Proof of Theorem 7.1 This follows by Definition/Proposition 7.2 and Propositions 7.7, 7.10 and 7.12. □

Remark 7.13 There is a generalization of the construction above. Let $B_1\xleftarrow{f}B\xrightarrow{g}B_2$ be maps between closed oriented manifolds. Then there is a flow morphism $\mathfrak{H}$ from $\mathcal{C}\times B_2$ to $\mathcal{C}\times B_1$ with $\mathcal{H}_{i,j}:=\mathcal{M}_{i,j}\times[0,j-i]\times B$, where the source and target map are induced by g and f. The homotopy type of the induced cochain map is determined by the oriented bordism group $\Omega^*_{SO}(B_1,B_2)$, which is defined as follows: an element in $\Omega^n_{SO}(B_1,B_2)$ is represented by a closed oriented n–manifold M and two maps f and g from M to B_1 and B_2. The triples (M,f,g) and (N,f',g') are equivalent if and only if there is an oriented bordism D from M to N and two maps F and G from D to B_1 and B_2 extending f, g, f' and g'.

7.2 Equivariant cohomology

The functor $\mathcal{C}\times$ is not very interesting, because it is quite independent of the flow category $\mathcal{C}$. However, if $\mathcal{C}$ has a compact Lie group G acting on it, then the Borel construction, which is just a product modulo the G–action, merges some information of $\mathcal{C}$ into the "homotopy quotient". Thus nontrivial phenomena may arise from this construction. The first step towards the Borel construction is to upgrade Theorem 7.1:

Theorem 7.14 *Let the compact Lie group G act on $\mathcal{C}$ in an orientation-preserving way (Definition 7.15). Then there is a contravariant functor*

$$\mathcal{C}\times_G\colon\mathcal{K}(\mathcal{P}rin_G)\to\mathcal{K}(\mathcal{C}h),$$

where $\mathcal{K}(\mathcal{P}rin_G)$ is the category whose objects are closed oriented principal G–bundles and morphisms are G–equivariant homotopy classes of G–equivariant maps.

The classifying space $EG\to BG$ can be approximated by a sequence of closed oriented G–bundles $E_n\to B_n$ such that $\cdots\subset E_n\subset E_{n+1}\subset\cdots$. Note that $EG\to BG$ can be understood as the "G–equivariant homotopy colimit" of the diagram $\cdots\subset E_n\subset E_{n+1}\subset\cdots$. The classical Borel construction of the equivariant cohomology [38] suggests that the equivariant cochain complex of a flow category should be the composition of a homotopy limit and the functor $C\times_G$ to the diagram $\cdots\subset E_n\subset E_{n+1}\subset\cdots$. We will construct this theory in this subsection. In particular, we will show that such a construction is independent of the approximation $\{E_n\to B_n\}$.

7.2.1 The functor $\mathcal{C}\times_G$

Definition 7.15 A G–action on an oriented flow category $\mathcal{C}$ consists of left G–actions on C_i and $\mathcal{M}_{i,j}$ such that the source, target and multiplication maps are G–equivariant. We say the G–action preserves the orientation if the G–actions on C_i and $\mathcal{M}_{i,j}$ preserve the orientations.

Let $E\to B$ be an oriented G–bundle. Assume G acts on $\mathcal{C}$ in a orientation-preserving manner. Then G acts from the right on $C_i\times E$ and $\mathcal{M}_{i,j}\times E$ by $(x,e)\cdot g=(g^{-1}\cdot c,e\cdot g)$. Let $C_i\times_G E$ and $\mathcal{M}_{i,j}\times_G E$ denote quotients of the respective G–actions. If we orient B, $C_i\times_G E$ and $\mathcal{M}_{i,j}\times_G E$ by $[B][G]=[E]$, $[C_i\times_G E][G]=[C_i][E]$ and $[\mathcal{M}_{i,j}\times_G E][G]=(-1)^{\dim B}[\mathcal{M}_{i,j}][E]$, then Definition/Proposition 7.2 can be generalized to the following statement by an analogous proof:

Proposition 7.16 *If G acts on the oriented flow category $\mathcal{C}$ and preserves orientation, then $\mathcal{C}\times_G E=\{C_i\times_G E,\mathcal{M}_{i,j}\times_G E\}$ is an oriented flow category.*

Moreover, Propositions 7.7, 7.10 and 7.12 can be generalized to the equivariant settings:

Proposition 7.17 *Assume G acts on the oriented flow category $\mathcal{C}$ and preserves the orientation. Let $E_1\to B_2$ and $E_2\to B_2$ be two oriented G–principal bundles.*

1. *Let f be a smooth G–equivariant map $E_1\to E_2$. Then there is an oriented flow morphism $\mathfrak{H}_G(f)$ from $\mathcal{C}\times_G E_2$ to $\mathcal{C}\times_G E_1$.*
2. *Let g be another G–equivariant map $E_1\to E_2$ and $D\colon[0,1]\times E_1\to E_2$ an equivariant homotopy between f and g. Then there is an oriented flow homotopy $\mathscr{Y}_G(D)$ between $\mathfrak{H}_G(f)$ and $\mathfrak{H}_G(g)$.*
3. *Let $h\colon E_2\to E_3$ be another equivariant map between two oriented G–principal bundles. Then there is an oriented flow homotopy $\mathscr{Y}_G^{\mathcal{C}}$ from $\mathfrak{H}_G(h)\circ\mathfrak{H}_G(f)$ to $\mathfrak{H}_G(h\circ f)\circ\mathfrak{I}$.*

Then Theorem 7.14 follows from Propositions 7.16 and 7.17.

7.2.2 Approximations of classifying spaces

Definition 7.18 Let G be a compact Lie group. An approximation of the classifying space $EG\to BG$ is a sequence of oriented principal G–bundles $E_n\to B_n$ such that $E_n\subset E_{n+1}$ equivariantly. Moreover, for each $k\in\mathbb{N}$, there exists $N_k\in\mathbb{N}$ such that for all $n\geq N_k$, E_n is k–connected.

Given an approximation of the classifying space, we can compute the equivariant cohomology for G–actions:

Theorem 7.19 [38] *Let M be a compact manifold with a smooth G–action and $E_n\to B_n$ an approximation of the classifying space $EG\to BG$. Then*

$$\varprojlim H^*(M\times_G E_n)=H^*(M\times_G EG)=H^*_G(M).$$

Approximations of the classifying spaces can be constructed as follows. Fix an embedding $G \subset U(m)$. By $H(n,m)$, we mean the set of m orthogonal vectors in $\mathbb{C}^n$, which is a compact orientable smooth manifold. $U(m)$ acts on it with quotient the Grassmannian $\mathrm{Gr}(n,m)$, and $\{H(n,m) \to \mathrm{Gr}(n,m)\}$ serves as a finite-dimensional approximation of the classifying principal bundle $EU(m) \to BU(m)$ as $n \to \infty$. Then $EG \to BG$ can be approximated by $H(n,m) \to H(n,m)/G$. It was checked in [38] that this construction is an approximation in the sense of Definition 7.18.

7.2.3 Homotopy limit Since our construction uses an approximation, we need to take a limit in the end. Consider a directed system of cochain-complexes

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0.$$

Then the limit $\varprojlim A_i$ is also a cochain complex. However, this limit is not very nice from the homotopy-theoretic point of view. If we change the maps in the directed system by homotopic maps, then the homotopy type of $\varprojlim A_i$ may change. In our setting, the cochain map is constructed only up to homotopy (Section 3.6), thus we need to apply a better limit called the homotopy limit, whose homotopy type is invariant under the replacement of homotopic maps. We recall some of the basic definitions and properties of homotopy limits from [60].

Let $\mathbb{N}^{\mathrm{op}}$ be the inverse directed set $\{\cdots \to 2 \to 1 \to 0\}$ and $\{A_n, \mu_{nm} \colon A_n \to A_m\}$ an inverse system of cochain complexes over this directed set:

$$\cdots \xrightarrow{\mu_4} A_3 \xrightarrow{\mu_3} A_2 \xrightarrow{\mu_2} A_1 \xrightarrow{\mu_1} A_0.$$

Then there is a map $v \colon \prod A_i \to \prod A_i$ such that $v(a_n) = \mu_n(a_n)$ over the basis $a_n \in A_n$. Then $\mathrm{holim}\, A_n$ is defined to be the homotopy kernel of $1 - v$, that is, $\Sigma^{-1} C(1-v)$, where $C(\cdot)$ denotes the mapping cone and Σ is shifting by 1.[26] Then we have a triangle in $\mathcal{K}(\mathcal{C}h)$:

$$\begin{tikzcd} \prod A_n \arrow[rr, "1-v"] & & \prod A_n \arrow[dl, "+1"] \\ & \mathrm{holim}\, A_n \arrow[ul] & \end{tikzcd} \tag{7-1}$$

This construction is the infinite telescope construction. Thus it is clear that the homotopy limits of any final subsets of $\mathbb{N}^{\mathrm{op}}$ are homotopic to each other, and changing μ_i up to homotopy does not affect the homotopy type of the homotopy limit. There is a commutative diagram in $\mathcal{K}(\mathcal{C}h)$,

$$\begin{tikzcd} \mathrm{holim}\, A_n \arrow[r] & \prod A_n \\ \varprojlim A_n \arrow[u] \arrow[ur] & \end{tikzcd} \tag{7-2}$$

[26] We assume everything is graded by $\mathbb{Z}/2$ for simplicity. If everything is ungraded, then shifting just means multiplying the differential by -1. This also enters into the definition of mapping cone in the ungraded case.

When $\varprojlim^1 A_n = 0$, ie the Mittag-Leffler condition holds for A_n, then $\varprojlim A_n \to \operatorname{holim} A_n$ is a quasi-isomorphism [60, Remark 27]. This is the reason why sometimes we can use the limit instead of homotopy limit in applications, eg [16]. The long exact sequence from the triangle (7-1) implies we have the short exact sequence

$$0 \to \varprojlim^1 H^{*-1}(A_n) \to H^*(\operatorname{holim} A_n) \to \varprojlim H^*(A_n) \to 0.$$

7.2.4 Equivariant cochain complexes Now, we are ready to define the equivariant cochain complex of a flow category with a group action. Pick an approximation $E_0 \subset \cdots \subset E_i \subset \cdots$ of the classifying space such that E_i is oriented and G preserves the orientation. Then applying the functor $\mathcal{C} \times_G$ to this sequence, we get an inverse system in $\mathcal{K}(\mathcal{C}h)$,

$$\cdots \to \mathrm{BC}(\mathcal{C} \times_G E_2) \to \mathrm{BC}(\mathcal{C} \times_G E_1) \to \mathrm{BC}(\mathcal{C} \times_G E_0).$$

Definition 7.20 The equivariant cochain complex BC_G is defined as $\operatorname{holim} \mathrm{BC}(\mathcal{C} \times_G E_n)$.

Results in Section 3.6 imply that the homotopy type of BC_G is independent of the auxiliary defining data. To get a canonical theory, we still need to check that BC_G does not depend on the choice of the approximation $E_n \to B_n$.

7.2.5 Independence of approximations With another approximation $E'_n \to B'_n$ of the classifying space, we claim that we can form a new sequence of approximations containing both $E'_n \to B'_n$ and $E_n \to B_n$ as final subsets. As preparation, we state two propositions; the first is a simple application of obstruction theory.

Proposition 7.21 *Let $Y \to X$ be a smooth fiber bundle, where the fiber F is k–connected and X is a k–dimensional manifold. Then there is a cross-section for $Y \to X$, and any two cross-sections are homotopic.*

By this proposition, [38, Proposition 1.1.1.] can be modified into the following:

Proposition 7.22 *Let $E \to B$ be a G–principal bundle, with E k–connected. Then, for any closed manifold M with $\dim M \leq k$, the G–principal bundles over M are classified by $[M, B]$ (the set of homotopy classes of maps from M to B).*

Therefore by Definition 7.18 and Proposition 7.22, there exists $n_1 \in \mathbb{N}$ such that there is an equivariant map $E_1 \to E'_{n_1}$. Moreover, there exists $m_1 \in \mathbb{N}$ such that there is an equivariant map $E'_{n_1} \to E_{m_1}$ and the composition $E_1 \to E'_{n_1} \to E_{m_1}$ is equivariantly homotopic to $E_1 \subset E_{m_1}$. We can keep applying this argument to get a directed system in the equivariant homotopy category of spaces

$$E_1 \to E'_{n_1} \to E_{m_1} \to E'_{n_2} \to E_{m_2} \to \cdots,$$

which is also compatible with the two approximations $\{E_{m_i}\}$ and $\{E'_{n_i}\}$ up to equivariant homotopy. Then Theorem 7.14 implies that there is a well-defined inverse directed system in the homotopy category of cochain complexes,

$$\cdots \to \mathrm{BC}(\mathcal{C} \times_G E_{m_2}) \to \mathrm{BC}(\mathcal{C} \times_G E'_{n_2}) \to \mathrm{BC}(\mathcal{C} \times_G E_{m_1}) \to \mathrm{BC}(\mathcal{C} \times_G E'_{n_1}) \to \mathrm{BC}(\mathcal{C} \times_G E_1). \tag{7-3}$$

Let H denote the homotopy limit of (7-3). Since both $\mathrm{BC}(\mathcal{C} \times_G E'_{n_i})$ and $\mathrm{BC}(\mathcal{C} \times_G E_{m_i})$ are final in the inverse directed systems above,

$$\operatorname{holim} \mathrm{BC}(\mathcal{C} \times_G E'_n) = \operatorname{holim} \mathrm{BC}(\mathcal{C} \times_G E'_{n_i}) = H = \operatorname{holim} \mathrm{BC}(\mathcal{C} \times_G E_{m_i}) = \operatorname{holim} \mathrm{BC}(\mathcal{C} \times_G E_m).$$

Therefore the homotopy type of BC_G is independent of the approximation, giving the following theorem:

Theorem 7.23 *Let $\mathcal{C}$ be an oriented flow category. Assume the compact Lie group G acts on $\mathcal{C}$ and preserves the orientation. Then the homotopy type of the equivariant cochain complex BC_G in Definition 7.20 is well defined, ie independent of all the choices, particularly the choice of the approximation $\{E_n \to B_n\}$.*

7.2.6 Spectral sequences From (7-1), the homotopy limit is the shifted mapping cone of $1 - v$. Thus the action spectral sequence in Proposition 4.1 on $\mathrm{BC}(\mathcal{C} \times_G E_n)$ induces a spectral sequence on the homotopy limit. In particular, we need to apply the following result:

Proposition 7.24 [75, Exercise 5.4.4] *Let $f: B \to C$ be a map of filtered cochain complexes. For a fixed integer $r \geq 0$, there is a filtration on the mapping cone $C(f)$, defined by*

$$F_p C(f) := F_{p+r} B_{n+1} \oplus F_p C_n.$$

Then the r^{th} page $E_r(C(f))$ of the induced spectral sequence is the mapping cone of the map on the r^{th} page $f^r: E_r(B) \to E_r(C)$.

Let $r = 1$. By Proposition 7.24, there is a spectral sequence for BC_G induced from the action filtration on $\Pi \mathrm{BC}^{\mathcal{C} \times_G E_n}$. Since $E_1^p(\Pi \mathrm{BC}(\mathcal{C} \times_G E_n)) = \Pi H^*(C_p \times_G E_n)$ with the differential coming from the d_1 term in (3-15) for each $\mathcal{C} \times_G E_n$, again by Proposition 7.24 $E_1(\mathrm{BC}_G)$ is the (shifted) mapping cone of the cochain morphism

$$1 - v : \prod_n \varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*(C_p \times_G E_n) \to \prod_n \varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*(C_p \times_G E_n).$$

Since $\varprojlim^1 H^*(C_p \times_G E_n) = 0$, ie the Mittag-Leffler condition holds for inverse system

$$\cdots \to H^*(C_p \times_G E_n) \to H^*(C_p \times_G E_{n-1}) \to \cdots,$$

the natural map (7-2)

$$\varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H_G^*(C_p) = \varprojlim_n \varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H^*(C_p \times_G E_n) \to E_1(\mathrm{BC}_G)$$

is a quasi-isomorphism. The induced differential d_1^G on $\varinjlim_q \prod_{p=q}^{\infty} H_G^*(C_p)$ is the limit of d_1 for $\mathcal{C} \times_G E_n$. Since d_1 comes from the moduli spaces without boundary (the pullback and pushforward on cohomology),

d_1^G is $t_* \circ s^* \colon H_G^*(C_p) \to H_G^*(C_{p+1})$ up to sign (the pullback and pushforward on equivariant cohomology). The polyfold theoretic version of d_1^G is the analog of the equivariant fundamental class in [77].

Corollary 7.25 *There is a spectral sequence for* BC_G *such that*

$$E_2^p(\mathrm{BC}_G) \simeq H^*\Bigl(\varinjlim_{q \to -\infty} \prod_{p=q}^{\infty} H_G^*(C_p), d_1^G\Bigr).$$

8 A basic example: finite-dimensional Morse–Bott cohomology

The aim of this section is to construct a flow category for the finite-dimensional Morse–Bott theory. The existence of such a flow category is a folklore theorem, stated in various places, eg [3; 33]. The Morse version of the flow category was introduced in [19], and [74] provided a detailed construction for the flow category of a Morse function for metrics which are standard near critical points. In this section, we prove that there is a flow category for any Morse–Bott function if we choose a suitable metric. The local analysis in our case is just a family version of the analysis in [74].

In the Morse case, [2, Section 3.4] provides an argument to reduce constructions of continuation maps and homotopies to counting gradient flow lines on some larger manifolds. Similarly, we can construct the flow morphisms and flow homotopies by looking at flow categories arising from some larger manifolds with suitable Morse–Bott functions. With all of these established, just like the Morse case, we can prove that the cohomology of the flow category is independent of the Morse–Bott function. The main theorem of this chapter is the following:

Theorem 8.1 *Let f be a Morse–Bott function on a closed manifold M. Then there exists a metric g such that the compactified moduli spaces of* (*unparametrized*) *gradient flow lines form a flow category with an orientation structure. The cohomology of the flow category is independent of the Morse–Bott function and is equal to the regular cohomology $H^*(M, \mathbb{R})$.*

Let f be a Morse–Bott function on M throughout this section, and let the critical manifolds $C_1, \dots, C_n$ be such that $f(C_i) < f(C_j)$ if and only if $i < j$. We can fix a real number $\delta > 0$ such that δ is strictly smaller than the absolute values of the nonzero eigenvalues of $\mathrm{Hess}(f)$ over all critical manifolds C_i.

8.1 The Fredholm property for the finite-dimensional Morse–Bott theory

Like the Morse case, the moduli spaces of *parametrized* gradient flow lines from C_i to C_j is a zero set of a Fredholm operator over some Banach space $\mathcal{B}_{i,j}$. The construction of $\mathcal{B}_{i,j}$ was included in the appendix of [32] as part of the Banach manifolds of the cascades construction; we review the construction briefly.

First we fix an auxiliary metric g_0 on M. Let γ be a smooth curve defined over $\mathbb{R}$ such that

$$\lim_{t\to-\infty}\gamma(t)=x\in C_i,\qquad \lim_{t\to+\infty}\gamma(t)=y\in C_j, \tag{8-1}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma\right|_{g_0}<Ce^{-\delta|t|}\quad\text{for } |t|\gg 0 \text{ and some constant } C. \tag{8-2}$$

Let $P(C_i,C_j)$ be the space of continuous paths defined over $\mathbb{R}$ connecting C_i and C_j. The Banach manifold $\mathcal{B}_{i,j}$ will be a subspace of $P(C_i,C_j)$. We will first describe the neighborhood of γ in $\mathcal{B}_{i,j}$. For this purpose:

(1) Fix a smooth function $\chi\colon\mathbb{R}\to\mathbb{R}$ such that $\chi(t)=|t|$ for $|t|\gg 0$. Then we can define the weighted Sobolev space $H^k_\delta(\mathbb{R},\gamma^*TM)$ with norm $|f|_{H^k_\delta}:=|e^{\delta\chi(t)}f|_{H^k}$, for $k\ge 1$.

(2) Fix local charts of M near x and y such that C_i near x is a radius-r ball in the $x_1,\dots,x_{c_i}$ coordinates, and C_j near y is a radius-r ball in the $y_1,\dots,y_{c_j}$ coordinates.

(3) $\rho_\pm(t)$ are smooth functions which are 1 near $\pm\infty$ and 0 near $\mp\infty$ such that (8-3) makes sense using the local charts above.

There exists a positive number K such that when $f\in H^k_\delta(\mathbb{R},\gamma^*TM)$ with $|f|_{H^k_\delta}<K$, we have that $|f|$ is pointwise smaller than the injective radius of the metric g_0. Let exp denote the exponential map associated to the metric g_0. Then there is a map

$$\begin{gathered} B_K(H^k_\delta(\mathbb{R},\gamma^*TM))\times B_r(\mathbb{R}^{c_i})\times B_r(\mathbb{R}^{c_j})\to P(C_i,C_j),\\ (f,x_1,\dots,x_{c_1},y_1,\dots,y_{c_j})\mapsto \exp_\gamma f+\sum_1^{c_i}\rho_-x_i+\sum_1^{c_j}\rho_+y_i. \end{gathered} \tag{8-3}$$

$\mathcal{B}_{i,j}$ consists of images of all such maps in $P(C_i,C_j)$ for all curves γ satisfying (8-1) and (8-2). Let $\mathcal{E}_{i,j}\to\mathcal{B}_{i,j}$ be the vector bundle, where the fiber over $\gamma\in\mathcal{B}_{i,j}$ is $H^{k-1}_\delta(\mathbb{R},\gamma^*TM)$.

Proposition 8.2 [32] *$\mathcal{B}_{i,j}$ is a Banach manifold and $\mathcal{E}_{i,j}\to\mathcal{B}_{i,j}$ is a Banach bundle.*

Since the evaluation maps $\mathcal{B}_{i,j}\to C_i\times C_j$ are submersions for all $i<j$, the fiber products $\mathcal{B}_{i,j}\times_j\cdots\times_k\mathcal{B}_{k,l}$ are Banach manifolds. Moreover, $\mathcal{E}_{i_0,i_1}\times_{i_1}\cdots\times_{i_{k-1}}\mathcal{E}_{i_{k-1},i_k}\to\mathcal{B}_{i_0,i_1}\times_{i_1}\cdots\times_{i_{k-1}}\mathcal{B}_{i_{k-1},i_k}$ are Banach bundles for all $i_0<i_1<\cdots<i_k$. Given a metric g, then there is a section $s_{i,j}\colon\mathcal{B}_{i,j}\to\mathcal{E}_{i,j}$ defined by $s(\gamma)=\gamma'-\nabla_g f(\gamma)$.

Proposition 8.3 [32] *The section $s_{i,j}$ is a Fredholm operator with index $d_j-d_i+c_i+c_j$, where d_i is the dimension of the negative eigenspace of* Hess(f) *on C_i (d_i is the grading structure for our flow category).*

Proposition 8.4 *For a generic metric g, $s_{i,j}$ is transverse to 0 and, for all $i_0<\cdots<i_k$, the fiber products $s^{-1}_{i_0,i_1}(0)\times_{i_1}\cdots\times_{i_{k-1}}s^{-1}_{i_{k-1},i_k}(0)$ are cut out transversely.*

Proof The proof follows from a standard Sard–Smale argument by considering the universal moduli space of all metrics. The result for the fiber products follows from applying the Sard–Smale argument to $s_{i_0,i_1}\times_{i_1}\cdots\times_{i_{k-1}}s_{i_{k-1},i_k}\colon\mathcal{B}_{i_0,i_1}\times_{i_1}\cdots\times_{i_{k-1}}\mathcal{B}_{i_{k-1},i_k}\to\mathcal{E}_{i_0,i_1}\times_{i_1}\cdots\times_{i_{k-1}}\mathcal{E}_{i_{k-1},i_k}$. □

We call such a pair (f,g) a *Morse–Bott–Smale pair* (this is weaker than the Morse–Bott–Smale condition in Remark 2.17). Let $M_{i,j}$ denote $s_{i,j}^{-1}(0)/\mathbb{R}$. Then $\mathcal{M}_{i,j} := \bigcup_{i<i_1<\cdots<i_k<j} M_{i,i_1} \times_{i_1} \cdots \times_{i_k} M_{i_k,j}$ can be made into a compact topological space. The topology on this space is completely analogous to the Gromov–Floer topology on the set of broken flow lines in the Morse case; see [2; 74] for details.

8.2 Flow categories of Morse–Bott functions

The main theorem of this section is that we can put smooth structures on $\mathcal{M}_{i,j}$ such that the following holds:

Theorem 8.5 *The set $\{C_i, \mathcal{M}_{i,j}\}$ is a flow category with an orientation structure.*

To prove this theorem, we need to equip $\mathcal{M}_{i,j}$ with a smooth structure with boundaries and corners. One strategy is using a gluing map [69], which can be generalized to Floer theories. This method requires certain compatibility between gluing maps to guarantee a smooth structure.[27] In the context of Lagrangian Floer theory, such a construction was carried out in [6]. Another method is finding an (M–)polyfold description of the moduli spaces. Then the manifold structures with boundaries and corners come from those of the ambient (M–)polyfolds; see [24; 44]. In this section, we will adopt a more elementary method from [2; 19; 74], so that the smooth structure on the moduli spaces is inherited from some ambient manifolds.

Lemma 8.6 [61] *Let C_i be a critical manifold of the Morse–Bott function f. Then there is a tubular neighborhood of C_i in M diffeomorphic to the normal bundle N of C_i. Moreover, N can be decomposed into stable and unstable bundles N^s and N^u, and there are metrics g^s and g^u on N^s and N^u such that $f(v)|_N = f(C_i) - |v^s|^2_{g^s} + |v^u|^2_{g^u}$, where $v \in N$, and v^s and v^u are the stable and unstable components of v.*

If we fix a connection on N, then g^s and g^u can be understood as bilinear forms on TN. Let g_{C_i} be a metric on C_i. If a metric g near C_i has the form $\pi^* g_{C_i} + g^s + g^u$, where π is the projection $N \to C_i$, we say the metric g is standard near C_i. In fact, we can require the Morse–Bott–Smale pair to have standard metric near all critical manifolds, as we can obtain transversality by perturbing the metric away from critical manifolds. For a standard metric, the gradient vector in N is contained in the fibers of the tubular neighborhood. Therefore the local picture of the gradient flow is just a family of the Morse flow lines in each fiber. When restricted to a fiber F with coordinate $x_1, \dots, x_s, y_1, \dots, y_u$, the pair (f,g) is standard and is in the form

$$f|_F = -x_1^2 - \cdots - x_s^2 + y_1^2 + \cdots + y_u^2 + C,$$
$$g|_F = \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \cdots + \mathrm{d}x_s \otimes \mathrm{d}x_2 + \mathrm{d}y_1 \otimes \mathrm{d}y_1 + \cdots + \mathrm{d}y_u \otimes \mathrm{d}y_u.$$

Inside the fiber F, we define

$$S_s^r := \{(x_1, \dots, x_s) \mid x_1^2 + \cdots + x_s^2 = r^2\}, \quad S_u^r := \{(y_1, \dots, y_u) \mid y_1^2 + \cdots + y_u^2 = r^2\},$$
$$D_s^r := \{(x_1, \dots, x_s) \mid x_1^2 + \cdots + x_s^2 < r^2\}, \quad D_u^r := \{(y_1, \dots, y_u) \mid y_1^2 + \cdots + y_u^2 < r^2\}.$$

[27] One condition that guarantees compatibility is the so-called "associative gluing" [74].

Let $\mathcal{M}$ be the moduli space of gradient flow lines and broken gradient flow lines of $(f|_F, g|_F)$ from $S_s^r \times D_u^r$ to $D_s^r \times S_u^r$. Let ev_- and ev_+ be the two evaluation maps at the two ends defined on $\mathcal{M}$. Then the following lemma is essentially contained in [74]:

Lemma 8.7 *The image* $\mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(\mathcal{M}) \subset (S_s^r \times D_u^r) \times (D_s^r \times S_u^r)$ *is a submanifold with boundary inside the fiber* F*.*

Proof The gradient flow lines are $(e^{-2t}x, e^{2t}y)$, and thus the images of unbroken flow lines are $(x, y, (|y|/r)x, (r/|y|)y)$, which is a submanifold in $(S_s^r \times D_u^r) \times (D_s^r \times S_u^r)$. The images of broken flow lines are $(x, 0, 0, y)$, which is also a submanifold in $(S_s^r \times D_u^r) \times (D_s^r \times S_u^r)$. The boundary chart is given by $(t, x, 0, 0, y) \to (x, ty, tx, y)$ for $t \in [0, 1)$; thus the lemma is proven. ◻

Remark 8.8 Lemma 4.4 of [74] composes the map $\mathrm{ev}_- \times \mathrm{ev}_+$ with the projection $(x, y', x', y) \to ((|x|' + |y'|)/(2r), x, y)$ to get a homeomorphism from $\mathcal{M}$ to $[0, 1) \times S_s^r \times S_u^r$. This was used in [74] to construct a smooth structure with boundaries and corners on $\mathcal{M}$. Since the projection restricted to $\mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(\mathcal{M})$ is a diffeomorphism, we can also use the smooth structure on $\mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(\mathcal{M})$ to make $\mathcal{M}$ into a manifold with boundaries and corners.

Since $S_s^r \times D_u^r$ and $D_s^r \times S_u^r$ are transverse to the gradient flow, Lemma 8.7 also holds if we replace $S_s^r \times D_u^r$ and $D_s^r \times S_u^r$ by open sets in $f|_F^{-1}(C-\epsilon)$ and $f|_F^{-1}(C+\epsilon)$. Now we return to the Morse–Bott case with a standard metric near C_i. Let ϕ^t be the flow for ∇f. Then the stable manifold S_i of C_i is defined to be

$$S_i = \{x \in M \mid \lim_{t\to\infty} \phi^t(x) \in C_i\},$$

and the unstable manifold U_i is defined to be

$$U_i = \{x \in M \mid \lim_{t\to-\infty} \phi^t(x) \in C_i\}.$$

Both S_i and U_i are equipped with smooth evaluation maps to C_i. Then we have the family version of Lemma 8.7 as follows:

Lemma 8.9 *Given a standard metric near* C_i*, let* N_r *be the radius-*r *open tube of* C_i*. Suppose* ϵ *is a small positive real number, and* $v_i^{\pm\epsilon}$ *denotes* $f(C_i) \pm \epsilon$*. Let* $\mathcal{M}_{i,\epsilon,r}$ *denote the moduli space of flow lines and broken flow lines from* $f^{-1}(v_i^{-\epsilon}) \cap N_r$ *to* $f^{-1}(v_i^{+\epsilon}) \cap N_r$*. Then there exist* $\epsilon, r > 0$ *such that the image of* $\mathrm{ev}_- \times \mathrm{ev}_+|_{\mathcal{M}_{i,\epsilon,r}}$ *is a submanifold with boundary in* $(f^{-1}(v_i^{-\epsilon}) \cap N_r) \times (f^{-1}(v_i^{+\epsilon}) \cap N_r)$*, and the boundary is* $(S_i \cap f^{-1}(v_i^{-\epsilon})) \times_{C_i} (U_i \cap f^{-1}(v_i^{+\epsilon}))$*.*

Proposition 8.10 $M_{i,j} \times_j M_{j,k} \cup M_{i,k}$ *can be given the structure of a manifold with boundary.*

Proof Since we have diffeomorphisms

$$M_{i,j} \simeq U_i \cap S_j \cap f^{-1}(v_j^{-\epsilon}) \quad \text{and} \quad M_{j,k} \simeq U_j \cap S_k \cap f^{-1}(v_j^{+\epsilon}),$$

the Morse–Bott–Smale condition implies that the intersections are transverse. On the other hand, let $M_{i,k} \cap \mathcal{M}_{j,\epsilon,r}$ be the set of flow lines in $M_{i,k}$ which contains a flow line in $\mathcal{M}_{j,\epsilon,r}$. Then it is an open set of $M_{i,k}$, and we have the embedding

$$\mathrm{ev}_- \times \mathrm{ev}_+ \colon M_{i,k} \cap \mathcal{M}_{j,\epsilon,r} \to (f^{-1}(v_j^{-\epsilon}) \cap N_r) \times (f^{-1}(v_j^{+\epsilon}) \cap N_r).$$

The image is

$$\mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(M_{i,k} \cap \mathcal{M}_{j,\epsilon,r}) = \mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(\partial_0 \mathcal{M}_{j,\epsilon,r}) \cap \big((U_i \cap f^{-1}(v_j^{-\epsilon})) \times (S_k \cap f^{-1}(v_j^{+\epsilon}))\big),$$

where $\partial_0 \mathcal{M}_{j,\epsilon,r}$ is the interior (depth-0 boundary, Definition 2.1) of $\mathcal{M}_{j,\epsilon,r}$. The Morse–Bott–Smale condition implies that the intersection is transverse. Moreover, since the fiber product $M_{i,j} \times_j M_{j,k}$ is transverse, $\partial\, \mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(\mathcal{M}_{j,\epsilon,r}) = (S_j \cap f^{-1}(v_j^{-\epsilon})) \times_{C_j} (U_j \cap f^{-1}(v_j^{+\epsilon}))$ is also transverse to $(U_i \cap f^{-1}(v_j^{-\epsilon})) \times (S_k \cap f^{-1}(v_j^{+\epsilon}))$. Thus $\mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(M_{i,k} \cap \mathcal{M}_{j,\epsilon,r})$ can be completed by the boundary structure of $\mathrm{im}(\mathrm{ev}_- \times \mathrm{ev}_+)(\mathcal{M}_{j,\epsilon,r})$. That is, we can add in

$$(U_i \cap S_j \cap f^{-1}(v_j^{-\epsilon})) \times_{C_j} (S_k \cap U_j \cap f^{-1}(v_j^{+\epsilon})) \simeq M_{i,j} \times_j M_{j,k}$$

as the boundary of $M_{i,k} \cap \mathcal{M}_{j,\epsilon,r}$. The topology check is analogous to [74]. □

Therefore we have a smooth boundary structure on $M_{i,j} \times_j M_{j,k} \subset \mathcal{M}_{i,k}$. We still need to construct corner structures near curves with multiple breaking and prove the compatibility of smooth structures. The proof is very similar, and the corner structure will be inherited from (fiber) products of the manifolds with boundary in Lemma 8.9.

Proposition 8.11 *$M_{i,j} \times_j M_{j,k} \times_k M_{k,l} \cup M_{i,k} \times_k M_{k,l} \cup M_{i,j} \times_j M_{j,l} \cup M_{i,l}$ can be given the structure of manifold with boundaries and corners, which is compatible with structure given in Proposition 8.10.*

Proof Let $N_{*,r}$ denote the radius-r open tube around C_*. We use $\mathcal{M}_{j,k,\epsilon,r}$ to denote the moduli space of gradient flow lines from $f^{-1}(v_j^{-\epsilon}) \cap N_{j,r}$ to $f^{-1}(v_k^{+\epsilon}) \cap N_{k,r}$, passing through $f^{-1}(v_j^{+\epsilon}) \cap N_{j,r}$ and $f^{-1}(v_k^{-\epsilon}) \cap N_{k,r}$, such that the only breaking allowed is at C_j or C_k, or both. Then $\mathrm{ev}_{-,+,-,+} := \mathrm{ev}_- \times \mathrm{ev}_+ \times \mathrm{ev}_- \times \mathrm{ev}_+$ defines an embedding

$$\mathcal{M}_{j,k,\epsilon,r} \to (f^{-1}(v_j^{-\epsilon}) \cap N_{j,r}) \times (f^{-1}(v_j^{+\epsilon}) \cap N_{j,r}) \times (f^{-1}(v_k^{-\epsilon}) \cap N_{k,r}) \times (f^{-1}(v_k^{+\epsilon}) \cap N_{k,r}).$$

We define $V \subset f^{-1}(v_j^{+\epsilon}) \cap N_{j,r}, U \subset f^{-1}(v_k^{-\epsilon}) \cap N_{k,r}$ be the sets such that the flow lines from V will end in U without breaking. Then V and U are both open subsets and there is a diffeomorphism $\phi \colon V \to U$ defined using the gradient flow, and so $\mathrm{im}(\mathrm{ev}_{-,+,-,+})$ is contained inside the fiber product $(f^{-1}(v_j^{-\epsilon}) \cap N_{j,r}) \times V \times_\phi U \times (f^{-1}(v_k^{+\epsilon}) \cap N_{k,r})$. By a little abuse of notation, we use $V \cap \mathcal{M}_{j,\epsilon,r}$ to denote $\mathrm{ev}_+^{-1}(V) \subset \mathcal{M}_{j,\epsilon,r}$ and $U \cap \mathcal{M}_{k,\epsilon,r}$ to denote $\mathrm{ev}_-^{-1}(U) \subset \mathcal{M}_{k,\epsilon,r}$, which are both open subsets and inherit the structure of a manifold with boundary from Lemma 8.9. Then $\mathrm{im}(\mathrm{ev}_{-,+,-,+}) = \mathrm{ev}_{-,+}(V \cap \mathcal{M}_{j,\epsilon,r}) \times_\phi \mathrm{ev}_{-,+}(U \cap \mathcal{M}_{k,\epsilon,r})$. The Morse–Bott–Smale condition implies that the fiber product $\mathrm{ev}_{-,+}(V \cap \mathcal{M}_{j,\epsilon,r}) \times_\phi \mathrm{ev}_{-,+}(U \cap \mathcal{M}_{k,\epsilon,r})$ is cut out transversely as a manifold with boundaries and corners. Therefore $\mathcal{M}_{j,k,\epsilon,r}$ inherits the structure of a manifold with corners from its image under $\operatorname{im}\mathrm{ev}_{-,+,-,+}$, whose depth-1 boundary is

$$(\mathrm{ev}_{-,+}(V\cap\partial_1\mathcal{M}_{j,\epsilon,r})\times_\phi \mathrm{ev}_{-,+}(U\cap\partial_0\mathcal{M}_{k,\epsilon,r}))\cup(\mathrm{ev}_{-,+}(V\cap\partial_0\mathcal{M}_{j,\epsilon,r})\times_\phi \mathrm{ev}_{-,+}(U\cap\partial_1\mathcal{M}_{k,\epsilon,r})),$$

and depth-2 boundary (corner) is $\mathrm{ev}_{-,+}(V\cap\partial_1\mathcal{M}_{j,\epsilon,r})\times_\phi \mathrm{ev}_{-,+}(U\cap\partial_1\mathcal{M}_{k,\epsilon,r})$.

We define $M_{i,l}\cap\mathcal{M}_{j,k,\epsilon,r}$ to be the open subset of $M_{i,l}$ consisting of flow lines with a portion in $\mathcal{M}_{j,k,\epsilon,r}$. Similar to the proof of Proposition 8.10, we can use the boundary and corner structures on $\mathcal{M}_{j,k,\epsilon,r}$ to give a corner structure near $M_{i,l}\cap\mathcal{M}_{j,k,\epsilon,r}$ by intersecting the unstable and stable manifolds of C_i and C_l with $\mathrm{im}(\mathrm{ev}_{-,+,-+})$ inside $(f^{-1}(v_j^{-\epsilon})\cap N_{j,r})\times(f^{-1}(v_j^{+\epsilon})\cap N_{j,r})\times(f^{-1}(v_k^{-\epsilon})\cap N_{k,r})\times(f^{-1}(v_k^{+\epsilon})\cap N_{k,r})$. More explicitly, we get a corner structure near $M_{i,j}\times_j M_{j,k}\times_k M_{k,l}$, which also gives a boundary structure near $M_{i,j}\times_j(M_{j,l}\cap(U\cap\partial_0\mathcal{M}_{k,\epsilon,r}))$ and $(M_{i,k}\cap(V\cap\partial_0\mathcal{M}_{j,\epsilon,r}))\times_k M_{k,l}$. Moreover, the boundary structure is exactly the one constructed in Proposition 8.10. This finishes the proof. □

Proof of Theorem 8.5 Following the same proof as that of Proposition 8.11, we can prove that $\mathcal{M}_{i,j}$ is endowed with the structure of compact manifold with boundaries and corners. Let o_i be the determinant line bundle of the stable bundle N^s over C_i. Then $\{C_i,\mathcal{M}_{i,j}\}$ defines a flow category $\mathcal{C}_{f,g}$ with an orientation structure following the construction in Section 5.1.2. □

8.3 Morphisms and homotopies

To derive the flow morphisms between different Morse–Bott functions and flow homotopies between them, we will use the argument from [2] to reduce the construction of flow morphisms and flow homotopies back to flow categories.

8.3.1 Flow morphisms [2, Theorem 3.4.2, first step] Let (f_1,g_1) and (f_2,g_2) be two locally standard Morse–Bott–Smale pairs, and let $\mathcal{C}^1=\{C_i^1,\mathcal{M}_{i,j}^1\}$ and $\mathcal{C}^2=\{C_i^2,\mathcal{M}_{i,j}^2\}$ denote the associated flow categories. We can find a smooth function $F:\mathbb{R}\times M\to\mathbb{R}$ such that

$$F(t,x)=\begin{cases} f_1(x) & \text{if } t<\frac{1}{3},\\ f_2(x) & \text{if } t>\frac{2}{3}.\end{cases}$$

We consider a Morse function h on $\mathbb{R}$ that only has two critical points: one local minimum at 0 and one local maximum at 1. Also, h satisfies

$$\frac{\partial F}{\partial t}+\frac{dh}{dt}>0\quad\text{for all } x\in M \text{ and } t\in(0,1).$$

Then $F+h$ defines a Morse–Bott function on $\mathbb{R}\times M$ with critical manifolds $\{C_i^1\times\{0\}\}$ and $\{C_i^2\times\{1\}\}$. We can find a locally standard metric G such that

$$G(t,x)=\begin{cases} g_1+\mathrm{d}t\otimes\mathrm{d}t & \text{if } t<\frac{1}{3},\\ g_2+\mathrm{d}t\otimes\mathrm{d}t & \text{if } t>\frac{2}{3}.\end{cases}$$

We can assume (F,G) is a locally standard Morse–Bott–Smale pair. Then by Theorem 8.5, we can associate to $(F+h,G)$ a flow category with an orientation structure. Let $\mathcal{F}_{i,j}$ dquadenote the compactified moduli space of flow lines from $C_i^1 \times \{0\}$ to $C_j^2 \times \{1\}$. Then $\mathcal{F}_{i,j}$ forms a flow morphism $\mathfrak{F}$ from $\mathcal{C}^1$ to $\mathcal{C}^2$. When $F(t,x) = f(x)$, we can choose metric $g + dt^2$. Then $F_{i,i} = C_i$ and $F_{i,j} \simeq \mathcal{M}_{i,j} \times [0, j-i] \simeq I_{i,j}$ for $i < j$, that is, the construction gives the identity flow morphism [2, Theorem 3.4.2. second step].

8.3.2 Flow homotopies [2, Theorem 3.4.2, third step] Assume we have continuations F, G and H from f_1 to f_2, f_2 to f_3 and f_1 to f_3, respectively. Then we can find $K\colon \mathbb{R}_s \times \mathbb{R}_t \times M \to \mathbb{R}$ such that

$$K(s,t,x) = \begin{cases} H(t,x) & \text{if } s < \frac{1}{3}, \\ F(s,x) & \text{if } t < \frac{1}{3}, \\ G(t,x) & \text{if } s > \frac{2}{3}, \\ f_3(x) & \text{if } t > \frac{2}{3}. \end{cases}$$

We can find h with one local minimum at 0 and local maximum at 1 such that

$$\frac{\partial K}{\partial s} + h'(s) > 0 \quad \forall (s,t,x) \in (0,1) \times \mathbb{R} \times M \qquad \text{and} \qquad \frac{\partial K}{\partial t} + h'(t) > 0 \quad \forall (s,t,x) \in \mathbb{R} \times (0,1) \times M.$$

Then $K + h(s) + h(t)$ defines a Morse–Bott function, with critical manifolds $\{C_i^1 \times \{(0,0)\}\}$, $\{C_i^2 \times \{(1,0)\}\}$, $\{C_i^3 \times \{(0,1)\}\}$ and $\{C_i^3 \times \{(1,1)\}\}$, and we can find a locally standard Morse–Bott–Smale metric extending the locally standard metrics used in F, G, H and f_3. Then the flow lines from $C_i^1 \times \{(0,0)\}$ to $C_j^3 \times \{(1,1)\}$ give rise to a flow homotopy between $\mathfrak{G} \circ \mathfrak{F}$ and $\mathfrak{I} \circ \mathfrak{H}$.

Proof of Theorem 8.1 By Theorem 8.5, we have a flow category $\mathcal{C}_{f,g}$ with an orientation structure for any locally standard Morse–Bott–Smale pair (f,g). Using the flow morphisms and flow homotopies above, we can see that the cohomology of $\mathcal{C}_{f,g}$ does not depend on (f,g). Thus we can let $f \equiv C$, and g be any metric. Then (f,g) is a locally standard Morse–Bott–Smale pair. The object space and morphism space of the corresponding flow category are both M; thus the cohomology of the flow category equals the cohomology $H^*(M,\mathbb{R})$. □

A Morse–Smale pair is a special case of a Morse–Bott–Smale pair, and our definition of the minimal Morse–Bott cochain complex recovers the Morse cochain complex when the function is Morse. As a corollary, the $\mathbb{R}$ coefficient Morse cohomology equals the de Rham cohomology of M.

8.4 Noncompact case

Let M be a noncompact manifold of finite type, as introduced in Definition 6.1, throughout this subsection. That is, M is the set of interior points of a compact manifold with nonempty boundary. Let ∂_r be a nonzero outward-pointing vector field on the collar neighborhood of the end of M. In the following, we will only consider two types of Morse–Bott functions:

(1) Morse–Bott functions f such that $\partial_r f > 0$ on the collar,

(2) constant functions.

In (1), we have a flow category $\mathcal{C}_f$ by Theorem 8.5. In (2), the flow category is a single space M, which is a proper flow category. Next we will show how to associate a flow morphism between flow categories from different Morse–Bott functions and flow homotopy between them. Once they are set up like the compact case, we have that the cohomology of the flow category is independent of the Morse–Bott function. In particular, one can choose a constant, and hence the cohomology is the regular cohomology.

8.4.1 Flow morphisms and homotopies Given two admissible Morse–Bott functions f_1 and f_2 on M, the homotopy between them is a smooth function $F\colon \mathbb{R} \times M \to \mathbb{R}$ such that

$$F(t,x) = \begin{cases} f_1(x) & \text{if } t < \frac{1}{3}, \\ f_2(x) & \text{if } t > \frac{2}{3}, \end{cases}$$

and when $t \in \left(\frac{1}{3}, \frac{2}{3}\right)$, we have $\partial_r F(t,x) > 0$ on the collar. Then $h + F$ defines a Morse–Bott function on $\mathbb{R} \times M$, and we claim that the associated flow category defines a proper flow morphism from $\mathcal{C}_{f_1}$ to $\mathcal{C}_{f_2}$. We may assume the metric on $\mathbb{R} \times M$ has the property that the gradient for the collar coordinate $r \in (-1, 0)$ is ∂_r on the collar. Then $\partial_r F(t,x) \geq 0$ for all t implies that $\partial_r F(t,x) = \partial_r(h + F(t,x)) = \langle \nabla r, \nabla(h + F(t,x)) \rangle \geq 0$. Therefore any gradient flow line from a critical point of f_1 to a critical point f_2 has the property that if it touches the collar then it stays in the collar after the touching point. In addition to the argument in Section 8.3, we need to show the properness of the target maps in order to prove the claim. We divide it into the following cases.

(i) **Both f_1 and f_2 are of type (1)** Any gradient flow line that touches the collar neighborhood cannot return to the interior side. Hence the construction in Section 8.3 gives compact moduli spaces and a flow morphism from $\mathcal{C}_{f_1}$ to $\mathcal{C}_{f_2}$.

(ii) **f_1 is of type (2) and f_2 is of type (1)** The same argument as in case (i) holds.

(iii) **f_1 is of type (1) and f_2 is of type (2)** Let $K \subset M = \mathrm{Crit}(f_2)$ be a compact subset. For points outside the collar, we define $r = -1$. Let $R := \max\{r(x) \mid x \in K\}$. Then $R < 0$ and all gradient flow lines from critical points of f_1 to a point in K stay inside the domain $[0,1] \times \{r \leq R\}$, and hence the space of such flow lines is compact. This shows that the target maps are proper.

(iv) **Both f_1 and f_2 are of type (2)** The same argument as in case (iii) holds.

Remark 8.12 If we replace the condition on the collar by $\partial_r F(t,x) < 0$, this would force f_1 and f_2 to have the property that $\partial_r f_1, \partial_r f_2 < 0$ if they are not constant. In this case, the gradient flow lines in $\mathbb{R} \times M$ will shrink on the collar neighborhood instead of expanding, and hence the source map is proper and the target map is not. We can similarly define a cochain complex using the compactly supported cohomology in this case. The cohomology of the cochain complex is the compactly supported cohomology, which is isomorphic to the homology.

The asymmetry of the flow morphism prevents us from constructing a flow morphism from $\mathcal{C}_f$ to $\mathcal{C}_f$. Assume $f > 0$ without loss of generality. There exists a flow morphism from $\mathcal{C}_f$ to $\mathcal{C}_{2f}$ constructed from $F(t,x) = \phi(t) f(x)$, where $\phi(t)$ is an increasing function with $\phi(t) = 1$ for $t \leq 0$ and $\phi(t) = 2$ for $t \geq 1$. The flow morphism is diffeomorphic to the identity flow morphism when we use the metric $g + dt^2$. The flow homotopy follows from the same argument as if we require the increasing property on the collar when constructing the homotopy of homotopy. Therefore we have the invariance of the cohomology with respect to the Morse–Bott function:

Theorem 8.13 *If M is a noncompact manifold of finite type and f is a Morse–Bott function of type* (1) *or* (2), *then the flow category $\mathcal{C}_f$ is proper and has a local system such that the cohomology is $H^*(M;\mathbb{R})$.*

8.4.2 The Gysin exact sequence Let M be an n–dimensional manifold of finite type. Assume f is a Morse–Bott function on M and, when M is noncompact, f is one of the two admissible types (1) or (2). Let g be a metric such that (f,g) is a locally standard Morse–Bott–Smale pair. Then we have a (proper) flow category $\mathcal{C}_f=\{C_i,\mathcal{M}_{i,j}\}$. Let $\pi\colon E\to M$ be a oriented k–sphere bundle. Then π^*f is a Morse–Bott function on E with critical manifolds $\{\pi^{-1}(C_i)\}$. We pick a metric g_F on the fibers of E, (a metric only defined on the subbundle of fiber directions T^vE of TE). Fix a connection of $TE=T^vE\oplus T^hE$. Then g_F can be understood as a semipositive bilinear form on TE vanishing on T^hE, and $g_F+\pi^*g$ is a metric on E. It can be verified directly that a gradient flow line $\tilde{\gamma}$ of $(\pi^*f, g_F+\pi_*g)$ is a parallel lift of a gradient flow line γ of (f,g). Hence $(\pi^*f, g_F+\pi_*g)$ is again a Morse–Bott–Smale pair, and the induced flow category $\mathcal{C}_{\pi^*f}$ is given by

$$\mathrm{Obj}(\mathcal{C}_{\pi^*f})=\{E_i:=\pi^{-1}(C_i)\}\quad\text{and}\quad \mathrm{Mor}(\mathcal{C}_{\pi^*f})=\{\mathcal{M}^E_{i,j}=s^*_{i,j}E_i\}.$$

The source map is the natural map and the target map is given by the parallel transportation along flow lines in $\mathcal{M}_{i,j}$. As a consequence, we have an oriented k–sphere bundle $\mathcal{C}_{\pi^*f}\to\mathcal{C}_f$. The flow morphisms and flow homotopies defined in the previous discussions can be lifted to the sphere bundle level by the same parallel transportation construction. Therefore the induced Gysin exact sequence is independent of the function f. In particular, one may choose f to be constant, and hence the Gysin exact sequence will become the usual Gysin exact sequence by Proposition 6.24. Therefore we have the following isomorphism of long exact sequences:

Theorem 8.14 *Let M be an n–dimensional manifold of finite type and $\pi\colon E\to M$ a k–sphere bundle. Suppose f is an admissible Morse–Bott function on M. Then we have the following isomorphic long exact sequences*:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(\mathcal{C}_f) \arrow[r] \arrow[d] & H^i(\mathcal{C}_{\pi^*f}) \arrow[r] \arrow[d] & H^{i-k}(\mathcal{C}_f) \arrow[r] \arrow[d] & H^{i+1}(\mathcal{C}_f) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^i(M) \arrow[r, "\pi^*"] & H^i(E) \arrow[r, "\pi_*"] & H^{i-k}M \arrow[r, "\wedge(-1)^{\dim C+1}e"] & H^{i+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

9 Transversality by polyfold theory

With the theory on flow categories developed in the previous sections, we now want to get flow categories in applications, ie we need to solve the transversality problems. For this purpose, we will adopt the polyfold theory developed by Hofer, Wysocki and Zehnder [40; 42; 43; 41; 44]. This section outlines some ideas on combining our construction with polyfold theory; details will appear in a future work.

9.1 Polyflow categories

The main result of Section 3 is that, for any oriented flow category, we can construct a well-defined cochain complex up to homotopy. If we want to write down a representative cochain complex of the homotopy class, we need to fix defining data Θ. In applications, take Hamiltonian Floer cohomology as an example, the flow category consists of the zero sets of some sc–Fredholm sections over a family of polyfolds [73]. A natural idea is that we replace every manifold $\mathcal{M}_{i,j}$ in the flow category by strong polyfold bundle $W_{i,j} \to Z_{i,j}$ with an sc–Fredholm section $\kappa_{i,j}$ such that all $W_{i,j} \to Z_{i,j}, \kappa_{i,j}$ are organized just like a flow category. When all $\kappa_{i,j}$ are transverse to 0, then $\kappa_{i,j}^{-1}(0)$ defines a flow category. In this case, we expect to assign a well-defined cochain complex to such a system of polyfolds up to homotopy. When we need to write down an explicit representative cochain complex for the homotopy class, we need to fix a family of perturbations that are compatible with category structure and defining data (on C_i), which does not depend on the perturbation. We first give a preliminary definition of such a system:

Definition 9.1 A *polyflow category* is a small category $\mathcal{Z}$ with following properties:

(1) The object space $\mathrm{Obj}(\mathcal{Z}) = C := \bigsqcup_{i\in\mathbb{Z}} C_i$ is the disjoint union of manifolds C_i such that each connected component of C_i is a manifold of finite type (Definition 6.1).

(2) The morphism space $\mathrm{Mor}(\mathcal{Z}) = Z$ is a polyfold. The source and target maps $s, t\colon Z \to C$ are sc–smooth. Let $Z_{i,j}$ denote $(s\times t)^{-1}(C_i \times C_j)$.

(3) $Z_{i,i} \simeq C_i$ (the identity morphisms), $Z_{i,j} = \varnothing$ for $j < i$, and $Z_{i,j}$ is a polyfold for $j > i$.

(4) The fiber product $Z_{i_0,i_1} \times_{i_1} Z_{i_1,i_2} \times_{i_2} \cdots \times_{i_{k-1}} Z_{i_{k-1},i_k}$ is cut transversely, for all increasing sequences $i_0 < i_1 < \cdots < i_k$.

(5) The composition $m\colon Z_{i,j} \times_j Z_{j,k} \to Z_{i,k}$ is an sc–smooth injective map into the boundary of $Z_{i,k}$. Moreover, $\partial Z_{j,k} = \bigcup_{i<j<k} m(Z_{i,j} \times_j Z_{j,k})$ and $d(x)+d(y)+1 = d(m(x,y))$ for $(x,y) \in Z_{i,j} \times_j Z_{j,k}$, where d is the degeneracy index [44, Definition 2.4.1]. When restricted to any stratum of fixed degeneracy index, m is a local sc–diffeomorphism to a stratum with a fixed degeneracy index.

(6) There are strong polyfold bundles $W_{i,j} \to Z_{i,j}$ and sc–Fredholm sections $\kappa_{i,j}$ such that both bundles and sections are compatible with m, ie $m^* W_{i,k}|_{Z_{i,j}\times_j Z_{j,k}} = W_{i,j} \times W_{j,k}$ and $\kappa_{i,k}|_{m(Z_{i,j}\times_j Z_{j,k})} = m(s_{i,j}, s_{j,k})$.

(7) $\kappa_{i,j}^{-1}(0) \cap t_{i,j}^{-1}(K)$ is compact for every compact set $K \cap C_j$.

Remark 9.2 (i) Condition (4) can be replaced by the more convenient condition that the $(s\times t)|_{Z_{i,j}}$ are submersions. Then (4) follows from [25].

(ii) The index $\operatorname{ind} s_{i,j}$ plays the role of $m_{i,j}$. Orientation structures defined in Section 5 can be generalized to polyflow categories such that orientation structures are enough to give coherent orientations or local systems on flow categories from perturbations in Claim 9.3.

(iii) Condition (5) is stronger than Definition 2.9(4). When we define operators from a flow category, we use integration and Stokes' theorem. Hence an almost identification on the boundary is enough. However, in the polyflow category, we need to perturb $Z_{i,j}$ inductively in a coherent way, which requires a finer identification of all the boundary and corner structures.

When all sections $\kappa_{i,j}$ are transverse to 0, the zero sets form a proper flow category. Hence our goal is to find a family of sc^+–perturbations $\tau_{i,j}$ such that $s_{i,j}+\tau_{i,j}$ is transverse in general position and consistent with the composition m. The consistency depends on the combinatorics of the problem in general. In the case of polyflow categories, the combinatorics are relatively simple and we expect to have a perturbation scheme.

Claim 9.3 *There exist coherent perturbations $\tau_{i,j}$ such that $\kappa_{i,j}+\tau_{i,j}$ is transverse to* 0 *and in general position* [44, Definition 5.3.9].

Remark 9.4 The claim does not hold when there are inner symmetries that we want to preserve. To be more precise, assume we have a strong polyfold bundle $W \to Z$ with two submersive evaluation maps $s,t\colon Z\to C$. Let $\kappa\colon Z\to W$ be a Fredholm section. When $\dim C>0$, given any transverse perturbation $\tau\colon Z\to W$, it is not necessarily true that (τ,τ) is a transverse perturbation to (κ,κ) on the fiber product $Z_t\times_s Z$. In fact, it is possible that there is no transverse perturbation to (κ,κ) on $Z_t\times_s Z$ in the form of (τ,τ) for a perturbation $\tau\colon Z\to W$. Such phenomena can appear in a polyflow category, eg we may have $C_i=C_j=C_k$, $W_{i,j}=W_{j,k}$ and $\kappa_{i,j}=\kappa_{j,k}$. If we require $\tau_{i,j}=\tau_{j,k}$, then we run into this problem. In applications, for example Hamiltonian Floer cohomology, we see this when the Novikov coefficient has to be used. The requirement of symmetry in perturbations guarantees the cochain complex is a module over the Novikov field. In the S^1–Morse theory case, this also causes problems (self-gluing) in the homotopy argument. The homotopy argument can be viewed as a Morse–Bott problem with critical manifolds copies of $\mathbb{R}$. In these two explicit examples, special methods can be adopted to overcome the challenge. In the most general case, under certain assumptions[28] of the polyflow category, we can actually perturb the source and target maps consistently to destroy all the inner symmetries. We will discuss this in detail in our future work.

Although the polyfold perturbation only produces weighted branched suborbifolds as the transverse zero sets, it causes no problem, since the convergence results (Lemmas 3.7 and 3.14), are local in nature. The only thing we need about $\mathcal{M}_{i,j}$ is Stokes' theorem, which was proven in [43]. Thus all the proofs

[28] Basically, we require a collar neighborhood near the boundaries and corners of polyfolds. Such assumptions are satisfied in all known examples.

in Section 3 apply to the weighted branched suborbifold case. Similar to Definition 9.1, we can define polyflow morphisms and polyflow homotopies by replacing the manifolds by polyfolds with sc–Fredholm sections. Once the perturbation scheme is given for those structures, we can generate flow morphisms and flow homotopies.

Remark 9.5 To generalize the identity flow category (Definition/Lemma 3.23) to the polyfold case, the naive construction of multiplying by an interval does not work, because the product with an interval does not have the right boundary and corner structures to apply an inductive perturbation scheme. However, there is a more natural construction of the identity (poly)flow category which has the right boundary and corner structures. The construction is closely related to the geometric realization of the category, which will be discussed in a future work.

The enrichment to polyflow categories causes more choices, ie the choice of perturbation. We would like to have the cohomology independent of the perturbation. Such invariance can be proven using the identity polyflow category or a homotopy argument.

Claim 9.6 *Let $\mathcal{Z}$ be a polyflow category with orientation structures. If there is no inner symmetry,*[29] *then we can associate it with a Morse–Bott cochain complex* $(\mathrm{BC}(\mathcal{Z}), d_{\mathrm{BC}})$ *such that the homotopy type of the cochain complex is independent of defining data and* sc^+*–perturbations.*

9.2 Equivariant theory

In Section 7, we discuss the equivariant theory when the flow category is equipped with a group action. However, requiring G symmetry on the flow category is equivalent to requiring G–equivariant transversality on the background polyflow category. Since G–equivariant transversality is often obstructed, the construction in Section 7 cannot be applied directly. However, the construction in Section 7 can be generalized to polyflow categories. Hence we can apply the Borel construction on the level of polyfolds.

Definition 9.7 Let $\mathcal{Z}$ be a polyflow category. A compact Lie group G acts on $\mathcal{Z}$ if and only if G acts on C_i and $W_{i,j} \to Z_{i,j}$ in the sense of [78, Definition 3.66] so that all sc–Fredholm sections $\kappa_{i,j}$ and the structure maps s, t and m are G–equivariant.

Assume G acts a polyflow category $\mathcal{Z}$. If we fix an approximation E_n of EG, then we can form a sequence of polyflow categories $\mathcal{Z} \times_G E_n$ by the quotient construction in [78]. Using the identity polyflow morphism and the construction in Section 7, we have a sequence of polyflow morphisms connecting different $\mathcal{Z} \times_G E_n$. Then we have a directed system in the "category" of polyflow categories. We can get an inverse system of cochain complexes by applying Claim 9.6. Then the equivariant cochain complex will be the homotopy limit of such an inverse system. Details of the construction will appear in a future work.

[29] Or collar neighborhood assumptions on the polyfolds hold, if there are inner symmetries.

Appendix A Convergence

This section proves the convergence results used in Section 3. We will see that transversality of fiber products is not only natural from the polyfold point of view as explained in Section 9, but also essential in proving the convergence results, especially Lemma 3.14.

A.1 The Thom class

We review the construction of Thom classes in [11, Section 6]. Let $\pi\colon E \to M$ be an oriented vector bundle with a metric over an oriented manifold. The fiber F, the base manifold M and the total space E are oriented in the manner of $[M][F]=[E]$. If $S(E)$ denotes the sphere bundle of E, then we can find a form ψ (an angular form) on $S(E)$ such that the integration over each fiber is 1, and $\mathrm{d}\psi=-\pi^*e$, where e is the Euler class of the sphere bundle. Then we pick smooth functions $\rho_n\colon \mathbb{R}^+ \to \mathbb{R}$ such that ρ_n is increasing, supported in $[0,1/n]$ and is -1 near 0; see Figure 4.

Then $\mathrm{d}(\rho_n\psi)$ defines a form on $\mathbb{R}^+\times S(E)$, and it is π^*e on an open neighborhood of $\{0\}\times S(E)$. Thus $\mathrm{d}(\rho_n\psi)$ is a lift of some form on E, that is, $\mathrm{d}(\rho_n\psi)=p^*\delta^n$ for $\delta^n\in\Omega^*(E)$, where p is the natural map $\mathbb{R}^+\times S(E)\to E$. This δ^n is a Thom class of $\pi\colon E\to M$. The next lemma asserts that δ^n actually represent the zero section not only in the cohomological sense, but also in a stronger sense of currents. Let δ_M denote the Dirac current of the zero section: $\delta_M(\alpha)=\int_M i^*\alpha$ for $\alpha\in\Omega^*(E)$, where $i\colon M\to E$ is the zero section.

Lemma A.1 (Lemma 3.1) *We have $\delta^n\to\delta_M$ in the sense of currents, ie for all $\alpha\in\Omega^*(E)$,*

$$\lim_{n\to\infty}\int_E \alpha\wedge\delta^n \to \delta_M(\alpha).$$

Proof Let $F\simeq\mathbb{R}^n$ be a fiber of the bundle. Since δ^n is compactly supported, the integration over a fiber is

$$\int_F\delta^n=\int_{F-\{0\}}\delta^n=\int_{(0,\infty)\times S^{n-1}}p^*\delta^n=\int_{[0,\infty)\times S^{n-1}}p^*\delta^n=\int_{[0,\infty)\times S^{n-1}}\mathrm{d}(\rho_n\psi)=-\int_{\{0\}\times S^{n-1}}\psi=1.$$

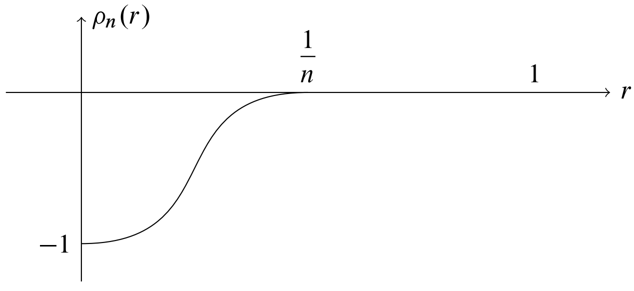


Figure 4: The graph of ρ_n.

Let $\alpha \in \Omega^*(E)$. Since $\int_F \delta^n = 1$ for any fiber F,

$$\int_E \pi^* i^* \alpha \wedge \delta^n = \int_M \int_F \pi^* i^* \alpha \wedge \delta^n = \int_M i^* \alpha.$$

Therefore, it is enough to show

$$\lim_{n \to \infty} \int_E (\alpha - \pi^* i^* \alpha) \wedge \delta^n = 0.$$

We will prove this by partition of unity. Let $\{U_i\}$ be an open cover of M and $\{p_i\}$ a partition of unity subordinated to this open cover. We fix trivializations over each U_i. Then over $\pi^{-1}(U_i)$,

$$(\pi^* p_i) \cdot (\alpha - \pi^* i^* \alpha) = \sum f^{I,J} \mathrm{d}x^I \wedge \mathrm{d}y^J,$$

where x are the coordinates in U_i and y are the coordinates in the fiber direction. I and J are sets of indices. Since α and $\pi^* i^* \alpha$ are the same when restricted to the zero section, $\lim_{r \to 0} f^{I,\varnothing} = 0$, where r is the radius coordinate in the fiber direction. Hence

$$\begin{aligned}\lim_{n \to \infty} \int_{\pi^{-1}(U_i)} f^{I,\varnothing} \mathrm{d}x^I \wedge \delta^n &= \lim_{n \to \infty} \int_{\mathbb{R}^+ \times S^{n-1} \times U_i} f^{I,\varnothing} \mathrm{d}x^I \wedge \mathrm{d}\rho_n \wedge \psi - f^{I,\varnothing} \mathrm{d}x^I \wedge \rho_n \pi^* e \\ &= \lim_{n \to \infty} \int_0^{1/n} \int_{S(E)|_{U_i}} \pm f^{I,\varnothing} \mathrm{d}\rho_n \wedge \psi \wedge \mathrm{d}x^I \pm \rho_n f^{I,\varnothing} \pi^* e \wedge \mathrm{d}x^I.\end{aligned}$$

Since $|\rho_n|$ is supported in $[0, 1/n]$ and bounded by 1, $\int_0^{1/n} |\mathrm{d}\rho_n| = 1$, $\lim_{r \to 0} f^{I,\varnothing} = 0$ and ψ is bounded on $S(E)$, we have

$$\lim_{n \to \infty} \int_{\pi^{-1}(U)} f^{I,\varnothing} \, \mathrm{d}x^I \wedge \delta^n = 0.$$

When the cardinality $|J|$ of J is greater than 0, using the spherical coordinate in the fiber direction, $\mathrm{d}y^I = C r^{|J|} \mathrm{d}\theta^J + D r^{|J|-1} \, \mathrm{d}r \wedge \mathrm{d}\theta^{J-1}$, where $\mathrm{d}\theta^J$ and $\mathrm{d}\theta^{J-1}$ are forms on the sphere of degree $|J|$ and $|J|-1$ and C, D are bounded functions. Because $\mathrm{d}\rho_n$ is purely in the $\mathrm{d}r$ direction,

$$\begin{aligned}&\lim_{n \to \infty} \int_{\pi^{-1}(U_i)} f^{I,J} \, \mathrm{d}x^I \wedge \mathrm{d}y^J \wedge \delta^n \\ &\quad = \lim_{n \to \infty} \int_0^{1/n} \int_{S(E)|_{U_i}} f^{I,J} C r^{|J|} \, \mathrm{d}x^I \wedge \mathrm{d}\theta^J \wedge \mathrm{d}\rho_n \wedge \psi \\ &\qquad - \lim_{n \to \infty} \int_0^{1/n} \int_{S(E)|_{U_i}} f^{I,J} C r^{|J|} \psi \wedge \mathrm{d}x^I \wedge \mathrm{d}\theta^J \wedge \rho_n \pi^* e \qquad \text{(A-1)} \\ &\qquad - \lim_{n \to \infty} \int_0^{1/n} \int_{S(E)|_{U_i}} f^{I,J} D r^{|J|-1} \wedge \psi \wedge \mathrm{d}x^I \wedge \mathrm{d}r \wedge \mathrm{d}\theta^{J-1} \wedge \rho_n \pi^* e. \qquad \text{(A-2)}\end{aligned}$$

Because $f^{I,J}$ and C are bounded, $\mathrm{d}\theta^J$ is bounded on $S(E)$, $\int_0^{1/n} |\mathrm{d}\rho_n| = 1$ and $\lim_{r \to 0} r^{|J|} = 0$, the first term limits to zero. Since everything in (A-1) and (A-2) is uniformly bounded and ρ_n is supported in $[0, 1/n]$, (A-1) and (A-2) have limit zero. Hence

$$\lim_{n \to \infty} \int_{\pi^{-1}(U_i)} \pi^* p_i (\alpha_i - \pi^* i^* \alpha) \wedge \delta^n = 0.$$

Therefore

$$\lim_{n \to \infty} \int_E (\alpha_i - \pi^* i^* \alpha) \wedge \delta^n = \lim_{n \to \infty} \sum_i \int_{\pi^{-1}(U_i)} (\pi^* p_i) \cdot (\alpha_i - \pi^* i^* \alpha) \wedge \delta^n = 0. \qquad \square$$

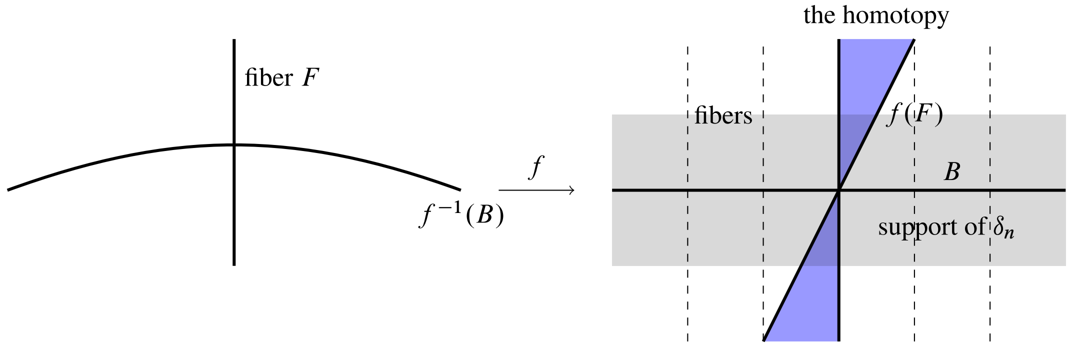


Figure 5: The pullback of Thom classes.

Next we will show that Lemma A.1 is preserved under pullback, when transversality conditions are met.

Lemma A.2 *Let M be a compact manifold with boundaries and corners and $E \to B$ a vector bundle over a* ***closed*** *manifold B. If $f: M \to E$ is transverse to B and we orient $f^{-1}(B)$ by $[f^{-1}(B)]f^*[E] = [TM|_{f^{-1}(B)}]$, then for $\alpha \in \Omega^*(C)$,*

$$\lim_{n\to\infty} \int_M \alpha \wedge f^*\delta^n = \int_{f^{-1}(B)} \alpha|_{f^{-1}(B)}.$$

Proof Fix a tubular neighborhood $\pi\colon N \to f^{-1}(B)$. For n big enough, $f^*\delta^n$ is the Thom class of $f^{-1}(B)$, ie $f^*\delta^n$ has integration 1 along each fiber. This is because the fiber F of $f^{-1}(B)$ is diffeomorphic to a submanifold homotopic to a fiber of $E \to B$ though the map f. Since δ^n is closed and has a small enough support, Stokes' theorem implies $\int_F f^*\delta_n = \int_{f(F)} \delta^n = \int_{\text{fiber of } E} \delta^n = 1$. Then by the same argument as in the proof of Lemma A.1, we only need to prove

$$\lim_{n\to\infty} \int_N (\alpha - \pi^* i^* \alpha) \wedge f^*\delta^n = 0.$$

Picking a point $x \in f^{-1}(B)$, by the implicit function theorem, we can find a local chart of x in M,

$$\phi\colon \mathbb{R}^k_+ \times \mathbb{R}^n \to M, \quad \phi(0) = x,$$

and local trivialization of $E \to B$ over $f(x)$,

$$\psi\colon \mathbb{R}^i \times \mathbb{R}^j \to E, \quad \psi(0,0) = (f(x), 0),$$

such that

$$\psi^{-1} \circ f \circ \phi(x_1, \dots, x_k, y_1, \dots, y_{n-j}, z_{n-j+1}, \dots, z_n) = (f_1, \dots, f_i, z_{n-j+1}, \dots, z_n),$$

where $f_1, \dots, f_i$ are functions of x_*, y_* and z_*. Replacing the z coordinates by spherical coordinates, the pullback of $\mathrm{d}(\rho_n\psi)$ through f is $\mathrm{d}(\rho_n\tilde{\psi})$, where $\tilde{\psi}$ is defined on $\mathbb{R}^k_+ \times \mathbb{R}^{n-j} \times S^{j-1} \times \mathbb{R}_+$ and uniformly bounded. Then the proof of Lemma A.1 can be applied to prove the claim. □

A.2 Proof of Lemmas 3.7 and 3.14

Following the discussion in Section 3.1, we pick representatives $\{\theta_{i,a}\}$ of a basis of $H^*(C_i)$ in $\Omega^*(C_i)$ to get a quasi-isomorphic embedding

$$H^*(C_i) \to \Omega^*(C_i),$$

and denote the dual basis by $\{\theta^*_{i,a}\}$ such that $\{\theta^*_{i,a}\}$ are in the image of the chosen embedding $H^*(C_i) \to \Omega^*(C_i)$ and $(-1)^{\dim C_i |\theta_i^b|} \int_{C_i} \theta^*_{i,a} \wedge \theta_{i,b} = \delta_{ab}$. Then by Proposition 3.2, the Thom class $\delta_i^n = \mathrm{d}(\rho_n \psi_i)$ of $\Delta_i \subset C_i \times C_i$ and $\sum_a \pi_1^* \theta_{i,a} \wedge \pi_2^* \theta^*_{i,a}$ both represent the Poincaré dual of the diagonal Δ_i, thus they are cohomologous in $\Omega^*(C_i \times C_i)$. Therefore we can find f_i^n such that $\mathrm{d} f_i^n = \delta_i^n - \sum_a \pi_1^* \theta_{i,a} \wedge \pi_2^* \theta^*_{i,a}$ and

$$f_i^n - f_i^m = (\rho_n - \rho_m)\psi_i. \tag{A-3}$$

Thus the support of $f_i^n - f_i^m$ converges to a measure-zero set. To show the convergence results (Lemmas 3.7 and 3.14), we need to show that f_i^n is uniformly bounded. The uniform boundedness is not necessarily true in $C_i \times C_i$, but it holds if we use spherical coordinates near the diagonal Δ_i. To apply spherical coordinates in an intrinsic way, we recall blow-ups of real submanifolds:

Definition A.3 [58, Chapter 5] Let $p\colon E \to M$ be vector bundle over a manifold. Then the blow-up of E along M is the manifold

$$\mathrm{Bl}_M E = \{(v,e) \in E \times S(E) \mid p(v) = p(e) \text{ and } ae = v \text{ for some } a \geq 0\},$$

where $S(E)$ is the sphere bundle $(E \backslash \{0_M\})/\mathbb{R}^+$, and 0_M is the zero section of $E \to M$.

Then one can define a blow-up of a submanifold $N \subset M$ in the sense of Definition 2.2 by blowing up N in the tubular neighborhood which is identified with the normal bundle. Moreover, the blow-up of the submanifold N can be described intrinsically as

$$\mathrm{Bl}_N M := (M \backslash N) \cup S(TM/TN|_N),$$

where $S(TM/TN|_N)$ is the sphere bundle of the quotient bundle (normal bundle) $TM/TN|_N$ over N. The smooth structure on $\mathrm{Bl}_N M$ can be given using an auxiliary tubular neighborhood and it is independent of

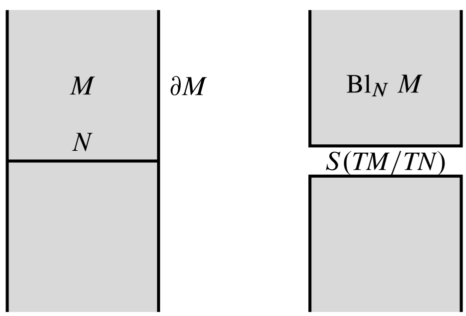


Figure 6: Blowing up one submanifold.

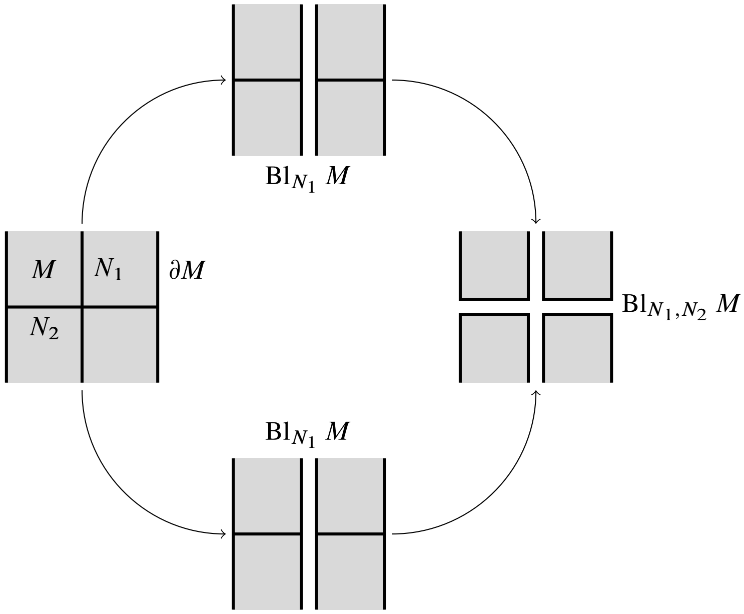


Figure 7: Blowing up two submanifolds.

the tubular neighborhood [58, Chapter 5]. The natural map $\mathrm{Bl}_N M \to M$ is smooth and is a diffeomorphism up to measure-zero sets. Thom classes $\delta_i^n = \mathrm{d}(\rho_n \psi_i)$ can be pulled back to $\mathrm{Bl}_{\Delta_i} C_i \times C_i$, and the primitives $\rho_n \psi_i$ are uniformly bounded on $\mathrm{Bl}_{\Delta_i} C_i \times C_i$.

Using this intrinsic description, when a smooth map $f : M \times N \to C \times C$ is transverse to the diagonal Δ, there is a natural map $\mathrm{Bl}_\Delta f : \mathrm{Bl}_{M \times_C N} M \times N \to \mathrm{Bl}_{\Delta_C} C \times C$ induced by $f : M \times N \to C \times C$. Moreover, we have the following commutative diagram of smooth maps:

$$\begin{array}{ccc} \mathrm{Bl}_{M\times_C N} M \times N & \xrightarrow{\mathrm{Bl}_\Delta f} & \mathrm{Bl}_{\Delta_C} C \times C \\ \downarrow & & \downarrow \\ M \times N & \xrightarrow{\quad f \quad} & C \times C \end{array}$$

If we have two submanifolds N_1 and N_2 of M such that N_1 is transverse to N_2 in the sense of Definition 2.4, then we can blow up N_1 and N_2. It was shown in [58, Chapter 5] that the order of blowing up does not influence the diffeomorphism type. The resulting blow-up is denoted by $\mathrm{Bl}_{N_1,N_2} M$. Similarly, if we have a sequence of submanifolds $N_1, N_2, \ldots, N_k$ such that $\left(\bigcap_{\alpha \in A} N_\alpha\right)$ is transverse to N_β for $\beta \notin A$, then we can blow up all $N_1, \ldots, N_k$. The diffeomorphism type does not depend on the order; let $\mathrm{Bl}_{N_1,\ldots,N_k} M$ denote the blow-up.

In the setting of a flow category (Definition 2.9), any fiber product $\mathcal{M}_{i_0,i_1} \times_{i_1} \mathcal{M}_{i_1,i_2} \times_{i_2} \cdots \times_{i_n} \mathcal{M}_{i_n,i_{n+1}}$ is cut out transversely in $\mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,\iota_2} \times \cdots \times \mathcal{M}_{i_n,i_{n+1}}$. Therefore

$$N_j := \mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \cdots \times \mathcal{M}_{i_{j-1},i_j} \times_{i_j} \mathcal{M}_{i_j,i_{j+1}} \times \cdots \times \mathcal{M}_{i_n,i_{n+1}}$$

are submanifolds in the product $\mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \cdots \times \mathcal{M}_{i_n,i_{n+1}}$ such that $\left(\bigcap_{\alpha \in A} N_\alpha\right)$ is transverse to N_β for $\beta \notin A$. Then we have a blow-up $\mathrm{Bl}_n := \mathrm{Bl}_{N_1,\ldots,N_n} \mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \cdots \times \mathcal{M}_{i_n,i_{n+1}}$ and a similar commutative diagram of smooth maps

$$\begin{tikzcd} \mathrm{Bl}_n \arrow[r, "\mathrm{Bl}_{\Delta_i}(t\times s)"] \arrow[d] & \mathrm{Bl}_{\Delta_{i_j}} C_{i_j} \times C_{i_j} \arrow[d] \\ \mathcal{M}_{i_0,i_1} \times \mathcal{M}_{i_1,i_2} \times \cdots \times \mathcal{M}_{i_n,i_{n+1}} \arrow[r, "t\times s"] & C_{i_j} \times C_{i_j} \end{tikzcd} \tag{A-4}$$

Now we start to prove Lemmas 3.7 and 3.14. The definition of $\mathcal{M}^{s,k}_{i_1,\ldots,i_r}[\alpha, f^{n_1}_{s+i_1}, \ldots, f^{n_r}_{s+i_r}, \gamma]$ is (3-10).

Lemma A.4 (Lemma 3.7) *For every $\alpha \in \Omega^*(C_v)$ and $\gamma \in \Omega^*(C_{v+k})$, and any defining data Θ, $\lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, f^n_{v+i_r}, \gamma]$ exists.*

Proof Since $\mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^{n_1}_{v+i_1}, \ldots, f^{n_r}_{v+i_r}, \gamma]$ is an integration over $\mathcal{M}^{v,k}_{i_1,\ldots,i_r}$, and $\bigcup_j \mathcal{M}^{v,k}_{i_1,\ldots,\bar{i}_j,\ldots,i_r}$ is a measure-zero set in $\mathcal{M}^{v,k}_{i_1,\ldots,i_r}$, we can restrict the integral to

$$\mathcal{M}^{v,k}_{i_1,\ldots,i_r} - \bigcup_j \mathcal{M}^{v,k}_{i_1,\ldots,\bar{i}_j,\ldots,i_r}$$

to get the same value.

We have a blow-up $\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}$ by blowing up all $\mathcal{M}^{v,k}_{i_1,\ldots,\bar{i}_j,\ldots,i_r}$ for $1 \le j \le r$. The primitives f^n_i can be lifted to $\mathrm{Bl}_{\Delta_i} C_i \times C_i$ and $t \times s$ can be lifted to the blow-ups to $\mathrm{Bl}_{\Delta_i}(t \times s)$. We define $\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^{n_1}_{v+i_1}, \ldots, f^{n_r}_{v+i_r}, \gamma]$ to be the result of integrating the wedge product of pullbacks of $\alpha, f^{n_1}_{v+i_1}, \ldots, f^{n_r}_{v+i_r}, \gamma$ to $\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}$. Because $\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}$ and $\mathcal{M}^{v,k}_{i_1,\ldots,i_r} - \bigcup_j \mathcal{M}^{v,k}_{i_1,\ldots,\bar{i}_j,\ldots,i_r}$ also differ by a measure-zero set, by the commutative diagram (A-4),

$$\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, f^n_{v+i_r}, \gamma] = \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, f^n_{v+i_r}, \gamma].$$

Then

$$\begin{aligned} &\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, f^n_{v+i_r}, \gamma] - \mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^m_{v+i_1}, \ldots, f^m_{v+i_r}, \gamma] \\ &\qquad = \sum_{p=1}^{r} \mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^m_{v+i_1}, \ldots, f^m_{v+i_{p-1}}, f^n_{v+i_p} - f^m_{v+i_p}, f^n_{v+i_{p+1}}, \ldots, f^n_{v+i_r}, \gamma]. \end{aligned} \tag{A-5}$$

Note that the $f^n_{v+i_j}$ are uniformly bounded over $\mathrm{Bl}_{\Delta_{v+i_j}} C_{v+i_j} \times C_{v+i_j}$ for every $n \in \mathbb{N}$, and the support of $f^n_{v+i_j} - f^m_{v+i_j}$ converges to a measure-zero set in $\mathrm{Bl}_{\Delta_{v+i_j}} C_{v+i_j} \times C_{v+i_j}$ when $n, m \to \infty$. By (A-4), the pullbacks of $f^n_{v+i_j}$ to $\mathrm{Bl}_r \mathcal{M}^{v,k}_{i_1,\ldots,i_r}$ have the same properties. Thus (A-5) implies the convergence. $\square$

Lemma A.5 (Lemma 3.14) *For an oriented flow category $\mathcal{C}$ and any defining data, we have*

$$\lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, \delta^n_{v+i_p}, \ldots, f^n_{v+i_r}, \gamma] = (-1)^* \lim_{n\to\infty} \mathcal{M}^{v,k}_{i_1,\ldots,i_{p-1},\bar{i}_p,i_{p+1},\ldots,i_r}[\alpha, f^n_{v+i_1}, \ldots, f^n_{v+i_r}, \gamma],$$

where $* = (|\alpha| + m_{v,v+i_p})c_{v+i_p}$.

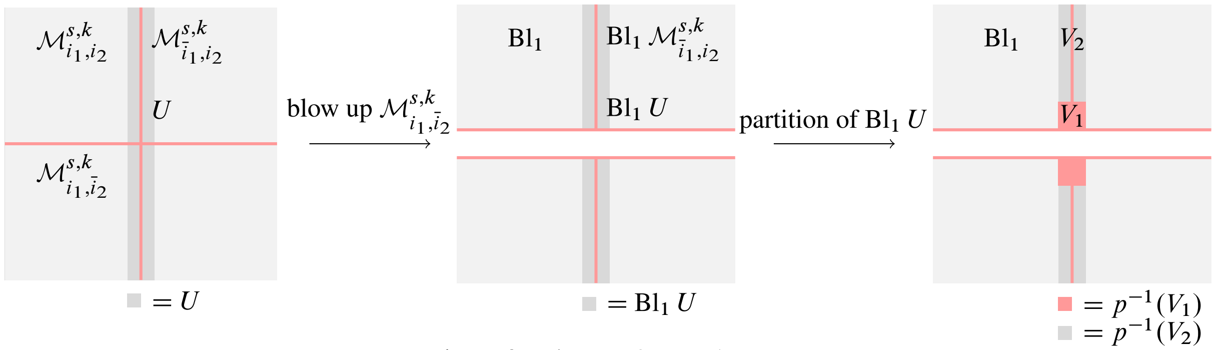


Figure 8: The $r = 2$, $p = 1$ case.

Proof The limit $\lim_{n\to\infty}\mathcal{M}^{v,k}_{i_1,\dots,i_{p-1},\bar{i}_p,i_{p+1},\dots,i_r}[\alpha, f^n_{v+i_1},\dots,f^n_{v+i_r},\gamma]$ exists by the same argument used in the proof of Lemma A.4. To prove the limit on the left-hand side exists, we can blow up everything except for $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ to get Bl_{r-1}. Assume that the pullback of $\delta^n_{v+i_p}$ is supported in the tubular neighborhood U of $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ in $\mathcal{M}^{v,k}_{i_1,\dots,i_r}$. Then U can be lifted to the blow-up Bl_{r-1} to get $\mathrm{Bl}_{r-1}\,U$ (see Figure 7). For simplicity, we suppress the wedge and pullback notation. Then we have

$$\lim_{n\to\infty}\int_{\mathcal{M}^{v,k}_{i_1,\dots,i_r}}\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma = \lim_{n\to\infty}\int_{\mathrm{Bl}_{r-1}\,U}\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma.$$

Let $\mathrm{Bl}_{r-1}\,\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ denote the lift of $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ in Bl_{r-1}. Then $\mathrm{Bl}_{r-1}\,U$ is still a tubular neighborhood of $\mathrm{Bl}_{r-1}\,\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$. Let $p\colon \mathrm{Bl}_{r-1}\,U\to \mathrm{Bl}_{r-1}\,\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ denote the projection of the tubular neighborhood. Then we can divide $\mathrm{Bl}_{r-1}\,\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ into two parts, V_1 and V_2, such that V_1 is a small open set containing the blow-up domain, and V_2 is the complement. Then $p^{-1}(V_1)$ and $p^{-1}(V_2)$ are partitions of $\mathrm{Bl}_{r-1}\,U$ (see Figure 8). Using the same local coordinates as in Lemma A.2, if we integrate the fiber direction of the tubular neighborhood, because $f^n_{v+i_1},\dots,f^n_{v+i_{p-1}},f^n_{v+i_{p+1}},\dots,f^n_{v+i_r}$ are uniformly bounded over Bl_{r-1}, we have

$$\left|\int_{p^{-1}(V_1)}\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma\right|\le K\,\mathrm{vol}(V_1), \tag{A-6}$$

where K is a constant. Over $p^{-1}(V_2)$, the pullbacks of $f^n_{v+i_1},\dots,f^n_{v+i_{p-1}},f^n_{v+i_{p+1}},\dots,f^n_{v+i_r}$ do not change for n large enough, because $p^{-1}(V_2)$ stays away from the blown-up area. Thus the only thing that varies over $p^{-1}(V_2)$ is $\delta^n_{v+i_p}$. Note that

$$\begin{aligned}\lim_{n\to\infty}\int_{p^{-1}(V_2)}&\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma\\ &= (-1)^{(|\alpha|+\sum_{j<p}(c_{v+i_j}-1))c_{v+i_p}}\lim_{n\to\infty}\int_{p^{-1}(V_2)}\delta^n_{v+i_p}\alpha f^n_{v+i_1}\cdots f^n_{v+i_r}.\end{aligned}$$

By Definition 2.15, the orientation relation on $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}\supset V_2$ satisfies

$$[N_{v+i_p}][\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}] = (-1)^{(\sum_{j\le p}m_{v+i_{j-1},v+i_j}).c_{v+i_p}}[\mathcal{M}^{v,k}_{i_1,\dots,i_r}]$$

Combining with Lemma A.2 and

$$\Big(|\alpha|+\sum_{j<p}(c_{v+i_j}-1)\Big)c_{v+i_p}+\Big(\sum_{j\le p}m_{v+i_{j-1},v+i_j}\Big)c_{v+i_p} = (|\alpha|+m_{v,v+i_p})c_{v+i_p}\mod 2,$$

we can conclude that

$$\text{(A-7)}\quad \lim_{n\to\infty}\int_{p^{-1}(V_2)}\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma = \lim_{n\to\infty}(-1)^{(|\alpha|+m_{v,v+i_p})c_{v+i_p}}\int_{V_2}\alpha f^n_{v+i_1}\cdots f^n_{v+i_{p-1}}f^n_{v+i_{p+1}}\cdots f^n_{v+i_r}\gamma.$$

By (A-6) and (A-7), since V_1 can be arbitrarily small, $\lim_{n\to\infty}\mathcal{M}^{s,k}_{i_1,\dots,i_r}[\alpha, f^n_{v+i_1},\dots,\delta^n_{v+i_p},\dots,f^n_{v+i_r},\gamma]$ exists. Since $f^n_{v+i_1},\dots,f^n_{v+i_{p-1}},f^n_{v+i_{p+1}},\dots,f^n_{v+i_r}$ are uniformly bounded over $\mathrm{Bl}_{r-1}\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$,

$$\text{(A-8)}\quad \left|\int_{V_1}\alpha f^n_{v+i_1}\cdots f^n_{v+i_{i_{p-1}}}f^n_{v+i_{p+1}}\cdots f^n_{v+i_r}\gamma\right| < K'\operatorname{vol}(V_1).$$

Since $\mathrm{Bl}_{r-1}\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ and $\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}$ differ by a measure-zero set,

$$\text{(A-9)}\quad \int_{\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}}\alpha f^n_{v+i_1}\cdots f^n_{v+i_{p-1}}f^n_{v+i_{p+1}}\cdots f^n_{v+i_r}\gamma = \int_{\mathrm{Bl}_{r-1}\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}}\alpha f^n_{v+i_1}\cdots f^n_{v+i_{p-1}}f^n_{v+i_{p+1}}\cdots f^n_{v+i_r}\gamma = \int_{V_1\cup V_2}\alpha f^n_{v+i_1}\cdots f^n_{v+i_{p-1}}f^n_{v+i_{p+1}}\cdots f^n_{v+i_r}\gamma.$$

Therefore by (A-6), (A-7), (A-8) and (A-9),

$$\left|\lim_{n\to\infty}\left(\int_{\mathcal{M}^{v,k}_{i_1,\dots,i_r}}\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma-(-1)^{(|\alpha|+m_{v,v+i_p})c_{s+i_p}}\int_{\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}}\alpha f^n_{v+i_1}\cdots f^n_{v+i_r}\gamma\right)\right| \le (K+K')\operatorname{vol}(V_1).$$

Thus, since V_1 can be arbitrarily small,

$$\lim_{n\to\infty}\int_{\mathcal{M}^{v,k}_{i_1,\dots,i_r}}\alpha f^n_{v+i_1}\cdots\delta^n_{v+i_p}\cdots f^n_{v+i_r}\gamma = \lim_{n\to\infty}(-1)^{(|\alpha|+m_{v,v+i_p})c_{v+i_p}}\int_{\mathcal{M}^{v,k}_{i_1,\dots,\bar{i}_p,\dots,i_r}}\alpha f^n_{v+i_1}\cdots f^n_{v+i_r}\gamma. \qquad\square$$

Appendix B Proof of Proposition 6.21

Proposition B.1 (Proposition 6.21) *Let $\pi\colon E\to C$ be an oriented k–sphere bundle over an oriented closed manifold. Let $A=A^*$ be the reduction on $\Omega^*(E)$ built from the discussion after the statement of Theorem 6.19* (*in particular, we choose ψ_i such that $\mathrm{d}\psi_i=0$ if k is even*)*. Suppose T is the closed form in $\pi_1^*A\wedge\pi_2^*A$ representing the diagonal by the definition of reduction. Then there exist approximations $\delta^{E,n}$ of the Dirac current of the diagonal Δ_E such that*:

(1) *There exist forms $f^{E,n}$ on $E\times E$ such that*

$$\mathrm{d}f^{E,n}=\delta^{E,n}-T.$$

(2) *Lemmas 3.7 and 3.14 hold for $f^{E,n}$. In particular, the construction in Section 6.2 works for $f^{E,n}$.*

(3) *Let $\pi_{1,2}$ denote the projection $E \times E \to C \times C$. Then $f^{E,n}$ can be written as sums of differential forms in the form $(\pi_{1,2}^*\alpha) \wedge \beta$ with $\alpha \in \Omega^*(C \times C)$ and $\deg(\beta) \le k$ (the fiber degree of $f^{E,n}$ is at most k). In other words, if $v_1, \dots, v_{k+1}$ are $k+1$ vertical vectors in $T_p(E \times E)$ for $p \in C \times C$, then $f^{E,n}(v_1 \wedge \cdots \wedge v_{k+1} \wedge \cdots) = 0$.*

Proof Let $\delta^{C,n}$ be the Thom classes of $\Delta_C \subset C \times C$ constructed using (3-4) with the angular form Ψ_C of the normal bundle. Let $\delta^{S^k,n}$ be the Thom classes of $\Delta_E \subset E \times_C E$ constructed using (3-4). We define $p\colon U \to E \times_C E$ to be a projection in a tubular neighborhood U of $E \times_C E$ in $E \times E$. Then $\pi_{1,2}(U)$ is a tubular neighborhood of $\Delta_C \subset C \times C$. By the same argument as in Lemma 3.1, $\lim_{n\to\infty} \pi_{1,2}^*\delta^{C,n} \wedge p^*\delta^{S^k,n}$ is the Dirac current of the diagonal $\Delta_E \subset E \times E$. Since, for $n \gg 0$, the support of $\pi_{1,2}^*\delta^{C,n}$ is contained in U, the $\pi_{1,2}^*\delta^{C,n} \wedge p^*\delta^{S^k,n}$ are cohomologous to each other and represent Thom classes of Δ_E for $n \gg 0$.

Next, we show that we can find the desired primitives $f^{E,n}$. Let $p_1, p_2\colon E \times_C E \to E$ be the projections to the first and second components, respectively. Then $(-1)^k p_1^*\psi + p_2^*\psi$ is a closed form on $E \times_C E$ because $\mathrm{d}((-1)^k p_1^*\psi + p_2^*\psi) = (-1)^{k+1}q^*e - q^*e = 0$ for any k (when k is even, e is zero by assumption), where $q\colon E \times_C E \to C$ is the projection. We claim $(-1)^k p_1^*\psi + p_2^*\psi$ is cohomologous to $\delta^{S^k,n}$: there are $f^{S^k,n} \in \Omega^{k-1}(E \times_C E)$ such that

$$\delta^{S^k,n} - (-1)^k p_1^*\psi - p_2^*\psi = \mathrm{d} f^{S^k,n}. \tag{B-1}$$

We first proceed assuming (B-1). Let Π_1 and Π_2 be the two projections $E \times E \to E$. Note that $(-1)^k\Pi_1^*\psi + \Pi_2^*\psi$ is not closed on U. We have $\mathrm{d}((-1)^k\Pi_1^*\psi + \Pi_2^*\Psi) = \pi_{1,2}^*((-1)^{k+1}e \otimes 1 - 1 \otimes e)$, and the closed form $(-1)^{k+1}e \otimes 1 - 1 \otimes e$ is zero on Δ_C. Hence $(-1)^{k+1}e \otimes 1 - 1 \otimes e$ is exact on $\pi_{1,2}(U)$. Therefore we can find $h \in \Omega^k(\pi_{1,2}(U))$ with $h|_{\Delta_C} = 0$ and $(-1)^k\Pi_1^*\psi + \Pi_2^*\Psi + \pi_{1,2}^*h$ is closed on U. Since $((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h)|_{E\times_C E} = (-1)^k p_1^*\psi + p_2^*\psi$, we know that there exists $g \in \Omega^{k-1}(U)$ such that

$$p^*((-1)^k p_1^*\psi + p_2^*\psi) - (-1)^k\Pi_1^*\psi - \Pi_2^*\psi = \mathrm{d}g + \pi_{1,2}^*h.$$

Now we make any extension of h to $C \times C$; the extended form is still denoted by h. We have

$$\begin{aligned}\pi_{1,2}^*\delta^{C,n} \wedge p^*\delta^{S^k,n} &= \pi_{1,2}^*\delta^{C,n} \wedge p^*((-1)^k p_1^*\psi + p_2^*\psi) + \pi_{1,2}^*\delta^{C,n} \wedge p^*\mathrm{d} f^{S^k,n} \\ &= \pi_{1,2}^*\delta^{C,n} \wedge \big(((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h) + \pi_{1,2}^*\delta^{C,n} \wedge (\mathrm{d}g + p^*\mathrm{d} f^{S^k,n})\big).\end{aligned}$$

If we write $\mathrm{d} f^{C,n} = \delta^{C,n} - \sum_a \pi_1^*\theta_a \wedge \pi_2^*\theta_a^*$, then

$$\begin{aligned}&\pi_{1,2}^*\delta^{C,n} \wedge ((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h) \\ &= \pi_{1,2}^*(\mathrm{d} f^{C,n} + \sum_a \pi_1^*\theta_a \wedge \pi_2\theta_a^*) \wedge ((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h) \\ &= \mathrm{d}\big(\pi_{1,2}^* f^{C,n} \wedge ((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h)\big) + \pi_{1,2}^*\Big(\sum_a \pi_1^*\theta_a \wedge \pi_2\theta_a^*\Big) \wedge ((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h) \\ &\qquad + (-1)^{\dim C}\pi_{1,2}^* f^{C,n} \wedge \mathrm{d}((-1)^k\Pi_1^*\psi + \Pi_2^*\psi + \pi_{1,2}^*h).\end{aligned}$$

Let S_n denote the last two terms. Then $S_n - S_m = 0$ for $n, m \gg 0$ as $\operatorname{supp}(f^{C,n} - f^{C,m}) \subset (\pi_{1,2})(U)$ and $\mathrm{d}((-1)^k \Pi_1^* \psi + \Pi_2^* \psi + \pi_{1,2}^* h)$ is zero on U.

Next, recall from Lemma 6.20 that $A = A^*$ has a basis of the form

$$\langle \pi^*\theta_1, \dots, \pi^*\theta_k, \xi_1 := \pi^*\theta_1^* \wedge \psi - \pi^*\eta_1, \dots, \xi_k := \pi^*\theta_k^* \wedge \psi - \pi^*\eta_k \rangle$$

such that the dual basis is $\langle \xi_1, \dots, \xi_k, \pi^*\theta_1, \dots, \pi^*\theta_k \rangle$, up to sign. It is easy to check that $S_n - T$ is in the form $\pi_{1,2}^* \alpha$ for $\alpha \in \Omega^*(C \times C)$. Since T and $\pi_{1,2}^* \delta^{C,n} \wedge p^* \delta^{S^k,n}$ both represent Δ_E, we have that $S_n - T$ is exact. Therefore α is a closed class in $\Omega^*(C \times C)$ such that $[\pi_{1,2}^* \alpha] = 0$. As a consequence, $[\alpha] = \sum_i ([\alpha_i] \wedge [e]) \otimes [\beta_i] + \sum_j [\alpha'_j] \otimes ([\beta'_j] \wedge [e])$ on cohomology. Therefore there exist $\alpha_0, \alpha_1, \alpha_2 \in \Omega^*(C \times C)$ such that

$$S_n - T = \pi_{1,2}^* \alpha = \mathrm{d}(\pi_{1,2}^* \alpha_0 \wedge \Pi_1^* \Psi + \pi_{1,2}^* \alpha_1 \wedge \Pi_2^* \Psi + \pi_{1,2}^* \alpha_2) = \mathrm{d}w.$$

So we can take $\delta^{E,n} := \pi_{1,2}^* \delta^{C,n} \wedge p^* \delta^{S^k,n}$ and

(B-2) $$f^{E,n} := w + f^{C,n} \wedge ((-1)^k \Pi_1^* \psi + \Pi_2^* \psi + \pi_{1,2}^* h) + (-1)^{\dim C} (\pi_1 \times \pi_2)^* \delta^{C,n} \wedge (g + p^* f^{S^k,n}).$$

Since f_C^n and $f_{S^k}^n$ can be chosen so that (3-7) holds, Lemmas 3.7 and 3.14 hold for $f^{E,n}$ using the same argument as in Appendix A. By (B-2), the third property of the proposition holds, since each component has the property. □

Proof of (B-1) Note that $p_1 \colon E \times_C E \to E$ is also a sphere bundle (it is the pullback of the bundle $\pi \colon E \to C$ through π itself). Then $p_2^* \psi$ is the angular form of p_1. After fixing representatives $\{\alpha_1, \dots, \alpha_m\}$ of a basis of $H^*(E)$, we get a reduction of $\Omega^*(E \times_C E)$ by the same argument as the one after the statement of Theorem 6.19:

$$B = B^* = \langle p_1^* \alpha_1, \dots, p_1^* \alpha_m, \chi_1 := p_1^* \alpha_1 \wedge p_2^* \psi - p_1^* f_1, \dots, \chi_m := p_1^* \alpha_m \wedge p_2^* \psi - p_1^* f_m \rangle.$$

Since d is closed on B and the cohomology is the cohomology of $E \times_C E$ (since it is a reduction), it suffices to prove that, for any $\beta \in B$,

$$\int_{E \times_C E} \beta \wedge ((-1)^k p_1^* \psi + p_2^* \psi) = \int_{\Delta_E} \beta.$$

If $\beta = p_1^* \alpha_i$, then

$$\int_{E \times_C E} p_1^* \alpha_i \wedge ((-1)^k p_1^* \psi + p_2^* \psi) = \int_{E \times_C E} (-1)^k p_1^* (\alpha_i \wedge \psi) + \int_{E \times_C E} p_1^* \alpha_i \wedge p_2^* \psi.$$

The first term is clearly zero, and the second term is $\int_E \alpha_i = \int_{\Delta_E} (p_1^* \alpha_i)|_{\Delta_E}$ by integration along the fiber of p_1. If $\beta = \chi_i = p_1^* \alpha_i \wedge p_2^* \psi - p_1^* f_i$, then by the same argument as above, we have

$$\int_{E \times_C E} \chi_i \wedge ((-1)^k p_1^* \psi + p_2^* \psi) = \int_{E \times_C E} (p_1^* \alpha_i \wedge p_2^* \psi) \wedge ((-1)^k p_1^* \psi + p_2^* \psi) + \int_{\Delta E} (p_1^* f_i)|_{\Delta_E}.$$

The first term is $\int_{E \times_C E} p_1^* \alpha_i \wedge p_1^* \psi \wedge p_2^* \psi = \int_E \alpha_i \wedge \psi = \int_{\Delta_E} (p_1^* \alpha_i \wedge p_2^* \psi)|_{\Delta_E}$. □

References

[1] **M Abouzaid**, *Symplectic cohomology and Viterbo's theorem*, from "Free loop spaces in geometry and topology" (J Latschev, A Oancea, editors), IRMA Lect. Math. Theor. Phys. 24, Eur. Math. Soc., Zürich (2015) 271–485 MR Zbl

[2] **M Audin**, **M Damian**, *Morse theory and Floer homology*, Springer (2014) MR Zbl

[3] **D M Austin**, **P J Braam**, *Morse–Bott theory and equivariant cohomology*, from "The Floer memorial volume" (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 123–183 MR Zbl

[4] **A Banyaga**, **D E Hurtubise**, *Morse–Bott homology*, Trans. Amer. Math. Soc. 362 (2010) 3997–4043 MR Zbl

[5] **A Banyaga**, **D E Hurtubise**, *Cascades and perturbed Morse–Bott functions*, Algebr. Geom. Topol. 13 (2013) 237–275 MR Zbl

[6] **J-F Barraud**, **O Cornea**, *Lagrangian intersections and the Serre spectral sequence*, Ann. of Math. 166 (2007) 657–722 MR Zbl

[7] **P Biran**, **O Cornea**, *A Lagrangian quantum homology*, from "New perspectives and challenges in symplectic field theory" (M Abreu, F Lalonde, L Polterovich, editors), CRM Proc. Lecture Notes 49, Amer. Math. Soc., Providence, RI (2009) 1–44 MR Zbl

[8] **R Bott**, *Nondegenerate critical manifolds*, Ann. of Math. 60 (1954) 248–261 MR Zbl

[9] **R Bott**, *An application of the Morse theory to the topology of Lie-groups*, Bull. Soc. Math. France 84 (1956) 251–281 MR Zbl

[10] **R Bott**, **H Samelson**, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. 80 (1958) 964–1029 MR Zbl

[11] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Math. 82, Springer (1982) MR Zbl

[12] **F Bourgeois**, *A Morse–Bott approach to contact homology*, PhD thesis, Stanford University (2002) Available at `https://www.proquest.com/docview/305591502`

[13] **F Bourgeois**, **K Mohnke**, *Coherent orientations in symplectic field theory*, Math. Z. 248 (2004) 123–146 MR Zbl

[14] **F Bourgeois**, **A Oancea**, *An exact sequence for contact- and symplectic homology*, Invent. Math. 175 (2009) 611–680 MR Zbl Correction in 200 (2015) 1065–1076

[15] **F Bourgeois**, **A Oancea**, *Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces*, Duke Math. J. 146 (2009) 71–174 MR Zbl

[16] **F Bourgeois**, **A Oancea**, *The Gysin exact sequence for S^1–equivariant symplectic homology*, J. Topol. Anal. 5 (2013) 361–407 MR Zbl

[17] **K Cieliebak**, **K Mohnke**, *Symplectic hypersurfaces and transversality in Gromov–Witten theory*, J. Symplectic Geom. 5 (2007) 281–356 MR Zbl

[18] **K Cieliebak**, **E Volkov**, *Chern–Simons theory and string topology*, preprint (2023) arXiv 2312.05922

[19] **R L Cohen**, **J D S Jones**, **G B Segal**, *Floer's infinite-dimensional Morse theory and homotopy theory*, from "The Floer memorial volume" (H Hofer, C H Taubes, A Weinstein, E Zehnder, editors), Progr. Math. 133, Birkhäuser, Basel (1995) 297–325 MR Zbl

[20] **L Diogo**, **S T Lisi**, *Morse–Bott split symplectic homology*, J. Fixed Point Theory Appl. 21 (2019) art. id. 77 MR Zbl

[21] **L Diogo**, **S T Lisi**, *Symplectic homology of complements of smooth divisors*, J. Topol. 12 (2019) 967–1030 MR Zbl

[22] **S Dostoglou**, **D A Salamon**, *Self-dual instantons and holomorphic curves*, Ann. of Math. 139 (1994) 581–640 MR Zbl

[23] **Y Eliashberg**, **A Givental**, **H Hofer**, *Introduction to symplectic field theory*, from "Visions in mathematics" (N Alon, J Bourgain, A Connes, M Gromov, V Milman, editors), volume 2, Birkhäuser (= GAFA special volume), Boston, MA (2000) 560–673 MR Zbl

[24] **O Fabert**, **J W Fish**, **R Golovko**, **K Wehrheim**, *Polyfolds: a first and second look*, EMS Surv. Math. Sci. 3 (2016) 131–208 MR Zbl

[25] **B Filippenko**, *Polyfold regularization of constrained moduli spaces*, J. Symplectic Geom. 19 (2021) 241–350 MR Zbl

[26] **J Fish**, **H Hofer**, *Applications of polyfold theory*, *II*: *The polyfolds of symplectic field theory*, in preparation

[27] **A Floer**, *An instanton-invariant for* 3*–manifolds*, Comm. Math. Phys. 118 (1988) 215–240 MR Zbl

[28] **A Floer**, *Morse theory for Lagrangian intersections*, J. Differential Geom. 28 (1988) 513–547 MR Zbl

[29] **A Floer**, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. 120 (1989) 575–611 MR Zbl

[30] **A Floer**, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. 30 (1989) 207–221 MR Zbl

[31] **A Floer**, **H Hofer**, *Coherent orientations for periodic orbit problems in symplectic geometry*, Math. Z. 212 (1993) 13–38 MR Zbl

[32] **U Frauenfelder**, *The Arnold–Givental conjecture and moment Floer homology*, Int. Math. Res. Not. 2004 (2004) 2179–2269 MR Zbl

[33] **K Fukaya**, *Floer homology of connected sum of homology* 3*–spheres*, Topology 35 (1996) 89–136 MR Zbl

[34] **K Fukaya**, **Y-G Oh**, **H Ohta**, **K Ono**, *Lagrangian intersection Floer theory: anomaly and obstruction, II*, AMS/IP Stud. Adv. Math. 46.2, Amer. Math. Soc., Providence, RI (2009) MR Zbl

[35] **K Fukaya**, **Y-G Oh**, **H Ohta**, **K Ono**, *Kuranishi structure*, *pseudo-holomorphic curve*, *and virtual fundamental chain*, *I*, preprint (2015) arXiv 1503.07631

[36] **P Griffiths**, **J Harris**, *Principles of algebraic geometry*, Wiley, New York (1978) MR Zbl

[37] **M Gromov**, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985) 307–347 MR Zbl

[38] **V W Guillemin**, **S Sternberg**, *Supersymmetry and equivariant de Rham theory*, Springer (1999) MR Zbl

[39] **F R Harvey**, **H B Lawson, Jr**, *Finite volume flows and Morse theory*, Ann. of Math. 153 (2001) 1–25 MR Zbl

[40] **H Hofer**, **K Wysocki**, **E Zehnder**, *A general Fredholm theory, I: A splicing-based differential geometry*, J. Eur. Math. Soc. 9 (2007) 841–876 MR Zbl

[41] **H Hofer**, **K Wysocki**, **E Zehnder**, *A general Fredholm theory, II: Implicit function theorems*, Geom. Funct. Anal. 19 (2009) 206–293 MR Zbl

[42] **H Hofer**, **K Wysocki**, **E Zehnder**, *A general Fredholm theory, III: Fredholm functors and polyfolds*, Geom. Topol. 13 (2009) 2279–2387 MR Zbl

[43] **H Hofer**, **K Wysocki**, **E Zehnder**, *Integration theory on the zero sets of polyfold Fredholm sections*, Math. Ann. 346 (2010) 139–198 MR Zbl

[44] **H Hofer**, **K Wysocki**, **E Zehnder**, *Polyfold and Fredholm theory*, Ergebnisse der Math. 72, Springer (2021) MR Zbl

[45] **D E Hurtubise**, *Multicomplexes and spectral sequences*, J. Algebra Appl. 9 (2010) 519–530 MR Zbl

[46] **E-N Ionel**, **T H Parker**, *A natural Gromov–Witten virtual fundamental class*, preprint (2013) arXiv 1302.3472

[47] **D Joyce**, *On manifolds with corners*, from "Advances in geometric analysis" (S Janeczko, J Li, D H Phong, editors), Adv. Lect. Math. 21, International, Somerville, MA (2012) 225–258 MR Zbl

[48] **D Joyce**, *A new definition of Kuranishi space*, preprint (2014) arXiv 1409.6908

[49] **M Kontsevich**, **Y Soibelman**, *Homological mirror symmetry and torus fibrations*, from "Symplectic geometry and mirror symmetry" (K Fukaya, Y-G Oh, K Ono, G Tian, editors), World Sci., River Edge, NJ (2001) 203–263 MR Zbl

[50] **P Kronheimer**, **T Mrowka**, *Monopoles and three-manifolds*, New Math. Monogr. 10, Cambridge Univ. Press (2007) MR Zbl

[51] **J Latschev**, *Gradient flows of Morse–Bott functions*, Math. Ann. 318 (2000) 731–759 MR Zbl

[52] **J Li**, **G Tian**, *Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds*, from "Topics in symplectic 4–manifolds" (R J Stern, editor), First Int. Press Lect. Ser. 1, International, Cambridge, MA (1998) 47–83 MR Zbl

[53] **F Lin**, *A Morse–Bott approach to monopole Floer homology and the triangulation conjecture*, Mem. Amer. Math. Soc. 1221, Amer. Math. Soc., Providence, RI (2018) MR Zbl

[54] **R Lipshitz**, **S Sarkar**, *A Khovanov stable homotopy type*, J. Amer. Math. Soc. 27 (2014) 983–1042 MR Zbl

[55] **J McCleary**, *User's guide to spectral sequences*, Math. Lect. Ser. 12, Publish or Perish, Wilmington, DE (1985) MR Zbl

[56] **D McDuff**, **D Salamon**, *J–holomorphic curves and symplectic topology*, Amer. Math. Soc. Colloq. Publ. 52, Amer. Math. Soc., Providence, RI (2004) MR Zbl

[57] **D McDuff**, **K Wehrheim**, *Smooth Kuranishi atlases with isotropy*, Geom. Topol. 21 (2017) 2725–2809 MR Zbl

[58] **R Melrose**, *Differential analysis on manifolds with corners*, unpublished manuscript (1996) Available at `https://math.mit.edu/~rbm/book.html`

[59] **M Morse**, *The calculus of variations in the large*, Amer. Math. Soc. Colloq. Publ. 18, Amer. Math. Soc., Providence, RI (1932)

[60] **D Murfet**, *Derived categories*, *I*, preprint (2006) Available at `http://therisingsea.org/notes/DerivedCategories.pdf`

[61] **L I Nicolaescu**, *An invitation to Morse theory*, Springer (2007) MR Zbl

[62] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR Zbl

[63] **J Pardon**, *An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves*, Geom. Topol. 20 (2016) 779–1034 MR Zbl

[64] **J Pardon**, *Contact homology and virtual fundamental cycles*, J. Amer. Math. Soc. 32 (2019) 825–919 MR Zbl

[65] **L Qin**, *On moduli spaces and CW structures arising from Morse theory on Hilbert manifolds*, J. Topol. Anal. 2 (2010) 469–526 MR Zbl

[66] **J Robbin**, **D Salamon**, *The Maslov index for paths*, Topology 32 (1993) 827–844 MR Zbl

[67] **Y Ruan**, **G Tian**, *A mathematical theory of quantum cohomology*, J. Differential Geom. 42 (1995) 259–367 MR Zbl

[68] **F Schmäschke**, *Floer homology of Lagrangians in clean intersection*, preprint (2016) arXiv 1606.05327

[69] **M Schwarz**, *Morse homology*, Progr. Math. 111, Birkhäuser, Basel (1993) MR Zbl

[70] **P Seidel**, *A biased view of symplectic cohomology*, from "Current developments in mathematics" (B Mazur, T Mrowka, W Schmid, R Stanley, S-T Yau, editors), International, Somerville, MA (2008) 211–253 MR Zbl

[71] **P Seidel**, *Fukaya categories and Picard–Lefschetz theory*, Eur. Math. Soc., Zürich (2008) MR Zbl

[72] **E G Sklyarenko**, *The Thom isomorphism for nonorientable bundles*, Fundam. Prikl. Mat. 9 (2003) 55–103 MR Zbl In Russian; translated in J. Math. Sci. 136 (2006) 4166–4200

[73] **K Wehrheim**, *Fredholm notions in scale calculus and Hamiltonian Floer theory* (2012) arXiv 1209.4040 To appear in J. Symplectic Geom.

[74] **K Wehrheim**, *Smooth structures on Morse trajectory spaces, featuring finite ends and associative gluing*, from "Proceedings of the Freedman Fest" (R Kirby, V Krushkal, Z Wang, editors), Geom. Topol. Monogr. 18, Geom. Topol. Publ., Coventry (2012) 369–450 MR Zbl

[75] **C A Weibel**, *An introduction to homological algebra*, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press (1994) MR Zbl

[76] **E Witten**, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982) 661–692 MR Zbl

[77] **Z Zhou**, *Equivariant fundamental class and localization theorem*, in preparation

[78] **Z Zhou**, *Quotient theorems in polyfold theory and S^1–equivariant transversality*, Proc. Lond. Math. Soc. 121 (2020) 1337–1426 MR Zbl

[79] **Z Zhou**, *On the cohomology ring of symplectic fillings*, Algebr. Geom. Topol. 23 (2023) 1693–1724 MR Zbl

[80] **A Zinger**, *The determinant line bundle for Fredholm operators: construction, properties, and classification*, Math. Scand. 118 (2016) 203–268 MR Zbl

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The localization spectral sequence in the motivic setting

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We construct and study a motivic lift of a spectral sequence associated to a stratified scheme, recently discovered by Petersen in the context of mixed Hodge theory and ℓ–adic Galois representations. The original spectral sequence expresses the compactly supported cohomology of an open stratum in terms of the compactly supported cohomology of the closures of strata and the combinatorics of the poset underlying the stratification. Some of its special cases are classical tools in the study of arrangements of subvarieties and configuration spaces. Our motivic lift lives in the triangulated category of étale motives and takes the shape of a Postnikov system. We describe its connecting morphisms and study some of its functoriality properties.

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Introduction

For a topological space X, an open subspace U and a complementary closed subspace Z, the compactly supported cohomology groups of X, U and Z are related by a localization long exact sequence

$$\cdots \to H_c^\bullet(U) \to H_c^\bullet(X) \to H_c^\bullet(Z) \to H_c^{\bullet+1}(U) \to \cdots . \tag{1}$$

This can typically be used for two different purposes: either to compute the compactly supported cohomology of X knowing that of U and Z, or to compute the compactly supported cohomology of U knowing that of X and Z.

More generally, let X be a topological space equipped with a stratification, ie a partition by locally closed subspaces called strata such that the closure of a stratum is a union of strata; we assume for simplicity that there is a unique open stratum X_0. The specialization relation turns the set of strata into a finite poset whose least element is X_0. One may either want to understand the space X in terms of the strata, or to understand the open stratum X_0 in terms of the closures of the strata. In the former case, the localization long exact sequence can be generalized to a spectral sequence in an obvious way. In the latter case, however, this was explained only recently by Petersen [2017] who devised a spectral sequence converging to the compactly supported cohomology of X_0, whose first page is expressed in terms of the compactly supported cohomology of the closures of strata, and of the combinatorics of the poset of strata. We refer the reader to the introduction of [loc. cit.] for a clear interpretation in terms of inclusion-exclusion.

A precursor of Petersen's spectral sequence (or rather, of its Poincaré dual version) is Deligne's spectral sequence appearing in mixed Hodge theory [Deligne 1971, 3.2.4.1] where the stratification is induced by a normal crossing divisor inside a smooth projective complex variety. Several other special cases are classical tools in the study of more combinatorially involved contexts such as arrangements of subvarieties [Bibby 2016; Björner and Ekedahl 1997; Dupont 2015; Goresky and MacPherson 1988; Looijenga 1993] and configuration spaces [Cohen and Taylor 1978; Getzler 1999; Kříž 1994; Totaro 1996]. In the general case, Petersen proves that his spectral sequence is compatible with mixed Hodge structures when X is a complex algebraic variety equipped with an algebraic stratification. It also has an étale ℓ–adic variant which is compatible with Galois actions. The proofs are sheaf-theoretic and involve filtering well-chosen resolutions in abelian categories of sheaves.

The goal of this article is to lift Petersen's spectral sequence to a motivic setting. Let now X be a scheme equipped with a stratification (see Section 3 for the relevant assumptions) with a unique open stratum X_0, and let $j : X_0 \hookrightarrow X$ denote the open immersion. We also denote by $i^X_{\overline{S}} : \overline{S} \hookrightarrow X$ the closed immersion of the closure of a stratum S. We denote by $\widehat{P}$ the poset of strata and fix a strictly increasing map $\sigma\colon \widehat{P} \to \mathbb{Z}$ such that $\sigma(X_0) = 0$. We fix a ring of coefficients $\mathbb{K}$. To every stratum $S \in \widehat{P}$ is associated a cochain complex of $\mathbb{K}$–modules $C^\bullet(S)$ which computes the reduced cohomology of the open interval (X_0, S) in the poset $\widehat{P}$.

We work in the context of the triangulated category of étale motives (or motivic sheaves) over X with coefficients in $\mathbb{K}$, denoted by $\mathbb{DA}_X$ [Ayoub 2007a; 2007b; 2014a; Cisinski and Déglise 2016; 2019]. The lack of an abelian-categorical formalism for motivic sheaves forces us to depart from Petersen's original techniques. In the triangulated setting, the notion of a filtration has to be replaced with that of a Postnikov system, that is, a sequence of distinguished triangles where each triangle has a vertex in common with the next one. The main result of this article is as follows (see Theorems 3.3 and 3.16 for more precise statements).

Main Theorem *For $\mathscr{F} \in \mathbb{DA}_X$ there is a Postnikov system in $\mathbb{DA}_X$,*

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & F^2 \arrow[rr] \arrow[dl] & & F^1 \arrow[rr] \arrow[dl] & & F^0 = j_! j^! \mathscr{F} \arrow[dl] \\
& G^2 \arrow[ul, "+1"] & & G^1 \arrow[ul, "+1"] & & G^0 \arrow[ul, "+1"] &
\end{tikzcd}$$

where the graded objects are given by

$$G^k = \bigoplus_{\substack{S \in \widehat{P} \\ \sigma(S)=k}} (i^X_{\overline{S}})_* (i^X_{\overline{S}})^* \mathscr{F} \otimes C^\bullet(S).$$

The connecting morphisms $G^k \to G^{k+1}[1]$ are explicitly computed. This Postnikov system is functorial in $\mathscr{F}$ and functorial with respect to a class of stratified morphisms.

In the case of the constant motivic sheaf $\mathscr{F} = \mathbb{K}_X$, this theorem expresses the compactly supported motive of X_0 in terms of the compactly supported motives of the closures of strata $\overline{S}$ and the complexes $C^\bullet(S)$. For instance, if the stratification consists of an open $j : U \hookrightarrow X$ and its closed complement $i : Z \hookrightarrow X$, the Postnikov system reduces to the localization triangle

$$j_! \mathbb{K}_U \to \mathbb{K}_X \to i_* \mathbb{K}_Z \xrightarrow{+1}$$

which is the motivic lift of the localization long exact sequence (1).

One can recover Petersen's spectral sequence(s) along with a description of the d_1 differential from our main theorem, by applying (compactly supported) cohomological realization functors. In a genuinely motivic setting, an application to the study of classical polylogarithm motives will appear as a joint article of the first author with J Fresán [Dupont and Fresán 2023]. There, it is crucial to have a Postnikov system that is functorial with respect to a group action on a stratified scheme, which is a special case of the functoriality statement in our theorem.

One of the main difficulties in the proof of our main theorem is to construct the Postnikov system in a way that makes it obviously functorial. For this we cannot simply work in the context of a triangulated category, where cones are not functorial. Rather, we are led to work in the enhanced setting of triangulated derivators. Another reason for this choice is that we rely on the six functor formalism for étale motives, developed by Ayoub [2007a; 2007b] and written in the language of algebraic derivators, a geometric enrichment of the notion of a triangulated derivator. From the standpoint of homotopy theory, it is natural to expect our main theorem to lift to the stable ∞–category of motives; this would require an ∞–categorical enhancement of Ayoub's six functor formalism.

We also study a dual version of our main theorem (Theorem 3.9) where we are interested in describing the object $j_* j^* \mathscr{F}$. Due to the lack of duality in the general setting of algebraic derivators, we cannot simply repeat the proof. Instead, we rely on applying Poincaré–Verdier duality, but the latter is available at the motivic level only under certain assumptions (see Section 3.4). Note that, if we gave up on functoriality, then we would not need to work in the setting of algebraic derivators and could prove the dual statement (without functoriality) in full generality. This strongly suggests that the dual statement (with functoriality) is true in full generality, even though we are not able to prove it with our methods. In any case, if one is only interested in working with realizations, one can first apply a realization functor to the main theorem and then dualize.

Perspectives

A natural direction of research would be to try and apply our main theorem to prove motivic representation stability results in the spirit of the homological representation stability results of Petersen [2017]. Also, it would be desirable to clarify the general functoriality properties of our construction, beyond those already explored here.

A motivation for this project is the possibility to study a more general geometric setting mixing $j_!$ and j_* extensions, depending on the strata. The corresponding motives can be viewed as relative cohomology motives on some blow-up of the ambient variety and are ubiquitous in the geometric study of periods (see eg [Goncharov 2002] and the introduction of [Dupont 2017]).

Outline

In Section 1 we review classical definitions and properties of poset (co)homology; to the best of our knowledge, the only original content is the introduction of connecting morphisms relating poset (co)homology complexes of different intervals in a poset. In Section 2 we work in the setting of triangulated derivators and collect some tools to produce and study functorial Postnikov systems. In Section 3 we apply those tools to our geometric setting and prove the main results.

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1 Poset (co)homology

In this section we review poset (co)homology. To the best of our knowledge, the only original content is the introduction of connecting morphisms relating poset (co)homology complexes of different intervals in a poset. We fix a commutative ring with unit $\mathbb{K}$ for the rest of this article, that will serve as a ring of coefficients.

1.1 Definition

Let P be a finite poset. We will sometimes make use of the extension $\widehat{P} = \{\hat{0}\} \cup P$ where $\hat{0} < p$ for all $p \in P$. For any element $x \in P$ we let $C_\bullet(x)$, denoted by $C_\bullet^P(x)$ when we want to make dependence on P explicit, be the chain complex whose degree n component is the free $\mathbb{K}$–module on chains

$$[x_1 < \cdots < x_{n-1} < x_n = x],$$

and whose differential $\partial\colon C_n(x) \to C_{n-1}(x)$ is given by

$$\partial[x_1 < \cdots < x_{n-1} < x_n = x] = \sum_{i=1}^{n-1} (-1)^{i-1} [x_1 < \cdots < \hat{x}_i < \cdots < x_{n-1} < x_n = x].$$

We let $h_\bullet(x)$ denote the homology of $C_\bullet(x)$. Up to a shift, $C_\bullet(x)$ is the (reduced) normalized chain complex of the nerve of the poset $P_{<x} = \{p \in P \mid p < x\}$ and thus we have

$$h_n(x) = H_n(C_\bullet(x)) = \widetilde{H}_{n-2}(P_{<x}).$$

We let $C^\bullet(x)$, or $C_P^\bullet(x)$ when we want to make dependence on P explicit, denote the cochain complex dual to $C_\bullet(x)$ and use the same notation for the basis of chains and the (dual) basis of cochains. The differential $d\colon C^n(x) \to C^{n+1}(x)$ is given by

$$d[x_1 < \cdots < x_{n-1} < x_n = x] = \sum_{i=1}^{n} (-1)^{i-1} \sum_{x_{i-1}<y<x_i} [x_1 < \cdots < x_{i-1} < y < x_i < \cdots < x_{n-1} < x_n = x],$$

where by convention we have $x_0 = \hat{0}$ in $\widehat{P}$. We let $h^\bullet(x)$ denote the cohomology of $C^\bullet(x)$ and we have

$$h^n(x) = H^n(C^\bullet(x)) = \widetilde{H}^{n-2}(P_{<x}).$$

The following lemma is classical.

Lemma 1.1 *If P has a least element a then $C_\bullet(x)$ and $C^\bullet(x)$ are contractible for all $x > a$.*

Proof The nerve of $P_{<x} = [a, x)$ is a cone over the nerve of the open interval (a, x) and thus contractible. Concretely, a contracting homotopy $c\colon C_\bullet(x) \to C_{\bullet+1}(x)$ is provided by the formula

$$c[x_1 < \cdots < x_{n-1} < x_n = x] = \begin{cases} 0 & \text{if } x_1 = a, \\ [a < x_1 < \cdots < x_{n-1} < x_n = x] & \text{if } x_1 > a. \end{cases}$$

The transpose of c is a contracting homotopy for $C^\bullet(x)$. □

It is sometimes convenient to extend the definitions to $\widehat{P}$ by defining $C_\bullet(\hat{0})$ and $C^\bullet(\hat{0})$ to be $\mathbb{K}$ concentrated in degree zero.

Remark 1.2 The complexes $C_\bullet$ have a certain functoriality property with respect to morphisms of posets. In this article we will only deal with functoriality with respect to isomorphisms (and in particular with respect to group actions). For $\alpha\colon P \to P'$ an isomorphism of posets we have for every $x \in P$ a natural isomorphism of chain complexes $C_\bullet(\alpha)\colon C_\bullet^P(x) \to C_\bullet^{P'}(x')$ for $x' = \alpha(x)$. They satisfy $C_\bullet(\mathrm{id}) = \mathrm{id}$ and $C_\bullet(\beta \circ \alpha) = C_\bullet(\beta) \circ C_\bullet(\alpha)$. Dually we have natural isomorphisms of cochain complexes $C^\bullet(\alpha)\colon C_{P'}^\bullet(x') \to C_P^\bullet(x)$ that satisfy $C^\bullet(\mathrm{id}) = \mathrm{id}$ and $C^\bullet(\beta \circ \alpha) = C^\bullet(\alpha) \circ C^\bullet(\beta)$.

1.2 The connecting maps

For $x < y$ in P we define a map

$$b_x^y\colon C_{\bullet+1}(y) \to C_\bullet(x)$$

by setting

$$b_x^y[x_1 < \cdots < x_n < x_{n+1} = y] = \begin{cases} (-1)^n [x_1 < \cdots < x_n = x] & \text{if } x_n = x, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.3 $(\partial b_x^y + b_x^y\partial)[x_1 < \cdots < x_n < x_{n+1} = y] = \begin{cases} [x_1 < \cdots < x_{n-1} = x] & \text{if } x_{n-1} = x, \\ 0 & \text{otherwise.} \end{cases}$

Proof We compute, for $X = [x_1 < \cdots < x_n < x_{n+1} = y]$,

$$b_x^y\partial X = \sum_{i=1}^{n-1} (-1)^{i-1} b_x^y [x_1 < \cdots < \hat{x}_i < \cdots < x_n < x_{n+1} = y] + (-1)^{n-1} b_x^y [x_1 < \cdots < x_{n-1} < x_{n+1} = y].$$

If $x_{n-1} = x$ then $x_n \neq x$ and we have $b_x^y\partial X = [x_1 < \cdots < x_{n-1} = x]$ and $\partial b_x^y X = \partial 0 = 0$, which proves the first part of the claim. If $x_{n-1} \neq x$ and $x_n \neq x$ then all terms vanish and we get $b_x^y\partial X = \partial b_x^y X = 0$. If $x_{n-1} \neq x$ and $x_n = x$ then

$$b_x^y\partial X = \sum_{i=1}^{n-1} (-1)^{n-i} [x_1 < \cdots < \hat{x}_i < \cdots < x_n = x] = -\partial b_x^y X. \qquad \square$$

We write $x \lessdot y$ when y covers x in P, ie when $x < y$ and there is no $z \in P$ such that $x < z < y$.

Lemma 1.4 (1) *For $x \lessdot y$ in P, $b_x^y \colon C_{\bullet+1}(y) \to C_\bullet(x)$ is a morphism of complexes.*

(2) *Let $x < z$ in P be such that every $y \in (x, z)$ satisfies $x \lessdot y \lessdot z$. Then the morphism of complexes*

$$\sum_{x<y<z} b_x^y b_y^z \colon C_{\bullet+2}(z) \to C_\bullet(x)$$

is homotopic to zero.

The first part of the lemma implies that we get connecting morphisms $b_x^y \colon h_{\bullet+1}(y) \to h_\bullet(x)$ in homology, for $x \lessdot y$.

Proof (1) For $x_{n-1} < x_n < x_{n+1} = y$ we cannot have $x_{n-1} = x$ since y covers x. Then Lemma 1.3 implies that $\partial b_x^y = -b_x^y\partial$, thus b_x^y is a morphism of complexes.

(2) We have

$$\sum_{x<y<z} b_x^y b_y^z [x_1 < \cdots < x_{n+1} < x_{n+2} = z] = \begin{cases} -[x_1 < \cdots < x_n = x] & \text{if } x_n = x, \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to Lemma 1.3 this can be rewritten as

$$\sum_{x<y<z} b_x^y b_y^z = -\partial b_x^z - b_x^z\partial. \qquad \square$$

By duality we get a map that we denote by the same symbol, since there is no risk of confusion,

$$b_x^y \colon C^\bullet(x) \to C^{\bullet+1}(y).$$

It is defined by the formula

$$b_x^y [x_1 < \cdots < x_n = x] = (-1)^n [x_1 < \cdots < x_n = x < x_{n+1} = y].$$

Lemma 1.5 (1) *For $x \lessdot y$ in P, $b_x^y\colon C^\bullet(x) \to C^{\bullet+1}(y)$ is a morphism of complexes.*

(2) *Let $x < z$ in P be such that every $y \in (x, z)$ satisfies $x \lessdot y \lessdot z$. Then the morphism of complexes*

$$\sum_{x<y<z} b_y^z b_x^y \colon C^\bullet(x) \to C^{\bullet+2}(z)$$

is homotopic to zero.

Proof This is the dual of Lemma 1.4. □

It is sometimes convenient to extend the definitions to $\widehat{P}$. Indeed, for $\hat{0} \lessdot y$, ie for y a minimal element of P, we can define $b_{\hat{0}}^y\colon C_{\bullet+1}(y) \to C_\bullet(\hat{0})$ to be the natural (iso)morphism of complexes. The same goes in cohomology for $b_{\hat{0}}^y\colon C^\bullet(\hat{0}) \to C^{\bullet+1}(y)$. One easily checks that Lemmas 1.4 and 1.5 also apply to the case $x = \hat{0}$.

Remark 1.6 Let us assume for simplicity that the poset $\widehat{P}$ is graded, ie any two maximal chains between two elements $x < y$ in $\widehat{P}$ have the same length. For $x \in \widehat{P}$ let $\mathrm{rk}(x)$ denote the length of a maximal chain from $\hat{0}$ to x. In many geometric cases we have, for every $x \in \widehat{P}$,

$$h_n(x) = 0 \quad \text{for } n \neq \mathrm{rk}(x),$$

and we simply write $h(x) = h_{\mathrm{rk}(x)}(x)$. (This implies that the cohomology of $C^\bullet(x)$ is concentrated in degree $\mathrm{rk}(x)$ and that $h^{\mathrm{rk}(x)}(x) \simeq h(x)^\vee$.) This condition is satisfied, eg if the poset $\widehat{P}$ is Cohen–Macaulay [Baclawski 1980; Björner et al. 1982]. In this case we get a chain complex (h, b) where

$$h_n = \bigoplus_{\substack{x \in \widehat{P} \\ \mathrm{rk}(x) = n}} h(x)$$

and $b\colon h_{n+1} \to h_n$ is induced by the connecting maps b_x^y for $x < y$. One can also prove that these connecting maps induce acyclic complexes of $\mathbb{K}$–modules, for every $x \in P$,

$$0 \to h(x) \to \bigoplus_{\substack{y \in \widehat{P}, y < x \\ \mathrm{rk}(y) = \mathrm{rk}(x) - 1}} h(y) \to \bigoplus_{\substack{z \in \widehat{P}, z < x \\ \mathrm{rk}(z) = \mathrm{rk}(x) - 2}} h(z) \to \cdots \to h(\hat{0}) \to 0.$$

This allows one to define $h(x)$ together with the connecting morphisms b_u^x by induction on $\mathrm{rk}(x)$.

A typical example of a Cohen–Macaulay poset is the poset of flats of a matroid (for instance, the poset of strata of a central hyperplane arrangement); in this case (h, b) is the underlying chain complex of the Orlik–Solomon algebra of the matroid [Orlik and Solomon 1980; Orlik and Terao 1992].

1.3 Interpretation of poset cohomology as homotopy limit

We will now consider the abelian category of representations of the finite poset P, ie the category $(\mathbb{K}\text{–Mod})^P$ of functors from P viewed as a category to the category of $\mathbb{K}$–modules. Since $\mathbb{K}$–Mod is abelian, it admits finite limits, so we have a limit functor $\lim_P\colon (\mathbb{K}\text{–Mod})^P \to \mathbb{K}\text{–Mod}$, which is right adjoint to the constant functor $\mathbb{K}\text{–Mod} \to (\mathbb{K}\text{–Mod})^P$; since it has a left adjoint, it is left exact, and we may consider the right derived functor $\mathrm{R}\lim_P\colon D((\mathbb{K}\text{–Mod})^P) \to D(\mathbb{K}\text{–Mod})$. In anticipation of the next section, we will call it homotopy limit and denote it by holim_P. We now prove and discuss the following interpretation of the complexes $C^\bullet(x)$ (see also [Tosteson 2016] for a similar discussion).

Proposition 1.7 *For $x \in P$ we denote by $\mathbb{K}_x$ the representation of P defined by $\mathbb{K}_x(y) = \mathbb{K}$ if $y = x$ and zero otherwise. We have a canonical isomorphism in $D(\mathbb{K}\text{–Mod})$,*

$$\mathrm{holim}_P\, \mathbb{K}_x \simeq C^{\bullet+1}(x).$$

In order to compute the functor holim_P we introduce convenient $\lim_P$–acyclic representations of P. For $x \in P$ and $M \in \mathbb{K}\text{–Mod}$, we let $M_{\leqslant x} \in (\mathbb{K}\text{–Mod})^P$ denote the representation given by $M_{\leqslant x}(y) = M$ if $y \leqslant x$ and zero otherwise, the transition morphisms being the identity of M or zero.

Lemma 1.8 *The representation $M_{\leqslant x}$ is $\lim_P$–acyclic.*

Proof The functor

$$(-)_{\leqslant x}\colon \mathbb{K}\text{–Mod} \to (\mathbb{K}\text{–Mod})^P, \quad M \mapsto M_{\leqslant x},$$

is exact and sends injectives to injectives. Indeed, for $T \in (\mathbb{K}\text{–Mod})^P$ we have an isomorphism

$$\mathrm{Hom}_{(\mathbb{K}\text{–Mod})^P}(T, M_{\leqslant x}) \simeq \mathrm{Hom}_{\mathbb{K}\text{–Mod}}(T(x), M),$$

and thus the functor $\mathrm{Hom}_{(\mathbb{K}\text{–Mod})^P}(-, M_{\leqslant x})$ is exact if M is injective. Thus, we have isomorphisms

$$\mathrm{R}\lim_P(M_{\leqslant x}) \simeq \mathrm{R}\lim_P \circ R(-)_{\leqslant x}(M) \simeq R(\lim_P \circ (-)_{\leqslant x})(M) \simeq M \simeq \lim_P(M_{\leqslant x}).$$

The first isomorphism follows from the fact that $(-)_{\leqslant x}$ is exact, the second follows from the fact that it sends injectives to injectives, the third and fourth from the equality $\lim_P \circ (-)_{\leqslant x} = \mathrm{Id}_{\mathbb{K}\text{–Mod}}$. □

Proof of Proposition 1.7 For $z \leqslant y$ we have a canonical morphism $\mathbb{K}_{\leqslant y} \to \mathbb{K}_{\leqslant z}$. Moreover, those morphisms compose functorially. Using them we can form a resolution

$$0 \to \mathbb{K}_x \to \mathbb{K}_{\leqslant x} \to \bigoplus_{y<x} \mathbb{K}_{\leqslant y} \to \bigoplus_{z<y<x} \mathbb{K}_{\leqslant z} \to \cdots.$$

More precisely, we set

$$R_x^n = \bigoplus_{[x_1<\cdots<x_n<x_{n+1}=x]} \mathbb{K}_{\leqslant x_1}.$$

In analogy with the construction of the complexes $C^\bullet(x)$ of Section 1.1, we define a differential $d\colon R_x^n \to R_x^{n+1}$. Its component indexed by chains $[x_1 < \cdots < x_n < x_{n+1} = x]$ on the source and $[x_1 < \cdots < x_{i-1} < y < x_i < \cdots < x_n < x_{n+1} = x]$ on the target equals $(-1)^i$ times the natural map (the latter being the identity for $i > 1$ and the canonical morphism $\mathbb{K}_{\leqslant x_1} \to \mathbb{K}_{\leqslant y}$ for $i = 1$). The other components are zero. One easily checks that we get a complex $R_x^\bullet$ of representations of P which is such that

$$R_x^\bullet(a) = \begin{cases} \mathbb{K} & \text{if } a = x, \\ C_{[a,x]}^{\bullet+1}(x) & \text{if } a < x, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1.1, the complex $C_{[a,x]}^\bullet(x)$ is contractible and we thus get a resolution $\mathbb{K}_x \xrightarrow{\sim} R_x^\bullet$.

By Lemma 1.8 this resolution is $\lim_P$–acyclic. Hence, it can be used to compute $\operatorname{holim}_P \mathbb{K}_x = \mathrm{R}\lim_P \mathbb{K}_x$. Since each $\lim_P \mathbb{K}_{\leqslant x_1}$ is just $\mathbb{K}$, applying $\lim_P$ to the resolution gives $\lim_P R_x^\bullet \simeq C^{\bullet+1}(x)$, and the result follows. □

Remark 1.9 The resolution appearing in the proof of Proposition 1.7 is a Bousfield–Kan resolution [1972, Chapter XI].

We now turn to the interpretation of the connecting morphisms b_x^y. For $x < y$ in P we let $\mathbb{K}_x^y$ denote the representation of P defined by $\mathbb{K}_x^y(z) = \mathbb{K}$ if $z \in \{x, y\}$ and zero otherwise, the transition morphism $\mathbb{K}_x^y(x) \to \mathbb{K}_x^y(y)$ being the identity. We have a short exact sequence in $(\mathbb{K}\text{–Mod})^P$,

$$0 \to \mathbb{K}_y \to \mathbb{K}_x^y \to \mathbb{K}_x \to 0,$$

which induces a distinguished triangle $\mathbb{K}_y \to \mathbb{K}_x^y \to \mathbb{K}_x \xrightarrow{+1}$ in $D((\mathbb{K}\text{–Mod})^P)$. We denote by

$$a_x^y : \mathbb{K}_x \to \mathbb{K}_y[1]$$

the connecting morphism.

Proposition 1.10 *Assume that $x \lessdot y$. We have a commutative square in $D(\mathbb{K}\text{–Mod})$,*

$$\begin{tikzcd} \operatorname{holim}_P \mathbb{K}_x \arrow[r, "\operatorname{holim}_P a_x^y"] \arrow[d, "\simeq"'] & \operatorname{holim}_P \mathbb{K}_y[1] \arrow[d, "\simeq"] \\ C^{\bullet+1}(x) \arrow[r, "b_x^y[1]"'] & C^{\bullet+2}(y) \end{tikzcd}$$

where the vertical isomorphisms are those of Proposition 1.7.

Proof Let $R_x^\bullet$ and $R_y^\bullet$ denote the resolutions of $\mathbb{K}_x$ and $\mathbb{K}_y$ described in the proof of Proposition 1.7. By mimicking the definition of b_x^y and the proof of Lemma 1.5 1) we get a morphism of complexes $R_x^\bullet \to R_y^{\bullet+1}$. We let $S^\bullet$ denote its cone shifted by -1, so that $S^\bullet = R_x^\bullet \oplus R_y^\bullet$ as graded P–representations. We consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{K}_y \arrow[r] \arrow[d] & \mathbb{K}_x^y \arrow[r] \arrow[d, dashed] & \mathbb{K}_x \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R_y^\bullet \arrow[r] & S^\bullet \arrow[r] & R_x^\bullet \arrow[r] & 0 \end{tikzcd}$$

where both rows are short exact sequences. The dotted arrow $\mathbb{K}_x^y \to S^0 = \mathbb{K}_{\leqslant x} \oplus \mathbb{K}_{\leqslant y}$ is defined so that its value at y is the identity of $\mathbb{K}$ and its value at x is the diagonal morphism $\mathbb{K} \to \mathbb{K} \oplus \mathbb{K}$. It is a morphism of representations of P because $x \lessdot y$. The composite $\mathbb{K}_x^y \to S^0 \to S^1$ is zero, as one can check on the values at x and y. In the above commutative diagram, the bottom row is thus a $\lim_P$–acyclic resolution of the top row, by the five lemma. This implies that the connecting morphism $\operatorname{holim}_P \mathbb{K}_x \to \operatorname{holim}_P \mathbb{K}_y[1]$ is computed by the connecting morphism in the long exact sequence associated to the short exact sequence

$$0 \to \lim_P R_y^\bullet \to \lim_P S^\bullet \to \lim_P R_x^\bullet \to 0.$$

By construction, this is nothing but the short exact sequence for the cone of the morphism

$$b_x^y[-1]\colon C^{\bullet-1}(x) \to C^\bullet(y),$$

and the connecting morphism is b_x^y. □

Remark 1.11 Let $\alpha\colon P \to P'$ be an isomorphism of posets, let $x \in P$ and $x' = \alpha(x) \in P'$. One easily proves that the natural isomorphism

$$C_{P'}^{\bullet+1}(x') \simeq \operatorname{holim}_{P'} \mathbb{K}_{x'} \simeq \operatorname{holim}_P \mathbb{K}_x \simeq C_P^{\bullet+1}(x)$$

is the isomorphism of complexes denoted by $C^{\bullet+1}(\alpha)$ in Remark 1.2.

2 Triangulated derivators

In this section we collect some tools about triangulated derivators and natural Postnikov systems arising in this context. The main result is Proposition 2.20.

2.1 The framework of triangulated derivators

We work within the framework of triangulated derivators, introduced by Grothendieck [1991] and developed by several authors; see [Ayoub 2007a; Cisinski and Neeman 2008; Franke 1996; Groth 2013; Heller 1988; Maltsiniotis 2001]. Broadly speaking, triangulated derivators are like triangulated categories with well-defined homotopy limits and colimits (and more generally homotopy Kan extensions).

We work with Ayoub's notion [2007a] of a triangulated derivator in order to be able to use the notion of an algebraic derivator from [loc. cit.] in the next section. There a 2–category of "diagrams" is fixed, which is a full 2–subcategory of the 2–category of (small) categories satisfying the axioms D0, D1 and D2 of [Ayoub 2007a, section 2.1.2]; we choose it to be the 2–category of finite posets, since those are the only diagrams that we will need. All 2–categories in this paper are strict, and our notion of a 2–functor between two 2–categories is the weak one, ie that of a pseudofunctor in the sense of [Borceux 1994, 7.5].

2.1.1 Finite posets A finite poset P is viewed as a category with a unique morphism from x to y if $x \leqslant y$, and none otherwise. Finite posets thus form a full 2–subcategory of the 2–category of (small) categories. A functor between finite posets is an order-preserving map and is simply called a morphism of posets. For two such morphisms $f, g\colon P \to Q$, there is a unique natural transformation from f to g if $f(x) \leqslant g(x)$ for every $x \in P$, and none otherwise.

We denote by e the poset with one element. For P a finite poset, we denote by p or $p_P\colon P \to e$ the morphism to a point. An inclusion between posets $Q \subset P$ is denoted $i_Q^P\colon Q \to P$. For P a finite poset and $x \in P$, we use the notation i_x or $i_x^P\colon e \to P$ for the inclusion of x.

2.1.2 Triangulated prederivators

Definition 2.1 A *triangulated prederivator* $\mathbb{D}$ is a 1–contravariant and 2–covariant 2–functor from the 2–category of finite posets to the 2–category of triangulated categories. In other words, it associates

- (0) to every finite poset P, a triangulated category $\mathbb{D}(P)$;
- (1) to every morphism $f\colon P \to Q$ of finite posets, a triangulated functor $f^*\colon \mathbb{D}(Q) \to \mathbb{D}(P)$;
- (2) to every pair of morphisms $f, g\colon P \to Q$ such that $f(x) \leqslant g(x)$ for every $x \in P$, a natural transformation of triangulated functors $f^* \to g^*$;

in a way that is compatible with horizontal and vertical composition.

Remark 2.2 The triangulated category $\mathbb{D}(e)$ is called the *ground category*. For a finite poset P, an element $x \in P$ and an object $\mathscr{F} \in \mathbb{D}(P)$, the pullback $(i_x)^*\mathscr{F} \in \mathbb{D}(e)$ is called the *value* of $\mathscr{F}$ at x. For elements $x, y \in P$ such that $x \leqslant y$ we have two morphisms $i_x, i_y\colon e \to P$ such that $i_x(\cdot) \leqslant i_y(\cdot)$ and thus a natural transformation $(i_x)^* \to (i_y)^*$. Thus, the functors $(i_x)^*$ induce an *underlying diagram functor*

$$\mathbb{D}(P) \to \mathbb{D}(e)^P \tag{2}$$

which is not an equivalence in general. The category $\mathbb{D}(P)$ should be thought of as the category of "homotopy coherent" P–shaped diagrams of objects of the ground category $\mathbb{D}(e)$, whereas the category $\mathbb{D}(e)^P$ consists of "homotopy incoherent" diagrams. More generally we have "partial" underlying diagram functors, for finite posets P and E,

$$\mathbb{D}(E \times P) \to \mathbb{D}(E)^P$$

and diagrams in $\mathbb{D}(E)^P$ can be called "partially homotopy incoherent".

Remark 2.3 Our variance convention slightly differs from that of [Ayoub 2007a] since there prederivators are 1–contravariant and 2–contravariant, which makes the underlying diagram functor land in $\mathbb{D}(e)^{P^{\mathrm{op}}}$.

2.1.3 Triangulated derivators A *triangulated derivator* [Ayoub 2007a, définition 2.1.34] is a triangulated prederivator that satisfies a certain number of axioms, including the following that we mention for future reference:

(1) We have $\mathbb{D}(\varnothing) = 0$.

(2) The underlying diagram functor (2) is conservative for every finite poset P; it is a triangulated equivalence if P is discrete.

(3) For every morphism $f\colon P \to Q$ of finite posets, the functor $f^*\colon \mathbb{D}(Q) \to \mathbb{D}(P)$ admits right and left adjoints,

$$f_*\colon \mathbb{D}(P) \to \mathbb{D}(Q), \quad f_!\colon \mathbb{D}(P) \to \mathbb{D}(Q),$$

respectively, which are automatically triangulated functors. They play the role of homotopy right and left Kan extension functors; in the special case of $p\colon P \to e$, the projection to a point, they are a homotopy limit and colimit functors and we write $p_* = \mathrm{holim}$ and $p_! = \mathrm{hocolim}$.

(4) For a morphism $f : P \to Q$ and some element $y \in Q$, let $y/P \subset P$ denote the subposet consisting of elements $x \in P$ such that $y \leqslant f(x)$. We have a natural transformation

$$(p_{y/P})^*(i_y^Q)^* \to (i_{y/P}^P)^* f^*$$

associated by 2–functoriality to the two morphisms $(i_y^Q) \circ p_{y/P}$ and $f \circ (i_{y/P}^P)$ from y/P to Q. By using the units and counits of the adjunctions we can obtain from it a natural transformation

$$(i_y^Q)^* f_* \to (p_{y/P})_*(i_{y/P}^P)^*$$

which is $(i_y^Q)^* f_* \to (p_{y/P})_*(p_{y/P})^*(i_y^Q)^* f_* \to (p_{y/P})_*(i_{y/P}^P)^* f^* f_* \to (p_{y/P})_*(i_{y/P}^P)^*$. We require this last natural transformation to be invertible. In the same vein, let $P/y \subset P$ denote the subposet consisting of elements $x \in P$ such that $f(x) \leqslant y$. Then we have a natural transformation

$$(p_{P/y})_!(i_{P/y}^P)^* \to (i_y^Q)^* f_!$$

that we require to be invertible.

Remark 2.4 The axioms listed above are similar to the axioms 1–4 of [Ayoub 2007a, définition 2.1.34], albeit slightly less complete for (2) and (4). In [loc. cit.] two more axioms, 5 and 6, relate the triangulated structures on the categories $\mathbb{D}(P)$ with the homotopy Kan extension functors f_* and $f_!$ and will not be used in the rest of this article.

Remark 2.5 If $\mathscr{A}$ is a Grothendieck abelian category, eg $\mathscr{A} = \mathbb{K}\text{–Mod}$, then we have a derivator $\mathbb{D}_{\mathscr{A}}$ such that $\mathbb{D}_{\mathscr{A}}(P)$ is the derived category of the diagram category $\mathscr{A}^P$ for every finite poset P. The pullback functors f^* are the obvious ones and their adjoints are obtained by deriving the usual Kan extension functors.

2.1.4 Monoidal structure The triangulated derivators that we will deal with all have a unital symmetric monoidal structure in the sense of [Ayoub 2007a, section 2.1.6]. This means that for every finite poset P the triangulated category $\mathbb{D}(P)$ is endowed with the structure of a unital symmetric monoidal triangulated category and that for every morphism $f : P \to Q$ the functor $f^*\colon \mathbb{D}(Q) \to \mathbb{D}(P)$ is endowed with the structure of a unital symmetric monoidal functor. The triangulated derivator $\mathbb{D}_{\mathbb{K}\text{–Mod}}$ of the abelian category $\mathbb{K}$–Mod is symmetric monoidal.

Let $\mathbb{D}$ be a unital symmetric monoidal derivator. Then we have, for every morphism of finite posets $f : P \to Q$ and for $\mathscr{F} \in \mathbb{D}(P)$, $\mathscr{G} \in \mathbb{D}(Q)$, a natural morphism

$$\mathscr{G} \otimes f_*\mathscr{F} \to f_*(f^*\mathscr{G} \otimes \mathscr{F}). \tag{3}$$

It is obtained as the composition $\mathscr{G}\otimes f_*\mathscr{F}\to f_*f^*(\mathscr{G}\otimes f_*\mathscr{F})\xrightarrow{\sim} f_*(f^*\mathscr{G}\otimes f^*f_*\mathscr{F})\to f_*(f^*\mathscr{G}\otimes\mathscr{F})$, where the first and last steps involve the unit and the counit of the adjunction, and the middle step uses that f^* is monoidal. In the same way, we have a natural morphism

$$f_!(f^*\mathscr{G}\otimes\mathscr{F})\to\mathscr{G}\otimes f_!\mathscr{F}. \tag{4}$$

Neither (3) nor (4) is an isomorphism in general.

2.1.5 Coefficients In the remainder of this section we fix a unital symmetric monoidal triangulated derivator $\mathbb{D}$ equipped with a morphism of unital symmetric monoidal triangulated derivators $\mathbb{D}_{\mathbb{K}\text{–Mod}}\to\mathbb{D}$. Such an object can be called a *unital symmetric monoidal triangulated derivator with coefficients in* $\mathbb{K}$.

We will allow ourselves to interpret complexes of $\mathbb{K}$–modules as objects of $\mathbb{D}(e)$ without specific reference to the morphism $\mathbb{D}_{\mathbb{K}\text{–Mod}}\to\mathbb{D}$.

2.2 Extension by zero

We start with a classical lemma.

Lemma 2.6 *Let P be a finite poset with projection $p\colon P\to e$.*

(1) *If P has a least element x then we have isomorphisms $p_*\simeq(i_x)^*$ and $p^*\simeq(i_x)_!$. The natural morphism $p_!p^*\to\mathrm{id}_{\mathbb{D}(e)}$ is an isomorphism.*
(2) *If P has a greatest element y then we have isomorphisms $p_!\simeq(i_y)^*$ and $p^*\simeq(i_y)_*$. The natural morphism $\mathrm{id}_{\mathbb{D}(e)}\to p_*p^*$ is an isomorphism.*

Proof We prove the first point (the second is proved dually). The fact that x is the least element of P may be expressed by the fact that (i_x,p) is an adjoint pair of functors. It follows that $(p^*,(i_x)^*)$ is also an adjoint pair of functors. Now $(i_x)^*$ being a right adjoint to p^* means that it is equal to p_*, and p^* being a left adjoint to $(i_x)^*$ means that it is equal to $(i_x)_!$.

For the second assertion, note that $pi_x=\mathrm{id}_e$, hence $p_!(i_x)_!\simeq\mathrm{id}_{\mathbb{D}(e)}$, and the isomorphism $p^*=(i_x)_!$ identifies this with the adjunction morphism $p_!p^*\to\mathrm{id}_{\mathbb{D}(e)}$. □

Lemma 2.7 *Let $i\colon Q\hookrightarrow P$ denote the inclusion of a subposet. For every $\mathscr{G}\in\mathbb{D}(Q)$ the natural morphisms*

$$i^*i_*\mathscr{G}\to\mathscr{G}\quad\text{and}\quad\mathscr{G}\to i^*i_!\mathscr{G}$$

are isomorphisms.

Proof We prove that the first morphism is an isomorphism (the second case is proved dually). For every $x\in Q$ we have a sequence of isomorphisms

$$(i_x^Q)^*i^*i_*\mathscr{G}\simeq(i_x^P)^*i_*\mathscr{G}\simeq(p_{x/Q})_*(i_{x/Q}^Q)^*\mathscr{G}\simeq(i_x^{x/Q})^*(i_{x/Q}^Q)^*\mathscr{G}\simeq(i_x^Q)^*\mathscr{G},$$

where the second isomorphism follows from Section 2.1.3(4) and the third isomorphism follows from Lemma 2.6 since x is the least element of x/Q. One checks that the composition of these isomorphisms is the composition of $(i_x^Q)^*$ with the natural morphism $i^*i_*\mathcal{G} \to \mathcal{G}$. By Section 2.1.3(2) this proves the claim. □

Definition 2.8 Let P be a finite poset.

(1) A *sieve* in P is a subset $U \subset P$ such that for every $x \leqslant y$ in P, $y \in U$ implies $x \in U$.

(2) A *cosieve* in P is a subset $V \subset P$ such that for every $x \leqslant y$ in P, $x \in V$ implies $y \in V$.

The complement of a sieve is a cosieve and the complement of a cosieve is a sieve. We also call a sieve (resp. cosieve) the functor of posets given by the inclusion of a sieve (resp. cosieve). The following lemma is classical and says that the functor u_* (resp. $v_!$) deserves the name "extension by zero" if u is a sieve (resp. if v is a cosieve).

Lemma 2.9 *Let P be a finite poset.*

(1) *Let $u\colon U \hookrightarrow P$ be a sieve. For $\mathcal{F} \in \mathbb{D}(P)$, the natural morphism $\mathcal{F} \to u_*u^*\mathcal{F}$ is an isomorphism if and only if $(i_x)^*\mathcal{F} = 0$ for all $x \in P \setminus U$.*

(2) *Let $v\colon V \hookrightarrow P$ be a cosieve. For $\mathcal{F} \in \mathbb{D}(P)$, the natural morphism $v_!v^*\mathcal{F} \to \mathcal{F}$ is an isomorphism if and only if $(i_x)^*\mathcal{F} = 0$ for all $x \in P \setminus V$.*

Proof We prove the first point (the second is proved dually). Let us assume that the natural morphism $\mathcal{F} \to u_*u^*\mathcal{F}$ is an isomorphism. Then for $x \in P \setminus U$ we have an isomorphism

$$(i_x)^*\mathcal{F} \simeq (i_x)^*u_*u^*\mathcal{F} \simeq (p_{x/U})_*(i_{x/U}^U)^*u^*\mathcal{F},$$

where the second isomorphism follows from Section 2.1.3(4). By assumption, we have $x/U = \varnothing$ and Section 2.1.3(1) implies that $(i_x)^*\mathcal{F} = 0$. Conversely, if $(i_x)^*\mathcal{F} = 0$ for all $x \in P \setminus U$ then the same argument shows that the natural morphism $(i_x)^*\mathcal{F} \to (i_x)^*u_*u^*\mathcal{F}$ is an isomorphism. The fact that it is an isomorphism also for $x \in U$ follows from the same kind of reasoning as in the proof of Lemma 2.7. Thanks to Section 2.1.3(2) we conclude that the morphism $\mathcal{F} \to u_*u^*\mathcal{F}$ is an isomorphism. □

The next lemma explains the compatibility between extension by zero and pullback.

Lemma 2.10 (1) *Consider the cartesian diagram in the category of finite posets,*

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "u'"] \arrow[d, "g"'] & Q \arrow[d, "f"] \\ U \arrow[r, "u"'] & P \end{tikzcd}$$

where u is a sieve. Then we have a canonical isomorphism $f^*u_* \xrightarrow{\sim} (u')_*g^*$.

(2) *Consider the cartesian diagram in the category of posets*,

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "v'"] \arrow[d, "h"'] & Q \arrow[d, "f"] \\ V \arrow[r, "v"'] & P \end{tikzcd}$$

where v is a cosieve. Then we have a canonical isomorphism $(v')_! h^* \xrightarrow{\sim} f^* v_!$.

Proof We prove the first point (the second is proved dually). The morphism $f^* u_* \to (u')_* g^*$ is the composite $f^* u_* \to (u')_*(u')^* f^* u_* \xrightarrow{\sim} (u')_* g^* u^* u_* \xrightarrow{\sim} (u')_* g^*$. The fact that it is an isomorphism follows from Lemma 2.9 and the fact that u and u' are sieves. □

The next lemma provides a projection formula for the "extension by zero" functors.

Lemma 2.11 *Let P be a finite poset.*

(1) *Let $u\colon U \hookrightarrow P$ be a sieve. For $\mathscr{F} \in \mathbb{D}(P)$ and $\mathscr{G} \in \mathbb{D}(U)$, the natural morphism*

$$\mathscr{F} \otimes u_* \mathscr{G} \to u_*(u^* \mathscr{F} \otimes \mathscr{G})$$

defined in Section 2.1.4(3) *is an isomorphism.*

(2) *Let $v\colon V \hookrightarrow P$ be a cosieve. For $\mathscr{F} \in \mathbb{D}(P)$ and $\mathscr{G} \in \mathbb{D}(V)$, the a natural morphism*

$$v_!(v^* \mathscr{F} \otimes \mathscr{G}) \to \mathscr{F} \otimes v_! \mathscr{G}$$

defined in Section 2.1.4(4) *is an isomorphism.*

Proof We prove the first point (the second is proved dually). Let $c\colon P \setminus U \hookrightarrow P$ denote the cosieve complementary to u. Then $c^*(\mathscr{F} \otimes u_* \mathscr{G}) \simeq c^* \mathscr{F} \otimes c^* u_* \mathscr{G} = 0$ since $c^* u_* = 0$ by Lemma 2.9. Using that same lemma and also Lemma 2.7, we see that each step in the definition of the morphism Section 2.1.4(3) is an isomorphism. □

2.3 Localization triangles

Let P be a finite poset. Let $u\colon U \hookrightarrow P$ be a sieve and $v\colon V \hookrightarrow P$ denote the complementary cosieve.

Lemma 2.12 *For $\mathscr{F} \in \mathbb{D}(P)$ there is a unique distinguished triangle in $\mathbb{D}(P)$,*

$$v_! v^* \mathscr{F} \to \mathscr{F} \to u_* u^* \mathscr{F} \xrightarrow{+1}, \tag{5}$$

such that the first two maps are the counit and unit respectively. It is functorial in $\mathscr{F}$ and we call it a ***localization triangle****.*

Proof Let C denote a cone of the counit morphism $v_! v^* \mathscr{F} \to \mathscr{F}$, so that we have a distinguished triangle in $\mathbb{D}(P)$,

$$v_! v^* \mathscr{F} \to \mathscr{F} \to C \xrightarrow{+1}. \tag{6}$$

By applying the triangulated functor v^* to (6) and using Lemma 2.7 we get a distinguished triangle in $\mathbb{D}(V)$,

$$v^*\mathscr{F} \xrightarrow{\mathrm{id}} v^*\mathscr{F} \to v^*C \xrightarrow{+1}.$$

We thus have $v^*C = 0$ and Lemma 2.9 implies that we have an isomorphism $C \simeq u_*u^*C$. By applying the triangulated functor u^* to (6) and using $u^*v_! = 0$, which follows from Lemma 2.9, we get a distinguished triangle in $\mathbb{D}(U)$,

$$0 \to u^*\mathscr{F} \to u^*C \xrightarrow{+1},$$

and deduce that we have an isomorphism $C \simeq u_*u^*\mathscr{F}$. This implies the existence of a distinguished triangle whose first two edges are the counit $v_!v^*\mathscr{F} \to \mathscr{F}$ and the unit $\mathscr{F} \to u_*u^*\mathscr{F}$. By adjunction and $v^*u_* = 0$, which follows from Lemma 2.9, we have $\mathrm{Hom}_{\mathbb{D}(P)}(v_!v^*\mathscr{F}, u_*u^*\mathscr{F}[-1]) = 0$, and [Beĭlinson et al. 1982, corollaire 1.1.10] implies that the remaining edge of the triangle is unique. This implies that the triangle is functorial in $\mathscr{F}$. □

Remark 2.13 The output of the above lemma, as well as the results of the rest of this section, is a diagram in the triangulated category $\mathbb{D}(P)$, and is thus a partially incoherent diagram from the point of view of derivators (see Remark 2.2). It is of course possible to lift it to a coherent diagram living in $\mathbb{D}(P \times [3])$, where $[n]$ denotes the poset $(\{0, 1, \dots, n\}, \leqslant)$ with n consecutive arrows. We choose not to phrase our results (and in particular Proposition 2.20 below) in this totally coherent way but rather in a way that is more appealing to readers familiar with the setting of triangulated categories.

However, let us sketch a way to do so in the particular example of the above lemma. The first step is to lift the counit morphism $v_!v^*\mathscr{F} \to \mathscr{F}$ to an object of $\mathbb{D}(P \times [1])$. For this we can consider the cosieve $v'\colon V' \hookrightarrow P \times [1]$ where V' consists of those elements (x, i) such that $x \in V$ if $i = 0$. If $f\colon P \times [1] \to P$ denotes the natural projection, then we can consider the object

$$(v')_!(v')^* f^*\mathscr{F} \in \mathbb{D}(P \times [1])$$

and check that its underlying morphism in $\mathbb{D}(P)$ is indeed the counit morphism $v_!v^*\mathscr{F} \to \mathscr{F}$. One can then proceed as in [Groth 2013, Section 4.2] (see also [Ayoub 2007a, remarque 2.1.38]) to produce a coherent lift of the triangle (6), and the same arguments as in the proof above identify it to a coherent lift of the triangle (5).

The next lemma explains the compatibility between the localization triangles and pullback.

Lemma 2.14 *Let $f\colon Q \to P$ be a morphism of finite posets and introduce a sieve $u'\colon f^{-1}(U) \hookrightarrow Q$ and a cosieve $v'\colon f^{-1}(V) \hookrightarrow Q$. For $\mathscr{F} \in \mathbb{D}(P)$ we have the following isomorphism of distinguished triangles, where the first triangle is obtained by applying f^* to* (5) *and the second triangle is the localization triangle* (5) *of $f^*\mathscr{F}$ with respect to u' and v':*

$$\begin{tikzcd}
f^*v_!v^*\mathscr{F} \arrow[r] & f^*\mathscr{F} \arrow[r] \arrow[d, equal] & f^*u_*u^*\mathscr{F} \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\
(v')_!(v')^*f^*\mathscr{F} \arrow[r] \arrow[u, "\simeq"] & f^*\mathscr{F} \arrow[r] & (u')_*(u')^*f^*\mathscr{F} \arrow[r, "+1"] & {}
\end{tikzcd}$$

Proof It is obtained from the diagram

$$
\begin{tikzcd}
f^*v_!v^*\mathscr{F} \arrow[r] & f^*\mathscr{F} \arrow[r] \arrow[dd, equal] & f^*u_*u^*\mathscr{F} \arrow[r, "+1"] \arrow[d, "\simeq"] & {} \\
(v')_!h^*v^*\mathscr{F} \arrow[u, "\simeq"'] \arrow[d, leftrightarrow, "\simeq"'] & & (u')_*g^*u^*\mathscr{F} \arrow[d, leftrightarrow, "\simeq"] & \\
(v')_!(v')^*f^*\mathscr{F} \arrow[r] & f^*\mathscr{F} \arrow[r] & (u')_*(u')^*f^*\mathscr{F} \arrow[r, "+1"] & {}
\end{tikzcd}
$$

where the notation is borrowed from Lemma 2.10. The isomorphisms between the first and second rows follow from Lemma 2.10. The two visible squares of the diagram commute, and the remaining square commutes by the uniqueness statement in Lemma 2.12. □

Lemma 2.15 *For $\mathscr{F},\mathscr{F}'\in\mathbb{D}(P)$ we have the following isomorphism of distinguished triangles, where the rows are (induced by) localization triangles:*

$$
\begin{tikzcd}
v_!v^*(\mathscr{F}\otimes\mathscr{F}') \arrow[r] \arrow[d, "\simeq"'] & \mathscr{F}\otimes\mathscr{F}' \arrow[r] \arrow[d, equal] & u_*u^*(\mathscr{F}\otimes\mathscr{F}') \arrow[r, "+1"] & {} \\
\mathscr{F}\otimes v_!v^*\mathscr{F}' \arrow[r] & \mathscr{F}\otimes\mathscr{F}' \arrow[r] & \mathscr{F}\otimes u_*u^*\mathscr{F}' \arrow[r, "+1"] \arrow[u, "\simeq"'] & {}
\end{tikzcd}
$$

Proof It is obtained from the diagram

$$
\begin{tikzcd}
v_!v^*(\mathscr{F}\otimes\mathscr{F}') \arrow[r] \arrow[d, leftrightarrow, "\simeq"'] & \mathscr{F}\otimes\mathscr{F}' \arrow[r] \arrow[dd, equal] & u_*u^*(\mathscr{F}\otimes\mathscr{F}') \arrow[r, "+1"] \arrow[d, leftrightarrow, "\simeq"] & {} \\
v_!(v^*\mathscr{F}\otimes v^*\mathscr{F}') \arrow[d, "\simeq"'] & & u_*(u^*\mathscr{F}\otimes u^*\mathscr{F}') & \\
\mathscr{F}\otimes v_!v^*\mathscr{F}' \arrow[r] & \mathscr{F}\otimes\mathscr{F}' \arrow[r] & \mathscr{F}\otimes u_*u^*\mathscr{F}' \arrow[r, "+1"] \arrow[u, "\simeq"'] & {}
\end{tikzcd}
$$

where the isomorphisms between the second and third rows follow from Lemma 2.11, the two visible squares of the diagram commute, and the remaining square commutes by the uniqueness statement in Lemma 2.12. □

For $x<y$ in P and $\mathscr{F}\in\mathbb{D}(P)$ let us denote by $(i_{x<y})^*\mathscr{F}\colon(i_x)^*\mathscr{F}\to(i_y)^*\mathscr{F}$ the corresponding morphism in $\mathbb{D}(e)$ in the underlying diagram (see Remark 2.2). Recall from Section 1.3 the morphism $a_x^y\colon\mathbb{K}_x\to\mathbb{K}_y[1]$ in $\mathbb{D}_{\mathbb{K}\text{–Mod}}(P)$.

Lemma 2.16 *Assume that U and V are discrete posets. Then the connecting morphism in the localization triangle* (5) *reads*

$$u_*u^*\mathscr{F}\simeq\bigoplus_{x\in U}p^*(i_x)^*\mathscr{F}\otimes\mathbb{K}_x\to\bigoplus_{y\in V}p^*(i_y)^*\mathscr{F}\otimes\mathbb{K}_y[1]\simeq v_!v^*\mathscr{F}[1],$$

where the component indexed by $x\in U$ and $y\in V$ is $p^*(i_{x<y})^*\mathscr{F}\otimes a_x^y$ if $x<y$ and zero otherwise.

Note that the object $p^*(i_x)^*\mathscr{F}\otimes\mathbb{K}_x\in\mathbb{D}(P)$ has value $(i_x)^*\mathscr{F}$ at x and zero at every other point.

Proof We proceed in two steps.

(1) Assume that we work in the derivator $\mathbb{D}_{\mathbb{K}\text{–Mod}}$ and that $\mathcal{F} = p^*\mathbb{K} \in \mathbb{D}(P)$ is the constant object with values $\mathbb{K}$. Since U and V are discrete posets we have, by Section 2.1.3(2), isomorphisms

$$u_*u^*p^*\mathbb{K} \simeq \bigoplus_{x\in U} \mathbb{K}_x \quad \text{and} \quad v_!v^*p^*\mathbb{K} \simeq \bigoplus_{y\in V} \mathbb{K}_y.$$

For $x \in U$ and $y \in V$, we can apply Lemma 2.14 to $Z = \{x, y\}$, to reduce the computation of the connecting morphism to the case where $P = Z$ has two elements. If $x < y$ then the connecting morphism is a_x^y by definition. Otherwise P is itself discrete and Section 2.1.3(2) implies that we have $\mathcal{F} \simeq u_*u^*p^*\mathbb{K} \oplus v_!v^*p^*\mathbb{K}$, and the connecting morphism is zero.

(2) We now work in the general case of the lemma. We write $\mathcal{F} = \mathcal{F} \otimes p^*\mathbb{K}$. By applying Lemma 2.15 for $\mathcal{F}' = p^*\mathbb{K}$ and using the first step of the proof, we get a commutative diagram

$$\begin{tikzcd}
u_*u^*\mathcal{F} \arrow[r] & v_!v^*\mathcal{F}[1] \arrow[d, "\simeq"] \\
\mathcal{F} \otimes u_*u^*p^*\mathbb{K} \arrow[r] \arrow[u, "\simeq"] & \mathcal{F} \otimes v_!v^*p^*\mathbb{K}[1] \\
\bigoplus_{x\in U} \mathcal{F} \otimes \mathbb{K}_x \arrow[r] \arrow[u, leftrightarrow, "\simeq"] & \bigoplus_{y\in V} \mathcal{F} \otimes \mathbb{K}_y[1] \arrow[u, leftrightarrow, "\simeq"]
\end{tikzcd}$$

where the component of the bottom morphism indexed by $x \in U$ and $y \in V$ is $\mathrm{id}_{\mathcal{F}} \otimes a_x^y$ if $x < y$ and zero otherwise. Let us now fix $x \in U$ and $y \in V$ with $x < y$. By 2–functoriality we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, equal] & \mathcal{F} \arrow[d] \\
(i_x)_!(i_x)^*\mathcal{F} \arrow[u] \arrow[d, equal] & (i_y)_*(i_y)^*\mathcal{F} \arrow[d, equal] \\
(i_x)_!(i_x)^*p^*(i_x)^*\mathcal{F} \arrow[d] & (i_y)_*(i_y)^*p^*(i_y)^*\mathcal{F} \\
p^*(i_x)^*\mathcal{F} \arrow[r, "p^*(i_{x<y})^*\mathcal{F}"'] & p^*(i_y)^*\mathcal{F} \arrow[u]
\end{tikzcd}$$

where the values at x of the vertical arrows on the left are isomorphisms and the values at y of the vertical arrows on the right are isomorphisms. We then conclude that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F} \otimes \mathbb{K}_x \arrow[r, "\mathrm{id}\otimes a_x^y"] & \mathcal{F} \otimes \mathbb{K}_y[1] \\
p^*(i_x)^*\mathcal{F} \otimes \mathbb{K}_x \arrow[r, "p^*(i_{x<y})^*\mathcal{F}\otimes a_x^y"'] \arrow[u, leftrightarrow, "\simeq"] & p^*(i_y)^*\mathcal{F} \otimes \mathbb{K}_y[1] \arrow[u, leftrightarrow, "\simeq"']
\end{tikzcd}$$

and the claim follows. □

2.4 Postnikov systems from derivators

Let P be a finite poset and let $\sigma\colon P \to \mathbb{Z}_{\geqslant 1}$ be a strictly increasing map. This defines a finite decreasing filtration of P by cosieves $V^k = \{x \in P \mid \sigma(x) \geqslant k\}$ such that each complement $V^k \setminus V^{k+1}$ is a discrete poset (an antichain in P). We let $v^k\colon V^k \hookrightarrow P$.

Lemma 2.17 *Let $\mathscr{F} \in \mathbb{D}(P)$.*

(1) *We set $F^k\mathscr{F} = (v^k)_!(v^k)^*\mathscr{F}$. We have a Postnikov system in $\mathbb{D}(P)$,*

$$\begin{tikzcd}
\cdots \arrow[rr] & & F^3\mathscr{F} \arrow[rr] \arrow[dl] & & F^2\mathscr{F} \arrow[rr] \arrow[dl] & & F^1\mathscr{F} = \mathscr{F} \arrow[dl] \\
& G^3\mathscr{F} \arrow[ul, "+1"] & & G^2\mathscr{F} \arrow[ul, "+1"] & & G^1\mathscr{F} \arrow[ul, "+1"] &
\end{tikzcd}$$

where the graded objects are given by

$$G^k\mathscr{F} \simeq \bigoplus_{\sigma(x)=k} p^*(i_x)^*\mathscr{F} \otimes \mathbb{K}_x.$$

(2) *For every integer k, the connecting morphism $G^k\mathscr{F} \to G^{k+1}\mathscr{F}[1]$ has its component indexed by x and y with $\sigma(x) = k$ and $\sigma(y) = k+1$, given by*

$$p^*(i_x)^*\mathscr{F} \otimes \mathbb{K}_x \xrightarrow{p^*(i_{x<y})^*\mathscr{F} \otimes a_x^y} p^*(i_y)^*\mathscr{F} \otimes \mathbb{K}_y[1]$$

if $x < y$, and zero otherwise.

(3) *The above Postnikov system is functorial in $\mathscr{F}$.*

Proof (1) The morphism $F^{k+1}\mathscr{F} \to F^k\mathscr{F}$ is defined as the composite

$$(v^{k+1})_!(v^{k+1})^*\mathscr{F} \simeq (v^k)_!v_!v^*(v^k)^*\mathscr{F} \to (v^k)_!(v^k)^*\mathscr{F}$$

where $v\colon V^{k+1} \hookrightarrow V^k$ is a cosieve with complementary sieve $u\colon V^k \setminus V^{k+1} \hookrightarrow V^k$. According to Lemma 2.12 this morphism fits into a distinguished triangle

$$F^{k+1}\mathscr{F} \to F^k\mathscr{F} \to G^k\mathscr{F} \xrightarrow{+1}$$

with $G^k\mathscr{F} = (v^k)_!u_*u^*(v^k)^*\mathscr{F}$. Since $V^k \setminus V^{k+1}$ is a discrete poset we have, as in the proof of Lemma 2.16, an isomorphism

$$G^k\mathscr{F} \simeq \bigoplus_{\sigma(x)=k} \mathscr{F} \otimes \mathbb{K}_x \simeq \bigoplus_{\sigma(x)=k} p^*(i_x)^*\mathscr{F} \otimes \mathbb{K}_x.$$

(2) Applying Lemma 2.14 to $Z = \{x \in P \mid \sigma(x) \in \{k, k+1\}\}$ we are reduced to the two-step case where $\sigma(P) \subset \{1, 2\}$. In this case the claim is Lemma 2.16 and we are done.

(3) The functoriality statement follows from the functoriality of localization triangles (Lemma 2.12). $\square$

Remark 2.18 In the spirit of Remark 2.13 let us sketch a way to lift the partially incoherent Postnikov system of the above lemma to a totally coherent diagram.[1] The first step is to lift the horizontal morphisms to an object of $\mathbb{D}(P \times [n])$ where n is an integer such that $\sigma(P) \subset \{1, \dots, n\}$. For this we consider the cosieve $v' \colon V' \hookrightarrow P \times [n]$ consisting of elements (x, i) such that $x \in V^{i+1}$. If $f : P \times [n] \to P$ denotes the natural projection then the object $(v')_!(v')^* f^* \mathscr{F} \in \mathbb{D}(P \times [n])$ is a coherent lift of the composable morphisms $F^{k+1}\mathscr{F} \to F^k \mathscr{F}$ in $\mathbb{D}(P)$. One can then produce the remainder of the Postnikov system in a coherent way as in Remark 2.13.

In the next section we will apply the functor p_* to a Postnikov system as in Lemma 2.17. For this reason we now recast poset cohomology in the context of a general monoidal triangulated derivator.

Lemma 2.19 *Let P be a finite poset and let $x \in P$. For $M \in \mathbb{D}(e)$ we have a functorial isomorphism*

$$p_*(p^* M \otimes \mathbb{K}_x) \simeq M \otimes C^{\bullet+1}(x).$$

Proof Call $\mathscr{F} \in \mathbb{D}(P)$ *admissible* if for any $M \in \mathbb{D}(e)$, the natural morphism

$$M \otimes p_* \mathscr{F} \to p_*(p^* M \otimes \mathscr{F}) \tag{7}$$

defined in Section 2.1.4(3) is an isomorphism. Admissible objects satisfy the following properties:

(a) If P has a greatest element then for every $N \in \mathbb{D}(e)$, $p^* N$ is admissible. Indeed by Lemma 2.6 we have $p_* p^* \simeq \mathrm{id}_{\mathbb{D}(e)}$ and (7) is isomorphic to the identity of $M \otimes N$.

(b) If $u : U \hookrightarrow P$ is a sieve and $\mathscr{G} \in \mathbb{D}(U)$ is admissible, then $u_* \mathscr{G}$ is admissible. Indeed, let $v : P \setminus U \hookrightarrow P$ denote the cosieve complementary to U. Then $v^*(p^* M \otimes u_* \mathscr{G}) \simeq v^* p^* M \otimes v^* u_* \mathscr{G} = 0$ since $v^* u_* = 0$. By Lemma 2.9 we thus have an isomorphism

$$p^* M \otimes u_* \mathscr{G} \simeq u_* u^*(p^* M \otimes u_* \mathscr{G}) \simeq u_*((p \circ u)^* M \otimes \mathscr{G}),$$

and (7) is isomorphic to the natural morphism

$$M \otimes (p \circ u)_* \mathscr{G} \to (p \circ u)_*((p \circ u)^* M \otimes \mathscr{G}),$$

which is an isomorphism because $\mathscr{G}$ is admissible by assumption.

(c) By the naturality of (7), an extension of admissible objects (and in particular a finite direct sum of admissible objects) is admissible. A shift of an admissible object is admissible.

We now note that we have, as in the proof of Proposition 1.7, a resolution $\mathbb{K}_x \xrightarrow{\sim} R_x^\bullet$ with

$$R_x^n = \bigoplus_{[x_1 < \cdots < x_n < x_{n+1} = x]} \mathbb{K}_{\leqslant x_1}.$$

For every $y \leqslant x$ we have $\mathbb{K}_{\leqslant y} \simeq (u_{\leqslant y})_*(p_{\leqslant y})^* \mathbb{K}$, where $u_{\leqslant y} \colon P_{\leqslant y} \hookrightarrow P$ and $p_{\leqslant y} \colon P_{\leqslant y} \to e$ are the inclusion and projection maps of the subposet $P_{\leqslant y} = \{a \in P \mid a \leqslant y\}$. Since y is the greatest element

[1] This was suggested to us by Martin Gallauer.

of $P_{\leqslant y}$, we get by (a) above that $(p_{\leqslant y})^*\mathbb{K}$ is admissible. Since $P_{\leqslant y}$ is a sieve in P, we get by (b) above that $\mathbb{K}_{\leqslant y}$ is admissible. By (c) above we thus get that every R^n_x is admissible and then that $\mathbb{K}_x$ is admissible. The claim then follows from Proposition 1.7 since p_* is the homotopy limit functor. $\square$

The next proposition will be our main tool in the next section. It computes a homotopy limit in the shape of a Postnikov system.

Proposition 2.20 *Let $\mathscr{F} \in \mathbb{D}(P)$.*

(1) *We set $F^k p_*\mathscr{F} = p_*(v^k)_!(v^k)^*\mathscr{F}$. We have a functorial Postnikov system in $\mathbb{D}(e)$,*

$$\begin{tikzcd}[column sep=small] \cdots \arrow[rr] & & F^2 p_*\mathscr{F} \arrow[rr] \arrow[dl] & & F^1 p_*\mathscr{F} = p_*\mathscr{F} \arrow[dl] \\ & G^2 p_*\mathscr{F} \arrow[ul, "+1"] & & G^1 p_*\mathscr{F} \arrow[ul, "+1"] & \end{tikzcd}$$

where the graded objects are given by

$$G^k p_*\mathscr{F} \simeq \bigoplus_{\sigma(x)=k} (i_x)^*\mathscr{F} \otimes C^{\bullet+1}(x).$$

(2) *For every integer k, the connecting morphism $G^k p_*\mathscr{F} \to G^{k+1} p_*\mathscr{F}[1]$ has its component indexed by x and y with $\sigma(x) = k$ and $\sigma(y) = k+1$, given by*

$$(i_x)^*\mathscr{F} \otimes C^{\bullet+1}(x) \xrightarrow{(i_{x<y})^*\mathscr{F}\otimes b^y_x[1]} (i_y)^*\mathscr{F} \otimes C^{\bullet+2}(y)$$

if $x < y$, and zero otherwise.

(3) *The above Postnikov system is functorial in $\mathscr{F}$.*

Proof This follows from applying the triangulated functor p_* to the Postnikov system of Lemma 2.17 and setting $F^k p_*\mathscr{F} := p_* F^k\mathscr{F}$ and $G^k p_*\mathscr{F} := p_* G^k\mathscr{F}$. The description of the graded objects follows from Lemma 2.19. The description of the connecting morphisms follows from Proposition 1.10. $\square$

Remark 2.21 The Postnikov system of Proposition 2.20 is functorial with respect to isomorphisms of posets in the following sense. Let $\alpha\colon P \to P'$ be an isomorphism of posets; we set $\sigma' = \sigma \circ \alpha^{-1}$. For $\mathscr{F}' \in \mathbb{D}(P')$ there is a natural isomorphism $(p')_*\mathscr{F}' \xrightarrow{\sim} p_*\alpha^*\mathscr{F}'$ and a natural isomorphism between the Postnikov system corresponding to $\mathscr{F}' \in \mathbb{D}(P')$ and the one corresponding to $\alpha^*\mathscr{F}' \in \mathbb{D}(P)$. The corresponding isomorphism at the level of graded objects has component indexed by $x' \in P'$ and $x \in P$ given by

$$(i_{x'})^*\mathscr{F}' \otimes C^{\bullet+1}_{P'}(x') \xrightarrow[\sim]{\mathrm{id}\otimes C^{\bullet+1}(\alpha)} (i_x)^*\alpha^*\mathscr{F}' \otimes C^{\bullet+1}_P(x)$$

if $\alpha(x) = x'$ and zero otherwise, where $C^{\bullet+1}(\alpha)$ was defined in Remark 1.2. This follows easily from Remark 1.11.

3 The main theorem

3.1 Categories of motives

3.1.1 Conventions on schemes In what follows we fix a noetherian base scheme B and write "scheme" for "separated scheme over B".

3.1.2 Motives over a scheme For every scheme X we have, following Morel and Voevodsky [1999] and Ayoub [2007a; 2007b], a unital symmetric monoidal triangulated derivator $\mathbb{DA}_X$ of étale motives over X with coefficients in $\mathbb{K}$. It is a particular case of a stable homotopical functor $\mathbb{SH}^T_{\mathfrak{M}}$ constructed in [Ayoub 2007b, définition 4.5.21], taking for the model category $\mathfrak{M}$ (the category of "coefficients") the category of complexes of $\mathbb{K}$–modules, for T the Tate motive (the stabilization consists in formally inverting the functor $T \otimes -$), and considering the étale topology; the axioms of a unital symmetric monoidal triangulated derivator are proved to hold in [Ayoub 2007b, section 4.5]. Other constructions lead to equivalent (under certain assumptions) categories of motives, such as Beilinson motives, étale motives with transfers, and h–motives; see [Ayoub 2014b, théorème B.1; Cisinski and Déglise 2016, Corollary 5.5.5; 2019, Section 16.2].

Remark 3.1 By making other choices of $\mathfrak{M}$ and T one is led to other categories such as the Morel–Voevodsky stable $\mathbb{A}^1$–homotopy categories of schemes $\mathbb{SH}$, where our results below still hold.

There is a natural morphism of unital symmetric monoidal triangulated derivators $\mathbb{D}_{\mathbb{K}\text{–Mod}} \to \mathbb{DA}_X$, so that the derivator $\mathbb{D} = \mathbb{DA}_X$ satisfies the assumptions of Section 2.1.5. In what follows we will make an abuse of notation and simply write $\mathbb{DA}_X$ for the ground category $\mathbb{DA}_X(e)$.

Let us note that $X \mapsto \mathbb{DA}_X$ satisfies the "six functor formalism", for which we will not give a definition here but rather refer to Ayoub. This means that it has the same formal functoriality properties as derived categories of sheaves in familiar contexts. In particular, it underlies a cross functor [Ayoub 2007a, définition 1.2.12, scholie 1.4.2]. This notion (defined in [loc. cit., section 1.2]) abstracts the properties of the exchange morphisms between ! and $*$ pullbacks and/or pushforwards (such as the morphism appearing in the proper base change theorem).

Another important feature that we will use is the existence of functorial localization triangles [Ayoub 2007a, section 1.4.4] for $\mathscr{F} \in \mathbb{DA}_X$, where $i : Z \hookrightarrow X$ denotes a closed immersion and $j : X \setminus Z \hookrightarrow X$ denotes the complementary open immersion,

$$j_! j^! \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \xrightarrow{+1} . \tag{8}$$

3.1.3 Motives over a diagram of schemes In the proof of the main theorem below we will make use of categories of motives over diagrams of schemes, introduced by Ayoub. A *diagram of schemes* $(P, \mathscr{X})$ is the datum of a finite poset P along with a functor $\mathscr{X}\colon P^{\mathrm{op}} \to \mathrm{Sch}$. (Our convention is actually opposed to Ayoub's, see Remark 3.2 below.) For X a scheme we have the constant diagram of schemes (P, X)

where all the transition maps are the identity of X. We view a scheme as the constant diagram of schemes on the poset with one element: $X = (e, X)$. Diagrams of schemes form a 2–category [Ayoub 2007a, définition 2.4.4] in which a morphism $\alpha\colon (P,\mathscr{X}) \to (Q,\mathscr{Y})$ consists of a morphism of posets $\alpha\colon P \to Q$ along with a natural transformation $\mathscr{X} \Rightarrow \mathscr{Y} \circ \alpha$.

Ayoub defines a (1–contravariant, 2–covariant) 2–functor

$$(P,\mathscr{X}) \mapsto \mathbb{DA}(P,\mathscr{X})$$

from the 2–category of diagrams of schemes to the 2–category of triangulated categories which extends the derivator $P \mapsto \mathbb{DA}(P, X) = \mathbb{DA}_X(P)$ for every scheme X. This functor satisfies the axioms of an *algebraic derivator* [Ayoub 2007a, 2.4.2] that we will not discuss here. We simply note that for $\alpha\colon (P,\mathscr{X}) \to (Q,\mathscr{Y})$ a morphism of diagrams of schemes, the natural morphism $\alpha^*\colon \mathbb{DA}(Q,\mathscr{Y}) \to \mathbb{DA}(P,\mathscr{X})$ admits a right adjoint $\alpha_*\colon \mathbb{DA}(P,\mathscr{X}) \to \mathbb{DA}(Q,\mathscr{Y})$. The existence of left adjoints is more constrained.

Remark 3.2 Our convention for diagrams of schemes and for the variance of $\mathbb{DA}$ is opposed to Ayoub's but is consistent with our variance convention for derivators (see Remark 2.3) and with the convention for posets of strata introduced in the next subsection.

3.2 The main theorem

Let X be a scheme and let X_0 be a dense open subscheme of X with complement Z. We denote by $j\colon X_0 \hookrightarrow X$ and $i\colon Z \hookrightarrow X$ the corresponding open and closed immersions. Let us be given a (finite) *stratification* of Z, ie a finite partition of Z by locally closed subschemes called *strata* such that the Zariski closure of each stratum is a union of strata. The set P of strata is naturally endowed with the structure of a poset where for strata $S, T \in P$,

$$S \leqslant T \iff \overline{S} \supset T.$$

We thus get a stratification of X indexed by the extended poset $\widehat{P} = \{X_0\} \cup P$ with $X_0 < S$ for all $S \in P$.

For $S \in P$ we have defined (see Section 1.1) a complex of $\mathbb{K}$–modules $C^\bullet(S)$ which computes the reduced cohomology groups of the poset $P_{<S}$. For strata $S, T \in P$ with $S \lessdot T$ we have defined (see Section 1.2) a morphism of complexes $b_S^T\colon C^\bullet(S) \to C^\bullet(T)[1]$. We also define $C^\bullet(X_0)$ to be the complex $\mathbb{K}$ concentrated in degree zero. For a minimal stratum $S \in P$, ie such that $X_0 \lessdot S$ in $\widehat{P}$, we have a natural (iso)morphism of complexes $b_{X_0}^S\colon C^\bullet(X_0) \to C^\bullet(S)[1]$.

We fix a strictly increasing map $\sigma\colon \widehat{P} \to \mathbb{Z}$, and we assume that $\sigma(X_0) = 0$. Such a map always exists. If P is graded then we may take $\sigma = \mathrm{rk}$, the rank function.

In the statement of the next theorem, we will use the following "restriction" morphisms of functors (for strata $S \leqslant T$):

$$\rho_S^T\colon (i_{\overline{S}}^X)_*(i_{\overline{S}}^X)^* \to (i_{\overline{S}}^X)_*(i_{\overline{T}}^{\overline{S}})_*(i_{\overline{T}}^{\overline{S}})^*(i_{\overline{S}}^X)^* \simeq (i_{\overline{T}}^X)_*(i_{\overline{T}}^X)^*. \tag{9}$$

Theorem 3.3 *Let $\mathscr{F} \in \mathbb{DA}_X$ and set $\mathscr{G} = j_! j^! \mathscr{F}$.*

(1) *There is a Postnikov system in $\mathbb{DA}_X$,*

$$\begin{tikzcd}[column sep=small] \cdots \arrow[rr] & & F^2\mathscr{G} \arrow[rr] \arrow[dl] & & F^1\mathscr{G} \arrow[rr] \arrow[dl] & & F^0\mathscr{G} = \mathscr{G} \arrow[dl] \\ & G^2\mathscr{G} \arrow[ul, "+1"] & & G^1\mathscr{G} \arrow[ul, "+1"] & & G^0\mathscr{G} \arrow[ul, "+1"] & \end{tikzcd}$$

where the graded objects are given by

$$G^k\mathscr{G} = \bigoplus_{\sigma(S)=k} (i_{\overline{S}}^X)_*(i_{\overline{S}}^X)^*\mathscr{F} \otimes C^\bullet(S).$$

(2) *For every integer k, the connecting morphism $G^k\mathscr{G} \to G^{k+1}\mathscr{G}[1]$ has its component indexed by S and T with $\sigma(S) = k$ and $\sigma(T) = k+1$, given by*

$$(i_{\overline{S}}^X)_*(i_{\overline{S}}^X)^*\mathscr{F} \otimes C^\bullet(S) \xrightarrow{\rho_S^T\mathscr{F} \otimes b_S^T} (i_{\overline{T}}^X)_*(i_{\overline{T}}^X)^*\mathscr{F} \otimes C^\bullet(T)[1]$$

if $S < T$ and zero otherwise.

(3) *The above Postnikov system is functorial in $\mathscr{F}$.*

Proof We proceed in three steps.

(a) We construct the first triangle. The (rotated) localization triangle (8) reads

$$i_*i^*\mathscr{F}[-1] \to j_!j^!\mathscr{F} \to \mathscr{F} \xrightarrow{+1}$$

and provides the first triangle of the Postnikov system, with $F^1\mathscr{G} = i_*i^*\mathscr{F}[-1]$ and $G^0\mathscr{G} = \mathscr{F}$. It is functorial in $\mathscr{F}$.

(b) We work with motives over diagrams of schemes. We consider the diagram of schemes $(P, \mathscr{Z})$ where $\mathscr{Z}\colon P^{\mathrm{op}} \to \mathrm{Sch}$ is defined by $S \mapsto \overline{S}$ and where the transition morphisms are the natural closed immersions. We have a natural morphism of diagram of schemes $s\colon (P, \mathscr{Z}) \to Z$ induced by the closed immersions $\overline{S} \hookrightarrow Z$. This was previously considered by Ayoub and Zucker [2012, Lemma 1.18] who proved that the natural counit $\mathrm{id}_{\mathbb{DA}_Z} \to s_*s^*$ is an isomorphism. We thus have an isomorphism in $\mathbb{DA}_Z$,

$$i_*i^*\mathscr{F} \simeq i_*s_*s^*i^*\mathscr{F}.$$

Let us recall that (P, X) denotes a constant diagram of schemes. We have a natural morphism of diagrams of schemes $r\colon (P, \mathscr{Z}) \to (P, X)$ induced by the closed immersions $\overline{S} \hookrightarrow X$. If we also denote by $p\colon (P, X) \to (e, X) = X$ the projection to a point, we have the following commutative diagram:

$$\begin{tikzcd} (P, \mathscr{Z}) \arrow[r, "r"] \arrow[d, "s"'] & (P, X) \arrow[d, "p"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

We thus have an isomorphism

$$F^1\mathscr{G} \simeq p_*\mathscr{H}[-1]$$

where we set $\mathscr{H} = r_*r^*p^*\mathscr{F} \in \mathbb{DA}(P,X) = \mathbb{DA}_X(P)$. It is easy to see, using the axiom DerAlg 3d in [Ayoub 2007a, définition 2.4.12], that the value of $\mathscr{H}$ at a stratum S is $(i^X_{\overline{S}})_*(i^X_{\overline{S}})^*\mathscr{F}$. Moreover, for strata $S \leqslant T$ the transition map from the value at S to the value at T is the restriction morphism $\rho^T_S\mathscr{F}$ defined in (9).

(c) We construct the Postnikov system. By applying Proposition 2.20(1) to the object $\mathscr{H} \in \mathbb{DA}_X(P)$ we get a Postnikov system in $\mathbb{DA}_X$,

$$\begin{tikzcd}
\cdots \arrow[rr] & & F^2 p_*\mathscr{H} \arrow[dl] \arrow[rr] & & F^1 p_*\mathscr{H} = p_*\mathscr{H} = F^1\mathscr{G}[1] \arrow[dl] \\
& G^2 p_*\mathscr{H} \arrow[ul, "+1"] & & G^1 p_*\mathscr{H} \arrow[ul, "+1"] &
\end{tikzcd}$$

with

$$G^k p_*\mathscr{H} \simeq \bigoplus_{\sigma(S)=k} (i^X_{\overline{S}})_*(i^X_{\overline{S}})^*\mathscr{F} \otimes C^{\bullet+1}(S).$$

This is, up to a shift, the remainder of the Postnikov system promised in the theorem, ie we set, for $k \geqslant 1$,

$$F^k\mathscr{G} = F^k p_*\mathscr{H}[-1] \quad \text{and} \quad G^k\mathscr{G} = G^k p_*\mathscr{H}[-1].$$

The description of the connecting morphisms follows from Proposition 2.20(2). (The connecting morphism $G^0\mathscr{F} \to G^1\mathscr{F}[1]$ needs to be treated separately; it is the composite $\mathscr{F} \to i_*i^*\mathscr{F} \to \bigoplus_{\sigma(S)=1}(i^X_{\overline{S}})_*(i^X_{\overline{S}})^*\mathscr{F}$ which is the sum of the morphisms $\rho^S_{X_0}\mathscr{F}$.) The functoriality statement follows from Proposition 2.20(3). $\square$

For any (B–)scheme X let us denote by $a_X \colon X \to B$ its structural map. The next corollary expresses the "compactly supported cohomology" of a motivic sheaf $\mathscr{F}$ on the open X_0 in terms of "compactly supported cohomology" of $\mathscr{F}$ on all the closures of strata.

Corollary 3.4 *Let $\mathscr{F} \in \mathbb{DA}_X$ and set $M = (a_{X_0})_!j^!\mathscr{F} \in \mathbb{DA}_B$.*

(1) *There is a Postnikov system in $\mathbb{DA}_B$,*

$$\begin{tikzcd}
\cdots \arrow[rr] & & F^2 M \arrow[dl] \arrow[rr] & & F^1 M \arrow[dl] \arrow[rr] & & F^0 M = M \arrow[dl] \\
& G^2 M \arrow[ul, "+1"] & & G^1 M \arrow[ul, "+1"] & & G^0 M \arrow[ul, "+1"] &
\end{tikzcd}$$

where the graded objects are given by

$$G^k M = \bigoplus_{\sigma(S)=k} (a_{\overline{S}})_!(i^X_{\overline{S}})^*\mathscr{F} \otimes C^\bullet(S).$$

(2) *For every integer* k, *the connecting morphism* $G^k M \to G^{k+1} M[1]$ *has its component indexed by* S *and* T *with* $\sigma(S) = k$ *and* $\sigma(T) = k+1$, *given by*

$$(a_{\overline{S}})_!(i^X_{\overline{S}})^*\mathscr{F} \otimes C^\bullet(S) \xrightarrow{\rho^T_S \mathscr{F} \otimes b^T_S} (a_{\overline{T}})_!(i^X_{\overline{T}})^*\mathscr{F} \otimes C^\bullet(T)[1]$$

if $S < T$ *and zero otherwise.*

(3) *The above Postnikov system is functorial in* $\mathscr{F}$.

Proof This follows from applying the functor $(a_X)_!$ to the Postnikov system of Theorem 3.3. By the projection formula we have an isomorphism

$$(a_X)_!\big((i^X_{\overline{S}})_*(i^X_{\overline{S}})^*\mathscr{F} \otimes C^\bullet(S)\big) = (a_X)_!\big((i^X_{\overline{S}})_*(i^X_{\overline{S}})^*\mathscr{F} \otimes (a_X)^* C^\bullet(S)\big) \simeq (a_X)_!(i^X_{\overline{S}})_*(i^X_{\overline{S}})^*\mathscr{F} \otimes C^\bullet(S),$$

and this equals $(a_{\overline{S}})_!(i^X_{\overline{S}})^*\mathscr{F} \otimes C^\bullet(S)$ since $(a_X)_!(i^X_{\overline{S}})_* = (a_X)_!(i^X_{\overline{S}})_! = (a_{\overline{S}})_!$. □

Remark 3.5 One can also apply the functor $(a_X)_*$ to the Postnikov system of Theorem 3.3 and get a Postnikov system expressing the relative motive of the pair (X, Z) with coefficients in a motivic sheaf $\mathscr{F}$. It is a motivic refinement of the classical long exact sequence in relative cohomology.

3.3 Localization spectral sequences

We recover the spectral sequences of [Petersen 2017] by applying realization functors.

3.3.1 Betti realization We now consider a finite type scheme X over $\mathbb{C}$. We have the Betti realization functor [Ayoub 2010]

$$\mathbb{DA}_X \to D(X^{\mathrm{an}}),$$

whose target is the derived category of the category of sheaves of $\mathbb{K}$–modules on the analytification X^{an}. This functor is compatible with the operations f^*, f_*, $f_!$ and $\otimes$, and we thus get from Theorem 3.3 (resp. Corollary 3.4) a Postnikov system in $D(X^{\mathrm{an}})$ (resp. $D(B^{\mathrm{an}})$). We can then derive a spectral sequence by applying a cohomological functor such as the "cohomology sheaves" functor $\mathscr{H}^0 \colon D(B^{\mathrm{an}}) \to \mathrm{Sh}(B^{\mathrm{an}})$.

Remark 3.6 We may also apply other natural cohomological functors when available. For instance, if the Betti realization of $\mathscr{F}$ is a complex of sheaves with constructible cohomology sheaves, almost all of which are zero (eg if $\mathscr{F}$ is a constant sheaf), then one can also apply the perverse cohomology functor ${}^p\mathrm{H}^0$ with target the category of perverse sheaves ${}^p\mathrm{Perv}(B^{\mathrm{an}})$ for any perversity function p [Beĭlinson et al. 1982].

In the case $B = \mathrm{Spec}(\mathbb{C})$, the spectral sequence reads:

$$E_1^{p,q} = \bigoplus_{\sigma(S)=p} H^{p+q}(R\Gamma_c(i^X_{\overline{S}})^*\mathscr{F} \otimes C^\bullet(S)) \Rightarrow H_c^{p+q}(X_0, j^!\mathscr{F}).$$

We can make it more explicit under some extra assumptions as in [Petersen 2017, Section 3], and we get for instance the following corollary [Petersen 2017, Theorem 3.3(ii)]. We recall the notation $h^n(S) = H^n(C^\bullet(S))$ from Section 1.1.

Corollary 3.7 *Assume that $\mathbb{K}$ is a hereditary ring (eg $\mathbb{K}$ is a field or $\mathbb{K} = \mathbb{Z}$) and that for every stratum S and every integer n the cohomology group $h^n(S)$ is a torsion-free $\mathbb{K}$–module. Then we have a spectral sequence of $\mathbb{K}$–modules*

$$E_1^{p,q} = \bigoplus_{\substack{\sigma(S)=p \\ i+j=p+q}} H_c^i(\overline{S}, (i_{\overline{S}}^X)^*\mathscr{F}) \otimes h^j(S) \Rightarrow H_c^{p+q}(X_0, j^!\mathscr{F}).$$

Proof Since $C^\bullet(S)$ is a complex of free $\mathbb{K}$–modules, the tensor product by $C^\bullet(S)$ is also the derived tensor product. Moreover, since $\mathbb{K}$ is hereditary, the complex $C^\bullet(S)$ is quasi-isomorphic to its cohomology. Finally, since that cohomology is assumed to be torsion-free, the Künneth formula applies without the Tor correction term. $\square$

Remark 3.8 In the context of Remark 1.6 we can simplify further since most cohomology groups $h^j(S)$ vanish: we get a spectral sequence

$$E_1^{p,q} = \bigoplus_{\mathrm{rk}(S)=p} H_c^q(\overline{S}, (i_{\overline{S}}^X)^*\mathscr{F}) \otimes h(S)^\vee \Rightarrow H_c^{p+q}(X_0, j^!\mathscr{F}).$$

The differential $d_1^{p,q}$ has component indexed by strata S and T, with $\mathrm{rk}(S) = p$ and $\mathrm{rk}(T) = p+1$, given by

$$H_c^q(\overline{S}, (i_{\overline{S}}^X)^*\mathscr{F}) \otimes h(S)^\vee \xrightarrow{\rho_S^T\mathscr{F}\otimes b_S^T} H_c^q(\overline{T}, (i_{\overline{T}}^X)^*\mathscr{F}) \otimes h(T)^\vee$$

if $S < T$, and zero otherwise.

3.3.2 Hodge realization In the case $\mathbb{K} = \mathbb{Q}$, the Betti realization functor can be enriched into a Hodge realization functor in the constructible case. Following [Ayoub 2014a, Definition 2.11] we define $\mathbb{DA}_X^{\mathrm{ct}}$ to be the smallest triangulated subcategory of $\mathbb{DA}_X$ stable under direct summands and Tate twists and containing the motives $f_*\mathbb{K}_Y$ for $f: Y \to X$ of finite presentation. Objects of $\mathbb{DA}_X^{\mathrm{ct}}$ are called *constructible*.

Thanks to [Ivorra 2016] we have Hodge realization functors

$$\mathbb{DA}_X^{\mathrm{ct}} \to D^b(\mathrm{MHM}(X))$$

which are compatible with the six functor formalism, where $\mathrm{MHM}(X)$ is Saito's category of mixed Hodge modules on X [Saito 1990]. This proves that the spectral sequence of Corollary 3.7 is compatible with mixed Hodge structures if X has finite type over $\mathrm{Spec}(\mathbb{C})$ and $\mathscr{F}$ is constructible, eg $\mathscr{F} = \mathbb{Q}_X$ the constant sheaf. This was already noted by Petersen [2017, Theorem 3.3(ii)].

3.3.3 Étale (and ℓ–adic) realization Let us assume that $B = \mathrm{Spec}(k)$ for some field k. We fix a prime ℓ invertible in k and set $\mathbb{K} = \mathbb{Q}_\ell$. By [Ayoub 2014b, sections 5 and 9; Cisinski and Déglise 2016, Section 7.2], we have an étale (or ℓ–adic) realization functor

$$\mathbb{DA}_X^{\mathrm{ct}} \to D_c^b(X^{\text{ét}})$$

compatible with the six operations, where $D_c^b(X^{\text{ét}})$ is Ekedahl's triangulated category of ℓ–adic systems [Ekedahl 1990].

This implies that we have a spectral sequence in étale cohomology analogous to that of Corollary 3.7 with $\mathbb{Q}_\ell$ coefficients, with values in the category of continuous representations of the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. This was already noted by Petersen [2017, Theorem 3.3(ii)].

3.4 The dual version

We start with the "dual" variant of Theorem 3.3, where we consider the same geometric situation but study the object $j_*j^*\mathscr{F}$ instead of $j_!j^!\mathscr{F}$. We will derive one from the other by using Verdier duality in the motivic setting (see Remark 3.10 below for a discussion of this strategy).

For simplicity we assume that the base scheme B is of finite type over a characteristic zero field. Then we have a Verdier duality functor [Ayoub 2014a, Theorem 3.10]

$$\mathrm{D}_X : (\mathbb{DA}_X^{\mathrm{ct}})^{\mathrm{op}} \to \mathbb{DA}_X^{\mathrm{ct}}$$

which satisfies the usual compatibilities $\mathrm{D}_X \circ \mathrm{D}_X \simeq \mathrm{id}$ and $\mathrm{D}_Y \circ f_* \simeq f_! \circ \mathrm{D}_X$ for $f : X \to Y$ a morphism of schemes.

Recall from Sections 1.1 and 1.2 the homological complexes $C_\bullet(S)$, for $S \in P$, that we now treat with cohomological conventions (ie with negative cohomological degrees) and the connecting morphisms $b_S^T : C_{\bullet+1}(T) \to C_\bullet(T)$ for $S \lessdot T$, which in cohomological conventions read $b_S^T : C_\bullet(T) \to C_\bullet(S)[1]$. As in the previous paragraph we set $C_\bullet(X_0) = \mathbb{K}$ concentrated in degree 0, and for $S \in P$ a minimal element, we consider the natural (iso)morphism $b_{X_0}^S : C_\bullet(S) \to C_\bullet(X_0)[1]$.

In the statement of the next theorem we will use the following "Gysin-type" morphisms of functors, which are dual to restriction morphisms ρ_S^T (for strata $S \leqslant T$):

$$\gamma_S^T : (i_T^X)_!(i_T^X)^! \simeq (i_S^X)_!(i_T^{\overline{S}})_!(i_T^{\overline{S}})^!(i_S^X)^! \to (i_S^X)_!(i_S^X)^!. \tag{10}$$

Theorem 3.9 *Let $\mathscr{F} \in \mathbb{DA}_X^{\mathrm{ct}}$ be a constructible object and let us set $\mathscr{G} = j_*j^*\mathscr{F}$.*

(1) *There is a Postnikov system in $\mathbb{DA}_X$,*

$$\begin{tikzcd}
\mathscr{G} = F_0\mathscr{G} \arrow[rr] & & F_1\mathscr{G} \arrow[rr] \arrow[dl, "+1"] & & F_2\mathscr{G} \arrow[rr] \arrow[dl, "+1"] & & \cdots \arrow[dl, "+1"] \\
& G_0\mathscr{G} \arrow[ul] & & G_1\mathscr{G} \arrow[ul] & & G_2\mathscr{G} \arrow[ul] &
\end{tikzcd}$$

where the graded objects are given by

$$G_k\mathscr{G} = \bigoplus_{\sigma(S)=k} (i_S^X)_!(i_S^X)^!\mathscr{F} \otimes C_\bullet(S).$$

(2) *For every integer k, the connecting morphism $G_{k+1}\mathcal{G} \to G_k\mathcal{G}[1]$ has its component indexed by S and T with $\sigma(S) = k$ and $\sigma(T) = k+1$, given by*

$$(i_{\overline{T}}^X)_!(i_{\overline{T}}^X)^!\mathcal{F} \otimes C_\bullet(T) \xrightarrow{\gamma_S^T\mathcal{F} \otimes b_S^T} (i_{\overline{S}}^X)_!(i_{\overline{S}}^X)^!\mathcal{F} \otimes C_\bullet(S)[1]$$

if $S < T$, and zero otherwise.

(3) *The above Postnikov system is functorial in $\mathcal{F}$.*

Proof We apply Theorem 3.3 to the Verdier dual of $\mathcal{F}$ and dualize the Postnikov system obtained in this way. The only thing that needs to be checked is the description of $G_k\mathcal{G}$ and the connecting morphisms. Let $\omega_X \in \mathbb{DA}_X^{\mathrm{ct}}$ denote the dualizing object. For any object $\mathcal{U} \in \mathbb{DA}_X^{\mathrm{ct}}$,

$$\mathrm{D}_X(\mathcal{U} \otimes C^\bullet(S)) = \underline{\mathrm{Hom}}_{\mathbb{DA}_X^{\mathrm{ct}}}(C^\bullet(S) \otimes \mathcal{U}, \omega_X) \simeq \underline{\mathrm{Hom}}_{\mathbb{DA}_X^{\mathrm{ct}}}(C^\bullet(S), \mathrm{D}_X\,\mathcal{U}) \simeq \mathrm{D}_X\,\mathcal{U} \otimes C_\bullet(S).$$

In the last step we have used the fact that $C_\bullet(S)$ is the strong dual of $C^\bullet(S)$ in the monoidal category $\mathbb{D}_{\mathbb{K}\text{–Mod}}$ because it is a bounded complex of free $\mathbb{K}$–modules of finite rank. By applying this to

$$\mathcal{U} = (i_{\overline{S}}^X)_*(i_{\overline{S}}^X)^*\,\mathrm{D}_X\,\mathcal{F},$$

using the compatibility between Verdier duality and the functors i_* and $i_!$, and the fact that $\mathrm{D}_X \circ \mathrm{D}_X\,\mathcal{F} \simeq \mathcal{F}$, we get an isomorphism

$$\mathrm{D}_X\big((i_{\overline{S}}^X)_*(i_{\overline{S}}^X)^*\,\mathrm{D}_X\,\mathcal{F} \otimes C^\bullet(S)\big) \simeq (i_{\overline{S}}^X)_!(i_{\overline{S}}^X)^!\mathcal{F} \otimes C_\bullet(S).$$

This implies the description of $G_k\mathcal{G}$ as in the statement of the theorem. The fact that the Gysin morphisms γ_S^T defined in (10) and the restriction morphisms ρ_S^T defined in (9) are Verdier dual to each other is clear, and the claim follows. □

Remark 3.10 Theorem 3.9 is most certainly true without the assumption that $\mathcal{F}$ is constructible and without the assumption that B is a finite type scheme over a characteristic zero field. In fact, as noted in the introduction, we can prove it without the functoriality statement using only the language of triangulated categories. However, it seems that the tools that we are using do not allow us to do it functorially. Indeed, we cannot simply repeat the proof of Theorem 3.3 since the existence of a left adjoint to the functor s^* appearing in the proof is not guaranteed in the context of an algebraic derivator.

Remark 3.11 As in Corollary 3.4 and Remark 3.5 one may apply the functors $(a_X)_*$ or $(a_X)_!$ to the Postnikov system of Theorem 3.9 to get localization Postnikov systems in $\mathbb{DA}_B$. In the case of $(a_X)_*$ this computes $(a_{X_0})_*j^*\mathcal{F}$, the cohomology of X_0 with coefficients in the restriction of $\mathcal{F}$; a particularly interesting case is when $\mathcal{F} = \mathbb{K}_X$ is a constant motivic sheaf. There the main difficulty is to be able to compute the graded objects of the Postnikov system, ie the objects $(a_{\overline{S}})_*(i_{\overline{S}}^X)^!\mathbb{K}_X$ for all strata S. Luckily, if $\overline{S}$ is smooth of codimension c in X, then by purity we have an isomorphism

$$(i_{\overline{S}}^X)^!\mathbb{K}_X \simeq \mathbb{K}_{\overline{S}}[-2c](-c),$$

and the localization Postnikov system is expressed in terms of the motives of the closures of strata.

Remark 3.12 By applying realization functors and cohomological functors one gets spectral sequences from Theorem 3.9 as in Section 3.3. We only state one special case that is important for applications. Let $\mathscr{F} = \mathbb{K}_X$, and assume that we are in the context of Corollary 3.7 and Remark 3.8. Further assume that for every stratum S the closure $\overline{S}$ is smooth of codimension c_S in X. Then we get by the previous remark a (second quadrant) spectral sequence in mixed Hodge structures:

$$E_1^{-p,q} = \bigoplus_{\mathrm{rk}(S)=p} H^{q-2c_S}(\overline{S})(-c_S) \otimes h(S) \Rightarrow H^{-p+q}(X_0). \tag{11}$$

A special case of interest is when the stratification is induced by a normal crossing divisor, in which case $c_S = \mathrm{rk}(S)$ and $h(S)$ has rank one for every stratum S; one then recovers Deligne's spectral sequence [1971, 3.2.4.1]. The other classical spectral sequences cited in the introduction [Bibby 2016; Björner and Ekedahl 1997; Cohen and Taylor 1978; Dupont 2015; Getzler 1999; Goresky and MacPherson 1988; Kříž 1994; Looijenga 1993; Totaro 1996] are all special cases of (11).

3.5 Functoriality

We now turn to the functoriality of our main theorem with respect to morphisms of schemes. With a little more work it should be easy to treat more general cases.

3.5.1 A category of stratified schemes For simplicity we restrict to morphisms between stratified schemes whose underlying combinatorial datum is an isomorphism of posets.

Definition 3.13 Let X and X' be two stratified schemes with posets of strata $\widehat{P}$ and $\widehat{P}'$ as in Section 3.2. A *stratified morphism* from X to X' is a pair (α, f) where $\alpha\colon \widehat{P} \to \widehat{P}'$ is an isomorphism of posets and $f: X \to X'$ is a morphism of schemes such that

$$f(\overline{S}) \subset \overline{\alpha(S)} \quad \text{for all } S \in \widehat{P}.$$

Note that for a stratified morphism (α, f), the morphism f does not determine α in general. However, for an isomorphism of schemes $f: X \to X'$ such that the image by f of every stratum of X is a stratum of X', there is a unique $\alpha\colon \widehat{P} \to \widehat{P}'$ such that (α, f) is a stratified isomorphism.

Our notion of stratified morphism is more easily understood in the context of the category of diagrams of schemes. For a stratified scheme X with poset of strata $\widehat{P}$ we have a natural diagram of schemes $(\widehat{P}, \mathscr{X})$ where $\mathscr{X}\colon \widehat{P}^{\mathrm{op}} \to \mathrm{Sch}$ sends S to $\overline{S}$. A stratified morphism (α, f) as above gives rise to a morphism of diagrams of schemes

$$(\alpha, f)\colon (\widehat{P}, \mathscr{X}) \to (\widehat{P}', \mathscr{X}').$$

One can thus view our category of stratified schemes as a subcategory of the category of diagrams of schemes. It is not a full subcategory since we only consider morphisms (α, f) for which α is an isomorphism of posets.

3.5.2 Functoriality of the localization triangle The first step in the construction of the Postnikov system is just the localization triangle (8). So let us consider a morphism of pairs $f: (X, Z) \to (X', Z')$, where Z and Z' are closed subschemes and $f(Z) \subset Z'$. If we denote by X_0 and X'_0 the open complements, then $f^{-1}(X'_0) \subset X_0$. We have the diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "f"'] & X_0 \arrow[l, "j"'] & f^{-1}(X'_0) \arrow[l, "j_0"'] \arrow[d, "f"] \\ Z' \arrow[r, "i'"'] & X' & & X'_0 \arrow[ll, "j'"] \end{tikzcd}$$

where the left square is commutative and the rectangle on the right is cartesian. Given an object $\mathscr{F}' \in \mathbb{DA}_{X'}$, we want to define a morphism between the localization triangle for $\mathscr{F}'$ and f_* of the localization triangle for $f^*\mathscr{F}'$:

$$\begin{tikzcd} (i')_*(i')^*\mathscr{F}'[-1] \arrow[r] \arrow[d] & (j')_!(j')^!\mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F}' \arrow[r, "+1"] \arrow[d] & {} \\ f_*i_*i^*f^*\mathscr{F}'[-1] \arrow[r] & f_*j_!j^!f^*\mathscr{F}' \arrow[r] & f_*f^*\mathscr{F}' \arrow[r, "+1"] & {} \end{tikzcd}$$

Let us now define the three vertical morphisms:

- The right morphism is of course the adjunction unit $\mathscr{F}' \to f_*f^*\mathscr{F}'$.
- The left morphism is given by the composition
$$(i')_*(i')^*\mathscr{F}'[-1] \to (i')_*f_*f^*(i')^*\mathscr{F}'[-1] \xrightarrow{\sim} f_*i_*i^*f^*\mathscr{F}'[-1],$$
where the first arrow is induced by the adjunction unit, and the isomorphism on the right follows from the commutativity of the left square in the diagram above.
- The middle morphism is given by the composition
$$(j')_!(j')^!\mathscr{F}' \to (j')_!f_*f^*(j')^!\mathscr{F}' \to f_*j_!(j_0)_!(j_0)^!j^!f^*\mathscr{F}' \to f_*j_!j^!f^*\mathscr{F}',$$
where the first arrow is induced by the adjunction unit, the second arrow induced by two exchange morphisms (which are part of the cross functor structure; see [Ayoub 2007a, section 1.2]) for the cartesian square on the right of the diagram above, and the third arrow is induced by the adjunction counit.

We leave it to the reader to check that this defines indeed a morphism of triangles. The commutativity of the left square is easy, the commutativity of the right square is a nice exercise on using the axioms of a cross functor, and the commutativity of the third square follows from [Beĭlinson et al. 1982, proposition 1.1.9].

Remark 3.14 Assume that $B = \mathrm{Spec}(\mathbb{C})$ and denote by $a\colon X \to B$ and $a'\colon X' \to B$ the structure morphisms. If f is proper, we have $a'_! f_* = a'_! f_! = a_!$. Consequently, taking $\mathscr{F}' = \mathbb{Q}_{X'}$, applying the functor $a'_!$ and taking the Betti realization, we get the functoriality (for proper morphisms) of the localization long exact sequence of the introduction:

$$\begin{tikzcd} \cdots \arrow[r] & H_c^\bullet(X'_0) \arrow[r] \arrow[d] & H_c^\bullet(X') \arrow[r] \arrow[d] & H_c^\bullet(Z') \arrow[r] \arrow[d] & H_c^{\bullet+1}(X'_0) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_c^\bullet(X_0) \arrow[r] & H_c^\bullet(X) \arrow[r] & H_c^\bullet(Z) \arrow[r] & H_c^{\bullet+1}(X_0) \arrow[r] & \cdots \end{tikzcd}$$

Similarly, using a'_* instead, we get the functoriality of the long exact sequence in relative cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & H^\bullet(X',Z') \arrow[r] \arrow[d] & H^\bullet(X') \arrow[r] \arrow[d] & H^\bullet(Z') \arrow[r] \arrow[d] & H^{\bullet+1}(X',Z') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^\bullet(X,Z) \arrow[r] & H^\bullet(X) \arrow[r] & H^\bullet(Z) \arrow[r] & H^{\bullet+1}(X,Z) \arrow[r] & \cdots
\end{tikzcd}$$

In this case we do not need to assume that f is proper; we always have $a'_* f_* = a_*$.

3.5.3 Functoriality of the localization spectral sequence To express the functoriality of Theorem 3.3 with respect to stratified morphisms, we adopt a more meaningful notation:

- For an object $\mathscr{H} \in \mathbb{DA}_X(P)$ we denote by $\widetilde{\Pi}(\mathscr{H})$ the Postnikov system in $\mathbb{DA}_X$ described in Proposition 2.20.
- For an object $\mathscr{F} \in \mathbb{DA}_X$ we denote by $\Pi(\widehat{P}, X; \mathscr{F})$ the Postnikov system in $\mathbb{DA}_X$ described in Theorem 3.3.

Borrowing notation from the proof of Theorem 3.3 we have that $\Pi(\widehat{P}, X; \mathscr{F})$ is obtained by appending $\widetilde{\Pi}(r_*r^*p^*\mathscr{F})[-1]$ to the first (localization) triangle.

We start with a general lemma explaining the compatibility between the Postnikov systems $\widetilde{\Pi}$ and certain pushforwards. We recall (see Remark 1.2) that an isomorphism of posets $\alpha\colon P \to P'$ induces isomorphisms of complexes denoted by

$$C^\bullet(\alpha)\colon C^\bullet_{P'}(S') \to C^\bullet_P(S)$$

for elements $S \in P$ and $S' \in P'$ such that $S' = \alpha(S)$. If $\sigma\colon \widehat{P} \to \mathbb{Z}$ is a strictly increasing map such that $\sigma(\hat{0}) = 0$ and if $\alpha\colon \widehat{P} \to \widehat{P}'$ is an isomorphism of posets then we denote by $\sigma'\colon \widehat{P}' \to \mathbb{Z}$ the composite $\sigma' = \sigma \circ \alpha^{-1}$. In the next lemma, for $\mathscr{H} \in \mathbb{DA}_X(P)$ and $S \in P$ we denote by $\mathscr{H}_S \in \mathbb{DA}_X$ the value of $\mathscr{H}$ at S.

Lemma 3.15 *Let $\alpha\colon P \to P'$ be an isomorphism of posets, let $f : X \to X'$ be a morphism of schemes, and let us denote by $(\alpha, f)\colon (P, X) \to (P', X')$ the corresponding morphism of (constant) diagrams of schemes. For $\mathscr{H} \in \mathbb{DA}_X(P)$ we have an isomorphism*

$$\widetilde{\Pi}((\alpha, f)_*\mathscr{H}) \xrightarrow{\sim} f_*\widetilde{\Pi}(\mathscr{H}).$$

At the level of graded objects it reads

$$\bigoplus_{\sigma(S')=k} f_*\mathscr{H}_{\alpha^{-1}(S')} \otimes C^{\bullet+1}_{P'}(S') \xrightarrow{\sim} \bigoplus_{\sigma(S)=k} f_*(\mathscr{H}_S \otimes C^{\bullet+1}_P(S)) \simeq \bigoplus_{\sigma(S)=k} f_*\mathscr{H}_S \otimes C^{\bullet+1}_P(S),$$

and its component indexed by S' and S is given by $\mathrm{id} \otimes C^{\bullet+1}(\alpha)$ if $S = \alpha(S')$ and zero otherwise.

Proof Since $(\alpha, f) = (\mathrm{id}, f) \circ (\alpha, \mathrm{id})$ it is enough to do the proof in the case $\alpha = \mathrm{id}$ and in the case $f = \mathrm{id}$. In the former case it follows from the fact that $(\mathrm{id}, f)_*\colon \mathbb{DA}_X \to \mathbb{DA}_{X'}$ is a morphism of derivators. In the latter case it is the content of Remark 2.21. □

In the statement of the next theorem we will use the following "pullback" morphisms of functors in the context of a morphism of schemes $f\colon X \to X'$ and two strata S and S' such that $f(\overline{S}) \subset \overline{S}'$, where $f_{\overline{S}}^{\overline{S}'}\colon \overline{S} \to \overline{S}'$ denotes the morphism induced by f:

$$\eta_S^{S'}(f)\colon (i_{\overline{S}'}^{X'})_*(i_{\overline{S}'}^{X'})_* \to (i_{\overline{S}'}^{X'})_*(f_{\overline{S}}^{\overline{S}'})_*(f_{\overline{S}}^{\overline{S}'})^*(i_{\overline{S}'}^{X'})^* \simeq f_*(i_{\overline{S}}^{X})_*(i_{\overline{S}}^{X})^* f^*.$$

Theorem 3.16 (1) *The Postnikov system of Theorem 3.3 is functorial with respect to stratified morphisms. More precisely, for every morphism $(\alpha, f)\colon (\widehat{P}, X) \to (\widehat{P}', X')$ and every object $\mathscr{F}' \in \mathbb{DA}_{X'}$ we have a morphism of Postnikov systems*

$$\Pi(\alpha, f; \mathscr{F}')\colon \Pi(\widehat{P}', X'; \mathscr{F}') \to f_*\Pi(\widehat{P}, X; f^*\mathscr{F}').$$

They satisfy $\Pi(\mathrm{id}, \mathrm{id}; \mathscr{F}') = \mathrm{id}$ and the equality

$$\Pi(\beta \circ \alpha, g \circ f; \mathscr{F}'') = g_*\Pi(\alpha, f; g^*\mathscr{F}'') \circ \Pi(\beta, g; \mathscr{F}'')$$

for composable morphisms

$$(\widehat{P}, X) \xrightarrow{(\alpha,f)} (\widehat{P}', X') \xrightarrow{(\beta,g)} (\widehat{P}'', X'')$$

and $\mathscr{F}'' \in \mathbb{DA}_{X''}$.

(2) *For every integer k, the morphism $\Pi(\alpha, f; \mathscr{F}')$ reads, at the level of graded objects,*

$$\bigoplus_{\sigma'(S')=k} (i_{\overline{S}'}^{X'})_*(i_{\overline{S}'}^{X'})^*\mathscr{F}' \otimes C_{P'}^\bullet(S') \to \bigoplus_{\sigma(S)=k} f_*(i_{\overline{S}}^{X})_*(i_{\overline{S}}^{X})^* f^*\mathscr{F}' \otimes C_P^\bullet(S)$$

and has its component indexed by S' and S given by $\eta_S^{S'}(f)\mathscr{F}' \otimes C^\bullet(\alpha)$ if $S' = \alpha(S)$ and zero otherwise.

(3) *The morphism $\Pi(\alpha, f; \mathscr{F}')$ is functorial in $\mathscr{F}'$.*

Proof We proceed in three steps as in the proof of Theorem 3.3.

(a) The first triangle of the Postnikov system is the localization triangle and its functoriality follows from the discussion of Section 3.5.2.

(b) Following the proof of Theorem 3.3 we consider the following commutative diagram in the category of diagrams of schemes:

$$\begin{tikzcd}
(P,\mathscr{X}) \arrow[r, "r"] \arrow[d, "s"'] \arrow[drr, "(\alpha,f)"] & (P,X) \arrow[d, "p"'] \arrow[drr, "(\alpha,f)"] & & \\
Z \arrow[r, "i"] \arrow[drr, "f"] & X \arrow[drr, "f"] & (P',\mathscr{X}') \arrow[r, "r'"] \arrow[d, "s'"'] & (P',X') \arrow[d, "p'"] \\
& & Z' \arrow[r, "i'"'] & X'
\end{tikzcd}$$

The morphism $(\alpha, f)\colon (P, X) \to (P, X')$ is induced by α at the level of posets and by $f : X \to X'$ at the level of schemes. The morphism $(\alpha, f)\colon (P, \mathscr{X}) \to (P, \mathscr{X}')$ is induced by α at the level of posets and by the maps $\overline{S} \to \overline{\alpha(S)}$ induced by f at the level of schemes. We have the commutative diagram in $\mathbb{DA}_{X'}$,

$$\begin{tikzcd}
(i')_*(i')^*\mathscr{F}' \arrow[rr] \arrow[d, "\sim"'] & & f_*i_*i^*f^*\mathscr{F}' \arrow[d, "\sim"] \\
(i')_*(s')_*(s')^*(i')^*\mathscr{F}' & & f_*i_*s_*s^*i^*f^*\mathscr{F}' \\
(p')_*(r')_*(r')^*(p')^*\mathscr{F}' \arrow[u, "\sim"] \arrow[r, "(p')_*\varphi"'] & (p')_*(\alpha, f)_*r_*r^*(\alpha, f)^*(p')^*\mathscr{F}' & f_*p_*r_*r^*p^*f^*\mathscr{F}' \arrow[u, "\sim"'] \arrow[l, "\sim"']
\end{tikzcd}$$

where the vertical arrows $\xrightarrow{\sim}$ are isomorphisms by [Ayoub and Zucker 2012, Lemma 1.18] as in the proof of Theorem 3.3. We have the objects

$$\mathscr{H}' = (r')_*(r')^*(p')^*\mathscr{F}' \quad \text{and} \quad \mathscr{H} = r_*r^*(\alpha, f)^*(p')^*\mathscr{F}' \simeq r_*r^*p^*f^*\mathscr{F}'$$

of $\mathbb{DA}_{X'}(P')$ and $\mathbb{DA}_X(P)$, respectively, and the natural morphism $\varphi\colon \mathscr{H}' \to (\alpha, f)_*\mathscr{H}$ appearing in the above diagram. For $S' \in P'$, the value of $\mathscr{H}'$ at S' is $(i^{X'}_{\overline{S'}})_*(i^{X'}_{\overline{S'}})^*\mathscr{F}'$, that of $(\alpha, f)_*\mathscr{H}$ is $f_*(i^X_{\overline{S}})_*(i^X_{\overline{S}})^*f^*\mathscr{F}'$, for $S' = \alpha(S)$, and the value of φ is $\eta^{S'}_S(f)\mathscr{F}'$.

(c) We define the remainder of $\Pi(\alpha, f; \mathscr{F}')$ to be the composite

$$\widetilde{\Pi}(\mathscr{H}') \xrightarrow{\widetilde{\Pi}(\varphi)} \widetilde{\Pi}((\alpha, f)_*\mathscr{H}) \xrightarrow{\sim} f_*\widetilde{\Pi}(\mathscr{H})$$

where the second arrow is described in Lemma 3.15. The compatibility with composition is left to the reader. The description of $\Pi(\alpha, f; \mathscr{F}')$ at the level of graded objects follows from Lemma 3.15 and the description of the values of φ in (b). The functoriality in $\mathscr{F}'$ is obvious. $\square$

Remark 3.17 By applying Poincaré–Verdier duality one gets the dual statement that the Postnikov system of Theorem 3.9 is functorial with respect to stratified morphisms.

References

[Ayoub 2007a] **J Ayoub**, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, I*, Astérisque 314, Soc. Math. France, Paris (2007) MR Zbl

[Ayoub 2007b] **J Ayoub**, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, II*, Astérisque 315, Soc. Math. France, Paris (2007) MR Zbl

[Ayoub 2010] **J Ayoub**, *Note sur les opérations de Grothendieck et la réalisation de Betti*, J. Inst. Math. Jussieu 9 (2010) 225–263 MR Zbl

[Ayoub 2014a] **J Ayoub**, *A guide to (étale) motivic sheaves*, from "Proceedings of the International Congress of Mathematicians, II" (S Y Jang, Y R Kim, D-W Lee, I Ye, editors), Kyung Moon Sa, Seoul (2014) 1101–1124 MR Zbl

[Ayoub 2014b] **J Ayoub**, *La réalisation étale et les opérations de Grothendieck*, Ann. Sci. École Norm. Sup. 47 (2014) 1–145 MR Zbl

[Ayoub and Zucker 2012] **J Ayoub**, **S Zucker**, *Relative Artin motives and the reductive Borel–Serre compactification of a locally symmetric variety*, Invent. Math. 188 (2012) 277–427 MR Zbl

[Baclawski 1980] **K Baclawski**, *Cohen–Macaulay ordered sets*, J. Algebra 63 (1980) 226–258 MR Zbl

[Beĭlinson et al. 1982] **A A Beĭlinson**, **J Bernstein**, **P Deligne**, *Faisceaux pervers*, from "Analysis and topology on singular spaces, I", Astérisque 100, Soc. Math. France, Paris (1982) 5–171 MR Zbl

[Bibby 2016] **C Bibby**, *Cohomology of abelian arrangements*, Proc. Amer. Math. Soc. 144 (2016) 3093–3104 MR Zbl

[Björner and Ekedahl 1997] **A Björner**, **T Ekedahl**, *Subspace arrangements over finite fields: cohomological and enumerative aspects*, Adv. Math. 129 (1997) 159–187 MR Zbl

[Björner et al. 1982] **A Björner**, **A M Garsia**, **R P Stanley**, *An introduction to Cohen–Macaulay partially ordered sets*, from "Ordered sets" (I Rival, editor), NATO Adv. Study Inst. Ser. C: Math. Phys. Sci. 83, Reidel, Dordrecht (1982) 583–615 MR Zbl

[Borceux 1994] **F Borceux**, *Handbook of categorical algebra, I: Basic category theory*, Encycl. Math. Appl. 50, Cambridge Univ. Press (1994) MR Zbl

[Bousfield and Kan 1972] **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304, Springer (1972) MR Zbl

[Cisinski and Déglise 2016] **D-C Cisinski**, **F Déglise**, *Étale motives*, Compos. Math. 152 (2016) 556–666 MR Zbl

[Cisinski and Déglise 2019] **D-C Cisinski**, **F Déglise**, *Triangulated categories of mixed motives*, Springer (2019) MR Zbl

[Cisinski and Neeman 2008] **D-C Cisinski**, **A Neeman**, *Additivity for derivator K–theory*, Adv. Math. 217 (2008) 1381–1475 MR Zbl

[Cohen and Taylor 1978] **F R Cohen**, **L R Taylor**, *Computations of Gelfand–Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, from "Geometric applications of homotopy theory, I" (M G Barratt, M E Mahowald, editors), Lecture Notes in Math. 657, Springer (1978) 106–143 MR Zbl

[Deligne 1971] **P Deligne**, *Théorie de Hodge, II*, Inst. Hautes Études Sci. Publ. Math. 40 (1971) 5–57 MR Zbl

[Dupont 2015] **C Dupont**, *The Orlik–Solomon model for hypersurface arrangements*, Ann. Inst. Fourier (Grenoble) 65 (2015) 2507–2545 MR Zbl

[Dupont 2017] **C Dupont**, *Relative cohomology of bi-arrangements*, Trans. Amer. Math. Soc. 369 (2017) 8105–8160 MR Zbl

[Dupont and Fresán 2023] **C Dupont**, **J Fresán**, *A construction of the polylogarithm motive*, preprint (2023) arXiv 2305.00789

[Ekedahl 1990] **T Ekedahl**, *On the adic formalism*, from "The Grothendieck Festschrift, II" (P Cartier, L Illusie, N M Katz, G Laumon, K A Ribet, editors), Progr. Math. 87, Birkhäuser, Boston, MA (1990) 197–218 MR Zbl

[Franke 1996] **J Franke**, *Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence*, preprint (1996)

[Getzler 1999] **E Getzler**, *Resolving mixed Hodge modules on configuration spaces*, Duke Math. J. 96 (1999) 175–203 MR Zbl

[Goncharov 2002] **A B Goncharov**, *Periods and mixed motives*, preprint (2002) arXiv math/0202154

[Goresky and MacPherson 1988] **M Goresky**, **R MacPherson**, *Stratified Morse theory*, Ergebnisse der Math. 14, Springer (1988) MR Zbl

[Groth 2013] **M Groth**, *Derivators, pointed derivators and stable derivators*, Algebr. Geom. Topol. 13 (2013) 313–374 MR Zbl

[Grothendieck 1991] **A Grothendieck**, *Les dérivateurs*, unpublished manuscript (1991) Available at `https://webusers.imj-prg.fr/~georges.maltsiniotis/groth/Derivateurs.html`

[Heller 1988] **A Heller**, *Homotopy theories*, Mem. Amer. Math. Soc. 383, Amer. Math. Soc., Providence, RI (1988) MR Zbl

[Ivorra 2016] **F Ivorra**, *Perverse, Hodge and motivic realizations of étale motives*, Compos. Math. 152 (2016) 1237–1285 MR Zbl

[Kříž 1994] **I Kříž**, *On the rational homotopy type of configuration spaces*, Ann. of Math. 139 (1994) 227–237 MR Zbl

[Looijenga 1993] **E Looijenga**, *Cohomology of $\mathcal{M}_3$ and $\mathcal{M}_3^1$*, from "Mapping class groups and moduli spaces of Riemann surfaces" (C-F Bödigheimer, R M Hain, editors), Contemp. Math. 150, Amer. Math. Soc., Providence, RI (1993) 205–228 MR Zbl

[Maltsiniotis 2001] **G Maltsiniotis**, *Introduction à la théorie des dérivateurs*, preprint (2001) Available at `https://webusers.imj-prg.fr/~georges.maltsiniotis/ps/m.pdf`

[Morel and Voevodsky 1999] **F Morel**, **V Voevodsky**, $\mathbb{A}^1$*–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR Zbl

[Orlik and Solomon 1980] **P Orlik**, **L Solomon**, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. 56 (1980) 167–189 MR Zbl

[Orlik and Terao 1992] **P Orlik**, **H Terao**, *Arrangements of hyperplanes*, Grundl. Math. Wissen. 300, Springer (1992) MR Zbl

[Petersen 2017] **D Petersen**, *A spectral sequence for stratified spaces and configuration spaces of points*, Geom. Topol. 21 (2017) 2527–2555 MR Zbl

[Saito 1990] **M Saito**, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990) 221–333 MR Zbl

[Tosteson 2016] **P Tosteson**, *Lattice spectral sequences and cohomology of configuration spaces*, preprint (2016) arXiv 1612.06034

[Totaro 1996] **B Totaro**, *Configuration spaces of algebraic varieties*, Topology 35 (1996) 1057–1067 MR Zbl

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Complex hypersurfaces in direct products of Riemann surfaces

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We study smooth complex hypersurfaces in direct products of closed hyperbolic Riemann surfaces and give a classification in terms of their fundamental groups. This answers a question of Delzant and Gromov on subvarieties of products of Riemann surfaces in the smooth codimension one case. We also answer Delzant and Gromov's question of which subgroups of a direct product of surface groups are Kähler for two classes: subgroups of direct products of three surface groups, and subgroups arising as the kernel of a homomorphism from the product of surface groups to $\mathbb{Z}^3$. These results will be a consequence of answering the more general question of which subgroups of a direct product of surface groups are the image of a homomorphism from a Kähler group, which is induced by a holomorphic map, for the same two classes. This provides new constraints on Kähler groups.

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1 Introduction

A *Kähler group* is a group that can be realized as fundamental group of a compact Kähler manifold.

Convention Throughout this work, S_g will denote a closed orientable surface of genus $g \geq 2$ and $\Gamma_g = \pi_1(S_g)$ its fundamental group. Furthermore, a *surface group* will always be a group isomorphic to Γ_g for some $g \geq 2$.

Kähler groups have attracted much interest over the last decades and have been studied from many different points of view. An important motivation for studying them is that they are closely linked to the study of the topology of smooth complex projective varieties. Historically, a key technique for understanding Kähler groups is through their homomorphisms onto surface groups. For some examples of how surface groups are used in the study of Kähler groups, as well as for general background on Kähler groups, we refer the reader to [Amorós et al. 1996] (and also [Biswas and Mj 2017; Burger 2011] for more recent developments).

A central objective of this work will be to develop new constraints on homomorphisms from Kähler groups onto surface groups by studying complex hypersurfaces in direct products of Riemann surfaces. More precisely, we will address the following questions, raised by Delzant and Gromov [2005] in their fundamental work on cuts in Kähler groups:

Question 1 [Delzant and Gromov 2005] *Which subgroups of direct products of surface groups are Kähler?*

Question 2 [Delzant and Gromov 2005] *Given a subgroup $G \leq \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r})$, when does there exist an algebraic variety $V \subset S_{g_1} \times \cdots \times S_{g_r}$ of a given dimension n such that the image of the fundamental group of V is G?*

Question 2 can be seen as a more general version of Question 1. This is particularly apparent from the following group-theoretic reformulation:

Question 3 *When is a subgroup $G \leq \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r})$ the image of a homomorphism $\pi_1(X) \to \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r})$ which is induced by a holomorphic map $X \to S_{g_1} \times \cdots \times S_{g_r}$ from a compact Kähler manifold X?*

Answers to these questions in concrete situations provide new constraints on Kähler groups and can thus have interesting applications. Indeed, one such application of Theorem 1.1 has been provided recently by Llosa Isenrich and Py [2021]. They apply it to obtain constraints on Kodaira fibrations admitting more than two fiberings, thereby making progress on the question [Salter 2015; Catanese 2017] of whether such Kodaira fibrations can exist.

Delzant and Gromov [2005] give criteria for when a Kähler group admits a homomorphism to a direct product of surface groups. These results have been extended by [Py 2013; Delzant and Py 2019]. A key consequence of their works is that many actions of Kähler groups on CAT(0) cube complexes factor through homomorphisms to direct products of surface groups. Combined with the important role that CAT(0) cube complexes have played in recent advances in geometric group theory and low-dimensional topology (eg [Agol 2013]), this motivates Delzant and Gromov's questions.

The first nontrivial examples of Kähler subgroups of direct products of surface groups were constructed by Dimca, Papadima and Suciu [Dimca et al. 2009] with the purpose of showing that there is a Kähler group which does not have a classifying space which is a quasiprojective variety. They arise as fundamental groups of generic fibres of holomorphic maps from a direct product of Riemann surfaces onto an elliptic curve, which restrict to ramified coverings of degree two on the factors. These examples have been generalized by Llosa Isenrich [2019] and Biswas, Mj and Pancholi [Biswas et al. 2014]. All of these examples are fundamental groups of smooth complex hypersurfaces in direct products of closed Riemann surfaces. More general classes of Kähler subgroups of direct products of surface groups have been constructed from holomorphic maps onto higher-dimensional tori [Llosa Isenrich 2020]. They include examples coming from subvarieties of all possible codimensions. On the other hand, Kähler subgroups of direct products of surface groups must satisfy strong constraints and the same remains true for subgroups arising as images of homomorphisms which are induced by holomorphic maps [Llosa Isenrich 2020]. We will provide more details on these results in Section 2.

The combination of the diversity of examples and constraints reveals the subtle conditions that a complete answer to Delzant and Gromov's question needs to satisfy. However, as discussed above, solutions even in specific cases provide new tools for studying Kähler groups, enabling interesting applications. This work is thus concerned with finding natural situations in which complete answers can be obtained. For this we combine insights from previous works with Albanese maps and a careful analysis of complex hypersurfaces in direct products of closed Riemann surfaces.

Our first result is an answer to Question 3 for direct products of three surface groups.

Definition For a direct product $G_1 \times \cdots \times G_r$ of groups, denote by $p_i : G_1 \times \cdots \times G_r \to G_i$ the projection onto the i^{th} factor. A subgroup $H \le G_1 \times \cdots \times G_r$ is called

- *subdirect* if $p_i(H) = G_i$ for $1 \le i \le r$, and
- *full* if $H \cap G_i := H \cap (1 \times \cdots \times 1 \times G_i \times 1 \times \cdots \times 1)$ is nontrivial for $1 \le i \le r$.

Theorem 1.1 *Let $G = \pi_1(X)$ be the fundamental group of a compact Kähler manifold X, and let $\phi : G \to \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ be a homomorphism with finitely presented full subdirect image $\overline{G} := \phi(G)$ of infinite index. Assume that* $\ker(p_i \circ \phi)$ *is finitely generated for* $1 \le i \le 3$.

Then there are finite-index subgroups $\Gamma_{\gamma_i} \le \Gamma_{g_i}$, a complex elliptic curve E and a holomorphic map

$$f = \sum_{i=1}^{3} f_i : S_{\gamma_1} \times S_{\gamma_2} \times S_{\gamma_3} \to E,$$

induced by branched holomorphic coverings $f_i : S_{\gamma_i} \to E$, such that $\overline{G}_0 = \ker(f_*) \cong \pi_1(H) \le \overline{G}$ is a finite-index subgroup, where H is the smooth generic fibre of f and $f_* : \Gamma_{\gamma_1} \times \Gamma_{\gamma_2} \times \Gamma_{\gamma_3} \to \pi_1(E)$ is the induced map on fundamental groups.

We emphasize that the condition that $\ker(p_i \circ \phi)$ is finitely generated in Theorem 1.1 implies that the homomorphism ϕ is induced by a holomorphic map, and, conversely, that every homomorphism to a surface group induced by a holomorphic map will have finitely generated kernel, after possibly passing to a finite ramified cover. Thus, our result does really provide an answer to Question 3 for direct products of three surface groups.

Remark 1.2 Theorem 1.1 also provides constraints on homomorphisms to products of more than three surface groups satisfying the remaining assumptions of the theorem. To see this, we use that, for subdirect products of surface groups, finite presentability is equivalent to satisfying the virtual surjection to pairs property (VSP) [Bridson et al. 2013, Theorem D]. Thus, finite presentability is preserved under projections to factors, allowing us to apply Theorem 1.1 to every composition of such a homomorphism with a projection to three of the surface group factors.

We also give a description of all possible images of homomorphisms with ϕ as in Theorem 1.1 when the image is not a full subdirect product (see Theorem 4.3). However, in this case the homomorphism will not always be induced by a holomorphic map.

As a consequence of Theorem 4.3, we obtain the following answer to Question 1 in the three factor case:

Corollary 1.3 *Let $G=\pi_1(X)\leq\Gamma_{g_1}\times\Gamma_{g_2}\times\Gamma_{g_3}$ for X a compact Kähler manifold. Then there is a finite-index subgroup $G_0\leq G$ such that either*

(1) *$G_0\cong\mathbb{Z}^{2k}\times\Gamma_{h_1}\times\cdots\times\Gamma_{h_s}$ for $h_1,\ldots h_s\geq 2$ and $0\leq 2k+s\leq 3$; or*

(2) *G_0 is the kernel of an epimorphism $\psi\colon\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}\to\mathbb{Z}^2$ which is induced by a surjective holomorphic map $f=\sum_{i=1}^3 f_i\colon S_{\gamma_1}\times S_{\gamma_2}\times S_{\gamma_3}\to E$ with the same properties as the map f in Theorem 1.1.*

Conversely, every group which satisfies one of the conditions (1) *and* (2) *is Kähler.*

We remark that Theorem 1.1 and Corollary 1.3 will hold for any choice of compact Kähler manifold X with $G=\pi_1(X)$. However, the complex structures on E and S_{γ_i} obtained in the proof will depend on the complex structure of X, since we will make use of the fact that there is a holomorphic map $X\to S_{g_1}\times S_{g_2}\times S_{g_3}$ which realizes the homomorphism $G\to\Gamma_{g_1}\times\Gamma_{g_2}\times\Gamma_{g_3}$. Both results will be consequences of the more general criterion provided by Theorem 3.1. Theorem 3.1 also allows us to classify connected smooth complex hypersurfaces in a direct product of r closed Riemann surfaces in terms of the image of their fundamental groups, thus providing a complete answer to Question 2 for this case.

Theorem 1.4 *Let $X\subset S_{g_1}\times\cdots\times S_{g_r}$ be a connected smooth complex hypersurface in a product of closed Riemann surfaces of genus $g_i\geq 2$. Then there are finite unramified covers $X_0\to X$ and $S_{\gamma_i}\to S_{g_i}$, and a holomorphic embedding $\iota\colon X_0\hookrightarrow S_{\gamma_1}\times\cdots\times S_{\gamma_r}$ such that one of the following holds*:

(1) *ι_* is surjective on fundamental groups.*

(2) *X_0 is a direct product of $r-1$ Riemann surfaces.*

(3) *There is $3\leq s\leq r$, an elliptic curve E and surjective holomorphic maps $h_i\colon S_{\gamma_i}\to E$ for $1\leq i\leq s$ such that $X_0=H\times S_{g_{s+1}}\times\cdots\times S_{g_r}$ for H the smooth generic fibre of $h=\sum_{i=1}^s h_i\colon S_{\gamma_1}\times\cdots\times S_{\gamma_s}\to E$.*

Moreover, if (3) *holds, then h induces a short exact sequence*

$$1\to\pi_1(H)\to\pi_1(S_{\gamma_1})\times\cdots\times\pi_1(S_{\gamma_s})\to\pi_1(E)\to 1.$$

Finally, the techniques used to prove Theorem 3.1 can be adapted to give a complete classification of Kähler subgroups of direct products of surface groups arising as kernels of homomorphisms to $\mathbb{Z}^3$, hence also answering Question 1 for this case. We refer to Section 6 for the precise statement and results.

Structure

In Section 2 we will give some additional background and motivation for this work. In Section 3 we will prove Theorem 3.1, which is the main technical result of this work. We apply this result in Section 4 to prove Theorem 1.1 and Corollary 1.3 and in Section 5 to prove Theorem 1.4. In Section 6 we explain how the techniques used in the proof of Theorem 3.1 can be applied to kernels of homomorphisms from direct products of surface groups to $\mathbb{Z}^3$.

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2 Background

When approaching Delzant and Gromov's questions, it is helpful to use our understanding of the nature of subgroups of direct products of surface groups from geometric group theory. The work of Bridson, Howie, Miller and Short [Bridson et al. 2009; 2013] and other authors (eg [Kochloukova 2010; Kuckuck 2014]) shows that finiteness properties play a key role in this context. We say that a group has finiteness type $\mathscr{F}_k$ if it has a classifying CW–complex with finitely many cells of dimension $\leq k$. Note that type $\mathscr{F}_1$ is equivalent to being finitely generated, while type $\mathscr{F}_2$ is equivalent to being finitely presented. A subgroup of type $\mathscr{F}_r$ of a direct product of r surface groups is virtually a direct product of finitely many free groups and surface groups [Bridson et al. 2009; 2013]. Thus, all "nontrivial" subgroups of such a product must have exotic finiteness properties. Moreover, for groups which are not of type $\mathscr{F}_r$, stronger finiteness properties mean stronger constraints on the type of group. For more details we refer to [Bridson et al. 2009; 2013; Kochloukova 2010; Kuckuck 2014].

As explained in the introduction, finding a complete answer to Delzant and Gromov's question is far from trivial. However, there are interesting subclasses of direct products of surface groups in which finding an answer seems more feasible. Indeed, a first class are the subgroups G of type $\mathscr{F}_\infty$: since any such G is virtually a direct product of surface groups and free groups, one deduces readily that G being Kähler is equivalent to G being virtually a product $\mathbb{Z}^{2k} \times \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_s})$ for $k \geq 0$, $s \geq 0$ and $g_1, \ldots, g_s \geq 2$.

In terms of finiteness properties, the first nontrivial class of subgroups of a direct product of r surface groups is given by the ones which are of type $\mathscr{F}_{r-1}$ but not $\mathscr{F}_r$. The examples constructed in [Dimca et al. 2009] show the existence of Kähler groups of this type for every $r \geq 3$. They are obtained as fundamental groups of complex hypersurfaces in direct products of Riemann surfaces. Their construction was subsequently generalized in [Biswas et al. 2014; Llosa Isenrich 2019]. All known examples of this type can be obtained from the following result:

Theorem 2.1 [Dimca et al. 2009; Llosa Isenrich 2019] *Let $r \geq 3$, let E be an elliptic curve and let $f_i \colon S_{g_i} \to E$ be branched covers for $1 \leq i \leq r$. Define the map $f := \sum_{i=1}^{r} \colon S_{g_1} \times \cdots \times S_{g_r} \to E$ using the additive structure in E. Assume that the induced map f_* on fundamental groups is surjective and let H be the smooth generic fibre of f. Then f induces a short exact sequence*

$$1 \to \pi_1(H) \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \pi_1(E) \to 1.$$

The group $\pi_1(H)$ is Kähler of type $\mathscr{F}_{r-1}$ but not of type $\mathscr{F}_r$. Moreover, $\pi_1(H) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is an irreducible full subgroup.

When passing to subgroups with more general finiteness properties, the situation turns out to be more subtle. Indeed, the class of Kähler subgroups of direct products of surface groups that one can then obtain is much larger: they can attain any possible finiteness properties and can arise from subvarieties of all codimensions [Llosa Isenrich 2020]. Moreover, there is no apparent correlation between the codimension of a smooth subvariety realizing a subgroup and its finiteness properties (see [Llosa Isenrich 2020, Theorems 1.2 and 4.1] for precise statements of these results).

On the other hand, it is not hard to see that Kähler subgroups of a direct product of surface groups have to satisfy many restrictions. It is well known that a Kähler subgroup of a direct product of surface groups must be isomorphic to a subdirect product of a free abelian group of even rank and finitely many surface groups. Even among subgroups of this form, strong constraints hold [Llosa Isenrich 2020, Sections 6–9]. For instance, every Kähler full subdirect product of r surface groups which is of type $\mathscr{F}_k$ with $k > \frac{1}{2}r$ must virtually be isomorphic to the kernel of an epimorphism $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^{2m}$ for some $m \geq 0$ and $g_1, \ldots, g_r \geq 2$; a similar result holds for finitely presented images of homomorphisms from Kähler groups to direct products of surface groups which are induced by holomorphic maps.

Given the explicit nature of Theorem 2.1, one may now wonder if these constraints can be strengthened to show that all Kähler subgroups of direct products of r surface groups are of the form of this theorem if they are of type $\mathscr{F}_{r-1}$ but not $\mathscr{F}_r$. Theorem 1.1, Corollary 1.3 and Theorem 6.4 show that this is indeed the case after imposing additional assumptions and that the same remains true even when we consider images of homomorphisms to direct products of surface groups. The common key to these results is that our assumptions will allow us to reduce to situations in which all interesting Kähler subgroups are fundamental groups of smooth complex hypersurfaces.

We now turn to explaining in more detail why the condition that $\ker(p_i \circ \phi)$ is finitely generated in Theorem 1.1 arises naturally. For this recall the following classical result about Kähler groups:

Theorem 2.2 *Let $G = \pi_1(X)$, for X a compact Kähler manifold. Fix $h \geq 2$. The following properties are equivalent*:

(1) *There exists a surjective homomorphism $\phi \colon G \twoheadrightarrow \Gamma_h$.*

(2) *There exists $g \geq h$ and a holomorphic map $f \colon X \to S_g$ with connected fibres.*

(3) *There exists $g \geq h$ and a holomorphic map $\hat{f}: X \to S_{g,\underline{n}}$ with connected and nonmultiple fibres such that the kernel of the induced homomorphism $\hat{f}_*: G \to \pi_1^{\mathrm{orb}}(S_{g,\underline{n}})$ is finitely generated, where $S_{g,\underline{n}}$ is a closed hyperbolic Riemann orbisurface with cone points of orders $\underline{n} = (n_1, \dots, n_k)$.*

Moreover, if (1) *is satisfied, then we can choose a map f satisfying* (2) *such that ϕ factors through $f_*: \pi_1(X) \to \Gamma_g$. Similarly, if* (2) *is satisfied, then we can choose a map $\hat{f}$ satisfying* (3) *such that f factors through $\hat{f}$.*

The equivalence of (1) and (2) is due to Siu [1987] and Beauville [1988], while the orbifold version was proved by Catanese [2003] (although it seems to have been known earlier; see [Kotschick 2012] for further details). Conversely, every homomorphism from a Kähler group onto a closed hyperbolic orbisurface group with finitely generated kernel is induced by a holomorphic map (see [Catanese 2008; Delzant 2016, Theorem 2]). For further background and definitions on maps from compact Kähler manifolds to hyperbolic orbisurfaces, we refer the reader to [Delzant 2016, Section 2].

Note that every hyperbolic orbisurface group has a finite-index subgroup which is a surface group. Considering that all of the main results in this paper require us to pass to finite-index subgroups, we will thus restrict ourselves to considering surface groups for the remainder of this work.

We conclude this section by fixing some notation and definitions which we will require later. For a direct product $G_1 \times \cdots \times G_r$ of groups and $1 \leq i_1 < \cdots < i_k \leq r$, we denote by $p_{i_1,\dots,i_k}: G_1 \times \cdots \times G_r \to G_{i_1} \times \cdots \times G_{i_k}$ the projection homomorphism. We say that a subgroup $K \leq G_1 \times \cdots \times G_r$ *surjects onto k–tuples* if $p_{i_1,\dots,i_k}(K) = G_{i_1} \times \cdots \times G_{i_k}$, *virtually surjects onto k–tuples* if $p_{i_1,\dots,i_k}(K) \leq G_{i_1} \times \cdots \times G_{i_k}$ is a finite-index subgroup, and *virtually surjects onto pairs* (VSP) if K virtually surjects onto 2–tuples for all $1 \leq i_1 < \cdots < i_k \leq r$.

We call a subgroup $K \leq G_1 \times \cdots \times G_r$ *coabelian* if it is the kernel of an epimorphism $\psi: G_1 \times \cdots \times G_r \to \mathbb{Z}^k$ for some $k \geq 0$, and *coabelian of even rank* if k is even.

Moreover, for a product of surfaces $S_{g_1} \times \cdots \times S_{g_r}$ and $1 \leq i_1 < \cdots < i_k \leq r$, we will denote by $q_{i_1,\dots,i_k}: S_{g_1} \times \cdots \times S_{g_r} \to S_{g_{i_1}} \times \cdots \times S_{g_{i_k}}$ the projection. We say that a subset $X \subset S_{g_1} \times \cdots \times S_{g_r}$ *geometrically surjects onto k–tuples* if $q_{i_1,\dots,i_k}(X) = S_{g_{i_1}} \times \cdots \times S_{g_{i_k}}$ for all $1 \leq i_1 < \cdots < i_k \leq r$. We say that X is *geometrically subdirect* if it geometrically surjects onto 1–tuples.

3 From homomorphisms to complex hypersurfaces

In this section we will prove the main result of this work. The results described in the introduction will be consequences of this result and the techniques developed in its proof.

Theorem 3.1 *Let $r \geq 3$, let X be a compact Kähler manifold and let $G = \pi_1(X)$. Let $\phi: G \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ be a homomorphism with full subdirect image which can be realized by a holomorphic* *map $f: X \to S_{g_1} \times \dots \times S_{g_r}$. Assume that*

- *$\phi(G)$ is coabelian and a proper subgroup of $\Gamma_{g_1} \times \dots \times \Gamma_{g_r}$; and*
- *for $1 \le i_1 < \dots < i_{r-1} \le r$, the composition $q_{i_1,\dots,i_{r-1}} \circ f: X \to S_{g_{i_1}} \times \dots \times S_{g_{i_{r-1}}}$ is surjective.*

Then there is an elliptic curve B and branched covers $h_i: S_{g_i} \to B$ such that $\phi(G) = \pi_1(H)$, where H is the connected smooth generic fibre of the holomorphic map $h = \sum_{i=1}^r h_i: S_{g_1} \times \dots \times S_{g_r} \to B$.

Moreover, $f(X)$ is a (possibly singular) fibre of h.

The proof of Theorem 3.1 uses the following simple and well-known result:

Lemma 3.2 *Let X and Y be complex tori and let $f: X \to Y$ be a surjective holomorphic homomorphism. Then $f_*(\pi_1(X)) \le \pi_1(Y)$ is a finite-index subgroup.*

Proof of Theorem 3.1 Let $A(X)$ be the Albanese torus of X, let $A_i = A(S_{g_i})$ be the Albanese torus of S_{g_i} for $1 \le i \le r$, and denote by $a_X: X \to A(X)$ and $a_i: S_{g_i} \to A_i$ the respective Albanese maps. By the universal property of the Albanese map, we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_X"'] & S_{g_1} \times \dots \times S_{g_r} \arrow[d, "(a_1,\dots,a_r)"'] \arrow[dr, "h"] & \\ A(X) \arrow[r, "\bar{f}"] & A_1 \times \dots \times A_r \arrow[r] & B \end{tikzcd} \tag{3-1}$$

where B is the complex torus $(A_1 \times \dots \times A_r)/\bar{f}(A(X))$ (this quotient is well defined, since the induced map on complex tori is a holomorphic homomorphism with image a complex subtorus). Denote by $b: A_1 \times \dots \times A_r \to B$ the quotient map. It is the sum $b = \sum_{i=1}^r b_i$ of the restrictions $b_i: A_i \to B$.

Surjectivity of the map $q_{1,\dots,r-1} \circ f: X \to S_{g_1} \times \dots \times S_{g_{r-1}}$ implies that, for every $(s_1, \dots, s_{r-1}) \in S_{g_1} \times \dots \times S_{g_{r-1}}$, there are $x \in X$ and $s_{x,r} \in S_{g_r}$ with $f(x) = (s_1, \dots, s_{r-1}, s_{x,r})$. By commutativity of (3-1), we obtain that

$$(t_1, \dots, t_{r-1}, t_{x,r}) := (a_1(s_1), \dots, a_{r-1}(s_{r-1}), a_r(s_{x,r})) = \bar{f}(a_X(x)).$$

Denote by $\Sigma_i := b_i(a_i(S_{g_i}))$ the image of S_{g_i} in B. Since $\bar{f}(A(X)) = \ker(b)$, we obtain that $b(t_1, \dots, t_{r-1}, t_{x,r}) = 0 \in B$ and hence $\sum_{i=1}^{r-1} b_i(t_i) = -b_r(t_{x,r}) \in -\Sigma_r$. Irreducibility of S_{g_i} implies that Σ_i is an irreducible subvariety of dimension at most one in B. Thus, the holomorphic map

$$\sum_{i=1}^{r-1} b_i: a_1(S_{g_1}) \times \dots \times a_{r-1}(S_{g_{r-1}}) \to -\Sigma_r, \quad (t_1, \dots, t_{r-1}) \mapsto \sum_{i=1}^{r-1} b_i(t_i),$$

is either trivial or surjective. It follows that the image $b_i(a_i(S_{g_i}))$ is either a point or a translate of $-\Sigma_r$ for $1 \le i \le r-1$. If, moreover, at least one of the images $b_i(a_i(S_{g_i}))$ is nontrivial, then $-\Sigma_r \subset B$ is nontrivial and therefore an irreducible subvariety of dimension one. A repeated application of the same argument to all $j \in \{1, \dots, r\}$ shows that, if at least one of the images Σ_i of S_{g_i} in B is one-dimensional, then all of the Σ_i are one-dimensional and translates of each other.

It follows that either

(1) Σ_i is a point for all $i \in \{1, \dots, r\}$, or

(2) Σ_i is a one-dimensional irreducible projective variety and Σ_i is a translate of Σ_j for all $i, j \in \{1, \dots, r\}$.

Consider the case when the image of all of the S_{g_i} is one-dimensional in B. Then the restriction of the holomorphic map

$$\sum_{i=1}^{r-1} b_i \circ a_i \colon S_{g_1} \times \cdots \times S_{g_{r-1}} \to -\Sigma_r$$

to $\{(s_1, \dots, s_{j-1})\} \times S_{g_j} \times \{(s_{j+1}, \dots, s_{r-1})\}$ is a surjective holomorphic map for every $j \in \{1, \dots, r-1\}$, $(s_1, \dots, s_{j-1}) \in S_{g_1} \times \cdots \times S_{g_{j-1}}$ and $(s_{j+1}, \dots, s_r) \in S_{g_{j+1}} \times \cdots \times S_{g_{r-1}}$. By symmetry, the same holds for $\sum_{i=1, i\neq j}^{r} b_i \circ a_i$ for $1 \leq j \leq r$.

By assumption, $r \geq 3$. It follows that, for any choice of points $s_{1,0} \in S_{g_1}$ and $s_{r,0} \in S_{g_r}$, we have

$$\begin{aligned} -\Sigma_r + b_r(a_r(s_{r,0})) &= h(S_{g_1} \times \cdots \times S_{g_{r-1}} \times \{s_{r,0}\}) \\ &= h(\{s_{1,0}\} \times S_{g_2} \times \cdots \times S_{g_{r-1}} \times \{s_{r,0}\}) \\ &= h(\{s_{1,0}\} \times S_{g_2} \times \cdots \times S_{g_r}) = b_1(a_1(s_{1,0})) - \Sigma_1. \end{aligned}$$

Hence, $-\Sigma_r + b_r(a_r(s_{r,0})) = b_1(a_1(s_{1,0})) - \Sigma_1$ is independent of $s_{1,0}$ and $s_{r,0}$ and therefore the image $h(S_{g_1} \times \cdots \times S_{g_r}) = b_r(a_r(s_{r,0})) - \Sigma_r$ is one-dimensional and a translate of $-\Sigma_r$. Furthermore, the restriction $h|_{\{(s_1,\dots,s_{j-1})\} \times S_{g_j} \times \{(s_{j+1},\dots,s_r)\}}$ maps onto $b_r(a_r(s_{r,0})) - \Sigma_r$ for every $j \in \{1, \dots, r\}$, $(s_1, \dots, s_{j-1}) \in S_{g_1} \times \cdots \times S_{g_{j-1}}$ and $(s_{j+1}, \dots, s_r) \in S_{g_{j+1}} \times \cdots \times S_{g_r}$.

Choose $s_{1,0} \in S_{g_1}$ such that there is an open neighbourhood $U \subset S_{g_1}$ of $s_{1,0}$ in which the restriction $b_1 \circ a_1 \colon U \to b_1(a_1(U)) \subset \Sigma_1$ is biholomorphic. In particular, $b_1(a_1(U))$ is a smooth one-dimensional complex manifold.

Surjectivity of the restriction $\beta|_{\{(s_{1,0})\} \times S_{g_2} \times \{(s_3,\dots,s_r)\}}$ for every $(s_3, \dots, s_r) \in S_{g_3} \times \cdots \times S_{g_r}$ implies that, for every $z \in a_r(b_r(s_{r,0})) - \Sigma_r$, there is a point $s_{2,z} \in S_{g_2}$ such that $h(s_1, s_{2,z}, s_3, \dots, s_{r,0}) = z$. Then the map

$$U \to a_r(b_r(s_{r,0})) - \Sigma_r, \quad u \mapsto b_1(a_1(u)) + b_2(a_2(s_{2,z})) + \sum_{i=3}^{r} b_i(a_i(s_i))$$

is a biholomorphic map from U onto a neighbourhood of $z \in \Sigma_r$. Hence, z is a smooth point of Σ_r and it follows that Σ_r is a smooth connected projective variety of dimension one.

The S_{g_i} are finite-sheeted branched coverings of the closed Riemann surface $a_r(b_r(s_{r,0})) - \Sigma_r$ and thus the image of $\pi_1(S_{g_i})$ in $\pi_1\big(a_r(b_r(s_{r,0}))-\Sigma_r\big)$ is a finite-index subgroup for $1 \le i \le r$. Since $r \ge 2$, there is a $\mathbb{Z}^2$ subgroup in $\pi_1\big(a_r(b_r(s_{r,0})) - \Sigma_r\big)$ and the only closed Riemann surface with a $\mathbb{Z}^2$ subgroup in its fundamental group is an elliptic curve. Thus, $a_r(b_r(s_{r,0})) - \Sigma_r$ is an elliptic curve.

Surjectivity of the maps $a_{i*}\colon \pi_1(S_{g_i}) \to \pi_1(A_i)$ on fundamental groups and the fact that the fibres of the quotient map $A_1 \times \cdots \times A_r \to B$ are connected imply that the map h is surjective on fundamental groups. Hence, $a_r(b_r(s_{r,0})) - \Sigma_r = B$, h is surjective holomorphic, and the restrictions $h|_{S_{g_j}}$ for $1 \le j \le r$ are branched covers. Theorem 2.1 implies that h induces a short exact sequence

$$1 \to \pi_1(H) \to \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r}) \xrightarrow{h_*} \pi_1(B) = \mathbb{Z}^2 \to 1$$

on fundamental groups, where H is the connected smooth generic fibre of h.

Since $\phi(G) \le \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is coabelian, we obtain a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \phi(G) \arrow[r] \arrow[d] & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \arrow[r] \arrow[d] & \mathbb{Z}^l \arrow[r] \arrow[d] & 1 \\ & (\phi(G))_{\mathrm{ab}} \arrow[r] & (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} \arrow[r] & \mathbb{Z}^l \arrow[r] & 1 \end{tikzcd} \tag{3-2}$$

where the lower sequence is exact by right-exactness of abelianization.

We now use the same line of argument as in the proof of [Llosa Isenrich 2020, Lemma 6.1] to show that $l = \mathrm{rk}_{\mathbb{Z}}(\pi_1(B))$. Since it is short, we include it here for the readers convenience:

By definition of the Albanese map, the commutative diagram (3-1) induces a commutative diagram

$$\begin{tikzcd} \pi_1(X) \arrow[r, "f_*"] \arrow[d] & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \arrow[r] \arrow[d] \arrow[dr, two heads, "h_*"] & \mathbb{Z}^l \arrow[r] & 1 \\ \pi_1(A(X)) = (\pi_1(X))_{\mathrm{ab}} \arrow[r, "\bar{f}_* = f_{*,\mathrm{ab}}"] & \pi_1(A_1) \times \cdots \times \pi_1(A_r) = (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} \arrow[r, two heads] & \pi_1(B) \end{tikzcd} \tag{3-3}$$

The map $\phi\colon \pi_1(X) \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ factors through $\phi(G)$; thus, the map $(\pi_1(X))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}$ factors through $(\phi(G))_{\mathrm{ab}}$. It follows that

$$\mathrm{im}\big((\pi_1(X))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}\big) = \mathrm{im}\big((\phi(G))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}\big),$$

and exactness of the bottom horizontal sequence in (3-2) implies that

$$(\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} / \mathrm{im}\big((\pi_1(X))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}\big) \cong \mathbb{Z}^l.$$

The commutative diagram (3-3) can be extended to a commutative diagram

$$\begin{tikzcd} \pi_1(X) \arrow[r, "f_*"] \arrow[d] & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \arrow[r, two heads] \arrow[d] \arrow[dr, two heads] & \mathbb{Z}^l \arrow[r] \arrow[d, two heads] & 1 \\ \pi_1(A(X)) = (\pi_1(X))_{\mathrm{ab}} \arrow[r, "\bar{f}_* = f_{*,\mathrm{ab}}"] & \pi_1(A_1) \times \cdots \times \pi_1(A_r) = (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} \arrow[r, two heads] \arrow[ur, two heads] & \pi_1(B) \end{tikzcd}$$

Hence, the fundamental group $\pi_1(B)$ is a quotient of $\mathbb{Z}^l$. By Lemma 3.2, we have $\mathrm{rk}_{\mathbb{Z}}\,\bar{f}_*\big(\pi_1(A(X))\big) = \mathrm{rk}_{\mathbb{Z}}\pi_1\big(\bar{f}(A(X))\big)$. Thus, we obtain

$$\begin{aligned}\mathrm{rk}_{\mathbb{Z}}(\pi_1(B)) &= 2\cdot\dim_{\mathbb{C}} B = 2\cdot\dim_{\mathbb{C}}(A_1\times\cdots\times A_r) - 2\cdot\dim_{\mathbb{C}}\bar{f}(A(X))\\ &= \mathrm{rk}_{\mathbb{Z}}(\Gamma_{g_1}\times\cdots\times\Gamma_{g_r})_{\mathrm{ab}} - \mathrm{rk}_{\mathbb{Z}}\,\bar{f}_*\big(\pi_1(A(X))\big) = l.\end{aligned}$$

It follows that the epimorphism $\mathbb{Z}^l \to \pi_1(B)$ is an isomorphism and therefore we obtain an isomorphism of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \phi(G) \arrow[r] \arrow[d, "\cong"] & \Gamma_{g_1}\times\cdots\times\Gamma_{g_r} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}^l \arrow[r] \arrow[d, "\cong"] & 1\\ 1 \arrow[r] & \pi_1(H) \arrow[r] & \Gamma_{g_1}\times\cdots\times\Gamma_{g_r} \arrow[r, "h_*"] & \pi_1(B) \arrow[r] & 1 \end{tikzcd}$$

If Σ_i is a point, then the same argument shows that B is a point and the isomorphism of short exact sequences implies that $\phi(G)\cong\pi_1(H)\cong\Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$ is not a proper subgroup.

Finally, observe that, since $h\circ f\colon X\to B$ factors through the Albanese torus $A(X)$ of X, the image of X in B is trivial. Hence, $f(X)$ is contained in a fibre of h. Since $f(X)$ is the image of a smooth complex manifold under a proper holomorphic map, it is an irreducible subvariety of a fibre of h. The map h has isolated singularities, since the restriction of h to every surface factor is a branched covering of B, and its fibres (singular or nonsingular) are connected.

If $f(X)$ is contained in a smooth generic fibre of h, then it is equal to this fibre, since smooth projective varieties are irreducible. So assume that $f(X)$ is contained in one of the finitely many singular fibres H_s of h and let $z\in H_s$ be a singular point. By Milnor's theory [1968] of isolated hypersurface singularities, a neighbourhood of z in H_s is homeomorphic to a cone over a smooth manifold K (called the link of the singularity). Furthermore, K is $(n-2)$–connected for n the complex dimension of H_s. In particular, K is connected if $n\geq 2$. Since the complex dimension of H_s is $r-1\geq 2$, it follows that K is connected. Thus, the complement of the cone point in the cone over K is connected. Connectedness of H_s then implies that the complement of the finite set of singular values in H_s is a connected smooth complex manifold. It follows that H_s is an irreducible variety and thus $H_s = f(X)$. □

4 The three factor case

By combining Theorem 3.1 with the following results, we can complete the classification of Kähler subgroups of direct products of three surface groups up to passing to finite-index subgroups.

Proposition 4.1 [Llosa Isenrich 2020, Proposition 9.5] *Let $r\geq 2$, let X be a compact Kähler manifold and let $G=\pi_1(X)$. Let $\phi\colon G\to\Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$ be a homomorphism with finitely presented full subdirect image such that the projections $p_i\circ\phi\colon G\to\Gamma_{g_i}$, $1\leq i\leq r$, have finitely generated kernel.*

Then ϕ is induced by a holomorphic map $f\colon X\to S_{g_1}\times\cdots\times S_{g_r}$ and the composition $q_{i,j}\circ f\colon X\to S_{g_i}\times S_{g_j}$ is surjective for $1\leq i<j\leq r$.

Theorem 4.2 [Llosa Isenrich 2020, Theorem 6.13] *Let X be a compact Kähler manifold and let $G=\pi_1(X)$. Let $\psi\colon G\to\Gamma_{g_1}\times\Gamma_{g_2}\times\Gamma_{g_3}$ be a homomorphism such that the projection $p_i\circ\psi$ has finitely generated kernel for $1\le i\le r$ and the image $\overline{G}:=\psi(G)$ is finitely presented. Then one of the following holds*:

(1) *$\overline{G}=\pi_1(R)$ for R a closed Riemann surface of genus ≥ 0.*

(2) *$\overline{G}=\mathbb{Z}^k$ for $k\in\{1,2,3\}$.*

(3) *$\overline{G}$ is virtually a direct product $\mathbb{Z}^k\times\Gamma_{h_1}\times\Gamma_{h_2}$ for $h_1,h_2\ge 2$ and $k\in\{0,1\}$.*

(4) *$\overline{G}$ is virtually $\mathbb{Z}^k\times\Gamma_h$ for $h\ge 2$ and $k\in\{1,2\}$.*

(5) *$\overline{G}$ is virtually subdirect and coabelian of even rank.*

As a consequence one can obtain a constraint on Kähler subgroups of direct products of surface groups by imposing the evenness condition on the first Betti number for (1)–(5) in Theorem 4.2. Note that, while groups of the form $\pi_1(R)$, $\Gamma_{h_1}\times\Gamma_{h_2}$ and $\mathbb{Z}^2\times\Gamma_h$ are Kähler, the same turns out to not be true in general for coabelian subgroups of $\Gamma_{h_1}\times\Gamma_{h_2}\times\Gamma_{h_3}$ of even rank. In fact, many such subgroups are not even the image of a homomorphism from a Kähler group which is induced by a holomorphic map. As an application of Theorem 3.1, we can make this statement precise and thus prove Theorem 1.1 and Corollary 1.3.

Theorem 4.3 *Let $G=\pi_1(X)$ be Kähler and let $\psi\colon G\to\Gamma_{g_1}\times\Gamma_{g_2}\times\Gamma_{g_3}$ be a homomorphism such that the projections $p_i\circ\psi\colon G\to\Gamma_{g_i}$ have finitely generated kernel for $1\le i\le 3$ and the image is finitely presented. Then there is a finite-index subgroup $\overline{G}_0\le\overline{G}=\psi(G)$ such that either*

(1) *$\overline{G}_0\cong\mathbb{Z}^k\times\Gamma_{h_1}\times\cdots\times\Gamma_{h_s}$ with $0\le k+s\le 3$; or*

(2) *there are finite-index subgroups $\Gamma_{\gamma_i}\le\Gamma_{g_i}$, an elliptic curve E and branched holomorphic coverings $f_i\colon S_{\gamma_i}\to E$ for $1\le i\le 3$ such that $\overline{G}_0\cong\pi_1(H)\cong\ker(f_*)$, where H is the smooth generic fibre of the surjective holomorphic map $f=\sum_{i=1}^3 f_i$.*

Conversely, any group satisfying one of the conditions (1) *and* (2) *is the image of a homomorphism satisfying the above hypotheses.*

Proof By Theorem 4.2, it suffices to consider the case when $\overline{G}$ is virtually coabelian of even rank. Then there are finite-index subgroups $\Gamma_{\gamma_i}\le\Gamma_{g_i}$ for $l\ge 0$ and an epimorphism $\phi\colon\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}\to\mathbb{Z}^{2l}$ such that $\overline{G}_0:=\ker\phi\le G$ is a finite-index subgroup and $\overline{G}_0\le\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}$ is a finitely presented full subdirect product. We may further assume that $\overline{G}_0\le\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}$ is a proper subgroup (if not, then (1) holds with $k=0$ and $s=3$).

Let $X_0\to X$ be the finite-sheeted holomorphic cover corresponding to the subgroup $\psi^{-1}(\overline{G}_0)\le G$. Then X_0 is a compact Kähler manifold with $\psi(\pi_1(X_0))=\overline{G}_0=\ker\phi$ and the projections

$$p_i\circ\psi|_{\pi_1(X_0)}\colon\pi_1(X_0)\to\Gamma_{\gamma_i}$$

have finitely generated kernel. Proposition 4.1 implies that $\psi|_{\pi_1(X_0)}$ is induced by a holomorphic map $f\colon X_0\to S_{\gamma_1}\times S_{\gamma_2}\times S_{\gamma_3}$ with the property that $q_{i,j}\circ f\colon X_0\to S_{\gamma_i}\times S_{\gamma_j}$ is a surjective holomorphic map for $1 \le i < j \le 3$. Hence, all assumptions of Theorem 3.1 are satisfied. It follows that $\overline{G}_0$ satisfies (2). The converse direction follows easily by taking quotients of Kähler groups of the form $\mathbb{Z}^{2s} \times \Gamma_{h_1} \times \cdots \times \Gamma_{h_s}$ and from Theorem 2.1. □

Proof of Theorem 1.1 If in Theorem 4.3 the group $\overline{G}$ is a full subdirect product, then (1) can only hold if $\overline{G}_0 \le \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ has finite index. Hence, we must be in case (2). □

To reduce Corollary 1.3 to Theorem 4.3, we will apply the following result of Bridson and Miller:

Theorem 4.4 [Bridson and Miller 2009, Theorem 4.6] *Let $\Gamma_{g\ge 2}$ be a surface group, let A be any group and let $G \le \Gamma_g \times A$. Assume that G is finitely presented and that the intersection $G \cap \Gamma_g$ is nontrivial. Then $G \cap A$ is finitely generated.*

Proof of Corollary 1.3 Let $G = \pi_1(X) \le \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ be a nontrivial Kähler group; in particular, G is finitely presented. Let $\psi : \pi_1(X) \hookrightarrow \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ be the canonical inclusion. To apply Theorem 4.3, we need to show that $\ker(p_i \circ \psi)$ is finitely generated for $1 \le i \le 3$.

Assume first that $G \cap \Gamma_{g_i}$ is nontrivial for $1 \le i \le 3$. Then Theorem 4.4 implies that $\ker(p_1 \circ \psi) = G \cap (\Gamma_{g_2} \times \Gamma_{g_3})$ is finitely generated and that, similarly, $\ker(p_2 \circ \psi)$ and $\ker(p_3 \circ \psi)$ are finitely generated. If some of the intersections $G \cap \Gamma_{g_i}$ are trivial, then, by reordering factors and projecting away from factors with trivial intersection, we may assume that G is a full subgroup of $\Gamma_{g_1} \times \cdots \times \Gamma_{g_s}$ with $1 \le s \le 2$. In particular, we may assume that the embedding of G in $\Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ has trivial projection to the last $3 - s$ factors. For $s = 1$, it is now trivially true that $\ker(p_i \circ \psi)$ is finitely generated for $1 \le i \le 3$, and for $s = 2$ the same follows from another application of Theorem 4.4.

Thus, we can apply Theorem 4.3 in all cases. The first part of the result is then a direct consequence of the fact that Kähler groups have even first Betti number.

Conversely, groups satisfying condition (1) and having even first Betti number are clearly Kähler and $\pi_1(H)$ in (2) is Kähler as the fundamental group of H. □

Remark 4.5 Corollary 1.3 provides a classification of Kähler subgroups of direct products of three surface groups up to passing to finite-index subgroups. This statement can be made more precise in the cases corresponding to (1): when $k = 0$, finite extensions of these groups are Kähler if they are subdirect products of surface groups; and when $k = 2$, the group G is either a finite-index subgroup of a direct product $\mathbb{Z}^2 \times \Gamma_{h'}$ with $h \ge h' \ge 2$ or $\cong \mathbb{Z}^2$.

The following example shows that it may be necessary to pass to finite-index subgroups:

Example 4.6 Let $\Gamma_{g_1} \times \Gamma_{g_2}$ be a direct product of surface groups. For $m \ge 2$, consider the canonical epimorphisms $\nu_i : H_1(\Gamma_{g_i}, \mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}$ obtained by mapping a basis of $H_1(\Gamma_{g_i}, \mathbb{Z})$ to $1 \in \mathbb{Z}/m\mathbb{Z}$. Denote by $\hat{\nu}_i : \Gamma_{g_1} \to \mathbb{Z}/m\mathbb{Z}$ the composition of ν_i with the abelianization map and define $\hat{\nu} := \nu_1 + \nu_2 : \Gamma_{g_1} \times \Gamma_{g_2} \to \mathbb{Z}/m\mathbb{Z}$. The finite-index subgroup $\ker \hat{\nu} \le \Gamma_{g_1} \times \Gamma_{g_2}$ is Kähler and virtually a direct product $\ker \nu_1 \times \ker \nu_2$ of surface groups, but is not itself a direct product of surface groups.

5 Complex hypersurfaces

In this section we prove Theorem 1.4. We consider an embedded connected smooth complex hypersurface $\iota\colon X \hookrightarrow S_{g_1}\times\cdots\times S_{g_r}$ in a direct product of closed Riemann surfaces of genus $g_i \geq 2$. Observe that we may assume that all projections $q_i\circ\iota\colon X\to S_{g_i}$ are nonconstant. Indeed, if one of the projections $q_i\circ\iota\colon X\to S_{g_i}$ in Lemma 5.2 is constant, say $q_r\circ\iota$, then $X = S_{g_1}\times\cdots\times S_{g_{r-1}}$ is a direct product of $r-1$ surfaces. Hence, we do not lose much by excluding this case.

Lemma 5.1 *Let $r\geq 2$ and let $\iota_X\colon X\hookrightarrow S_{g_1}\times\cdots\times S_{g_r}$ be a geometrically subdirect embedding of a connected smooth complex hypersurface in a direct product of closed Riemann surfaces. Then there is $2\leq s\leq r$ such that $X = Y\times S_{g_{s+1}}\times\cdots\times S_{g_r}$ with $\iota_Y\colon Y\hookrightarrow S_{g_1}\times\cdots\times S_{g_s}$ an embedded smooth complex hypersurface which geometrically surjects onto $(s-1)$–tuples.*

Proof The result follows by induction on the number of factors $r\geq 2$. For $r=2$, the result holds due to the assumption that the embedding is geometrically subdirect. If X does not geometrically surject onto $(r-1)$–tuples, then there is an $(r-1)$–tuple $1\leq i_1<\cdots<i_{r-1}\leq r$ such that the irreducible variety $\overline{X} = q_{i_1,\dots,i_{r-1}}(X)$ is $(r-2)$–dimensional; we may assume $i_j=j$. Hence, the smooth generic fibre of $q_{1,\dots,r-1}\colon X\to S_{g_1}\times\cdots\times S_{g_{r-1}}$ is one-dimensional and therefore equal to S_{g_r}. Let $\overline{X}^*\subset\overline{X}$ be the locus of nonsingular values. Then $\overline{X}^*\times S_{g_r}\subset X$ is an open dense submanifold. It follows that $X=\overline{X}\times S_{g_r}$ with $\overline{X}\hookrightarrow S_{g_1}\times\cdots\times S_{g_{r-1}}$ a connected smooth embedded hypersurface. Clearly $\overline{X}$ is geometrically subdirect. The result follows by induction. □

Lemma 5.2 *Let $r\geq 1$ and let $\iota\colon X\hookrightarrow S_{g_1}\times\cdots\times S_{g_r}$ be a connected smooth complex hypersurface such that the projections $q_i\circ\iota\colon X\to S_{g_i}$ are nontrivial. Then there are finite regular covers $S_{h_i}\to S_{g_i}$ for $1\leq i\leq r$ such that ι lifts to an embedding $j\colon X\hookrightarrow S_{h_1}\times\cdots\times S_{h_r}$ with $i_*(\pi_1(X))\cong j_*(\pi_1(X))\leq \Gamma_{h_1}\times\cdots\times\Gamma_{h_r}$ a subdirect product.*

Proof The projections $q_i\circ\iota\colon X\to S_{g_i}$ are proper holomorphic maps between compact Kähler manifolds. Thus, $\Gamma_{h_i} := (q_i\circ\iota)_*(\pi_1(X))\leq\pi_1(S_{g_i})$ is a finite-index subgroup for $1\leq i\leq r$. Let $f_i\colon S_{h_i}\to S_{g_i}$ be the associated unramified coverings. Then ι factors through a continuous map $j\colon X\to S_{h_1}\times\cdots\times S_{h_r}$ making the diagram

$$\begin{tikzcd} & S_{h_1}\times\cdots\times S_{h_r} \arrow[d] \\ X \arrow[ur, "j"] \arrow[r, "\iota"] & S_{g_1}\times\cdots\times S_{g_r} \end{tikzcd}$$

commutative. Since ι and the f_i are holomorphic, the map j defines a holomorphic embedding and, by choice of the f_i, the group $j_*(\pi_1(X))\leq\Gamma_{h_1}\times\cdots\times\Gamma_{h_r}$ is subdirect. □

We may in fact assume that the image $\iota_*(\pi_1(X))\leq\Gamma_{h_1}\times\cdots\times\Gamma_{h_r}$ is full subdirect.

Lemma 5.3 *Let $r \geq 2$ and let $\iota\colon X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ be an embedded connected smooth complex hypersurface such that $\Lambda := \iota_*(\pi_1(X)) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is a subdirect product. If Λ is not full in $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$, then (after possibly reordering factors) X is biholomorphic to $R_\gamma \times S_{g_3} \times \cdots \times S_{g_r}$ with $j\colon R_\gamma \hookrightarrow S_{g_1} \times S_{g_2}$ an embedded Riemann surface such that $j_*(\pi_1(R_\gamma)) \cong \Gamma_{g_2}$, the projection $R_\gamma \to S_{g_i}$ for $i = 1, 2$ is a branched covering, and $\Gamma_{g_1} \cap j_*(\pi_1(R_\gamma)) = \{1\}$.*

Proof After applying Lemma 5.1 and splitting off direct surface factors from X, we may assume that X geometrically surjects onto $(r-1)$–tuples for $r \geq 2$. If Λ is not full, then there is a factor Γ_{g_i} with $\Gamma_{g_i} \cap \Lambda = \{1\}$, say $i = 1$. Hence, the projection $q_{2,\dots,r}\colon S_{g_1} \times \cdots \times S_{g_r} \to S_{g_2} \times \cdots \times S_{g_r}$ induces an isomorphism $\Lambda \cong q_{2,\dots,r,*}(\Lambda) =: \overline{\Lambda} \leq \Gamma_{g_2} \times \cdots \times \Gamma_{g_r}$. Since X geometrically surjects onto $(r-1)$–tuples, the map $q_{2,\dots,r}\colon X \to S_{g_2} \times \cdots \times S_{g_r}$ is a surjective holomorphic map between closed complex manifolds. It follows that $\overline{\Lambda} \leq \Gamma_{g_2} \times \cdots \times \Gamma_{g_r}$ is a finite-index subgroup and thus a full subdirect product.

The epimorphism $p_1\colon \Lambda \to \Gamma_{g_1}$ induces an epimorphism $\bar{p}_1\colon \overline{\Lambda} \to \Gamma_{g_1}$. By the universal property of full subdirect products of limit groups (see [Bridson et al. 2013, Theorem C(3)]), $\bar{p}_1$ is induced by a homomorphism $\Gamma_{g_2} \times \cdots \times \Gamma_{g_r} \to \Gamma_{g_1}$ and thus factors through the projection $\Gamma_{g_2} \times \cdots \times \Gamma_{g_r} \to \Gamma_{g_i}$ for some $2 \leq i \leq r$ (else the image Γ_{g_1} would contain an element with noncyclic centralizer), say $i = 2$. It follows that the projection $\Lambda \to \Gamma_{g_1} \times \Gamma_{g_2}$ factors through the projection to Γ_{g_2} and thus has image isomorphic to Γ_{g_2}. However, this contradicts geometric surjection to $(r-1)$–tuples unless $r = 2$ (since, as above, $q_{1,\dots,r-1,*}(\Lambda) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_{r-1}}$ is a finite-index subgroup).

This leaves us with the situation when $X = R_\gamma$ is a closed Riemann surface of genus $\gamma \geq 2$ with the property that $\Lambda = \iota_*(\pi_1(X)) \cong \Gamma_{g_2}$. Since $\iota_*(\pi_1(X))$ is subdirect, the projections onto factors induce finite-sheeted branched coverings $R_\gamma \to S_{g_i}$ for $i = 1, 2$. □

Proof of Theorem 1.4 If X is not geometrically subdirect, then (2) holds. Hence, we can assume that X is geometrically subdirect. By Lemma 5.1, reduce to the case that $X = Y \times S_{g_{s+1}} \times \cdots \times S_{g_r}$ with $j\colon Y \hookrightarrow S_{g_1} \times \cdots \times S_{g_s}$ an embedded smooth complex hypersurface that geometrically surjects onto $(s-1)$–tuples. If $s = 1$, then Y is a point and we are in case (2). If $s = 2$ then Y is a smooth Riemann surface and we are again in case (2). Hence, we may assume that $s \geq 3$. By Lemmas 5.2 and 5.3, we may further assume that $\Lambda := j_*(\pi_1(Y)) \leq \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_s})$ is a full subdirect product.

Since Y geometrically surjects onto $(s-1)$–tuples, the projections

$$q_{1,\dots,i-1,i+1,\dots,s} \circ j\colon Y \to S_{g_1} \times \cdots \times S_{g_{i-1}} \times S_{g_{i+1}} \times \cdots \times S_{g_s}$$

are surjective holomorphic maps between closed complex manifolds of the same dimension. Hence, $(q_{1,\dots,i-1,i+1,\dots,s,*} \circ j)(\pi_1(Y)) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_{i-1}} \times \Gamma_{g_{i+1}} \times \cdots \times \Gamma_{g_s}$ is a finite-index subgroup for $1 \leq i \leq s$. Hence, Corollary 3.6 of [Kuckuck 2014] implies that there are finite-index subgroups $\Gamma_{\gamma_i} \leq \Gamma_{g_i}$ and an epimorphism $\phi\colon \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s} \to \mathbb{Z}^k$ such that $\Lambda_0 := \ker\phi = \Lambda \cap (\Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s}) \leq \Lambda$ is a finite-index subgroup and the restriction of ϕ to every factor is surjective. Note that, in particular, $\Lambda_0 \leq \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s}$ is a full subdirect product.

Denote by $Y_0 \to Y$ the finite-sheeted covering associated to the finite-index subgroup $j_*^{-1}(\Lambda_0) \leq \pi_1(Y)$. Then there is a holomorphic embedding $\iota\colon Y_0 \hookrightarrow S_{\gamma_1} \times \cdots \times S_{\gamma_s}$ making the diagram

$$\begin{tikzcd} Y_0 \arrow[r, "\iota"] \arrow[d] & S_{\gamma_1} \times \cdots \times S_{\gamma_s} \arrow[d] \\ Y \arrow[r, "j"] & S_{g_1} \times \cdots \times S_{g_s} \end{tikzcd}$$

commutative. By construction, we have $\iota_*(\pi_1(Y_0)) = \Lambda_0$ and that Y_0 geometrically surjects onto $(s-1)$–tuples

If $\Lambda_0 \leq \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s}$ is a finite-index subgroup, then we are in case (1). Hence, we may assume that Λ_0 has infinite index. In particular, $k \geq 1$ and all conditions of Theorem 3.1 are satisfied. Hence, there is an elliptic curve E and branched covers $h_i\colon S_{\gamma_i} \to E$ such that Y_0 is equal to a fibre of the holomorphic map $h = \sum_{i=1}^{s} h_i\colon S_{\gamma_1} \times \cdots \times S_{\gamma_s} \to E$.

The map h has isolated singularities and all fibres are irreducible varieties by the proof of Theorem 3.1. In particular, the map h is a submersion in all but finitely many points. It follows that h has reduced fibres and thus the fibres of h over singular values are singular varieties and, in particular, cannot be smooth manifolds (see eg [Milnor 1968, page 13]). Since Y_0 is a smooth subvariety of $S_{\gamma_1} \times \cdots \times S_{\gamma_s}$, it follows that Y_0 is a smooth generic fibre of h. □

Remark 5.4 We want to mention that case (2) in Theorem 1.4 splits into three cases (after reordering factors):

(i) X_0 has trivial image in one factor, say S_{γ_r}, and thus $X_0 = S_{\gamma_1} \times \cdots \times S_{\gamma_{r-1}}$.

(ii) $\iota_*(\pi_1(X_0)) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is not full. In this case, the proof of Lemma 5.3 shows that $X_0 = R_h \times S_{\gamma_3} \times \cdots \times S_{\gamma_r}$ with $R_h \hookrightarrow S_{\gamma_1} \times S_{\gamma_2}$ an embedded curve and $\iota_*(\pi_1(X_0)) \cong \Gamma_{\gamma_2} \times \cdots \times \Gamma_{\gamma_r}$.

(iii) $s = 2$, $X_0 = R_h \times S_{\gamma_3} \times \cdots \times S_{\gamma_r}$ with $R_h \hookrightarrow S_{\gamma_1} \times S_{\gamma_2}$ an embedded curve and $\iota_*(\pi_1(X_0)) = \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r}$ This happens for instance when R_h is a generic hyperplane section of $S_{g_1} \times S_{g_2}$. Note that in this case ι_* is not injective and furthermore this is precisely the case when (1) and (2) both hold in Theorem 1.4.

Remark 5.5 In case (1) of Theorem 1.4, the epimorphism $\iota\colon \pi_1(X_0) \to \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r}$ is not necessarily injective. For instance, X_0 can be as in Remark 5.4(iii). However, it can be an isomorphism: Take X to be a smooth generic hyperplane section of $S_{g_1} \times \cdots \times S_{g_r}$. If $r \geq 3$ the Lefschetz hyperplane theorem implies that $X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ induces an isomorphism on fundamental groups.

Remark 5.6 In the light of Theorem 1.4, it is natural to ask if one can also classify smooth subvarieties X of codimension $k \geq 2$ in a direct product of Riemann surfaces $S_{g_1} \times \cdots \times S_{g_r}$ in terms of their fundamental groups. The examples constructed in [Llosa Isenrich 2020] show that the class of fundamental groups of such subvarieties will be much larger. Furthermore, the Lefschetz hyperplane theorem will allow us to realize any fundamental group of a smooth subvariety of codimension $l < k$ as the fundamental group of a smooth subvariety of codimension k whenever $k \leq r-2$. These two observations show that any such classification will have to allow a much wider variety of fundamental groups. One observation that seems worth mentioning is that, for $k < \frac{1}{2}r$, the image of $\pi_1(X)$ in $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ has to be isomorphic to a virtually coabelian subgroup of even rank in a direct product of $\leq r$ surface groups (we might need to get rid of some factors and replace others by finite-index subgroups).

To see this, we first split off direct factors, using the same methods as above, to obtain a codimension k subvariety X_0 in a product of $s \leq r$ surfaces which geometrically surjects onto $(s-k)$–tuples. Then we combine results of Kuckuck [2014] with the fact that the inclusion $X_0 \hookrightarrow S_{g_1} \times \cdots \times S_{g_s}$ is holomorphic and thus the images $q_{i_1,\dots,i_{s-k},*}(\pi_1(X)) \leq \Gamma_{g_{i_1}} \times \cdots \times \Gamma_{g_{i_{s-k}}}$ are finite-index subgroups for $1 \leq i_1 < \cdots < i_{s-k} \leq s$ (see [Llosa Isenrich 2020, Sections 5 and 6] for details, in particular Proposition 6.3).

6 Maps to $\mathbb{Z}^3$

Another situation in which we can give a complete answer to Delzant and Gromov's question is the case of coabelian subgroups of rank two. Our proof will make use of [Bridson et al. 2013].

Theorem 6.1 [Bridson et al. 2013, Theorem D] *Let $G \leq \Lambda_1 \times \cdots \times \Lambda_r$ be a finitely generated full subdirect product of nonabelian limit groups Λ_i for $1 \leq i \leq r$.*

Then G is finitely presented if and only if G virtually surjects onto pairs.

Theorem 6.2 *Let X be compact Kähler, let $G = \pi_1(X)$ and let $\phi\colon G \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ be a homomorphism with finitely presented full subdirect image which is induced by a holomorphic map $f\colon X \to S_{g_1} \times \cdots \times S_{g_r}$. Assume that there is an epimorphism $\psi\colon \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^2$ such that $\ker\psi = \phi(G)$.*

Then (after possibly reordering factors) there is $s \geq 3$, an elliptic curve E and branched covering maps $f_i\colon S_{g_i} \to E$ for $1 \leq i \leq s$ such that $\phi(G) = \pi_1(H) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$, where H is the connected smooth generic fibre of the holomorphic map $f = \sum_{i=1}^{s} f_i\colon S_{g_1} \times \cdots \times S_{g_s} \to E$, $f_* = \psi|_{\Gamma_{g_1} \times \cdots \times \Gamma_{g_s}}$, and $\psi|_{\Gamma_{g_i}}$ trivial for $i \geq s+1$.

Proof With the same notation as in the proof of Theorem 3.1, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a_X"'] & S_{g_1} \times \cdots \times S_{g_r} \arrow[d, "(a_1,\dots,a_r)"'] \arrow[dr, "h"] & \\ A(X) \arrow[r, "\bar f"] & A_1 \times \cdots \times A_r \arrow[r] & B \end{tikzcd}$$

Arguing as in the proof of Theorem 3.1 (see diagram (3-3) and subsequent discussion) we obtain that $\mathrm{rk}_{\mathbb{Z}}\pi_1(B) = 2$ and that the map ψ is induced by the holomorphic map $h\colon S_{g_1} \times \cdots \times S_{g_r} \to B$. Since the restriction $h|_{S_{g_i}}\colon S_{g_i} \to B$ is a holomorphic map, either it is surjective or $h(S_{g_i})$ is a point.

A surjective holomorphic map between closed Riemann surfaces is a branched covering. Hence, there is $1 \leq s \leq r$ such that (after reordering factors):

- $h\colon S_{g_i} \to B$ is a branched holomorphic covering for $1 \leq i \leq s$.
- $h(S_{g_i})$ is a point for $s+1 \leq i \leq r$.

It follows that

$$\begin{aligned}\phi(G) = \ker h_* &= \ker((h|_{S_{g_1}\times\cdots\times S_{g_s}})_*) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r} \\ &= \ker \psi = \ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}.\end{aligned}$$

Since $\Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$ is finitely generated and $\phi(G)$ is finitely presented, the full subdirect product $\ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) \cong \phi(G)/(\Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_s}$ is finitely presented.

If $s = 1$, then being a full subdirect product implies that $\ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) = \Gamma_{g_1}$, and, if $s = 2$, then Theorem 6.1 implies that the group $\ker(\psi|_{\Gamma_{g_1}\times\Gamma_{g_2}}) \leq \Gamma_{g_1} \times \Gamma_{g_2}$ is a finite-index subgroup. However, ψ is an epimorphism onto the infinite group $\mathbb{Z}^2$. It follows that $s \geq 3$.

Hence, the restriction $h|_{S_{g_1}\times\cdots\times S_{g_s}}$ satisfies all conditions of Theorem 2.1, so $\ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) = \pi_1(H)$ for H the smooth generic fibre of the restriction $h|_{S_{g_1}\times\cdots\times S_{g_s}}$. Thus, $\phi(G) = \pi_1(H) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$. □

As a consequence of Theorem 6.2, we can now classify all Kähler subgroups arising as kernels of homomorphisms from a direct product of surface groups to $\mathbb{Z}^3$. For this we will require the following result:

Theorem 6.3 [Llosa Isenrich 2020, Corollary 1.6] *Let $k \geq 0$ and $g_1, \dots, g_r \geq 2$. If $\phi\colon \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^{2k+1}$ is a surjective homomorphism, then $\ker \phi$ is not Kähler.*

Theorem 6.4 *Let $r \geq 1$, let $\phi\colon \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^3$ be a homomorphism, let $G = \ker \phi \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ and let $p_i(G) = \Gamma_{\gamma_i} \leq \Gamma_{g_i}$ be the projection of G to the i^{th} factor. Then the following are equivalent:*

1. *G is Kähler.*
2. *Either $G = \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$, or there is $r \geq s \geq 3$, an elliptic curve E and surjective holomorphic maps $f_i\colon S_{\gamma_i} \to E$ for $1 \leq i \leq s$ such that $G = \pi_1(H) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$ (after possibly reordering factors), where H is the connected smooth generic fibre of the holomorphic map $f = \sum_{i=1}^{s} f_i \colon S_{\gamma_1} \times \cdots \times S_{\gamma_s} \to E$, $f_* = \phi|_{\Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_s}} \colon \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s} \to \pi_1(E) \cong \phi(\Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s})$ and $\phi|_{\Gamma_{g_i}}$ is trivial for $i \geq s+1$.*

Theorem 6.4 shows in particular that the image of ϕ is either trivial or isomorphic to $\mathbb{Z}^2$.

Proof By Theorem 2.1, (2) implies (1). Assume that G is Kähler. If ϕ is trivial, then $G = \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is Kähler, and, if $\mathrm{im}(\phi) \leq \mathbb{Z}^3$ has odd rank, then, by Theorem 6.3, G is not Kähler. Thus, we may assume that G is a finitely presented full subdirect product of $\Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r}$ which is the kernel of an epimorphism $\phi\colon \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r} \to \mathbb{Z}^2 = \mathrm{im}(\phi)$, where, by slight abuse of notation, ϕ now denotes the restriction of ϕ to $\Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r}$.

Since G is finitely presented, we apply Theorem 4.4 as in the proof of Corollary 1.3 to show that the kernels of the projections of $\ker(\phi)$ to factors are finitely generated. Let X be a compact Kähler manifold with $G = \pi_1(X)$. Then Proposition 4.1 implies that ϕ is induced by a holomorphic map $f\colon X \to S_{\gamma_1} \times \cdots \times S_{\gamma_s} \times S_{g_{s+1}} \times \cdots \times S_{g_r}$. Hence, all conditions of Theorem 6.2 are satisfied and we obtain (2). □

References

[Agol 2013] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 MR Zbl

[Amorós et al. 1996] **J Amorós**, **M Burger**, **K Corlette**, **D Kotschick**, **D Toledo**, *Fundamental groups of compact Kähler manifolds*, Math. Surv. Monogr. 44, Amer. Math. Soc., Providence, RI (1996) MR Zbl

[Beauville 1988] **A Beauville**, letter to F Catanese (1988) MR Zbl Appendix to F Catanese, *Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations*, Invent. Math. 104 (1991) 263–289

[Biswas and Mj 2017] **I Biswas**, **M Mj**, *A survey of low dimensional (quasi) projective groups*, from "Analytic and algebraic geometry" (A Aryasomayajula, I Biswas, A S Morye, A J Parameswaran, editors), Hindustan, New Delhi (2017) 49–65 MR Zbl

[Biswas et al. 2014] **I Biswas**, **M Mj**, **D Pancholi**, *Homotopical height*, Int. J. Math. 25 (2014) art. id. 1450123 MR Zbl

[Bridson and Miller 2009] **M R Bridson**, **C F Miller, III**, *Structure and finiteness properties of subdirect products of groups*, Proc. Lond. Math. Soc. 98 (2009) 631–651 MR Zbl

[Bridson et al. 2009] **M R Bridson**, **J Howie**, **C F Miller, III**, **H Short**, *Subgroups of direct products of limit groups*, Ann. of Math. 170 (2009) 1447–1467 MR Zbl

[Bridson et al. 2013] **M R Bridson**, **J Howie**, **C F Miller, III**, **H Short**, *On the finite presentation of subdirect products and the nature of residually free groups*, Amer. J. Math. 135 (2013) 891–933 MR Zbl

[Burger 2011] **M Burger**, *Fundamental groups of Kähler manifolds and geometric group theory*, from "Séminaire Bourbaki 2009/2010", Astérisque 339, Soc. Math. France, Paris (2011) Exposé 1022, 305–321 MR Zbl

[Catanese 2003] **F Catanese**, *Fibred Kähler and quasi-projective groups*, Adv. Geom. 2003 (2003) S13–S27 MR Zbl

[Catanese 2008] **F Catanese**, *Differentiable and deformation type of algebraic surfaces, real and symplectic structures*, from "Symplectic 4–manifolds and algebraic surfaces" (F Catanese, G Tian, editors), Lecture Notes in Math. 1938, Springer (2008) 55–167 MR Zbl

[Catanese 2017] **F Catanese**, *Kodaira fibrations and beyond: methods for moduli theory*, Jpn. J. Math. 12 (2017) 91–174 MR Zbl

[Delzant 2016] **T Delzant**, *Kähler groups, $\mathbb{R}$–trees, and holomorphic families of Riemann surfaces*, Geom. Funct. Anal. 26 (2016) 160–187 MR Zbl

[Delzant and Gromov 2005] **T Delzant**, **M Gromov**, *Cuts in Kähler groups*, from "Infinite groups: geometric, combinatorial and dynamical aspects" (L Bartholdi, T Ceccherini-Silberstein, T Smirnova-Nagnibeda, A Zuk, editors), Progr. Math. 248, Birkhäuser, Basel (2005) 31–55 MR Zbl

[Delzant and Py 2019] **T Delzant**, **P Py**, *Cubulable Kähler groups*, Geom. Topol. 23 (2019) 2125–2164 MR Zbl

[Dimca et al. 2009] **A Dimca**, **Ş Papadima**, **A I Suciu**, *Non-finiteness properties of fundamental groups of smooth projective varieties*, J. Reine Angew. Math. 629 (2009) 89–105 MR Zbl

[Kochloukova 2010] **D H Kochloukova**, *On subdirect products of type* FP_m *of limit groups*, J. Group Theory 13 (2010) 1–19 MR Zbl

[Kotschick 2012] **D Kotschick**, *The deficiencies of Kähler groups*, J. Topol. 5 (2012) 639–650 MR Zbl

[Kuckuck 2014] **B Kuckuck**, *Subdirect products of groups and the* n–$(n+1)$–$(n+2)$ *conjecture*, Q. J. Math. 65 (2014) 1293–1318 MR Zbl

[Llosa Isenrich 2019] **C Llosa Isenrich**, *Branched covers of elliptic curves and Kähler groups with exotic finiteness properties*, Ann. Inst. Fourier (Grenoble) 69 (2019) 335–363 MR Zbl

[Llosa Isenrich 2020] **C Llosa Isenrich**, *Kähler groups and subdirect products of surface groups*, Geom. Topol. 24 (2020) 971–1017 MR Zbl

[Llosa Isenrich and Py 2021] **C Llosa Isenrich**, **P Py**, *Mapping class groups, multiple Kodaira fibrations, and* CAT(0) *spaces*, Math. Ann. 380 (2021) 449–485 MR Zbl

[Milnor 1968] **J Milnor**, *Singular points of complex hypersurfaces*, Ann. of Math. Stud. 61, Princeton Univ. Press (1968) MR Zbl

[Py 2013] **P Py**, *Coxeter groups and Kähler groups*, Math. Proc. Cambridge Philos. Soc. 155 (2013) 557–566 MR Zbl

[Salter 2015] **N Salter**, *Surface bundles over surfaces with arbitrarily many fiberings*, Geom. Topol. 19 (2015) 2901–2923 MR Zbl

[Siu 1987] **Y T Siu**, *Strong rigidity for Kähler manifolds and the construction of bounded holomorphic functions*, from "Discrete groups in geometry and analysis" (R Howe, editor), Progr. Math. 67, Birkhäuser, Boston, MA (1987) 124–151 MR Zbl

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The $K(\pi,1)$ conjecture and acylindrical hyperbolicity for relatively extra-large Artin groups

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Let A_Γ be an Artin group with defining graph Γ. We introduce the notion of A_Γ being extra-large relative to a family of arbitrary parabolic subgroups. This generalizes a related notion of A_Γ being extra-large relative to two parabolic subgroups, one of which is always large type. Under this new condition, we show that A_Γ satisfies the $K(\pi,1)$ conjecture whenever each of the distinguished subgroups do. In addition, we show that A_Γ is acylindrically hyperbolic under only mild conditions.

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Let Γ be a finite simplicial graph whose edges are labeled with (finite) integers, each at least 2. For vertices s, t of Γ connected by an edge, let $m(s,t)$ denote the label of the edge between s and t. Let $S = \mathrm{Vert}(\Gamma)$. Since Γ is simplicial, we use the convention that an edge of Γ is the same as an unordered pair $\{s,t\}$ of vertices of Γ. The Artin group defined by Γ is

$$A_\Gamma = \langle S \mid \mathrm{prod}(s,t;m(s,t)) = \mathrm{prod}(t,s;m(s,t)) \text{ for } \{s,t\} \text{ an edge of } \Gamma\rangle,$$

where $\mathrm{prod}(a,b;n)$ is the alternating word in a and b starting with a of length n (eg $aba\ldots$). We call the pair (A,S) an *Artin–Tits system*.

There is a Coxeter group also naturally associated with this defining graph; namely,

$$W_\Gamma = \langle S \mid (st)^{m(s,t)} = 1 \text{ for } \{s,t\} \text{ an edge of } \Gamma \text{ and } s^2 = 1 \text{ for } s \in S\rangle.$$

It is well known that there is a natural surjective homomorphism $A_\Gamma \to W_\Gamma$ induced by the identity map on S. Recall that, if W_Γ is finite, then we call W_Γ *spherical* and call A_Γ *spherical type*. In this case, we may sometimes refer to Γ itself as *spherical type*.

By [van der Lek 1983], if Γ' is a full (or "induced") subgraph of Γ, then the natural map from the Artin group $A_{\Gamma'}$ to A_Γ is an injection. (Recall that a subgraph Γ' of Γ is called *full* if, for any pair of vertices v, w of Γ' which span an edge $\{v,w\}$ in Γ, we also have that $\{v,w\}$ is an edge of Γ'.) We call such a subgroup of A_Γ a (*standard*) *parabolic subgroup*. Sometimes, if $T = \mathrm{Vert}(\Gamma')$, we write A_T for $A_{\Gamma'}$.

It is also well known that the Artin group A_Γ is the fundamental group of a space $N(W)$ which is the quotient of a complement of a certain complexified hyperplane arrangement by a natural W_Γ–action. (See [Paris 2014] for more details.) The long-standing $K(\pi,1)$ conjecture states that $N(W)$ is aspherical (ie has contractible universal cover). Currently, the $K(\pi,1)$ conjecture is known to be true when

(1) A_Γ is spherical type [Deligne 1972];

(2) A_Γ is affine type (meaning W_Γ has a finite-index subgroup which acts properly by isometries on a Euclidean space), proven in general in [Paolini and Salvetti 2021];

(3) if Γ' is a full spherical-type subgraph of Γ, then $|\mathrm{Vert}(\Gamma')| = 2$ (in which case A_Γ is called 2–dimensional) [Charney and Davis 1995], or, more generally, if A_Γ is locally reducible [Charney 2000],

(4) every full complete subgraph of Γ is spherical type (in which case A_Γ is called FC type) [Charney and Davis 1995]; and

(5) some combination criteria are satisfied, including results by Godelle and Paris [2012] and Ellis and Sköldberg [2010].

We present a new criterion based on the following familiar condition: an Artin group A_Γ is *extra-large type* if every edge of Γ has label at least 4. In this case, A_Γ is 2–dimensional, and thus satisfies the $K(\pi,1)$ conjecture. Juhász [2018] introduced the following condition: Let $H = A_{\Gamma'}$ be a standard parabolic subgroup of A (with $\Gamma' \subseteq \Gamma$ a full subgraph). Then A is *extra-large relative to H* (or Γ'*–relatively extra-large*) if

(1) for every edge $\{s,t\}$ of Γ with $s \in \Gamma'$ and $t \notin \Gamma'$, we have $m(s,t) \geq 4$; and

(2) for every edge $\{t,t'\}$ of Γ with $t,t' \notin \Gamma'$, we have $m(t,t') \geq 3$.

It is then shown that A_Γ satisfies the word problem or $K(\pi,1)$ conjecture whenever H does. It is in this spirit that we make the following generalization.

Let $\{\Gamma_i\}$ be a finite family consisting of disjoint, nonempty full subgraphs of Γ with vertex sets $S = \mathrm{Vert}(\Gamma)$ and $S_i = \mathrm{Vert}(\Gamma_i)$. Suppose also that $S = \bigcup S_i$. In direct analogy to the relatively extra-large condition, we consider:

(REL) Every edge of Γ between Γ_i and Γ_j for some $i \neq j$ has label at least 4.

If this condition is satisfied, we say that A_Γ is $\{\Gamma_i\}$*–relatively extra-large*. We establish the following theorem regarding such Artin groups:

Theorem *Suppose A_Γ is $\{\Gamma_i\}$–relatively extra-large. Then A_Γ satisfies the $K(\pi,1)$ conjecture if and only if each A_{Γ_i} does.*

In fact, a somewhat stronger fact can be established using our methods. Instead of (REL), consider:

(REL′) If e is an edge of Γ between Γ_i and Γ_j for some $i \neq j$ and e shares a vertex with a distinct edge between Γ_i and Γ_k for some $i \neq k$, then e has label at least 4.

Specifically, this allows edges which are isolated among those edges between the subgraphs in the family $\{\Gamma_i\}$ to have label 2 or 3. We show:

Theorem A *Suppose Γ and $\{\Gamma_i\}$ satisfy* (REL′)*. Then A_Γ satisfies the $K(\pi,1)$ conjecture if and only if each A_{Γ_i} does.*

In addition to this, we are able to show under mild hypotheses that Artin groups satisfying (REL′) are acylindrically hyperbolic. Acylindrical hyperbolicity is a property of interest for many groups, including Artin groups. Some of the classes for which acylindrical hyperbolicity is known for include

(1) right-angled Artin groups ($m(s,t)=2$ for each edge of Γ) which are not cyclic or a direct product of nontrivial subgroups [Osin 2016],

(2) spherical-type Artin groups [Calvez and Wiest 2017],

(3) type FC Artin groups whose defining graph has diameter at least 3 [Chatterji and Martin 2019],

(4) extra-extra-large type Artin groups (meaning $m(s,t)\geq 5$ for each edge $\{s,t\}$ of the defining graph) of rank at least 3 [Haettel 2022],

(5) Artin groups A_Γ such that Γ is not a join of two subgraphs Γ_1 and Γ_2 [Charney and Morris-Wright 2019],

(6) affine-type Artin groups [Calvez 2022],

(7) 2–dimensional Artin groups of hyperbolic type (meaning the associated Coxeter group is hyperbolic) [Martin and Przytycki 2022], and

(8) 2–dimensional Artin groups [Vaskou 2022].

We show acylindrical hyperbolicity in our setting as well:

Theorem B *Suppose A_Γ and $\{\Gamma_i\}_{i=1}^n$, $n\geq 2$ satisfy* (REL′). *In addition, assume $|\mathrm{Vert}(\Gamma)|\geq 3$ and not all edges between the family $\{\Gamma_i\}$ have label* 2. *Then A_Γ is acylindrically hyperbolic.*

We note that the conditions in Theorem A include the original relatively extra-large condition of Juhász as a special case. Suppose A_Γ is Γ'–relatively extra-large (in the sense of [Juhász 2018]). Let Γ'' be the full subgraph on the vertices of Γ which are not in Γ'. Then A_Γ is $\{\Gamma',\Gamma''\}$–relatively extra-large in our sense. The condition (2) in the definition of Γ–relatively extra-large is equivalent to requiring that $A_{\Gamma''}$ be large type (ie all edge labels are at least 3). Then $A_{\Gamma''}$ satisfies the $K(\pi,1)$ conjecture as $A_{\Gamma''}$ is 2–dimensional. Thus according to our result, A_Γ satisfies the $K(\pi,1)$ conjecture if and only if $A_{\Gamma'}$ does.

Our theorems include many new examples for which the $K(\pi,1)$ conjecture and/or acylindrical hyperbolicity was not previously known. As one example, consider two graphs Γ_1 and Γ_2 of type $\widetilde{C}_3$ (see Figure 1). These defining graphs generate an affine Artin group, and thus satisfy the $K(\pi,1)$ conjecture.

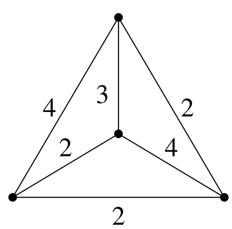

Figure 1: A defining graph of type $\widetilde{C}_3$.

Then let Γ be the join of Γ_1 and Γ_2 with each of the new edges labeled by 4 (or greater). It is quickly checked that A_Γ satisfies none of the previously listed conditions. But Γ and $\{\Gamma_1, \Gamma_2\}$ satisfy (REL$'$), and each A_{Γ_i} satisfies the $K(\pi, 1)$ conjecture, so A_Γ does. In addition, none of the edges between Γ_1 and Γ_2 are labeled 2, so A_Γ is acylindrically hyperbolic. More generally, if a clique Γ with at least three vertices is extra-large relative to a family $\{\Gamma_i\}$, then A_Γ is acylindrically hyperbolic, and, if each Γ_i satisfies the $K(\pi, 1)$ conjecture, then A_Γ does as well.

We also note that our methodology for proving Theorem A differs from Juhász's original work, allowing us to drop his condition (2) and treat more general defining graphs. This also allows us to easily prove acylindrical hyperbolicity. We hope that this method may be adapted for other similar restrictions on A_Γ. Namely, our strategy for proving Theorem A is as follows. In Section 1, we construct a simplicial complex as a variation of the usual Deligne complex. We show the complex is CAT(0), and hence contractible, in Section 2. Then, in Section 3, we show that this complex is homotopy equivalent to the universal cover of $N(W)$ by a result of Godelle and Paris [2012]. In Section 4, we prove Theorem B using recent results of Vaskou [2022].

We would also like to note that the conditions (REL) and (REL$'$) can be naturally relaxed to allow edges with label at least 3, which would define a *relatively large type* condition. This case is also currently of interest to the author; however, it is somewhat more complicated.

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1 The Deligne-like complex

Before we define our complex, we wish to establish a lemma in Artin groups similar to a well-known property of cosets of standard parabolic subgroups of Coxeter groups. We include a proof for the reader's convenience. We make heavy use of this result in the subsequent sections.

Lemma 1.1 *Suppose (A, S) is an Artin–Tits system, $\alpha, \alpha' \in A$ and $T, T' \subseteq S$. Then, if $\alpha A_T \subseteq \alpha' A_{T'}$, we have $\alpha^{-1}\alpha' \in A_{T'}$ and $T \subseteq T'$.*

Proof Let w and w' be the image of α and α', respectively, under the quotient homomorphism $A_\Gamma \to W_\Gamma$. The inclusion $\alpha A_T \subseteq \alpha' A_{T'}$ is preserved under the quotient map, giving us the relation $wW_T \subseteq w'W_{T'}$ in W_Γ. So, by [Bourbaki 2002, Chapter IV, Section 8, Theorem 2(iii)], $T \subseteq T'$ as subsets of W_Γ. Since the quotient map is bijective on the generators, this gives $T \subseteq T'$ viewed in A_Γ.

To see that α and α' must be in the same $A_{T'}$–coset, note that $\alpha A_T \subseteq \alpha A_{T'}$ as well as $\alpha A_T \subseteq \alpha' A_{T'}$, so $\varnothing \neq \alpha A_T \subseteq \alpha A_{T'} \cap \alpha' A_{T'}$. Since cosets partition the group and these cosets have nonempty intersection, they must be the same. □

We also briefly give a restatement of a result of van der Lek.

Lemma 1.2 *If (A, S) is an Artin–Tits system and $s \in S$, then s cannot be written as a product of the elements of $S \setminus \{s\}$.*

Proof By [van der Lek 1983],

$$A_{\{s\}} \cap A_{S\setminus\{s\}} \cong A_{\{s\}\cap S\setminus\{s\}} = A_{\varnothing} = 1.$$

Thus, in particular, $s \notin A_{S\setminus\{s\}}$. Since $A_{S\setminus\{s\}}$ is the collection of all possible products of the generators $S \setminus \{s\}$, the result follows. □

1.1 Definition of the complex

Through the rest of the paper, we let $A = A_\Gamma$ be an Artin group such that Γ and $\{\Gamma_i\}$ satisfy (REL′), with $S_i = \mathrm{Vert}(\Gamma_i)$ and $A_i = A_{\Gamma_i}$.

We now introduce a simplicial complex based on our distinguished subgroups A_i of A analogous to the Deligne complex. To do this, we mimic the construction of the Deligne complex in [Charney and Davis 1995], but replace the poset of spherical generating sets with the following set:

Definition 1.3 Let $\mathcal{S}^\ell$ be the set of all $T \subseteq S$ satisfying either

(1) $T = \varnothing$ (in which case $A_T = 1$, the trivial subgroup of A),

(2) $T = S_i$,

(3) $T = \{s_i, s_j\}$ for vertices $s_i \in S_i$ and $s_j \in S_j$ of an edge between Γ_i and Γ_j with $i \neq j$, or

(4) $T = \{s\}$ for a vertex s of an edge between Γ_i and Γ_j with $i \neq j$.

With this, we define

$$A\mathcal{S}^\ell = \{\alpha A_T : \alpha \in A,\ T \in \mathcal{S}^\ell\},$$

and order these sets by inclusion. We then let X denote the geometric realization of the derived complex of $\mathcal{S}^\ell$ and $\widehat{\Phi}$ denote the geometric realization of the derived complex of $A\mathcal{S}^\ell$ (recall that the derived complex of a poset is the set of chains in the poset ordered by inclusion of chains).

We will denote an n–simplex of $\widehat{\Phi}$ by

$$[\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \dots, \alpha_n A_{T_n}],$$

where $\alpha_0 A_{T_0} < \alpha_1 A_{T_1} < \cdots < \alpha_n A_{T_n}$ is a chain in $A\mathcal{S}^\ell$. We use similar notation for simplices of X. Notice that $\widehat{\Phi}$ inherits a natural left action of A with fundamental domain isomorphic to X via the simplicial map induced by the set map $T \mapsto A_T$.

We note that, if one replaces $\mathcal{S}^\ell$ by $\mathcal{S}^f$, the set of $T \subseteq S$ such that A_T is spherical type, then the definition of the (modified) Deligne complex of [Charney and Davis 1995] is recovered. To further borrow their notation, we will let K denote the geometric realization of the derived complex of $\mathcal{S}^f$, let $A\mathcal{S}^f$ denote the cosets of A_T for $T \in \mathcal{S}^f$, and let $\Phi_M = \Phi_M(A_\Gamma)$ denote the geometric realization of the derived complex of $A\mathcal{S}^f$.

The rest of this section and the next is dedicated to showing that $\widehat{\Phi}$ is CAT(0). First we show that $\widehat{\Phi}$ is simply connected, then endow it with a metric of nonpositive curvature.

To show that $\widehat{\Phi}$ is simply connected, we will use basic facts about complexes of groups. We will only need the fact that the action of A_Γ on $\widehat{\Phi}$ has a complex of groups structure briefly, so we will summarize the basic argument here, and refer the reader to [Haefliger 1992] for more details on complexes of groups.

Lemma 1.4 *The complex $\widehat{\Phi}$ is simply connected.*

Proof The stabilizer of a vertex $[\alpha A_T]$ of $\widehat{\Phi}$ is the subgroup $\alpha A_T \alpha^{-1}$ of A. Thus, A acts on $\widehat{\Phi}$ without inversion. The complex X is homeomorphic to the quotient $\widehat{\Phi}/A$ via the simplicial map induced by $T \mapsto A_T$. In addition, X is simply connected, as $[\varnothing]$ is a cone point in X. This information determines a complex of groups [Haefliger 1992, Section 2.1], which we denote by $A(X)$. The edge maps are the usual inclusion maps $A_T \hookrightarrow A_{T'}$. Note that this complex is developable by definition.

Since X is simply connected, $\pi_1(A(X))$ is the colimit of the groups A_T along the inclusion maps [Haefliger 1992, Section 2.7], implying $\pi_1(A(X)) = A$. It follows that the classifying space of $A(X)$ is $BA(X) = \widehat{\Phi} \times_A EA$ [Haefliger 1992, Proposition 3.2.3], and thus the universal cover is

$$\widetilde{BA}(X) = \widehat{\Phi} \times EA,$$

which is homotopy equivalent to $\widehat{\Phi}$. This shows that $\widehat{\Phi}$ is simply connected. □

1.2 The metric on $\widehat{\Phi}$

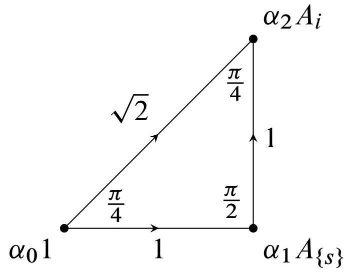
In order to put a metric on $\widehat{\Phi}$, we first note the following:

Lemma 1.5 *The complex $\widehat{\Phi}$ is 2–dimensional.*

Proof Suppose we have a 3–simplex $[\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \alpha_2 A_{T_2}, \alpha_3 A_{T_3}]$ of $\widehat{\Phi}$. By Lemma 1.1, we then have a chain $T_0 < T_1 < T_2 < T_3$. In particular, $|T_2| \geq 2$. The only sets of $\mathcal{S}^\ell$ with cardinality at least 2 are either S_i for some i or an edge $\{s_i, s_j\}$. But, in either case, there is no element of $\mathcal{S}^\ell$ containing T_2, a contradiction. □

As a consequence of the proof of the lemma, there are only two kinds of top-dimensional simplices of $\widehat{\Phi}$: the first is $[\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$ for a vertex $s \in S_i$ of an edge between Γ_i and some Γ_j, and the second is $[\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i, s_j\}}]$ for $\{s_i, s_j\}$ an edge between Γ_i and Γ_j.

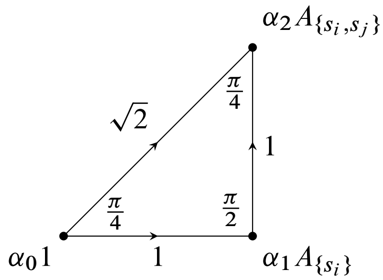
We now put a metric on the two kinds of 2–simplices of $\widehat{\Phi}$. First consider $[\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$. We give this simplex the metric of a Euclidean isosceles right triangle with right angle at $\alpha_1 A_{\{s\}}$ and whose legs have length 1. Pictorially, we have



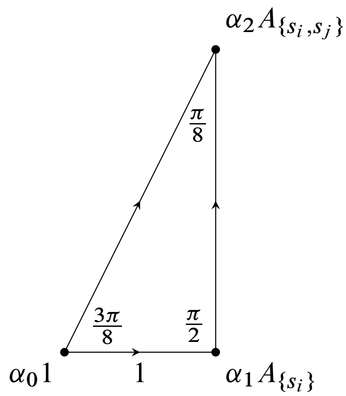
The arrows here denote the inclusion of the relevant groups.

Now consider a simplex of the form $\Delta = [\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i, s_j\}}]$ for $e = \{s_i, s_j\}$ an edge between Γ_i and Γ_j.

1.2.1 Case 1: a disjoint edge Suppose that e is disjoint from all other edges between any Γ_k and Γ_ℓ. We then put a similar metric on Δ as in the previous case, namely



1.2.2 Case 2: a nondisjoint edge Now suppose that e shares a vertex with some other edge between Γ_i and Γ_j. Then we still put the metric of a Euclidean right triangle on Δ, but it will no longer be isosceles. Specifically, the metric we put on Δ still assigns a right angle to the vertex $\alpha_1 A_{\{s_i\}}$, but now places an angle of $\frac{3\pi}{8}$ to $\alpha_0 1$ and an angle of $\frac{\pi}{8}$ to $\alpha_2 A_{\{s_i, s_j\}}$. Moreover, importantly, the 1–simplex $[\alpha_0 1, \alpha_1 A_{\{s_i\}}]$ is given length 1. The diagram for this case is



In order to show this properly defines a piecewise Euclidean metric on $\widehat{\Phi}$, we examine the gluings between adjacent simplices. We begin with a simplex of the form $\Delta = [\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$. The only type of simplex Δ can be adjacent to which is not of the same type is one of the form $\Delta' = [\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2' A_{\{s,t\}}]$ with $\{s,t\}$ an edge between Γ_i and Γ_j and t a vertex of Γ_j. These simplices are glued only along the edge $[\alpha_0 1, \alpha_1 A_{\{s\}}]$, and within both simplices we have assigned this edge a length of 1.

Now consider $\Delta = [\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i,s_j\}}]$ for $\{s_i, s_j\}$ an edge between Γ_i and Γ_j. The case where Δ is adjacent to a simplex of the form $[\alpha_0 1, \alpha_1 A_{\{s\}}, \alpha_2 A_i]$ was covered above. So consider an adjacent simplex of the form $[\alpha_0' 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i,s_j\}}]$ or $[\alpha_0 1, \alpha_1' A_{\{s_i'\}}, \alpha_2 A_{\{s_i,s_j\}}]$. In either case, the metric put on the simplices is the same as that of Δ as this metric only depended on the edge $\{s_i, s_j\}$, so there is no issue with the gluing.

It remains to check the simplices of the form $\Delta' = [\alpha_0 1, \alpha_1 A_{\{s_i\}}, \alpha_2 A_{\{s_i',s_k\}}]$ for an edge $\{s_i', s_j'\}$ and $s_k \in \Gamma_k$ for some $k \neq i$. By Lemma 1.1, since $\alpha_1 A_{\{s_i\}} \subseteq \alpha_2 A_{\{s_i',s_k\}}$, we have $\{s_i\} \subseteq \{s_i', s_k\}$, and since $s_k \in \Gamma_k$ we must have $s_i' = s_i$. Thus, if this is to be a simplex distinct from Δ, we must have $s_k \neq s_j$, so $\{s_i, s_k\}$ and $\{s_i, s_j\}$ are both edges which are not distinct. Thus, the metrics on Δ and Δ' are the same, so they may be glued as required.

2 Links

The purpose of this section is to show the following:

Proposition 2.1 *The complex* $\widehat{\Phi}$ (*with the above metric*) *is* CAT(0) (*and hence contractible*).

To do this, we compute the link at each relevant vertex of $\widehat{\Phi}$ and show that the link condition is satisfied. Let us briefly recall the relevant definitions. (For more details, see [Bridson and Haefliger 1999].)

Definition 2.2 (link of a vertex) Let K be a polyhedral complex and v a vertex of K. Then the link of v in K, denoted by $\mathrm{lk}_K(v)$, is the ε–sphere of K centered at v. We give the link a cell structure coming from the intersection of the sphere with the cell structure of K. The link is endowed with a natural spherical metric inherited from the ε–sphere.

In the case of the geometric realization of an abstract simplicial complex (such as $\widehat{\Phi}$), we can give an explicit description of the link of a vertex using the underlying set. Let $[\alpha A_T]$ be a vertex of $\widehat{\Phi}$ (so $\alpha A_T \in A\mathcal{S}^{\ell}$). Then the vertex set of $\mathrm{lk}_{\widehat{\Phi}}([\alpha A_T])$ is

$$\{\alpha' A_{T'} : \alpha' A_{T'} \subseteq \alpha A_T\} \cup \{\alpha'' A_{T''} : \alpha'' A_{T''} \supseteq \alpha A_T\}.$$

But, by Lemma 1.1, this is the same as the set

$$\{\alpha' A_{T'} : \alpha' A_{T'} \subseteq \alpha A_T\} \cup \{\alpha A_{T''} : T'' \supseteq T\}.$$

A collection of vertices $\alpha_0 A_{T_0} < \cdots < \alpha_j A_{T_j} < \alpha A_{T'_0} < \cdots < \alpha A_{T'_k}$ spans a $(j+k)$–simplex of $\mathrm{lk}_{\widehat{\Phi}}([\alpha A_T])$ if and only if

$$[\alpha_0 A_{T_0} < \cdots < \alpha_j A_{T_j} < \alpha A_T < \alpha A_{T'_0} < \cdots < \alpha A_{T'_k}]$$

is a $(j+k+1)$–simplex of $\widehat{\Phi}$. In the case of $\widehat{\Phi}$, we can say slightly more than this. Our complex $\widehat{\Phi}$ is 2–dimensional, so the link of any vertex is 1–dimensional. Moreover, the link of a simplicial complex is itself a simplicial complex, so the link here is always a simplicial graph.

We can also explicitly describe the spherical metric on each link in $\widehat{\Phi}$. If $[\alpha A_T]$ is a vertex of $\widehat{\Phi}$ and $e = [\alpha_0 A_{T_0}, \alpha_1 A_{T_1}]$ is an edge of $\mathrm{lk}_{\widehat{\Phi}}([\alpha A_T])$, then the length of e is the angle assigned above to the vertex corresponding to αA_T in the simplex of $\widehat{\Phi}$ spanned by the vertices αA_T, $\alpha_0 A_{T_0}$ and $\alpha_1 A_{T_1}$.

Definition 2.3 We say that a polyhedral complex K satisfies the *link condition* if, for each vertex v of K, the link $\mathrm{lk}_K(v)$ is a $\mathrm{CAT}(1)$ space (under the induced spherical metric).

To show $\widehat{\Phi}$ is $\mathrm{CAT}(0)$, we make use of the following criterion, proven in [Bridson and Haefliger 1999]:

Lemma 2.4 *If K is a Euclidean polyhedral complex* (*meaning each cell of K has the metric of a Euclidean polytope*) *and K is simply connected, then K is* $\mathrm{CAT}(0)$ *if and only if it satisfies the link condition.*

Since our complex $\widehat{\Phi}$ is 2–dimensional, to verify our links are $\mathrm{CAT}(1)$, we can use the following equivalent condition, also proven in [Bridson and Haefliger 1999]:

Lemma 2.5 *A 2–dimensional Euclidean simplicial complex K satisfies the link condition if and only if, for each vertex v of K, every embedded closed loop in $\mathrm{lk}_K(v)$ has length at least 2π.*

We now turn to examining the links of our complex in detail. Since each vertex of $\widehat{\Phi}$ is a translate of one of the cosets A_T, it suffices to just compute the link at A_T for $T \in \mathcal{S}^\ell$.

2.1 Case 1: $T = S_i$

Let us first examine the link of A_i for fixed i. The vertex set of this link can be decomposed as

$$\{\alpha 1 : \alpha \in A_i\} \quad \text{and} \quad \{\alpha A_{\{s\}} : \alpha \in A_i,\, s \in S_i\}.$$

It is easily seen that there is no edge between any two vertices which are in the same set, meaning the link is a bipartite graph. By definition, we can only have an edge when $\alpha 1 \subseteq \alpha' A_{\{s\}}$, or, in other words, when $\alpha \in \alpha' A_{\{s\}}$.

To show that the shortest embedded closed loop in A_i has length at least 2π, we claim that any embedded closed loop in $\mathrm{lk}_{\widehat{\Phi}}(A_i)$ must have at least eight edges. Since the link is a bipartite graph, we know the edge length of any cycle is even and at least 4. So we only need to verify that there are no cycles of edge length 4 or 6.

Suppose we have a loop with four edges. Then, by our discussion regarding the possible edges in the link, this loop must have the form

$$\begin{tikzcd} & \alpha_1 1 \arrow[dl] \arrow[dr] & \\ \alpha' A_{\{s'\}} & & \alpha'' A_{\{s''\}} \\ & \alpha_2 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

(The arrows correspond to inclusions; the paths we consider are not directed.) This gives us equations of the form

$$\alpha'(s')^{k_1} = \alpha_1 = \alpha''(s'')^{j_1} \quad \text{and} \quad \alpha'(s')^{k_2} = \alpha_2 = \alpha''(s'')^{j_2}.$$

Or, rewriting,

$$(s'')^{j_1}(s')^{-k_1} = (\alpha'')^{-1}\alpha' = (s'')^{j_2}(s')^{-k_2},$$

implying

$$(s'')^{j_1 - j_2} = (s')^{k_1 - k_2}.$$

Since we're assuming the loop is embedded, $s' \neq s''$ (otherwise two cosets of the same subgroup $A_{s'} = A_{s''}$ would intersect nontrivially, and thus be the same), and $\alpha_1 \neq \alpha_2$, so $k_1 \neq k_2$ and $j_1 \neq j_2$. However, these are distinct generators, so this cannot happen by Lemma 1.2. Thus, this loop is not embedded.

Now suppose we have a loop with six edges. This loop has the form

$$\begin{tikzcd} & & \beta_1 1 \arrow[dl] \arrow[dr] & & \\ & \alpha_1 A_{\{s_1\}} & & \alpha_3 A_{\{s_3\}} & \\ \beta_2 1 \arrow[ur] \arrow[rr] & & \alpha_2 A_{\{s_2\}} & & \beta_3 1 \arrow[ul] \arrow[ll] \end{tikzcd}$$

Since the loop is embedded, each β_i is distinct and at most one of the β_i can be the identity, so assume $\beta_1 \neq 1$ and $\beta_2 \neq 1$. Then, since $\beta_1 \neq 1$, we must have $s_1 \neq s_3$ (as before, if we did have $s_1 = s_3$, then the cosets $\alpha_1 A_{\{s_1\}}$ and $\alpha_3 A_{\{s_3\}}$ would be cosets of the same subgroup $A_{\{s_1\}} = A_{\{s_3\}}$ which intersect nontrivially, and thus would be the same coset). Similarly, $s_1 \neq s_2$.

From our diagram, we see that

$$\alpha_1 s_1^{k_1} = \beta_1 = \alpha_3 s_3^{k_3}, \quad \alpha_2 s_2^{j_2} = \beta_2 = \alpha_1 s_1^{j_1}, \quad \alpha_3 s_3^{\ell_3} = \beta_3 = \alpha_2 s_2^{\ell_2}$$

for some $k_i, j_i, \ell_i \in \mathbb{Z}$. Then

$$s_1^{j_1 - k_1} = s_1^{-k_1} s_1^{j_1} = (\alpha_1^{-1}\beta_1)^{-1}(\alpha_1^{-1}\beta_2) = \beta_1^{-1}\beta_2,$$

and, similarly,

$$s_2^{\ell_2 - j_2} = \beta_2^{-1}\beta_3, \quad s_3^{k_3 - \ell_3} = \beta_3^{-1}\beta_1.$$

Note that, since the β_i are distinct, none of these exponents are zero. But

$$s_1^{j_1-k_1} = \beta_1^{-1}\beta_2 = \beta_1^{-1}(\beta_3\beta_3^{-1})\beta_2 = (\beta_3^{-1}\beta_1)^{-1}(\beta_2^{-1}\beta_3)^{-1} = s_3^{\ell_3-k_3}s_2^{j_2-\ell_2}.$$

This means $s_1^{j_1-k_1} \in A_{\{s_2,s_3\}}$, and so $A_{\{s_1\}} \cap A_{\{s_2,s_3\}} \neq 1$ since $j_1 - k_1 \neq 0$. But then, by [van der Lek 1983], this would mean $\{s_1\} \cap \{s_2, s_3\} \neq \varnothing$, a contradiction. Thus, this loop cannot be embedded.

Therefore, each embedded loop in $\mathrm{lk}_{\widehat{\Phi}}(A_i)$ has at least eight edges. The spherical metric on the link assigns each of these edges a length of $\frac{\pi}{4}$, so the shortest possible length of an embedded loop is 2π.

2.2 Case 2: $T = \{s\}$

Now we look at the link of $A_{\{s\}}$ with $s \in \mathrm{Vert}(\Gamma_i)$ a vertex of an edge between Γ_i and Γ_j. In this case the link is again a bipartite graph: the vertices can be divided into the sets

$$\{\alpha 1 : \alpha \in A_{\{s\}}\} \quad \text{and} \quad \{A_i\} \cup \big\{A_{\{s,s_k\}} : \{s, s_k\} \text{ is an edge between } \Gamma_i \text{ and } \Gamma_k \text{ with } k \neq i\big\}.$$

So every embedded loop has at least four edges. The spherical metric on the link assigns a length of $\frac{\pi}{2}$ to each of these edges, implying the length of every embedded loop is at least 2π.

2.3 Case 3: $T = \{s_i, s_j\}$

The link of A_T for $T = \{s_i, s_j\}$ with $s_i \in S_i$ and $i \neq j$, is slightly different, as there are two cases to consider. However, in both cases the minimal number of edges in an embedded loop are the same.

Lemma 2.6 *If $T = \{s_i, s_j\}$ is an edge between Γ_i and Γ_j for $i \neq j$, then each embedded loop in $\mathrm{lk}_{\widehat{\Phi}}([A_T])$ has at least $4m(s_i, s_j)$ edges.*

Proof The link of A_T has vertex set which can be split into

$$\{\alpha 1 : \alpha \in A_T\} \quad \text{and} \quad \{\alpha A_{s_k} : \alpha \in A_T, k = i, j\},$$

on which the link is a bipartite graph. By applying the natural A_T–action on the link, we may consider only loops which contain the vertex 1. Namely, we may consider only loops of the form

$$\begin{tikzcd}
\alpha_1 A_{t_1} & \beta_1 \arrow[l] \arrow[r] & \alpha_2 A_{t_2} & \beta_2 \arrow[l] \arrow[d] \\
1 \arrow[u] \arrow[d] & & & \vdots \\
\alpha_n A_{t_n} & \beta_{n-1} \arrow[l] \arrow[r] & \alpha_n A_{t_{n-1}} & \beta_{n-2} \arrow[l] \arrow[u]
\end{tikzcd}$$

where each $\alpha_k \in A_T$ and each $t_k \in T$. This loop has $2n$ edges. Moreover, assuming this loop is embedded, this gives rise to a (reduced) word in s_i and s_j of syllable length at least n (see [Appel and Schupp 1983, Section 4] for the definition of syllable length) which is equal to the identity in A_T. By [Appel and Schupp 1983, Lemma 6], this means $n \geq 2m(s_i, s_j)$. Thus, this loop has at least $4m(s_i, s_j)$ edges. □

Now we can compute the length of these loops in each given link.

2.3.1 Case 3a: a disjoint edge If $\{s_i, s_j\}$ is disjoint from every other edge between the subgraphs in $\{\Gamma_k\}$, then the spherical metric on the link of A_T implies that the length of each edge here is $\frac{\pi}{4}$. So the length of any embedded loop is at least $(4m(s_1, s_2))\left(\frac{\pi}{4}\right) = \pi m(s_1, s_2)$. Since $m(s_1, s_2) \geq 2$, this loop has length at least 2π, as required.

2.3.2 Case 3b: a nondisjoint edge If this edge is not disjoint from every other edge between the subgraphs in $\{\Gamma_k\}$, the metric we have assigned implies that the length of each edge is $\frac{\pi}{8}$. So the length of any embedded loop is at least $(4m(s_1, s_2))\left(\frac{\pi}{8}\right) = \frac{\pi}{2} m(s_1, s_2)$. But in this case we have also assumed $m(s_1, s_2) \geq 4$, so the length of this loop is still at least 2π.

2.4 Case 4: $T = \varnothing$

It remains to check the link of the trivial coset 1. Note again that this link is bipartite, with a partition of the vertices given by

$$\{A_{\{s\}} : s \text{ a vertex of an edge between the subgraphs in } \{\Gamma_k\}\}$$

and

$$\{A_i\} \cup \left\{A_{\{s_i, s_j\}} : \{s_i, s_j\} \text{ an edge between } \Gamma_i \text{ and } \Gamma_j \text{ with } j \neq i\right\}.$$

We first verify that there are no embedded loops with four edges. Suppose we had such an embedded loop, say

$$\begin{tikzcd}
& A_{\{s\}} \arrow[dl] \arrow[dr] & \\
A_{T_1} & & A_{T_2} \\
& A_{\{s'\}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Since this loop is embedded, $s \neq s'$. Thus, by Lemma 1.1, both T_1 and T_2 contain $\{s, s'\}$. If s and s' are in the same vertex set S_i, then $T_1 = T_2 = S_i$ by our definition of $\mathcal{S}^\ell$. Similarly, if they are in distinct vertex sets, then both T_1 and T_2 must exactly be the edge $\{s, s'\}$. In either case, we have a contradiction.

It is entirely possible that we have embedded loops of length 6. Suppose

$$\begin{tikzcd}
& A_{\{s_1\}} \arrow[dl] \arrow[dr] & \\
A_{T_1} & & A_{T_3} \\
A_{\{s_2\}} \arrow[u] \arrow[dr] & & A_{\{s_3\}} \arrow[u] \arrow[dl] \\
& A_{T_2} &
\end{tikzcd}$$

is such a loop. If each pair $\{s_i, s_j\}$ is an edge between the family of subgraphs $\{\Gamma_i\}$, then the T_i must be the edges

$$T_1 = \{s_1, s_2\}, \quad T_2 = \{s_2, s_3\}, \quad T_3 = \{s_3, s_1\}$$

since these are the only sets of $\mathscr{S}^\ell$ which satisfy the containments implied by the diagram. But none of these edges are disjoint, so the metric we've put on $\widehat{\Phi}$ assigns the following edge lengths to this path:

$$\begin{tikzcd}
 & A_{\{s_1\}} \arrow[dl, "\frac{3\pi}{8}"'] \arrow[dr, "\frac{3\pi}{8}"] & \\
A_{T_1} & & A_{T_3} \\
A_{\{s_2\}} \arrow[u, "\frac{3\pi}{8}"] \arrow[dr, "\frac{3\pi}{8}"'] & & A_{\{s_3\}} \arrow[u, "\frac{3\pi}{8}"'] \arrow[dl, "\frac{3\pi}{8}"] \\
 & A_{T_2} &
\end{tikzcd}$$

And thus this loop has length at least 2π. Now suppose two of the vertices s_i are in the same vertex set and the other is in a distinct vertex set. Without loss of generality, we can take $s_1, s_2 \in S_i$ and $s_3 \in S_j$ with $i \neq j$. Then the only set $T_1 \in \mathscr{S}^\ell$ containing both s_1 and s_2 is $T_1 = S_i$, and so

$$T_1 = S_i, \quad T_2 = \{s_2, s_3\}, \quad T_3 = \{s_3, s_1\}.$$

The metric on $\widehat{\Phi}$ then assigns the edge lengths

$$\begin{tikzcd}
 & A_{\{s_1\}} \arrow[dl, "\frac{\pi}{4}"'] \arrow[dr, "\frac{3\pi}{8}"] & \\
A_{T_1} & & A_{T_3} \\
A_{\{s_2\}} \arrow[u, "\frac{\pi}{4}"] \arrow[dr, "\frac{3\pi}{8}"'] & & A_{\{s_3\}} \arrow[u, "\frac{3\pi}{8}"'] \arrow[dl, "\frac{3\pi}{8}"] \\
 & A_{T_2} &
\end{tikzcd}$$

which is still at least 2π total. We note that it is not possible to have $s_1, s_2, s_3 \in S_i$ for any i, since then $T_1 = T_2 = T_3 = S_i$, and this loop would not be embedded.

Finally, if we have a loop with eight edges in this link, then the length of each edge under our metric is at least $\frac{\pi}{4}$, and thus the length of this loop is at least 2π as well.

This concludes every possibility for T, so it follows that $\widehat{\Phi}$ satisfies the link condition by Lemma 2.5. By Lemma 1.4, $\widehat{\Phi}$ is simply connected, so, by Lemma 2.4, $\widehat{\Phi}$ is CAT(0) and thus contractible, as desired.

3 The $K(\pi,1)$ conjecture

In this section only, we assume that each A_{Γ_i} satisfies the $K(\pi,1)$ conjecture. In addition, we assume that A_Γ is not spherical type. (Since the $K(\pi,1)$ conjecture is known for spherical-type Artin groups, there is no loss of generality in making this assumption.)

We will use the following:

Definition 3.1 [Godelle and Paris 2012] Let (A, S) be an Artin–Tits system and let $\mathcal{S}$ be a family of subsets of S. Then $\mathcal{S}$ is *complete and* $K(\pi, 1)$ if the following are satisfied:

(1) If $T \in \mathcal{S}$ and $T' \subseteq T$, then $T' \in \mathcal{S}$.

(2) (A_T, T) satisfies the $K(\pi, 1)$ conjecture for each $T \in \mathcal{S}$.

(3) If A_T is spherical type, then $T \in \mathcal{S}$.

Then let
$$A\mathcal{S} = \{\alpha A_T : \alpha \in A,\ T \in \mathcal{S}\}$$
and let $\Phi(A, \mathcal{S})$ denote the geometric realization of the derived complex of $A\mathcal{S}$.

The relevant result for us is:

Theorem 3.2 [Godelle and Paris 2012, Theorem 3.1] *Let (A, S) be an Artin–Tits system and let $\mathcal{S}$ be a complete and $K(\pi, 1)$ family of subsets of S. Then $\Phi(A, \mathcal{S})$ has the same homotopy type as the universal cover of $N(W)$.*

Our family $\mathcal{S}^\ell$ is not itself complete and $K(\pi, 1)$, so we cannot directly apply this result. Instead we show that $\hat{\Phi}$ is homotopic to $\Phi := \Phi(A, \overline{\mathcal{S}})$ for a certain complete and $K(\pi, 1)$ collection $\overline{\mathcal{S}}$, which we define as follows: the sets of $\overline{\mathcal{S}}$ are the subsets of S consisting of the sets in $\mathcal{S}^\ell$ and every subset of S_i.

Lemma 3.3 *$\overline{\mathcal{S}}$ is a complete and $K(\pi, 1)$ family of subsets of S.*

Proof First we note that (1) and (2) are satisfied immediately by our definition of $\overline{\mathcal{S}}$ (to see (2), note that a standard parabolic subgroup satisfies the $K(\pi, 1)$ conjecture whenever the original group does by [Godelle and Paris 2012, Corollary 2.4]). It remains to show that $\overline{\mathcal{S}}$ contains all spherical-type generating sets.

Suppose Γ' is a full subgraph of Γ such that $A_{\Gamma'}$ is spherical type, and let $T = \mathrm{Vert}(\Gamma')$. If $T \subseteq S_i$, then we already have $T \in \overline{\mathcal{S}}$. So suppose there are $t_1, t_2 \in T$ with t_1 and t_2 in distinct vertex sets, say $t_1 \in S_i$ and $t_2 \in S_j$ with $i \neq j$.

If $T = \{t_1, t_2\}$, then, since we're assuming $A_{\Gamma'}$ is spherical type, $\{t_1, t_2\}$ is an edge of Γ, and thus $T \in \overline{\mathcal{S}}$. In other words, whenever $|T| = 2$ and $T \not\subseteq S_k$ for any k, we must have that T is an edge of Γ, so $T \in \overline{\mathcal{S}}$.

Suppose $|T| > 2$ and let $t_3 \in T$ be distinct from t_1 and t_2. If any of $\{t_1, t_2\}$, $\{t_2, t_3\}$ or $\{t_3, t_1\}$ were not an edge of Γ, then $A_{\Gamma'}$ would not be spherical, so each of these are edges. There are three cases to consider: t_3 is in either S_1, S_2 or neither. By symmetry, we may consider only the cases where $t_3 \in S_1$ and t_3 is in neither. In both of these cases, $\{t_1, t_2\}$ and $\{t_3, t_2\}$ are distinct nondisjoint edges between the family $\{\Gamma_i\}$, so, by the (REL') condition, both of their labels must be at least 4. By the classification of finite Coxeter groups, then, $A_{\Gamma'}$ is not spherical type. Thus, if $T \not\subseteq S_i$, we cannot have $|T| > 2$. □

Thus, Φ is homotopy equivalent to the universal cover of $N(W)$. It remains to show that:

Theorem 3.4 *There is a deformation retract from Φ to $\widehat{\Phi}$.*

Proof Note that there is a natural embedding of $\widehat{\Phi}$ into Φ induced by the inclusion of $\mathcal{S}^{\ell}$ in $\overline{\mathcal{S}}$.

We establish the deformation retract directly by describing the maps on each simplex. Let Δ be a maximal simplex of Φ (ie one which is not a face of any other simplex). There are two types of simplices to consider. The first is

$$\Delta = [\alpha_0 1, \alpha_0 A_{\{s\}}, \alpha_1 A_{\{s,t\}}]$$

for an edge $\{s,t\}$ between the family $\{\Gamma_i\}$. This is already a maximal simplex of $\widehat{\Phi}$, so we leave it unchanged. In the other case,

$$\Delta = [\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \dots, \alpha_{n-1} A_{T_{n-1}}, \alpha_n A_{S_i}]$$

for some S_i. Since Δ is maximal, $T_0 = \varnothing$ and $T_1 = \{s\}$ for some $s \in S_i$. There are two subcases to consider. If s is a vertex of an edge between Γ_i and some Γ_j, then there is a natural deformation retract from Δ to the simplex $[\alpha_0 A_{T_0}, \alpha_1 A_{T_1}, \alpha_n A_{S_i}]$ of $\widehat{\Phi}$. Otherwise, there is a natural deformation retract from Δ to the simplex $[\alpha_0 A_{T_0}, \alpha_n A_{S_i}]$ of $\widehat{\Phi}$. Moreover, these can easily be parametrized so that we can glue deformation retracts of adjacent maximal simplices to attain a deformation retract on the entire complex Φ. $\square$

We have therefore proven:

Theorem A *Suppose Γ and $\{\Gamma_i\}$ satisfy* (REL$'$). *Then A_Γ satisfies the $K(\pi,1)$ conjecture if and only if each A_{Γ_i} does.*

Proof First suppose A_Γ satisfies the $K(\pi,1)$ conjecture. Then, by [Godelle and Paris 2012, Corollary 2.4], each A_{Γ_i} also does.

Now suppose each A_{Γ_i} satisfies the $K(\pi,1)$ conjecture. Combining Theorem 3.2 and Lemma 3.3, Φ is homotopy equivalent to the universal cover of $N(W)$, and, by Theorem 3.4, $\widehat{\Phi}$ is homotopy equivalent to Φ. Thus, by Proposition 2.1, the universal cover of $N(W)$ is contractible. $\square$

4 Acylindrical hyperbolicity

We conclude by showing the following:

Theorem B *Suppose A_Γ and $\{\Gamma_i\}_{i=1}^n$, $n \geq 2$ satisfy* (REL$'$). *In addition, assume $|\mathrm{Vert}(\Gamma)| \geq 3$ and not all edges between the family $\{\Gamma_i\}$ have label 2. Then A_Γ is acylindrically hyperbolic.*

For the full definition of acylindrical hyperbolicity, we refer the reader to [Bowditch 2008].

First, if there are no edges between the family $\{\Gamma_i\}$, then A_Γ is a free product of the A_{Γ_i}, and thus is acylindrically hyperbolic by considering the action of A_Γ on its Bass–Serre tree. So we assume there is an edge between the family $\{\Gamma_i\}$, say $e = \{s_i, s_j\}$ with $s_i \in S_i$ and $s_j \in S_j$. By our assumptions on Γ, we may take e to have label at least 3. In this case, we make use of the following adaptation of a theorem from [Martin 2017]:

Theorem 4.1 [Martin 2017, Theorem B] *Let X be a $\mathrm{CAT}(0)$ simplicial complex and G a group acting on X by simplicial isomorphisms. Suppose there is a vertex v of X with stabilizer G_v satisfying*:

(1) *The orbits of G_v on the link $\mathrm{lk}_X(v)$ are unbounded in the associated spherical metric.*

(2) *G_v is weakly malnormal in G (ie there exists an element $g \in G$ such that $G_v \cap gG_vg^{-1}$ is finite).*

Then G is either virtually cyclic or acylindrically hyperbolic.

Remark 4.2 This is a strictly weaker statement than the one given in [Martin 2017], which allows X to be a polyhedral complex satisfying the "strong concatenation property". By [Martin 2017, Example 2.9 and Lemma 2.11], $\mathrm{CAT}(0)$ simplicial complexes always satisfy this property.

We use the action of A_Γ on our Deligne-like complex $\widehat{\Phi}$, which we have shown is $\mathrm{CAT}(0)$ for any Artin group satisfying (REL'). We claim that $v_e := [A_{\{s_i,s_j\}}]$ is a vertex of $\widehat{\Phi}$ which satisfies the conditions of Theorem 4.1.

Since we have assumed $|\mathrm{Vert}(\Gamma)| \geq 3$, there is at least one vertex s of Γ distinct from s_i and s_j. If there is no such s for which either $\{s, s_i\}$ or $\{s, s_j\}$ is an edge of Γ, then A_Γ is a free product and thus acylindrically hyperbolic by our previous remarks. In the other case, take s such that one of $\{s, s_i\}$ or $\{s, s_j\}$ is an edge of Γ, and define $\Delta = \{s, s_i, s_j\}$. Then the full subgraph of Γ on vertices Δ is connected and, by the (REL') condition, A_Δ is a 2–dimensional Artin group. Moreover, we have assumed $m(s_i, s_j) > 2$, so A_Δ is not a right-angled Artin group. Thus, we may use the following:

Theorem 4.3 [Vaskou 2022, Lemma 5.7] *Let A_Λ be a 2–dimensional Artin group of rank at least 3, and suppose that Λ is connected and A_Λ is not a right-angled Artin group. Then there exists an Artin subgroup $A_{\{a,b\}}$ with coefficient $3 \leq m(a,b) < \infty$ and an element $g \in A_\Lambda$ such that $A_{\{a,b\}} \cap gA_{\{a,b\}}g^{-1} = \{1\}$.*

Applying this to A_Δ, we have $a, b \in \Delta$ and $g \in A_\Delta$ such that $A_{\{a,b\}} \cap gA_{\{a,b\}}g^{-1} = \{1\}$. The proof of the theorem implies that we may take $\{a, b\} = \{s_i, s_j\}$. This shows that v_e satisfies (2). To show v_e satisfies (1), we use:

Theorem 4.4 [Vaskou 2022, Lemma 4.5] *Consider an Artin group $A_{\{a,b\}}$ with coefficient $3 \leq m(a,b) \leq \infty$. Then*

$$\{\ell_{\mathcal{S}}(g) : g \in A_{\{a,b\}}\}$$

is unbounded (where $\ell_{\mathcal{S}}(g)$ is the syllable length of g).

By the same analysis in the case of loops, if $\{a,b\}$ is an edge between the family $\{\Gamma_i\}$, then reduced words in a and b correspond to paths in $\mathrm{lk}_{\hat{\Phi}}([A_{\{a,b\}}])$, and vice versa. The edge length of such a path is at least the syllable length of the given word. So, since $m(s_i,s_j) \geq 3$, the action of $A_{\{s_i,s_j\}}$ on $\mathrm{lk}_{\hat{\Phi}}([A_{\{s_i,s_j\}}]) = \mathrm{lk}_{\hat{\Phi}}(v_e)$ is unbounded. Therefore, v_e also satisfies (1), and thus A_Γ is acylindrically hyperbolic.

References

[Appel and Schupp 1983] **K I Appel**, **P E Schupp**, *Artin groups and infinite Coxeter groups*, Invent. Math. 72 (1983) 201–220 MR Zbl

[Bourbaki 2002] **N Bourbaki**, *Lie groups and Lie algebras*, *Chapters* 4–6, Springer (2002) MR Zbl

[Bowditch 2008] **B H Bowditch**, *Tight geodesics in the curve complex*, Invent. Math. 171 (2008) 281–300 MR Zbl

[Bridson and Haefliger 1999] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR Zbl

[Calvez 2022] **M Calvez**, *Euclidean Artin–Tits groups are acylindrically hyperbolic*, Groups Geom. Dyn. 16 (2022) 963–983 MR Zbl

[Calvez and Wiest 2017] **M Calvez**, **B Wiest**, *Acylindrical hyperbolicity and Artin–Tits groups of spherical type*, Geom. Dedicata 191 (2017) 199–215 MR Zbl

[Charney 2000] **R Charney**, *The Tits conjecture for locally reducible Artin groups*, Int. J. Algebra Comput. 10 (2000) 783–797 MR Zbl

[Charney and Davis 1995] **R Charney**, **M W Davis**, *The $K(\pi,1)$–problem for hyperplane complements associated to infinite reflection groups*, J. Amer. Math. Soc. 8 (1995) 597–627 MR Zbl

[Charney and Morris-Wright 2019] **R Charney**, **R Morris-Wright**, *Artin groups of infinite type: trivial centers and acylindrical hyperbolicity*, Proc. Amer. Math. Soc. 147 (2019) 3675–3689 MR Zbl

[Chatterji and Martin 2019] **I Chatterji**, **A Martin**, *A note on the acylindrical hyperbolicity of groups acting on* CAT(0) *cube complexes*, from "Beyond hyperbolicity" (M Hagen, R Webb, H Wilton, editors), Lond. Math. Soc. Lect. Note Ser. 454, Cambridge Univ. Press (2019) 160–178 MR Zbl

[Deligne 1972] **P Deligne**, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972) 273–302 MR Zbl

[Ellis and Sköldberg 2010] **G Ellis**, **E Sköldberg**, *The $K(\pi,1)$ conjecture for a class of Artin groups*, Comment. Math. Helv. 85 (2010) 409–415 MR Zbl

[Godelle and Paris 2012] **E Godelle**, **L Paris**, *$K(\pi,1)$ and word problems for infinite type Artin–Tits groups, and applications to virtual braid groups*, Math. Z. 272 (2012) 1339–1364 MR Zbl

[Haefliger 1992] **A Haefliger**, *Extension of complexes of groups*, Ann. Inst. Fourier (Grenoble) 42 (1992) 275–311 MR Zbl

[Haettel 2022] **T Haettel**, *XXL type Artin groups are* CAT(0) *and acylindrically hyperbolic*, Ann. Inst. Fourier (Grenoble) 72 (2022) 2541–2555 MR Zbl

[Juhász 2018] **A Juhász**, *Relatively extra-large Artin groups*, Groups Geom. Dyn. 12 (2018) 1343–1370 MR Zbl [van der Lek 1983] **H van der Lek**, *The homotopy type of complex hyperplane complements*, PhD thesis, Katholieke Universiteit Nijmegen (1983) Available at https://hdl.handle.net/2066/148301

[Martin 2017] **A Martin**, *On the acylindrical hyperbolicity of the tame automorphism group of* $\mathrm{SL}_2(\mathbb{C})$, Bull. Lond. Math. Soc. 49 (2017) 881–894 MR Zbl

[Martin and Przytycki 2022] **A Martin**, **P Przytycki**, *Acylindrical actions for two-dimensional Artin groups of hyperbolic type*, Int. Math. Res. Not. 2022 (2022) 13099–13127 MR Zbl

[Osin 2016] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR Zbl

[Paolini and Salvetti 2021] **G Paolini**, **M Salvetti**, *Proof of the* $K(\pi,1)$ *conjecture for affine Artin groups*, Invent. Math. 224 (2021) 487–572 MR Zbl

[Paris 2014] **L Paris**, $K(\pi,1)$ *conjecture for Artin groups*, Ann. Fac. Sci. Toulouse Math. 23 (2014) 361–415 MR Zbl

[Vaskou 2022] **N Vaskou**, *Acylindrical hyperbolicity for Artin groups of dimension* 2, Geom. Dedicata 216 (2022) art. id. 7 MR Zbl

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The localization of orthogonal calculus with respect to homology

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For a set of maps of based spaces S we construct a version of Weiss's orthogonal calculus which depends only on the S–local homotopy type of the functor involved. We show that S–local homogeneous functors of degree n are equivalent to levelwise S–local spectra with an action of the orthogonal group $O(n)$ via a zigzag of Quillen equivalences between appropriate model categories. Our theory specialises to homological localizations and nullifications at a based space. We give a variety of applications including a reformulation of the telescope conjecture in terms of our local orthogonal calculus and a calculus version of Postnikov sections. Our results also apply when considering the orthogonal calculus for functors which take values in spectra.

55P60, 55P65; 55N20, 55P42

1 Introduction

1.1 Motivation

Weiss's orthogonal calculus [1995] studies functors from the category of real inner product spaces and isometries to the category of based spaces or spectra. The motivation for such a version of functor calculus comes from a desire to study geometric and differential topology through a homotopy-theoretic lens. For example, Arone, Lambrechts and Volić [Arone et al. 2007] and Arone [2009] utilised Weiss's calculus to provide a comprehensive study of the (stable) homotopy type of spaces of embeddings $\mathsf{Emb}(M, N \times \mathbb{R}^k)$ where M and N are fixed smooth manifolds. More recently, Krannich and Randal-Williams [2021] have studied the Weiss tower of the classifying space $\mathsf{BTOP}(\mathbb{R}^k)$ of the group of homeomorphisms of $\mathbb{R}^k$ to understand the homotopy type of the space of diffeomorphisms of discs. In all of these cases, the authors are only able to ascertain geometric information up to rational homotopy via ad-hoc means. These vastly varying approaches highlight the need for a comprehensive account of the interactions between orthogonal calculus and localizations.

The theory of localizations at homology theories are ubiquitous and have had wide applications; of particular note is *chromatic homotopy theory* which among other things gives a spectrum level interpretation for the periodic families appearing in the stable homotopy groups of spheres. An extensive amount of effort has been geared toward understanding how localization at homology theories — particularly the chromatic localizations — interact with Goodwillie's calculus of functors [Arone and Mahowald 1999;

Kuhn 2004; 2006a; 2006b]; see eg [Kuhn 2007] for a survey. Analogous questions remain in Weiss's orthogonal calculus, and we propose a noticeably different approach than those applied to the Goodwillie calculus.

Overview

Given a functor F from the category of Euclidean spaces to the category of based spaces or spectra, the calculus assigns a tower of functors

$$\begin{tikzcd} & & & F \arrow[dll, bend right] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[drr, bend left] & & \\ \cdots \arrow[r] & T_nF \arrow[r] & T_{n-1}F \arrow[r] & \cdots \arrow[r] & T_1F \arrow[r] & T_0F \end{tikzcd}$$

called the *Weiss tower* for F. The functor T_nF is a categorification of the n^{th} Taylor polynomial from differential calculus. The n^{th} layer of the tower D_nF is the homotopy fibre of the map $T_nF \to T_{n-1}F$, and is a categorification of homogeneous functions from differential calculus. Orthogonal calculus is synonymous with being the most computationally challenging flavour of functor calculus due to the interaction between the highly "geometric" nature of the objects of study and the highly homotopical constructions.

Let $\mathcal{C}$ denote either the category of based spaces or spectra. Given a set S of maps in $\mathcal{C}$ we produce an *S–local Weiss tower* of the form

$$\begin{tikzcd} & & & F \arrow[dll, bend right] \arrow[dl, bend right] \arrow[dr, bend left] \arrow[drr, bend left] & & \\ \cdots \arrow[r] & T_n^SF \arrow[r] & T_{n-1}^SF \arrow[r] & \cdots \arrow[r] & T_1^SF \arrow[r] & T_0^SF \\ & D_n^SF \arrow[u] & D_{n-1}^SF \arrow[u] & & D_1^SF \arrow[u] & \end{tikzcd}$$

To understand the S–local Weiss tower, we utilise Bousfield's [1975; 1979] interpretation of localizations in terms of model structures on $\mathcal{C}$. We begin by constructing a model structure, denoted by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C})$, which captures the homotopy theory of functors which are S–locally polynomial of degree less than or equal n. Under some assumptions on the set of localizing objects the composite T_nL_S is a fibrant replacement functor, hence satisfying the necessary universal property.

We further construct a model structure, denoted by $\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C})$, which captures the homotopy theory of functors which are S–locally homogeneous of degree n. Through a zigzag of Quillen equivalences we characterise the S–local n–homogeneous functors in terms of appropriately S–local spectra with an action of $O(n)$.

Theorem (Corollary 5.16) *Let S be a set of maps of based spaces and $n \geq 0$. There is a zigzag of Quillen equivalences*
$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)],$$
where $\mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)]$ is the category of levelwise S–local spectra with an action of $O(n)$.

Theorem (Corollary 5.18) *Let S be a set of maps of spectra and $n \geq 0$. There is a zigzag of Quillen equivalences*
$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Sp}) \simeq_Q L_S\mathsf{Sp}[O(n)],$$
where $L_S\mathsf{Sp}[O(n)]$ is the category of S–local spectra with an action of $O(n)$.

In particular, an S–local n–homogeneous functor F is determined by and determines an appropriately S–local spectrum with an $O(n)$–action, denoted by $\partial_n^S F$. On the derived level, we obtain a computationally accessible classification theorem for S–local homogeneous of degree n functors.

Theorem (Theorem 5.20) *Let S be a set of maps of in $\mathcal{C}$ and $n \geq 1$.*

(1) *A Top_*–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*
$$V \mapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)}],$$
and any functor of the above form is objectwise S–local and n–homogeneous.

(2) *An Sp–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*
$$V \mapsto (S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)},$$
and any functor of the above form is objectwise S–local and n–homogeneous.

Applications

We envision that the applications of this local version of orthogonal calculus are vast. For example, extending the rational computations of [Arone 2009; Arone et al. 2007; Krannich and Randal-Williams 2021] to higher chromatic height or another perspective on the full understanding of the Weiss tower of $\mathsf{BO}(-)$ in v_n–periodic homotopy theory achieved by Arone [2002] using computations of Arone and Mahowald [1999].

Very little of our results use the fact that the target category is based spaces or spectra. The largest hurdle to having a theory of localizations of orthogonal calculus with target any (simplicial cofibrantly generated) model category is the development of orthogonal calculus in this realm. We hope that our exposé of orthogonal calculus with target space a localization of spaces or spectra will motivate the construction of orthogonal calculus based on more general homotopy theories such as arbitrary model categories or ∞–categories.

In the last part of this paper, we give several initial applications, which we survey here.

Bousfield classes Bousfield [1979] introduced an equivalence relation on the stable homotopy category that has turned out to be of extreme importance. Define the Bousfield class $\langle E\rangle$ of a spectrum E to be the collection of E–acyclic spectra, and say that E and E' are Bousfield equivalent if and only if $\langle E\rangle = \langle E'\rangle$. These Bousfield classes assemble into a lattice, the understanding of which has been a major task in stable homotopy theory. For example, the nilpotence theorem of Devanitz, Hopkins and Smith [Devinatz et al. 1988; Hopkins and Smith 1998] is equivalent to a classification of the Bousfield classes for *finite spectra*. The Bousfield lattice has many interesting interactions with homological localizations of orthogonal calculus.

Theorem (Example 6.6) *Let E and E' be spectra. The E–local orthogonal calculus is equivalent to the E'–local orthogonal calculus if and only if E and E' are Bousfield equivalent.*

Fix a prime p. Ravenel's height n telescope conjecture [1984, Conjecture 10.5] is the statement that the height n Morava K–theory, $K(n)$, is Bousfield equivalent to $T(n)$, the telescope of any v_n–self map on a finite type n complex. The telescope conjecture is trivial at height $n=0$, has been verified at height $n=1$ and at all primes by Bousfield [1979], Mahowald [1981] and Miller [1981], but in general, is widely believed to be false.

Theorem (Corollary 6.9) *The height n telescope conjecture holds if and only if the $K(n)$–local orthogonal calculus and the $T(n)$–local orthogonal calculus are equivalent.*

The Weiss tower of a functor F produces a spectral sequence as it is a tower of fibrations. We call this spectral sequence the *Weiss spectral sequence*. From a computational perspective we obtain the following relation between the telescope conjecture and the local Weiss spectral sequences.

Theorem (Lemma 6.10) *If the height n telescope conjecture holds, then for all $r \geq 0$, the r^{th} page of the $T(n)$–local Weiss spectral sequence is isomorphic to the r^{th} page of the $K(n)$–local Weiss spectral sequence.*

Nullifications For functors from the category of Euclidean spaces to the category of based spaces we also consider localization at a based space W, which is sometimes referred to as *nullification*. In this setting W–local objects are also called W–periodic, following Bousfield [1994] and Dror Farjoun [1996].

We give alternative constructions for the n–polynomial and n–homogenous model structures when the localization is a nullification. These alternative constructions yield an identical n–polynomial model structure but sheds new light on some of the formal properties of the model structure, and yield an n–homogeneous model structure which is Quillen equivalent to the original W–local model structure via the identity functor. These alternative descriptions are particularly useful when considering Postnikov sections of orthogonal calculus.

The results obtained for nullifications do not hold for more general localizations as the techniques employed rely crucially on a right properness condition on the model categories. We show in Proposition 7.2 that the right proper condition is satisfied if and only if the localization is a nullification. This is an extension of a remark of Bousfield [2001].

Postnikov sections Considering nullifications with respect to the spheres produces a theory of Postnikov sections in orthogonal calculus. We prove that our S^{k+1}–local projective model structure on the category of functors from Euclidean spaces to based spaces is identical to the model structure of k–types in the category of functors from Euclidean spaces to based spaces in the sense of k–types in an arbitrary model category developed by Gutiérrez and Roitzheim [2017, Section 4].

Theorem (Proposition 8.2) *Let $k \geq 0$. The model structure of k–types in orthogonal functors is identical to the S^{k+1}–local model structure; that is, there is an equality of model structures,*

$$P_k \operatorname{Fun}(\mathcal{J}_0, \mathsf{Top}_*) := L_{W_k} \operatorname{Fun}(\mathcal{J}_0, \mathsf{Top}_*) = \operatorname{Fun}(\mathcal{J}_0, L_{S^{k+1}} \mathsf{Top}_*).$$

As an application we produce a tower of model categories

$$\cdots \to \mathsf{Homog}^n(\mathcal{J}_0, P_k \mathsf{Top}_*) \to \cdots \to \mathsf{Homog}^n(\mathcal{J}_0, P_0 \mathsf{Top}_*),$$

where $P_k \mathsf{Top}_*$ denotes the S^{k+1}–local model structure on based spaces. By applying the theory of homotopy limits of model categories, we show that the n–homogeneous model structure of Barnes and Oman [2013, Proposition 6.9] is the homotopy limit of this tower, in the following sense.

Theorem (Corollary 8.13) *There is a Quillen equivalence*

$$\mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \simeq_Q \operatorname*{holim}_k \mathsf{Homog}^n(\mathcal{J}_0, P_k \mathsf{Top}_*).$$

Relation to other work

This work is intimately related to the rational orthogonal calculus developed by Barnes [2017]; by replacing our generalised homology theory E_* with rational homology one recovers Barnes' theory.

Unstable chromatic homotopy theory can be described algebraically, via Heuts's [2021] *algebraic model* for v_n–periodic spaces via an equivalence (of ∞–categories) with Lie algebras in $T(n)$–local spectra. This model indicated that there is likely a relationship between v_n–periodic orthogonal calculus and orthogonal calculus of Heuts's Lie algebra models. Such an equivalence at chromatic height zero suggests a relationship between rational orthogonal calculus and the algebraic models for rational homotopy theory of Sullivan [1977] and Quillen [1969]. This together with Barnes' [2017] model for rational n–homogeneous functors using the classification of rational spectra with an $O(n)$–action as torsion modules over the rational cohomology ring of $\mathsf{BSO}(n)$ of Greenlees and Shipley [2014] suggests the existence of *algebraic model calculi*. We plan to return to this in future work.

This work also forms part of an extensive program to go "beyond orthogonal calculus" which was initiated in the PhD thesis of the author [Taggart 2020], together with a series of articles exploring extensions of the orthogonal calculus and the relations between these [Taggart 2021; 2022a; 2022b; 2023]. This extensive project hopes to illuminate our understanding of orthogonal calculus which (at least relative to Goodwillie calculus) remains largely unexplored.

The future applications of the homological localization of orthogonal calculus are abounding. For example in the recent work of Beaudry, Bobkova, Pham and Xu [Beaudry et al. 2022], the authors compute the tmf–homology of $\mathbb{R}P^2$, where tmf denotes the connective spectrum of topological modular forms. Their computation for $\mathbb{R}P^2$ and the tmf–local Weiss tower for the functor $V \mapsto \mathbb{R}P(V)$ should yield a calculation of the tmf–homology of $\mathbb{R}P^k$ for all k. Such a connection would, for example, feed into a chromatic understanding of block structures; see eg [Macko 2007].

Conventions

We work extensively with model categories and refer the reader to the survey article [Dwyer and Spaliński 1995] and the textbooks [Hovey 1999; Hirschhorn 2003] for a detailed account of the theory. We further assume the reader has familiarity with orthogonal calculus, references for which include [Barnes and Oman 2013; Weiss 1995].

The category Top_* will always denote the category of based compactly generated weak Hausdorff spaces, and we will, for brevity, call the objects of this category "based spaces". The category of based spaces will always be equipped with the Quillen model structure unless specified otherwise. The weak equivalences are the weak homotopy equivalences and fibrations are Serre fibrations. This is a cellular, proper and topological model category with sets of generating cofibrations and acyclic cofibrations denoted by I and J, respectively.

Unless otherwise stated the word "spectra" is synonymous with the phrase "orthogonal spectra", details of which can be found in [Mandell et al. 2001] in the nonequivariant case, and [Mandell and May 2002] in the equivariant situation.

We will denote by $\mathcal{C}$ either the category of based spaces or of orthogonal spectra.

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Part I Local orthogonal calculus

2 Orthogonal functors

Denote by $\mathcal{C}$ the category Top_* of based topological spaces or the category Sp of (orthogonal) spectra. Define $\mathcal{J}$ to be the category with finite-dimensional inner product subspaces of $\mathbb{R}^\infty$ as objects and with the linear isometries as morphisms. Define $\mathcal{J}_0$ to be the category with the same objects and $\mathcal{J}_0(U,V)=\mathcal{J}(U,V)_+$. The morphism set $\mathcal{J}(U,V)$ may be topologised as the Stiefel manifold of $\dim(U)$–frames in V. As such, $\mathcal{J}$ is a topologically enriched category, and $\mathcal{J}_0$ is enriched in based spaces. Since the functor

$$\Sigma^\infty\colon \mathsf{Top}_*\to\mathsf{Sp}$$

is symmetric monoidal — see eg [Mandell and May 2002, Lemma II.4.8] — we may enhance the topological enrichment of $\mathcal{J}_0$ to a spectral enrichment, resulting in a category $\mathcal{J}_0^{\mathsf{Sp}}$, whose class of objects agrees with the class of objects in $\mathcal{J}_0$, and morphism spectrum

$$\mathcal{J}_0^{\mathsf{Sp}}(V,W)=\Sigma^\infty\mathcal{J}_0(V,W).$$

We will omit the superscript "Sp" when confusion is unlikely to occur.

The category $\mathsf{Fun}(\mathcal{J}_0,\mathcal{C})$ of $\mathcal{C}$–enriched functors from $\mathcal{J}_0$ to $\mathcal{C}$ is the category of input functors for orthogonal calculus. We will refer to such functors as *$\mathcal{C}$–valued orthogonal functors* or simply *orthogonal functors* when confusion is unlikely. Examples of orthogonal functors are abound in geometry, topology and homotopy theory, and examples of Top_*–valued orthogonal functors include

(1) the one-point compactification functor $\mathbb{S}\colon V\mapsto S^V$;

(2) the functor $\mathsf{BO}(-)\colon V\mapsto\mathsf{BO}(V)$ which sends an inner product space to the classifying space of its orthogonal group;

(3) the functor $\mathsf{BTOP}(-)\colon V\mapsto\mathsf{BTOP}(V)$, which sends an inner product space V to $\mathsf{BTOP}(V)$, the classifying space of the space of self-homeomorphisms of V;

(4) the functor $\mathsf{BDiff}^b(M\times -)\colon V\mapsto\mathsf{BDiff}^b(M\times V)$, which for a fixed smooth and compact manifold M sends an inner product space V to the classifying space of the group of bounded diffeomorphisms from $M\times V$ to $M\times V$ which are the identity on $\partial M\times V$; and

(5) the restriction of an endofunctor on based spaces to evaluation on spheres.[1]

The category of orthogonal functors may be equipped with a projective model structure.

[1]Endofunctors of based spaces are particularly interesting from a homotopy-theoretic point of view when you restrict to the values on spheres; see eg [Arone 2002; Arone and Mahowald 1999; Behrens 2012]. In particular for F the identity functor, the Weiss tower of $F\circ\mathbb{S}=\mathbb{S}$ and the Goodwillie tower for F agree up to weak equivalence [Barnes and Eldred 2016]; hence orthogonal calculus is intimately related to understanding the (stable) homotopy groups of spheres.

Proposition 2.1 *There is a model category structure on the category of orthogonal functors* $\mathsf{Fun}(\mathcal{J}_0, \mathcal{C})$ *with weak equivalences and fibrations defined objectwise. This model structure is cellular, proper and topological, and in the case of* Sp*–valued orthogonal functors, this model structure is spectral and stable.*

2.1 Local input functors

The "base" model structure for the S–local orthogonal calculus will be the S–local model structure on the category of orthogonal functors.

Proposition 2.2 *Let S be a set of maps in $\mathcal{C}$. There is model structure on the category of orthogonal functors such that a map is a weak equivalence or fibration if it is an objectwise S–local equivalence or an objectwise S–local fibration in $\mathcal{C}$, respectively. This model structure is cellular, left proper and topological, and in the case of* Sp*–valued orthogonal functors this model structure is spectral. We call this model structure the S–local projective model structure and denote it by* $\mathsf{Fun}(\mathcal{J}_0, L_S\mathcal{C})$.

Proof This model structure is an instance of a projective model structure on a category of functors; see eg [Hirschhorn 2003, Theorem 11.6.1]. □

Example 2.3 For E_* a generalised homology theory, the model structure of Proposition 2.2 has weak equivalences the objectwise E_*–isomorphisms, and fibrant objects objectwise E_*–local objects. This follows since the E_*–localization of spaces and spectra exist by work of Bousfield [1975; 1979].

3 Polynomial functors

3.1 Polynomial functors

Polynomial functors behave in many ways like polynomial functions from classical calculus; eg a functor which is polynomial of degree less than or equal to n is polynomial of degree less than or equal to $n+1$. We give only the necessary details here and refer the reader to [Weiss 1995] or [Barnes and Oman 2013] for more details on polynomial functors in orthogonal calculus.

Definition 3.1 An orthogonal functor F is *polynomial of degree less than or equal n* if F is objectwise fibrant and for each $U \in \mathcal{J}_0$, the canonical map

$$F(U) \to \underset{0 \neq V \subseteq \mathbb{R}^{n+1}}{\operatorname{holim}} F(U \oplus V) =: \tau_n F(U)$$

is a weak homotopy equivalence. Functors which are polynomial of degree less than or equal to n will sometimes be referred to as *n–polynomial* functors.

Remark 3.2 Given an orthogonal functor F and an inner product space U we can restrict the orthogonal functor $F(U \oplus -)$ to a functor

$$F(U \oplus -)\colon \mathcal{P}(\mathbb{R}^{n+1}) \to \mathsf{Top}_*,$$

where $\mathcal{P}(\mathbb{R}^{n+1})$ is the poset of finite-dimensional inner product subspaces of $\mathbb{R}^{n+1}$. Such functors are deserving of the name $\mathbb{R}^{n+1}$*–cubes* by analogy with cubical homotopy theory. The orthogonal functor F being n–polynomial is equivalent to asking that for each U this restricted functor is homotopy cartesian. Informally speaking, orthogonal calculus can be thought of as calculus built from $\mathbb{R}^n$*–cubical homotopy theory* in a similar way to how Goodwillie calculus is built from cubical homotopy theory.

There is a functorial assignment of a universal (up to homotopy) n–polynomial functor to any orthogonal functor F. It is the *n–polynomial approximation* of F, and is defined as

$$T_n F(U) = \operatorname{hocolim}(F(U) \to \tau_n F(U) \to \cdots \to \tau_n^k F(U) \to \cdots).$$

Barnes and Oman [2013, Propositions 6.5 and 6.6] construct a localization of the projective model structure on the category of orthogonal functors which captures the homotopy theory of n–polynomial functors, in particular the n–polynomial approximation functor is a fibrant replacement. There are two equivalent ways to consider this model structure; as the Bousfield–Friedlander localization of $\mathsf{Fun}(\mathcal{J}_0, \mathcal{C})$ at the n–polynomial approximation endofunctor

$$T_n\colon \mathsf{Fun}(\mathcal{J}_0, \mathcal{C}) \to \mathsf{Fun}(\mathcal{J}_0, \mathcal{C}),$$

or as the left Bousfield localization at the set

$$\mathcal{S}_n = \{S\gamma_{n+1}(U,V)_+ \to \mathcal{J}_0(U,V) \mid U, V \in \mathcal{J}_0\}$$

for Top_*–valued orthogonal functors, or the set $\Sigma^\infty \mathcal{S}_n = \{\Sigma^\infty f \mid f \in \mathcal{S}_n\}$ for Sp–valued orthogonal functors, where $S\gamma_{n+1}(V,W)$ is the sphere bundle of the $(n+1)$–fold Whitney sum of the orthogonal complement bundle over the space of linear isometries $\mathcal{J}(V,W)$.

Proposition 3.3 [Barnes and Oman 2013, Proposition 6.5] *There is a model category structure on the category of orthogonal functors with weak equivalences the T_n–equivalences*[2] *and fibrations those objectwise fibrations $f\colon X \to Y$ such that the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & T_n X \arrow[d] \\ Y \arrow[r] & T_n Y \end{tikzcd}$$

is a homotopy pullback in the projective model structure. This model structure is cellular, proper and topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this the n–polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathcal{C})$.

[2]A map $f\colon X \to Y$ is a T_n–equivalence if $T_n(f)\colon T_n X \to T_n Y$ is an objectwise weak equivalence.

3.2 Local polynomial functors

The definition of an S–locally n–polynomial functor is the analogous definition of an n–polynomial functor when the base model category is $L_S\mathcal{C}$, ie an objectwise fibrant functor which satisfies a cartesian $\mathbb{R}^{n+1}$–cube condition.

Definition 3.4 Let S be a set of maps in $\mathcal{C}$. An orthogonal functor is *S–locally n–polynomial* if it is objectwise S–local and n–polynomial.

The S–locally n–polynomial model structure is an iterated left Bousfield localization involving the set $\mathcal{S}_n$ and the set
$$J_S = \{\mathcal{J}_0(U,-) \wedge j \mid U \in \mathcal{J},\ j \in J_{L_S\mathcal{C}}\},$$
as this iterative localization will necessarily have the S–locally n–polynomial functors as fibrant objects. This model structure was first constructed by Barnes [2017] for the rationalization of Top_*–valued orthogonal functors.

Proposition 3.5 *Let S be a set of maps in $\mathcal{C}$. There is model category structure on the category of orthogonal functors with cofibrations the projective cofibrations, and fibrant objects the S–locally n–polynomial functors. This model structure is cellular, left proper, topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this model structure the S–local n–polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C})$.*

Proof The process of left Bousfield localizations may be iterated and it follows that the J_S–localization of the n–polynomial model structure and the $\mathcal{S}_n$–localization of the S–local projective model structure are identical, and have as cofibrations the projective cofibrations.

For the fibrant objects, notice that the model structure is equivalently described as the left Bousfield localization of the projective model structure with respect to the set of maps $\mathcal{S}_n \cup J_S$. By definition an object X is $\mathcal{S}_n \cup J_S$–local if and only if it is both $\mathcal{S}_n$–local and J_S–local, and hence the fibrant objects are precise those S–locally n–polynomial functors. □

The S–local n–polynomial model structure behaves precisely like a left Bousfield localization of the n–polynomial model structure in the following sense.

Lemma 3.6 *Let S be a set of maps in $\mathcal{C}$. The adjoint pair*
$$\mathbb{1} \colon \mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathcal{C}) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C}) : \mathbb{1}$$
is a Quillen adjunction.

Proof The left adjoint preserves cofibrations since the classes of cofibrations are identical. The right adjoint is right Quillen since it preserves fibrant objects as every S–locally n–polynomial functor is necessarily n–polynomial. □

The composite $T_n L_S$ need not be a fibrant replacement functor in the S–local n–polynomial model structure since the class of S–local objects need not be closed under filtered homotopy colimits. Imposing a condition on the set S which forces $T_n L_S$ to be S–local in turn forces $T_n L_S$ to be a functorial fibrant replacement.

Proposition 3.7 *Let S be a set of maps in $\mathcal{C}$. If the class of S–local objects is closed under sequential homotopy colimits, then the weak equivalences of the S–local n–polynomial model structure are those maps $f\colon X \to Y$ such that the induced map*

$$T_n L_S f\colon T_n L_S X \to T_n L_S Y$$

is an S–local equivalence. In particular, The composite $T_n L_S$ is a functorial fibrant replacement in the S–local n–polynomial model structure.

Proof We apply [Barnes 2017, Lemma 5.5], which shows that a map $f\colon X \to Y$ is weak equivalence in the iterated left Bousfield localization if and only if

$$L_S f\colon L_S X \to L_S Y$$

is an $\mathcal{S}_n$–local equivalence. This last is equivalent to $L_S f\colon L_S X \to L_S Y$ being a T_n–equivalence, ie $T_n L_S f\colon T_n L_S X \to T_n L_S Y$ being an objectwise weak equivalence. Since both the domain and codomain of this map are S–local, checking this map is an objectwise weak equivalence is equivalent to checking that it is an S–local equivalence by the S–local Whitehead theorem. □

Remark 3.8 Let S be a set of maps in $\mathcal{C}$. To ease notation, we will denote the composite $T_n L_S$ by T_n^S. In particular, for E a spectrum we denote the composite functor $T_n L_E$ by T_n^E. In general, T_n^S need not be S–local, but will be when the class of S–local objects is closed under sequential homotopy colimits.

Example 3.9 (1) For a finite cell complex W, $T_n^W F$ is W–local (or W–periodic) for all Top_*–valued orthogonal functors F.

(2) For localization at the Eilenberg–Mac Lane spectrum associated to a subring R of the rationals, $T_n^{HR} F$ is HR–local for all orthogonal functors F.

(3) For E a spectrum such that the associated localization of spectra is smashing, $T_n^E F$ is E–local for all Sp–valued orthogonal functors F.

4 Differentiation

The analogy between orthogonal calculus and differential calculus (Taylor's version) indicated the existence of an inductive "formula" for the n–polynomial approximation. The building blocks of such a "formula" are the derivatives of the functor under consideration.

4.1 The derivatives

The orthogonal complement of the pullback of the tautological bundle to the Stiefel manifold $\mathcal{J}_0(V, W)$ is a vector bundle $\gamma_1(V, W)$ with fibre over an isometry f given by $f(V)^\perp$. For $n \geq 0$, we denote the n–fold Whitney sum of $\gamma_1(V, W)$ by $\gamma_n(V, W)$. Define $\mathcal{J}_n$ to be the category with the same objects as $\mathcal{J}$ and morphism space $\mathcal{J}_n(U, V)$ given as the Thom space of $\gamma_n(U, V)$. Define $\mathcal{J}_n^{\mathsf{Sp}}$ to be the spectral enriched version of $\mathcal{J}_n$, ie the category with the same objects but morphism spectrum given by

$$\mathcal{J}^{\mathsf{Sp}}(V, W) = \Sigma^\infty \mathcal{J}_n(V, W).$$

The standard action of $O(n)$ on $\mathbb{R}^n$ via the regular representation induces an action on the vector bundles that is compatible with the composition; hence $\mathcal{J}_n$ is naturally enriched over based spaces with an $O(n)$–action.

Recall that $\mathcal{C}$ denotes the category of based spaces or spectra. We denote by $\mathcal{C}[O(n)]$ the category of $O(n)$–objects in $\mathcal{C}$. For $\mathcal{C} = \mathsf{Top}_*$, this recovers the category of $O(n)$–spaces, and for $\mathcal{C} = \mathsf{Sp}$, this is the category of spectra with an $O(n)$–action. Let $0 \leq m \leq n$. The inclusion $i_m^n \colon \mathbb{R}^m \to \mathbb{R}^n$ induces a functor $i_m^n \colon \mathcal{J}_m \to \mathcal{J}_n$. Postcomposition with i_m^n induces a topological functor

$$\mathsf{res}_m^n \colon \mathsf{Fun}(\mathcal{J}_n, \mathcal{C}) \to \mathsf{Fun}(\mathcal{J}_m, \mathcal{C}),$$

which by [Weiss 1995, Proposition 2.1] has a right adjoint

$$\mathsf{ind}_m^n \colon \mathsf{Fun}(\mathcal{J}_m, \mathcal{C}) \to \mathsf{Fun}(\mathcal{J}_n, \mathcal{C}),$$

the right Kan extension along i_m^n, and is given by

$$\mathsf{ind}_m^n F(U) = \mathsf{nat}_m(\mathcal{J}_n(U, -), F),$$

where $\mathsf{nat}_m(-, -)$ denotes the space of natural transformations in $\mathsf{Fun}(\mathcal{J}_m, \mathcal{C})$ and $\mathcal{J}_n(U, -)$ is considered as an object of $\mathsf{Fun}(\mathcal{J}_m, \mathcal{C})$ by restriction. Combining the restriction and induction functors with change of group adjunctions from [Mandell and May 2002], we obtain an adjoint pair

$$\mathsf{res}_m^n / O(n-m) \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)]) \rightleftarrows \mathsf{Fun}_{O(m)}(\mathcal{J}_m, \mathcal{C}[O(m)]) : \mathsf{ind}_m^n \mathsf{CI}$$

(see [Barnes and Oman 2013, Section 4]), where $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$ is the category of $\mathcal{C}[O(n)]$–enriched functors from $\mathcal{J}_n$ to $\mathcal{C}[O(n)]$. We refer to this category as the n^{th} *intermediate category* on the point of its role as an intermediate in the classification of n–homogeneous functors; see Section 5.

Definition 4.1 Let F be an orthogonal functor. For $n \geq 0$, the n^{th} *derivative of* F is given by $\mathsf{ind}_0^n \mathsf{CI} F$. For this we write $\mathsf{ind}_0^n \varepsilon^* F$ or $F^{(n)}$.

Restricted evaluation in the n^{th} intermediate category induces structure maps of the form

$$X(V) \wedge S^{\mathbb{R}^n \otimes W} \to X(V \oplus W)$$

for $X \in \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$ and $V, W \in \mathcal{J}_n$; see eg [Barnes and Oman 2013, Section 7]. It is thus reasonable to think of the objects of the n^{th} intermediate category as *spectra of multiplicity* n; see eg

[Weiss 1995, Section 9]. This idea leads to an object Z in the n^{th} intermediate category being called an *$n\Omega$–spectrum* if the adjoint structure maps

$$Z(V) \to \Omega^{\mathbb{R}^n \otimes W} Z(V \oplus W)$$

are weak equivalences in $\mathcal{C}$, and a map $f\colon X \to Y$ in the n^{th} intermediate category being called an n–stable equivalence if the induced map

$$f^*\colon [Y, Z] \to [X, Z]$$

on objectwise homotopy classes of maps is an isomorphism for all $n\Omega$–spectra Z. With these definitions we get an n–stable model structure on the n^{th} intermediate category analogous to the stable model structure on spectra; see eg [Barnes and Oman 2013, Section 7].

Proposition 4.2 [Barnes and Oman 2013, Proposition 7.14] *There is a model category structure on the n^{th} intermediate category with weak equivalences the n–stable equivalences and fibrations the objectwise fibrations $X \to Y$ such that the square*

$$\begin{array}{ccc} X(U) & \longrightarrow & \Omega^{\mathbb{R}^n \otimes V} X(U \oplus V) \\ \downarrow & & \downarrow \\ Y(U) & \longrightarrow & \Omega^{\mathbb{R}^n \otimes V} Y(U \oplus V) \end{array}$$

is a homotopy pullback in $\mathcal{C}$ for all $U, V \in \mathcal{J}_n$. The fibrant objects are the $n\Omega$–spectra. This model structure is cellular, proper, stable and topological, and in the case of Sp–valued orthogonal functors, this model structure is spectral. We call this the n–stable model structure and denote it by $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$.

4.2 The local n–stable model structure

We now equip the n^{th} intermediate category with an S–local model structure which will be intermediate in our classification of S–local n–homogeneous functors as appropriately[3] S–local spectra with an action of $O(n)$. This model structure was first defined by Barnes [2017] for the rationalization of Top_*–valued orthogonal functors.

Proposition 4.3 *Let S be a set of maps in $\mathcal{C}$. There is a model category structure on the n^{th} intermediate category with cofibrations the cofibrations of the n–stable model structure and fibrant objects the $n\Omega$–spectra which are objectwise S–local. This model structure is cellular, left proper and topological, and in the case of Sp–valued orthogonal functors, this model structure is spectral. We call this the S–local n–stable model structure and denote it by $L_S \,\mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)])$.*

Proof This model structure is the left Bousfield localization of the n–stable model structure at the set

$$\mathcal{Q}_n = \{O(n)_+ \wedge \mathcal{J}_n(U, -) \wedge j \mid U \in \mathcal{J}, j \in J_{L_S\mathcal{C}}\}. \qquad \square$$

[3] Here “appropriately” means levelwise S–local spectra for Top_*–valued orthogonal functors and S–local spectra for Sp–valued orthogonal functors.

We record the following fact which will prove useful later.

Lemma 4.4 *Let S be a set of maps in $\mathcal{C}$. If F is an S–local functor, then $F^{(n)} = \mathsf{ind}_0^n F$ is S–local.*

Proof The objectwise smash product

$$(-) \wedge (-) \colon \mathsf{Fun}(\mathcal{J}_n, L_S\mathcal{C}) \times L_S\mathcal{C} \to \mathsf{Fun}(\mathcal{J}_n, L_S\mathcal{C})$$

is a Quillen bifunctor, and the result follows from the definition of $\mathsf{ind}_0^n F$. □

4.3 The derivatives as spectra

The n^{th} derivative ($n \geq 0$) is naturally an object of the n^{th} intermediate category, ie is a spectrum of multiplicity n. This multiplicity may be reduced to $n = 1$ through a Quillen equivalence

$$(\alpha_n)_! \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*$$

in the topological case (see eg [Barnes and Oman 2013, Section 8]) and by a series of Quillen equivalences

$$\mathsf{Fun}(\mathcal{J}_n, \mathsf{Sp}[O(n)]) \underset{(\alpha_n)^*}{\overset{(\alpha_n)_!}{\rightleftarrows}} \mathsf{Sp}(\mathsf{Sp}[O(n)]) \underset{\mathrm{Ev}_0}{\overset{F_0}{\leftrightarrows}} \mathsf{Sp}[O(n)]$$

in the spectral case (see eg [Barnes and Oman 2013, Section 11]). Here $\mathsf{Sp}(\mathsf{Sp}[O(n)])$ denotes the category of spectrum objects in spectra with an $O(n)$–action or equivalently, orthogonal bispectra with an $O(n)$–action, and is Quillen equivalent to orthogonal spectra by arguments similar to [Hovey 2001, Theorem 5.1] or [Schwede and Shipley 2003, Theorem 3.8.2].

Example 4.5 The (spectrum representing the) n^{th} derivative of the Top_*–valued orthogonal functor $\mathsf{BO}(-)$ have been completely calculated by Arone [2002]. Weiss [1995] calculated the first few examples by hand, for instance the first derivative is the sphere spectrum with trivial $O(1)$–action, the second derivative is the shifted sphere spectrum $\mathbb{S}^{-1}$ with trivial action, and the third derivative is the 2–fold loops on the mod-3 Moore spectrum $\Omega^2(\mathbb{S}/3)$. Higher derivatives have a striking resemblance with the Goodwillie derivatives of the identity functors on based spaces.

We now prove that this result holds S–locally for any set S of maps in our category $\mathcal{C}$. Since the adjunctions are slightly different, we prove each separately.

Theorem 4.6 *Let S be a set of maps of based spaces. The adjoint pair*

$$(\alpha_n)_! \colon L_S\, \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)] : (\alpha_n)^*$$

is a Quillen equivalence between the S–local model structures.

Proof For the Quillen adjunction apply [Hirschhorn 2003, Theorem 3.3.20(1)], noting that there is an isomorphism

$$(\alpha_n)_!(O(n)_+ \wedge \mathcal{J}_n(U,-) \wedge j) \cong O(n)_+ \wedge \mathcal{J}_1(\mathbb{R}^n \otimes U,-) \wedge j$$

for j a generating acyclic cofibration for the S–local model structure on based spaces.

By [Barnes and Oman 2013, Proposition 8.3], the adjoint pair

$$(\alpha_n)_! \colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*$$

is a Quillen equivalence. To show that the adjunction between the S–local model structures is a Quillen equivalence, it suffices by [Hovey 2001, Proposition 2.3] to show that if Y is fibrant in $\mathsf{Sp}[O(n)]$ such that $(\alpha_n)^*Y$ is fibrant in the S–local n–stable model structure, then Y is fibrant in the S–local model structure on $\mathsf{Sp}[O(n)]$. This follows readily from the definitions of fibrant objects in both model structures. □

The category of orthogonal bispectra with an $O(n)$–action, or equivalently the category of (orthogonal) spectrum objects in spectra with an $O(n)$–action may be equipped with an L_S–local model structure, similar to Proposition 4.3. For S a set of maps of spectra, the S–local model structure $L_S\mathsf{Sp}(\mathsf{Sp}[O(n)])$ is the left Bousfield localization of the stable model structure at the set

$$\{\mathcal{J}_1(V,-) \wedge j \mid V \in \mathcal{J},\ j \in J_{L_S\mathsf{Sp}[O(n)]}\}$$

since the category $\mathsf{Sp}(\mathsf{Sp}[O(n)])$ may also be described as the category of $O(n)$–objects in $\mathsf{Fun}(\mathcal{J}_1, \mathsf{Sp})$. In particular, the fibrant objects of the S–local model structure on $\mathsf{Sp}(\mathsf{Sp}[O(n)])$ are $O(n)$–objects $X \in \mathsf{Fun}(\mathcal{J}_1, \mathsf{Sp})$ such that $X(V)$ is an S–local spectrum for each $V \in \mathcal{J}_1$.

Theorem 4.7 *Let S be a set of maps of spectra. The adjoint pairs*

$$L_S\,\mathsf{Fun}(\mathcal{J}_n, \mathsf{Sp}[O(n)]) \underset{(\alpha_n)^*}{\overset{(\alpha_n)_!}{\rightleftarrows}} L_S\mathsf{Sp}(\mathsf{Sp}[O(n)]) \underset{\mathrm{Ev}_0}{\overset{F_0}{\leftrightarrows}} L_S\mathsf{Sp}[O(n)]$$

are Quillen equivalences between the S–local model structures.

Proof Identifying the category of spectrum objects in spectra with an $O(n)$–action with the category of $O(n)$–objects in $\mathsf{Fun}(\mathcal{J}_1, \mathsf{Sp})$, the proof that the adjunction

$$(\alpha_n)_! \colon L_S\,\mathsf{Fun}(\mathcal{J}_n, \mathsf{Sp}[O(n)]) \rightleftarrows L_S\mathsf{Sp}(\mathsf{Sp}[O(n)]) : (\alpha_n)^*,$$

is a Quillen equivalence follows analogously to Theorem 4.6.

For the adjunction

$$F_0 \colon L_S\mathsf{Sp}[O(n)] \rightleftarrows L_S\mathsf{Sp}(\mathsf{Sp}[O(n)]) : \mathrm{Ev}_0,$$

note that the composite functor

$$\mathsf{Sp}[O(n)] \xrightarrow{F_0} \mathsf{Sp}(\mathsf{Sp}[O(n)]) \xrightarrow{\mathbb{1}} L_S\mathsf{Sp}(\mathsf{Sp}[O(n)])$$

is left Quillen, and to extend to a left Quillen functor from $L_S\mathsf{Sp}[O(n)]$, it suffices by [Hirschhorn 2003, Proposition 3.3.18(1) and Theorem 3.1.6(1)] to exhibit that the right adjoint preserves S–local objects, which follows immediately from the definition of S–local objects in the respective model structures.

To see that the adjunction is a Quillen equivalence, we apply [Hovey 2001, Proposition 2.3], which reduces the problem to showing that if Y is an Ω–spectrum object in $\mathsf{Sp}[O(n)]$ (ie fibrant in $\mathsf{Sp}(\mathsf{Sp}[O(n)])$) such that $\mathrm{Ev}_0(Y)$ is S–local, then Y is S–local. This follows from the Ω–spectrum structure and the interaction of homotopy function complexes with the suspension-loops adjunction. □

5 Homogeneous functors and their classification

5.1 Homogeneous functors

The layers of the Weiss tower associated to an orthogonal functor F are the homotopy fibres of maps $T_n F \to T_{n-1} F$ and have two interesting properties: first, they are polynomial of degree less than or equal to n; and second, their $(n-1)$–polynomial approximation is trivial. We denote the n^{th} layer of the Weiss tower of F by $D_n F$.

Definition 5.1 For $n \geq 0$, an orthogonal functor F is said to be *n–reduced* if its $(n-1)$–polynomial approximation is objectwise weakly equivalent to the terminal object. An orthogonal functor F is said to be *homogeneous of degree n* if it is both polynomial of degree less than or equal n and n–reduced. We will sometimes refer to a functor which is homogeneous of degree n as being *n–homogeneous*.

There is a model structure on the category of orthogonal functors which contains the n–homogeneous functors as the bifibrant objects. This model structure is a right Bousfield localization of the n–polynomial model structure.

Proposition 5.2 [Barnes and Oman 2013, Proposition 6.9] *There is a model category structure on the category of orthogonal functors with weak equivalences the D_n–equivalences and fibrations the fibrations of the n–polynomial model structure. The cofibrant objects are the n–reduced projectively cofibrant objects and the fibrant objects are the n–polynomial functors. In particular, cofibrant-fibrant objects of this model structure are the projectively cofibrant n–homogeneous functors. This model structure is cellular, proper, stable and topological, and in the case of* Sp*–valued orthogonal functors this model structure is spectral. We call this the n–homogeneous model structure and denote it by* $\mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C})$.

Remark 5.3 The model structure of [Barnes and Oman 2013, Proposition 6.9] has as weak equivalences those maps which induce objectwise weak equivalences on the n^{th} derivatives of their n–polynomial approximations. We showed in [Taggart 2022a, Proposition 8.2] that the class of such equivalences is precisely the class of D_n–equivalences. The proof of [Taggart 2022a, Proposition 8.2] is valid for Sp–valued orthogonal functors since Sp–valued n–homogeneous functors admit and analogous classification in terms of spectral with an $O(n)$–action; see eg [Barnes and Oman 2013, Section 11].

The n–homogeneous model structure is (zigzag) Quillen equivalent to spectra with an action of $O(n)$.

Proposition 5.4 [Barnes and Oman 2013, Proposition 8.3, Theorems 10.1 and 11.3, and Corollary 11.4] *Let $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C}) \simeq_Q \mathsf{Sp}[O(n)].$$

On the homotopy category level, the Barnes–Oman zigzag of Quillen equivalences recovers Weiss's characterisation of homogeneous functors of degree n.

Proposition 5.5 [Weiss 1995, Theorem 7.3; Barnes and Oman 2013, Theorem 11.5] *Let $n \geq 1$.*

(1) *An n–homogeneous functor F is determined by and determines a spectrum $\partial_n F$ with an $O(n)$–action.*

(2) *A Top_*–valued n–homogeneous functor F is objectwise weak homotopy equivalent to the functor*

$$V \mapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n F)_{hO(n)}],$$

and any functor of the above form is homogeneous of degree n.

(3) *An Sp–valued n–homogeneous functor F is objectwise weak homotopy equivalent to the functor*

$$V \mapsto (S^{\mathbb{R}^n \otimes V} \wedge \partial_n F)_{hO(n)},$$

and any functor of the above form is homogeneous of degree n.

5.2 Local homogeneous functors

Definition 5.6 Let S be a set of maps in $\mathcal{C}$. An orthogonal functor F is *S–locally homogeneous of degree n* if it is objectwise S–local and n–homogeneous.

Lemma 5.7 *Let S be a set of maps of in $\mathcal{C}$, and F and orthogonal functor. For $n \geq 1$, there is a homotopy fibre sequence*

$$D_n^S F \to T_n^S F \to T_{n-1}^S F$$

in which $D_n^S(F)$ is

(1) *homogeneous of degree n; and*

(2) *S–locally n–homogeneous if, in addition, the class of S–local objects is closed under sequential homotopy colimits.*

Proof By [Weiss 1995, Lemma 5.5] the homotopy fibre of a map between n–polynomial functors is n–polynomial; hence $D_n^S F$ is n–polynomial. Applying T_{n-1} to the homotopy fibre sequence, yields that the $(n-1)$–polynomial approximation of $D_n^S F$ is objectwise weakly contractible, proving (1).

For (2), observe that the homotopy fibre of a map between S–local objects is S–local and when the class of S–local objects is closed under sequential homotopy colimits, $T_n L_S F$ is S–local for all n. □

Example 5.8 (1) For homological localization at the Eilenberg–Mac Lane spectrum associated to a subring R of the rationals, $D_n^{HR} F$ is HR–locally n–homogeneous for any orthogonal functor F.

(2) For nullification at a based finite cell complex W, $D_n^W F$ is W–locally n–homogeneous for any Top_*–valued orthogonal functor F.

(3) For a spectrum E whose associated localization of spectra is smashing, $D_n^E F$ is E–locally n–homogeneous for any Sp–valued orthogonal functor.

Proposition 5.9 *Let S be a set of maps in $\mathcal{C}$. There is model category structure on the category of orthogonal functors with cofibrations the cofibrations of the n–homogeneous model structure and fibrant objects the n–polynomial functors whose n^{th} derivative is objectwise S–local in the n^{th} intermediate category. This model structure is cellular, left proper and topological, and in the case of Sp–valued orthogonal functors this model structure is spectral. We call this the S–local n–homogeneous model structure and denote it by $\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C})$.*

Proof We left Bousfield localize the n–homogeneous model structure at the set of maps

$$\mathcal{K}_n = \{\mathcal{J}_n(U,-)\wedge j \mid U \in \mathcal{J},\, j \in J_{L_S\mathcal{C}}\}.$$

This left Bousfield localization exists since the n–homogeneous model structure is cellular and left proper by [Barnes 2017, Lemma 6.1]. The description of the cofibrations follows immediately.

The fibrant objects are the $\mathcal{K}_n$–local objects which are also fibrant in the n–homogeneous model structure, ie those n–polynomial functors Z for which the induced map

$$[\mathcal{J}_n(U,-)\wedge B, Z] \to [\mathcal{J}_n(U,-)\wedge A, Z]$$

is an isomorphism for all maps $\mathcal{J}_n(U,-)\wedge A \to \mathcal{J}_n(U,-)\wedge B$ in $\mathcal{K}_n$. A straightforward adjunction argument and the definition of the n^{th} derivative of an orthogonal functor yield the required characterisation of the fibrant objects. □

Corollary 5.10 *Let S be a set of maps in $\mathcal{C}$. The cofibrant objects of the S–local n–homogeneous model structure are the projectively cofibrant functors which are n–reduced.*

Proof The S–local n–homogeneous model structure is a particular left Bousfield localization of the n–homogeneous model structure, hence has the same cofibrant objects. The result follows by the orthogonal calculus version of [Taggart 2022a, Corollary 8.6]. □

The S–local n–homogeneous model structure behaves like a right Bousfield localization of the S–local n–polynomial model structure in the following sense.

Lemma 5.11 *Let S be a set of maps in $\mathcal{C}$. The adjoint pair*

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C}) : \mathbb{1}$$

is a Quillen adjunction.

Proof The cofibrations of the S–local n–homogeneous model structure are the cofibrations of the n–homogeneous model structure, which are contained in the cofibrations of the n–polynomial model structure, which in turn are precisely the cofibrations of the S–local n–polynomial model structure, hence

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) \to \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C})$$

preserves cofibrations.

On the other hand, to show that the right adjoint is right Quillen it suffices to show that the identity functor sends fibrant objects in the S–local n–polynomial model structure to fibrant objects in the S–local n–homogeneous model structure. This follows from Lemma 4.4 since the fibrant objects in the S–local n–polynomial model structure are the S–locally n–polynomial functors by Proposition 3.5 and the fibrant objects of the S–local n–homogeneous model structure are the n–polynomial functors with S–local n^{th} derivative by Proposition 5.9. □

5.3 Characterisations for stable localizations

We obtain a characterisation of the fibrations of the S–local n–homogeneous model structure when the localizing set S is stable in the sense of [Barnes and Roitzheim 2014, Definition 4.2], ie when the class of S–local spaces is closed under suspension. For the statement of the following result recall the definition of the n^{th} derivative of an orthogonal functor from Definition 4.1.

Proposition 5.12 *If S is a set of maps in $\mathcal{C}$ which is stable, then the fibrations of the S–local n–homogeneous model structure are those maps $f\colon X \to Y$ which are fibrations in the n–polynomial model structure such that*

$$X^{(n)} \to Y^{(n)}$$

is an objectwise fibration in $L_S\mathcal{C}$.

Proof We first given an explicit characterisation of the acyclic cofibrations since the fibrations are characterised by the right lifting property against these maps. The maps in $\mathcal{K}_n$ are cofibrations between cofibrant objects since $\mathcal{J}_n(U,-)$ is cofibrant in $\mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C})$ and the maps in $J_{L_S\mathcal{C}}$ are cofibrations of the S–local model structure on $\mathcal{C}$. Moreover, since the localizing set S is stable, it follows the set of generating acyclic cofibrations $J_{L_S\mathcal{C}}$ is stable and in turn that the set $\mathcal{K}_n$ is stable. Hence by [Barnes and Roitzheim 2014, Theorem 4.11], the generating acyclic cofibrations are given by the set $J_{\mathsf{Homog}^n} \cup \Lambda(\mathcal{K}_n)$, where J_{Homog^n} is the set of the generating acyclic cofibrations of the n–homogeneous model structure and $\Lambda(\mathcal{K}_n)$ the set of horns on $\mathcal{K}_n$ in the sense of [Hirschhorn 2003, Definition 4.2.1]. As horns in topological model categories are given by pushouts and $\mathcal{K}_n$ is a set of cofibrations between cofibrant objects it suffices to use the set $J_{\mathsf{Homog}^n} \cup \mathcal{K}_n$ as the generating acyclic cofibrations of the S–local n–homogeneous model structure.

If $f\colon X \to Y$ is a map with the right lifting property with respect to $J_{\mathsf{Homog}^n} \cup \mathcal{K}_n$, then f has the right lifting property with respect to J_{Homog^n} and the right lifting property with respect to $\mathcal{K}_n$ independently. Having the right lifting property with respect to J_{Homog^n} is equivalent to being a fibration in the n–

polynomial model structure. On the other hand, a map in $\mathcal{K}_n$ is of the form $\mathcal{J}_n(U,-)\wedge A \to \mathcal{J}_n(U,-)\wedge B$ for $A \to B$ a generating acyclic cofibration of the S–local model structure on $\mathcal{C}$. A lift in the diagram

$$\begin{tikzcd} \mathcal{J}_n(U,-)\wedge A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{J}_n(U,-)\wedge B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(indicated by the dotted arrow) exists if and only if the lift in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathsf{nat}_0(\mathcal{J}_n(U,-), X) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathsf{nat}_0(\mathcal{J}_n(U,-), Y) \end{tikzcd}$$

exists, which is equivalent to the statement that $X^{(n)} \to Y^{(n)}$ is an objectwise fibration of S–local objects in $\mathcal{C}$; see Section 4.1. □

This specialises to homological localizations.

Corollary 5.13 *Let E be a spectrum. The fibrations of the E–local n–homogeneous model structure are those maps $f\colon X \to Y$ which are fibrations in the n–polynomial model structure such that*

$$X^{(n)} \to Y^{(n)}$$

is an objectwise fibration in $L_E\mathcal{C}$.

Proof Combine Proposition 5.12 with [Barnes and Roitzheim 2014, Example 4.3]. □

Corollary 5.14 *Let E be a spectrum. An orthogonal functor F is fibrant in the E–local n–homogeneous model structure if and only if F is n–polynomial and $F^{(n)}$ is objectwise E–local. In particular, the bifibrant objects are the projectively cofibrant n–homogeneous functors with E–local n^{th} derivative.*

Proof Apply Corollary 5.13 to the map $F \to *$. □

5.4 Differentiation as a Quillen functor

The n^{th} derivative is a right Quillen functor as part of a Quillen equivalence between the n–homogeneous model structure and the n^{th} intermediate category; the adjunction

$$\mathsf{res}_0^n/O(n)\colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)]) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, \mathcal{C}) : \mathsf{ind}_0^n\varepsilon^*$$

is a Quillen equivalence [Barnes and Oman 2013, Theorem 10.1]. We now show that this extends to the S–local situation.

Theorem 5.15 *Let S be a set of maps in $\mathcal{C}$. The adjoint pair*

$$\mathsf{res}_0^n/O(n)\colon L_S\,\mathsf{Fun}_{O(n)}(\mathcal{J}_n, \mathcal{C}[O(n)]) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) : \mathsf{ind}_0^n\varepsilon^*$$

is a Quillen equivalence between the S–local model structures.

Proof The left adjoint applied to the localizing set of the S–local n–stable model structure is precisely the localization set of the S–local n–homogeneous model structure, hence the result follows from [Hirschhorn 2003, Theorem 3.3.20(1)]. □

Corollary 5.16 *Let S be a set of maps of based spaces, and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_S\mathsf{Top}_*)[O(n)].$$

Example 5.17 Let R be a subring of the rationals. Then there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, L_{HR}\mathsf{Top}_*) \simeq_Q \mathsf{Sp}_{HR}[O(n)]$$

between HR–local n–homogeneous functors and HR–local[4] spectra with an action of $O(n)$.

Corollary 5.18 *Let S be a set of maps of spectra, and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\mathsf{Sp}) \simeq_Q L_S\mathsf{Sp}[O(n)].$$

Example 5.19 Let E be a spectrum. Then there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, L_E\mathsf{Sp}) \simeq_Q \mathsf{Sp}_E[O(n)]$$

between E–local n–homogeneous functors and E–local spectra with an action of $O(n)$.

5.5 The classification

As in the classical theory, any S–locally n–homogeneous functor may be expressed concretely in terms of a levelwise S–local spectrum with an action of $O(n)$. The proof of which follows as in the classical setting [Weiss 1995, Theorem 7.3] and can be realised through the derived equivalence of homotopy categories provided by our zigzag of Quillen equivalences.

Theorem 5.20 *Let S be a set of maps of in $\mathcal{C}$ and $n \geq 1$.*

(1) *An S–local n–homogeneous functor F is determined by and determines an appropriately S–local spectrum with an $O(n)$–action, denoted by $\partial_n^S F$.*

(2) *A Top_*–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*

$$V \mapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)}],$$

and any functor of the above form is objectwise S–local and n–homogeneous.

(3) *An Sp–valued S–local n–homogeneous functor F is objectwise weakly equivalent to the functor*

$$V \mapsto (S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)},$$

and any functor of the above form is objectwise S–local and n–homogeneous.

[4] In particular, the HR–local model structure on spectra is identical to the levelwise HR–local model structure since a spectrum is HR–local if and only if it is levelwise HR–local; see eg [Barnes and Roitzheim 2011, Lemma 8.6].

Part II Applications

6 Bousfield classes

6.1 Bousfield classes

For a spectrum E, the *Bousfield class* of E, denoted by $\langle E\rangle$, is the equivalence class of E under the relation: $E \sim E'$ if for any spectrum X, $E \wedge X = 0$ if and only if $E' \wedge X = 0$. If $\langle E\rangle = \langle E'\rangle$, then the classes of E_*–isomorphisms and E'_*–isomorphisms agree and hence the localization functors (on spaces or spectra) agree. The collection of all Bousfield classes forms a lattice, with partial ordering $\langle E\rangle \leq \langle E'\rangle$ given by reverse containment, ie if and only if the class of E'–acyclic spectra is contained in the class of E–acyclic spectra, in particular, the partial ordering induces a natural transformation $L_{E'} \to L_E$. Bousfield classes have been studied at length; see eg [Bousfield 1979; Ravenel 1984].

A similar story remains true unstably. Given a based space W the *unstable Bousfield class* of W, or the *nullity class* of W, is the equivalence class $\langle W\rangle$ of all spaces W' such that the class of W–periodic[5] spaces agrees with the class of W'–periodic spaces. There is a partial ordering $\langle W\rangle \leq \langle W'\rangle$ given by reverse containment, ie if and only if every W'–periodic space is W–periodic. In particular, the relation $\langle W\rangle \leq \langle W'\rangle$ implies that every W–local equivalence is a W'–local equivalence and there is a natural transformation $P_W \to P_{W'}$, which is a W'–localization. Nullity classes have also been studied at length; see eg [Bousfield 1994; Dror Farjoun 1996].

Remark 6.1 It is worth noting that in both cases there is a choice of ordering of the equivalence classes, and our choices have been made to align with the predominant references on the subject, which unfortunately means the "stable" and "unstable" directions are dual. The choice of ordering used by Bousfield and that of Dror Farjoun also differ, adding further confusion to the literature on these matters.

Theorem 6.2 *Let S and S' be sets of maps in $\mathcal{C}$. The class of S–local objects agrees with the class of S'–local objects if and only if for every orthogonal functor F, the S–local Weiss tower of F is objectwise weakly equivalent to the S'–local Weiss tower of F.*

Proof If the class of S–local objects agrees with the class of S'–local objects, then the localization functors L_S and $L_{S'}$ agree on $\mathcal{C}$ and hence on the level of orthogonal functors. In particular, for every orthogonal functor F, the canonical map[6]

$$L_S F \to L_{S'} F$$

[5] W–periodic spaces are precisely W–local spaces. This change in terminology is classical; see eg [Bousfield 1994; Dror Farjoun 1996].

[6] This map is induced from the S–local objects being contained in the S'–local objects. We could also use the canonical $L_{S'} F \to L_S F$ since the S–local objects also contained the S'–local objects.

is an objectwise weak equivalence. Now, consider the commutative diagram

$$\begin{tikzcd} D_n^S F \arrow[r] \arrow[d] & T_n^S F \arrow[r] \arrow[d] & T_{n-1}^S F \arrow[d] \\ D_n^{S'} F \arrow[r] & T_n^{S'} F \arrow[r] & T_{n-1}^{S'} F \end{tikzcd}$$

in which the rows are homotopy fibre sequences. For each $n \geq 0$, the map

$$T_n^S F \to T_n^{S'} F$$

is an objectwise weak equivalence since polynomial approximation preserves objectwise weak equivalences. It follows that the leftmost vertical arrow is also an objectwise weak equivalence and that the S–local Weiss tower is objectwise weakly equivalent to the S'–local Weiss tower.

The converse is immediate from specialising for every object $C \in \mathcal{C}$ to the constant functor at C. □

Example 6.3 (1) Let E and E' be spectra. For every orthogonal functor F the E–local Weiss tower of F and the E'–local Weiss tower of F agree if and only if E and E' are Bousfield equivalent.

(2) Let W and W' be based spaces. For every Top_*–valued orthogonal functor F the W–local Weiss tower of F and the W'–local Weiss tower of F agree if and only if W and W' have the same nullity class.

6.2 Bousfield classes and model categories for orthogonal calculus

On the model category level, we have the following.

Theorem 6.4 *Let S and S' be sets of maps of maps in $\mathcal{C}$. The class of S–local objects in $\mathcal{C}$ agrees with the class of S'–local objects in $\mathcal{C}$ if and only if there are equalities of model structures making the diagram*

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{J}_0, L_S\mathcal{C}) \arrow[r, shift left, "\mathbb{1}"] \arrow[d, equal] & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\mathcal{C}) \arrow[l, shift left, "\mathbb{1}"] \arrow[r, shift right, "\mathbb{1}"'] \arrow[d, equal] & \mathsf{Homog}^n(\mathcal{J}_0, L_S\mathcal{C}) \arrow[l, shift right, "\mathbb{1}"'] \arrow[d, equal] \\ \mathsf{Fun}(\mathcal{J}_0, L_{S'}\mathcal{C}) \arrow[r, shift left, "\mathbb{1}"] & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S'}\mathcal{C}) \arrow[l, shift left, "\mathbb{1}"] \arrow[r, shift right, "\mathbb{1}"'] & \mathsf{Homog}^n(\mathcal{J}_0, L_{S'}\mathcal{C}) \arrow[l, shift right, "\mathbb{1}"'] \end{tikzcd}$$

commute.

Proof For one direction assume that the class of S–local objects agrees with the class of S'–local objects. Then the S–local model structure and the S'–local model structure on $\mathcal{C}$ agree as they have the same cofibrations and fibrant objects. This equality lifts to the local projective model structures on the category of orthogonal functors. As left Bousfield localization does not alter the cofibrations, the cofibrations of the S–local n–polynomial model structure agree with the cofibrations of the S'–local n–polynomial model structure. These model structures also have the same fibrant objects since a functor is S–locally n–polynomial if and only if it is S'–local n–polynomial under our assumption.

For the local n–homogeneous model structures, recall that these are certain left Bousfield localizations of the n–homogeneous model structure (see Proposition 5.9), hence have the same cofibrations. As before, these model structures have the same fibrant objects since our assumption together with Lemma 4.4 implies that the n^{th} derivative of a functor is S–local if and only if it is S'–local, and the fibrant objects are the n–polynomial functors with local derivatives; see Proposition 5.9.

For the converse note that since the S–local model structure on the category of orthogonal functors agrees with the S'–local model structure, the objectwise S–local equivalences are precise the objectwise S–local equivalences. It follows that the local model structures on $\mathcal{C}$ must agree. □

6.3 The partial ordering of Bousfield classes

Lemma 6.5 *Let S and S' be sets of maps in $\mathcal{C}$ and F an orthogonal functor. If the class of S'–local objects of $\mathcal{C}$ is contained in the class of S–local objects, then*

(1) *there is an S'–local equivalence $D_n^S F \to D_n^{S'} F$; and*

(2) *if F is reduced, then the S–local Weiss tower of F is S'–locally equivalent to the S'–local Weiss tower of F.*

Proof For (1), note that the map on derivatives $\partial_n^S F \to \partial_n^{S'} F$ induced by the natural transformation $L_S \to L_{S'}$ is an S'–local equivalence; hence the n–homogeneous functors which correspond to these spectra are S'–locally equivalent, ie the map $D_n^S F \to D_n^{S'} F$ is an S'–local equivalence. For (2), since F is reduced, [Weiss 1995, Corollary 8.3] implies that there is a commutative diagram

$$\begin{tikzcd} T_n^S F \arrow[r] \arrow[d] & T_{n-1}^S F \arrow[r] \arrow[d] & R_n^S F \arrow[d] \\ T_n^{S'} F \arrow[r] & T_{n-1}^{S'} F \arrow[r] & R_n^{S'} F \end{tikzcd}$$

in which both rows are homotopy fibre sequences. The map $R_n^S F \to R_n^{S'} F$ is an S'–local equivalence by part (1), and the map $T_0^S F \to T_0^{S'} F$ is also an S'–local equivalence since F is reduced. An induction argument on the degree of polynomials yields the result. □

Example 6.6 (1) Let E and E' be spectra and F an orthogonal functor. If $\langle E\rangle \leq \langle E'\rangle$, then

- (a) there is an E–local equivalence $D_n^{E'} F \to D_n^E F$; and
- (b) if F is reduced, then the E'–local Weiss tower of F is E–locally equivalent to the E–local Weiss tower of F.

(2) Let W and W' be based spaces and F a Top_*–valued orthogonal functor. If $\langle W\rangle \leq \langle W'\rangle$, then

- (a) there is an W'–local equivalence $D_n^W F \to D_n^{W'} F$; and
- (b) if F is reduced, then the W–local Weiss tower of F is W'–locally equivalent to the W'–local Weiss tower of F.

6.4 The telescope conjecture

The height n telescope conjecture of Ravenel [1984, Conjecture 10.5] asserts that the $T(n)$–localization and $K(n)$–localization of spectra agree. There are numerous equivalent formalisations of the conjecture — see eg [Barthel 2020, Proposition 3.6] — and we choose the following as it best suits any possible interaction with the calculus.

Conjecture 6.7 (the height n telescope conjecture) *Let $n \geq 0$. The Bousfield class of $T(n)$ agrees with the Bousfield class of $K(n)$.*

Corollary 6.8 *Let $n \geq 0$. The validity of the height n telescope conjecture implies equality of model structures*

$$\begin{tikzcd}
\mathsf{Fun}(\mathcal{J}_0, L_{T(n)}\mathcal{C}) \arrow[r, shift left, "\mathbb{1}"] \arrow[d, equal] & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{T(n)}\mathcal{C}) \arrow[l, shift left, "\mathbb{1}"] \arrow[r, shift right, "\mathbb{1}"'] \arrow[d, equal] & \mathsf{Homog}^{n}(\mathcal{J}_0, L_{T(n)}\mathcal{C}) \arrow[l, shift right, "\mathbb{1}"'] \arrow[d, equal] \\
\mathsf{Fun}(\mathcal{J}_0, L_{K(n)}\mathcal{C}) \arrow[r, shift left, "\mathbb{1}"] & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{K(n)}\mathcal{C}) \arrow[l, shift left, "\mathbb{1}"] \arrow[r, shift right, "\mathbb{1}"'] & \mathsf{Homog}^{n}(\mathcal{J}_0, L_{K(n)}\mathcal{C}) \arrow[l, shift right, "\mathbb{1}"']
\end{tikzcd}$$

Proof The telescope conjecture implies that the Bousfield class of $T(n)$ and the Bousfield class of $K(n)$ agree, hence the result follows by Theorem 6.4. □

The following is an immediate corollary to Theorem 6.2.

Corollary 6.9 *Let $n \geq 0$. The height n telescope conjecture holds if and only if for every orthogonal functor F the $K(n)$–local Weiss tower of F and the $T(n)$–local Weiss tower of F agree.*

This provides new insight into the height n telescope conjecture. For example, to find a counterexample it now suffices to find an orthogonal functor such that one corresponding term in the $K(n)$–local and $T(n)$–local Weiss towers disagree. This can also be seen through the spectral sequences associated to the local Weiss towers. The $K(n)$–local and $T(n)$–local Weiss towers of an orthogonal functor F produce two spectral sequences,

$$\pi_{t-s} D_s^{K(n)} F(V) \cong \pi_{t-s}((S^{\mathbb{R}^s \otimes V} \wedge \partial_s^{K(n)} F)_{hO(n)}) \Rightarrow \pi_* \operatorname*{holim}_d T_d^{K(n)} F(V),$$
$$\pi_{t-s} D_s^{T(n)} F(V) \cong \pi_{t-s}((S^{\mathbb{R}^s \otimes V} \wedge \partial_s^{T(n)} F)_{hO(n)}) \Rightarrow \pi_* \operatorname*{holim}_d T_d^{T(n)} F(V).$$

These are closely related to the telescope conjecture as follows.

Lemma 6.10 *Let F be an orthogonal functor. If the height n telescope conjecture holds, then for all $r \geq 1$, the E_r–page of the $T(n)$–local Weiss spectral sequence is isomorphic to the E_r–page of the $K(n)$–local Weiss spectral sequence.*

Proof It suffices to prove the claim for $r = 1$. The validity of the height n telescope conjecture implies that there is a natural transformation $L_{K(n)} \to L_{T(n)}$. This natural transformation induces a map $D_d^{K(n)} F \to D_d^{T(n)} F$, which by Corollary 6.9 is an objectwise weak equivalence. It hence suffices to show that the natural map $D_d^{K(n)} F \to D_d^{T(n)} F$ induces a map on the E_1–pages of the spectral sequences, that is, we have to show that the induced diagram

$$\begin{tikzcd} \pi_{t-s} D_s^{K(n)} F(V) \arrow[r, "d_1^{K(n)}"] \arrow[d] & \pi_{t-s+1} D_{s+1}^{K(n)} F(V) \arrow[d] \\ \pi_{t-s} D_s^{T(n)} F(V) \arrow[r, "d_1^{T(n)}"'] & \pi_{t-s+1} D_{s+1}^{T(n)} F(V) \end{tikzcd}$$

commutes for all s and t. This follows from the commutativity of the induced diagram of long exact sequences induced by the diagram of homotopy fibre sequences

$$\begin{tikzcd} D_s^{K(n)} F(V) \arrow[r] \arrow[d] & T_s^{K(n)} F(V) \arrow[r] \arrow[d] & T_{s-1}^{K(n)} F(V) \arrow[d] \\ D_s^{T(n)} F(V) \arrow[r] & T_s^{T(n)} F(V) \arrow[r] & T_{s-1}^{T(n)} F(V) \end{tikzcd}$$

and the construction of the d_1–differential in the homotopy spectral sequence associated to a tower of fibrations. □

7 The calculus for nullifications

7.1 Nullifications of orthogonal functors

Bousfield, Dror Farjoun and others — see eg [Bousfield 1994; 1996; Casacuberta 1994; Dror Farjoun 1996] — have extensively studied the nullification of the category of based spaces at a based space W. This nullification is functorial, giving a functor

$$P_W \colon \mathsf{Top}_* \to \mathsf{Top}_*,$$

and the Bousfield–Friedlander localization of Top_* at the endofunctor P_W defines a model structure which we call the W–periodic model structure, and denote by $P_W \mathsf{Top}_*$. This model structure is precisely the left Bousfield localization at the set $S = \{* \to W\}$, ie the W–periodic and W–local model structures agree.

The endofunctor $P_W \colon \mathsf{Top}_* \to \mathsf{Top}_*$ extends objectwise to a functor

$$P_W \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*),$$

and the W–periodic model structure on spaces — see eg [Bousfield 2001, Section 9.8] — extends in a canonical way to give the Bousfield–Friedlander localization of the category of orthogonal functors at the functor P_W, which we denote by $\mathsf{Fun}(\mathcal{J}_0, P_W\mathsf{Top}_*)$, and call the W–periodic model structure. This model structure agrees with the S–local model structure on orthogonal functors for $S = \{* \to W\}$.

In this section we give an alternative construction of the model structures for W–local orthogonal calculus. The key to this is that the W–periodic model structure on based spaces is right proper.

Remark 7.1 The process of left Bousfield localization can interfere with other model categorical properties, for instance left Bousfield localization need not preserve right properness. For example if $E = H\mathbb{Q}$, then the $H\mathbb{Q}$–local model structure on based spaces is not right proper since there is a pullback square

$$\begin{array}{ccc} K(\mathbb{Q}/\mathbb{Z},0) & \longrightarrow & P \\ \downarrow & & \downarrow \\ K(\mathbb{Z},1) & \xrightarrow{\simeq H\mathbb{Q}} & K(\mathbb{Q},1) \end{array}$$

in which the right-hand vertical map is a fibration, P is contractible and the lower horizontal map is a $H\mathbb{Q}$–equivalence but the left-hand vertical map is not. Another example is provided by Quillen [1969, Remark 2.9].

The property of being right proper has many advantages including the ability to right Bousfield localize. As such we investigate when the S–local model structure is right proper. It suffices to examine when the f–local model structure is right proper for some map $f\colon X \to Y$ of based spaces.

The following has motivation in [Bousfield 2001, Remark 9.11], which notes that the f–local model structure cannot be right proper unless the localization functor L_f is equivalent to a nullification. We extend Bousfield's remark by showing that his nullification condition is both necessary and sufficient in a stronger sense than originally proposed by Bousfield. This result depends on two constructions also due to Bousfield: the first is the construction of a based space $A(f)$ associated to a map $f\colon X \to Y$ of based spaces (see [Bousfield 1997, Theorem 4.4]); the second is the nullification functor $P_W\colon \mathsf{Top}_* \to \mathsf{Top}_*$ associated to any based space W (see [Bousfield 1994, Theorem 2.10]). This nullification functor has two key properties which we would also like to highlight: first, when W is connected, P_W preserves disjoint unions (see eg [Bousfield 2001, Theorem 9.9]); and second, P_W is contractible when W is not connected (see eg [Bousfield 1994, Example 2.3]). For example, if f is the map which induces localization with respect to integral homology, then $P_{A(f)}$ is Quillen's plus construction; see eg [Dror Farjoun 1996, 1.E.5].

Proposition 7.2 *Let $f\colon X \to Y$ be a map of based spaces. The f–local model structure on based spaces is right proper if and only if there exists a based space $A(f)$ and equality of model structures*

$$L_f\mathsf{Top}_* = P_{A(f)}\mathsf{Top}_*,$$

where $P_{A(f)}\mathsf{Top}_*$ is the Bousfield–Friedlander localization [Bousfield 2001, Theorem 9.3], *at the nullification endofunctor*

$$P_{A(f)}\colon \mathsf{Top}_* \to \mathsf{Top}_*.$$

Proof By [Bousfield 1997, Theorem 4.4], there exists a based space $A(f)$ such that the classes of $A(f)$–acyclic and f–acyclic spaces agree, and every $P_{A(f)}$–equivalence is an f–local equivalence.

Assume that the f–local model structure is right proper. For a connected based space X, the path fibration over $L_f X$ is an f–local fibration; hence the homotopy fibre of the map $X \to L_f X$ is f–acyclic, and hence $A(f)$–acyclic. It follows by [Bousfield 1994, Corollary 4.8(i)], the map $X \to L_f X$ is a $P_{A(f)}$–equivalence; hence every f–local equivalences of connected spaces is a $P_{A(f)}$–equivalence. Since the functor $P_{A(f)}$ on based spaces comes from a functor on unbased spaces which preserves disjoint unions when $A(f)$ is connected and which takes contractible values when $A(f)$ is not connected, every f–local equivalence must be a $P_{A(f)}$–equivalence. It follows that the class of f–local equivalences agrees with the class of $P_{A(f)}$–equivalences. The equality of the model structures follows immediately since both model structures have the same cofibrations inherited from the Quillen model structure on the category of based spaces.

For the converse, assume that the f–local model structure agrees with the $A(f)$–local model structure. The latter model structure is right proper by [Bousfield 2001, Theorem 9.9], and since both model structures have the same weak equivalences and fibrations, the f–local model structure must also be right proper. □

Remark 7.3 The property of being right proper is completely determined by the weak equivalence class of the model structure; if two model structures have the same weak equivalences, then one is right proper if and only if the other is; see eg [Balchin 2021, Remark 2.5.6].

7.2 Nullifications and polynomial functors

Recall from Proposition 3.5 that we have minimal control over the W–local n–polynomial model structure, in particular, unless the localization is well-behaved with respect to sequential homotopy colimits, $T_n L_W$ is not a fibrant replacement functor. We construct a W–periodic n–polynomial model structure as the Bousfield–Friedlander localization at the composite

$$T_n \circ P_W : \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$$

and show that this model structure is precisely the W–local n–polynomial model structure.

We begin with a lemma which deals with fibrant objects in the Bousfield–Friedlander localization of orthogonal functors at the endofunctor P_W, which we call the W–periodic projective model structure.

Lemma 7.4 *For a finite cell complex W and an orthogonal functor F, the functor $T_n P_W F$ is fibrant in the Bousfield–Friedlander localization of the category of orthogonal functors at the functor P_W. In particular, the map*

$$\omega_{T_n P_W F} : T_n P_W F \to P_W T_n P_W F$$

is an objectwise weak homotopy equivalence.

Proof The Bousfield–Friedlander localization of based spaces at the endofunctor P_W is identical to the left Bousfield localization of based spaces at the map $* \to W$, since both model structures have the same cofibrations and fibrant objects. It follows that the Bousfield–Friedlander localization of the category of orthogonal functors at the endofunctor P_W is identical to the W–local projective model structure. In particular, we see that $P_W F$ is fibrant and hence $\tau_n P_W F$ is also fibrant, since the class of W–local objects is closed under homotopy limits. The result follows since local objects for a nullification are closed under sequential homotopy colimits by [Dror Farjoun 1996, 1.D.6]. $\square$

Proposition 7.5 *For a finite cell complex W the Bousfield–Friedlander localization of the category of orthogonal functors at the endofunctor*

$$T_n \circ P_W \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$$

exists. This model structure is proper and topological. We call this the W–periodic n–polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W \mathsf{Top}_*)$.

Proof We verify the axioms of [Bousfield 2001, Theorem 9.3]. First note that since P_W and T_n both preserve objectwise weak equivalences so does their composite, hence verifying [Bousfield 2001, Theorem 9.3(A1)].

The natural transformation from the identity to the composite $T_n \circ P_W$ is given in components as the composite

$$F \xrightarrow{\omega_F} P_W F \xrightarrow{\eta_{P_W F}} T_n P_W F,$$

where $\omega \colon \mathbb{1} \to P_W$ and $\eta \colon \mathbb{1} \to T_n$; hence at $T_n P_W F$, we obtain the composite

$$T_n P_W F \xrightarrow{\omega_{T_n P_W F}} P_W T_n P_W F \xrightarrow{\eta_{P_W T_n P_W F}} T_n P_W T_n P_W F.$$

Since the domain is fibrant in the W–periodic projective model structure the first map in the composite is an objectwise weak equivalence; see Lemma 7.4. The second map is also a weak equivalence. To see this, note that since $T_n P_W F$ is polynomial of degree less than or equal n, the functor $P_W T_n P_W F$ is also polynomial of degree less than or equal n by the commutativity of the diagram

$$\begin{tikzcd}
T_n P_W F \arrow[r] \arrow[d] & \tau_n T_n P_W F \arrow[d] \\
P_W T_n P_W F \arrow[r] & \tau_n P_W T_n P_W F
\end{tikzcd}$$

and the fact that homotopy limits preserve objectwise weak equivalences. It follows that the natural transformation $\eta \colon T_n P_W F \to T_n P_W T_n P_W F$ is an objectwise weak equivalence, as a composite of two objectwise weak equivalences.

The map $T_n P_W(\eta)\colon T_n P_W F \to T_n P_W T_n P_W F$ is also an objectwise weak equivalence. To see this, note that there is a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\omega_F"] \arrow[d, "\omega_F"'] \arrow[dr, phantom, "(1)"] & P_W F \arrow[r, "\eta_{P_W F}"] \arrow[d, "\omega_{P_W F}"] \arrow[dr, phantom, "(2)"] & T_n P_W F \arrow[d, "\omega_{T_n P_W F}"] \\
P_W F \arrow[r, "P_W \omega_F"] \arrow[d, "\eta_{P_W F}"'] \arrow[dr, phantom, "(3)"] & P_W P_W F \arrow[r, "P_W \eta_{P_W F}"] \arrow[d, "\eta_{P_W P_W F}"] \arrow[dr, phantom, "(4)"] & P_W T_n P_W F \arrow[d, "\eta_{P_W T_n P_W F}"] \\
T_n P_W F \arrow[r, "T_n P_W \omega_F"'] & T_n P_W P_W F \arrow[r, "T_n P_W \eta_{P_W F}"'] & T_n P_W T_n P_W F
\end{tikzcd}$$

in which the required map is given by the lower horizontal composite. Since P_W is a homotopically idempotent functor, $P_W \omega_F$ is an objectwise weak equivalence. It follows that the bottom horizontal map

$$T_n P_W \omega_F\colon T_n P_W F \to T_n P_W P_W F$$

of (3) is a weak equivalence since T_n preserves weak equivalences.

Moreover, P_W being homotopically idempotent yields that the vertical map

$$\omega_{P_W F}\colon P_W F \to P_W P_W F$$

in (2) is an objectwise weak equivalence. The right-hand vertical map in this square is also an equivalence by Lemma 7.4. By [Weiss 1995, Theorem 6.3], the top right-hand horizontal map

$$\eta_{P_W F}\colon P_W F \to T_n P_W F$$

is an approximation of order n in the sense of [Weiss 1995, Definition 5.16]. By commutativity of (2), the lower horizontal map

$$P_W \eta_{P_W F}\colon P_W P_W F \to P_W T_n P_W F$$

is an approximation of order n. The proof of [Weiss 1995, Theorem 6.3] also demonstrates that the vertical maps in (4) are approximations of order n, and since three out of the four maps in the lower right square are approximations of order n, so too is the lower right-hand horizontal map

$$T_n P_W \eta_{P_W F}\colon T_n P_W P_W F \to T_n P_W T_n P_W F.$$

An application of [Weiss 1995, Theorem 5.15] yields that this map is an objectwise weak equivalence as both source and target are polynomial of degree less than or equal n. This concludes the proof that the map

$$T_n P_W(\eta)\colon T_n P_W F \to T_n P_W T_n P_W F$$

is an objectwise weak equivalence, and verifies [Bousfield 2001, Theorem 9.3(A2)].

Finally we verify [Bousfield 2001, Theorem 9.3(A3)]. Let

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

be a pullback square with f an objectwise fibration between W–local n–polynomial functors, and $T_n P_W h\colon T_n P_W C \to T_n P_W D$ an objectwise weak equivalence. By [Bousfield 2001, Theorem 9.9], we see that the fibre of k is P_W–acyclic, ie $P_W(\mathsf{fib}(k))$ is objectwise weakly contractible. Since T_n preserves objectwise weak equivalences, we see that $T_n P_W(\mathsf{fib}(k))$ is objectwise weakly contractible, and hence k is a $T_n P_W$–equivalence.

The fact that the resulting model structure is topological follows from [Bousfield 2001, Theorem 9.1]. □

This Bousfield–Friedlander localization results in an identical model structure to the W–local n–polynomial model structure of Proposition 3.5

Proposition 7.6 *For a finite cell complex W there is an equality of model structures*

$$\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_W \mathsf{Top}_*) = \mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W \mathsf{Top}_*),$$

that is, the W–local n–polynomial model structure and the W–periodic n–polynomial model structure agree. In particular, these model structures are cellular, proper and topological.

Proof Both model structures have the same cofibrations, namely the projective cofibrations. It suffices to show that they share the same fibrant objects. Working through the definition of a fibrant object in the Bousfield–Friedlander localization we see that an orthogonal functor F is fibrant if and only if the canonical map $F \to T_n P_W F$ is an objectwise weak equivalence. It follows that F must be W–local and n–polynomial, hence fibrant in the W–local n–polynomial model structure. Conversely, if F is fibrant in the W–local n–polynomial model structure, then the map $F \to P_W F$ is an objectwise weak equivalence and there is a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & P_W F \arrow[d] \\ T_n F \arrow[r] & T_n P_W F \end{tikzcd}$$

in which three out of the four arrows are objectwise weak equivalences, hence so too is the right-hand vertical arrow. It follows that F is fibrant in the Bousfield–Friedlander localization. □

Remark 7.7 The nullification condition here is necessary. The above lemma does not hold in general. To see this, consider the (smashing) localization at the spectrum $E = H\mathbb{Q}$. The $H\mathbb{Q}$–local model structure is not right proper (see Remark 7.1), yet if this were expressible as a Bousfield–Friedlander localization it would necessarily be right proper [Bousfield 2001, Theorem 9.3].

Corollary 7.8 *For a finite cell complex W, a map $f : X \to Y$ is a fibration in the W–local n–polynomial model structure if and only if f is a fibration in the projective model structure and the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & T_n P_W X \arrow[d] \\ Y \arrow[r] & T_n P_W Y \end{tikzcd}$$

is a homotopy pullback square in the projective model structure on $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$.

Remark 7.9 It is highly unlikely that this result holds in more general localizations than nullifications. Let $\mathcal{C}$ be a model category and S a set of maps in $\mathcal{C}$ such that the left Bousfield localization of $\mathcal{C}$ at S exists. By [Hirschhorn 2003, Proposition 3.4.8(1)] right properness of $\mathcal{C}$ and $L_S\mathcal{C}$ is sufficient for a map $f : X \to Y$ being a fibration in $L_S\mathcal{C}$ if and only if f is a fibration in $\mathcal{C}$ and the square

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "f"'] & \widehat{X} \arrow[d, "\hat{f}"] \\ Y \arrow[r, "j_Y"'] & \widehat{Y} \end{tikzcd}$$

is a homotopy pullback square, where $\hat{f} : \widehat{X} \to \widehat{Y}$ is a S–localization of f in the sense of [Hirschhorn 2003, Definition 3.2.16]. In our situation, Proposition 7.2 guarantees that a homological localization is right proper if and only if it is a nullification. However, it is not clear in general if right properness of the base model category and the localized model category is a necessary condition for the above description of the fibrations in $L_S\mathcal{C}$.

7.3 Nullifications and homogeneous functors

In the case of a nullification, the W–local n–homogeneous model structure of Proposition 5.9 is not the only way of constructing a model structure with the correct homotopy category. Since the W–local model structure on based spaces is right proper, so too is the W–local n–polynomial model structure and hence we can also follow the more standard procedure and preform a right Bousfield localization at the set

$$\mathcal{K}'_n = \{\mathcal{J}_n(U, -) \mid U \in \mathcal{J}\},$$

to obtain a local n–homogeneous model category structure.

Proposition 7.10 *For a finite cell complex W there exists a model structure on the category of orthogonal functors with weak equivalences those maps $X \to Y$ such that*

$$(T_n P_W X)^{(n)} \to (T_n P_W X)^{(n)}$$

is an objectwise weak equivalence and with fibrations the fibrations of the W–local n–polynomial model structure. This model structure cellular, proper, stable and topological. We call this the W–periodic n–homogeneous model structure and denote it by $\mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*)$.

Proof This is the right Bousfield localization of the W–local n–polynomial model structure. The proof of which follows exactly as in [Barnes and Oman 2013, Proposition 6.9]. Note that this right Bousfield localization exists since the W–local n–polynomial model structure in right proper and cellular when the localization is a nullification; see Proposition 7.6. □

This right Bousfield localization behaves like a left Bousfield localization of the n–homogeneous model structure in the following sense.

Lemma 7.11 *For a finite cell complex W, the adjoint pair*

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction.

Proof Since the acyclic cofibrations of the n–homogeneous model structure are precisely the acyclic cofibrations of the n–polynomial model structure and similarly, the acyclic cofibrations of W–periodic n–homogeneous model structure are precisely the acyclic cofibrations of the W–local n–polynomial model structure, the identity functor preserves acyclic cofibrations by Lemma 3.6.

On the other hand, by [Hirschhorn 2003, Proposition 3.3.16(2)], cofibrations between cofibrant objects in a right Bousfield localization are cofibrations in the underlying model structure; hence Lemma 3.6 shows that the identity functor preserves cofibrations between cofibrant objects. The result follows by [Dugger 2001, Corollary A.2]. □

An analogous Quillen equivalence is obtained between the W–local intermediate category and the W–periodic n–homogeneous model structure of Proposition 7.10, which we recall is obtained as a right Bousfield localization of the W–local n–polynomial model structure. The proof is all but identical to [Barnes and Oman 2013, Theorem 10.1].

Theorem 7.12 *For a finite cell complex W, the adjoint pair*

$$\mathsf{res}_0^n/O(n)\colon L_W\,\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*) : \mathsf{ind}_0^n\varepsilon^*$$

is a Quillen equivalence.

Propositions 5.9 and 7.10 provide two different model structures which both capture the homotopy theory of W–locally n–homogeneous functors. However, these model structures are not identical. For instance, the W–local model structure of Proposition 5.9 has fibrant objects the n–polynomial functors which have W–local n^{th} derivative, whereas the fibrant objects of the W–periodic n–homogeneous model structure (Proposition 7.10) are the W–local n–polynomial functors. However, they are Quillen equivalent via the identity functor.

Corollary 7.13 *For a finite cell complex W, the adjoint pair*

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, L_W\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen equivalence.

Proof Since cofibrations between cofibrant objects in $\mathsf{Homog}^n(\mathcal{J}_0, L_W\mathsf{Top}_*)$ are projective cofibrations which are T_n–equivalences, and the cofibrations between cofibrant objects of $\mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*)$ are the projective cofibrations, it follows that the identity functor

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, L_W\mathsf{Top}_*) \to \mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*)$$

necessarily preserves cofibrations between cofibrant objects. On the other hand, the identity functor

$$\mathbb{1}\colon \mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*) \to \mathsf{Homog}^n(\mathcal{J}_0, L_W\mathsf{Top}_*)$$

preserves fibrant objects since if X is objectwise W–local, $\mathsf{ind}_0^n X$ is objectwise W–local, by Lemma 4.4. It follows that the adjunction is a Quillen adjunction. To see that the adjunction is a Quillen equivalence, there is a commutative square

$$\begin{tikzcd}
L_W\,\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \arrow[r, shift left, "\mathsf{res}_0^n/O(n)"] \arrow[d, shift right, "\mathbb{1}"'] & \mathsf{Homog}^n(\mathcal{J}_0, L_W\mathsf{Top}_*) \arrow[l, shift left, "\mathsf{ind}_0^n\varepsilon^*"] \arrow[d, shift right, "\mathbb{1}"'] \\
L_W\,\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \arrow[r, shift left, "\mathsf{res}_0^n/O(n)"] \arrow[u, shift right, "\mathbb{1}"'] & \mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*) \arrow[l, shift left, "\mathsf{ind}_0^n\varepsilon^*"] \arrow[u, shift right, "\mathbb{1}"']
\end{tikzcd}$$

of Quillen adjunctions, in which three out of four are Quillen equivalences by Theorems 5.15 and 7.12. Hence the remaining Quillen adjunction must also be a Quillen equivalence. □

It follows that there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, P_W\mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_W\mathsf{Top}_*)[O(n)],$$

whenever both model structures exist.

8 Postnikov sections

Given a based space A, the k^{th} Postnikov section of A is the nullification of A at S^{k+1}, ie $P_kA = P_{S^{k+1}}A$. Given a diagram of (simplicial, left proper, combinatorial) model categories, Barwick [2010, Section 5, Application 1] and Bergner [2012] develop a general machinery for producing a model structure which captures the homotopy theory of the homotopy limit of the diagram of model categories. Gutiérrez and Roitzheim [2016, Section 4] applied this to the study of Postnikov sections for model categories, which recovers the classical theory when $\mathcal{C}$ is the Kan–Quillen model structure on simplicial sets. We consider the relationship between Postnikov sections and orthogonal calculus via our local calculus.

8.1 A combinatorial model for calculus

The current theory of homotopy limits of model categories requires that the model categories in question be combinatorial, ie locally presentable and cofibrantly generated. Since the category of based compactly generated weak Hausdorff spaces is not locally presentable, the Quillen model structure is not combinatorial and hence none of our model categories for orthogonal functors are either. We invite the reader to take for granted that all of our cellular model categories may be replaced by combinatorial model categories by starting with a combinatorial model for the Quillen model structure on based spaces, and hence skip directly to Section 8.2.

We give the details of these combinatorial replacements here. We replace compactly generated weak Hausdorff spaces with Δ–*generated spaces*; a particular full subcategory of the category of topological spaces, which were developed by Vogt [1971] and unpublished work of Smith, and are surveyed by Dugger [2003]. The category of Δ–generated spaces may be equipped with a model structure analogous to the Quillen model structure on compactly generated weak Hausdorff spaces with weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations. This model structure is combinatorial, proper and topological. The existence of the model structure follows from [Dugger 2003, Section 1.9]. The locally presentable (and hence combinatorial) property follows from [Fajstrup and Rosický 2008, Corollary 3.7]. The Quillen equivalence may be extracted from [Dugger 2003, Section 1.9].

This combinatorial model for spaces transfers to categories of functors and we obtain a projective model structure on the category of orthogonal functors which is Quillen equivalent to our original projective model structure but is now combinatorial. A left or right Bousfield localization of a combinatorial model category is again combinatorial; hence the n–polynomial, n–homogeneous and local versions of these model categories are all combinatorial when we begin with the combinatorial model for the projective model structure on orthogonal functors.

Hypothesis 8.1 *For the remainder of this section, we will assume that all our model structures are combinatorial, since they are all Quillen equivalent to combinatorial model categories using the combinatorial model for based spaces.*

8.2 The model structure of k–types in orthogonal functors

Denote by I the set of generating cofibrations of the projective model structure of orthogonal functors, and denote by W_k the set of maps of the form

$$B \wedge S^{k+1} \amalg_{A \wedge S^{k+1}} A \wedge D^{k+2} \to B \wedge D^{k+2},$$

where $A \to B$ is a map in I. The model category of k–types in $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ is the left Bousfield localization of the projective model structure at $I \,\square\, \{S^{k+1} \to D^{k+2}\}$ used by Gutiérrez and Roitzheim [2016] to model Postnikov sections.

Proposition 8.2 *Let $k \geq 0$. Under Hypothesis 8.1, the model structure of k–types in the category of orthogonal functors is identical to the S^{k+1}–local model structure, that is, there is an equality of model structures,*

$$P_k \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) := L_{W_k} \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) = \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}} \mathsf{Top}_*).$$

Proof It suffices to show that both model structures have the same fibrant objects since the cofibrations in both model structures are identical. To see this, note that by examining the pushout product we can rewrite the set W_k as

$$W_k = \{\mathcal{J}_0(U,-) \wedge S_+^{n+k+1} \to \mathcal{J}_0(U,-) \wedge D_+^{n+k+2} \mid n \geq 0,\, U \in \mathcal{J}_0\}.$$

It follows by an adjunction argument that an orthogonal functor Z is W_k–local if and only if $\pi_i Z(U)$ is trivial for all $i \geq k+1$ and all $U \in \mathcal{J}_0$. This last condition is equivalent to being objectwise S^{k+1}–local. □

8.3 The model structure of k–types in spectra

Taking I_{Sp} to be the set of generating cofibrations of the stable model structure on Sp and denoting again by W_k the relevant pushout product maps, we obtain a similar characterisation of the category of k–types in spectra.

Proposition 8.3 *Let $k \geq 0$. Under Hypothesis 8.1, there is an equality of model structures between the model category of k–types in spectra, and the stabilisation of S^{k+1}–local spaces, that is,*

$$P_k \mathsf{Sp} := L_{W_k} \mathsf{Sp} = \mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*).$$

Proof Both model structures can be described as particular left Bousfield localizations of the stable model structure on spectra, hence have the same cofibrations. The proof reduces to the fact that the model structures have the same fibrant objects. To see this, note that the fibrant objects of $P_k\mathsf{Sp}$ are the k–truncated Ω–spectra, and the fibrant objects of $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$ are the levelwise k–truncated Ω–spectra. Since both fibrant objects are Ω–spectra a connectivity style argument yields that an Ω–spectrum is k–truncated if and only if it is levelwise k–truncated, and hence both model structures have the same fibrant objects. □

Remark 8.4 Given a compact Lie group G, a similar procedure shows that there is an equality of model structures

$$P_k \mathsf{Sp}[G] := L_{W_k} \mathsf{Sp}[G] = \mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)[G].$$

8.4 Postnikov reconstruction of orthogonal functors

The collection of S^{k+1}–local model structures on the category of orthogonal functors assembles into a tower of model categories[7]

$$\mathsf{P}_\bullet \colon \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}} \mathsf{Top}_*),$$

[7] A tower of model categories is a special instance of a left Quillen presheaf, that is a diagram of the form $F \colon \mathcal{J}^{\mathrm{op}} \to \mathsf{MCat}$ for some small indexing category $\mathcal{J}$.

where MCat denotes the category of model categories and left Quillen functors. The homotopy limit of this tower of model categories recovers the projective model structure on orthogonal functors. The existence of a model structure which captures the homotopy theory of the limit of these model categories follows from [Gutiérrez and Roitzheim 2016, Proposition 2.2]. In particular, the homotopy limit model structure is a model structure on the category of sections[8] of the diagram $\mathsf{P}_\bullet$ formed by right Bousfield localizing the injective model structure in which a map of sections is a weak equivalence or cofibration if it is an objectwise weak equivalence or cofibration respectively.

Lemma 8.5 [Gutiérrez and Roitzheim 2016, Theorem 1.3 and Proposition 2.2] *There is a combinatorial model structure on the category of sections of $\mathsf{P}_\bullet$ where a map $f_\bullet\colon X_\bullet \to Y_\bullet$ is a fibration if and only if f_0 is a fibration in $\mathsf{Fun}(\mathcal{J}_0, L_{S^1}\mathsf{Top}_*)$ and for every $k \geq 1$ the induced map*

$$\begin{tikzcd}
X_k \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y_k \times_{Y_{k-1}} X_{k-1} \arrow[r] \arrow[d] & X_{k-1} \arrow[d] \\
& Y_k \arrow[r] & Y_{k-1}
\end{tikzcd}$$

indicated by a dotted arrow in the above diagram is a fibration in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$. A section $X_\bullet$ is cofibrant if and only if X_n is cofibrant in $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ and for every $k \geq 0$, the map $X_{k+1} \to X_k$ is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$. A map of cofibrant sections is a weak equivalence if and only if the map is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$ for each $k \geq 0$. We will refer to this model structure as the homotopy limit model structure and denote it by $\operatorname{holim}\mathsf{P}_\bullet$.

Proposition 8.6 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const}\colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \operatorname{holim}\mathsf{P}_\bullet :\lim$$

is a Quillen equivalence.

Proof The adjoint pair exists, and is a Quillen adjunction by [Gutiérrez and Roitzheim 2016, Lemma 2.4].

To see that the adjoint pair is a Quillen equivalence let $X_\bullet$ be a cofibrant and fibrant section in the homotopy limit model structure. Showing that

$$\mathrm{const}\lim \mathrm{X}_\bullet \to \mathrm{X}_\bullet$$

is a weak equivalence is equivalent to showing that the map $\lim X_\bullet \to X_k$ is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$ for all $k \geq 0$. This is in turn, equivalent to the map $(\lim X_\bullet)(U) \to X_k(U)$ being a

[8] A section $X_\bullet$ of the tower $\mathsf{P}_\bullet$ is a sequence $\cdots \to X_k \to X_{k+1} \to \cdots \to X_0$ of orthogonal functors, and a morphism of sections $f\colon X_\bullet \to Y_\bullet$ is given by maps of orthogonal functors $f_k\colon X_k \to Y_k$ for all $k \geq 0$ subject to a commutative ladder condition.

weak equivalence in $L_{S^{k+1}}\mathsf{Top}_*$ for all $k \geq 0$. Since limits in functor categories are computed objectwise, the fact that the unit is a weak equivalence follows from [Gutiérrez and Roitzheim 2016, Theorem 2.5]. A similar argument shows that the counit is also a weak equivalence. □

8.5 Postnikov reconstruction for spectra with an $O(n)$–action

The aim is to show that similar reconstruction theorems may be obtained for the n–homogeneous model structures. We first start by investigating analogous theorems for spectra and show that such reconstructions are compatible with the zigzag of Quillen equivalences between spectra with an $O(n)$–action and the n–homogeneous model structure. Proposition 8.3 and [Gutiérrez and Roitzheim 2016, Section 2.1] imply that the functor

$$\mathsf{P}_\bullet^{\mathsf{Sp}}\colon \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$$

defines a left Quillen presheaf.[9] This left Quillen presheaf is "convergent" in the following sense.

Proposition 8.7 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const}\colon \mathsf{Sp} \rightleftarrows \operatorname{holim} \mathsf{P}_\bullet^{\mathsf{Sp}} : \lim$$

is a Quillen equivalence.

Proof The fact that the adjoint pair is a Quillen adjunction follows from [Gutiérrez and Roitzheim 2016, Lemma 2.4].

The left adjoint reflects weak equivalences between cofibrant objects. Indeed, if $X \to Y$ is a map between cofibrant spectra X and Y such that

$$\mathrm{const}(X) \to \mathrm{const}(Y)$$

is a weak equivalence in $\operatorname{holim} \mathsf{P}_\bullet^{\mathsf{Sp}}$, then

$$\mathrm{const}(X) \to \mathrm{const}(Y)$$

is a weak equivalence in $\mathrm{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathsf{Sp}})$ by the colocal Whitehead theorem and the fact that the left adjoint is left Quillen and thus preserves cofibrant objects. It follows that for each $k \in \mathbb{N}$, the induced map

$$\mathrm{const}(X)_k \to \mathrm{const}(Y)_k$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$, that is, $X \to Y$ is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all k. Unpacking the definition of a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ and using the fact that the right adjoint is a right Quillen functor and hence preserves weak equivalences between fibrant objects, we see that the induced map

$$\lim P_k X \to \lim P_k Y$$

is a weak equivalence in Sp, and hence, so is the map $X \to Y$.

[9] Alternatively, the adjunction $\mathbb{1}\colon \mathsf{Sp}(L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$, is a Quillen adjunction. This fact follows from the facts that both model structures have the same cofibrations and a S^{k+1}–local space is S^{k+2}–local as $\langle \Sigma W \rangle \leq \langle W \rangle$ for all based spaces W; see eg [Bousfield 1994, Section 9.9]. Hence $\mathsf{P}_\bullet^{\mathsf{Sp}}$ is a left Quillen presheaf.

It is left to show that the derived counit is an isomorphism. Let $Y_\bullet$ be bifibrant in holim $\mathsf{P}_\bullet^{\mathsf{Sp}}$. The condition that the counit applied to $Y_\bullet$ is a weak equivalence is equivalent to asking for the map

$$\lim_{\geq k} P_k Y_\bullet \to Y_k$$

to be a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all $k \in \mathbb{N}$. The structure maps of $Y_\bullet$ induce a map of towers

$$\begin{tikzcd}
\cdots \arrow[r] & Y_j \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_{k+3} \arrow[r] \arrow[d] & Y_{k+2} \arrow[r] \arrow[d] & Y_{k+1} \arrow[d] \\
\cdots \arrow[r] & Y_{k+1} \arrow[r] & \cdots \arrow[r] & Y_{k+1} \arrow[r] & Y_{k+1} \arrow[r] & Y_{k+1}
\end{tikzcd}$$

in which each vertical arrow is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$. This map of towers induces a map

$$\begin{tikzcd}
0 \arrow[r] & \lim^1_{\geq k}\ \pi_{i+1}(Y_\bullet) \arrow[r] \arrow[d] & \pi_i(\lim_{\geq k}\ Y_\bullet) \arrow[r] \arrow[d] & \lim_{\geq k}\ \pi_i(Y_\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \lim^1_{\geq k}\ \pi_{i+1}(Y_{k+1}) \arrow[r] & \pi_i(\lim_{\geq k}\ Y_{k+1}) \arrow[r] & \lim_{\geq k}\ \pi_i(Y_{k+1}) \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences. For $0 \leq i < n$ the left- and right-hand side maps are isomorphisms; hence the map

$$\lim_{\geq k}\ Y_\bullet \to Y_{k+1}$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all k, and it follows that the required map

$$\lim_{\geq k}\ Y_\bullet \to Y_{k+1} \to Y_k$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)$ for all k. □

A similar justification to before provides a left Quillen presheaf

$$\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} \colon \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)[O(n)],$$

where $\mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*)[O(n)]$ is the category of $O(n)$–objects in the category of k–types in spectra. This is equivalent to the category of k–types in spectra with an $O(n)$–action. As a corollary to Proposition 8.7, we obtain that the induced left Quillen presheaf on spectra with an $O(n)$–action is also suitably convergent.

Corollary 8.8 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const} \colon \mathsf{Sp}[\mathrm{O(n)}] \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet^{\mathsf{Sp}[\mathrm{O(n)}]} : \lim$$

is a Quillen equivalence.

8.6 Postnikov reconstruction for the intermediate categories

The functor

$$\mathsf{P}_\bullet^{\mathcal{J}_n} : \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \mapsto L_{S^{k+1}} \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)$$

defines a left Quillen presheaf, since there is an equality of model structures between the S^{k+1}–local n–stable model structure and the model structure of k–types in $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)$. The proof of which is completely analogous to the case for spectra; see Proposition 8.3. Since the S^{k+1}–local n–stable model structure agrees with the model structure of k–types, we will denote both model structures by $P_k \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)$.

The homotopy limit of this left Quillen presheaf agrees with the homotopy limit of the left Quillen presheaf on spectra with an $O(n)$–action in the sense that the homotopy limit model categories are Quillen equivalent. In detail, the adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*$$

of [Barnes and Oman 2013, Section 8] induces an adjunction

$$(\alpha_n)_!^{\mathbb{N}} : \mathsf{Fun}(\mathbb{N}, \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*)) \rightleftarrows \mathsf{Fun}(\mathbb{N}, \mathsf{Sp}[O(n)]) : (\alpha_n^*)^{\mathbb{N}}$$

where $(\alpha_n^*)^{\mathbb{N}} = (\alpha_n)^* \circ (-)$. This adjunction in turn induces an adjunction

$$(\alpha_n)_!^{\mathbb{N}} : \operatorname{holim} \mathsf{P}_\bullet^{\mathcal{J}_n} \rightleftarrows \operatorname{holim} \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} : (\alpha_n^*)^{\mathbb{N}}.$$

Proposition 8.9 *Under Hypothesis 8.1 the adjoint pair*

$$(\alpha_n)_!^{\mathbb{N}} : \operatorname{holim} \mathsf{P}_\bullet^{\mathcal{J}_n} \rightleftarrows \operatorname{holim} \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} : (\alpha_n^*)^{\mathbb{N}}$$

is a Quillen equivalence.

Proof Fibrations of the homotopy limit model structure of $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$ are precisely the fibrations of the injective model structure on the category of sections of $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$ since the homotopy limit model structure is a right Bousfield localization of the injective model structure. A similar characterisation holds for the left Quillen presheaf $\mathsf{P}_\bullet^{\mathcal{J}_n}$; hence to show that the right adjoint preserves fibrations it suffices to show that the left adjoint preserves acyclic cofibrations of the injective model structure on the categories of sections. To see this, note that the adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*$$

is a Quillen adjunction, and hence so too is the induced adjunction on the injective model structures on the categories of sections.

To show that the left adjoint preserves cofibrations it suffices to show that cofibrations between cofibrant objects are preserved. As the homotopy limit model structures are right Bousfield localizations [Hirschhorn

2003, Proposition 3.3.16(2)] implies that cofibrations between cofibrant objects are cofibrations of the injective model structures on the categories of sections which by the analogous reasoning as above are preserved by the left adjoint. This yields that the adjunction in question is a Quillen adjunction.

To show that the adjunction is a Quillen equivalence notice that the right adjoint reflects weak equivalences between cofibrant objects by the colocal Whitehead theorem [Hirschhorn 2003, Theorem 3.2.13(2)], and the fact that the induced adjunction on the injective model structures on the categories of sections is a Quillen equivalence since for $B_\bullet \in \mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathcal{J}_n})$ and $X_\bullet \in \mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]})$, a map $B_\bullet \to (\alpha_n^*)^{\mathbb{N}} X_\bullet$ is a weak equivalence if and only if for each $k \in \mathbb{N}$, the map $B_k \to (\alpha_n^*)^{\mathbb{N}} X_k$ is a weak equivalence of spectra, which in turn happens if and only if the adjoint map $(\alpha_n)_! B_k \to X_k$ is an n–stable equivalence, which is precisely the condition that the adjoint map $(\alpha_n)_!^{\mathbb{N}} B_\bullet \to X_\bullet$ is a weak equivalence.

It is left to show that the derived counit is an isomorphism. Let $Y_\bullet$ be bifibrant in the homotopy limit model structure of the left Quillen presheaf $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$. Then the derived counit

$$(\alpha_n)_!^{\mathbb{N}} \hat{c}((\alpha_n^*)^{\mathbb{N}} Y_\bullet) \to Y_\bullet$$

is a map between cofibrant objects, hence a weak equivalence in the homotopy limit model structure if and only if a weak equivalence in the injective model structure on the category of sections, ie if and only if for each $k \in \mathbb{N}$, the induced map

$$(\alpha_n)_!(\alpha_n)^* Y_k \to Y_k$$

is a weak equivalence, which it always is by [Barnes and Oman 2013, Proposition 8.3]. □

As a corollary, we see that the left Quillen presheaf $\mathsf{P}_\bullet^{\mathcal{J}_n}$ is convergent.

Corollary 8.10 *Under Hypothesis 8.1 the adjoint pair*

$$\mathrm{const}\colon \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \rightleftarrows \operatorname{holim} \mathsf{P}_\bullet^{\mathcal{J}_n} :\lim$$

is a Quillen equivalence.

Proof Consider the commutative diagram

$$\begin{tikzcd}
\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\mathsf{Top}_*) \arrow[r, shift left, "(\alpha_n)_!"] \arrow[d, shift right, "\mathrm{const}"'] & \mathsf{Sp}[O(n)] \arrow[l, shift left, "(\alpha_n)^*"] \arrow[d, shift right, "\mathrm{const}"'] \\
\operatorname{holim} \mathsf{P}_\bullet^{\mathcal{J}_n} \arrow[r, shift left, "(\alpha_n)_!^{\mathbb{N}}"] \arrow[u, shift right, "\lim"'] & \operatorname{holim} \mathsf{P} \arrow[l, shift left, "(\alpha_n^*)^{\mathbb{N}}"] \arrow[u, shift right, "\lim"']
\end{tikzcd}$$

of Quillen adjunctions in which three out of the four adjoint pairs are Quillen equivalences by [Barnes and Oman 2013, Proposition 8.3], Corollary 8.8 and Proposition 8.9. It follows since Quillen equivalences satisfy the two-out-of-three property, that the remaining Quillen adjunction is a Quillen equivalence. □

8.7 Postnikov reconstruction for homogeneous functors

The same approach as we have just employed for moving from spectra with an $O(n)$–action to the intermediate categories yields similar results for the homogeneous model structures. We choose to model S^{k+1}–local n–homogeneous functors by the S^{k+1}–periodic n–homogeneous model structures of Proposition 7.10.

Lemma 8.11 *The functor*

$$\mathsf{P}_\bullet^{\mathsf{Homog}^n} : \mathbb{N}^{\mathrm{op}} \to \mathsf{MCat}, \quad k \to \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+1}}\mathsf{Top}_*)$$

defines a left Quillen presheaf.

Proof It suffices to show that the adjoint pair

$$\mathbb{1} : \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction. The adjoint pair

$$\mathbb{1} : \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction since the composite of Quillen adjunctions is a Quillen adjunction, so the adjunction

$$\mathbb{1} : \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction, and by [Hirschhorn 2003, Proposition 3.3.18(1) and Theorem 3.1.6(1)], this composite Quillen adjunction extends to the S^{k+2}–local n–polynomial model structure since S^{k+1}–local n–polynomial functors are S^{k+2}–locally n–polynomial.

An application of [Hirschhorn 2003, Theorem 3.3.20(2)(a)] yields the desired result about the n–homogeneous model structures. □

Similar proofs to Proposition 8.9 and Corollary 8.10 yield the following results relating the n–homogeneous model structure to the homotopy limit of the tower of S^{k+1}–local n–homogeneous model structures.

Proposition 8.12 *Under Hypothesis 8.1 the adjunction*

$$(\mathsf{res}_0^n / O(n))^{\mathbb{N}} : \operatorname{holim} \mathsf{P}_\bullet^{\mathsf{Homog}^n} \rightleftarrows \operatorname{holim} \mathsf{P}_\bullet^{\mathcal{J}_n} : (\mathsf{ind}_0^n \varepsilon^*)^{\mathbb{N}}$$

is a Quillen equivalence.

Corollary 8.13 *Under Hypothesis 8.1 the adjunction*

$$\mathrm{const} : \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \operatorname{holim} \mathsf{P}_\bullet^{\mathsf{Homog}^n} : \lim$$

is a Quillen equivalence.

References

[Arone 2002] **G Arone**, *The Weiss derivatives of $BO(-)$ and $BU(-)$*, Topology 41 (2002) 451–481 MR Zbl

[Arone 2009] **G Arone**, *Derivatives of embedding functors, I: The stable case*, J. Topol. 2 (2009) 461–516 MR Zbl

[Arone and Mahowald 1999] **G Arone**, **M Mahowald**, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. 135 (1999) 743–788 MR Zbl

[Arone et al. 2007] **G Arone**, **P Lambrechts**, **I Volić**, *Calculus of functors, operad formality, and rational homology of embedding spaces*, Acta Math. 199 (2007) 153–198 MR Zbl

[Balchin 2021] **S Balchin**, *A handbook of model categories*, Algebra Appl. 27, Springer (2021) MR Zbl

[Barnes 2017] **D Barnes**, *Rational orthogonal calculus*, J. Homotopy Relat. Struct. 12 (2017) 1009–1032 MR Zbl

[Barnes and Eldred 2016] **D Barnes**, **R Eldred**, *Comparing the orthogonal and homotopy functor calculi*, J. Pure Appl. Algebra 220 (2016) 3650–3675 MR Zbl

[Barnes and Oman 2013] **D Barnes**, **P Oman**, *Model categories for orthogonal calculus*, Algebr. Geom. Topol. 13 (2013) 959–999 MR Zbl

[Barnes and Roitzheim 2011] **D Barnes**, **C Roitzheim**, *Local framings*, New York J. Math. 17 (2011) 513–552 MR Zbl

[Barnes and Roitzheim 2014] **D Barnes**, **C Roitzheim**, *Stable left and right Bousfield localisations*, Glasg. Math. J. 56 (2014) 13–42 MR Zbl

[Barthel 2020] **T Barthel**, *A short introduction to the telescope and chromatic splitting conjectures*, from "Bousfield classes and Ohkawa's theorem" (T Ohsawa, N Minami, editors), Springer Proc. Math. Stat. 309, Springer (2020) 261–273 MR Zbl

[Barwick 2010] **C Barwick**, *On left and right model categories and left and right Bousfield localizations*, Homology Homotopy Appl. 12 (2010) 245–320 MR Zbl

[Beaudry et al. 2022] **A Beaudry**, **I Bobkova**, **V-C Pham**, **Z Xu**, *The topological modular forms of $\mathbb{R}P^2$ and $\mathbb{R}P^2 \wedge \mathbb{C}P^2$*, J. Topol. 15 (2022) 1864–1926 MR Zbl

[Behrens 2012] **M Behrens**, *The Goodwillie tower and the EHP sequence*, Mem. Amer. Math. Soc. 1026, Amer. Math. Soc., Providence, RI (2012) MR Zbl

[Bergner 2012] **J E Bergner**, *Homotopy limits of model categories and more general homotopy theories*, Bull. Lond. Math. Soc. 44 (2012) 311–322 MR Zbl

[Bousfield 1975] **A K Bousfield**, *The localization of spaces with respect to homology*, Topology 14 (1975) 133–150 MR Zbl

[Bousfield 1979] **A K Bousfield**, *The localization of spectra with respect to homology*, Topology 18 (1979) 257–281 MR Zbl

[Bousfield 1994] **A K Bousfield**, *Localization and periodicity in unstable homotopy theory*, J. Amer. Math. Soc. 7 (1994) 831–873 MR Zbl

[Bousfield 1996] **A K Bousfield**, *Unstable localization and periodicity*, from "Algebraic topology: new trends in localization and periodicity" (C Broto, C Casacuberta, G Mislin, editors), Progr. Math. 136, Birkhäuser, Basel (1996) 33–50 MR Zbl

[Bousfield 1997] **A K Bousfield**, *Homotopical localizations of spaces*, Amer. J. Math. 119 (1997) 1321–1354 MR Zbl

[Bousfield 2001] **A K Bousfield**, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. 353 (2001) 2391–2426 MR Zbl

[Casacuberta 1994] **C Casacuberta**, *Recent advances in unstable localization*, from "The Hilton Symposium 1993: Topics in topology and group theory" (G Mislin, editor), CRM Proc. Lecture Notes 6, Amer. Math. Soc., Providence, RI (1994) 1–22 MR Zbl

[Devinatz et al. 1988] **E S Devinatz**, **M J Hopkins**, **J H Smith**, *Nilpotence and stable homotopy theory, I*, Ann. of Math. 128 (1988) 207–241 MR Zbl

[Dror Farjoun 1996] **E Dror Farjoun**, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Math. 1622, Springer (1996) MR Zbl

[Dugger 2001] **D Dugger**, *Replacing model categories with simplicial ones*, Trans. Amer. Math. Soc. 353 (2001) 5003–5027 MR Zbl

[Dugger 2003] **D Dugger**, *Notes on delta-generated spaces*, preprint (2003) `https://pages.uoregon.edu/ddugger/delta.html`

[Dwyer and Spaliński 1995] **W G Dwyer**, **J Spaliński**, *Homotopy theories and model categories*, from "Handbook of algebraic topology" (I M James, editor), North-Holland, Amsterdam (1995) 73–126 MR Zbl

[Fajstrup and Rosický 2008] **L Fajstrup**, **J Rosický**, *A convenient category for directed homotopy*, Theory Appl. Categ. 21 (2008) 7–20 MR Zbl

[Greenlees and Shipley 2014] **J P C Greenlees**, **B Shipley**, *An algebraic model for free rational G–spectra*, Bull. Lond. Math. Soc. 46 (2014) 133–142 MR Zbl

[Gutiérrez and Roitzheim 2016] **J J Gutiérrez**, **C Roitzheim**, *Towers and fibered products of model structures*, Mediterr. J. Math. 13 (2016) 3863–3886 MR Zbl

[Gutiérrez and Roitzheim 2017] **J J Gutiérrez**, **C Roitzheim**, *Bousfield localisations along Quillen bifunctors*, Appl. Categ. Structures 25 (2017) 1113–1136 MR Zbl

[Heuts 2021] **G Heuts**, *Lie algebras and v_n–periodic spaces*, Ann. of Math. 193 (2021) 223–301 MR Zbl

[Hirschhorn 2003] **P S Hirschhorn**, *Model categories and their localizations*, Math. Surv. Monogr. 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[Hopkins and Smith 1998] **M J Hopkins**, **J H Smith**, *Nilpotence and stable homotopy theory, II*, Ann. of Math. 148 (1998) 1–49 MR Zbl

[Hovey 1999] **M Hovey**, *Model categories*, Math. Surv. Monogr. 63, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[Hovey 2001] **M Hovey**, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra 165 (2001) 63–127 MR Zbl

[Krannich and Randal-Williams 2021] **M Krannich**, **O Randal-Williams**, *Diffeomorphisms of discs and the second Weiss derivative of $B\mathrm{Top}(-)$*, preprint (2021) arXiv 2109.03500

[Kuhn 2004] **N J Kuhn**, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math. 157 (2004) 345–370 MR Zbl

[Kuhn 2006a] **N J Kuhn**, *Localization of André–Quillen–Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. 201 (2006) 318–378 MR Zbl

[Kuhn 2006b] **N J Kuhn**, *Mapping spaces and homology isomorphisms*, Proc. Amer. Math. Soc. 134 (2006) 1237–1248 MR Zbl

[Kuhn 2007] **N J Kuhn**, *Goodwillie towers and chromatic homotopy: an overview*, from "Proceedings of the Nishida Fest" (M Ando, N Minami, J Morava, W S Wilson, editors), Geom. Topol. Monogr. 10, Geom. Topol. Publ., Coventry (2007) 245–279 MR Zbl

[Macko 2007] **T Macko**, *The block structure spaces of real projective spaces and orthogonal calculus of functors*, Trans. Amer. Math. Soc. 359 (2007) 349–383 MR Zbl

[Mahowald 1981] **M Mahowald**, *bo–resolutions*, Pacific J. Math. 92 (1981) 365–383 MR Zbl

[Mandell and May 2002] **M A Mandell**, **J P May**, *Equivariant orthogonal spectra and S–modules*, Mem. Amer. Math. Soc. 755, Amer. Math. Soc., Providence, RI (2002) MR Zbl

[Mandell et al. 2001] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. Lond. Math. Soc. 82 (2001) 441–512 MR Zbl

[Miller 1981] **H R Miller**, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space*, J. Pure Appl. Algebra 20 (1981) 287–312 MR Zbl

[Quillen 1969] **D Quillen**, *Rational homotopy theory*, Ann. of Math. 90 (1969) 205–295 MR Zbl

[Ravenel 1984] **D C Ravenel**, *Localization with respect to certain periodic homology theories*, Amer. J. Math. 106 (1984) 351–414 MR Zbl

[Schwede and Shipley 2003] **S Schwede**, **B Shipley**, *Stable model categories are categories of modules*, Topology 42 (2003) 103–153 MR Zbl

[Sullivan 1977] **D Sullivan**, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977) 269–331 MR Zbl

[Taggart 2020] **N Taggart**, *Beyond orthogonal calculus*: *the unitary and real cases*, PhD thesis, Queen's University Belfast (2020) `https://pure.qub.ac.uk/en/studentTheses/beyond-orthogonal-calculus`

[Taggart 2021] **N Taggart**, *Comparing the orthogonal and unitary functor calculi*, Homology Homotopy Appl. 23 (2021) 227–256 MR Zbl

[Taggart 2022a] **N Taggart**, *Unitary calculus: model categories and convergence*, J. Homotopy Relat. Struct. 17 (2022) 419–462 MR Zbl

[Taggart 2022b] **N Taggart**, *Unitary functor calculus with reality*, Glasg. Math. J. 64 (2022) 197–230 MR Zbl

[Taggart 2023] **N Taggart**, *Recovering unitary calculus from calculus with reality*, J. Pure Appl. Algebra 227 (2023) art. id. 107416 MR Zbl

[Vogt 1971] **R M Vogt**, *Convenient categories of topological spaces for homotopy theory*, Arch. Math. (Basel) 22 (1971) 545–555 MR Zbl

[Weiss 1995] **M Weiss**, *Orthogonal calculus*, Trans. Amer. Math. Soc. 347 (1995) 3743–3796 MR Zbl

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Bounded subgroups of relatively finitely presented groups

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Let G be a finitely generated group that is relatively finitely presented with respect to a collection H_Λ of peripheral subgroups such that the corresponding relative Dehn function is well defined. We prove that every infinite subgroup H of G that is bounded in the relative Cayley graph of G with respect to H_Λ is conjugate into a peripheral subgroup. As an application, we obtain a trichotomy for subgroups of relatively hyperbolic groups. Moreover we prove the existence of the relative exponential growth rate for all subgroups of limit groups.

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1 Introduction

The notion of a group G that is hyperbolic relative to a finite set H_Λ of its subgroups was introduced by Gromov [10] as a generalization of a word hyperbolic group. In his definition, the groups $H \in H_\Lambda$ appear as stabilizers of points at infinity of a certain hyperbolic space X the group G acts on. Since then, the study of relatively hyperbolic groups has remained an active field of research, and several characterizations of relative hyperbolicity were introduced by Bowditch [2], Farb [8] and Osin [15]. In the last work, Osin uses the concept of relative presentations in order to define the relative hyperbolicity of a group G with respect to a set $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. To make this more precise, let $X \subseteq G$ be a symmetric subset such that G is generated by $\bigcup_{\lambda\in\Lambda} H_\lambda \cup X$. Then we obtain a canonical epimorphism

$$\varepsilon\colon F := \Big(\mathop{\ast}_{\lambda\in\Lambda} \widetilde{H}_\lambda \Big) \ast F(X) \to G,$$

where the groups $\widetilde{H}_\lambda$ are disjoint isomorphic copies of H_λ, and $F(X)$ denotes the free group over X. Consider a subset $\mathcal{R} \subseteq F$ whose normal closure is the kernel of ε. Then $\mathcal{R}$ gives rise to a so-called *relative presentation of* G *with respect to* H_Λ of the form

$$\Big\langle X, \mathcal{H} \;\Big|\; S = 1, S \in \bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda,\, R = 1, R \in \mathcal{R} \Big\rangle, \tag{1}$$

where $\mathcal{H} := \bigcup_{\lambda\in\Lambda} (\widetilde{H}_\lambda \setminus \{1\})$ and $\mathcal{S}_\lambda$ is the set of all relations over the alphabet $\widetilde{H}_\lambda$. In this framework, G is said to be *hyperbolic relative to* H_Λ if X and $\mathcal{R}$ can be chosen to be finite and (1) admits a *linear* *relative Dehn function*. That is, there is some $C > 0$ such that for every word w of length at most ℓ over $X \cup \mathcal{H}$ that represents the identity in G, there is an equality of the form

$$w =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i \tag{2}$$

that holds in F, where $k \leq C\ell$, $f_i \in F$ and $R_i \in \mathcal{R}$. Note that in general, there is no reason to expect that for every $\ell \in \mathbb{N}$ and every relation w of length at most ℓ there is a uniform upper bound $n \in \mathbb{N}$ such that w can be written as in (2) with $k \leq n$. Even if X and $\mathcal{R}$ are finite, in which case we say that (1) is a *finite relative presentation for* G, there are easy examples where there is no such n; see [15, Example 1.3].

Here we study groups G that admit a finite relative presentation as in (1) whose relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. This means that for every $\ell \in \mathbb{N}$ there is a minimal number $\delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell)$ such that for every relation w of length at most ℓ there is an expression of the form (2) with $k \leq \delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell)$. Examples of relatively finitely presented groups that admit a well-defined nonlinear relative Dehn function were considered by Hughes, Martínez-Pedroza and Sánchez Saldaña [12]. The study of groups with a well-defined relative Dehn function typically involves considerations in the so-called *relative Cayley graph* $\Gamma(G, X \cup \mathcal{H})$ of G. Since $X \cup \mathcal{H}$ can be (and usually is) infinite, it is natural to ask the following:

Question 1.1 *Which subgroups of G have bounded diameter in $\Gamma(G, X \cup \mathcal{H})$?*

Note that, aside from the finite subgroups of G, every subgroup of G that can be conjugated into some of the groups H_λ has bounded diameter in $\Gamma(G, X \cup \mathcal{H})$. It turns out that for finitely generated G, the existence of a well-defined relative Dehn function is enough to deduce that there are no further examples of subgroups of G whose diameter in $\Gamma(G, X \cup \mathcal{H})$ is finite.

Theorem 1.2 *Let G be a finitely generated group. Suppose that G is relatively finitely presented with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*
(ii) *K is infinite and conjugate to a subgroup of some H_λ.*
(iii) *K is unbounded in $\Gamma(G, X \cup \mathcal{H})$.*

Note for example that if one of the subgroups H_λ in Theorem 1.2 is infinite, then there is no subgroup $K \leq G$ that contains H_λ as a proper subgroup of finite index. This also follows from the fact that each H_λ is almost malnormal, which is shown in [15, Proposition 2.36].

Remark 1.3 The condition that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ in Theorem 1.2 is well defined cannot be removed. To see this, let G be the infinite cyclic group generated by an element a and let H be the subgroup of G that is generated by a^2. Then the relative Cayley graph of G with respect to $H_\Lambda = \{H\}$ is clearly bounded. In particular, G is a bounded subset of its relative Cayley graph while not being conjugate to a subgroup of H. On the other hand, it can be easily seen that G admits a finite relative presentation with respect to H.

If the group G in Theorem 1.2 is relatively hyperbolic with respect to H_Λ, then it is known that a subgroup $K \leq G$ with infinite diameter in $\Gamma(G, X \cup \mathcal{H})$ contains a loxodromic element; see Osin [16, Theorem 1.1 and Proposition 5.2]. Recall that an element $g \in G$ is called *loxodromic* if the map

$$\mathbb{Z} \to \Gamma(G, X \cup \mathcal{H}) \quad \text{given by } n \mapsto g^n$$

is a quasiisometric embedding. We therefore obtain the following classification of subgroups of relatively hyperbolic groups which, to the best of my knowledge, was not recorded before:

Corollary 1.4 *Let G be a finitely generated group. Suppose that G is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugate to a subgroup of some H_λ.*

(iii) *K contains a loxodromic element.*

As an application of Corollary 1.4, we consider relative exponential growth rates in finitely generated groups. Recall that for a finitely generated group G and a finite generating set X of G, the *growth function* $\beta_G^X\colon \mathbb{N} \to \mathbb{N}$ of G with respect to X is defined by $\beta_G^X(n) = |B_G^X(n)|$, where $B_G^X(n)$ denotes the set of all elements of G that are represented by words of length at most n in the generators of X and X^{-1}. Using Fekete's lemma, it is easy to see that the limit $\lim_{n\to\infty} \sqrt[n]{\beta_G^X(n)}$, known as the exponential growth rate of G with respect to X, always exist; see for example Milnor [13]. Given a subgroup $H \leq G$, a relative analogue of the exponential growth rate is obtained by counting the elements in the relative balls $B_H^X(n) := B_G^X(n) \cap H$. The resulting function

$$\beta_H^X\colon \mathbb{N} \to \mathbb{N} \quad \text{given by } n \mapsto |B_H^X(n)|$$

is called the *relative growth function* of H with respect to X. In [14, Remark 3.1], Olshanskii pointed out that, unlike in the nonrelative case, the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ does not exist in general. As a consequence, the *relative exponential growth rate of H in G with respect to X* is typically defined as $\limsup_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$. Nevertheless, in many cases where the relative exponential growth rate is studied in the literature (see for example Cohen [3], Grigorchuk [9], Olshanskii [14], Sharp [19], Coulon, Dal'Bo and Sambusetti [5] and Dahmani, Futer and Wise [7] where G is free or hyperbolic) the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ is known to exist, in which case we say that the relative exponential growth rate of H in G exists with respect to X. In the case where G is a free group, the existence of the relative exponential growth rate was proven by Olshanskii in [14], extending prior results of Cohen [3] and Grigorchuk [9] who have independently proven the existence for normal subgroups of G. More recently, these existence results were generalized by the author to the case where G is a finitely generated acylindrically hyperbolic group and H is a subgroup that contains a generalized loxodromic element of G; see [17]. By combining this with Corollary 1.4, we will be able conclude the following:

Theorem 1.5 *Let G be a finitely generated group that is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Suppose that each of the groups H_λ has subexponential growth. Then the relative exponential growth rate of every subgroup $H \leq G$ exists with respect to every finite generating set of G.*

By Osin [15, Theorem 1.1], each of the groups H_λ in Theorem 1.5 is finitely generated, so the assumption on subexponential growth indeed makes sense. Relatively hyperbolic groups G as in Theorem 1.5 include many naturally occurring examples of groups. A particularly interesting such class is given by limit groups, which were introduced by Sela in his solution of the Tarski problems [18], and naturally generalize the class of free groups. By work of Dahmani [6] and Alibegović [1], limit groups are known to be relatively hyperbolic with respect to a system of representatives for the conjugacy classes of their maximal abelian noncyclic subgroups. As a consequence, we obtain the following generalization of Olshanskii's existence result:

Corollary 1.6 *Let G be a limit group. Then the relative exponential growth rate of every subgroup $H \leq G$ exists with respect to every finite generating set of G.*

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2 Preliminaries

In this section we introduce some definitions and properties that will be relevant for our study of relatively finitely presented groups. More information about these groups can be found in [15].

2.1 Relative presentations

Let us fix a group G and a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of so-called *peripheral subgroups* of G. Let $X \subseteq G$ be a symmetric subset such that G is generated by $\bigcup_{\lambda\in\Lambda} H_\lambda \cup X$. Such an X will be referred to as a *relative generating* of G with respect to H_Λ. Note that this gives us a canonical epimorphism

$$\varepsilon\colon F := \Big(\mathop{\ast}_{\lambda\in\Lambda} \widetilde{H}_\lambda\Big) \ast F(X) \to G,$$

where the groups $\widetilde{H}_\lambda$ are pairwise disjoint isomorphic copies of H_λ and $F(X)$ denotes the free group over X. Let us also assume that $\widetilde{H}_\lambda \cap X = \varnothing$ for every $\lambda \in \Lambda$. Let N denote the kernel of ε and let $\mathcal{R} \subseteq N$ be a subset whose normal closure in F coincides with N. For each $\lambda \in \Lambda$ let $\mathcal{S}_\lambda$ be the set of words over $\widetilde{H}_\lambda \setminus \{1\}$ that represent the identity in G.

Definition 2.1 With the notation above, we say that a *relative presentation of G with respect to H_Λ* is a presentation of the form

$$\left\langle X, \mathcal{H} \;\middle|\; S = 1, S \in \bigcup_{\lambda \in \Lambda} \mathcal{S}_\lambda,\ R = 1,\ R \in \mathcal{R} \right\rangle, \tag{3}$$

where $\mathcal{H} := \bigcup_{\lambda \in \Lambda} (\widetilde{H}_\lambda \setminus \{1\})$. The relative presentation (3) is called finite if X and $\mathcal{R}$ are finite. In this case G is said to be *relatively finitely presented with respect to H_Λ*.

The following result will be crucial for us:

Theorem 2.2 [15, Theorem 1.1] *Let G be a finitely generated group and let $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ be a collection of its subgroups. Suppose that G is finitely presented with respect to H_Λ. Then the following conditions hold*:

(i) *The collection H_Λ is finite*, *ie* $|\Lambda| < \infty$.

(ii) *Each subgroup H_λ is finitely generated.*

2.2 Relative Dehn functions

Let G be a relatively finitely presented group with a finite relative presentation as in Definition 2.1. For each $\ell \in \mathbb{N}$, let N_ℓ denote the set of words of length at most ℓ over $X \cup \mathcal{H}$ that represent the identity in G. Given $w \in N_\ell$, let $\mathrm{vol}(w) \in \mathbb{N}$ be minimal with the property that there is an expression of the form

$$w =_F \prod_{i=1}^{\mathrm{vol}(w)} f_i^{-1} R_i^{\pm 1} f_i, \tag{4}$$

where the equality is taken in F, and $f_i \in F$ and $R_i \in \mathcal{R}$ for every $1 \le i \le \mathrm{vol}(w)$.

Definition 2.3 The *relative Dehn function* for the finite relative presentation (3) of G is defined by

$$\delta^{\mathrm{rel}}_{G,H_\Lambda} : \mathbb{N} \to \mathbb{N} \cup \{\infty\} \quad \text{given by } \ell \mapsto \sup\{\mathrm{vol}(w) \mid w \in N_\ell\}.$$

We say that $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined if $\delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell) < \infty$ for every $\ell \in \mathbb{N}$.

An important class of relatively finitely presented groups with a well-defined Dehn function consists of relatively hyperbolic groups, which can be defined in terms of the relative Dehn function.

Definition 2.4 A relatively finitely presented group G with a relative presentation (3) is called *relatively hyperbolic with respect to H_Λ* if there is some $C > 0$ such that $\delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell) \le C\ell$ for every $\ell \in \mathbb{N}$.

Of course, the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ depends on the finite relative presentation (3), and not just on H_Λ. But as for ordinary (nonrelative) Dehn functions of finitely presented groups, different finite relative presentations lead to asymptotically equivalent relative Dehn functions; see [15, Theorem 2.34]. In particular, the property of $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ being well defined or bounded above by a linear function does not depend on the choice of a finite relative presentation.

2.3 Geometry of the relative Cayley graph

Let us again consider a relatively finitely presented group G with a finite relative presentation as in Definition 2.1. The Cayley graph of G with respect to $X \cup \mathcal{H}$ is called *the relative Cayley graph* of G and will be denoted by $\Gamma(G, X \cup \mathcal{H})$. We will study the local geometry of $\Gamma(G, X \cup \mathcal{H})$. In order to do so, let us fix some terminology. Given an edge e of $\Gamma(G, X \cup \mathcal{H})$, we write $\partial_0(e)$ to denote the initial vertex of e and $\partial_1(e)$ to denote the terminal vertex of e. A sequence $p = (e_1, \dots, e_n)$ of edges in $\Gamma(G, X \cup \mathcal{H})$ is called a *path* if $\partial_1(e_i) = \partial_0(e_{i+1})$ for $1 \le i < n$. If moreover $\partial_0(e_1) = \partial_1(e_n)$, then p is said to be *cyclic*. The label of a path p will be denoted by $\mathrm{Lab}(p)$. Sometimes it is useful to forget about the initial vertex of a cyclic path $p = (e_1, \dots, e_n)$. To make this precise, we define the *loop* associated to p as the set $[p]$ of all paths of the form $(e_i, \dots, e_n, e_1, \dots, e_{i+1})$ for $1 \le i \le n$. A *subpath of a loop* $[p]$ is a subpath of some representative $p' \in [p]$. The algebraic counterpart of a loop is the set $[w]$ of all cyclic conjugates of a word w over $X \cup \mathcal{H}$, which will be referred to as a *cyclic word*. Accordingly, the label of a loop $[p]$ is defined as $\mathrm{Lab}([p]) := [\mathrm{Lab}(p)]$. Up to minor notational differences, the following definitions can be found in [15].

Definition 2.5 Let w be a word over $X \cup \mathcal{H}$. A subword v of w is a *λ–subword* if it consists of letters of $\widetilde{H}_\lambda$. If a λ–subword v of w is not properly contained in any other λ–subword of w, then v is called a *λ–syllable* of w. Similarly, we say that a word v over $X \cup \mathcal{H}$ is a λ–subword of a cyclic word $[w]$ if it is a λ–subword of some cyclic conjugate of w. If a λ–subword v of $[w]$ is not properly contained in any other λ–subword of $[w]$, then v is called a *λ–syllable* of $[w]$.

Let us now translate Definition 2.5 into conditions for paths in $\Gamma(G, X \cup \mathcal{H})$.

Definition 2.6 Let q be a path in $\Gamma(G, X \cup \mathcal{H})$. A subpath p of q is a *λ–subpath* if $\mathrm{Lab}(p)$ is a λ–subword of $\mathrm{Lab}(q)$. A λ–subpath p of q is called a *λ–component* of q if $\mathrm{Lab}(p)$ is a λ–syllable of $\mathrm{Lab}(q)$. Suppose now that q is cyclic, and consider the loop $[q]$ associated to q. We say that a subpath p of $[q]$ is a λ–subpath of $[q]$ if $\mathrm{Lab}(p)$ is a λ–subword of $\mathrm{Lab}([q])$. If moreover $\mathrm{Lab}(p)$ is a λ–syllable of $\mathrm{Lab}([q])$, then p is called a *λ–component* of $[q]$.

Definition 2.7 Let p_1 and p_2 be λ–components of a path p (resp. a loop $[q]$) in $\Gamma(G, X \cup \mathcal{H})$. We say that p_1 and p_2 are *connected*, if there is a path c in $\Gamma(G, X \cup \mathcal{H})$ that connects a vertex of p_1 with a vertex of p_2 and $\mathrm{Lab}(c)$ consists of letters of $\widetilde{H}_\lambda$. We say that p_1 is *isolated* in p (resp. $[q]$) if there are no further λ–components of p (resp. $[q]$) that are connected to p_1.

Let us now translate the notion of an isolated component of a path (loop) into a corresponding notion for syllables in (cyclic) words.

Definition 2.8 Let w be a word over $X \cup \mathcal{H}$ and let p be any path in $\Gamma(G, X \cup \mathcal{H})$ with $\mathrm{Lab}(p) = w$. We say that two λ–syllables v_1 and v_2 of w are *connected* (resp. *isolated*) if the corresponding λ–components p_1 and p_2 of p are connected (resp. isolated). If w represents the identity in G, and v_1 and v_2 are

λ–syllables of the cyclic word $[w]$, then v_1 and v_2 are *connected* (resp. *isolated*) if the corresponding λ–components p_1 and p_2 of the loop $[p]$ are connected (resp. isolated).

The following lemma is a direct consequence of [15, Lemma 2.27]. It will help us study the local structure of $\Gamma(G, X \cup \mathcal{H})$ and often lets us switch between the word metrics d_X and $d_{X\cup\mathcal{H}}$.

Lemma 2.9 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups, and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Then for every $n \in \mathbb{N}$ there is a finite subset $\Omega_n \subseteq G$ with the property that for every cyclic path q in $\Gamma(G, X \cup \mathcal{H})$ of length at most n and every isolated component p of the loop $[q]$, the label $\mathrm{Lab}(p)$ represents an element in Ω_n.*

3 The alternating growth condition

In this section we introduce the alternating growth condition, which will play a central role in our proof of Theorem 1.2.

3.1 Regular neighborhoods

Let us start by defining a condition for paths in graphs that can be thought of as a strong form of having no self-intersections.

Definition 3.1 Let Γ be a graph and let p be a path in Γ that consecutively traverses the sequence $v_0, \dots, v_n$ of vertices in Γ. We say that p has a *regular neighborhood in* Γ if every two vertices v_i and v_j that can be joined by an edge in Γ satisfy $|i - j| \leq 1$.

Example 3.2 If p is a geodesic path in a graph Γ then p has a regular neighborhood in Γ.

Example 3.3 If p is a nontrivial cyclic path in a graph Γ then p does not have a regular neighborhood in Γ.

Remark 3.4 Every path p that has a regular neighborhood in a graph Γ is locally 2–geodesic, ie the restriction of p to each subpath of length at most 2 is geodesic.

It will be useful for us to translate the concept of regular neighborhoods to words over some generating set of a group.

Definition 3.5 Let G be a group and let X be a generating set of G. A word w over X is called *regular* (with respect to X) if some path p in $\Gamma(G, X)$ with $\mathrm{Lab}(p) = w$ has a regular neighborhood in $\Gamma(G, X)$.

Remark 3.6 Let G be a group and let X be a generating set of G. From the definitions, it directly follows that a word w over X is regular if and only if every subword v of w of length at least 2 satisfies $|v|_X \geq 2$, where $|\cdot|_X$ denotes the word metric corresponding to X.

3.2 Sequences of alternating growth

We want to study sequences of regular words in the context of finitely generated relatively finitely presented groups. Let us therefore fix a finitely generated group G, a finite generating set X of G, and a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of peripheral subgroups of G. Suppose that G is relatively finitely presented with respect to H_Λ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. As in Section 2 we write $\widetilde{H}_\lambda$ to denote pairwise disjoint isomorphic copies of H_λ that also intersect trivially with X. Let us fix some notation in order to avoid ambiguities concerning the length and the evaluation of a word over $X \cup \mathcal{H}$, where as always $\mathcal{H} = \bigcup_{\lambda\in\Lambda}(\widetilde{H}_\lambda \setminus \{1\})$.

Notation 3.7 Let $w = w_1 \dots w_\ell$ be a word over $X \cup \mathcal{H}$. We write $\|w\| = \ell$ for the word length of w. The image of w in G will be denoted by $\bar{w}$. For any subset $Y \subseteq G$ we write $|\bar{w}|_Y$ for the length of a shortest word over Y that represents $\bar{w}$. If there is no such word, then we set $|\bar{w}|_Y = \infty$.

Definition 3.8 A sequence of words $(w_1^{(n)} \dots w_\ell^{(n)})_{n\in\mathbb{N}}$ of fixed length $\ell \geq 2$ over $X \cup \mathcal{H}$ satisfies the *alternating growth condition* if the following conditions are satisfied:

(I) If $w_i^{(n)} = x$ for some $1 \leq i \leq \ell$, $n \in \mathbb{N}$ and $x \in X$, then $w_i^{(m)} = x$ for every $m \in \mathbb{N}$. In this case we say that i is an *index of type* X.

(II) If $w_i^{(n)} \in \widetilde{H}_\lambda$ for some $1 \leq i \leq \ell$, $n \in \mathbb{N}$ and $\lambda \in \Lambda$, then $w_i^{(m)} \in \widetilde{H}_\lambda$ for every $m \in \mathbb{N}$. In this case we say that i is an *index of type* λ.

(III) The index 1 is not of type X.

(IV) Two consecutive indices are never of the same type.

(V) If i is of type λ, then $\bar{w}_i^{(n)} \notin H_\mu$ and $|\bar{w}_i^{(n)}|_X \geq n$ for every $\mu \in \Lambda \setminus \{\lambda\}$ and every $n \in \mathbb{N}$.

(VI) Each word $w_1^{(n)} \dots w_\ell^{(n)}$ is regular with respect to $X \cup \mathcal{H}$.

The following observation will be used frequently:

Remark 3.9 Given a regular word w over $X \cup \mathcal{H}$, it directly follows from the definitions that every syllable v in w is isolated and consists of a single edge.

3.3 Concatenating sequences of alternating growth

In what follows, we need to construct certain sequences $(w_1^{(n)} \dots w_\ell^{(n)})_{n\in\mathbb{N}}$ of words over $X \cup \mathcal{H}$ that satisfy the alternating growth condition so that ℓ can be chosen arbitrarily large. In order to do so, we will use the following lemma, which allows us to "concatenate" two sequences of words that satisfy the alternating growth condition so that the resulting sequence also satisfies the alternating growth condition.

Lemma 3.10 *With the notation above, suppose that there are two sequences $(v_1^{(n)}\dots v_M^{(n)})_{n\in\mathbb{N}}$ and $(w_1^{(n)}\dots w_N^{(n)})_{n\in\mathbb{N}}$ of words over $X\cup\mathcal{H}$ that satisfy the alternating growth condition. Let $\lambda\in\Lambda$ be such that $w_1^{(n)}\in\widetilde{H}_\lambda$ for some $n\in\mathbb{N}$.*

(i) *Suppose that $\bar{v}_M^{(n)}\notin H_\lambda$ for every $n\in\mathbb{N}$. Then there is a strictly increasing sequences of natural numbers $(s_n)_{n\in\mathbb{N}}$ such that*
$$(v_1^{(s_n)}\dots v_M^{(s_n)}w_1^{(s_n)}\dots w_N^{(s_n)})_{n\in\mathbb{N}}$$
satisfies the alternating growth condition.

(ii) *Suppose that $\bar{v}_M^{(n)}\in H_\lambda$ for every $n\in\mathbb{N}$. Then there are strictly increasing sequences of natural numbers $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that the sequence*
$$(v_1^{(s_n)}\dots v_{M-1}^{(s_n)}z^{(n)}w_2^{(t_n)}\dots w_N^{(t_n)})_{n\in\mathbb{N}},$$
where $z^{(n)}\in\widetilde{H}_\lambda$ is the element representing $\overline{v_M^{(s_n)}w_1^{(t_n)}}\in H_\lambda$, satisfies the alternating growth condition.

Proof Let us first prove (i). Suppose that there is no such sequence $(s_i)_{i\in\mathbb{N}}$. Then there are infinitely many $n\in\mathbb{N}$ such that $v_1^{(n)}\dots v_M^{(n)}w_1^{(n)}\dots w_N^{(n)}$ does not satisfy some of the conditions of Definition 3.8. Since (I)–(V) are clearly satisfied, it follows that $v_1^{(n)}\dots v_M^{(n)}w_1^{(n)}\dots w_N^{(n)}$ is not regular (with respect to $X\cup\mathcal{H}$) for infinitely many $n\in\mathbb{N}$. By restriction to a subsequence if necessary, we can assume that no word $v_1^{(n)}\dots v_M^{(n)}w_1^{(n)}\dots w_N^{(n)}$ is regular. Since $\bar{v}_M^{(n)}\notin H_\lambda$ for every $n\in\mathbb{N}$, none of the subwords $v_M^{(n)}w_1^{(n)}$ represent the trivial element in G. Along with the assumption that $v_1^{(n)}\dots v_M^{(n)}$ and $w_1^{(n)}\dots w_N^{(n)}$ are regular, it follows that there is a maximal index a_n such that
$$|v_{a_n}^{(n)}\dots v_M^{(n)}w_1^{(n)}\dots w_{b_n}^{(n)}|_{X\cup\mathcal{H}}=1 \tag{5}$$
for some index b_n. Suppose that each b_n is chosen to be minimal with respect to a_n. Then there are generators $u_n\in X\cup\mathcal{H}$ such that
$$q_n=v_{a_n}^{(n)}\dots v_M^{(n)}w_1^{(n)}\dots w_{b_n}^{(n)}u_n$$
represents the identity in G for every $n\in\mathbb{N}$. We want to argue that $w_1^{(n)}$ is an isolated λ–syllable in the cyclic word $[q_n]$. Suppose that this is not the case. Then there are three cases to consider:

Case 1 ($\overline{v_i^{(n)}\dots v_M^{(n)}w_1^{(n)}}\in H_\lambda$ for some $a_n\le i\le N$) Then $\overline{v_i^{(n)}\dots v_M^{(n)}}\in H_\lambda$, and since $v_1^{(n)}\dots v_M^{(n)}$ is regular, $i=M$. Thus $\bar{v}_M^{(n)}\in H_\lambda$, in contrast to our assumption $\bar{v}_M^{(n)}\notin H_\lambda$.

Case 2 ($\overline{w_1^{(n)}\dots w_i^{(n)}}\in H_\lambda$ for some $2\le i\le b_n$) This is a contradiction since $w_1^{(n)}\dots w_N^{(n)}$ is regular.

Case 3 ($\overline{w_1^{(n)}\dots w_{b_n}^{(n)}u_n}\in H_\lambda$) In this case we also have $\overline{v_{a_n}^{(n)}\dots v_M^{(n)}}\in H_\lambda$. Using again the assumption that $v_1^{(n)}\dots v_M^{(n)}$ is regular, $a_n=M$ and $\bar{v}_M^{(n)}\in H_\lambda$, which contradicts our assumption that $\bar{v}_M^{(n)}\notin H_\lambda$.

Thus $w_1^{(n)}$ is indeed an isolated λ–syllable in $[q_n]$. Moreover, $\|q_n\| \le M + N + 1$ for every $n \in \mathbb{N}$. From Lemma 2.9 it therefore follows that $\{\bar{w}_1^{(n)} \mid n \in \mathbb{N}\}$ is a finite subset of G. On the other hand, the alternating growth condition ensures that $|w_1^{(n)}|_X \ge n$ for every $n \in \mathbb{N}$. This finally gives us the contradiction that arose from our assumption that there is no sequence $(s_i)_{i \in \mathbb{N}}$ as in (i).

Let us now prove (ii). From the alternating growth condition, $|w_1^{(n)}|_X \ge n$ for every $n \in \mathbb{N}$. Thus we can choose strictly increasing sequences of natural numbers $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $|\overline{v_M^{(s_n)} w_1^{(t_n)}}|_X \ge n$ for every $n \in \mathbb{N}$. Note that Definition 3.8(I)–(V) are clearly satisfied for

$$(v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)})_{n\in\mathbb{N}},$$

where $z^{(n)} \in \widetilde{H}_\lambda$ is the element representing $\overline{v_M^{(s_n)} w_1^{(t_n)}}$. In order to prove the lemma it therefore suffices to show that $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$. To see this, let us first consider the subwords

$$v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} \quad \text{and} \quad z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}.$$

Suppose that there is some $1 \le i \le M-1$ with $|v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}|_{X\cup\mathcal{H}} \le 1$. Then there are two cases to consider:

Case 1 $(\overline{v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}} \in H_\lambda)$ Then we also have $\overline{v_i^{(s_n)} \dots v_{M-1}^{(s_n)}} \in H_\lambda$, and since $v_1^{(s_n)} \dots v_M^{(s_n)}$ is regular, it follows that $M-1 = 1$. But then $v_{M-1}^{(s_n)}$ and $v_M^{(s_n)}$ both represent elements of H_λ, which in turn contradicts the regularity of $v_1^{(s_n)} \dots v_M^{(s_n)}$.

Case 2 $(\overline{v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}} \notin H_\lambda)$ Then there is some $u_n \in X \cup \mathcal{H}$ that does not lie in $\widetilde{H}_\lambda$ such that $q_n := v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} u_n$ represents the identity in G. We claim that $z^{(n)}$ is an isolated syllable in the cyclic word $[q_n]$. Otherwise there would be some $i \le j \le M-1$ with

$$\overline{v_j^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}} \in H_\lambda,$$

which is impossible as we have seen in Case 1. Moreover, $\|q_n\| \le M$. From Lemma 2.9 it therefore follows that $\{\bar{z}_1^{(n)} \mid n \in \mathbb{N}\}$ is a finite subset of G. Since $|z^{(n)}|_X \ge n$, there are only finitely many $n \in \mathbb{N}$ such that $|v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}|_{X\cup\mathcal{H}} \le 1$ for some $1 \le i \le M-1$. Thus $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ is regular for all but finitely many $n \in \mathbb{N}$. Symmetric argument shows that $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$. By restriction to a subsequence if necessary, we can therefore assume that the words $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ and $w_2^{(t_n)} \dots w_N^{(t_n)}$ are regular for every n.

Now assume $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is not regular, and choose $1 \le a_n \le M-1$ and $2 \le b_n \le N$ such that $v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ is a minimal subword of $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ with

$$|v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}|_{X\cup\mathcal{H}} \le 1.$$

Case 1 $(q_n := v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ represents the identity in $G)$ Since $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ and $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ are regular, it follows that $z^{(n)}$ is an isolated syllable in the cyclic word $[q_n]$. In view of Lemma 2.9, there are only finitely many such n.

Case 2 ($v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ does not represent an element of H_λ) Then there is some $u_n \in \bigcup_{\mu\in\Lambda\setminus\{\lambda\}} (\widetilde{H}_\lambda \setminus \{1\}) \cup X$ such that

$$q_n := v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)} u_n$$

represents the trivial element in G. In particular, u_n is not part of a λ–syllable in the cyclic word $[q_n]$. Another application of Lemma 2.9 now reveals that there are only finitely many $n \in \mathbb{N}$ such that $v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ does not represent an element of H_λ.

Case 3 ($v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ represents a nontrivial element in H_λ) Then there is some $u_n \in \widetilde{H}_\lambda$ such that

$$q_n := v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)} u_n$$

represents the identity in G. Suppose $z^{(n)}$ is connected to some further λ–syllable in the cyclic word $[q_n]$. Since $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ and $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ are regular, $z^{(n)}$ has to be connected to u_n. Hence

$$\overline{z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)} u_n} \in H_\lambda,$$

which implies

$$\overline{w_2^{(t_n)} \dots w_{b_n}^{(t_n)}} \in H_\lambda.$$

From the regularity of $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ it therefore follows that $N = 2$. But then $\bar{w}_2^{(t_n)} \in H_\lambda$, which contradicts the regularity of $w_1^{(t_n)} w_2^{(t_n)} \dots w_N^{(t_n)}$. Thus u_n is an isolated syllable in $[q_n]$ and a final application of Lemma 2.9 proves that Case 3 can only occur finitely many times.

Altogether we have shown that $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$, which proves the lemma. □

Corollary 3.11 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$, and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Let $(w^{(n)})_{n\in\mathbb{N}}$ be a sequence of words over $X \cup \mathcal{H}$ that satisfies the alternating growth condition and let K be the subgroup of G generated by $\{\bar{w}_n \mid n \in \mathbb{N}\}$. Then there is some $C \in \mathbb{N}$ that satisfies the following. For every $L \in \mathbb{N}$ there is a sequence of words $(v_n)_{n\in\mathbb{N}}$ over $X \cup \mathcal{H}$ such that*:

(i) *$(v_n)_{n\in\mathbb{N}}$ satisfies the alternating growth condition.*

(ii) *The length of every word v_n is bounded by $L \le \|v_n\| \le L + C$.*

(iii) *Every word v_n represents an element of K.*

Proof Let us write $w^{(n)} = w_1^{(n)} \dots w_\ell^{(n)}$ for every $n \in \mathbb{N}$. From properties (II) and (III) of the alternating growth condition there is some $\lambda \in \Lambda$ such that $w_1^{(n)} \in \widetilde{H}_\lambda$ for every $n \in \mathbb{N}$. By restriction to a subsequence if necessary, we may assume that $(w_n)_{n\in\mathbb{N}}$ satisfies one of the following two conditions:

(i) $\bar{w}_\ell^{(n)} \notin H_\lambda$ for every $n \in \mathbb{N}$.

(ii) $\bar{w}_\ell^{(n)} \in H_\lambda$ for every $n \in \mathbb{N}$.

Suppose the first and let $k \in \mathbb{N}$. Then an inductive application of Lemma 3.10(i) provides us with subsequences

$$(w_1^{(s_i,n)} \dots w_\ell^{(s_i,n)})_{n\in\mathbb{N}}$$

of $w^{(n)}$ for each $1 \le i \le k$ such that the sequence of concatenated words

$$v_n := (w_1^{(s_1,n)} \dots w_\ell^{(s_1,n)})(w_2^{(s_2,n)} \dots w_\ell^{(s_2,n)}) \dots (w_1^{(s_k,n)} \dots w_\ell^{(s_k,n)})$$

has length $k\ell$ and satisfies the alternating growth condition. Thus the corollary is clearly satisfied for $C = \ell$.

Let us now consider (ii), and let $k \in \mathbb{N}$. Then an inductive application of Lemma 3.10(ii) provides us with subsequences

$$(w_1^{(s_i,n)} \dots w_\ell^{(s_i,n)})_{n\in\mathbb{N}}$$

of $w^{(n)}$ for each $1 \le i \le k$ such that the sequence of words v_n given by

$$(w_1^{(s_1,n)} \dots w_{\ell-1}^{(s_1,n)})z^{(t_1,n)}(w_2^{(s_2,n)} \dots w_{\ell-1}^{(s_2,n)})z^{(t_2,n)} \dots z^{(t_{k-1},n)}(w_2^{(s_k,n)} \dots w_\ell^{(s_k,n)}),$$

where $z^{(t_i,n)} \in \widetilde{H}_\lambda$ is the element representing $\overline{w_\ell^{(s_i,n)} w_1^{(s_{i+1},n)}} \in H_\lambda$, satisfies the alternating growth condition. In this case v_n has length $k(\ell-1)+1$ and we see that the corollary is satisfied for $C = \ell$. □

4 Dichotomy of infinite subgroups

Endowed with Corollary 3.11, we are now ready to study the subgroup of a relatively finitely presented group G that is generated by all the elements $\bar{w}_n$, where $(w_n)_{n\in\mathbb{N}}$ is a sequence that satisfies the alternating growth condition.

Lemma 4.1 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Suppose that $(w_n)_{n\in\mathbb{N}}$ is a sequence of words over $X \cup \mathcal{H}$ that satisfies the alternating growth condition. Then the subgroup $K \le G$ generated by $\{\bar{w}_n \in G \mid n \in \mathbb{N}\}$ is unbounded with respect to $d_{X\cup\mathcal{H}}$.*

Proof Suppose that K is bounded with respect to $d_{X\cup\mathcal{H}}$, ie that there is some $N \in \mathbb{N}$ with $|k|_{X\cup\mathcal{H}} \le N$ for every $k \in K$. Due to Corollary 3.11 there is a number $L \ge 4N$ and a sequence $(v_n)_{n\in\mathbb{N}}$ of words $v_n = v_1^{(n)} \dots v_L^{(n)}$ over $X \cup \mathcal{H}$ that satisfies the alternating growth condition such that each v_n represents an element of K. By restriction to a subsequence, we can assume that there is some $M \in \mathbb{N}$ with $|v_n|_{X\cup\mathcal{H}} = M \le N$ for every $n \in \mathbb{N}$. Let $u_1^{(n)} \dots u_M^{(n)}$ be a shortest word over $X \cup \mathcal{H}$ representing $\bar{v}_n^{-1}$. Then each word $q_n := v_1^{(n)} \dots v_L^{(n)} u_1^{(n)} \dots u_M^{(n)}$ represents the identity in G. The alternating growth condition ensures $v_1^{(n)} \dots v_L^{(n)}$ is regular and that two consecutive letters of v_n do not lie in X. It therefore follows that at least every second of its letters is an isolated syllable in v_n. Thus there are at least $2N$ isolated syllables in $v_n = v_1^{(n)} \dots v_L^{(n)}$. Note that for every $\lambda \in \Lambda$ and every λ–syllable of $u_1^{(n)} \dots u_M^{(n)}$, which necessarily consists of a single letter $u_i^{(n)}$, there is at most one λ–syllable $v_j^{(n)}$ in $v_1^{(n)} \dots v_L^{(n)}$ that is connected to $u_i^{(n)}$ in the cyclic word $[q_n]$. Otherwise there would be a connection between two different isolated λ–syllables of $v_1^{(n)} \dots v_L^{(n)}$ by a λ–word. This implies that there are at least $2N - M \geq N$ isolated syllables in $[q_n]$ that become arbitrarily large with respect to X as n goes to ∞. But this contradicts Lemma 2.9 since $\|q_n\| \leq M + L$ for every $n \in \mathbb{N}$. Thus K is an unbounded subset of $\Gamma(G, X \cup \mathcal{H})$. □

Lemma 4.2 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Let $K \leq G$ be an infinite subgroup that is bounded with respect to $d_{X\cup\mathcal{H}}$. Then there is an element $g \in G$ and an index $\eta \in \Lambda$ such that $|gKg^{-1} \cap H_\eta| = \infty$.*

Proof Since K is bounded with respect to $d_{X\cup\mathcal{H}}$, each of its conjugates gKg^{-1} is a bounded subset of $\Gamma(G, X \cup \mathcal{H})$. Let $m \in \mathbb{N}$ be minimal with the following property:

($*$) There is a conjugate $H := gKg^{-1}$ of K, a finite relative generating set Y of G, and an infinite sequence $(k_n)_{n\in\mathbb{N}}$ of pairwise distinct elements $k_n \in H$ with $|k_n|_{Y\cup\mathcal{H}} = m$ for every $n \in \mathbb{N}$.

Let g, Y and $(k_n)_{n\in\mathbb{N}}$ be as in ($*$). For each n let $u^{(n)} = u_1^{(n)} \dots u_m^{(n)}$ be a (shortest) word over $Y \cup \mathcal{H}$ that represents k_n. Due to the minimality of m, we can extend Y to any finite relative generating set Y' of G such that ($*$) is still satisfied for an appropriate subsequence of $(k_n)_{n\in\mathbb{N}}$. Since G is finitely generated, we can therefore assume that Y is a symmetric generating set of G.

Suppose first that $m = 1$. Then $u_1^{(n)} \in \mathcal{H} = \bigcup_{\lambda\in\Lambda}(\widetilde{H}_\lambda \setminus \{1\})$ for all but finitely many $n \in \mathbb{N}$. Since Λ is finite by Theorem 2.2, there is some $\eta \in \Lambda$ such that infinitely many pairwise distinct letters $u_1^{(n)}$ lie in $\widetilde{H}_\eta$. It therefore follows that $|gKg^{-1} \cap H_\eta| = \infty$.

Let us now consider the case $m \geq 2$. We want to modify Y and $u^{(n)}$ in such a way that some subsequence of $(u^{(n)})_{n\in\mathbb{N}}$ satisfies the alternating growth condition. This will be done inductively by going through the letters $u_i^{(n)}$ of $u^{(n)}$.

Suppose that $u_1^{(n)} \in Y$ for infinitely many $n \in \mathbb{N}$. Then we can choose some $x_1 \in Y$ and a subsequence $(k_{j_n})_{n\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$ with $u_1^{(j_n)} = x_1$ for every n. In this case we replace $(k_n)_{n\in\mathbb{N}}$ by $(k_{j_n})_{n\in\mathbb{N}}$.

Suppose next that $u_1^{(n)} \in \mathcal{H}$ for all but finitely many $n \in \mathbb{N}$. Since Λ is finite, there is some $\lambda_1 \in \Lambda$ with $u_1^{(n)} \in \widetilde{H}_{\lambda_1}$ for infinitely many $n \in \mathbb{N}$. We have to consider 2 cases:

Case 1 (there is some $\tilde{h}_1 \in \widetilde{H}_{\lambda_1}$ with $u_1^{(n)} = \tilde{h}_1$ for infinitely many $n \in \mathbb{N}$) Restrict $(k_n)_{n\in\mathbb{N}}$ to a subsequence $(k_{j_n})_{n\in\mathbb{N}}$ such that $u_1^{(j_n)} = \tilde{h}_1$ for every $n \in \mathbb{N}$. Moreover we add h_1 and h_1^{-1} to Y and replace the letter $u_1^{(j_n)} = \tilde{h}_1 \in \widetilde{H}_{\lambda_1}$ in $u^{(j_n)}$ by $h_1 \in Y$ for every $n \in \mathbb{N}$. Next we replace the resulting sequence by a subsequence that satisfies ($*$), which is possible by the choice of m.

Case 2 (there is no $\tilde{h}_1 \in \widetilde{H}_{\lambda_1}$ with $u_1^{(n)} = \tilde{h}_1$ for infinitely many $n \in \mathbb{N}$) Replace $(u^{(n)})_{n\in\mathbb{N}}$ by a subsequence $(u^{(j_n)})_{n\in\mathbb{N}}$ such that $|\bar{u}_1^{(j_n)}|_Y > n$ for every $n \in \mathbb{N}$.

We proceed analogously with the other indices $i \in \{2, \dots, m\}$. The resulting sequence of words over $Y \cup \mathcal{H}$ will be denoted by $(v_1^{(n)} \dots v_m^{(n)})_{n\in\mathbb{N}}$. Let $g_n \in H$ be the element represented by $v_1^{(n)} \dots v_m^{(n)}$.

Suppose that either two consecutive letters $v_i^{(n)}$ and $v_{i+1}^{(n)}$ or $v_1^{(n)}$ and $v_m^{(n)}$ both lie in Y. Then we could add $v_i^{(n)} v_{i+1}^{(n)}$ and $(v_i^{(n)} v_{i+1}^{(n)})^{-1}$ (resp. $v_m^{(n)} v_1^{(n)}$ and $(v_m^{(n)} v_1^{(n)})^{-1}$) to Y in order to obtain a shorter sequence of infinitely many pairwise distinct elements of H (resp. of $v_{-1}^{(n)} H v_1^{(n)}$) with respect to $d_{Y \cup \mathcal{H}}$. But this is a contradiction to the choice of m. Thus neither $v_i^{(n)}$ and $v_{i+1}^{(n)}$ nor $v_1^{(n)}$ and $v_m^{(n)}$ both lie in Y. In particular, we can replace $v_1^{(n)} \dots v_m^{(n)}$ by its inverse $(v_m^{(n)})^{-1} \dots (v_1^{(n)})^{-1}$ to ensure that the first letter does not lie in Y. Let us therefore assume that $v_1^{(n)}$ is never contained in Y. To prove that $(v_1^{(n)} \dots v_m^{(n)})_{n \in \mathbb{N}}$ satisfies the alternative growth condition, it remains to show that each $v_1^{(n)} \dots v_m^{(n)}$ is regular and that two consecutive letters $v_i^{(n)}$ and $v_{i+1}^{(n)}$ cannot lie in the same group $\widetilde{H}_\lambda$. But these properties are direct consequences of the condition $|g_n|_{Y \cup \mathcal{H}} = m$ from $(*)$, where k_n plays the role of g_n. Altogether we have shown that there is a conjugate H of K and a sequence $(g_n)_{n \in \mathbb{N}}$ of elements in H that can be represented by a sequence $(v_1^{(n)} \dots v_m^{(n)})_{n \in \mathbb{N}}$ of words over $Y \cup \mathcal{H}$ that satisfies the alternating growth condition. In this case, Lemma 4.1 tells us that H is an unbounded subset of $\Gamma(G, Y \cup \mathcal{H})$, which clearly contradicts our assumption that K is a bounded subset of $\Gamma(G, X \cup \mathcal{H})$. Hence $m = 1$, in which case we have already proven the lemma. □

We are now ready to prove our main theorem:

Theorem 4.3 *Let G be a finitely generated group and let X be a finite generating set of G. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugated to a subgroup of a peripheral subgroup.*

(iii) *K is unbounded in $\Gamma(G, X \cup \mathcal{H})$.*

Proof Suppose that K is infinite and bounded as a subset of $\Gamma(G, X \cup \mathcal{H})$. From Lemma 4.2 we know that there is an index $\eta \in \Lambda$ and an element $g \in G$ such that the $gKg^{-1} \cap H_\eta$ is infinite. We can therefore choose a sequence $(h_n)_{n \in \mathbb{N}}$ of distinct nontrivial elements $h_n \in gKg^{-1} \cap H_\eta$. Suppose that gKg^{-1} is not a subgroup of H_η and let $a \in gKg^{-1} \setminus H_\eta$. Let $\tilde{h}_n \in \widetilde{H}_\eta$ be the element representing h_n. Then, after adding $\{a, a^{-1}\}$ to X if necessary, we can consider the sequence of words $(\tilde{h}_n a)_{n \in \mathbb{N}}$ over $X \cup \mathcal{H}$. We claim that $(\tilde{h}_n a)_{n \in \mathbb{N}}$ has a subsequence that satisfies the alternating growth condition. The only property that is not directly evident is that $(\tilde{h}_n a)_{n \in \mathbb{N}}$ has a subsequence consisting of regular words. Suppose that this is not the case. Since Λ is finite by Theorem 2.2, it then follows that there is some $\mu \in \Lambda$ such that $\tilde{h}_n a$ represents an element in H_μ for infinitely many $n \in \mathbb{N}$. Then $\tilde{h}_m a (\tilde{h}_n a)^{-1} = \tilde{h}_m \tilde{h}_n^{-1}$ represents an element in $H_\mu \cap H_\eta$ whenever $\tilde{h}_m a$ and $\tilde{h}_n a$ both represent elements of H_μ. Since a was chosen outside of H_η, it moreover follows that $\tilde{h}_n a$ can never represent an element of H_η. In particular, $\eta \neq \mu$. But this is a contradiction to [15, Proposition 2.36], which says that $H_\mu \cap H_\eta$ is finite for $\mu \neq \eta$. Thus $(\tilde{h}_n a)_{n \in \mathbb{N}}$ has a subsequence that satisfies the alternating growth condition. In this case Lemma 4.1 tells us that the subgroup $\langle\{ah_n \mid n \in \mathbb{N}\}\rangle$ of gKg^{-1} is unbounded in $\Gamma(G, X \cup \mathcal{H})$, which contradicts our assumption that K is bounded in $\Gamma(G, X \cup \mathcal{H})$. Finally, this proves that gKg^{-1} is a subgroup of H_η. $\square$

Let us now consider the important special case of Theorem 1.2 where G is relatively hyperbolic with respect to H_Λ. Recall that an element $g \in G$ is called *loxodromic* if the map

$$\mathbb{Z} \to \Gamma(G, X \cup \mathcal{H}) \quad \text{given by } n \mapsto g^n$$

is a quasiisometric embedding. It is known that a subgroup $K \le G$ with infinite diameter in $\Gamma(G, X \cup \mathcal{H})$ contains a loxodromic element. This follows from a corresponding result for acylindrically hyperbolic groups [16, Theorem 1.1] and the fact that relatively hyperbolic groups act acylindrically on the (hyperbolic) graph $\Gamma(G, X \cup \mathcal{H})$ [16, Proposition 5.2].

Corollary 4.4 *Let G be a finitely generated group. Suppose that G is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Then every subgroup $K \le G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugate to a subgroup of some H_λ.*

(iii) *K contains a loxodromic element.*

4.1 A geometric proof of Corollary 4.4

As pointed out to the author by Jason Manning, there is a short and more geometric proof of Corollary 4.4 that uses the cusped space $\Omega = \Omega(G, H_\Lambda, X)$ associated to G, H_Λ and an appropriate finite generating set X of G (see [11, Definition 3.15], where cusped spaces for relatively hyperbolic groups were introduced). Indeed, according to [11, Remark 3.14 and Theorem 3.25], the space Ω is hyperbolic and admits a proper isometric action of G. Moreover it is evident from the construction of Ω that for each $x \in \Omega$ and every infinite subgroup $K \le \Omega$, the orbit $K.x$ has infinite diameter in Ω. Thus $K.x$ has at least one limit point $\xi \in \partial\Omega$. If $K.x$ has another limit point $\eta \in \partial\Omega$, then we can choose $g, h \in H$ such that the distances of $d_\Omega(g.x, x)$ and $d_\Omega(h.x, x)$ are arbitrarily large while the Gromov product $(g.x, h.x)_x$ is bounded. In this case, a standard argument tells us that at least one of the elements $g, h, gh \in K$ is loxodromic; see [4, Lemme 2.3]. In the remaining case, ξ is a fixed point of H and it is a consequence of the construction of Ω that H is conjugate to a subgroup of some H_λ.

5 Applications

As an application of the classification of subgroups of relatively hyperbolic groups given in Corollary 4.4, we prove the existence of the relative exponential growth rate for all subgroups of a large variety of relatively hyperbolic groups.

Definition 5.1 Let G be a finitely generated group and let X be a finite generating set of G. Given a subgroup $H \leq G$, we define the *relative growth function* of H in G with respect to X by

$$\beta_H^X : \mathbb{N} \to \mathbb{N}, \quad n \mapsto |B_H^X(n)|,$$

where $B_H^X(n)$ denotes the set of elements in H that are represented by words of length at most n over $X \cup X^{-1}$. The relative exponential growth rate of H in G with respect to X is defined by $\limsup_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$.

It is natural to ask whether lim sup can be replaced by lim, ie whether the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ exists. Unlike in the important special case $H = G$, in which it is well known that this limit exists (see eg [13]), it does not exist in general; see [14, Remark 3.1]. In the case where the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ does exist, we will say that the relative exponential growth rate of H in G exists with respect to X. The following result provides us with a large variety of finitely generated relatively hyperbolic groups G for which the relative exponential growth rate exists for each of its subgroups and generating sets.

Theorem 5.2 *Let G be a finitely generated group that is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Suppose that each of the groups H_λ has subexponential growth. Then the relative exponential growth rate of every subgroup $K \leq G$ exists with respect to every finite generating set of G.*

Proof Let X be a finite generating set of G. We go through the three cases of Corollary 4.4.

Suppose first that K is finite. Then β_K^X is eventually constant and it trivially follows that $\lim_{n\to\infty} \sqrt[n]{\beta_K^X(n)}$ exists and is equal to 1.

Let us next consider the case where K contains a loxodromic element k. By [16, Proposition 5.2], G acts acylindrically on the (hyperbolic) graph $\Gamma(G, X \cup \mathcal{H})$. In this case, [16, Theorem 1.1] tells us that either G is virtually cyclic, in which case the claim follows trivially, or G is acylindrically hyperbolic, in which case the claim is covered by [17, Theorem 5.8].

Consider now the remaining case, where K is infinite and conjugated to a subgroup of some peripheral subgroup. Thus there is some $g \in G$ and some $\lambda \in \Lambda$ such that $K \leq gH_\lambda g^{-1}$. By Theorem 2.2 each H_λ, and hence $gH_\lambda g^{-1}$, is finitely generated. We can therefore choose a finite generating set Y of $gH_\lambda g^{-1}$. Moreover, it follows from [15, Lemma 5.4] that each peripheral subgroup, and hence $gH_\lambda g^{-1}$, is undistorted in G. We can therefore choose a constant $C > 0$ such that

$$\beta_{gH_\lambda g^{-1}}^X(n) \leq \beta_{gH_\lambda g^{-1}}^Y(Cn) \tag{6}$$

for every $n \in \mathbb{N}$. By assumption, each H_λ, and therefore $gH_\lambda g^{-1}$, has subexponential growth. Thus we have $\lim_{n\to\infty} \beta_{gH_\lambda g^{-1}}^Y(n)/a^n = 0$ for every $a > 1$. In view of (6), this implies that

$$\lim_{n\to\infty} \beta_{gH_\lambda g^{-1}}^X(n)/a^n = 0.$$

Then $\lim_{n\to\infty} \sqrt[n]{\beta_K^X(n)} = 1$ since $\beta_K^X(n) \leq \beta_{gH_\lambda g^{-1}}^X(n)$ for $n \in \mathbb{N}$, and in particular the limit exists. □

References

[1] **E Alibegović**, *A combination theorem for relatively hyperbolic groups*, Bull. Lond. Math. Soc. 37 (2005) 459–466 MR Zbl

[2] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) art. id. 1250016 MR Zbl

[3] **J M Cohen**, *Cogrowth and amenability of discrete groups*, J. Funct. Anal. 48 (1982) 301–309 MR Zbl

[4] **M Coornaert**, **T Delzant**, **A Papadopoulos**, *Géométrie et théorie des groupes: les groupes hyperboliques de Gromov*, Lecture Notes in Math. 1441, Springer (1990) MR Zbl

[5] **R Coulon**, **F Dal'Bo**, **A Sambusetti**, *Growth gap in hyperbolic groups and amenability*, Geom. Funct. Anal. 28 (2018) 1260–1320 MR Zbl

[6] **F Dahmani**, *Combination of convergence groups*, Geom. Topol. 7 (2003) 933–963 MR Zbl

[7] **F Dahmani**, **D Futer**, **D T Wise**, *Growth of quasiconvex subgroups*, Math. Proc. Cambridge Philos. Soc. 167 (2019) 505–530 MR Zbl

[8] **B Farb**, *Relatively hyperbolic groups*, Geom. Funct. Anal. 8 (1998) 810–840 MR Zbl

[9] **R I Grigorchuk**, *Symmetrical random walks on discrete groups*, from "Multicomponent random systems" (R L Dobrushin, Y G Sinai, D Griffeath, editors), Adv. Probab. Related Topics 6, Dekker, New York (1980) 285–325 MR Zbl

[10] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR Zbl

[11] **D Groves**, **J F Manning**, *Dehn filling in relatively hyperbolic groups*, Israel J. Math. 168 (2008) 317–429 MR Zbl

[12] **S Hughes**, **E Martínez-Pedroza**, **L J Sánchez Saldaña**, *Quasi-isometry invariance of relative filling functions*, Groups Geom. Dyn. 17 (2023) 1483–1515 MR Zbl

[13] **J Milnor**, *A note on curvature and fundamental group*, J. Differential Geom. 2 (1968) 1–7 MR Zbl

[14] **A Y Olshanskii**, *Subnormal subgroups in free groups, their growth and cogrowth*, Math. Proc. Cambridge Philos. Soc. 163 (2017) 499–531 MR Zbl

[15] **D V Osin**, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. 843, Amer. Math. Soc., Providence, RI (2006) MR Zbl

[16] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR Zbl

[17] **E Schesler**, *The relative exponential growth rate of subgroups of acylindrically hyperbolic groups*, J. Group Theory 25 (2022) 293–326 MR Zbl

[18] **Z Sela**, *Diophantine geometry over groups, I: Makanin–Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. 93 (2001) 31–105 MR Zbl

[19] **R Sharp**, *Relative growth series in some hyperbolic groups*, Math. Ann. 312 (1998) 125–132 MR Zbl

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A topological construction of families of Galois covers of the line

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We describe a new construction of families of Galois coverings of the line using basic properties of configuration spaces, covering theory, and the Grauert–Remmert extension theorem. Our construction provides an alternative to a previous construction due to González-Díez and Harvey (which uses Teichmüller theory and Fuchsian groups) and, in the case the Galois group is nonabelian, corrects an inaccuracy therein. In the opposite case where the Galois group has trivial center, we recover some results due to Fried and Völklein.

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1 Introduction

The object of this note are families of Galois coverings of the line.

Let G be a finite group and let C and C' be smooth projective curves over the complex numbers endowed with a G–action. We say that C and C' are *topologically equivalent* or have the same (unmarked) *topological type* if there is an $\eta \in \operatorname{Aut} G$ and an orientation-preserving homeomorphism $f : C \to C'$ such that $f(g \cdot x) = \eta(g) \cdot f(x)$ for $x \in C'$ and $g \in G$. We say that C and C' are (unmarkedly) *G–isomorphic* if moreover f is a biholomorphism.

Given a G–covering $C \to \mathbb{P}^1$, it has been proved by González-Díez and Harvey [1992] that there exists an algebraic family of curves with a G–action

$$\pi : \mathscr{C} \to \mathsf{B}$$

such that

(1) every curve C' in the family is *topologically equivalent* to C;

(2) every curve with an action of the given topological type is *G–isomorphic* to some fiber of the family and to at most a finite number of fibers.

This result has been subsequently used in several papers, eg [Conti et al. 2022; Frediani et al. 2015; Frediani and Neumann 2003; Penegini 2015; Perroni 2022], just to mention a few.

The construction in [González-Díez and Harvey 1992] uses Teichmüller theory. Other approaches to this construction include [Fried and Völklein 1991; Li 2018; Völklein 1994]. In this paper we describe an alternative, explicit and mostly topological construction of such families. We expect this to be useful to make explicit computations on the family. For example, we expect this to allow a better understanding of the monodromy and the generic Hodge group for the natural variation of Hodge structure associated with the family, generalizing the results of [Rohde 2009] carried out in the cyclic case. Our motivation comes from the fact that these families and their variation of the Hodge structure are important in the study of Shimura subvarieties of the moduli space A_g (of principally polarized abelian varieties of dimension g) in relation with the Coleman–Oort conjecture; see eg [Moonen 2010; Moonen and Oort 2013; Frediani et al. 2015; Tamborini 2022]. The results presented here are, nevertheless, of independent interest.

1.1 We give a quick glance at our construction. For $n \geq 3$ let $\mathsf{M}_{0,n}$ denote the set of n–tuples

$$X = (x_1, \dots, x_n) \in (\mathbb{P}^1)^n$$

such that $x_i \neq x_j$ for $i \neq j$, $x_{n-2} = 0$, $x_{n-1} = 1$ and $x_n = \infty$. Consider the group

$$\Gamma_n = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle.$$

Let G be a finite group and let $\theta\colon \Gamma_n \to G$ be an epimorphism. Fix $X \in \mathsf{M}_{0,n}$. After choosing a base point $x_0 \in \mathbb{P}^1 - X$ and an isomorphism $\chi\colon \Gamma_n \cong \pi_1(\mathbb{P}^1 - X, x_0)$, Riemann's existence theorem yields a G–covering $C_X \to \mathbb{P}^1$ with monodromy $\theta \circ \chi^{-1}$ and branch locus X. Nevertheless this covering depends on the choices. Our goal is to make this construction for all $X \in \mathsf{M}_{0,n}$ together, in order to get a family of curves parametrized by $\mathsf{M}_{0,n}$. Consider the map

$$p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}, \quad p(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n).$$

We have $p^{-1}(X) = \mathbb{P}^1 - X$. Hence p can be thought as the universal family of genus 0 curves with n marked points. The basic idea of our construction is that the total space of our family should be a suitable G–covering of $\mathsf{M}_{0,n+1}$. For the construction of this covering, choose

(i) an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$;

(ii) an isomorphism $\chi\colon \Gamma_n \to \pi_1(\mathbb{P}^1 - X, x_0)$.

The following sequence is exact and splits:

$$1 \to \pi_1(\mathbb{P}^1 - X, x_0) \to \pi_1(\mathsf{M}_{0,n+1}, x) \to \pi_1(\mathsf{M}_{0,n}, X) \to 1.$$

Set for simplicity $N_x := \pi_1(\mathbb{P}^1 - X, x_0)$, $H_X := \pi_1(\mathsf{M}_{0,n}, X)$ and $f := \chi^{-1} \circ \theta$. Assume that we can find an extension $\tilde{f}$:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{P}^1 - X, x_0) \arrow[r] \arrow[dr, "f"'] & \pi_1(\mathsf{M}_{0,n+1}, x) \arrow[r] \arrow[d, dashed, "\tilde{f}"] & H_X \arrow[r] & 1 \\
 & & G & &
\end{tikzcd}$$

From $\tilde{f}$ we get a topological G–covering $\mathscr{C}^* \to \mathsf{M}_{0,n+1}$. By the Grauert–Remmert extension theorem (see Theorem 7.4 below) this compactifies to a branched covering $\mathscr{C} \to \mathbb{P}^1 \times \mathsf{M}_{0,n}$ of quasiprojective varieties. Composing with the projection to $\mathsf{M}_{0,n}$ we get a holomorphic family $\pi : \mathscr{C} \to \mathsf{M}_{0,n}$ satisfying properties (1) and (2).

1.2 Thus, if one is able to find the extension $\tilde{f}$, one can construct the families using only basic properties of configuration spaces, covering theory and the Grauert–Remmert extension theorem, avoiding Teichmüller theory and Fuchsian groups. In fact this strategy is not new, as it has already been used in exactly the same context in various papers by Michael D Fried and Helmut Völklein; see eg [Fried 1977; Fried and Völklein 1991; Völklein 1994].

If G is abelian, one is always able to find the extension $\tilde{f}$; see Section 10. In general however the extension $\tilde{f}$ does not exist, contrary to what is claimed in [González-Díez and Harvey 1992]. One can at least show that there are always finite-index subgroups $H_a \subset H_X$ such that f extends to a morphism $f_a \colon N_x \rtimes H_a \to G$. Geometrically passing from H_X to the subgroup H_a means that one builds a family satisfying (1) and (2) over a base which is no longer $\mathsf{M}_{0,n}$, but some finite cover Y_a of it. The pair (H_a, f_a) is far from unique, there are many of them and different choices yield families differing by finite étale pullback (see Section 7 for precise definitions.) So another problem arises: how is one supposed to choose the pair (H_a, f_a) in order to determine the family in a canonical way?

For a special class of groups, namely for groups G with trivial center, there is a canonical choice of (H_a, f_a), which allows to construct a canonical family of coverings. This case corresponds to the one studied in [Fried and Völklein 1991; Völklein 1994; 1996] where the condition that G be centerless plays a crucial role.

It is odd that for this problem the two special cases occur in opposite directions, namely for abelian and for centerless groups.

For general G one is not able to pick out a distinguished choice in a canonical way. This problem was already considered long ago in [Fried 1977, pages 57–58] where a cohomological interpretation of this difficulty is given.

Our approach instead is the following. Since we are stuck with a whole collection of pairs (H_a, f_a), each one giving rise to a family of coverings with base the cover Y_a of $\mathsf{M}_{0,n}$, we decide to consider the whole collection instead of the single families. This collection comes naturally with the structure of a directed set coming from the pullbacks among families. We are able to show that this collection with this structure is well defined and depends only on the topological data.

Summing up, our construction, which builds heavily on previous approaches, corrects an inaccuracy in [González-Díez and Harvey 1992], where it is erroneously claimed that one has always $\mathsf{Y}_a = \mathsf{M}_{0,n}$, confirms that $\mathsf{Y}_a = \mathsf{M}_{0,n}$ if G is abelian (Theorem 10.1), and allows to recover at least part of the results in the papers of Fried and Völklein quoted above, while generalizing them to arbitrary groups with nontrivial center.

1.3 The paper is organized as follows. In Section 2 we recall basic facts about the configuration spaces of $\mathbb{P}^1$. Section 3 deals with parallel transport for fiber bundles. This material is for sure known to the experts, but rather hard to locate in the literature. Since these arguments are quite useful and we like their geometric flavor, we prefer to expound them concisely. In Section 4 we recall some classical concepts of surface topology. After these preliminaries, in Section 5 we study the set $\mathcal{T}^n(G)$ of topological types of G–actions; the main result is Theorem 5.6, which gives a combinatorial description of the set of topological types. The proof of this well-known fact presents our ideas in a simple context. Section 6 is dedicated to the description of some technical tools for the construction of the families. In Section 7 we construct the collection of families $\{\mathscr{C}_a \to \mathsf{Y}_a\}_a$ as sketched above. In Section 8 we study the dependence of the collection on the choices (i) and (ii), and on the epimorphism $\theta\colon \Gamma_n \to G$. Also this point becomes quite neat using our approach. Section 9 is dedicated to the case where G has trivial center and Section 10 to case where G is abelian. Summing up our main theorem is the following:

Theorem 1.4 (1) *The topological types of G–curves C with $g(C) = g$, $g(C/G) = 0$ and n branch points are in bijection with the set $(\mathscr{D}^n(G)/\operatorname{Aut} G)/\operatorname{Out}^* \Gamma_n$ (see Corollary 5.7, Definitions 4.8 and 5.2, and* (4-1) *for notation).*

(2) *For any topological type there is a nonempty ordered set $(\mathscr{I}, \geq)$ and for any $a \in \mathscr{I}$ there is an algebraic family $\pi_a\colon \mathscr{C}_a \to \mathsf{Y}_a$ of genus g curves with a G–action. The following properties hold:*

- (a) *Every curve C in the family has the given topological type.*
- (b) *For any $a \in \mathscr{I}$ and for any G–curve C with $C/G \cong \mathbb{P}^1$, there is at least one fiber of $\pi_a\colon \mathscr{C}_a \to \mathsf{Y}_a$ which is G–isomorphic to C, and there are only finitely many such fibers.*
- (c) *Each Y_a is a finite étale cover of $\mathsf{M}_{0,n}$;*
- (d) *$(\mathscr{I}, \geq)$ is a directed set: for any $a, b \in \mathscr{I}$ there is a c with $c \geq a$ and $c \geq b$.*
- (e) *If $a \geq b$, there is an algebraic étale covering $v\colon \mathsf{Y}_a \to \mathsf{Y}_b$ such that $\mathscr{C}_a \cong v^*\mathscr{C}_b$.*
- (f) *All the families have the same moduli image.*
- (g) *If $Z(G) = \{1\}$, then $(\mathscr{I}, \geq)$ has a minimum; hence in this case we can associate to any topological type a single family instead of the whole collection.*
- (h) *If G is abelian, then there exists $a \in \mathscr{I}$ such that $Y_a = \mathsf{M}_{0,n}$.*

The precise statement can be found in Theorems 7.8 and 8.2. Roughly speaking one can say that for any topological type there is a "universal" family of G–curves with that topological type. Such a family is not unique, but only unique up to the equivalence relation generated by finite étale pullbacks.

1.5 The existence problem that we address in this paper can of course be generalized: instead of considering just Galois covers, one can ask for the construction of families satisfying (1) and (2) for all the coverings with a fixed Galois closure (equivalently with fixed monodromy). These kinds of problems have been studied a lot and they are extremely important also because of their relevance for the inverse Galois problem; see [Fried 1977; 2010; Fried and Jarden 1986; Fried and Völklein 1991; 1992; Völklein 1996]. In these cases it often happens that the "universal" family has more than one component. We stress that in this paper we restrict only to the Galois case and that in this case all families are connected. In fact, the base of each family is a (connected) cover of $\mathsf{M}_{0,n}$.

Another variant of the problem studied in this paper is obtained by letting G^* be a group such that $\operatorname{Inn} G \subset G^* \subset \operatorname{Aut} G$ and considering two data equivalent if and only if they belong to the same $G^* \times \operatorname{Out}^* \Gamma_n$–orbit. This also has attracted a lot of attention in the literature. Our case corresponds to the choice $G^* = \operatorname{Aut} G$. In this paper we restrict to this case since we are interested in the topological types.

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2 Configuration spaces

2.1 If M is a manifold, its configuration space is

$$\boldsymbol{F}_{0,n}\, M := \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

We use the following notation: $X = (x_1, \dots, x_n)$ is a point of $\boldsymbol{F}_{0,n}\, M$ and $x = (x_0, X) = (x_0, x_1, \dots, x_n)$ is a point of $\boldsymbol{F}_{0,n+1}\, M$. We set $M - X := M - \{x_1, \dots, x_n\}$. The group $\pi_1(\boldsymbol{F}_{0,n}\, M)$ is called the *pure braid group* with n strings of the manifold M.

2.2 If $n \geq 3$, then the group $\mathrm{PGL}(2, \mathbb{C})$ acts freely and holomorphically on $\boldsymbol{F}_{0,n}\, \mathbb{P}^1$. The quotient $\boldsymbol{F}_{0,n}\mathbb{P}^1/\mathrm{PGL}(2, \mathbb{C})$ is the moduli space of smooth curves of genus 0 with n marked points. Setting $\mathbb{C}^{**} := \mathbb{C} - \{0, 1\}$, the map

$$\boldsymbol{F}_{0,n-3}\, \mathbb{C}^{**} \to \boldsymbol{F}_{0,n}\mathbb{P}^1, \quad (z_1, \dots, z_{n-3}) \mapsto (z_1, \dots, z_{n-3}, 0, 1, \infty),$$

is a section for the action of $\mathrm{PGL}(2, \mathbb{C})$, ie its image intersects each orbit in exactly one point and it induces a biholomorphism of $\boldsymbol{F}_{0,n-3}\, \mathbb{C}^{**}$ onto the moduli space $\boldsymbol{F}_{0,n}\mathbb{P}^1/\mathrm{PGL}(2, \mathbb{C})$. We *define* $\mathsf{M}_{0,n}$ as the image of the section, ie we set

$$\mathsf{M}_{0,n} := \boldsymbol{F}_{0,n-3}\, \mathbb{C}^{**} \times \{(0, 1, \infty)\} = \{X = (x_1, \dots, x_n) \in \boldsymbol{F}_{0,n}\mathbb{P}^1 \mid x_{n-2} = 0, x_{n-1} = 1, x_n = \infty\}.$$

Points of $\mathsf{M}_{0,n}$ will be denoted by $X = (x_1, \dots, x_n)$ with the understanding that $x_{n-2} = 0$, $x_{n-1} = 1$ and $x_n = \infty$. Similarly we set

$$\mathsf{M}_{0,n+1} := \{x = (x_0, \dots, x_{n+1}) \in \boldsymbol{F}_{0,n+1}\, \mathbb{P}^1 \mid x_{n-2} = 0, x_{n-1} = 1, x_n = \infty\}.$$

It is often useful to compare the configuration space of $\mathbb{P}^1$ with that of the plane. Denote by $\mathrm{T}(2, \mathbb{C})$ the subset of elements of $\mathrm{PGL}(2, \mathbb{C})$ fixing ∞. The group $\mathrm{T}(2, \mathbb{C})$ acts on $\boldsymbol{F}_{0,n-1}\, \mathbb{C}$ and the map

$$\mathsf{M}_{0,n} \to \boldsymbol{F}_{0,n-1}\, \mathbb{C}, \quad X \mapsto (x_1, \dots, x_{n-3}, 0, 1), \tag{2-1}$$

is a section for this action; hence $\mathsf{M}_{0,n} \times \mathrm{T}(2,\mathbb{C}) \cong \boldsymbol{F}_{0,n-1}\,\mathbb{C}$. In particular, $\pi_1(\mathsf{M}_{0,n}) \subset \pi_1(\boldsymbol{F}_{0,n-1}\,\mathbb{C})$. Thus, when dealing with $\pi_1(\mathsf{M}_{0,n})$, we can work with the more classical braid group of the plane. The map

$$p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}, \quad p(x_0, X) := X \tag{2-2}$$

is a fiber bundle. In fact it is the restriction of the bundle $\boldsymbol{F}_{0,n}\,\mathbb{C} \to \boldsymbol{F}_{0,n-1}\,\mathbb{C}$; see [Birman 1974]. The fiber over X is $\mathbb{P}^1 - X = \mathbb{C}^{**} - \{x_1, \dots, x_{n-3}\}$. Hence (2-2) is the universal family of genus 0 curves with n ordered marked points.

2.3 Fix $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and let $\tilde{x} = (x_0, \widetilde{X}) \in \boldsymbol{F}_{0,n}\,\mathbb{C}$ be the corresponding point via (2-1). We have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(\mathbb{P}^1 - X, x_0) \arrow[r] \arrow[d, equal] & \pi_1(\mathsf{M}_{0,n+1}, x) \arrow[r] \arrow[d, hook] & \pi_1(\mathsf{M}_{0,n}, X) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \pi_1(\mathbb{C} - \widetilde{X}, x_0) \arrow[r] & \pi_1(\boldsymbol{F}_{0,n}\,\mathbb{C}, \tilde{x}) \arrow[r] & \pi_1(\boldsymbol{F}_{n-1}\,\mathbb{C}, \widetilde{X}) \arrow[r] & 1 \end{tikzcd} \tag{2-3}$$

The rows are the split exact sequence of the fibrations p and $\boldsymbol{F}_{0,n}\,\mathbb{C} \to \boldsymbol{F}_{0,n-1}\,\mathbb{C}$; see eg [Birman 1974, Corollary 1.8.1; Fadell 1962, Theorem 3.1]. A geometric way to exhibit the splitting is to produce a cross section as follows: given $x = (x_1, \dots, x_n) \in \mathsf{M}_{0,n}$ we set

$$f(x) := \tfrac{1}{2}\min\{1, |x_1|, \dots, |x_{n-3}|\}.$$

Then $s(x) := (f(x), x_1, \dots, x_n)$ is a section of $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$. (A similar idea is used in [Fadell 1962, Theorem 3.1].) The morphism $s_*\colon \pi_1(\mathsf{M}_{0,n}, X) \to \pi_1(\mathsf{M}_{0,n+1}, x)$ is a splitting. Setting

$$\begin{aligned} &\varepsilon\colon \pi_1(\mathsf{M}_{0,n}, X) \to \mathrm{Aut}(\pi_1(\mathbb{P}^1 - X, x_0)), \\ &\varepsilon([\alpha])([\gamma]) := s_*[\alpha] \cdot [\gamma] \cdot s_*[\alpha]^{-1} = [s \circ \alpha * \gamma * s \circ i(\alpha)], \end{aligned} \tag{2-4}$$

we get

$$\pi_1(\mathsf{M}_{0,n+1}, x) = \pi_1(\mathbb{P}^1 - X, x_0)) \rtimes_\varepsilon \pi_1(\mathsf{M}_{0,n}, X).$$

3 Parallel transport

In this section we recall a notion of parallel transport up to homotopy on any fiber bundle. In the sequel, we will use it for the bundle $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ to study the dependence of the construction of Section 7 from the choices made.

3.1 Given $b_0, b_1 \in B$ let $\Omega(B, b_0, b_1)$ denote the set of all paths α in B with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. We write $\alpha \sim \beta$ if $\alpha \simeq \beta$ rel $\{0, 1\}$. Let $\Pi_1(B)$ denote the fundamental groupoid of B; this is the small category whose objects are the points of B and with morphisms from b_0 to b_1 equal to $\Omega(B, b_0, b_1)/\!\sim$, composition being given by $[\alpha] \cdot [\beta] = [\alpha * \beta]$.

3.2 Let $p\colon E \to B$ be a fiber bundle (in the sense of [Spanier 1966, page 90], ie a locally trivial bundle). Assume that the base B is Hausdorff and paracompact. Then p is a fibration [Spanier 1966, Corollary 14, page 96], ie it has the homotopy lifting property for every topological space Z: if $H\colon Z \times [0,1] \to B$ is any map and $f\colon Z \to E$ lifts $H(\cdot,0)$, then there is a lift $\widetilde{H}$ of H with $\widetilde{H}(\cdot,0) = f$; see eg [Spanier 1966, page 66]. For any fiber bundle $p\colon E \to B$ one can define a sort of parallel transport up to homotopy, which is a contravariant functor T from $\Pi_1(B)$ to the homotopy category of topological spaces, denoted by h−TOP. For $b \in B$ set $T(b) := E_b = p^{-1}(b)$. Given $[\alpha] \in \Pi_1(B)(b_0,b_1)$ consider the map $H\colon E_{b_0} \times [0,1] \to B$ defined by $H(e,t) := \alpha(t)$. The inclusion $i\colon E_{b_0} \hookrightarrow E$ is a lift of $H(\cdot,0)$. By the homotopy lifting property there is $\widetilde{H}\colon E_{b_0} \times [0,1] \to E$ with $p\widetilde{H} = H$ and $\widetilde{H}(\cdot,0) = i$. Moreover the homotopy class of $\widetilde{H}(\cdot,1)$ is well defined. We call $T([\alpha]) = [\widetilde{H}(\cdot,1)] \in [E_{b_0}, E_{b_1}]$ the homotopy parallel transport along α; see eg [Spanier 1966, page 100f; May 1999, page 54].

3.3 If $p\colon E \to B$ is a differentiable fiber bundle one can say more. Recall the following basic fact from differential topology. Let M and N be smooth manifolds. An *isotopy* of M in N is a smooth map $f\colon M \times [0,1] \to N$ such that $f(\cdot,t)$ is an embedding for any t. If $M = N$, $f(\cdot,t)$ is a diffeomorphism of M for any t and $f(\cdot,0) = \mathrm{id}_M$; we say that f is an *ambient isotopy*.

Theorem 3.4 *If M is a compact submanifold of N, any isotopy $f\colon M \times [0,1] \to N$ such that $f(\cdot,0)$ is the inclusion $M \hookrightarrow N$ extends to an ambient isotopy.*

For a proof, see eg [Hirsch 1976, Theorem 1.3, page 180].

Lemma 3.5 *Assume that $p\colon E \to B$ is a differentiable bundle. Let α be a path in B from b_0 to b_1. Let σ be a path in E with $p\sigma = \alpha$ and set $x_0 = \sigma(0) \in E_{b_0}$ and $x_0' = \sigma(1) \in E_{b_1}$. Then there is a map $\widetilde{H}\colon E_{b_0} \times [0,1] \to E$ such that*

(1) $\widetilde{H}(\cdot,0)$ *is the inclusion* $E_{b_0} \hookrightarrow E$;

(2) $\widetilde{H}(\cdot,t)$ *is a diffeomorphism of* E_{b_0} *onto* $E_{\alpha(t)}$;

(3) $\widetilde{H}(x_0,t) = \sigma(t)$.

In particular, the map $f^\alpha := \widetilde{H}(\cdot,1)$ is a diffeomorphism of E_{b_0} onto E_{b_1} such that $f^\alpha(x_0) = x_0'$ and $T([\alpha]) = [f^\alpha]$. Moreover if G is a finite group acting fiberwise on E and the fiber is compact, then f^α can be chosen to be G–equivariant.

Proof If the fiber of E is compact, the argument is the usual proof of the Ehresmann theorem: pullback E to $[0,1]$, choose a lift to E of the vector field d/dt and integrate it; see eg [Voisin 2002]. A G–invariant lift gives the last statement. But we also need the case of noncompact fibers. This can be treated as follows. Denote by $\tilde{\alpha}\colon \alpha^*E \to E$ the bundle map covering α. Since $[0,1]$ is contractible, there is a (smooth) trivialization $\psi\colon E_{b_0} \times [0,1] \to \alpha^*E$ such that $\psi(x,0) = x$; see [Steenrod 1951, Corollary 11.6, page 53]. Given any such ψ the composition $\tilde{\alpha} \circ \psi\colon E_{b_0} \times [0,1] \to E$ is a possible choice for the map $\widetilde{H}$ in 3.2. We now modify ψ so that it matches conditions (1)–(3). First notice that if $\{h_t\}_{t\in[0,1]}$ is any path in $\mathrm{Diff}(E_{b_0})$ starting at the identity, then $\psi_t' := \psi_t h_t$ is a new trivialization of α^*E. Next observe that

$t \mapsto \psi_t^{-1}(\sigma(t))$ is a path in E_{b_0} from x_0 to $\psi_1^{-1}(x_0')$, ie an isotopy of $\{x_0\}$ in E_{b_0}. By Theorem 3.4 there is $\{h_t\}$ that extends this isotopy. Then $\psi_t' := \psi_t h_t$ is a trivialization and $\widetilde{H} := \tilde{\alpha} \circ \psi'$ satisfies (1)–(3). □

We now use this construction for the fiber bundle $\mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ and give a geometric interpretation of the morphism (2-4) in terms of parallel transport.

Proposition 3.6 *Let $x, x' \in \mathsf{M}_{0,n+1}$. Let $\beta\colon [0,1] \to \mathsf{M}_{0,n}$ be a path such that $\beta(0) = X$ and $\beta(1) = X'$. Let $\widetilde{H}$, f^β and $T([\beta])$ be as in Lemma 3.5. Assume that $f^\beta(x_0) = x_0'$. Set $\tilde{\beta}(t) := \widetilde{H}(t, x_0)$. Then for $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$ we have $f_*^\beta([\gamma]) = \tilde{\beta}_\#([\gamma])$.*

Proof Take $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. Consider the map

$$F\colon [0,1] \times [0,1] \to \mathsf{M}_{0,n+1}, \quad F(t,s) = \widetilde{H}(\gamma(s), t).$$

Then $F(0,s) = \widetilde{H}(\gamma(s), 0) = \gamma(s)$, $F(0,1) = \widetilde{H}(\gamma(s), 1) = f^\beta \circ \gamma(s)$ and

$$F(t,0) = F(t,1) = \widetilde{H}(x_0, t) = \tilde{\beta}(t).$$

It follows that $i(\tilde{\beta}) * \gamma * \tilde{\beta} \simeq f^\beta \circ \gamma$ rel $\{0,1\}$. Hence $f_*^\beta([\gamma]) = \tilde{\beta}_\#([\gamma])$ for any $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

Proposition 3.7 *Let $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$ and let $\widetilde{H}$, f^α and $T([\alpha])$ be as in Lemma 3.5. Assume that $\sigma(t) := \widetilde{H}(t, x_0) = s \circ \alpha$. Then $\varepsilon([\alpha]) = f_*^\alpha$.*

Proof By Proposition 3.6, we get $f_*^\alpha[\gamma] = \sigma_\#([\gamma]) = [\sigma * \gamma * i(\sigma)]$ for any $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. Hence f^α satisfies $f_*^\alpha[\gamma] = [s \circ \alpha * \gamma * s \circ i(\alpha)] = \varepsilon([\alpha])([\gamma])$ for every $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

4 Dehn–Nielsen theorems and consequences

We dedicate this section to fixing some notation and recalling some classical concepts of surface topology.

4.1 Let Σ be an oriented surface and set $\Sigma^* := \Sigma - \{y\}$ for some $y \in \Sigma$. Given $b_0, b_1 \in \Sigma$ let $\Omega(\Sigma, b_0, b_1)$ denote the set of all paths α in Σ with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. Fix $x_0 \in \Sigma^*$. Let $\tilde{\alpha} \in \Omega(\Sigma, x_0, y)$ be such that $\tilde{\alpha}(t) = y$ only for $t = 1$ and let D be a small disk around y. Let α be the loop that starts at x_0, travels along $\tilde{\alpha}$ till it reaches ∂D, then makes a complete tour of ∂D counterclockwise and finally goes back to x_0 again along $\tilde{\alpha}$. An important observation is that the conjugacy class of $[\alpha]$ in $\pi_1(\Sigma^*, x_0)$ is well defined. Indeed the choice of the disk does not change $[\alpha]$, while if a different path $\tilde{\beta} \in \Omega(\Sigma, x_0, y)$ is chosen, then $[\beta]$ and $[\alpha]$ are conjugate by the class of a loop in Σ^* that starts at x_0 travels along $\tilde{\alpha}$ up to ∂D, then along a piece of ∂D and finally goes back along $\tilde{\beta}$.

4.2 Fix a point $(x_0, X) \in \boldsymbol{F}_{0,n}\, S^2$. Consider a smooth regular arc $\tilde{\alpha}_i$ joining x_0 to x_{σ_i} (for some permutation σ). Assume that the paths $\tilde{\alpha}_i$ intersect only at x_0 and that the tangent vectors at x_0 are all distinct and follow each other in counterclockwise order (we orient S^2 by the outer normal). Now consider the loops α_i constructed as in 4.1 and assume that the circles are pairwise disjoint and that the intersection of the interior of the i^{th} circle with X reduces to x_{σ_i}.

Definition 4.3 Let $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$. We call a set of generators $\mathcal{B} = \{[\alpha_1], \dots, [\alpha_n]\}$ obtained as above a *geometric basis* of $\pi_1(S^2 - X, x_0)$. We say that a geometric basis $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ is *adapted* to x if it respects the order of the points in X, that is α_i turns around x_i, ie the permutation $\sigma = \mathrm{id}$.

Notice that, thanks to the permutation, the definition of geometric basis depends only on the set $\{x_1, \dots, x_n\}$, not on the ordering of the points. On the other hand the classes $\{[\alpha_i]\}$ have a fixed order.

4.4 For $n \geq 3$, set $\Gamma_n := \langle \gamma_1, \dots, \gamma_n \mid \prod_{i=1}^n \gamma_i = 1 \rangle$. From a geometric basis $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ we get an isomorphism

$$\chi \colon \Gamma_n \to \pi_1(S^2 - X, x_0)$$

such that $\chi(\gamma_i) = [\alpha_i]$. Assume that $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ and $\overline{\mathcal{B}} = \{[\bar{\alpha}_i]\}_{i=1}^n$ are two geometric bases for $\pi_1(S^2 - X, x_0)$. It follows from 4.1 that every $[\alpha_i]$ is conjugate to some $[\bar{\alpha}_j]$. If we denote by

$$\chi, \bar{\chi} \colon \Gamma_n \to \pi_1(S^2 - X, x_0)$$

the isomorphisms corresponding to the two bases, then $\mu := \bar{\chi} \circ \chi^{-1} \in \operatorname{Aut} \pi_1(S^2 - X, x_0)$ has the following properties:

(1) for every $i = 1, \dots, n$, $\mu([\alpha_i])$ is conjugate to $[\alpha_j]$ for some j;

(2) the induced homomorphism on $H^2(\pi_1(S^2 - X, x_0), \mathbb{Z})$ is the identity.

Definition 4.5 We denote by $\operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ the subgroup of elements of $\operatorname{Aut} \pi_1(S^2 - X, x_0)$ satisfying properties (1) and (2) above. By 4.4 this definition does not depend on the choice of the geometric basis $\mathcal{B}$.

4.6 Now assume that $\mathcal{B}$ and $\overline{\mathcal{B}}$ are adapted to X. In this case, for every $i = 1, \dots, n$, $[\alpha_i]$ is conjugate to $[\bar{\alpha}_i]$. As a consequence, the automorphism $\mu := \bar{\chi} \circ \chi^{-1}$ of $\pi_1(S^2 - X, x_0)$ belongs to the subgroup $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ defined as follows.

Definition 4.7 We denote by $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ the subgroup of $\operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ of elements that map $[\alpha_i]$ to a conjugate of $[\alpha_i]$ for every $i = 1, \dots, n$. This definition does not depend on the choice of the geometric basis $\mathcal{B}$ adapted to x.

Definition 4.8 Similarly, we denote by $\operatorname{Aut}^* \Gamma_n \subset \operatorname{Aut} \Gamma_n$ the subgroup of automorphisms ν satisfying:

(1) For $i = 1, \dots, n$ the element $\nu(\gamma_i)$ is conjugate to γ_j for some j.

(2) The automorphism of $H^2(\Gamma_n, \mathbb{Z})$ induced by ν is the identity.

We denote by $\operatorname{Aut}^{**} \Gamma_n \subset \operatorname{Aut}^* \Gamma_n$ the subgroup of automorphisms ν such that:

(1′) For $i = 1, \dots, n$ the element $\nu(\gamma_i)$ is conjugate to γ_i.

If $\chi \colon \Gamma_n \to \pi_1(S^2 - X, x_0)$ is induced from a geometric basis (not necessarily adapted to x), then $\nu \in \operatorname{Aut}^* \Gamma_n$ (resp. $\operatorname{Aut}^{**} \Gamma_n$) if and only if $\chi \nu \chi^{-1} \in \operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ (resp. $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$).

4.9 If G is a group and $a \in G$, then $\operatorname{inn}_a : G \to G$ denotes conjugation by a, ie $\operatorname{inn}_a(x) = axa^{-1}$. Notice that if $f : G \to H$ is a morphism, then $f \circ \operatorname{inn}_a = \operatorname{inn}_{f(a)} \circ f$. The group of inner automorphisms of G is denoted $\operatorname{Inn} G$. It is a normal subgroup of $\operatorname{Aut} G$. We set $\operatorname{Out} G := \operatorname{Aut} G / \operatorname{Inn} G$. For $(x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$, we observe that $\operatorname{Inn}(\pi_1(S^2 - X, x_0)) \subset \operatorname{Out}^{**}(\pi_1(S^2 - X, x_0))$ and $\operatorname{Inn} \Gamma_n \subset \operatorname{Aut}^{**} \Gamma_n$, and we define

$$\begin{aligned} \operatorname{Out}^* \pi_1(S^2 - X, x_0) &:= \frac{\operatorname{Aut}^* \pi_1(S^2 - X, x_0)}{\operatorname{Inn} \pi_1(S^2 - X, x_0)}, \\ \operatorname{Out}^{**} \pi_1(S^2 - X, x_0) &:= \frac{\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)}{\operatorname{Inn} \pi_1(S^2 - X, x_0)}, \\ \operatorname{Out}^* \Gamma_n &:= \frac{\operatorname{Aut}^* \Gamma_n}{\operatorname{Inn} \Gamma_r}. \end{aligned} \tag{4-1}$$

Using a geometric basis we immediately get $\operatorname{Out}^* \Gamma_n \cong \operatorname{Out}^* \pi_1(S^2 - X, x_0)$.

4.10 If $S_{g,n}$ is a topological surface of genus g with n punctures, the mapping class group of $S_{g,n}$ is denoted by $\operatorname{Mod}(S_{g,n})$, while $\operatorname{PMod}(S_{g,n})$ denotes the pure mapping class group of $S_{g,n}$, which is defined to be the subgroup of $\operatorname{Mod}(S_{g,n})$ of elements that fix each puncture individually.

4.11 In the sequel we will need the following variants of the Dehn–Nielsen–Baer theorem, for which see [Farb and Margalit 2012, Theorem 8.8, page 234; Ivanov 2002, Section 2.9; Zieschang et al. 1980, Theorem 5.7.1, page 197, and Theorem 5.13.1, page 214].

Theorem 4.12 (Dehn–Nielsen–Baer) *Let $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$. Then $\varphi \in \operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ if and only if there exists $\sigma \in \operatorname{Inn} \pi_1(S^2 - X, x_0)$ and an orientation-preserving homeomorphism*

$$h : S^2 - X \to S^2 - X$$

such that $h(x_0) = x_0$ and $\varphi = \sigma \circ h_*$. In other words, $\operatorname{Mod}(S^2 - X) \cong \operatorname{Out}^*(\pi_1(S^2 - X, x_0))$.

Corollary 4.13 *Let $x, y \in \boldsymbol{F}_{0,n+1}\, S^2$ and $\varphi : \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - Y, y_0)$ be a homomorphism that sends geometric bases to geometric bases. Then there exists $\sigma \in \operatorname{Inn}(\pi_1(S^2 - Y, y_0))$ and an orientation-preserving homeomorphism $h : S^2 - X \to S^2 - Y$ such that $h(x_0) = y_0$ and $\varphi = \sigma \circ h_*$.*

Proof Fix an orientation-preserving homeomorphism $f : S^2 - Y \to S^2 - X$ such that $f(y_0) = x_0$ and apply the Dehn–Nielsen–Baer theorem to $f_* \circ \varphi$. $\square$

Corollary 4.14 *Let $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$. Then $\varphi \in \operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ if and only if there exists $\sigma \in \operatorname{Inn}(\pi_1(S^2 - X, x_0))$ and an orientation-preserving self-homeomorphism h of S^2 such that $h(x_i) = x_i$ for $0 \le i \le n$ and $\varphi = \sigma \circ h_*$. In other words, $\operatorname{PMod}(S^2 - X) \cong \operatorname{Out}^{**} \pi_1(S^2 - X, x_0)$.*

Proof Applying the Dehn–Nielsen–Baer theorem we get the homeomorphism h of $S^2 - X$ and σ. It is elementary that h extends to a homeomorphism of S^2. Next assume $h(x_1) = x_j$ and fix a geometric basis $\mathscr{B} = \{[\alpha_i]\}$ adapted to x. Here α_i is a loop at x_0 that makes a counterclockwise turn around x_i as in 4.1.

Hence $[h\alpha_1]$ is a loop making a turn around $h(x_1) = x_j$. But $[h\alpha_1]$ is conjugate to $\sigma h_*([\alpha_1]) = \varphi([\alpha_1])$ which is conjugate to $[\alpha_1]$ since $\varphi \in \mathrm{Aut}^{**}\,\pi_1(S^2 - X, x_0)$. Since α_1 makes a turn around x_1 it follows that $h(x_1) = x_j = x_1$. Similarly $h(x_i) = x_i$ for any i. □

4.15 We conclude this section by interpreting some classical constructions in the theory of braid groups using parallel transport. We consider the (pure version of the) generalized Birman exact sequence associated with $\mathbb{C}^{**} = \mathbb{P}^1 - \{0, 1, \infty\}$ (see [Farb and Margalit 2012, Theorem 9.1, page 245])

$$1 \to \pi_1(\mathsf{M}_{0,n}, X) \xrightarrow{\text{Push}} \mathrm{PMod}(\mathbb{P}^1 - X) \xrightarrow{\text{Forget}} \mathrm{PMod}(\mathbb{C}^{**}) \to 1. \tag{4-2}$$

The map Forget is the natural homeomorphism obtained by filling in the punctures, ie it is the map induced by the inclusion $\mathbb{P}^1 - X \hookrightarrow \mathbb{C}^{**}$. The map Push is defined as follows; see [Farb and Margalit 2012, Section 4.2.1]. Let $\alpha = (\alpha_1, \dots, \alpha_n)\colon [0,1] \to \mathsf{M}_{0,n}$ be a pure braid in $\mathbb{P}^1$, with $\alpha(0) = \alpha(1) = X$. Thinking of α as an isotopy from X to X (sending each x_i to x_i) we get by Theorem 3.4 that it can be extended to an isotopy of the whole $\mathbb{P}^1$. Denoting by Φ_α the homeomorphism of $\mathbb{P}^1$ obtained at the end of the isotopy, we have that $\Phi_\alpha(x_i) = \alpha_i(1) = x_i$, and thus Φ_α can be regarded as an homeomorphism of $\mathbb{P}^1 - X$. Taking its isotopy class we get $\mathrm{Push}(\alpha) = [\Phi_\alpha] \in \mathrm{PMod}(\mathbb{P}^1 - X)$. This map is well defined, ie it does not depend on the choice of α within its homotopy class nor on the choice of the isotopy extension.

4.16 It is useful to reinterpret the morphism ε defined in (2-4) in this setting. In particular we note that $\mathrm{Im}\,\varepsilon \subset \mathrm{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))$. Fix $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$.

Arguing as in Proposition 3.7 note that f^α extends to a homeomorphism $f^\alpha\colon \mathbb{P}^1 \to \mathbb{P}^1$ that fixes every x_i individually. Hence $[f^\alpha] \in \mathrm{PMod}(\mathbb{P}^1 - X)$. Since $\varepsilon([\alpha]) = f^\alpha_*$, we have $\varepsilon([\alpha]) \in \mathrm{Aut}^{**}(\pi_1(\mathbb{P}^1 - X))$.

Let $\tilde{\varepsilon}\colon \pi_1(\mathsf{M}_{0,n}, X) \to \mathrm{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0))$ denote the composition of ε with the natural projection $\mathrm{Aut}^{**} \to \mathrm{Out}^{**}$. Also, denote by $F\colon \mathrm{PMod}(\mathbb{C}^{**} - X) \to \mathrm{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0))$ the isomorphism $F\colon [h] \mapsto [h_*]$ coming from Corollary 4.14 of the Dehn–Nielsen–Baer theorem. The following proposition is the analogue of [Birman 1974, Theorem 1.10] for configurations of points in $\mathbb{C}^{**}$ (instead of $\mathbb{C}$).

Proposition 4.17 *For $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$, let f^α be the parallel transport as in Lemma 3.5. Then* $\mathrm{Push}([\alpha]) = [f^\alpha]$. *Moreover, the following diagram commutes*:

$$\begin{tikzcd}
 & \mathrm{PMod}(\mathbb{P}^1 - X) \arrow[d, "F"] \\
\pi_1(\mathsf{M}_{0,n}, X) \arrow[ur, "\text{Push}"] \arrow[r, "\tilde{\varepsilon}"] & \mathrm{Out}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))
\end{tikzcd}$$

Proof Let $\alpha\colon [0,1] \to \mathsf{M}_{0,n}$ be a pure braid in $\mathbb{P}^1$, with $\alpha(0) = \alpha(1) = X$, that we think as an isotopy from X to X. Let $\widetilde{H}\colon (\mathbb{P}^1 - X) \times [0,1] \to \mathsf{M}_{0,n+1}$ and f^α be as in Lemma 3.5. Define a map $\psi\colon \mathbb{P}^1 \times [0,1] \to \mathbb{P}^1$ by $\psi(u,t) := \widetilde{H}(u,t)$ for $u \notin X$ and $\psi(x_i, t) := \alpha_i(t)$. So ψ is an ambient isotopy of $\mathbb{P}^1$ extending the isotopy α. This proves the result, since by Proposition 3.7 $\varepsilon([\alpha]) = f^\alpha_*$, so $\tilde{\varepsilon}([\alpha]) = f^\alpha_* \bmod \mathrm{Inn}\,\pi_1(\mathbb{P}^1 - X, x_0)$, while $\mathrm{Push}([\alpha]) = [f^\alpha]$. □

Remark 4.18 Considering configurations of points in $\mathbb{C}$ instead of $\mathbb{C}^{**}$, Proposition 4.17 corresponds to [Birman 1974, Theorem 1.10].

Proposition 4.19 *Let $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and let $\nu \in \operatorname{Aut}^{**} \pi_1(\mathbb{P}^1 - X, x_0)$. Then there is an $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$, a lift $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = x_0$, a parallel transport f_t^α such that $f_t^\alpha(x_0) = \tilde{\alpha}(t)$ and $z \in \pi_1(\mathbb{P}^1 - X, x_0)$ such that $\nu = \operatorname{inn}_z \circ f_*^\alpha$.*

Proof Since $\operatorname{PMod}(\mathbb{C}^{**})$ is trivial — see [Farb and Margalit 2012, Proposition 2.3] — it follows from (4-2) that Push (and thus $\tilde{\varepsilon}$) is an isomorphism. In particular, for every $\nu \in \operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))$, there exists $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$ and $\sigma \in \operatorname{Inn}(\pi_1(\mathbb{P}^1 - X, x_0))$ such that $f_*^\alpha = \varepsilon([\alpha]) = \nu \circ \sigma$. Thus $\nu = \operatorname{inn}_z \circ f_*^\alpha$ for some $z \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

5 Topological types of actions

Definition 5.1 Let G be a finite group and let Σ_1 and Σ_2 be oriented topological surfaces both endowed with an action of G. We say that the two actions are *topologically equivalent* if there is an $\eta \in \operatorname{Aut} G$ and an orientation-preserving homeomorphism $f : \Sigma_1 \cong \Sigma_2$ such that $f(g \cdot x) = \eta(g) \cdot f(x)$ for any $x \in \Sigma_1$ and any $g \in G$; see [González-Díez and Harvey 1992]. An equivalence class is called a *topological type* of G–action (sometimes this is called *unmarked topological type*).

Definition 5.2 Fix on S^2 the orientation by the outer normal. We let $\mathscr{T}^n(G)$ denote the set of topological types of G–actions on a topological surface Σ such that $\Sigma/G \cong S^2$ (as oriented surfaces) and the projection $\pi : \Sigma \to \Sigma/G$ has n branch points.

Definition 5.3 If G is a finite group an n–datum is an epimorphism $\theta : \Gamma_n \to G$ is such that $\theta(\gamma_i) \neq 1$ for $i = 1, \dots, n$. We let $\mathscr{D}^n(G)$ denote the set of all n–data associated with the group G.

5.4 Fix a point $x = (x_0, X) \in \boldsymbol{F}_{0,n+1} S^2$ and a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(S^2 - X, x_0)$. Denote by $\chi : \Gamma_n \cong \pi_1(S^2 - X, x_0)$ the corresponding isomorphism. If $\theta : \Gamma_n \to G$ is a n–datum, the epimorphism $\theta \circ \chi^{-1}$ gives rise to a topological G–covering $p : \Sigma_0^\theta \to S^2 - X$. By the topological part of the Riemann existence theorem, this can be completed to a branched G–cover $p : \Sigma^\theta \to S^2$. By taking the equivalence class of Σ^θ we get a topological type of G–action. We get a map

$$\mathscr{F}_{x,\mathscr{B}} : \mathscr{D}^n(G) \to \mathscr{T}^n(G), \quad (\theta : \Gamma_n \to G) \mapsto [\Sigma^\theta].$$

5.5 We now introduce an action on the set of data that will be very important for the rest of the paper. By the Dehn–Nielsen–Baer theorem, $\operatorname{Out}^* \Gamma_n \cong \operatorname{Out}^*(\pi_1(S^2 - X, x_0)) \cong \operatorname{Mod}(S^2 - X)$. The latter group has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\begin{aligned} &\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2, && (\sigma_1 \cdots \sigma_{n-1})^n = 1, \\ &\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i, && \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 = 1. \end{aligned} \tag{5-1}$$

See [Birman 1974, Theorem 4.5, page 164]. Consider the short exact sequence

$$1 \to \operatorname{Inn} \Gamma_n \hookrightarrow \operatorname{Aut}^* \Gamma_n \xrightarrow{\pi} \operatorname{Out}^* \Gamma_n \to 1.$$

Let $\tilde{\sigma}_i : \Gamma_r \to \Gamma_r$ be the automorphism defined by the rule

$$\tilde{\sigma}_i(\gamma_i) = \gamma_{i+1}, \qquad \tilde{\sigma}_i(\gamma_{i+1}) = \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}, \qquad \tilde{\sigma}_i(\gamma_j) = \gamma_j \quad \text{for } j \neq i, i+1.$$

These automorphisms belong to $\operatorname{Aut}^* \Gamma_n$ and satisfy the relations (5-1) up to inner automorphisms. Moreover, $\pi(\tilde{\sigma}_i) = \sigma_i$. The group $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$ acts on the set $\mathscr{D}^n(G)$ by the rule

$$(\eta, \nu) \cdot \theta := \eta \circ \theta \circ \nu^{-1},$$

where $(\eta, \nu) \in \operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$ and $\theta \in \mathscr{D}^n(G)$ is a datum. We can view this action as an iterated action: first $\operatorname{Aut} G$ acts on $\mathscr{D}^n(G)$, then $\operatorname{Aut}^* \Gamma_n$ acts on the quotient $\mathscr{D}^n(G)/\operatorname{Aut} G$. Observe also that for any $a \in \Gamma_n$,

$$\theta \circ (\operatorname{inn}_a)^{-1} = \operatorname{inn}_{\theta(a)^{-1}} \circ \theta.$$

So inner automorphisms of Γ_n can be absorbed in the action of $\operatorname{Aut} G$. Since the automorphisms $\tilde{\sigma}_i$ above satisfy the relations (5-1) up to inner automorphisms, it follows that they do satisfy them exactly when acting on $\mathscr{D}^n(G)/\operatorname{Aut} G$. In this way one gets an action of $\operatorname{Out}^* \Gamma_n$ on $\mathscr{D}^n(G)/\operatorname{Aut} G$. Finally we claim that the actions of $\operatorname{Aut}^* \Gamma_n$ and $\operatorname{Out}^* \Gamma_n$ on $\mathscr{D}^n(G)/\operatorname{Aut} G$ have the same orbits; hence

$$\mathscr{D}^n/(\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n) = (\mathscr{D}^n/\operatorname{Aut} G)/\operatorname{Aut}^* \Gamma_n = (\mathscr{D}^n/\operatorname{Aut} G)/\operatorname{Out}^* \Gamma_n.$$

The reason is the same as before: inner automorphisms of Γ_n can be absorbed in the action of $\operatorname{Aut} G$.

Theorem 5.6 *Let G be a finite group. Choose*

(1) *an element $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$;*

(2) *a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(S^2 - X, x_0)$.*

Then the map $\mathscr{F}_{x,\mathscr{B}}$ induces a bijection between $\mathscr{D}^n(G)/(\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n)$ and the set $\mathscr{T}^n(G)$ of topological types of G–actions. The bijection does not depend on the choices of the point $x \in \boldsymbol{F}_{0,n+1}\, S^2$ and of the geometric basis $\mathscr{B}$.

From the discussion in 5.5 we immediately get the following.

Corollary 5.7 *The topological types of G–actions on curves of genus g are in bijection with*

$$(\mathscr{D}^n/\operatorname{Aut} G)/\operatorname{Out}^* \Gamma_n.$$

The proof of Theorem 5.6 is based on the following two propositions.

Proposition 5.8 *The map $\mathscr{F}_{x,\mathscr{B}}$ is constant on the orbits of the action of* $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$.

Proof Let $\theta\colon \Gamma_n \to G$ be a datum and $(\eta, \nu) \in \operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$. Let $\theta' = \eta \circ \theta \circ \nu^{-1}$. We want to show that Σ^θ and $\Sigma^{\theta'}$ have the same topological type of G–action. Set $\bar{\nu} = \chi \circ \nu \circ \chi^{-1}$. Observe that $\bar{\nu} \in \operatorname{Aut}^*(\pi_1(S^2 - X, x_0))$ since $\nu \in \operatorname{Aut}^* \Gamma_n$. By the Dehn–Nielsen–Baer Theorem 4.12, there is $\sigma \in \operatorname{Inn}(\pi_1(S^2 - X, x_0))$ and an orientation-preserving diffeomorphism $h\colon (S^2 - X, x_0) \to (S^2 - X, x_0)$ such that $h(x_0) = x_0$ and $\sigma \circ h_* = \bar{\nu}$. Let p and p' denote the projections

$$\begin{tikzcd} \Sigma_0^\theta \arrow[d, "p"] & \Sigma_0^{\theta'} \arrow[d, "p'"] \\ (S^2 - X, x_0) \arrow[r, "h"] & (S^2 - X, x_0) \end{tikzcd}$$

Choose $\tilde{x}_0 \in \Sigma_0^\theta$ and $\tilde{x}_0' \in \Sigma_0^{\theta'}$ both over x_0. We have that

$$h_*(p_*(\pi_1(\Sigma_0^\theta, \tilde{x}_0))) = (\sigma^{-1} \circ \bar{\nu})(\ker(\theta \circ \chi^{-1})) = \sigma^{-1}(\ker(\theta \circ \chi^{-1} \circ \bar{\nu}^{-1})) = \ker(\theta \circ \chi^{-1} \circ \bar{\nu}^{-1}),$$

where the last equality holds because σ is an inner automorphism. Moreover, since $\eta \in \operatorname{Aut} G$,

$$\ker(\theta \circ \chi^{-1} \circ \bar{\nu}^{-1}) = \ker(\eta \circ \theta \circ \chi^{-1} \circ \bar{\nu}^{-1}).$$

Thus $h_*(p_*(\pi_1(\Sigma_0^\theta, \tilde{x}_0))) = \ker(\eta \circ \theta \circ \chi^{-1} \circ \bar{\nu}^{-1}) = (p')_*(\pi_1(\Sigma_0^{\theta'}))$. By the lifting theorem we get an oriented homeomorphism $\tilde{h}\colon \Sigma_0^\theta \to \Sigma_0^{\theta'}$ such that the diagram commutes and which extends to the compactifications. Hence the G–actions on Σ^θ and $\Sigma^{\theta'}$ have the same topological type. □

Proposition 5.9 *If $\theta \in \mathscr{D}^n(G)$, then $\mathscr{F}_{x,\mathscr{B}}(\theta)$ does not depend on the choices of the point $x \in \boldsymbol{F}_{0,n+1}\, S^2$ and of the geometric basis $\mathscr{B}$.*

Proof First fix x and consider two geometric bases $\mathscr{B}$ and $\overline{\mathscr{B}}$. Let $\chi, \bar{\chi}\colon \Gamma_n \to \pi_1(S^2 - X, x_0)$ denote the corresponding isomorphisms. Then $\nu := \chi^{-1} \circ \bar{\chi} \in \operatorname{Aut}^* \Gamma_n$. For a datum θ, we have $\theta \circ \bar{\chi}^{-1} = \theta \circ \nu^{-1} \circ \chi^{-1}$. So $\mathscr{F}_{x,\overline{\mathscr{B}}}(\theta) = \mathscr{F}_{x,\mathscr{B}}(\theta \circ \nu^{-1})$. By Proposition 5.8, $\mathscr{F}_{x,\mathscr{B}}(\theta \circ \nu^{-1}) = \mathscr{F}_{x,\mathscr{B}}(\theta)$. Hence $\mathscr{F}_{x,\overline{\mathscr{B}}}(\theta) = \mathscr{F}_{x,\mathscr{B}}(\theta)$, as desired. Now suppose that $x, y \in \boldsymbol{F}_{0,n+1}\, S^2$. Let $\chi\colon \Gamma_n \to \pi_1(S^2 - X, x_0)$ and $\bar{\chi}\colon \Gamma_n \to \pi_1(S^2 - Y, y_0)$ be the isomorphisms associated with two geometric bases $\mathscr{B}$ and $\overline{\mathscr{B}}$. Then

$$\nu := \bar{\chi} \circ \chi^{-1}\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - Y, y_0)$$

sends a geometric basis to a geometric basis. Hence, by Corollary 4.13, there is $\sigma \in \operatorname{Inn}(\pi_1(S^2 - Y, y_0))$ and an orientation-preserving homeomorphism $h\colon (S^2 - X, x_0) \to (S^2 - Y, y_0)$ such that $h(x_0) = y_0$ and $\sigma \circ h_* = \nu$. Given a datum θ, h_* maps the kernel of $\theta \circ \chi^{-1}$ to the kernel of $\theta \circ \bar{\chi}^{-1}$. By the lifting theorem there is an oriented diffeomorphism $\tilde{h}$ that extends to the compactifications. Hence the G–actions on Σ_x^θ and Σ_y^θ have the same topological type. □

We recall two basic facts about monodromy maps. Let $p\colon E \to B$ be a topological G–covering. For $b \in B$ and $e \in p^{-1}(b)$, we denote by $\mu_{p,e}$ the monodromy map $\mu_{p,e}\colon \pi_1(B, b) \to G$ such that $g = \mu_{p,e}[\alpha]$ maps e to $\alpha_e(1)$, where α_e is the lift of α with initial point e.

Lemma 5.10 *Let $p\colon E \to B$ be a topological G–covering. Fix $b_0, b_1 \in B$ and $e_i \in p^{-1}(b_i)$. Let δ be a path from e_0 to e_1 and $\gamma = p \circ \delta$. Then $\mu_{p,e_0} = \mu_{p,e_1} \circ \gamma_\#$. In particular, if $b_0 = b_1$ then μ_{p,e_0} and μ_{p,e_1} differ by an inner automorphism of $\pi_1(B, b_0)$ or — equivalently — of G.*

Lemma 5.11 *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be G–coverings. Let $\tilde{h}\colon E \to E'$ be a G–equivariant homeomorphism and denote by $h\colon B \to B'$ the induced homeomorphism. Fix $e_0 \in E$. Then $\mu_{p,e_0} = \mu_{p',\tilde{h}(e_0)} \circ h_*$.*

Proof of Theorem 5.6 By Proposition 5.8, $\mathcal{F}_{x,\mathcal{B}}$ induces a map between $\mathcal{D}^n(G)/\ (\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n)$ and $\mathcal{T}^n(G)$. To prove the statement we have to check that

(1) if two epimorphisms $\theta, \theta'\colon \Gamma_r \to G$ give rise to the same topological type of G–action, then θ and θ' are in the same orbit for the action of $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$;

(2) every topological type of G–action with n branch points can be constructed from a datum in $\mathcal{D}^n(G)$.

To prove (1), consider the branched covers $p\colon \Sigma \to S^2$ and $p'\colon \Sigma' \to S^2$ associated with $\theta \circ \chi^{-1}$ and $\theta' \circ \chi^{-1}$ and suppose that there exists $\eta \in \operatorname{Aut} G$ and an orientation-preserving homeomorphism $\tilde{h}\colon \Sigma \to \Sigma'$ such that $\tilde{h}(g \cdot e) = \eta(g)\tilde{h}(e)$. We get an induced homeomorphism $h\colon \Sigma/G \to \Sigma'/G$ and an isomorphism $h_*\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - X, h(x_0))$. Fix $e_0 \in p^{-1}(x_0)$. From Lemma 5.11 it follows that $\mu_{p,e_0} = \eta \circ \mu_{p',\tilde{h}(e_0)} \circ h_*$. Now fix $e_0' \in (p')^{-1}(x_0)$ and a path in Σ' from $\tilde{h}(e_0)$ to e_0'. Finally let $\gamma = p' \circ \delta$. By Lemma 5.10 we get that $\mu_{p',\tilde{h}(e_0)} = \mu_{p',e_0'} \circ \gamma_\#$. Thus

$$\mu_{p,e_0} = \eta \circ \mu_{p',e_0'} \circ \gamma_\# \circ h_*. \tag{5-2}$$

Observe that, since $\tilde{h}$ preserves the orientation, so does h; hence $\gamma_\# \circ h_*\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - X, x_0)$ lies in $\operatorname{Aut}^*(\pi_1(S^2 - X, x_0))$. Let $\nu := \chi^{-1} \circ (\gamma_\# \circ h_*) \circ \chi \in \operatorname{Aut}^* \Gamma_n$ be the corresponding automorphism in $\operatorname{Aut}^* \Gamma_n$. (Again we are using that χ comes from a geometric basis.) Also, observe that $\theta \circ \chi^{-1}$ coincides with μ_{p,e_0} up to an inner automorphism of G, and the same holds for $\theta' \circ \chi^{-1}$ and for $\mu_{p',e_0'}$. We get that there exists $\eta \in \operatorname{Aut} G$ such that (5-2) becomes

$$\theta \circ \chi^{-1} = \eta \circ \theta' \circ \chi^{-1} \circ (\gamma_\# \circ h_*) = \eta \circ \theta' \circ \nu^{-1} \circ \chi^{-1}.$$

Thus $(\eta, \nu).\theta' = \theta$; that is, they are in the same orbit for the action of $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$. To prove (2) assume that G acts effectively on a surface Σ in such a way that $\Sigma/G \cong S^2$. Up to diffeomorphism we can assume that the set of critical values of $p\colon \Sigma \to S^2$ coincides with X. Fix a point $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\theta := \mu_{p,\tilde{x}_0} \circ \chi\colon \Gamma_n \to G$ be the monodromy of the unramified cover. Since Σ_0^θ is connected θ is surjective, and $\theta(\gamma_i) \neq 1$ since all the points of X are branch points. So it is an n–datum. By construction the associated cover coincides with Σ. Finally, it follows from Proposition 5.9 that the bijection induced by $\mathcal{F}_{x,\mathcal{B}}$ does not depend on x and $\mathcal{B}$. □

6 Tools for the construction

This section is dedicated to some tools that we will need in the following section for the construction of the families. We start with some considerations from group theory, that will be at the basis of the construction of the ordered set $(\mathcal{I}, \geq)$ of Theorem 1.4.

Consider an exact sequence of groups

$$(*) \qquad 1 \to N \xrightarrow{i} K \xrightarrow{p} H \to 1$$

and an epimorphism

$$f : N \twoheadrightarrow G$$

onto a *finite* group G.

Definition 6.1 An *extension* a of $(*, f)$ is a pair $a = (H_a, f_a)$, whose first element is a subgroup H_a of H of *finite index*, and whose second element is a morphism $f_a \colon p^{-1}(H_a) \to G$ such that $f_a i = f$. We denote by $\mathscr{I}(*, f)$ the set of all extensions.

If $a = (H_a, f_a)$ is an extension, we set

$$K_a := p^{-1}(H_a).$$

K_a is a subgroup of K and f_a is defined on K_a.

On the set $\mathscr{I}(*, f)$ we introduce the order relation

$$a \geq b \iff H_a \subset H_b \text{ and } f_a = f_b|_{K_a}.$$

Proposition 6.2 *$(\mathscr{I}(*, f), \geq)$ is a directed set.*

Proof Given $a, b \in \mathscr{I}(*, f)$, set $H_c := \{h \in H_a \cap H_b \mid f_a(h) = f_b(h)\}$. Then H_c has finite index in H since G is finite. Set $f_c := f_a|_{H_c}$. Then $c := (H_c, f_c) \in \mathscr{I}(*, f)$, and $c \geq a$ and $c \geq b$. □

In the following lemmas we describe two natural bijections between the sets $\mathscr{I}(*, f)$, when f and $(*)$ change under some specific rule.

Lemma 6.3 *Given $f : N \twoheadrightarrow G$ and $\eta \in \operatorname{Aut} G$, set $\bar{f} := \eta \circ f$. Then*

$$\text{(6-1)} \qquad \Phi \colon \mathscr{I}(*, f) \to \mathscr{I}(*, \bar{f}), \quad \Phi(H_a, f_a) := (H_a, \eta \circ f_a).$$

is an order-preserving bijection.

The proof is immediate.

Lemma 6.4 *Consider a commutative diagram of groups*

$$\begin{tikzcd}
(*) \arrow[d] & 1 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\alpha"] & K \arrow[r, "p"] \arrow[d, "\gamma"] & H \arrow[r] \arrow[d, "\beta"] & 1 \\
(\bar{*}) & 1 \arrow[r] & \overline{N} \arrow[r, "\bar{i}"] & \overline{K} \arrow[r, "\bar{p}"] & \overline{H} \arrow[r] & 1
\end{tikzcd}$$

with exact rows and α, β and γ isomorphisms. In other words, $(*)$ and $(\bar{*})$ are isomorphic short exact sequences. Given $\bar{f} : \overline{N} \twoheadrightarrow G$, set $f := \bar{f} \circ \alpha : N \twoheadrightarrow G$. Then the map

$$\text{(6-2)} \qquad \Phi \colon \mathscr{I}(*, f) \to \mathscr{I}(\bar{*}, \bar{f}), \quad \Phi(H_a, f_a) := (\beta(H_a), f_a \circ \gamma^{-1}|_{\gamma(K_a)}),$$

is an order-preserving bijection.

Proof If $a = (H_a, f_a)$, set $K_a = p^{-1}(H_a)$ as above. Set $\overline{H}_{\bar a} := \beta(H_a)$. Then

$$\overline{K}_{\bar a} := \bar p^{-1}(\beta(H_a)) = (\beta^{-1}\bar p)^{-1}(H_a) = (p\gamma^{-1})^{-1}(H_a) = \gamma(K_a).$$

Set also $\bar{f_{\bar a}} := f_a \circ \gamma^{-1}|_{\overline{K}_{\bar a}}$. Then it is immediate to check that $\bar a := (\overline{H}_{\bar a}, \bar{f_{\bar a}}) = \Phi(a)$ belongs to $\mathcal{I}(\bar *, \bar f)$ and that Φ is an order-preserving bijection. □

Lemma 6.5 *Let N, H and G be groups and let $\varepsilon\colon H \to \operatorname{Aut} N$, $h \mapsto \varepsilon_h$, be a morphism. Let $f : N \to G$ and $\varphi\colon H \to G$ be morphisms. There is a morphism $f'\colon N \rtimes_\varepsilon H \to G$ extending both f and φ (when N and H are included in $N \rtimes_\varepsilon H$ in the obvious way) if and only if for any $h \in H$*

$$\operatorname{inn}_{\varphi(h)} \circ f = f \circ \varepsilon_h. \tag{6-3}$$

The proof is elementary.

Lemma 6.6 *Let N, H, G, $\varepsilon\colon H \to \operatorname{Aut} N$ and f be as above. Assume that f is surjective, that N is finitely generated and that G is finite. Then*

(a) *$H'' := \{h \in H \mid \varepsilon_h(\ker\theta) = \ker\theta\}$ is a finite-index subgroup of H;*

(b) *there is a morphism $\tilde\varepsilon\colon H'' \to \operatorname{Aut} G$ such that the diagram*

$$\begin{array}{ccc} N & \xrightarrow{\varepsilon_h} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\tilde\varepsilon_h} & G \end{array}$$

commutes for $h \in H''$;

(c) *$H' := \ker\tilde\varepsilon$ is a finite-index subgroup of H;*

(d) *there is a unique morphism $f'\colon N \rtimes_\varepsilon H' \to G$ that extends f and such that $f'|_{H'} \equiv 1$.*

Proof The subgroup $\ker f$ has index $d := |G| < \infty$ in N. Since N is finitely generated, there are a finite number of index d subgroups of N; see eg [Hall 1950, page 128; Kurosh 1960, page 56]. The natural action of $\operatorname{Aut} N$ on the subgroups of N preserves the index. Therefore the orbit of $\operatorname{Aut} N$ through $\ker f$ is finite. Hence $(\operatorname{Aut} N)_{\ker f}$ has finite index in $\operatorname{Aut} N$. Since H/H'' injects in $\operatorname{Aut} N/(\operatorname{Aut} N)_{\ker f}$, H'' also has finite index in H. The existence of $\tilde\varepsilon_h$ follows immediately from the inclusion $\varepsilon_h(\ker f) \subset \ker f$ for $h \in H''$. Since $\operatorname{Aut} G$ is finite, H' has finite index in H'' and in H. By construction, for any $h \in H'$ we have $f = \tilde\varepsilon_h \circ f = f \circ \varepsilon_h$, ie (6-3) holds with $\varphi\colon H' \to G$ the trivial morphism. □

Theorem 6.7 *If the sequence $(*)$ splits, then $\mathcal{I}(*, f) \neq \varnothing$.*

Proof By Lemma 6.4 we can assume that the split exact sequence $(*)$ is a semidirect product. The result then follows from Lemma 6.6. □

6.8 We dedicate the second part of this section to some considerations on coverings and fiber bundles, which will be fundamental tools for our construction.

In the following we assume that all the spaces considered are semilocally 1–connected. Let X be a connected space and let $x \in X$. For every subgroup $H \subset \pi_1(X,x)$ there is a pointed covering $p\colon (E,e) \to (X,x)$ such that $\operatorname{Im} p_* = H$. Moreover p is unique up to pointed isomorphism. If $\beta \in \Omega(X,x,x')$ and $\beta_\#\colon \pi_1(X,x) \to \pi_1(X,x')$ is the induced isomorphism, then the pointed coverings of X associated with $H \subset \pi_1(X,x)$ and with $\beta_\# H \subset \pi_1(X,x')$ are isomorphic. Indeed if β_e denotes the lift with $\beta_e(0) = e$ and $e' = \beta_e(1)$, then $p_*\pi_1(E,e') = \beta_\# H$, so E is associated with both subgroups. If X is a complex manifold, any covering has a unique complex structure such that p is holomorphic and the coverings associated to $H \subset \pi_1(X,x)$ and with $\beta_\# H \subset \pi_1(X,x')$ are biholomorphic.

Lemma 6.9 *Let $\bar{E}$, $\bar{B}$ and B be connected and locally arcwise connected topological spaces. Let $p\colon \bar{E} \to \bar{B}$ be a fiber bundle and $q\colon B \to \bar{B}$ be a covering. Let $E := q^*\bar{E}$ be the pullback bundle. Then in the diagram*

$$
\begin{array}{ccc}
(E,e) & \xrightarrow{\bar{q}} & (\bar{E},\bar{e}) \\
{\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
(B,b) & \xrightarrow{q} & (\bar{B},\bar{b})
\end{array}
\tag{6-4}
$$

$\bar{q}\colon E \to \bar{E}$ is also a covering. Moreover, if the fiber of p is arcwise connected, then

$$\bar{q}_*\pi_1(E,e) = p_*^{-1}(q_*\pi_1(B,b)).$$

Proof Fix $\bar{e} \in \bar{E}$, set $\bar{b} = p(\bar{e})$ and let $V \subset \bar{B}$ be an evenly covered open subset of $\bar{B}$, ie $q^{-1}(V) = \bigsqcup U_i$ and $q|_{U_i}$ is a homeomorphism of U_i onto V. We claim that $p^{-1}(V)$ is an evenly covered neighborhood of e. Indeed $\bar{q}^{-1}(p^{-1}(V)) = \bigsqcup \psi^{-1}(U_i)$. Moreover $\psi^{-1}(U_i) = (q|_{U_i})^*\bar{E}$ is mapped homeomorphically on $p^{-1}(V)$ by $\bar{q}$ since $q|_{U_i}$ is a homeomorphism onto V. This proves the first assertion. Next choose $e \in \bar{q}^{-1}(\bar{e})$ and set $b = \psi(e)$. Obviously $q(b) = \bar{b}$. Set $\bar{F} := p^{-1}(\bar{b})$ and $F := \psi^{-1}(\bar{b})$. The diagram (6-4) induces a morphism of the homotopy exact sequences of the bundles:

$$
\begin{array}{ccccccccccc}
\longrightarrow & \pi_2(B) & \longrightarrow & \pi_1(F,e) & \longrightarrow & \pi_1(E,e) & \xrightarrow{p_*} & \pi_1(B,b) & \longrightarrow & \pi_0(F) = 1 \\
& \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\bar{q}_*} & & \big\downarrow{\scriptstyle q_*} & & \big\downarrow{\scriptstyle\cong} \\
\longrightarrow & \pi_2(\bar{B}) & \longrightarrow & \pi_1(\bar{F},\bar{e}) & \longrightarrow & \pi_1(\bar{E},\bar{e}) & \xrightarrow{p_*} & \pi_1(\bar{B},\bar{b}) & \longrightarrow & \pi_0(F) = 1
\end{array}
$$

Set $H := q_*\pi_1(B,b) \subset \pi_1(\bar{B},\bar{b})$, and $K := p_*^{-1}(H) \subset \pi_1(\bar{E},\bar{e})$. In the lower row we can substitute $\pi_1(B,b)$ with H and $\pi_1(E,e)$ with K and the row remains exact. Clearly $\bar{q}_*$ maps into K since the diagram commutes. So we get the diagram

$$
\begin{array}{ccccccccccc}
\longrightarrow & \pi_2(B) & \longrightarrow & \pi_1(F,e) & \longrightarrow & \pi_1(E,e) & \xrightarrow{p_*} & \pi_1(B,b) & \longrightarrow & \pi_0(F) = 1 \\
& \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\bar{q}_*} & & \big\downarrow{\scriptstyle q_*} & & \big\downarrow{\scriptstyle\cong} \\
\longrightarrow & \pi_2(\bar{B}) & \longrightarrow & \pi_1(\bar{F},\bar{e}) & \longrightarrow & K & \xrightarrow{p_*} & H & \longrightarrow & \pi_0(F) = 1
\end{array}
$$

Now q_* is an isomorphism. Applying the short five lemma [Eilenberg and Steenrod 1952, page 16], we get that $K = \operatorname{Im} \bar{q}_*$, as desired. ◻

The following lemma is a sort of converse which will be needed later.

Lemma 6.10 *Let A, $\bar{E}$, E, B and $\bar{B}$ be connected and locally arcwise connected topological spaces. Consider the diagram*

(6-5)
$$\begin{tikzcd} (A,a) \arrow[r, "\tilde{q}"] \arrow[d, "\varphi"'] & (\bar{E},\bar{e}) \arrow[d, "p"] \\ (B,b) \arrow[r, "q"] & (\bar{B},\bar{b}) \end{tikzcd}$$

Assume that $\varphi\colon A \to B$ and $p\colon \bar{E} \to \bar{B}$ are fiber bundles with arcwise connected fibers, that q and $\tilde{q}$ are finite degree coverings, and that $\tilde{q}_*\pi_1(A,a) = p_*^{-1}(q_*\pi_1(B,b))$. Then A is isomorphic to $q^*\bar{E}$ as a fiber bundle over B.

Proof Apply Lemma 6.9. Using the same notation as in (6-4),

$$\bar{q}_*\pi_1(E,e) = p_*^{-1}(q_*\pi_1(B,b)) = \tilde{q}_*\pi_1(A,a).$$

Moreover $\bar{q}$ is also a covering. So there is $w\colon (A,a) \to (E,e)$ such that $\bar{q} \circ w = \tilde{q}$. It remains to show that $\psi \circ w = \varphi$. Combining (6-5) with (6-4) we get the commutative diagram

$$\begin{tikzcd} (A,a) \arrow[r, "w"] \arrow[rr, bend left, "\tilde{q}"] \arrow[dr, bend right, "\varphi"'] & (E,e) \arrow[r, "\bar{q}"] \arrow[d, "\psi"'] & (\bar{E},\bar{e}) \arrow[d, "p"] \\ & (B,b) \arrow[r, "q"] & (\bar{B},\bar{b}) \end{tikzcd}$$

From $\bar{q} \circ w = \tilde{q}$ we get $p \circ \bar{q} \circ w = p \circ \tilde{q}$; hence $q \circ \psi \circ w = q \circ \varphi$. So $\psi \circ w$ and φ lift the same map with respect to the covering q. Since $\psi \circ w(a) = \varphi(a)$, we conclude that $\psi \circ w = \varphi$ and the result follows. □

7 Construction of the families of G–curves

7.1 Fix an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and set

(7-1)
$$N_x := \pi_1(\mathbb{P}^1 - X, x_0), \quad K_x := \pi_1(\mathsf{M}_{0,n+1}, x), \quad H_X := \pi_1(\mathsf{M}_{0,n}, X).$$

Consider the split exact sequence in the top row of (2-3), namely

($*_x$)
$$1 \to N_x \xrightarrow{i_*} K_x \xrightarrow{p_*} H_X \to 1.$$

Here $i\colon \mathbb{P}^1 - X \hookrightarrow \mathsf{M}_{0,n+1}$ is the map $i(x') := (x', x_1, \dots, x_n)$ and $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ is the fibration. Now let G be a finite group and let $\theta\colon \Gamma_n \to G$ be a datum. Choose a geometric basis $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ of N_x. As in 4.4, let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathcal{B}$. We apply the group theoretical considerations of Section 6 to the exact sequence ($*_x$) with $f := \theta \circ \chi^{-1}\colon N_x \twoheadrightarrow G$. We get a directed set $\mathcal{I}(*_x, f)$, which is nonempty since $(*_x)$ splits. To stress the dependence from the choices made, we will set

$$\mathcal{I}(x, \mathcal{B}, \theta) := \mathcal{I}(*_x, \theta \circ \chi^{-1}).$$

Indeed χ contains the same information as the basis $\mathcal{B}$.

Definition 7.2 A *collection of families* is an indexed set $\{\mathscr{C}_a \to \mathsf{Y}_a\}_{a \in \mathcal{I}}$ where

(1) $(\mathcal{I}, \geq)$ is a directed set;

(2) (Y_a, y_a) is a pointed smooth complex quasiprojective variety;

(3) $\mathscr{C}_a \to \mathsf{Y}_a$ is a family of curves;

(4) if $a, b \in \mathcal{I}$ and $a \geq b$, then there is an étale cover of finite degree $v_{ab}\colon (\mathsf{Y}_a, y_a) \to (\mathsf{Y}_b, y_b)$ such that $\mathscr{C}_a \cong v_{ab}^* \mathscr{C}_b$.

In this section we construct a collection of families indexed by $\mathcal{I}(x, \mathcal{B}, \theta)$.

7.3 Fix $a = (H_a, f_a) \in \mathcal{I}(x, \mathcal{B}, \theta)$. Let $q_a\colon (\mathsf{Y}_a, y_a) \to (\mathsf{M}_{0,n}, X)$ be the pointed covering with $q_{a*}\pi_1(\mathsf{Y}_a, x_a) = H_a$. Endow Y_a with the unique structure of a complex manifold making q_a an unramified analytic cover. Consider the diagram

$$\begin{tikzcd}
(E_a := q_a^* \mathsf{M}_{0,n+1}, e_a) \arrow[r, "\bar{q}_a"] \arrow[d, "\psi_a"] & (\mathsf{M}_{0,n+1}, x) \arrow[d, "p"] \\
(\mathsf{Y}_a, y_a) \arrow[r, "q_a"] & (\mathsf{M}_{0,n}, X)
\end{tikzcd} \tag{7-2}$$

with $e_a := (y_a, x)$. Notice that $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ is the universal family of lines with n holes and hence $\psi_a\colon E_a \to \mathsf{Y}_a$ is also a holomorphic family of curves (lines with holes).

By Lemma 6.9 applied to the diagram (7-2), the map $\bar{q}_a\colon E_a \to \mathsf{M}_{0,n+1}$ is the covering such that $\bar{q}_{a*}\pi_1(E_a, e_a) = K_a := p_*^{-1}(H_a)$. Hence $f_a\colon K_a \to G$ gives a morphism $\pi_1(E_a, e_a) \to G$ and thus a pointed G–covering $u_a\colon (\mathscr{C}_a^*, z_a) \to (E_a, e_a)$ such that $\operatorname{Im} u_{a*} = (\bar{q}_{a*})^{-1}(\ker f_a)$. In other words, u_a is the covering such that

$$\operatorname{Im} u_{a*} = \bar{q}_{a*}^{-1}(\ker f_a). \tag{7-3}$$

Composing with ψ_a we finally get a holomorphic family of noncompact Riemann surfaces

$$\pi_a = \psi_a \circ u_a\colon \mathscr{C}_a^* \to \mathsf{Y}_a.$$

The following diagram describes the whole situation:

$$\begin{tikzcd}
(\mathscr{C}_a^*, z_a) \arrow[r, "u_a"] \arrow[dr, "\pi_a"'] & (E_a, e_a) \arrow[r, "\bar{q}_a"] \arrow[d, "\psi_a"] & (\mathsf{M}_{0,n+1}, x) \arrow[d, "p"] \\
& (\mathsf{Y}_a, y_a) \arrow[r, "q_a"] & (\mathsf{M}_{0,n}, X)
\end{tikzcd}$$

It might help to compare this diagram with the corresponding diagram of groups:

$$\begin{tikzcd} \ker f_a \arrow[r, hook] & K_a \arrow[r, hook] \arrow[d, "p_*"] & K_X \arrow[d, "p_*"] \\ & H_a \arrow[r, hook] & H_X \end{tikzcd}$$

Summing up: p is the universal family of lines with n holes, q_a is a covering used as a base change, ψ_a is the pullback family of lines with n holes, u_a is a Galois cover and π_a is a family of noncompact Riemann surfaces. Each fiber of π_a covers the corresponding fiber of ψ_a. More precisely, if $y \in \mathsf{Y}_a$ and $X = q_a(y) \in \mathsf{M}_{0,n}$, looking at the fibers over y we have the unramified G–covering

$$\mathscr{C}^*_{a,y} \to E_{a,y} = \mathbb{P}^1 - X. \tag{7-4}$$

The last step in the construction is the fiberwise compactification, which is an application of the Grauert–Remmert extension theorem; see [Grothendieck 1971, Chapter XII, Theorem 5.4, page 340].

Theorem 7.4 (Grauert–Remmert extension theorem) *Let Y be a connected complex manifold and $Z \subset Y$ a closed analytic subset such that $Y^\circ := Y - Z$ is dense in Y. Let $f^\circ \colon X^\circ \to Y^\circ$ be a finite unramified cover. Then up to isomorphism there exists a unique normal analytic space X and a unique analytic covering $f : X \to Y$ such that $X^\circ \subset X$ and $f^\circ = f|_{X^\circ}$.*

Corollary 7.5 *In the hypotheses above, if Z is a smooth divisor, then X is smooth.*

Proof Let D be the unit disc. Using a local chart $U \cong D^n$ of Y such that $U \cap Z = D^n \cap \{z_1 = 0\}$ we get a finite cover of $D^* \times D^{n-1}$. By the topological classification of coverings disc, it is of the form $(z_1, \dots, z_n) \mapsto (z_1^m, z_2, \dots, z_n)$ for some $m \geq 1$, hence extends to an analytic cover $D^n \to D^n$. So, by uniqueness, $f^{-1}(U) \cong D^n$. In particular, $f^{-1}(U)$ is smooth. □

Lemma 7.6 *The unramified covering $u_a \colon \mathscr{C}^*_a \to E_a$ extends uniquely to an algebraic ramified cover $u_a \colon \mathscr{C}_a \to \mathbb{P}^1 \times \mathsf{Y}_a$, with $\mathscr{C}_a$ and Y_a smooth and quasiprojective.*

Proof Consider $\mathbb{P}^1 \times \mathsf{M}_{0,n}$. Let $x_0 \in \mathbb{P}^1$ and $X = (x_1, \dots, x_n) \in \mathsf{M}_{0,n}$. Recall that this means that $x_{n-2} = 0$, $x_{n-1} = 1$, $x_n = \infty$ and $(x_1, \dots, x_{n-3}) \in \boldsymbol{F}_{0,n-3}\, \mathbb{C}^{**}$. Let $Z_i \subset \mathbb{P}^1 \times \mathsf{M}_{0,n}$ be the smooth divisor $Z_i := \{x_0 = x_i\}$ for $i = 1, \dots, n$. The divisors $Z_1, \dots, Z_n$ are pairwise disjoint, so their union, which we denote by Z, is a smooth divisor of $\mathbb{P}^1 \times \mathsf{M}_{0,n}$. The map $\bar{q}_a$ in (7-2) obviously extends to a map

$$\bar{q}_a \colon \mathbb{P}^1 \times \mathsf{Y}_a \to \mathbb{P}^1 \times \mathsf{M}_{0,n}.$$

Then $\bar{q}_a^* Z$ is a smooth divisor of $\mathbb{P}^1 \times \mathsf{Y}_a$. Since $\mathsf{M}_{0,n+1} = (\mathbb{P}^1 \times \mathsf{M}_{0,n}) - Z$, $E_a = (\mathbb{P}^1 \times \mathsf{Y}_a) - \bar{q}_a^* Z$. So we can apply the Grauert–Remmert extension theorem to the topological covering $u_a \colon \mathscr{C}^*_a \to E_a$, which can be thus completed to a ramified cover $u_a \colon \mathscr{C}_a \to \mathbb{P}^1 \times \mathsf{Y}_a$, with $\mathscr{C}_a$ smooth. To prove the quasiprojectivity one uses a similar argument. An étale analytic cover of a quasiprojective variety is quasiprojective and the covering map is algebraic; see eg [Grothendieck 1971, Chapter XII, Theorem 5.1, page 333]. Since $\mathsf{M}_{0,n}$ and $\mathsf{M}_{0,n+1}$ are quasiprojective, and q_a and $\bar{q}_a$ are étale, we get that Y_a and E_a are quasiprojective and q_a and $\bar{q}_a$ are algebraic morphisms. Let $\bar{\mathsf{Y}}_a$ be a projective manifold containing Y_a as an open subset. Then E_a is a Zariski open subset of $\mathbb{P}^1 \times \bar{\mathsf{Y}}_a$ and we can apply the Grauert–Remmert extension theorem to $u_a \colon \mathscr{C}_a^* \to E_a$, this time viewing E_a as an open subset of $\mathbb{P}^1 \times \overline{\mathsf{Y}}_a$. We obtain a ramified cover $\bar{u}_a \colon \overline{\mathscr{C}}_a \to \mathbb{P}^1 \times \overline{\mathsf{Y}}_a$. Since $\mathbb{P}^1 \times \overline{\mathsf{Y}}_a$ is projective, $\overline{\mathscr{C}}_a$ is also projective. By uniqueness, $\mathscr{C}_a = \bar{u}_a^{-1}(\mathbb{P}^1 \times \mathsf{Y}_a)$, so it is quasiprojective. □

7.7 Notice that the projection $\mathbb{P}^1 \times \mathsf{Y}_a \to \mathsf{Y}_a$ extends ψ_a in (7-2), while the composition

$$\mathscr{C}_a \xrightarrow{u_a} \mathbb{P}^1 \times \mathsf{Y}_a \to \mathsf{Y}_a$$

extends π_a. We denote the extensions by the same symbol. We claim that

$$\pi_a \colon \mathscr{C}_a \to \mathsf{Y}_a$$

is a submersion. Indeed, let $U \cong D^n$ be a local chart in $\mathbb{P}^1 \times \mathsf{Y}_a$ such that

$$U \cap \pi^* Z = U \cap \pi^* Z_i = \{x_0 - x_i = 0\}$$

for some $i = 1, \dots, n$ (with $x_{n-2} = 0$, $x_{n-1} = 1$ and $x_n = \infty$). Denoting $w = x_0 - x_i$, we get that $w, x_1, \dots, x_n$ are local coordinates on U and $\pi'|_{\pi'^{-1}(U)} \colon \pi'^{-1}(U) \to U$ is of the form

$$(w, x_1, \dots, x_n) \mapsto (w^m, x_1, \dots, x_n)$$

for some $m \geq 2$. We conclude that locally $\pi_a(w, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Thus π_a is a submersion onto a smooth base and its fibers are smooth curves.

If $y \in \mathsf{Y}_a$, the fiber $\mathscr{C}_{a,y} \to \mathbb{P}^1$ of π_a over y is the unique smooth compactification of the unramified cover (7-4), ie the one given by Riemann's existence theorem.

We call

$$\begin{tikzcd} \mathscr{C}_a \arrow[rr, "u_a"] \arrow[dr, "\pi_a"'] & & \mathbb{P}^1 \times \mathsf{Y}_a \arrow[dl, "\psi_a"] \\ & \mathsf{Y}_a & \end{tikzcd}$$

the family of G–coverings associated with the datum $\theta \in \mathscr{D}^n(G)$, the point $x = (x_0, X) \in \mathsf{M}_{0,n+1}$, the geometric basis $\mathscr{B}$ of $\pi_1(\mathbb{P}^1 - X, x_0)$ and the extension $a \in \mathscr{I}(x, \mathscr{B}, \theta)$.

Theorem 7.8 *If $x \in \mathsf{M}_{0,n+1}$, $\mathscr{B}$ is a basis of N_x and θ is an n–datum, then*

$$\mathfrak{K}(x, \mathscr{B}, \theta) := \{\mathscr{C}_a \to \mathsf{Y}_a\}_{a \in \mathscr{I}(x, \mathscr{B}, \theta)} \tag{7-5}$$

is a collection of families in the sense of Definition 7.2.

Proof It remains only to prove property (4). We start with an observation. If $p_i \colon (E_i, e_i) \to (B, b)$ are coverings and $\operatorname{Im} p_{1*} \subset \operatorname{Im} p_{2*}$, the unique continuous map $f \colon (E_1, e_1) \to (E_2, e_2)$ such that $p_2 \circ f = p_1$ is a covering map. Indeed let $f \colon (X, x) \to (E_2, e_2)$ be the covering with $\operatorname{Im} f_* = p_{2*}^{-1}(\operatorname{Im} p_{1*})$. Then $p_2 \circ f$ is a covering isomorphic to p_1, so we can assume $p_1 = p_2 \circ f$.

Now, given $a = (H_a, f_a)$ and $b = (H_b, f_b)$, $a \geq b$ means that $H_a \subset H_b \subset \pi_1(\mathsf{M}_{0,n}, X)$; hence $K_a = p_*^{-1}(H_a) \subset K_b = p_*^{-1}(H_b)$ and $f_a\colon K_a \to G$ is the restriction of f_b. We have coverings $q_i : (\mathsf{Y}_i, y_i) \to (\mathsf{M}_{0,n}, X)$ with $H_i = \operatorname{Im} q_{i*}$ for $i = a, b$. By the observation at the beginning there is a unique covering map $v\colon (\mathsf{Y}_a, y_a) \to (\mathsf{Y}_b, y_b)$ such that $q_b \circ v = q_a$ and $\operatorname{Im} v_* = q_{b*}^{-1}(H_a)$. For the same reason, since $\operatorname{Im} \bar{q}_{i*} = K_i$ for $i = a, b$, there is a covering $\bar{v}\colon (E_a, e_a) \to (E_b, e_b)$ such that $\bar{q}_b \circ \bar{v} = \bar{q}_a$. We claim that

$$\psi_b \bar{v} = v \psi_a. \tag{7-6}$$

Indeed, $q_b \psi_b \bar{v} = p \bar{q}_b \bar{v} = p \bar{q}_a = q_a \psi_a = q_b v \psi_a$. Hence $\psi_b \bar{v}$ and $v \psi_a$ lift the same map with respect to the covering q_b. Since $\psi_b \bar{v}(e_a) = y_b = v \psi_a(e_a)$ we conclude that $\psi_b \bar{v} = v \psi_a$ as claimed.

Finally we have the coverings $u_i : \mathscr{C}_i^* \to E_i$ such that $\operatorname{Im} u_{i*} = \bar{q}_{i*}^{-1}(\ker f_i)$; see (7-3). Since $\bar{v}_* = \bar{q}_{b*}^{-1} \circ \bar{q}_{a*}$ and $\ker f_a \subset \ker f_b$ we have $\bar{v}_*(\bar{q}_{a*}^{-1}(\ker f_a)) = \bar{q}_{b*}^{-1}(\ker f_a) \subset \bar{q}_{b*}^{-1}(\ker f_b)$. This means that

$$\operatorname{Im}(\bar{v} \circ u_a)_* = \bar{v}_*(\bar{q}_{a*}^{-1}(\ker f_a)) \subset \operatorname{Im} u_{b*}. \tag{7-7}$$

So we can apply once more the observation at the beginning and we get a covering $\tilde{v}\colon \mathscr{C}_a^* \to \mathscr{C}_b^*$ such that

$$u_b \tilde{v} = \bar{v} u_a, \quad \operatorname{Im} \tilde{v}_* = u_{b*}^{-1}(\operatorname{Im}(\bar{v} \circ u_a)_*). \tag{7-8}$$

Composing with ψ_a and ψ_b and using (7-6) we get a commutative diagram

$$\begin{tikzcd} \mathscr{C}_a^* \arrow[r, "\tilde{v}"] \arrow[d, "\pi_a"'] & \mathscr{C}_b^* \arrow[d, "\pi_b"] \\ \mathsf{Y}_a \arrow[r, "v"] & \mathsf{Y}_b \end{tikzcd} \tag{7-9}$$

with π_a and π_b bundles, and $\bar{v}$ and $\tilde{v}$ coverings. We claim that

$$\operatorname{Im} \tilde{v}_* = \pi_{b*}^{-1}(\operatorname{Im} v_*). \tag{7-10}$$

Indeed starting from (7-7) we compute

$$\operatorname{Im}(\bar{v} \circ u_a)_* = \bar{q}_{b*}^{-1}(\ker f_a) = \bar{q}_{b*}^{-1}(K_a \cap \ker f_b) = \bar{q}_{b*}^{-1}(K_a) \cap \bar{q}_{b*}^{-1}(\ker f_b),$$
$$\bar{q}_{b*}^{-1}(\ker f_b) = \operatorname{Im} u_{b*},$$
$$K_a = p_*^{-1}(H_a),$$
$$\bar{q}_{b*}^{-1}(K_a) = \bar{q}_{b*}^{-1} p_*^{-1}(H_a) = (p\bar{q}_{b*})_*^{-1}(H_a) = (q_{b*}\psi_{b*})^{-1}(H_a) = \psi_{b*}^{-1}(q_{b*}^{-1}(H_a)) = \psi_{b*}^{-1}(\operatorname{Im} v_*),$$
$$\operatorname{Im}(\bar{v} \circ u_a)_* = \psi_{b*}^{-1}(\operatorname{Im} v_*) \cap \operatorname{Im} u_{b*}.$$

So from (7-8) we get

$$\operatorname{Im} \tilde{v}_* = u_{b*}^{-1}(\operatorname{Im}(\bar{v} \circ u_a)_*) = u_{b*}^{-1}(\psi_{b*}^{-1}(\operatorname{Im} v_*) \cap \operatorname{Im} u_{b*}) = u_{b*}^{-1}(\psi_{b*}^{-1}(\operatorname{Im} v_*)) = \pi_{b*}^{-1}(\operatorname{Im} v_*).$$

This proves (7-10). Applying Lemma 6.10 to the diagram (7-9) we get that $\tilde{v}\colon \mathscr{C}_a^* \to v^*\mathscr{C}_b^*$ is an isomorphism of bundles over Y_a. The map $\tilde{v}$ is an isomorphism of the coverings $\mathscr{C}_a^* \to E_a$ and

$v^*\mathscr{C}_b^* \to E_a$. By the uniqueness statement in the Grauert–Remmert extension theorem, it extends to an isomorphism of the coverings $\mathscr{C}_a \to \mathbb{P}^1 \times \mathsf{Y}_a$ and $v^*\mathscr{C}_b \to \mathbb{P}^1 \times \mathsf{Y}_a$. This extension is an isomorphism of the families of curves, $\mathscr{C}_a \cong v^*\mathscr{C}_b$. □

8 Independence from the choices

In this section, we conclude the proof of our Theorem 1.4. We present two main arguments. The first one is Theorem 8.2, whose proof will take up most of the section. It states the independence of the collection $\mathfrak{K}(x, \mathscr{B}, \theta)$ on $x \in \mathsf{M}_{0,n+1}$, on the geometric basis $\mathscr{B}$, and on the $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit of θ. Secondly, we show (Theorem 8.6) that every curve in a family of the collection $\pi_a : \mathscr{C}_a \to \mathsf{Y}_a$ has the topological type associated with θ, and that, conversely, for any G–curve C with topological type $[\theta]$, there is at least one fiber of $\mathscr{C}_a \to \mathsf{Y}_a$ which is (unmarkedly) G–isomorphic to C (and there are only finitely many such fibers).

Definition 8.1 We say that two collections of families $\{\mathscr{C}_a \to \mathsf{Y}_a\}_{a \in \mathscr{I}}$ and $\{\overline{\mathscr{C}}_{\bar a} \to \overline{\mathsf{Y}}_{\bar a}\}_{\bar a \in \overline{\mathscr{I}}}$ are *equivalent* if there is an order-preserving bijection $a \mapsto \bar a$ of $\mathscr{I}$ onto $\overline{\mathscr{I}}$ and for every $a \in \mathscr{I}$ a biholomorphism $w_a : \mathsf{Y}_a \to \overline{\mathsf{Y}}_{\bar a}$ such that:

(1) $\mathscr{C}_a \cong w_a^* \overline{\mathscr{C}}_{\bar a}$.

(2) If $a, b \in \mathscr{I}$ and $a \ge b$, the following diagram commutes:

$$\begin{tikzcd} \mathsf{Y}_a \arrow[r, "w_a"] \arrow[d, "v_{ab}"'] & \overline{\mathsf{Y}}_{\bar a} \arrow[d, "\bar v_{\bar a \bar b}"] \\ \mathsf{Y}_b \arrow[r, "w_b"] & \overline{\mathsf{Y}}_{\bar b} \end{tikzcd}$$

In the following we conclude the independence of our collection from the choices made; different choices yield equivalent collections.

Theorem 8.2 *Up to equivalence, the collection of families $\mathfrak{K}(x, \mathscr{B}, \theta)$ is independent of the choices of x and $\mathscr{B}$ and only depends on the $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit of θ. In particular, the collection $\mathfrak{K}(x, \mathscr{B}, \theta)$ only depends on the topological type $[\theta]$.*

The proof of Theorem 8.2 is organized as follows: We start by showing that the action of $\operatorname{Aut} G$ on θ does not change the collection (Lemma 8.3); and then we prove that changing x and $\mathscr{B}$ by parallel transport leads to equivalent collections (Lemma 8.4). The combination of these two results implies that, up to equivalence, the collection of families $\mathfrak{K}(x, \mathscr{B}, \theta)$ does not change under the action $\operatorname{Aut} G \times \operatorname{Aut}^{**} \Gamma_n$ on θ (Lemma 8.5). Finally, we combine these results and complete the proof of Theorem 8.2.

Lemma 8.3 *Let $\theta \in \mathscr{D}^n(G)$ and $\eta \in \operatorname{Aut} G$. Set $\bar\theta := \eta \circ \theta$. Let $\mathscr{I}(x, \mathscr{B}, \theta) \to \mathscr{I}(x, \mathscr{B}, \bar\theta)$, $a \mapsto \bar a$, be the bijection of Lemma 6.3. Then $\mathsf{Y}_{\bar a} = \mathsf{Y}_a$ and $\mathscr{C}_{\bar a} = \mathscr{C}_a$. So $\mathfrak{K}(x, \mathscr{B}, \theta) = \mathfrak{K}(x, \mathscr{B}, \bar\theta)$. In particular, for $z \in N_x$, $\mathfrak{K}(x, \mathscr{B}, \theta) = \mathfrak{K}(x, \mathscr{B}, \theta \circ \operatorname{inn}_z)$.*

Proof Let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathscr{B}$. Set $f := \theta \circ \chi^{-1}$ and

$$\bar{f} := \bar{\theta} \circ \chi^{-1} = \eta \circ f.$$

By Lemma 6.3 we get a bijective correspondence $I(x, \mathscr{B}, \theta) \to I(x, \mathscr{B}, \bar{\theta})$ which sends $a = (H_a, f_a)$ to $\bar{a} := (H_{\bar{a}}, f_{\bar{a}})$, where $H_{\bar{a}} = H_a$, and $f_{\bar{a}} = \eta \circ f_a$. It follows that $K_{\bar{a}} = K_a$ and $\ker f_{\bar{a}} = \ker f_a$. Therefore $\mathsf{Y}_{\bar{a}} = \mathsf{Y}_a$, $E_{\bar{a}} = E_a$, $\mathscr{C}^*_{\bar{a}} = \mathscr{C}^*_a$ and $\mathscr{C}_{\bar{a}} = \mathscr{C}_a$. For the last statement, observe that $\theta \circ \mathrm{inn}_z = \mathrm{inn}_{\theta(z)} \circ \theta$. □

Lemma 8.4 *Let $\theta \in \mathscr{D}^n(G)$ and $x, x' \in \mathsf{M}_{0,n+1}$. Let the notation be as in Proposition 3.7: β is a path in $\mathsf{M}_{0,n}$ from X to X', f^β represents the parallel transport along β, $f^\beta(x_0) = x'_0$ and $\tilde{\beta}(t) = \widetilde{H}(t, x_0)$. Then the collections $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(x', f^\beta_*(\mathscr{B}), \theta)$ are equivalent.*

Proof Let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathscr{B}$. Set $f := \theta \circ \chi^{-1}$ and $\bar{f} := f \circ (f^\beta_*)^{-1}$. We show that if $a \in \mathscr{I}(x, \mathscr{B}, \theta)$ and $\bar{a} = \Phi(a)$, where Φ is the map in (6-2), then the families $\mathscr{C}_a \to \mathsf{Y}_a$ and $\mathscr{C}_{\bar{a}} \to \mathsf{Y}_{\bar{a}}$ are canonically isomorphic. Consider the diagram

$$\begin{tikzcd}
(*_x) \arrow[d] & 1 \arrow[r] & N_x \arrow[r, "i_*"] \arrow[d, "f^\beta_*"] & K_x \arrow[r, "p_*"] \arrow[d, "\tilde{\beta}_\#"] & H_X \arrow[r] \arrow[d, "\beta_\#"] & 1 \\
(*_{x'}) & 1 \arrow[r] & N_{x'} \arrow[r, "i_*"] & K_{x'} \arrow[r, "p_*"] & H_{X'} \arrow[r] & 1
\end{tikzcd}$$

Assume $a = (H_a, f_a)$ and $\bar{a} = (H_{\bar{a}}, f_{\bar{a}})$. By the definition of Φ we have $H_{\bar{a}} = \beta_\#(H_a)$, $K_{\bar{a}} = \tilde{\beta}_\#(K_a)$, $f_{\bar{a}} = f_a \circ (\tilde{\beta}_\#)^{-1}$ and $\ker f_{\bar{a}} = \tilde{\beta}_\#(\ker f_a)$. It follows from 6.8 that there are canonical isomorphisms $\mathsf{Y}_{\bar{a}} \cong \mathsf{Y}_a$, $E_{\bar{a}} \cong E_a$ and $\mathscr{C}^*_{\bar{a}} \cong \mathscr{C}^*_a$. By compactifying we get that the families $\mathscr{C}_a \to \mathsf{Y}_a$ and $\mathscr{C}_{\bar{a}} \to \mathsf{Y}_{\bar{a}}$ are isomorphic. □

Lemma 8.5 *Let $(\eta, \nu) \in \mathrm{Aut}\, G \times \mathrm{Aut}^{**}\,\Gamma_n$. Then the collections $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(x, \mathscr{B}, \eta \circ \theta \circ \nu^{-1})$ are equivalent.*

Proof We have $\bar{\nu} := \chi \circ \nu \circ \chi^{-1} \in \mathrm{Aut}^{**}\, N_x$. Set $\bar{\theta} := \eta \circ \theta \circ \nu^{-1}$, $f := \theta \circ \chi^{-1}\colon N_x \twoheadrightarrow G$ and $\bar{f} := \bar{\theta} \circ \chi^{-1} = \eta \circ f \circ \bar{\nu}^{-1}$. By Proposition 4.19, there is an $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$, a lift $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = x_0$, and a parallel transport f^α_t such that $f^\alpha_t(x_0) = \tilde{\alpha}(t)$ and $z \in \pi_1(\mathbb{P}^1 - X, x_0)$ such that $\bar{\nu} = \mathrm{inn}_z \circ f^\alpha_*$. Note that, in particular, $f^\alpha(x_0) = x_0$. We get $\bar{f} = \eta \circ f \circ (f^\alpha_*)^{-1} \circ \mathrm{inn}_{z^{-1}}$. The statement follows from the previous two lemmas. □

Proof of Theorem 8.2 Since changing geometric bases of N_x adapted to X corresponds to acting with $\mathrm{Aut}^{**}\,\Gamma_n$, by the previous lemma it follows that if the point x is fixed, changing the adapted basis does not matter. Next fix $x, \bar{x} \in \mathsf{M}_{0,n+1}$. Choose a path $\tilde{\beta}$ in $\mathsf{M}_{0,n+1}$ joining x to $\bar{x}$. Set $\beta := p \circ \tilde{\beta}$ and let f^β be a parallel transport such that $f^\beta(x_0) = \bar{x}_0$. Let $\mathscr{B}$ be an adapted basis at x. Then $f^\beta_*\mathscr{B}$ is an adapted basis at $\bar{x}$. By Lemma 8.4 we get that $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(\bar{x}, f^\beta_*\mathscr{B}, \theta)$ are equivalent. In other words we have independence from x and $\mathscr{B}$ as long as $\mathscr{B}$ is adapted to x. We also have that θ only matters through its $\operatorname{Aut} G \times \operatorname{Aut}^{**} \Gamma_n$–orbit by Lemma 8.5. It remains to show independence from the $\operatorname{Aut}^* \Gamma_n$–orbit. It follows from the definitions in 4.4 that this is equivalent to showing that if $x \in \mathsf{M}_{0,n+1}$, $\mathscr{B}$ is a basis adapted to x and $\overline{\mathscr{B}}$ is an arbitrary basis of $\pi_1(\mathbb{P}^1 - X, x_0)$, then the collections $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(x, \overline{\mathscr{B}}, \theta)$ are equivalent. Let us prove this statement. There is a permutation $\sigma \in S_n$ such that $\overline{\mathscr{B}}$ is adapted to $(x_0, x_{\sigma_1}, \dots, x_{\sigma_n})$. Define

$$\begin{aligned} \tau\colon \mathsf{M}_{0,n} \to \mathsf{M}_{0,n}, \qquad & \tau(x_1, \dots, x_n) := (x_{\sigma_1}, \dots, x_{\sigma_n}), \\ \tilde\tau\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n+1}, \quad & \tilde\tau(x_0, x_1, \dots, x_n) := (x_0, x_{\sigma_1}, \dots, x_{\sigma_n}). \end{aligned}$$

Set $\bar x = \tilde\tau(x)$ and $\overline{X} = \tau(X)$. By the previous results we know that $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(\bar x, \overline{\mathscr{B}}, \theta)$ are equivalent. It remains to check that also $\mathfrak{K}(\bar x, \overline{\mathscr{B}}, \theta)$ and $\mathfrak{K}(x, \overline{\mathscr{B}}, \theta)$ are equivalent. Consider the diagram

$$\begin{tikzcd} (*_x) \arrow[d] & 1 \arrow[r] & N_x \arrow[r, "i_*"] \arrow[d, "\mathrm{id}_{N_x}"] & K_x \arrow[r, "p_*"] \arrow[d, "\tilde\tau_*"] & H_X \arrow[r] \arrow[d, "\tau_*"] & 1 \\ (*_{\bar x}) & 1 \arrow[r] & N_{\bar x} \arrow[r, "i_*"] & K_{\bar x} \arrow[r, "p_*"] & H_{\overline{X}} \arrow[r] & 1 \end{tikzcd}$$

To check commutativity observe that $\tilde\tau$ sends the fiber over X to the fiber over $\overline{X} := (x_{\sigma_1}, \dots, x_{\sigma_n})$, ie $\tilde\tau(\mathbb{P}^1 - X) \times \{X\} = (\mathbb{P}^1 - X) \times \{\overline{X}\}$ and on the first factor it is the identity map. We use this diagram with $f = \bar f = \theta \circ \bar\chi^{-1}$. We get the usual correspondence $a \mapsto \bar a$, $\mathscr{I}(x, \overline{\mathscr{B}}, \theta) \to \mathscr{I}(\bar x, \overline{\mathscr{B}}, \theta)$, with

$$H_{\bar a} = \tau_*(H_a), \quad K_{\bar a} = \tilde\tau_*(K_a), \quad \ker f_{\bar a} = \tilde\tau_*(\ker f_a). \tag{8-1}$$

Consider the diagram

$$\begin{tikzcd} & (\mathscr{C}^*_{\bar a}, z_{\bar a}) \arrow[rr, "u_{\bar a}"] && (E_{\bar a}, e_{\bar a}) \arrow[rr, "\bar q_{\bar a}"] \arrow[dd, "\psi_{\bar a}"] && \mathsf{M}_{0,n+1} \arrow[dd, "p"] \\ (\mathscr{C}^*_a, z_a) \arrow[ur, dashed, "\hat w_a"] \arrow[rr, "u_a"] && (E_a, e_a) \arrow[ur, dashed, "\tilde w_a"] \arrow[rr, "\bar q_a"] \arrow[dd, "\psi_a"] && \mathsf{M}_{0,n+1} \arrow[ur, "\tilde\tau"] \arrow[dd, "p"] & \\ &&& (Y_{\bar a}, y_{\bar a}) \arrow[rr, "q_{\bar a}"] && (\mathsf{M}_{0,n}, \bar X) \\ && (Y_a, y_a) \arrow[ur, dashed, "w_a"] \arrow[rr, "q_a"'] && (\mathsf{M}_{0,n}, X) \arrow[ur, "\tau"] & \end{tikzcd}$$

By a repeated use of the lifting theorem and using (8-1) we can show the existence of homeomorphisms w_a, $\tilde w_a$ and $\hat w_a$ making the diagram commute. Indeed $(\operatorname{Im}(\tau \circ q_a)_*) = \tau_*(H_a) = \operatorname{Im} q_{\bar a *}$ by the first equation in (8-1). So w_a is the isomorphism between the pointed coverings $\tau \circ q_a$ and $q_{\bar a}$. By the same argument, using the second equation in (8-1), we get the isomorphism $\tilde w_a$. Consider the cube on the right in the diagram. All its faces (except the left one) commute. But then

$$q_{\bar a} \psi_{\bar a} \tilde w_a = p \bar q_{\bar a} \tilde w_a = p \tilde\tau \bar q_a = \tau p \bar q_a = \tau q_a \psi_a = q_{\bar a} w_a \psi_a.$$

So $\psi_{\bar a} \tilde w_a$ and $w_a \psi_a$ lift the same map with respect to $q_{\bar a}$. Since $\psi_{\bar a} \tilde w_a(e_a) = y_{\bar a} = w_a \psi_a(e_a)$ we conclude that $\psi_{\bar a} \tilde w_a = w_a \psi_a$.

Finally consider the horizontal square on the left of the diagram. We want to show that

$$\operatorname{Im}(\tilde{w}_a \circ u_a)_* = \operatorname{Im} u_{\bar{a}*}.$$

We compose with the injective morphism $\bar{q}_{\bar{a}*}$ and compute

$$\bar{q}_{\bar{a}*}(\operatorname{Im}(\tilde{w}_a \circ u_a)_*) = \tilde{\tau}_* q_{a*}(\operatorname{Im} u_{a*}) = \tilde{\tau}_*(\ker f_a).$$

By the third equation in (8-1) this equals $\ker f_{\bar{a}} = \bar{q}_{\bar{a}*}(\operatorname{Im} u_{\bar{a}*})$. Thus $\bar{q}_{\bar{a}*}(\operatorname{Im}(\tilde{w}_a \circ u_a)_*) = \bar{q}_{\bar{a}*}(\operatorname{Im} u_{\bar{a}*})$ and $\operatorname{Im}(\tilde{w}_a \circ u_a)_* = \operatorname{Im} u_{\bar{a}*}$. So the lifting theorem again yields existence of an isomorphism $\hat{w}_a$ making everything commutative. The homeomorphisms w_a, $\tilde{w}_a$ and $\hat{w}_a$ are in fact biholomorphisms as observed in 6.8. It follows that $\pi_{\bar{a}} \hat{w}_a = w_a \pi_a$. By the uniqueness statement in the Grauert–Remmert extension theorem, $\tilde{w}_a$ extends to a biholomorphism between $\mathscr{C}_a$ and $\mathscr{C}_{\bar{a}}$. Thus $\mathscr{C}_a \cong \tilde{w}_a^* \mathscr{C}_{\bar{a}}$.

Property (2) in Definition 8.1 follows again by the lifting theorem:

$$\begin{tikzcd}
\mathsf{Y}_a \arrow[rr, "w_a"] \arrow[drr, "v_{ab}"] \arrow[ddr, "q_a"'] & & \mathsf{Y}_{\bar{a}} \arrow[drr, "\bar{v}_{\bar{a}\bar{b}}"] \arrow[ddr, "q_{\bar{a}}"] & & \\
& & \mathsf{Y}_b \arrow[rr, "w_b"] \arrow[dl, "q_b"] & & \mathsf{Y}_{\bar{b}} \arrow[dl, "q_{\bar{b}}"] \\
& \mathsf{M}_{0,n} \arrow[rr, "\tau"'] & & \mathsf{M}_{0,n} &
\end{tikzcd}$$

We have $\bar{q}_{\bar{b}} \bar{v}_{\bar{a}\bar{b}} w_a = q_{\bar{a}} w_a = \tau q_a = \tau q_b v_{ab} = q_{\bar{b}} w_b v_{ab}$, so $\bar{v}_{ab} w_a$ and $w_a v_{ab}$ lift the same map. Moreover, $\bar{v}_{\bar{a}\bar{b}} w_a(y_a) = \bar{v}_{ab}(y_{\bar{a}}) = y_{\bar{b}} = w_b(y_b) = w_b v_{ab}(y_a)$, so the two maps coincide; $\bar{v}_{ab} w_a = w_a v_{ab}$. This proves (2). □

Theorem 8.6 *Let G be a finite group and $\theta \in \mathscr{D}^n(G)$. Choose a point $x \in \mathsf{M}_{0,n+1}$ and a geometric basis $\mathscr{B}$ of N_x. Let $\pi_a : \mathscr{C}_a \to \mathsf{Y}_a$ be any family in the collection $\mathfrak{K}(x, \mathscr{B}, \theta)$. Then every curve in the family has the topological type given by $[\theta] \in \mathscr{D}^n(G)/\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$. Conversely, every algebraic curve with a G–action of the topological type given by $[\theta]$ is (unmarkedly) G–isomorphic to some fiber. Moreover, there are only finitely many such fibers.*

Proof Consider $\pi_a : \mathscr{C}_a \to \mathsf{Y}_a$ and let $y, y' \in \mathsf{Y}_a$. Let β be a path in Y_a from y to y', and let f^{β} represent the parallel transport along β. By Lemma 3.5, we get a G–equivariant diffeomorphism $\mathscr{C}_{a,y} \to \mathscr{C}_{a,y'}$. Hence the G–actions on $\mathscr{C}_{a,y}$ and $\mathscr{C}_{a,y'}$ have the same topological type. This proves the first statement. Now let C be an algebraic curve such that G acts effectively on C in such a way that $C/G \cong \mathbb{P}^1$. We get the ramified covering $\pi : C \to \mathbb{P}^1$. By acting via $\mathrm{PGL}(2, \mathbb{C})$, one can move any three branch points of π to 0, 1 and ∞. We can thus assume that the set of critical values of $\pi : C \to \mathbb{P}^1$ coincides with $Y \in \mathsf{M}_{0,n}$. Set $C^* := \pi^{-1}(\mathbb{P}^1 - Y)$. Fix a point $y_0 \in \mathbb{P}^1 - Y$ and consider the monodromy $f : \pi_1(\mathbb{P}^1 - Y, y_0) \to G$ associated with $\pi|_{C^*} : C^* \to \mathbb{P}^1 - Y$. Finally fix a basis $\mathscr{B}'$ of $\pi_1(\mathbb{P}^1 - Y, y_0)$ to Y. Let $\chi : \Gamma_n \to \pi_1(\mathbb{P}^1 - Y, y_0)$ denote the associated isomorphism. Denote by $\theta' = f \circ \chi : \Gamma_n \to G$ the datum associated with C. We get a collection $\mathfrak{K}(y, \mathscr{B}', \theta')$. Assume that C has the same topological type of G–action as $[\theta]$, namely that $[\theta]=[\theta']\in\mathscr{D}^n(G)/\mathrm{Aut}^{**}\,\Gamma_n\times\mathrm{Aut}\,G$. By Theorem 8.2 the collections $\mathfrak{K}(x,\mathscr{B},\theta)$ and $\mathfrak{K}(y,\mathscr{B}',\theta')$ are equivalent. Thus there exist $\bar a\in\mathscr{I}(y,\mathscr{B}',\theta')$ and a biholomorphism $w_a\colon \mathsf{Y}_a\to\overline{\mathsf{Y}}_{\bar a}$ as in Definition 8.1. In particular, $\mathscr{C}_a\cong w_a^*\overline{\mathscr{C}}_{\bar a}$. It follows that C, which is the central fiber for $\pi_{\bar a}\colon\mathscr{C}_{\bar a}\to\mathsf{Y}_{\bar a}$, is G–isomorphic to some fiber of $\pi_a\colon\mathscr{C}_a\to\mathsf{Y}_a$. To check that only finitely many fibers can be G–isomorphic to C we argue as follows. For any $\sigma\in S_n$ there is a unique $g_\sigma\in\mathrm{Aut}\,\mathbb{P}^1$ such that $g_\sigma(y_{\sigma_{n-2}})=0$, $g_\sigma(y_{\sigma_{n-1}})=1$ and $g_\sigma(y_{\sigma_n})=\infty$. If $f\colon C\to\mathscr{C}_{a,y}$ is a G–isomorphism for some $y\in\mathsf{Y}_a$, then f descends to an isomorphism $\bar f\in\mathrm{Aut}\,\mathbb{P}^1$ that maps branch points to branch points. So if $X:=q_a(y)$, we have $\bar f(\{y_1,\dots,y_n\})=\{x_1,\dots,x_n\}$. Then there is a permutation σ such that $\bar f(y_{\sigma_i})=x_i$ for any $i=1,\dots,n$. So $\bar f=g_\sigma$ and $X=(g_\sigma(y_1),\dots,g_\sigma(y_n))$. This shows that there is a finite number of possibilities for X, so a finite number of possibilities for y since q_a is finite. □

9 The centerless case

If the group G has trivial center, the whole discussion in Sections 6, 7 and 8 is greatly simplified.

Indeed, let us go back to the setting at the beginning of Section 6 and let us consider again the sequence $(*)$.

Theorem 9.1 *If the sequence $(*)$ on page 1584 splits and $Z(G)=\{1\}$, then there exists a minimum $a_{\min}\in\mathscr{I}(*,f)$ and it is unique.*

Proof With the notation of Lemma 6.6, set $H''':=\{h\in H''\mid\tilde\varepsilon_h\in\mathrm{Inn}\,G\}$. Note that $H'\subset H'''\subset H''$ and that H''' has finite index in H'' and in H. By assumption the map $G\to\mathrm{Inn}\,G$ is bijective. So for every $h\in H'''$, there is a unique element of G, denoted by $\varphi(h)$, such that $\tilde\varepsilon_h=\mathrm{inn}_{\varphi(h)}$. We get a map $\varphi\colon H'''\to G$. Since $\tilde\varepsilon$ is a morphism, we have $\mathrm{inn}_{\varphi(hh')}=\mathrm{inn}_{\varphi(h)\varphi(h')}$ and, since $Z(G)=\{1\}$, this implies that φ is a morphism. Also, by construction, φ satisfies $\mathrm{inn}_{\varphi(h)}\circ f=f\circ\varepsilon_h$. Therefore, by Lemma 6.5, there exists a morphism $\tilde f\colon N\rtimes_\varepsilon H'''\to G$ extending f such that $\tilde f|_{H'''}=\varphi$. Thus $(H''',\tilde f)\in\mathscr{I}(*,f)$. Moreover, since φ is unique, so is $\tilde f$. Now let $a=(H_a,f_a)\in\mathscr{I}(*,f)$ and observe that, by Lemma 6.5, every $h\in H_a$ satisfies (6-3). It follows that $H_a\subset H'''$ and $\varphi_a=\varphi|_{H_a}$ and thus we conclude that $a=(H_a,f_a)\geq(H''',\tilde f)$. Uniqueness of the minimum in obvious in any ordered set. □

Next let N_x, K_x and H_X be as in (7-1) and consider the splitting exact sequence $(*_x)$. As usual, choose a geometric basis $\mathscr{B}=\{[\alpha_i]\}_{i=1}^n$ of N_x, let $\chi\colon\Gamma_n\to N_x$ be the isomorphism induced from the basis $\mathscr{B}$, and, for a datum $\theta\colon N_x\to G$, set $f:=\theta\circ\chi^{-1}\colon N_x\to G$. Theorem 9.1 applied to $(*_x)$ reads as follows:

Theorem 9.2 *If G has trivial center, then there exists a minimum $a_{\min}\in\mathscr{I}(x,\mathscr{B},\theta)$ and it is unique.*

Thus in this case by choosing the minimum we have a canonical choice of a family. Thus, if the center of G is trivial, the choice of a point $x\in\mathsf{M}_{0,n+1}$, a geometric basis $\mathscr{B}=\{[\alpha_i]\}_{i=1}^n$, and a datum $\theta\colon N_x\to G$ yields a well-defined minimum family

$$\pi_{(x,\mathscr{B},\theta)}\colon\mathscr{C}_{(x,\mathscr{B},\theta)}\to\mathsf{Y}_{(x,\mathscr{B},\theta)},$$

and we can forget about the whole collection. Moreover by Theorem 8.2 changing x or $\mathcal{B}$ or θ inside its $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit amounts to passing from a collection to an equivalent one. Since equivalence is order-preserving, it naturally maps the minimum to the minimum. This yields the following.

Theorem 9.3 *If G has trivial center, then up to isomorphism the family $\pi_{(x,\mathcal{B},\theta)}\colon \mathcal{C}_{(x,\mathcal{B},\theta)} \to \mathsf{Y}_{(x,\mathcal{B},\theta)}$ is independent of the choices of x and $\mathcal{B}$ and only depends on the $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit of θ. In particular, the family $\pi_{(x,\mathcal{B},\theta)}\colon \mathcal{C}_{(x,\mathcal{B},\theta)} \to \mathsf{Y}_{(x,\mathcal{B},\theta)}$ only depends on the topological type $[\theta]$.*

10 The abelian case

We conclude looking at the special case where the group G is abelian, the opposite of G being centerless.

Theorem 10.1 *If G is abelian, then there exists $a \in \mathcal{I}(x,\mathcal{B},\theta)$ such that $\mathsf{Y}_a = \mathsf{M}_{0,n}$.*

Proof Let N_x, K_x and H_X be as in (7-1) and consider the splitting exact sequence $(*_x)$, ie the top row of (2-3). Let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathcal{B}$. Set $f := \theta \circ \chi^{-1}\colon N_x \to G$. Now let $\varphi\colon H_X \to G$ be any morphism. Let

$$\varepsilon\colon \pi_1(\mathsf{M}_{0,n}) \to \operatorname{Aut}(\pi_1(\mathbb{P}^1 - X, x_0))$$

denote the morphism giving the semidirect product in $(*_x)$. By the considerations in 2.3, ε is just the restriction to $\pi_1(\mathsf{M}_{0,n})$ of the morphism $\tilde{\varepsilon}$ giving the splitting of the exact sequence in the second row of (2-3). In [Birman 1974, Corollary 1.8.3] it is explicitly described the image via $\tilde{\varepsilon}$ of the generators of the pure braid group of $n-1$ strings of the plane. To be more precise, the notation in [Birman 1974] corresponds to identify

$$\mathsf{M}_{0,n} \cong \{(x_1,\dots,x_{n-1}) \in \boldsymbol{F}_{0,n-1}\,\mathbb{C} \mid x_1 = 0, x_2 = 1\}$$

instead of (2-1). By this description one sees that, for a generator h of $\pi_1(\mathsf{M}_{0,n})$, ε_h sends a generator γ_j of $\pi_1(\mathbb{P}^1 - X, x_0)$ to a conjugate of it. In the setting of Lemma 6.5 we have $f \circ \varepsilon_h(\gamma_j) = f(\gamma_j)$ since G is abelian. Similarly $\operatorname{inn}_{\varphi(h)}$ is the identity since G is abelian. It follows immediately that there exists $f_a\colon K_x \to G$ extending both f and φ. Thus $(H_x, f_a) \in \mathcal{I}(x,\mathcal{B},\theta)$. □

10.2 The proof of Theorem 10.1 shows that, when G is abelian, for every morphism $\varphi\colon H_X \to G$ we can build $f_\varphi\colon K_x \to G$ extending both f and φ. We point out that this is the opposite of the uniqueness result in Theorem 9.2. Of course, $(H_X, f_\varphi) \in \mathcal{I}(x,\mathcal{B},\theta)$ is a minimal element for $(\mathcal{I}(x,\mathcal{B},\theta), \geq)$ since H_X is as big as possible, ie if $b \in \mathcal{I}(x,\mathcal{B},\theta)$ and $(H_X, f_\varphi) \geq b$, then $H_X = H_b$, so $b = (H_x, f_\varphi)$. But different choices of φ yield elements in $\mathcal{I}(x,\mathcal{B},\theta)$ that are not comparable with respect to the order relation $\geq$.

10.3 An important point to stress is that, in the general case, $H_a \subsetneq H_X$ and $\mathsf{Y}_a \neq \mathsf{M}_{0,n}$ for every $a \in \mathcal{I}(x,\mathcal{B},\theta)$. We now show this via an easy example. As in the proof of Theorem 10.1, we use the description in [Birman 1974] of image via $\tilde{\varepsilon}$ of the generators of the pure braid group of the plane and we show that, in general, there may not exist any morphism $\tilde{f}\colon \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H_X \to G$ extending f. Thus, in this case $H_a \subsetneq H_X$ for any $a \in \mathscr{I}$. Let $\theta\colon \Gamma_4 \to S_3$ be given by $\theta(\gamma_1) = (12)$, $\theta(\gamma_2) = (23)$, $\theta(\gamma_3) = (23)$ and $\theta(\gamma_4) = (12)$. With the notation in [Birman 1974], $\pi_1(\mathsf{M}_{0,4})$ is free on the generators A_{12} and A_{13}. We have $\theta(\varepsilon(A_{12})\gamma_1) = (23)$, $\theta(\varepsilon(A_{12})\gamma_2) = (13)$, $\theta(\varepsilon(A_{12})\gamma_3) = (23)$ and $\theta(\varepsilon(A_{12})\gamma_4) = (12)$. Now note that, on one side, $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$ and thus $\gamma_1\gamma_2\gamma_3\gamma_4 \in \ker\theta$, but on the other side $\theta(\varepsilon(A_{12})(\gamma_1\gamma_2\gamma_3\gamma_4)) = (23)(13) = (123) \neq 1$. With the notation of Lemma 6.6 it follows that $A_{12} \notin H''$, so $H'' \neq H_X$. It follows from Lemma 6.5 that for any $a \in \mathscr{I}$ we have $H_a \subset H''$. Thus in particular, $H_a \subset H'' \neq H_X$. Thus there is no morphism $\tilde{f}\colon \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H_X \to G$ extending f. Geometrically, one can interpret this fact as follows. On $\mathsf{M}_{0,4} \cong \mathbb{C}^{**}$ there is the universal family of elliptic curves $\mathscr{E} \to \mathsf{M}_{0,4}$. We denote by E_λ the fiber of $\mathscr{E} \to \mathsf{M}_{0,4}$ over $\lambda \in \mathbb{C}^{**}$. The family corresponding to θ shows that every elliptic curve has an effective action of S_3, which is built as follows: $S_3 = \mathbb{Z}/3 \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ is the multiplication by -1 on E and $\mathbb{Z}/3$ is a subgroup of the translations $(E, +)$. So to build such an action one has to choose a line inside $E_\lambda[3]$. If an extension $\tilde{f}\colon \pi_1(\mathsf{M}_{0,5}) \to S_3$ exists, then there is a family of lines $l_\lambda \subset E_\lambda[3] \cong H_1(E_\lambda, \mathbb{Z}/3)$ defined over $\mathsf{M}_{0,4}$. Equivalently, fixing a base point $\lambda_0 \in \mathsf{M}_{0,4}$, there is a line $l_{\lambda_0} \subset E_{\lambda_0}$ which is stable under the action of the monodromy of the family $\mathscr{E}$. But the image of this monodromy is Γ_2, the congruence subgroup of level 2, which fixes no line in $H_1(E_{\lambda_0}, \mathbb{Z}/3)$.

10.4 It follows from the previous remarks that, in the general case, Y_a cannot be $\mathsf{M}_{0,n}$ itself, but is necessarily a finite cover of it. As pointed out in the introduction, this corrects an inaccuracy in [González-Díez and Harvey 1992]. There it is claimed that $\mathsf{Y} = \mathsf{M}_{0,n}$ always. As $\mathsf{M}_{0,n}$ is birational to projective space, the authors concluded that the image of the family in M_g is always a unirational variety. By Theorem 10.1 their proof works for abelian covers, hence the moduli image of a family of abelian covers is always unirational. In the general case this argument fails and in fact the result is false. Indeed, Michael D Fried informed us that he recently found examples of families for which the moduli image is not unirational. In his work in progress [Fried $\geq$ 2024], Fried considers the moduli space of Galois covers of the line with fixed datum and fixed Nielsen class. When a component of this moduli space is of general type (ie a multiple of its canonical class gives an embedding), then the component is not unirational. When the datum is for covers with 4 branch points, and the equivalences include reduction by the action of Möbius transformations, there is an explicit formula for the genus of the components — see [Bailey and Fried 2002] — which in this case are one-dimensional and covers of the j–line. When that genus exceeds 1, these spaces have general type. For the group A_n, $n \equiv 1 \bmod 4$, and the branching type of the covers having all four conjugacy classes $(n+1)/2$–cycles, Fried has computed the components and their genuses. For n large, the genus is a nonconstant multiple of n^2. When the equivalence comes from the degree n permutation representation of A_n, the base Y_a of any family in the collection $\{\mathscr{C}_a \to \mathsf{Y}_a\}_{a\in\mathscr{I}}$ associated with the datum, has a natural map to one of these components. Thus its moduli image cannot be unirational.

References

[Bailey and Fried 2002] **P Bailey**, **M D Fried**, *Hurwitz monodromy, spin separation and higher levels of a modular tower*, from "Arithmetic fundamental groups and noncommutative algebra" (M D Fried, Y Ihara, editors), Proc. Sympos. Pure Math. 70, Amer. Math. Soc., Providence, RI (2002) 79–220 MR Zbl

[Birman 1974] **J S Birman**, *Braids, links, and mapping class groups*, Ann. of Math. Stud. 82, Princeton Univ. Press (1974) MR Zbl

[Conti et al. 2022] **D Conti**, **A Ghigi**, **R Pignatelli**, *Some evidence for the Coleman–Oort conjecture*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 116 (2022) art. id. 50 MR Zbl

[Eilenberg and Steenrod 1952] **S Eilenberg**, **N Steenrod**, *Foundations of algebraic topology*, Princeton Univ. Press (1952) MR Zbl

[Fadell 1962] **E Fadell**, *Homotopy groups of configuration spaces and the string problem of Dirac*, Duke Math. J. 29 (1962) 231–242 MR Zbl

[Farb and Margalit 2012] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Math. Ser. 49, Princeton Univ. Press (2012) MR Zbl

[Frediani and Neumann 2003] **P Frediani**, **F Neumann**, *Étale homotopy types of moduli stacks of algebraic curves with symmetries*, K–Theory 30 (2003) 315–340 MR Zbl

[Frediani et al. 2015] **P Frediani**, **A Ghigi**, **M Penegini**, *Shimura varieties in the Torelli locus via Galois coverings*, Int. Math. Res. Not. 2015 (2015) 10595–10623 MR Zbl

[Fried 1977] **M Fried**, *Fields of definition of function fields and Hurwitz families: groups as Galois groups*, Comm. Algebra 5 (1977) 17–82 MR Zbl

[Fried 2010] **M D Fried**, *Alternating groups and moduli space lifting invariants*, Israel J. Math. 179 (2010) 57–125 MR Zbl

[Fried $\geq$ 2024] **M D Fried**, *Modular curve-like towers of spaces attached to alternating groups*, in preparation

[Fried and Jarden 1986] **M D Fried**, **M Jarden**, *Field arithmetic*, Ergebnisse der Math. 11, Springer (1986) MR Zbl

[Fried and Völklein 1991] **M D Fried**, **H Völklein**, *The inverse Galois problem and rational points on moduli spaces*, Math. Ann. 290 (1991) 771–800 MR Zbl

[Fried and Völklein 1992] **M D Fried**, **H Völklein**, *The embedding problem over a Hilbertian PAC–field*, Ann. of Math. 135 (1992) 469–481 MR Zbl

[González-Díez and Harvey 1992] **G González-Díez**, **W J Harvey**, *Moduli of Riemann surfaces with symmetry*, from "Discrete groups and geometry" (W J Harvey, C Maclachlan, editors), Lond. Math. Soc. Lect. Note Ser. 173, Cambridge Univ. Press (1992) 75–93 MR Zbl

[Grothendieck 1971] **A Grothendieck**, *Revêtements étales et groupe fondamental* (SGA 1), Lecture Notes in Math. 224, Springer (1971) MR Zbl

[Hall 1950] **M Hall, Jr**, *A topology for free groups and related groups*, Ann. of Math. 52 (1950) 127–139 MR Zbl

[Hirsch 1976] **M W Hirsch**, *Differential topology*, Graduate Texts in Math. 33, Springer (1976) MR Zbl

[Ivanov 2002] **N V Ivanov**, *Mapping class groups*, from "Handbook of geometric topology" (R J Daverman, R B Sher, editors), North-Holland, Amsterdam (2002) 523–633 MR Zbl

[Kurosh 1960] **A G Kurosh**, *The theory of groups*, *I*, 2nd edition, Chelsea, New York (1960) MR Zbl

[Li 2018] **B Li**, *G–marked moduli spaces*, Commun. Contemp. Math. 20 (2018) art. id. 1750061 MR Zbl

[May 1999] **J P May**, *A concise course in algebraic topology*, Univ. of Chicago Press (1999) MR Zbl

[Moonen 2010] **B Moonen**, *Special subvarieties arising from families of cyclic covers of the projective line*, Doc. Math. 15 (2010) 793–819 MR Zbl

[Moonen and Oort 2013] **B Moonen**, **F Oort**, *The Torelli locus and special subvarieties*, from "Handbook of moduli, II" (G Farkas, I Morrison, editors), Adv. Lect. Math. 25, International, Somerville, MA (2013) 549–594 MR Zbl

[Penegini 2015] **M Penegini**, *Surfaces isogenous to a product of curves, braid groups and mapping class groups*, from "Beauville surfaces and groups" (I Bauer, S Garion, A Vdovina, editors), Springer Proc. Math. Stat. 123, Springer (2015) 129–148 MR Zbl

[Perroni 2022] **F Perroni**, *Smooth covers of moduli stacks of Riemann surfaces with symmetry*, Boll. Unione Mat. Ital. 15 (2022) 333–342 MR Zbl

[Rohde 2009] **J C Rohde**, *Cyclic coverings, Calabi–Yau manifolds and complex multiplication*, Lecture Notes in Math. 1975, Springer (2009) MR Zbl

[Spanier 1966] **E H Spanier**, *Algebraic topology*, McGraw-Hill, New York (1966) MR Zbl

[Steenrod 1951] **N Steenrod**, *The topology of fibre bundles*, Princeton Math. Ser. 14, Princeton Univ. Press (1951) MR Zbl

[Tamborini 2022] **C Tamborini**, *Symmetric spaces uniformizing Shimura varieties in the Torelli locus*, Ann. Mat. Pura Appl. 201 (2022) 2101–2119 MR Zbl

[Voisin 2002] **C Voisin**, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés 10, Soc. Math. France, Paris (2002) MR Zbl

[Völklein 1994] **H Völklein**, *Moduli spaces for covers of the Riemann sphere*, Israel J. Math. 85 (1994) 407–430 MR Zbl

[Völklein 1996] **H Völklein**, *Groups as Galois groups: an introduction*, Cambridge Stud. Adv. Math. 53, Cambridge Univ. Press (1996) MR Zbl

[Zieschang et al. 1980] **H Zieschang**, **E Vogt**, **H-D Coldewey**, *Surfaces and planar discontinuous groups*, Lecture Notes in Math. 835, Springer (1980) MR Zbl

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Braided Thompson groups with and without quasimorphisms

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We study quasimorphisms and bounded cohomology of a variety of braided versions of Thompson groups. Our first main result is that the Brin–Dehornoy braided Thompson group bV has an infinite-dimensional space of quasimorphisms and thus infinite-dimensional second bounded cohomology. This implies that, despite being perfect, bV is not uniformly perfect, in contrast to Thompson's group V. We also prove that relatives of bV like the ribbon braided Thompson group rV and the pure braided Thompson group bF similarly have an infinite-dimensional space of quasimorphisms. Our second main result is that, in stark contrast, the close relative of bV denoted by $\widehat{bV}$, which was introduced concurrently by Brin, has trivial second bounded cohomology. This makes $\widehat{bV}$ the first example of a left-orderable group of type F_∞ that is not locally indicable and has trivial second bounded cohomology. This also makes $\widehat{bV}$ an interesting example of a subgroup of the mapping class group of the plane minus a Cantor set that is nonamenable but has trivial second bounded cohomology, behavior that cannot happen for finite-type mapping class groups.

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1 Introduction

The braided Thompson group bV was introduced independently by Brin [2007] and Dehornoy [2006] as a braided version of the classical Thompson group V. This group and its relatives have proven to be important objects in geometric group theory, in particular thanks to their connections to big mapping class groups. Recall that a surface is said to be *of infinite type* if its fundamental group is not finitely generated, and to such a surface one can associate a mapping class group in the same way as for finite-type surfaces; such mapping class groups are called *big*. As an example of the connection, certain braided Thompson groups are dense in the big mapping class group of a compact surface minus a Cantor set [Skipper and Wu 2021, Corollary 3.20], and hence serve as finitely generated "approximations" of these big mapping class groups. For more on connections between braided Thompson groups and big mapping class groups, see eg [Aramayona et al. 2021; Aramayona and Funar 2021; Funar and Kapoudjian 2004; 2008; 2011; Genevois et al. 2022].

Here we are concerned with the question of which braided Thompson groups have an infinite-dimensional space of quasimorphisms, or second bounded cohomology, and which do not. A function $q\colon \Gamma \to \mathbb{R}$ is called a *quasimorphism* if the quantity $|q(g)+q(h)-q(gh)|$ is uniformly bounded; its supremum is called the *defect* of q and is denoted by $D(q)$. We denote by $Q(\Gamma)$ the space of quasimorphisms of Γ, modulo bounded functions (sometimes this notation is used to denote the space of homogeneous quasimorphisms, which is canonically isomorphic [Calegari 2009b, 2.2.2]). We may sometimes colloquially refer to a group as having "no quasimorphisms" if it only has bounded ones. The objects $Q(\Gamma)$ are of great interest in dynamics, geometric group theory, geometric topology and symplectic geometry. For example, they are intimately connected with bounded cohomology [Frigerio 2017] and stable commutator length [Calegari 2009b]. In this context, Thompson-like groups have played an important role: for instance, they have repeatedly served as the first finitely presented examples achieving certain values of stable commutator length [Ghys and Sergiescu 1987; Zhuang 2008; Fournier-Facio and Lodha 2023].

In addition to bV, we inspect the ribbon braided Thompson group rV, the pure braided Thompson group bF, the kernel bP of the projection $bV \to V$, and most importantly the group $\widehat{bV}$, which was introduced by Brin [2007] along with bV. One can view $\widehat{bV}$ as a braided analogue of a Cantor set point stabilizer in V. See Section 2 for the definitions of all these braided Thompson groups. The group $\widehat{bV}$, despite its strong similarities to bV, has extremely different behavior when it comes to quasimorphisms and bounded cohomology, as our two main results make clear:

Theorem 1.1 *For Γ any of the braided Thompson groups bV, rV, bF or bP, the space $Q(\Gamma)$ is infinite-dimensional, and thus also the second bounded cohomology $\mathrm{H}^2_b(\Gamma)$ is infinite-dimensional.*

Theorem 1.2 *We have $\mathrm{H}^2_b(\widehat{bV}) = 0$.*

Here $\mathrm{H}^2_b(\Gamma)$ denotes the second bounded cohomology of a group Γ, with trivial real coefficients. This invariant was introduced by Johnson [1972] and Trauber in the context of Banach algebras, and has since become a fundamental tool in geometric topology [Gromov 1982], dynamics [Ghys 1987] and rigidity theory [Burger and Monod 2002]. For every group Γ there is a map $Q(\Gamma) \to \mathrm{H}^2_b(\Gamma)$, whose kernel is the space of real-valued homomorphisms (Proposition 3.1). Using this, Theorem 1.2, together with the fact that the abelianization of $\widehat{bV}$ is isomorphic to $\mathbb{Z}$ (Corollary 2.14), implies:

Corollary 1.3 *$Q(\widehat{bV})$ is one-dimensional, spanned by the abelianization of $\widehat{bV}$.*

One consequence of Theorem 1.1 is that, despite being perfect [Zaremsky 2018a], bV is not uniformly perfect (Corollary 4.4). Recall that a group Γ is *uniformly perfect* if there exists $N \in \mathbb{N}$ such that every element in Γ can be written as a product of at most N commutators. This is in contrast to the fact that Thompson's group V is uniformly perfect, and even uniformly simple [Gal and Gismatullin 2017] — in fact, $\mathrm{H}^n_b(V) = 0$ for all $n \geq 1$ [Andritsch 2022]. Since bV is not uniformly perfect, the following natural question emerges:

Question 1.4 Which elements of bV have nonzero stable commutator length?

A characterization of this phenomenon in (finite-type) mapping class groups was given in [Bestvina et al. 2016]; see [Field et al. 2022] for some related results for big mapping class groups.

Theorem 1.2 has interesting consequences for subgroups of big mapping class groups. Pioneering work of Bestvina and Fujiwara [2002] showed that every subgroup of a (finite-type) mapping class group is either virtually abelian or has infinite-dimensional $Q(\Gamma)$; see also [Bestvina et al. 2016]. This can be viewed as a sort of Tits-like alternative, since every quasimorphism on an amenable group is at a bounded distance from a homomorphism [Brooks 1981], whereas groups with hyperbolic features typically have an infinite-dimensional space of quasimorphisms [Brooks 1981; Epstein and Fujiwara 1997; Hull and Osin 2013]. The question of whether something similar happens for the big mapping class group $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ for K a Cantor set was listed in the AIM problem list on big mapping class groups [AIM 2019, Question 4.7]. Namely, it is asked whether every subgroup $\Gamma \leq \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ is either amenable or has infinite-dimensional $Q(\Gamma)$. Theorem 1.2 provides a negative answer to this, since $\widehat{bV}$ is nonamenable (by virtue of containing braid groups), and embeds in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$; see Section 4.

In fact, we should point out that a negative answer to this question was already "almost" available in the literature. Indeed, by a result of Calegari and Chen [2021], every countable circularly orderable group Γ embeds in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$, and there are plenty of countable circularly orderable groups that are nonamenable and have a finite-dimensional space of quasimorphisms, or no quasimorphisms at all [Calegari 2007; Zhuang 2008; Fournier-Facio and Lodha 2023]. The most straightforward example is probably Thompson's group T, which has no quasimorphisms by virtue of being uniformly perfect (and even uniformly simple; see eg [Guelman and Liousse 2023]). In fact, when the groups are even left-orderable, many of them have vanishing second bounded cohomology [Fournier-Facio and Lodha 2023], and sometimes even vanishing bounded cohomology in every positive degree [Monod 2022]. As a remark, since the examples coming from the procedure in [Calegari and Chen 2021] act on the plane by fixing a radial coordinate and acting by rotations, which is really a "one-dimensional" picture, one can view $\widehat{bV}$ as providing the first truly "two-dimensional" example, ie one involving genuine braids.

In order to prove Theorem 1.1, we generally follow the approach used by Bavard [2016] to show that $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ has an infinite-dimensional space of quasimorphisms. Her proof in turn makes use of the approach of Bestvina and Fujiwara [2002] to finite-type mapping class groups, following suggestions of Calegari [2009a] from a blog post. Bavard's result prompted the study of analogues of curve graphs for big mapping class groups, and arguably initiated the recent surge of interest in big mapping class groups; see [Aramayona and Vlamis 2020] for more on the history of big mapping class groups.

In the course of proving Theorem 1.2, we also prove that $\widehat{bV}$ is of type F_∞, meaning it has a classifying space with finitely many cells in each dimension (Corollary 2.15); this is a stronger property than finite generation and finite presentability. It is known that bV and thus $\widehat{bV}$ are left-orderable [Ishida 2018], and that $\widehat{bV}$ contains a copy of bV (see Definition 2.9), which is finitely generated and perfect [Zaremsky 2018a]. Therefore $\widehat{bV}$ serves as the first example of a group with the following properties:

Corollary 1.5 *The group $\widehat{bV}$ is a left-orderable group of type F_∞ that is not locally indicable and has vanishing second bounded cohomology.* □

A finitely generated group is *indicable* if it admits a homomorphism onto $\mathbb{Z}$. A group is *locally indicable* if each of its finitely generated subgroups is indicable. The combination of these properties is interesting because it shows that in the celebrated Witte Morris theorem [2006] the hypothesis of amenability cannot be weakened to the vanishing of second bounded cohomology. The first finitely generated examples were found in [Fournier-Facio and Lodha 2023]; those examples have the additional property of being nonindicable, answering a question of Navas [2018]. Since $\widehat{bV}$ is indicable, the existence of type-F_∞ examples with these stronger properties is still open.

We will always stick to the "$n = 2$ case" to avoid getting bogged down in notation, but the reader should note that all of our results can be adapted to the braided Higman–Thompson groups bV_n (as in [Aroca and Cumplido 2022; Skipper and Wu 2023]) and their analogous subgroups $\widehat{bV_n}$, with appropriate small modifications to the arguments. It would be interesting to try and adapt our arguments to other more complicated Thompson-like groups related to asymptotically rigid mapping class groups, eg for positive genus surfaces [Aramayona and Funar 2021] or for higher-dimensional manifolds [Aramayona et al. 2021].

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2 Braided Thompson groups

The first braided Thompson group, which we denote by bV and which has also been denoted by BV, V_{br} and $\mathrm{br}\, V$ in the literature, was introduced independently by Brin [2007] and Dehornoy [2006], as a braided version of Thompson's group V. Other braided Thompson groups include the "F–like" pure braided Thompson groups bF [Brady et al. 2008], various "T–like" braided Thompson groups [Funar and Kapoudjian 2008; 2011; Witzel 2019], braided Higman–Thompson groups bV_n [Aroca and Cumplido 2022; Skipper and Wu 2023], braided Brin–Thompson groups sV_{br} [Spahn 2021], the "ribbon braided" Thompson group rV [Thumann 2017] and braided Röver–Nekrashevych groups $\mathrm{br}\, V_d(G)$ [Skipper and Zaremsky 2023]. Most relevant to our purposes here is a close relative $\widehat{bV}$ of bV, which was also introduced by Brin [2007] (there denoted by $\widehat{BV}$), and realized up to isomorphism as a concrete subgroup of bV by Brady, Burillo, Cleary and Stein [Brady et al. 2008]; see also [Burillo and Cleary 2009].

Let us recall the definitions of bV and $\widehat{bV}$ using the standard braided tree pair model, as in [Brady et al. 2008; Zaremsky 2018a]. By a *tree* we will always mean a finite rooted planar binary tree. An element of bV is represented by a *representative triple* (T_-, β, T_+), where T_- is a tree, T_+ is a tree with the same number of leaves as T_-, say n, and β is a braid in B_n. Elements of bV are equivalence classes

$[T_-, \beta, T_+]$ of representative triples, where the equivalence relation is given by the notion of expansion, which we now describe.

First, denote by $\rho_n : B_n \to S_n$ the usual map from the braid group to the symmetric group, recording how the numbering of the strands at the bottom changes when the strands move to the top. (We may write ρ for ρ_n when we do not need to care about n.) Let (T_-, β, T_+) be a representative triple, say with $T_\pm$ having n leaves and $\beta \in B_n$, and let $1 \le k \le n$. Let T'_+ be the tree obtained from T_+ by adding a caret to the k^{th} leaf, let T'_- be the tree obtained from T_- by adding a caret to the $\rho_n(\beta)(k)^{\text{th}}$ leaf, and let $\beta' \in B_{n+1}$ be the braid obtained from β by bifurcating the k^{th} strand (counting at the bottom) into two parallel strands.

Definition 2.1 (expansion, equivalence) With the above setup, call (T'_-, β', T'_+) the k^{th} *expansion* of (T_-, β, T_+). Declare that two representative triples are equivalent if one is an expansion of the other, and extend this to generate an equivalence relation on the set of representative triples.

The elements of the group bV are the equivalence classes $[T_-, \beta, T_+]$, and the group operation is described as follows. Given two elements $[T_-, \beta, T_+]$ and $[U_-, \gamma, U_+]$, up to expansions we can assume that $T_+ = U_-$. Now we define

$$[T_-, \beta, T_+][T_+, \gamma, U_+] := [T_-, \beta\gamma, U_+].$$

Some immediate subgroups of bV include Thompson's group F, which is the subgroup of elements of the form $[T_-, 1, T_+]$, and the pure braided Thompson group bF, which is the subgroup of elements of the form $[T_-, \beta, T_+]$ for β a pure braid. We will also be especially interested in the following subgroup:

Definition 2.2 (the group $\widehat{bV}$) For each $n \in \mathbb{N}$, let $\widehat{B}_n$ denote the standard copy of B_{n-1} inside B_n which only braids the first $n-1$ strands . Note that if (T'_-, β', T'_+) is an expansion of (T_-, β, T_+), say with $\beta \in B_n$ and $\beta' \in B_{n+1}$, then $\beta \in \widehat{B}_n$ if and only if $\beta' \in \widehat{B}_{n+1}$. Thus the equivalence classes $[T_-, \beta, T_+]$ for $\beta \in \widehat{B}_n$ form a well-defined subgroup of bV, denoted by $\widehat{bV}$.

There is a convenient way to picture elements of bV as (equivalence classes of) so-called *strand diagrams*. For an element $[T_-, \beta, T_+]$, we picture T_+ upside-down and below T_-, with β connecting the leaves of T_+ up to the leaves of T_-. See Figure 1 for an example of an element of bV, and an expansion.

To accurately model the equivalence relation coming from expansion, and the group operation, which amounts to stacking strand diagrams, some equivalences between strand diagrams naturally emerge. The three key equivalences are shown in Figure 2.

See [Brady et al. 2008; Zaremsky 2018a] for more details.

2.1 Using pure braids and ribbon braids

Some more subgroups of bV arise when we restrict to pure braids. As before, consider the standard projection $B_n \to S_n$ from the braid group B_n to the symmetric group S_n. The kernel of this map is the *pure braid group* PB_n. This leads us to the following definition of the pure braided Thompson group bF:

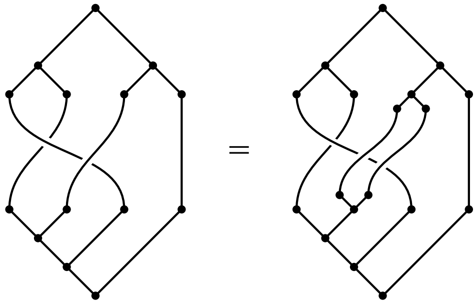


Figure 1: An element $[T_-, \beta, T_+]$ of bV. We draw T_+ upside down, with β as a braid from the leaves of T_+ up to the leaves of T_-. We have $\rho_4(\beta) = (1\ 2\ 3)$, so $\rho_4(\beta)(2) = 3$. Thus to perform the 2nd expansion, we add a caret to the 2nd leaf of T_+, a caret to the 3rd leaf of T_-, and bifurcate the 2nd strand of β (counting from the bottom) into two strands. Note that this element lies in the subgroup $\widehat{bV}$ since the rightmost strand does not braid with any of the others.

Definition 2.3 (the group bF) If (T_-', β', T_+') is an expansion of (T_-, β, T_+) then β' is pure if and only if β' is pure, so the equivalence classes $[T_-, \beta, T_+]$ for $\beta \in PB_n$ form a well-defined subgroup of bV, denoted by bF.

Note that bF is not normal in bV, but the following related subgroup is:

Definition 2.4 (the group bP) Let bP denote the subgroup of bV consisting of all $[T, \beta, T]$ such that β is pure.

For reference, the group bP was denoted by PBV in [Brady et al. 2008] and by P_{br} in [Zaremsky 2018a]. Note that the trees in $[T, \beta, T]$ must be the same, so bP is strictly smaller than bF. The quotient bV/bP is isomorphic to Thompson's group V, and the quotient bF/bP is isomorphic to Thompson's group F [Brady et al. 2008]. More precisely, upon passing to a quotient with kernel bP, the elements of bV change from being represented by triples (T_-, β, T_+) for $\beta \in B_n$ to being represented by triples (T_-, σ, T_+) for $\sigma \in S_n$. The notion of expansion has an obvious analogue for permutations, and we get equivalence classes $[T_-, \sigma, T_+]$, which are the elements of V. The image of $\widehat{bV}$ under the projection $bV \to V$ is a group called $\widehat{V}$, which was also considered in [Brin 2007]. The kernel $\widehat{bV} \cap bP$ of the projection $\widehat{bV} \to \widehat{V}$ has the following interesting property, which will be useful later:

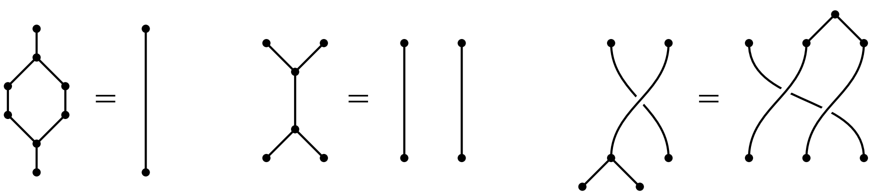


Figure 2: The three key equivalences for strand diagrams, which can occur anywhere inside a strand diagram representing an element of bV. Further equivalences are obtained by combining these, and by rotating and reflecting the one on the right.

Lemma 2.5 *There is an epimorphism* $\widehat{bV} \cap bP \to bP$*.*

Proof An element of $\widehat{bV} \cap bP$ is of the form $[T, \beta, T]$ for β a pure braid in which the rightmost strand does not braid with any of the others. Let T' be the subtree of T whose root is the left child of the root of T (or if T is trivial, just take T' to also be trivial). Let β' be the (pure) braid obtained from β by deleting any strands corresponding to leaves of T that are closer to the right child of the root than the left (so in particular, in the nontrivial case this includes the rightmost strand). Intuitively, $[T', \beta', T']$ is obtained by taking just the "left part" of $[T, \beta, T]$, from the point of view of the root of T. This operation is well defined up to expansions, and yields a well-defined homomorphism $[T, \beta, T] \to [T', \beta', T']$ from $\widehat{bV} \cap bP$ to bP, which is clearly surjective. □

Finally, let us discuss a "twisted" version of bV, called the *ribbon braided Thompson group* rV. This arises by treating the strands in a strand diagram as ribbons, which are allowed to twist. This first appeared officially in work of Thumann [2017, Section 3.5.3], where he proved that rV (there denoted by RV) is of type F_∞. The idea of using ribbons to represent strands in bV was actually already present in Brin's original paper [2007], but without twisting. We will mostly follow the approach from [Zaremsky 2018b, Example 4.2], which uses the notion of cloning systems from [Witzel and Zaremsky 2018] to provide a framework for elements of rV similar to the one we are using here for bV. An element of rV is represented by a triple $(T_-, \beta(m_1, \dots, m_n), T_+)$ where T_- and T_+ are trees with n leaves and $\beta(m_1, \dots, m_n) \in B_n \wr \mathbb{Z}$. More precisely, $\beta \in B_n$, $m_1, \dots, m_n \in \mathbb{Z}$ and $B_n \wr \mathbb{Z}$ denotes the wreath product $B_n \ltimes \mathbb{Z}^n$ with the action induced by the standard projection $B_n \to S_n$. (We write our wreath products with the acting group on the left, for convenience. Also, we may sometimes write $\beta(0, \dots, 0)$ as β and $1_{B_n}(m_1, \dots, m_n)$ as $(m_1, \dots, m_n)$ for the sake of notational elegance.) An *expansion* of this triple is another triple of the form

$$(T'_-, \beta' s_k^{m_k}(m_1, \dots, m_{k-1}, m_k, m_k, m_{k+1}, \dots, m_n), T'_+),$$

where T'_+ is T_+ with a caret added to the k^{th} leaf for some $1 \le k \le n$, β' is β with its k^{th} ribbon bifurcated into two parallel ribbons, and T'_- is T_- with a caret added to the $\rho(\beta)(k)^{\text{th}}$ leaf. Here s_k is the k^{th} standard generator of B_n, in the standard presentation

$$B_n = \langle s_1, \dots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i, \text{ and } s_i s_j = s_j s_i \text{ for all } i \text{ and } j \text{ with } |i-j| > 1 \rangle.$$

Let us adopt the convention that s_k crosses the k^{th} ribbon (counting at the bottom) under the $(k+1)^{\text{st}}$ ribbon, and a positive single twist of a ribbon involves the left side of the ribbon (looking at the bottom) twisting under the right side. These conventions make the definition of expansion look somewhat natural; see Figure 3.

By taking the equivalence relation generated by expansion, we get equivalence classes of the form $[T_-, \beta(m_1, \dots, m_n), T_+]$, which comprise the group rV. Just like in bV, the group operation is given, roughly, by first expanding until the right tree of the left element equals the left tree of the right element

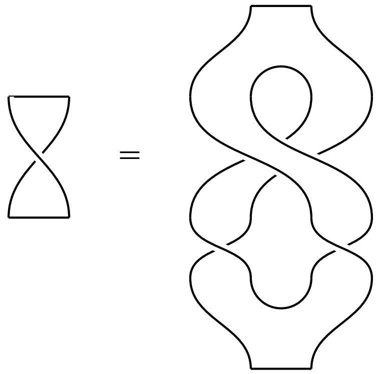


Figure 3: Expansion in rV. Here we see that $[\cdot, 1_{B_1}(1), \cdot] = [\wedge, s_1(1,1), \wedge]$, where $\cdot$ is the trivial tree and $\wedge$ is a single caret.

and then canceling these trees. We could consider various subgroups of rV by restricting to pure braids and/or full twists, but for our purposes we will just stick with all braids and all twists.

At this point we have

$$bP < bF < bV < rV,$$

where we view bV as the subgroup of rV consisting of elements $[T_-, \beta(0,\dots,0), T_+]$, that is, elements with no twisting. As we have said, bP is normal in bF and bV, and in fact it is even normal in rV, as we now show:

Lemma 2.6 *The subgroup bP is normal in rV.*

Proof Let $[U, \gamma, U] \in bP$ and $[T_-, \beta(m_1,\dots,m_n), T_+] \in rV$, expanding so that without loss of generality $U = T_+$. Then

$$\begin{aligned}
&[T_-, \beta(m_1,\dots,m_n), T_+][T_+, \gamma, T_+][T_-, \beta(m_1,\dots,m_n), T_+]^{-1} \\
&\qquad = [T_-, \beta(m_1,\dots,m_n), T_+][T_+, \gamma, T_+][T_+, (-m_1,\dots,-m_n)\beta^{-1}, T_-] \\
&\qquad = [T_-, \beta(m_1,\dots,m_n)\gamma(-m_1,\dots,-m_n)\beta^{-1}, T_-] = [T_-, \beta\gamma\beta^{-1}, T_-] \in bP.
\end{aligned}$$

The last equals sign holds because γ is pure, and hence $(m_1,\dots,m_n)\gamma = \gamma(m_1,\dots,m_n)$. □

As we have said, the quotients bV/bP and bF/bP are isomorphic to V and F, respectively. The quotient rV/bP is isomorphic to a Thompson-like group constructed analogously to rV but using $S_n \wr \mathbb{Z}$ instead of $B_n \wr \mathbb{Z}$; this could be made more precise by putting a cloning system, in the sense of [Witzel and Zaremsky 2018], on the family of groups $S_n \wr \mathbb{Z}$, but we will not need to worry about this here. Indeed, all we will need to use rV/bP for later is to relate quasimorphisms of rV to quasimorphisms of bV, bF and bP, and for this all we need to know about it is the following:

Lemma 2.7 *The quotient rV/bP is uniformly perfect.*

Proof Note that $bV/bP \cong V$ is uniformly perfect [Gal and Gismatullin 2017]. Choose $N \in \mathbb{N}$ such that every element of V is a product of at most N commutators. Set $M = 3N + 2$. We claim that every element of rV/bP is a product of at most M commutators. Let $[T_-, \beta(m_1, \dots, m_n), T_+] \in rV$, and write it as a product of three elements:

$$[T_-, \beta, T_+][T_+, (m_1, 0, \dots, 0), T_+][T_+, (0, m_2, \dots, m_n), T_+].$$

Modulo bP, we know that this first factor is a product of at most N commutators. The second factor is conjugate to $[T_+, (0, m_1, 0, \dots, 0), T_+]$ via the conjugator $[T_+, s_1, T_+]$, and this is of the same form as the third factor. Thus it suffices to focus on the third factor, and show that any element of the form $g = [T, (0, m_2, \dots, m_n), T]$ is, modulo bP, a product of at most $N + 1$ commutators.

Let T' be T with $n-1$ new carets added, one after the other, always attaching each new caret to the leftmost leaf. Thus $g = [T', (0, \dots, 0, m_2, \dots, m_n), T']$, where the number of 0s is n. Let T'' be T with $n-1$ new carets, one on each leaf other than the leftmost. Thus $g = [T'', \gamma(0, m_2, m_2, m_3, m_3, \dots, m_n, m_n), T'']$ for $\gamma \in B_{2n-1}$ the braid that arises from performing this expansion, namely $\gamma = s_2^{m_2} s_4^{m_3} \cdots s_{2n-2}^{m_n}$. Setting $h = [T'', \gamma, T''] \in bV$ we get $h^{-1}g = [T'', (0, m_2, m_2, m_3, m_3, \dots, m_n, m_n), T'']$. Now let $\alpha \in B_{2n-1}$ be any braid satisfying $\alpha(0, \dots, 0, m_2, \dots, m_n)\alpha^{-1} = (0, m_2, 0, m_3, \dots, 0, m_n, 0)$ in $B_{2n-1} \wr \mathbb{Z}$, and set $a = [T'', \alpha, T']$. We get

$$\begin{aligned} aga^{-1} &= [T'', \alpha, T'][T', (0, \dots, 0, m_2, \dots, m_n), T'][T', \alpha^{-1}, T''] \\ &= [T'', \alpha(0, \dots, 0, m_2, \dots, m_n)\alpha^{-1}, T''] = [T'', (0, m_2, 0, m_3, \dots, 0, m_n, 0), T'']. \end{aligned}$$

Hence $h^{-1}gag^{-1}a^{-1} = [T'', (0, 0, m_2, 0, m_3, \dots, m_{n-1}, 0, m_n), T'']$. Now using a similar trick as when we conjugated by a, this is conjugate to $[T', (0, \dots, 0, m_2, \dots, m_n), T']$, which equals g. Thus g is conjugate to $h^{-1}gag^{-1}a^{-1}$, and considered modulo bP this is an element of V times a commutator, so we are done. □

It is worth recording the following consequence:

Corollary 2.8 *The group rV is perfect.*

Proof We already know bV is perfect [Zaremsky 2018a], so the derived subgroup rV' contains bV. In particular it contains bP, and so rV/rV' is a quotient of rV/bP. This is perfect by Lemma 2.7, so we conclude that $rV = rV'$. □

2.2 Algebraic properties of $\widehat{bV}$

Our proof of Theorem 1.2 will rely on some algebraic properties of bV and $\widehat{bV}$, which are the focus of this subsection.

Definition 2.9 (right depth, $\widehat{bV}(1)$) Say that the *right depth* of a tree is the distance from its rightmost leaf to its root. Denote by $\widehat{bV}(1) \le \widehat{bV}$ the subgroup of elements that admit a representative of the form (T_-, β, T_+) such that T_- and T_+ both have right depth 1.

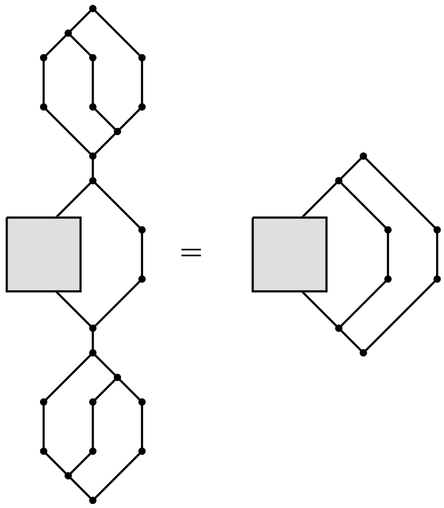


Figure 4: The proof of Lemma 2.11: conjugating an element of $\widehat{bV}(1)$ by x_0 yields another element of $\widehat{bV}(1)$.

Note that $\widehat{bV}(1)$ is naturally isomorphic to bV. Indeed, we have an isomorphism $bV \to \widehat{bV}(1)$ given by $[T_-, \beta, T_+] \to [U_-, \gamma, U_+]$, where U_- is obtained from T_- by adding a new caret whose left leaf is the root of T_-, U_+ is obtained from T_+ by adding a new caret whose left leaf is the root of T_+, and γ is obtained from β by adding one new unbraided strand on the right. This is also discussed in [Brady et al. 2008].

Definition 2.10 (homomorphism χ_1, subgroup $\widehat{D}$) Let $\chi_1 : \widehat{bV} \to \mathbb{Z}$ be the homomorphism sending $[T_-, \beta, T_+]$ to the right depth of T_- minus the right depth of T_+. Since expansions preserve this measurement, thanks to the rightmost strand of such a β not braiding, this is well defined, and is clearly a homomorphism. Denote by $\widehat{D}$ the kernel in $\widehat{bV}$ of χ_1.

We call this map χ_1 since its restriction to Thompson's group $F \leq \widehat{bV}$ coincides with a map usually denoted by χ_1. Note that $\widehat{D}$ consists of all $[T_-, \beta, T_+] \in \widehat{bV}$ such that T_- and T_+ have the same right depth. In particular $\widehat{D}$ contains $\widehat{bV}(1)$. We will see in Corollary 2.14 that $\widehat{D}$ equals the derived subgroup of $\widehat{bV}$.

Recall the usual first generator x_0 of Thompson's group F. This is the element $x_0 = [T_2, 1, T_1]$, where T_i is the tree consisting of a caret with a caret attached to its i^{th} leaf, and 1 is the identity in B_3. Note that $\chi_1(x_0) = 1$. Also note that $x_0^{-1} = [T_1, 1, T_2]$.

Lemma 2.11 *We have* $x_0^{-1} \cdot \widehat{bV}(1) \cdot x_0 \leq \widehat{bV}(1)$.

Proof This is clear using strand diagrams; see Figure 4. In the figure, we represent an element of $\widehat{bV}(1)$ by drawing the first carets of each tree and the last (unbraided) strand of the braid, and then drawing a gray box to represent the arbitrary remainder of the picture. Now conjugating by x_0 and applying some of the equivalence moves from Figure 2, we see that in the resulting strand diagram the trees again have right depth 1 and the rightmost strand is unbraided (in fact the two rightmost strands are both unbraided). □

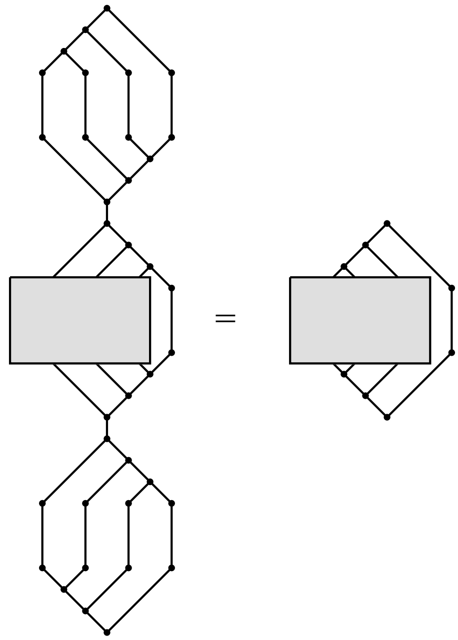


Figure 5: An example of the proof of Lemma 2.12 (for A having just one element): conjugating an element of $\widehat{D}$ in which the trees have right depth 3 by x_0^2 yields an element of $\widehat{bV}(1)$.

Lemma 2.12 *For any finite subset A of $\widehat{D}$, there exists $k \geq 0$ such that $x_0^{-k} \cdot A \cdot x_0^k \leq \widehat{bV}(1)$.*

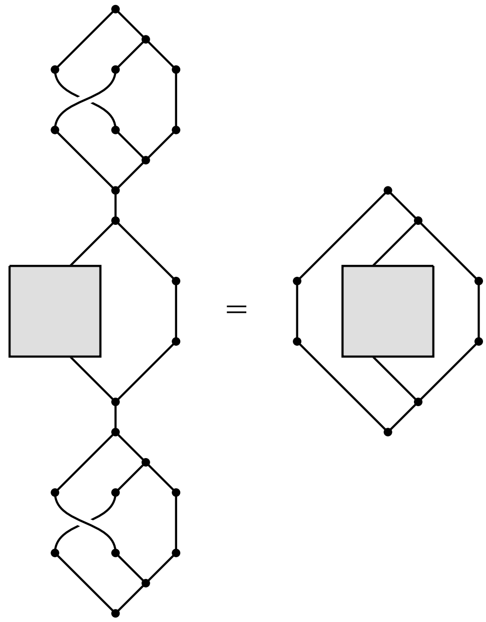
Proof Since A is finite, we can choose $k \geq 0$ such that every element of A can be represented by a triple (T_-, β, T_+) in which the right depth of T_- (and thus T_+) is at most $k+1$. For any such (T_-, β, T_+), it is clear that $x_0^{-k} \cdot [T_-, \beta, T_+] \cdot x_0^k \in \widehat{bV}(1)$. See Figure 5 for an example. □

Corollary 2.13 *The group $\widehat{bV}$ is isomorphic to an ascending HNN-extension of bV.*

Proof To get our result, we will verify the conditions in [Geoghegan et al. 2001, Lemma 3.1] using $\widehat{bV}(1)$ (which is isomorphic to bV) as the base and x_0 as the stable letter. Clearly no nontrivial power of x_0 lies in $\widehat{bV}(1)$. Lemma 2.11 shows that $x_0^{-1} \cdot \widehat{bV}(1) \cdot x_0 \leq \widehat{bV}(1)$. Finally, we need to show that $\widehat{bV}$ is generated by $\widehat{bV}(1)$ and x_0. Given $[T_-, \beta, T_+] \in \widehat{bV}$, up to right multiplication by a power of x_0 we can assume that $[T_-, \beta, T_+] \in \widehat{D}$, ie T_- and T_+ have the same right depth. Now Lemma 2.12 says we can conjugate by some power of x_0 so that our element lands in $\widehat{bV}(1)$. □

Corollary 2.14 *The derived subgroup $\widehat{bV}'$ equals $\widehat{D}$, so the abelianization of $\widehat{bV}$ is $\mathbb{Z}$, given by the map χ_1.*

Proof Since $\widehat{D}$ is the kernel of a map to $\mathbb{Z}$, it contains $\widehat{bV}'$. Conversely, since $\widehat{bV}(1)$ is isomorphic to bV, and bV is perfect [Zaremsky 2018a], Lemma 2.12 implies that any element of $\widehat{D}$ is conjugate in $\widehat{bV}$ to an element of a perfect subgroup of $\widehat{bV}$, which shows that every element of $\widehat{D}$ lies in $\widehat{bV}'$. This shows $\widehat{bV}' = \widehat{D}$, and the second statement follows since $\widehat{D}$ is the kernel of χ_1 in $\widehat{bV}$. □



$$=$$

Figure 6: An arbitrary conjugate of an element of $\widehat{bV}(1)$ by g. We see that it will commute with any element of $\widehat{bV}(1)$.

Brin [2006] showed that bV and $\widehat{bV}$ are finitely presented. In fact, bV is even of type F_∞ [Bux et al. 2016]. The techniques in [loc. cit.] could likely be used to show that $\widehat{bV}$ is also of type F_∞, but now, thanks to Corollary 2.13, we can prove this much more quickly:

Corollary 2.15 *The group $\widehat{bV}$ is of type F_∞.*

Proof It is a standard fact that an ascending HNN-extension of a group of type F_n is itself of type F_n; see eg [Baumslag et al. 1980, end of Section 2]. Since bV is of type F_∞ [Bux et al. 2016], Corollary 2.13 implies that $\widehat{bV}$ is as well. □

The key dynamical feature that will make bounded cohomology vanish is contained in the following lemma:

Lemma 2.16 *There exists $g \in \widehat{D}$ such that every element of $\widehat{bV}(1)$ commutes with every element of $g^{-1} \cdot \widehat{bV}(1) \cdot g$.*

Proof We define $g = [T_2, s_1, T_2]$, where as before T_2 is a caret with a second caret hanging on the right, and s_1 is the first standard generator of B_3, ie the element braiding the first two strands with a single twist. Since β does not braid the rightmost strand we have $g \in \widehat{bV}$, and since clearly $\chi_1([T_2, \beta, T_2]) = 0$ we have $g \in \widehat{D}$. We see in Figure 6 that, in any element of $g^{-1} \cdot \widehat{bV}(1) \cdot g$, the trees both have "left depth" 1 and the first strand does not braid with anything. Since elements of $\widehat{bV}(1)$ and $g^{-1} \cdot \widehat{bV}(1) \cdot g$ therefore braid disjoint sets of strands, they commute. □

3 Second bounded cohomology

We will work only with bounded cohomology with trivial real coefficients, and use the definition in terms of the bar resolution. We refer the reader to [Brown 1982; Frigerio 2017] for a general and complete treatment of ordinary and bounded cohomology of discrete groups, respectively. For the more general setting of locally compact groups, we refer the reader to [Monod 2001].

For every $n \geq 0$, denote by $\mathrm{C}^n(\Gamma)$ the set of real-valued functions on Γ^n. By convention, Γ^0 is a single point, so $\mathrm{C}^0(\Gamma) \cong \mathbb{R}$ consists only of constant functions. We define differential operators $\delta^\bullet\colon \mathrm{C}^\bullet(\Gamma) \to \mathrm{C}^{\bullet+1}(\Gamma)$ by $\delta^0 = 0$ and, for $n \geq 1$,

$$\delta^n(f)(g_1, \dots, g_{n+1}) = f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n).$$

One can check that $\delta^{\bullet+1}\delta^\bullet = 0$, so $(\mathrm{C}^\bullet(\Gamma), \delta^\bullet)$ is a cochain complex. We denote by $\mathrm{Z}^\bullet(\Gamma) := \ker(\delta^\bullet)$ the set of *cocycles*, and by $\mathrm{B}^\bullet(\Gamma) := \mathrm{im}(\delta^{\bullet-1})$ the set of *coboundaries*. The quotient $\mathrm{H}^\bullet(\Gamma) := \mathrm{Z}^\bullet(\Gamma)/\mathrm{B}^\bullet(\Gamma)$ is the *cohomology of* Γ *with trivial real coefficients*. We will also call this the *ordinary cohomology* to make a clear distinction from the bounded one, which we proceed to define.

Restricting to functions $f : \Gamma^\bullet \to \mathbb{R}$ that are bounded, meaning that their supremum $\|f\|_\infty$ is finite, leads to a subcomplex $(\mathrm{C}_b^\bullet(\Gamma), \delta^\bullet)$. We denote by $\mathrm{Z}_b^\bullet(\Gamma)$ the *bounded cocycles* and by $\mathrm{B}_b^\bullet(\Gamma)$ the *bounded coboundaries*. The vector space $\mathrm{H}_b^\bullet(\Gamma) := \mathrm{Z}_b^\bullet(\Gamma)/\mathrm{B}_b^\bullet(\Gamma)$ is the *bounded cohomology of* Γ with trivial real coefficients.

The inclusion of the bounded cochain complex into the ordinary one induces a linear map at the level of cohomology, called the *comparison map*:

$$c^\bullet\colon \mathrm{H}_b^\bullet(\Gamma) \to \mathrm{H}^\bullet(\Gamma).$$

This map is in general neither injective nor surjective. In degree 2, the kernel admits a description in terms of quasimorphisms:

Proposition 3.1 [Calegari 2009b, Theorem 2.50] *Let $Q(\Gamma)$ denote the space of quasimorphisms on Γ up to bounded distance, and $\mathrm{Z}^1(\Gamma)$ the space of homomorphisms $\Gamma \to \mathbb{R}$. Then the sequence*

$$0 \to \mathrm{Z}^1(\Gamma) \to Q(\Gamma) \xrightarrow{[\delta^1(-)]} \mathrm{H}_b^2(\Gamma) \xrightarrow{c^2} \mathrm{H}^2(\Gamma)$$

is exact. In particular, c^2 is injective if and only if every quasimorphism on Γ is at a bounded distance from a homomorphism.

While many applications of bounded cohomology in geometric group theory, eg the study of stable commutator length, are only concerned with quasimorphisms, in different settings the full knowledge of H_b^2 is of interest. Notable instances include the classification of circle actions [Ghys 1987], and the construction of manifolds with prescribed simplicial volume [Heuer and Löh 2021; 2023].

In order to prove Theorem 1.2, that $\widehat{bV}$ has vanishing second bounded cohomology, it is enough to prove this for the subgroup $\widehat{D}$ (from Definition 2.10), thanks to the following fact:

Proposition 3.2 ([Monod 2001, 8.6]; see also [Monod and Popa 2003]) *Let $n \geq 0$. Let Γ be a group and N a normal subgroup such that Γ/N is amenable. Then the inclusion $N \to \Gamma$ induces an injection in bounded cohomology $\mathrm{H}^n_b(\Gamma) \to \mathrm{H}^n_b(N)$. In particular, if $\mathrm{H}^n_b(N) = 0$ then $\mathrm{H}^n_b(\Gamma) = 0$.*

To prove vanishing of $\mathrm{H}^2_b(\widehat{D})$, we will use the following notion:

Definition 3.3 Let Γ be a group. We say that Γ has *commuting conjugates* if for every finitely generated subgroup $H \leq \Gamma$ there exists $g \in \Gamma$ such that every element of H commutes with every element of $g^{-1}Hg$.

Theorem 3.4 [Fournier-Facio and Lodha 2023] *If Γ is a group with commuting conjugates, then $\mathrm{H}^2_b(\Gamma) = 0$.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 Since $\widehat{bV}/\widehat{D} \cong \mathbb{Z}$ by definition, using Proposition 3.2 it suffices to show that $\mathrm{H}^2_b(\widehat{D}) = 0$. By Theorem 3.4 it suffices to show that $\widehat{D}$ has commuting conjugates. Let $H \leq \widehat{D}$ be a finitely generated subgroup. By Lemma 2.12 there exists $k \geq 0$ such that $x_0^{-k} \cdot H \cdot x_0^k \leq \widehat{bV}(1)$. Then by Lemma 2.16 there exists $g \in \widehat{D}$ such that every element of $g^{-1} \cdot x_0^{-k} \cdot H \cdot x_0^k \cdot g$ commutes with every element of $\widehat{bV}(1)$, and so in particular with every element of $x_0^{-k} \cdot H \cdot x_0^k$. Thus, every element of the conjugate of H by $x_0^k \cdot g \cdot x_0^{-k}$ commutes with every element of H. Finally, note that $x_0^k \cdot g \cdot x_0^{-k} \in \widehat{D}$ since $g \in \widehat{D}$, $x_0 \in \widehat{bV}$ and $\widehat{D}$ is normal in $\widehat{bV}$. This shows that $\widehat{D}$ has commuting conjugates and concludes the proof. □

4 Quasimorphisms on rV and bV

In this section we prove Theorem 1.1. We will first work with the ribbon braided Thompson group rV and prove that $Q(rV)$ is infinite-dimensional (Proposition 4.2), and then prove that unbounded quasimorphisms of rV restrict to unbounded quasimorphisms of bV, bF and bP (and indeed, any Γ satisfying $bP \leq \Gamma \leq rV$).

First we need to make the connection between rV and $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. This was done implicitly in [Aramayona and Funar 2021; Funar and Kapoudjian 2004], and more explicitly in [Skipper and Wu 2021, Theorem 3.24]. In short, rV is isomorphic to a certain subgroup of mapping classes of $S^2 \setminus K$, namely those that are "asymptotically quasirigid" with respect to some "rigid structure" involving choices of "admissible subsurfaces" and only act on half of $S^2 \setminus K$ in some sense; see [Skipper and Wu 2021, Definition 3.7] for all the details. We can view this as describing certain mapping classes of $D^2 \setminus K$ that do not require the boundary of D^2 to be fixed, but rather allow it to be half-twisted. For an example providing the intuition for how to view an element of bV as a mapping class, see Figure 7.

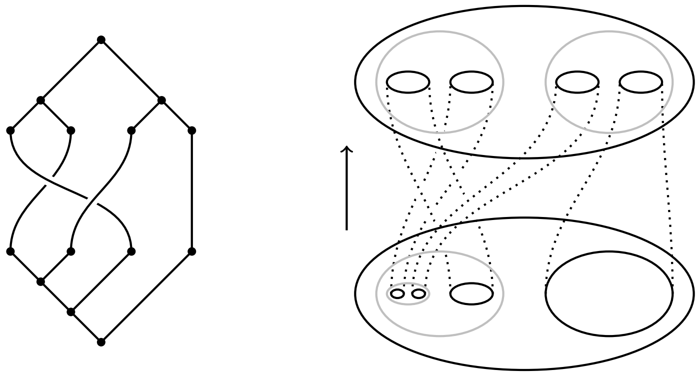


Figure 7: A visualization of the element $[T_-, \beta, T_+]$ of bV from Figure 1 as a mapping class on the disk. The bottom (domain) tree T_+ describes a decomposition of D^2 into the pieces shown, and the top (range) tree T_- describes another such decomposition. The braid β then treats the four smallest subdisks in the domain as "holes", with the range viewed similarly, and gives a homeomorphism from the former to the latter, indicated by the dotted lines. This element does not involve twists, but one could picture the holes twisting as well, yielding an element of rV.

Now we pass from this picture to $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ by viewing $D^2 \setminus K$ inside $\mathbb{R}^2 \setminus K$ at the expense of modding out the cyclic subgroup generated by a full twist around the boundary of D^2. This is represented by the element $[\,\cdot\,, 1_{B_1}(2), \cdot\,]$ of rV, which generates the center $Z(rV)$, so at this point we have embedded $rV/Z(rV)$ inside $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. In particular, we can work in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ to prove that $Q(rV/Z(rV))$ is infinite-dimensional, from which it will immediately follow that $Q(rV)$ is as well. It is also worth mentioning that $bV \cap Z(rV) = \{1\}$, so this provides an explicit embedding of bV into $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$.

4.1 Quasimorphisms on rV

The proof that $Q(rV/Z(rV))$ is infinite-dimensional closely follows Bavard's proof [2016, Théorème 4.8] that $Q(\mathrm{MCG}(\mathbb{R}^2 \setminus K))$ is infinite-dimensional.[1] We will especially use the constructions from [Bavard 2016, Section 4.1].

Bavard [2016] constructs the so-called *ray graph* X_r associated to the surface $\mathbb{R}^2 \setminus K$, and shows that it is hyperbolic. She proceeds to show that the action of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ on X_r satisfies the hypotheses of Bestvina and Fujiwara's main theorem [2002], which implies that $Q(\mathrm{MCG}(\mathbb{R}^2 \setminus K))$ is infinite-dimensional. To prove our Proposition 4.2, we will show that the action of $rV/Z(rV)$ also satisfies these properties, and make reference to [Bavard 2016, Section 4] throughout.

We start by reviewing Bavard's proof for $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. By the main theorem of [Bestvina and Fujiwara 2002], it suffices to exhibit elements $h_1, h_2 \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ with the following properties:

(1) h_1 and h_2 are hyperbolic elements for the action of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ on X_r, acting by translation on axes l_1 and l_2, which are equipped with the orientation of the action of the respective elements.

[1] This is Theorem 4.9 in the English translation.

(2) h_1 and h_2 are independent, meaning that their fixed point sets in ∂X_r are disjoint.

(3) There exist constants B and C such that for every segment w of l_2 longer than C, for every $g \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$, if the segment $g \cdot w$ is contained in the B–neighborhood of l_1, then it is oriented in the opposite direction.

Identify K with the set $\{0,1\}^{\mathbb{N}}$ of infinite words κ in the alphabet $\{0,1\}$, and for each finite word $w \in \{0,1\}^*$ let $K(w) := \{w\kappa \mid \kappa \in K\}$ be the *cone* corresponding to w. Then define

$$K_0 = K(00), \quad K_1 = K(010), \quad K_2 = K(0110), \quad \dots, \quad K_\infty = \{0\bar{1}\},$$
$$K_{-1} = K(11), \quad K_{-2} = K(101), \quad K_{-3} = K(1001), \quad \dots, \quad K_{-\infty} = \{1\bar{0}\}.$$

This provides a partition of K into sets K_i for $-\infty \le i \le \infty$, where each K_i for $i \in \mathbb{Z}$ is a clopen set and each $K_{\pm\infty}$ contains one point.

Let us now be more precise about how we would like K to live inside of $\mathbb{R}^2$. Assume that K lies on the horizontal axis $\mathbb{R}$ and is symmetric around $0 \notin K$, and that $K_i \subset \mathbb{R}_{<0}$ for all $0 \le i \le \infty$ and $K_i \subset \mathbb{R}_{>0}$ for all $-\infty \le i \le -1$. Let $I \subset \mathbb{R}$ be a symmetric open neighborhood of 0 that is disjoint from K. Finally, let $\mathcal{C} \subseteq \mathbb{R}^2$ be a homeomorphic copy of a circle, formed as the union of a segment in the horizontal axis $\mathbb{R}$ containing all of K and a semicircle in the upper half-plane. Let ϕ denote the homeomorphism of $\mathbb{R}^2$ that is a half-turn rotation about the origin, so ϕ stabilizes K, and denote by ϕ the mapping class of ϕ in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$.

Theorem 4.1 [Bavard 2016, Théorème 4.8] *Let $\tilde{t}_1$ be any homeomorphism of $\mathbb{R}^2$ that stabilizes $\mathcal{C}$, restricts to the identity on I and sends K_i to K_{i+1} for each $i \in \mathbb{Z}$. Let $t_1 \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ be the class of $\tilde{t}_1$, let $t_2 := \phi t_1 \phi^{-1}$, let $h_1 := t_1 t_2 t_1$ and let $h_2 := \phi h_1^{-1} \phi^{-1}$. Then the elements h_1 and h_2 satisfy the three properties above, and hence any subgroup of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ containing h_1 and h_2 has an infinite-dimensional space of quasimorphisms.*

As we have seen, rV maps to $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ with kernel $Z(rV) \cong \mathbb{Z}$. Note that the image in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ of the element $[\cdot, 1_{B_1}(1), \cdot] \in rV$, which is a single half-twist on one ribbon, is precisely the mapping class ϕ.

Proposition 4.2 *The space $Q(rV)$ is infinite-dimensional.*

Proof We will prove that $Q(rV/Z(rV))$ is infinite-dimensional, which implies our result. By Theorem 4.1 it suffices to show that the elements h_1 and h_2 can be realized inside of $rV/Z(rV)$. Since each of h_1 and h_2 is obtained as a product of conjugates of t_1 by ϕ and since $\phi \in rV/Z(rV)$, it suffices to show that t_1 can be realized inside $rV/Z(rV)$.

Recall that, in Theorem 4.1, the homeomorphism $\tilde{t}_1$ representing t_1 can be any homeomorphism of $\mathbb{R}^2$ satisfying

(1) $\tilde{t}_1$ stabilizes the topological circle $\mathcal{C}$,

(2) $\tilde{t}_1|_I$ is the identity, and

(3) $\tilde{t}_1(K_i) = K_{i+1}$ for each $i \in \mathbb{Z}$.

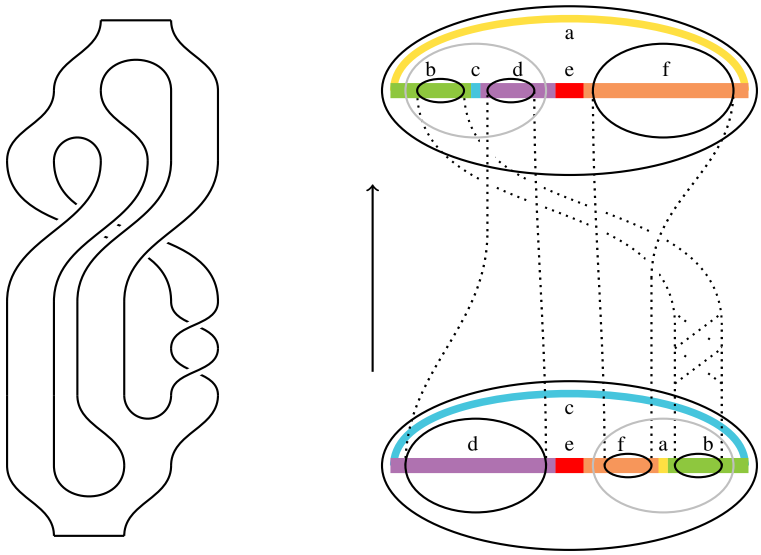


Figure 8: The desired element. Left: a ribbon strand diagram representing this element. Right: the corresponding mapping class, as in Figure 7. We indicate the circle $\mathcal{C}$ with rainbow colors and labels a–f, to make it clear how the mapping class acts on it. The fixed interval I is red (e).

Note that $\tilde{t}_1$ is defined on all of $\mathbb{R}^2$, so the image $\tilde{t}_1(K_i)$ is well defined, even if what we are interested in is a class $t_1 \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ — this is the usual equivocation between punctures and marked points.

Consider the element $[T_1, s_1^{-1}s_2^{-1}(0,0,-2), T_2]$ of rV represented as in Figure 8.

With $\mathcal{C}$ and I as indicated in the picture, it is clear that up to isotopy $\mathcal{C}$ is stabilized (thanks to the third strand twisting), and that I is fixed pointwise. One can also check that K_i is sent to K_{i+1} for each $i \in \mathbb{Z}$. We conclude that all the criteria are satisfied, and so we are done. □

4.2 Quasimorphisms on bV

The final step in the proof of Theorem 1.1 is to show that the quasimorphisms on rV constructed in the previous subsection restrict to nontrivial quasimorphisms on bV, bF and bP. This will be a consequence of the following general statement applied to rV and bP, which follows from left exactness of Q; see [Calegari 2009b, Remark 2.90].

Lemma 4.3 *Let Γ be a group and $\Lambda \leq \Gamma$ a normal subgroup, and suppose that $Q(\Gamma/\Lambda) = 0$. Then the restriction $Q(\Gamma) \to Q(\Lambda)$ is injective.*

Proof of Theorem 1.1 Let Γ be any group such that $bP \leq \Gamma \leq rV$, for instance any of the groups in the statement of the theorem. Note that the quotient rV/bP is uniformly perfect by Lemma 2.7, so every quasimorphism on rV/bP is bounded [Calegari 2009b, Lemma 2.2.4], ie $Q(rV/bP) = 0$. Hence Lemma 4.3 applies, and the restriction $Q(rV) \to Q(bP)$ is injective. Since this map factors through $Q(rV) \to Q(\Gamma)$, this restriction is also injective. We conclude by Proposition 4.2. □

Since every quasimorphism on a uniformly perfect group is bounded [Calegari 2009b, Lemma 2.2.4], an immediate corollary of Theorem 1.1 is the following:

Corollary 4.4 *The group bV is not uniformly perfect.* □

Of course we also conclude that rV is not uniformly perfect, despite being perfect (Corollary 2.8). Note that bF is not perfect (it has abelianization $\mathbb{Z}^4$), but it follows from [Zaremsky 2018a] that bF' is perfect. However, we can deduce in the same way:

Corollary 4.5 *The group bF' is not uniformly perfect.* □

We should also mention another group fitting between bP and rV and thus having infinite-dimensional space of quasimorphisms, namely the "braided T" group from [Witzel 2019]. This is the subgroup of bV consisting of elements $[T_-, \beta, T_+]$ such that $\rho(\beta) \in S_n$ is a cyclic permutation.

Let us discuss restricting quasimorphisms of bV to $\widehat{bV}$. If $\Gamma \leq \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ has a bounded orbit in X_r, then the quasimorphisms produced via this action are bounded on Γ. This is analogous to the behavior of finite-type mapping class groups, in particular for the braid group [Feller 2022], and was already noted by Calegari [2009a] for $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. We see that in fact this happens for $\widehat{bV}$, since it fixes the isotopy class of the ray going from the rightmost point of the Cantor set to infinity on the right. Thanks to Theorem 1.2, we can actually prove a stronger version of this statement. Namely, not only are these quasimorphisms bounded on $\widehat{bV}$, but the same is true for every quasimorphism of bV.

Corollary 4.6 *For every quasimorphism q of bV, the image of $\widehat{bV}$ under q is bounded.*

Proof Let $\psi = [\,\cdot\,, 1_{B_1}(1), \cdot\,]$. Then viewing $rV/Z(rV)$ as a subgroup of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$, we have $\psi Z(rV) = \phi$. Note that the conjugate $\psi^{-1}\widehat{bV}\psi$ equals the subgroup of bV consisting of all elements where the *leftmost* strand does not braid with anything. Let $\chi_0\colon \psi^{-1}\widehat{bV}\psi \to \mathbb{Z}$ be the map sending $\psi^{-1}g\psi$ to $\chi_1(g)$. Since conjugation by ψ is an isomorphism, Corollary 2.14 implies that the kernel of χ_0 equals the derived subgroup $(\psi^{-1}\widehat{bV}\psi)'$.

Let $g \in \widehat{bV}$, and choose $h \in \widehat{bV} \cap \psi^{-1}\widehat{bV}\psi$ such that $\chi_1(h) = \chi_1(g)$ and $\chi_0(h) = 0$. In particular, h lies in $(\psi^{-1}\widehat{bV}\psi)'$ and gh^{-1} lies in $\widehat{bV}'$. By Corollary 1.3, there exists a scalar $\lambda \in \mathbb{R}$ such that $q|_{\widehat{bV}}$ is at a bounded distance $r(q)$ from $\lambda \cdot \chi_1$, where χ_1 is the abelianization map of $\widehat{bV}$ (Corollary 2.14). It follows that

$$|q(gh^{-1})| \leq |\lambda \cdot \chi_1(gh^{-1})| + r(q) = r(q).$$

We also get $|q(h)| \leq r(q)$ by the same argument applied to $\psi^{-1}\widehat{bV}\psi$ (which is isomorphic to $\widehat{bV}$), up to taking a larger $r(q)$. Thus

$$|q(g)| \leq |q(gh^{-1})| + |q(h)| + D(q) \leq 2r(q) + D(q).$$

This shows that $q|_{\widehat{bV}}$ is bounded, which concludes the proof. □

Together with Theorem 1.1, this implies that, analogously to Corollary 4.4, there is no uniform-length factorization for elements of bV in terms of conjugates of elements in $\widehat{bV}$. Indeed, if such a factorization did exist, we could run a similar argument as in the proof of Corollary 4.6, and obtain that every quasimorphism of bV is bounded.

As one last indication of braided Thompson groups exhibiting unusual bounded cohomological behavior, consider the short exact sequence $1 \to \widehat{bV} \cap bP \to \widehat{bV} \to \widehat{V} \to 1$. By Theorem 1.2 $\mathrm{H}_b^2(\widehat{bV}) = 0$, and in fact the proof works using permutations instead of braids, mutatis mutandis, to show that $\mathrm{H}_b^2(\widehat{V}) = 0$ (also, it is true in general that a quotient of a group with vanishing second bounded cohomology itself has vanishing second bounded cohomology [Bouarich 1995]). However, by Theorem 1.1 $Q(bP)$ is infinite-dimensional, and thus so is $Q(\widehat{bV} \cap bP)$ since $\widehat{bV} \cap bP$ surjects onto bP (Lemma 2.5); in particular, $\mathrm{H}_b^2(\widehat{bV} \cap bP)$ is infinite-dimensional.

This gives a concrete example of the failure of a 2-out-of-3 property for vanishing of second bounded cohomology: if a group Γ has vanishing second bounded cohomology and a quotient Γ/N has the same property, then the kernel N can still have infinite-dimensional second bounded cohomology. For comparison, if $\mathrm{H}_b^2(N) = 0$, then $\mathrm{H}_b^2(\Gamma) = 0$ if and only if $\mathrm{H}_b^2(\Gamma/N) = 0$ [Moraschini and Raptis 2023, Corollary 4.2.2]. The failure of this 2-out-of-3 property was observed in [Fournier-Facio et al. 2023, Theorem 4.5] for every degree, but this is to our knowledge the first "naturally occurring" example in degree 2, as well as the first finitely generated one (and even type F_∞).

References

[AIM 2019] *Surfaces of infinite type*, online problem list, Amer. Inst. Math. (2019) Available at `http://aimpl.org/genusinfinity/`

[Andritsch 2022] **K Andritsch**, *Bounded cohomology of groups acting on Cantor sets* (2022) arXiv 2210.00459

[Aramayona and Funar 2021] **J Aramayona**, **L Funar**, *Asymptotic mapping class groups of closed surfaces punctured along Cantor sets*, Mosc. Math. J. 21 (2021) 1–29 MR Zbl

[Aramayona and Vlamis 2020] **J Aramayona**, **N G Vlamis**, *Big mapping class groups: an overview*, from "In the tradition of Thurston: geometry and topology" (K Ohshika, A Papadopoulos, editors), Springer (2020) 459–496 MR Zbl

[Aramayona et al. 2021] **J Aramayona**, **K-U Bux**, **J Flechsig**, **N Petrosyan**, **X Wu**, *Asymptotic mapping class groups of Cantor manifolds and their finiteness properties*, preprint (2021) arXiv 2110.05318

[Aroca and Cumplido 2022] **J Aroca**, **M Cumplido**, *A new family of infinitely braided Thompson's groups*, J. Algebra 607 (2022) 5–34 MR Zbl

[Baumslag et al. 1980] **G Baumslag**, **E Dyer**, **A Heller**, *The topology of discrete groups*, J. Pure Appl. Algebra 16 (1980) 1–47 MR Zbl

[Bavard 2016] **J Bavard**, *Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire*, Geom. Topol. 20 (2016) 491–535 MR Zbl Translated in arXiv 1802.02715

[Bestvina and Fujiwara 2002] **M Bestvina**, **K Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002) 69–89 MR Zbl

[Bestvina et al. 2016] **M Bestvina**, **K Bromberg**, **K Fujiwara**, *Stable commutator length on mapping class groups*, Ann. Inst. Fourier (Grenoble) 66 (2016) 871–898 MR Zbl

[Bouarich 1995] **A Bouarich**, *Suites exactes en cohomologie bornée réelle des groupes discrets*, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995) 1355–1359 MR Zbl

[Brady et al. 2008] **T Brady**, **J Burillo**, **S Cleary**, **M Stein**, *Pure braid subgroups of braided Thompson's groups*, Publ. Mat. 52 (2008) 57–89 MR Zbl

[Brin 2006] **M G Brin**, *The algebra of strand splitting, II: A presentation for the braid group on one strand*, Int. J. Algebra Comput. 16 (2006) 203–219 MR Zbl

[Brin 2007] **M G Brin**, *The algebra of strand splitting, I: A braided version of Thompson's group V*, J. Group Theory 10 (2007) 757–788 MR Zbl

[Brooks 1981] **R Brooks**, *Some remarks on bounded cohomology*, from "Riemann surfaces and related topics" (I Kra, B Maskit, editors), Ann. of Math. Stud. 97, Princeton Univ. Press (1981) 53–63 MR Zbl

[Brown 1982] **K S Brown**, *Cohomology of groups*, Graduate Texts in Math. 87, Springer (1982) MR Zbl

[Burger and Monod 2002] **M Burger**, **N Monod**, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2002) 219–280 MR Zbl

[Burillo and Cleary 2009] **J Burillo**, **S Cleary**, *Metric properties of braided Thompson's groups*, Indiana Univ. Math. J. 58 (2009) 605–615 MR Zbl

[Bux et al. 2016] **K-U Bux**, **M G Fluch**, **M Marschler**, **S Witzel**, **M C B Zaremsky**, *The braided Thompson's groups are of type* F_∞, J. Reine Angew. Math. 718 (2016) 59–101 MR Zbl Correction in 778 (2021) 219–221

[Calegari 2007] **D Calegari**, *Stable commutator length in subgroups of* $\mathrm{PL}^+(I)$, Pacific J. Math. 232 (2007) 257–262 MR Zbl

[Calegari 2009a] **D Calegari**, *Big mapping class groups and dynamics*, blog post (2009) Available at `https://lamington.wordpress.com/2009/06/22/big-mapping-class-groups-and-dynamics/`

[Calegari 2009b] **D Calegari**, *scl*, MSJ Mem. 20, Math. Soc. Japan, Tokyo (2009) MR Zbl

[Calegari and Chen 2021] **D Calegari**, **L Chen**, *Big mapping class groups and rigidity of the simple circle*, Ergodic Theory Dynam. Systems 41 (2021) 1961–1987 MR Zbl

[Dehornoy 2006] **P Dehornoy**, *The group of parenthesized braids*, Adv. Math. 205 (2006) 354–409 MR Zbl

[Epstein and Fujiwara 1997] **D B A Epstein**, **K Fujiwara**, *The second bounded cohomology of word-hyperbolic groups*, Topology 36 (1997) 1275–1289 MR Zbl

[Feller 2022] **P Feller**, *The slice-Bennequin inequality for the fractional Dehn twist coefficient*, preprint (2022) arXiv 2204.05288

[Field et al. 2022] **E Field**, **P Patel**, **A J Rasmussen**, *Stable commutator length on big mapping class groups*, Bull. Lond. Math. Soc. 54 (2022) 2492–2512 MR Zbl

[Fournier-Facio and Lodha 2023] **F Fournier-Facio**, **Y Lodha**, *Second bounded cohomology of groups acting on 1–manifolds and applications to spectrum problems*, Adv. Math. 428 (2023) art. id. 109162 MR Zbl

[Fournier-Facio et al. 2023] **F Fournier-Facio**, **C Löh**, **M Moraschini**, *Bounded cohomology and binate groups*, J. Aust. Math. Soc. 115 (2023) 204–239 MR Zbl

[Frigerio 2017] **R Frigerio**, *Bounded cohomology of discrete groups*, Math. Surv. Monogr. 227, Amer. Math. Soc., Providence, RI (2017) MR Zbl

[Funar and Kapoudjian 2004] **L Funar**, **C Kapoudjian**, *On a universal mapping class group of genus zero*, Geom. Funct. Anal. 14 (2004) 965–1012 MR Zbl

[Funar and Kapoudjian 2008] **L Funar**, **C Kapoudjian**, *The braided Ptolemy–Thompson group is finitely presented*, Geom. Topol. 12 (2008) 475–530 MR Zbl

[Funar and Kapoudjian 2011] **L Funar**, **C Kapoudjian**, *The braided Ptolemy–Thompson group is asynchronously combable*, Comment. Math. Helv. 86 (2011) 707–768 MR Zbl

[Gal and Gismatullin 2017] **Ś R Gal**, **J Gismatullin**, *Uniform simplicity of groups with proximal action*, Trans. Amer. Math. Soc. Ser. B 4 (2017) 110–130 MR Zbl

[Genevois et al. 2022] **A Genevois**, **A Lonjou**, **C Urech**, *Asymptotically rigid mapping class groups, I: Finiteness properties of braided Thompson's and Houghton's groups*, Geom. Topol. 26 (2022) 1385–1434 MR Zbl

[Geoghegan et al. 2001] **R Geoghegan**, **M L Mihalik**, **M Sapir**, **D T Wise**, *Ascending HNN extensions of finitely generated free groups are Hopfian*, Bull. Lond. Math. Soc. 33 (2001) 292–298 MR Zbl

[Ghys 1987] **É Ghys**, *Groupes d'homéomorphismes du cercle et cohomologie bornée*, from "The Lefschetz centennial conference, III" (A Verjovsky, editor), Contemp. Math. 58, Amer. Math. Soc., Providence, RI (1987) 81–106 MR Zbl

[Ghys and Sergiescu 1987] **É Ghys**, **V Sergiescu**, *Sur un groupe remarquable de difféomorphismes du cercle*, Comment. Math. Helv. 62 (1987) 185–239 MR Zbl

[Gromov 1982] **M Gromov**, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. 56 (1982) 5–99 MR Zbl

[Guelman and Liousse 2023] **N Guelman**, **I Liousse**, *Uniform simplicity for subgroups of piecewise continuous bijections of the unit interval*, Bull. Lond. Math. Soc. 55 (2023) 2341–2362 MR Zbl

[Heuer and Löh 2021] **N Heuer**, **C Löh**, *The spectrum of simplicial volume*, Invent. Math. 223 (2021) 103–148 MR Zbl

[Heuer and Löh 2023] **N Heuer**, **C Löh**, *Transcendental simplicial volumes*, Ann. Inst. Fourier (Grenoble) (online publication July 2023)

[Hull and Osin 2013] **M Hull**, **D Osin**, *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol. 13 (2013) 2635–2665 MR Zbl

[Ishida 2018] **T Ishida**, *Orderings of Witzel–Zaremsky–Thompson groups*, Comm. Algebra 46 (2018) 3806–3809 MR Zbl

[Johnson 1972] **B E Johnson**, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127, Amer. Math. Soc., Providence, RI (1972) MR Zbl

[Monod 2001] **N Monod**, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Math. 1758, Springer (2001) MR Zbl

[Monod 2022] **N Monod**, *Lamplighters and the bounded cohomology of Thompson's group*, Geom. Funct. Anal. 32 (2022) 662–675 MR Zbl

[Monod and Popa 2003] **N Monod**, **S Popa**, *On co-amenability for groups and von Neumann algebras*, C. R. Math. Acad. Sci. Soc. R. Can. 25 (2003) 82–87 MR Zbl

[Moraschini and Raptis 2023] **M Moraschini**, **G Raptis**, *Amenability and acyclicity in bounded cohomology*, Rev. Mat. Iberoam. 39 (2023) 2371–2404 MR Zbl

[Navas 2018] **A Navas**, *Group actions on 1–manifolds: a list of very concrete open questions*, from "Proceedings of the International Congress of Mathematicians, III" (B Sirakov, P N de Souza, M Viana, editors), World Sci. Publ., Hackensack, NJ (2018) 2035–2062 MR Zbl

[Skipper and Wu 2021] **R Skipper**, **X Wu**, *Homological stability for the ribbon Higman–Thompson groups*, preprint (2021) arXiv 2106.08751 To appear in Algebr. Geom. Topol.

[Skipper and Wu 2023] **R Skipper**, **X Wu**, *Finiteness properties for relatives of braided Higman–Thompson groups*, Groups Geom. Dyn. 17 (2023) 1357–1391 MR Zbl

[Skipper and Zaremsky 2023] **R Skipper**, **M C B Zaremsky**, *Braiding groups of automorphisms and almost-automorphisms of trees*, Canad. J. Math. (online publication March 2023)

[Spahn 2021] **R Spahn**, *Braided Brin–Thompson groups*, preprint (2021) arXiv 2101.03462

[Thumann 2017] **W Thumann**, *Operad groups and their finiteness properties*, Adv. Math. 307 (2017) 417–487 MR Zbl

[Witte Morris 2006] **D Witte Morris**, *Amenable groups that act on the line*, Algebr. Geom. Topol. 6 (2006) 2509–2518 MR Zbl

[Witzel 2019] **S Witzel**, *Classifying spaces from Ore categories with Garside families*, Algebr. Geom. Topol. 19 (2019) 1477–1524 MR Zbl

[Witzel and Zaremsky 2018] **S Witzel**, **M C B Zaremsky**, *Thompson groups for systems of groups, and their finiteness properties*, Groups Geom. Dyn. 12 (2018) 289–358 MR Zbl

[Zaremsky 2018a] **M C B Zaremsky**, *On normal subgroups of the braided Thompson groups*, Groups Geom. Dyn. 12 (2018) 65–92 MR Zbl

[Zaremsky 2018b] **M C B Zaremsky**, *A user's guide to cloning systems*, Topology Proc. 52 (2018) 13–33 MR Zbl

[Zhuang 2008] **D Zhuang**, *Irrational stable commutator length in finitely presented groups*, J. Mod. Dyn. 2 (2008) 499–507 MR Zbl

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Oriented and unitary equivariant bordism of surfaces

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Fix a finite group G. We study $\Omega_2^{\mathrm{SO},G}$ and $\Omega_2^{U,G}$, the unitary and oriented bordism groups of smooth G–equivariant compact surfaces, respectively, and we calculate them explicitly. Their ranks are determined by the possible representations around fixed points, while their torsion subgroups are isomorphic to the direct sum of the Bogomolov multipliers of the Weyl groups of representatives of conjugacy classes of all subgroups of G. We present an alternative proof of the fact that surfaces with free actions which induce nontrivial elements in the Bogomolov multiplier of the group cannot equivariantly bound. This result permits us to show that the 2–dimensional SK–groups (Schneiden und Kleben, or "cut and paste") of the classifying spaces of a finite group can be understood in terms of the bordism group of free equivariant surfaces modulo the ones that bound arbitrary actions.

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1 Introduction

Equivariant bordism groups have been a subject of ongoing research since the 1960s. Conner, Floyd, Landweber, Stong, Smith and tom Dieck, among others, laid the foundations for the extraordinary homology and cohomology theories obtained from equivariant bordism, and found many interesting properties of these groups. Given a finite group G, a particularly important problem is the explicit calculation of the oriented and complex G–equivariant bordism groups of a point, since they provide the coefficients for the theories. This turns out to be a complicated task.

Explicit calculations of the equivariant bordism groups for finite abelian groups (see Landweber [19], Ossa [26] and Stong [34]) led some to expect that, at least in the unitary case, equivariant bordism groups are always a free module over the unitary bordism ring for any finite group G; see Rowlett [28, page 1], May [21, Chapter XXVIII.5] and Greenlees and May [12, Conjecture 1.2]. This belief was confirmed for general abelian groups (see Löffler [20] and [21, Chapter XXVIII, Theorem 5.1]) and for metacyclic groups [28], and therefore it was conjectured that for any finite group this was the case. This conjecture remained dormant for some years and it was recalled Uribe in his 2018 ICM Lecture [35], where he named it "the evenness conjecture in equivariant unitary bordism".

When the evenness conjecture holds true for a group G, it implies that the G–equivariant unitary bordism ring is torsion-free. In particular, any unitary manifold with a free action of a finite group that generates a torsion class in the unitary bordism group of free actions would bound equivariantly. This has always been the first step for proving the evenness conjecture, namely, to construct explicit equivariant manifolds whose boundaries are the desired generators of the equivariant unitary bordism groups of free actions.

In the case of surfaces, the evenness conjecture would imply that all oriented surfaces with orientation-preserving free actions bound equivariantly (note that if an oriented surface with orientation-preserving free action does not bound equivariantly, then the class of the difference of this surface with G–times the quotient surface induces a nontrivial torsion class in the reduced G–equivariant unitary bordism group). Domínguez and Segovia [9] showed that indeed this is the case for abelian, dihedral, symmetric and alternating groups. Nevertheless, it fails to be true in general. It has been recently shown that there is an obstruction class for an oriented surface with an orientation-preserving free action to bound equivariantly (see Samperton [29; 30]), and this obstruction class lies in the Bogomolov multiplier of the group; see Bogomolov [3] and Kunyavskiĭ [18]. The Bogomolov multiplier of a finite group consists of the classes of the Schur multiplier $H^2(G,\mathbb{C}^*)$ that vanish once restricted to any abelian subgroup; the homological version of the Bogomolov multiplier is the quotient of the second integral homology of the group by the classes generated by 2–dimensional tori; see Moravec [22]. This result implies that indeed there are torsion classes in the equivariant unitary bordism groups, and therefore that the evenness conjecture in equivariant unitary bordism is false in general. The evenness conjecture might then be restated instead as a classification question, namely *which finite groups satisfy the evenness conjecture in equivariant unitary bordism*?

We focus on the calculation of the oriented and the unitary G–equivariant bordism groups for compact surfaces. We use the fixed-point construction methods developed by Rowlett [27] to determine the rank of the equivariant bordism groups, and then use the explicit generators of the equivariant bordism groups for adjacent families in dimension 3 in order to determine which equivariant surfaces bound. In Theorem 4.3 we present a generalization to all finite groups of the result shown by Samperton in [29] which states that the obstruction class for equivariantly bounding an oriented surface with free action is the element in the Bogomolov multiplier of the group that the surface defines. The Conner–Floyd spectral sequence will then allow us to determine the torsion group in the equivariant bordism group of surfaces. Our main result is:

Theorem 4.4 *Let G be a finite group and $\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G)$ the torsion subgroup of the unitary or oriented G–equivariant bordism of surfaces Ω_2^G. Then there is a canonical isomorphism*

$$\bigoplus_{(K)} \widetilde{B}_0(W_K) \cong \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G),$$

where (K) runs over all conjugacy classes of subgroups of G, $W_K = N_G K/K$ and $\widetilde{B}_0(W_K)$ is the homology version of the Bogomolov multiplier of the group W_K.

With the torsion group in hand, we describe explicitly in Theorem 4.5 the G–equivariant bordism groups of surfaces, unitary and oriented.

Since there are infinitely many groups with nontrivial Bogomolov multipliers, we conclude that there are infinitely many groups which do not satisfy the evenness conjecture in equivariant unitary bordism. On the other hand, there are also infinitely many groups G whose G–equivariant unitary bordism group of surfaces is a free abelian group, thus implying that these groups may still satisfy the evenness conjecture for equivariant unitary bordism.

We use our previous calculations to interpret which equivariant surfaces bound in terms of the SK–relation (cutting and pasting from the German *Schneiden und Kleben*). The study of invariants under cutting and pasting started with the characterization by Jänich [15; 14] of invariants with the additive properties of the Euler characteristic and the signature, and it was further developed with the introduction of the SK–groups of a space by Karras, Kreck, Neumann and Ossa [17]. The SK–groups of a space can be understood as the groups of equivalence classes of manifolds with continuous maps to the space subject to the equivalence relation given by cutting and pasting. The 2–dimensional SK–groups of BG can be understood in terms of cutting and pasting surfaces with free G–actions. The SK–groups of BG were studied in [17] and were identified by Neumann in [24, Theorem 2] with the second integral homology group of BG modulo the toral classes (as far as we know this is the first reference where the homological Bogomolov multiplier appears).

We conclude with the study of two explicit groups, of order 64 and 243, whose Bogomolov multipliers are nontrivial. We sketch why both groups possess nontrivial Bogomolov multipliers and give explicit homomorphisms from the fundamental group of a genus-2 surface to both groups that define the desired surfaces with free actions that do not bound equivariantly. These constructions allow us to give explicit generators for the torsion subgroup of the equivariant unitary bordism groups for both groups.

2 Preliminaries

2.1 Equivariant bordism

Let G be a finite group and consider compact manifolds endowed with smooth actions of the group G preserving either the orientation or the unitary (tangentially stable almost complex) structure.

Recall that a tangentially stable almost complex G–structure over the G–manifold M consists of a G–equivariant complex vector bundle ξ over M such that $TM \oplus \mathbb{R}^k \cong \xi$ as G–equivariant real vector bundles and k is some natural number; here G acts trivially on the stabilized part $\mathbb{R}^k$. Two tangentially stable almost complex structures are identified if they become isomorphic as complex vector bundles after stabilization with further G–trivial $\mathbb{C}$ summands.

With this definition at hand, if K is a subgroup of G, then the fixed-point set M^K is endowed with a canonical tangential stable almost complex W_K–structure with $W_K := N_G K/K$. This follows from the isomorphism of W_K–equivariant real bundles

(1) $$\xi^K \cong (TM \oplus \mathbb{R}^k)^K \cong (TM|_{M^K})^K \oplus \mathbb{R}^k = T(M^K) \oplus \mathbb{R}^k$$

and the fact that ξ^K becomes a W_K–equivariant complex vector bundle over M^K.

Now, as $N_G K$–equivariant real vector bundles, we have the isomorphism

$$\xi|_{M^K} \cong TM|_{M^K} \oplus \mathbb{R}^k \cong T(M^K) \oplus \nu(M^K, M) \oplus \mathbb{R}^k \cong \xi^K \oplus \nu(M^K, M), \tag{2}$$

where $\nu(M^K, M)$ denotes the normal bundle of the embedding $M^K \hookrightarrow M$. Since both $\xi|_{M^K}$ and ξ^K are $N_G K$–equivariant complex vector bundles over M^K, the normal bundle $\nu(M^K, M)$ is naturally endowed with the structure of an $N_G K$–equivariant complex vector bundle. The fact that the normal bundles of the fixed points M^K are endowed with complex structures plays an important role in the study of tangentially stable almost complex G–structures.

Tangentially stable almost complex G–structures are also called G–equivariant unitary structures, and the equivalence classes of manifolds under the bordism relation in the realm of G–equivariant unitary structures is called the G–equivariant unitary bordism group.

Following the notation of Stong [34], denote by Ω_*^G either the bordism ring $\Omega_*^{\mathrm{SO},G}$ of G–equivariant oriented manifolds or the bordism ring $\Omega_*^{U,G}$ of G–equivariant unitary (tangentially stable almost complex) manifolds. Whenever the upper script SO or U is not specified, it means that the construction and results apply to both homology theories.

For the explicit definitions of both unitary and oriented equivariant bordism rings see [34, Section 2], and for the properties of the tangentially stable almost complex manifolds defining the unitary equivariant bordism groups, including the ones presented above, see [21, XXVIII, Section 3; 13, Section 2; 2, Section 5].

2.2 Equivariant bordism for families

The study of the equivariant bordism groups led Conner and Floyd to restrict their attention to manifolds with prescribed isotropy groups [4; 5]. The allowed isotropy groups are therefore organized in families of subgroups of G which are closed under conjugation and under taking subgroups. For any such family of subgroups $\mathcal{F}$ there is a classifying G–space $E\mathcal{F}$ for actions whose isotropy groups lie on $\mathcal{F}$. This G–space is characterized by its properties on fixed-point sets, namely, the fixed-point set $E\mathcal{F}^H$ is contractible whenever $H \in \mathcal{F}$ and empty otherwise. The construction of $E\mathcal{F}$ can be carried out in such a way that an inclusion of families $\mathcal{F}' \subset \mathcal{F}$ induces a G–cofibration $E\mathcal{F}' \to E\mathcal{F}$ [8, Section 1.6].

The equivariant bordism groups $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ for a pair of families $\mathcal{F}' \subset \mathcal{F}$ are the bordism groups of G–equivariant compact manifolds with boundary $(M, \partial M)$ such that the isotropy groups of M lie in $\mathcal{F}$ and the isotropy groups of its boundary ∂M lie in $\mathcal{F}'$. Following [7, page 310] one may define the bordism of groups for a pair of G–spaces (X, A) and a pair of families by

$$\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(X, A) := \Omega_*^G(X \times E\mathcal{F}, X \times E\mathcal{F}' \cup A \times E\mathcal{F}), \tag{3}$$

or, equivalently, using a more geometrical description [34].

2.3 Long exact sequence for families

Whenever three families are related by the inclusions $\mathcal{F}'' \subset \mathcal{F}' \subset \mathcal{F}$ there is induced a long exact sequence in bordism [5, Theorem 5.1]

$$\cdots \to \Omega_*^G\{\mathcal{F}', \mathcal{F}''\} \to \Omega_*^G\{\mathcal{F}, \mathcal{F}''\} \to \Omega_*^G\{\mathcal{F}, \mathcal{F}'\} \xrightarrow{\partial} \Omega_{*-1}^G\{\mathcal{F}', \mathcal{F}''\} \to \cdots . \tag{4}$$

2.4 Conner–Floyd spectral sequence

More generally, associated to the families $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$ there is a spectral sequence converging to $\Omega_n^G\{\mathcal{F}\}$, whose filtration is

$$F_p \Omega_n^G\{\mathcal{F}\} := \mathrm{Im}(\Omega_n^G\{\mathcal{F}_p\} \to \Omega_n^G\{\mathcal{F}\}). \tag{5}$$

This spectral sequence is usually called the *Conner–Floyd spectral sequence*, its first page is given by

$$E^1_{p,q} \cong \Omega^G_{p+q}\{\mathcal{F}_p, \mathcal{F}_{p-1}\}, \tag{6}$$

and the differentials are induced by the boundary maps. The first page of this spectral sequence might be difficult to calculate, but whenever the pair of families $\mathcal{F}_{p-1} \subset \mathcal{F}_p$ are adjacent (see below for the definition), fixed-point methods together with the classification of the normal bundles can make them computable in terms of nonequivariant bordism groups.

2.5 Equivariant bordism for adjacent families

A pair of families $\mathcal{F}' \subset \mathcal{F}$ are called *adjacent* whenever they differ by the conjugacy class (K) of a subgroup K, in other words $\mathcal{F} - \mathcal{F}' = (K)$. A manifold $(M, \partial M)$ in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}$ is cobordant to the G–equivariant tubular neighborhood of the fixed-point set of all the subgroups of G conjugate to K (all isotropy groups in the complement of the tubular neighborhood belong to $\mathcal{F}'$; the explicit bordism can be found in [5, Lemma 5.2]). The fixed points M^K of K become a free $W_K := N_G K/K$ space and the G–equivariant tubular neighborhood can be reconstructed from a specific W_K–equivariant twisted bundle over M^K by extending the $N_G K$ space to a G space. Hence, if M^K is of dimension $n-k$ and M^K/W_K is connected, its tubular neighborhood can be recovered from a map $M^K \to C_{N_G K, K}(k)$ where $C_{N_G K, K}(k)$ is a W_K–space which classifies the $N_G K$–equivariant tubular neighborhoods of rank k around K–fixed points [35, (2.5)]. In the unitary case there is a decomposition in terms of nonequivariant unitary bordism groups [35, Theorem 2.8]

$$\Omega_n^{U,G}\{\mathcal{F}, \mathcal{F}'\} := \bigoplus_{2k \le n} \Omega^U_{n-2k}(C_{N_G K, K}(k) \times_{W_K} EW_K), \tag{7}$$

and a similar one in the case of oriented bordisms [1, Theorem 2.11] This localization theorem will become very useful in what follows once we apply it for the study of the equivariant bordism groups of surfaces.

2.6 G–fixed points

For every subgroup K of G denote by $\mathcal{A}K$ the family of all subgroups of K and its conjugates in G, and denote by $\mathcal{P}K$ the family $\mathcal{A}K-(K)$ of all *proper* subgroups of K, and its conjugates in G. The localization map at the fixed points of the whole group action

$$\Omega_*^G \to \Omega_*^G\{\mathcal{A}G, \mathcal{P}G\} \tag{8}$$

together with the decomposition into nonequivariant bordisms groups presented in (7), has been a powerful tool for determining the equivariant bordism groups for abelian groups (see for instance [6; 20; 13]). In this particular case, the bordism groups $\Omega_*^G\{\mathcal{A}G, \mathcal{P}G\}$ are isomorphic to the nonequivariant bordism groups of products of complex Grassmannians in the unitary case, and of products of real, complex and quaternionic Grassmannians in the oriented case.

2.7 Rowlett spectral sequence

We still need another spectral sequence suited for understanding the equivariant bordism groups of pairs of families. This spectral sequence was constructed by Rowlett in [27, Proposition 2.1], whence its name. Consider a pair of families $\mathcal{F}' \subset \mathcal{F}$ that are also families of subgroups of the normal subgroup K of G and $(M, \partial M)$ in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}$. Then it is easy to see that the classifying map $M/K \to EW_K$ of the free $W_K = G/K$ action of the quotient induces an isomorphism of bordism groups $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\} \xrightarrow{\cong} \Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(EW_K)$ by mapping M to the composition $M \to M/K \to EW_K$; the inverse is simply induced by the map $EW_K \to *$. The space EW_K can be constructed as a CW–complex whose n–skeleton $(EW_K)^n$ is constructed from $(EW_K)^{n-1}$ by attaching a finite number of copies of $W_K \times B^n$ with W_K acting trivially on the n–dimensional balls. One may filter $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(EW_K)$ by the images under the inclusion of the skeletons $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}((EW_K)^n)$, and therefore one obtains a spectral sequence converging to $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ whose first page becomes

$$E^1_{p,q} \cong \Omega^G_{p+q}\{\mathcal{F}, \mathcal{F}'\}((EW_K)^p, (EW_K)^{p-1}) \cong H_p((EW_K)^p, (EW_K)^{p-1}) \otimes_{W_K} \Omega_q^K\{\mathcal{F}, \mathcal{F}'\}, \tag{9}$$

and whose second page is

$$E^2_{p,q} \cong H_p(W_K, \Omega_q^K\{\mathcal{F}, \mathcal{F}'\}), \tag{10}$$

where the action of an element of W_K on a K–manifold M consists of the same manifold M endowed with the conjugate K–action. The zeroth column consists of the W_K–coinvariants

$$E^2_{0,q} \cong (\Omega_q^K\{\mathcal{F}, \mathcal{F}'\})_{W_K}, \tag{11}$$

and the edge homomorphism

$$\Omega_q^K\{\mathcal{F}, \mathcal{F}'\} \cong E^1_{0,q} \to E^2_{0,q} \to E^\infty_{0,q} \to \Omega_q^G\{\mathcal{F}, \mathcal{F}'\} \tag{12}$$

is simply the extension homomorphism factorizing through the coinvariants

$$\Omega_q^K\{\mathcal{F}, \mathcal{F}'\} \to (\Omega_q^K\{\mathcal{F}, \mathcal{F}'\})_{W_K} \to \Omega_q^G\{\mathcal{F}, \mathcal{F}'\} \quad \text{given by } M \mapsto M \times_K G. \tag{13}$$

In characteristic zero the spectral sequence collapses on the zeroth column of the second page. Since in characteristic zero the invariants and the coinvariants are isomorphic, we conclude that the extension homomorphism induces an isomorphism

$$\Omega_*^K\{\mathcal{F},\mathcal{F}'\}^{W_K}\otimes\mathbb{Q}\xrightarrow{\cong}\Omega_*^G\{\mathcal{F},\mathcal{F}'\}\otimes\mathbb{Q}. \tag{14}$$

In order to find the torsion classes in Ω_*^G we will construct the inverse map of the isomorphism (14) for every pair of adjacent families of groups. This map will be simply given by the localization at fixed points and will be the subject of the next section.

3 Localization at fixed points

For every subgroup K of G let us define the fixed-point homomorphism

$$f_K\circ r_K^G\colon \Omega_*^G\to\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\} \tag{15}$$

as the composition of the restriction homomorphism $r_K^G\colon\Omega_*^G\to\Omega_*^K$ with the localization at K–fixed points

$$f_K\colon\Omega_*^K\to\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}. \tag{16}$$

The composition $f_K\circ r_K^G$ takes a G–manifold and maps it to the tubular neighborhood N of the K–invariant points M^K. Since on the complement of N in M there are no points with isotropy K, the tubular neighborhood N and M become cobordant in $\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}$ [5, Lemma 5.2]. Since N_GK acts on the normal bundle N of M^K, the localization at K–fixed points lands in the W_K–fixed submodule. Therefore the fixed-point homomorphism becomes

$$f_K\circ r_K^G\colon\Omega_*^G\to\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_K}. \tag{17}$$

Also, for every pair of families of subgroups in G, we have the localized fixed-point homomorphism

$$\phi_*\colon\Omega_*^G\{\mathcal{F},\mathcal{F}'\}\to\bigoplus_{(K)\subset\mathcal{F}-\mathcal{F}'}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_K}. \tag{18}$$

This homomorphism applied to the pair of adjacent families $\{\mathcal{A}K,\mathcal{P}K\}$, composed with the edge homomorphism of the Rowlett spectral sequence (13), gives us the maps

$$\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}_{W_A}\to\Omega_*^G\{\mathcal{A}K,\mathcal{P}K\}\xrightarrow{\phi}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_A}. \tag{19}$$

In characteristic zero, this composition is an isomorphism and therefore we obtain the isomorphism

$$\phi_*\colon\Omega_*^G\{\mathcal{A}K,\mathcal{P}K\}\otimes\mathbb{Q}\xrightarrow{\cong}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_A}\otimes\mathbb{Q}, \tag{20}$$

which becomes the inverse of the map in (14) for adjacent families.

Applying the Conner–Floyd spectral sequence, we see that the fixed-point homomorphism (18) in characteristic zero becomes an isomorphism, and therefore we quote:

Theorem 3.1 [27, Theorem 1.1] *The fixed-point homomorphism in characteristic zero is an isomorphism*

$$\phi_*\otimes\mathbb{Q}\colon\Omega_*^G\otimes\mathbb{Q}\xrightarrow{\cong}\bigoplus_{(K)}\Omega_*^K\{\mathcal{A}K,\mathcal{P}K\}^{W_K}\otimes\mathbb{Q}. \tag{21}$$

We would like to remark that the rational isomorphism obtained in Theorem 3.1 by localizing on fixed points holds in general for any rational G–equivariant homology theory whose coefficients form a rational G–Mackey functor [11, Theorem A.16; 32, Corollary 3.4.28].

3.1 Kernel of fixed-point homomorphism

In the unitary case, the equivariant bordism group $\Omega_*^{U,K}\{\mathcal{A}K,\mathcal{P}K\}$ is isomorphic to the unitary bordism group of a disjoint union of products of complex Grassmannians [35, Theorem 2.8]. Therefore, the group $\Omega_*^{U,K}\{\mathcal{A}K,\mathcal{P}K\}$ is a free Ω_*^U–module on even-dimensional generators. Hence, by Theorem 3.1, we obtain the following result:

Lemma 3.2 *The group of torsion elements in $\Omega_*^{U,G}$ is isomorphic to the kernel of the fixed-point homomorphism ϕ of* (18):

$$\mathrm{Tor}_{\mathbb{Z}}(\Omega_*^{U,G}) = \mathrm{Ker}(\phi_*^U). \tag{22}$$

Whenever a group G satisfies the evenness conjecture in equivariant unitary bordism, the fixed-point homomorphism ϕ_*^U is automatically a monomorphism. This is the case for abelian [20] and metacyclic [28] groups. In the next section we will show that there are groups G such that the kernel of the fixed-point homomorphism is not trivial in dimension 2, thus defining torsion elements in $\Omega_2^{U,G}$. This fact refutes the evenness conjecture in the general case.

In the oriented case there are many torsion classes in the bordism ring Ω_*^{SO}, all of order 2 [36; 33]. Therefore we will be mainly interested in the torsion classes of the equivariant bordism group $\Omega_*^{\mathrm{SO},G}$ which are trivial under the fixed-point homomorphism ϕ_*^{SO}.

A very interesting and more general question associated to the equivariant oriented case is the following:

> *Are there G–equivariant oriented manifolds whose bordism class vanishes under the fixed-point homomorphism ϕ^{SO} which do not bound equivariantly?*

In the next section we answer this question for dimension 2. The 3–dimensional case (with its interesting application to Chern–Simons theory) remains open for the interested reader.

Note that the equivariant bordism group $\Omega_*^{\mathrm{SO},K}\{\mathcal{A}K,\mathcal{P}K\}$ is in general more difficult to calculate than the unitary one. On the one hand, the fixed-point set M^K need not be orientable, and on the other, the normal bundles are classified by products of real, complex and quaternionic Grassmannians.

Since we are mainly interested in the 2– and 3–dimensional bordism groups, we know that all fixed points are of real codimension 0 or 2 in the unitary case because the normal bundles are endowed with a complex structure, see (2), and 0, 2 or 3 in the oriented case, because there are no 1–dimensional real representations preserving the orientation. Here the real codimension of the fixed points matches the real dimension of the representation of the respective isotropy group.

In the case that the fixed points are of real codimension 2, the normal bundle is of complex dimension 1 in the unitary case and of real dimension 2 in the oriented case. Since the 2–dimensional oriented representations can be parametrized by the 1–dimensional complex representations, we may denote by $\mathrm{Irr}^1_{\mathbb{C}}(K)$ the set of 1–dimensional nontrivial irreducible complex representations of the group K. The complex conjugation map on $\mathrm{Irr}^1_{\mathbb{C}}(K)$ acts freely on the representations of complex type $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}$ and acts trivially on the representations of real type $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{R}}$. Denote by $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}$ the quotient of representations of complex type by complex conjugation and by $\mathrm{Irr}^3_{\mathbb{R},\mathrm{SO}}(K)$ the set of 3–dimensional irreducible real representations of K in the category of oriented representations.

Proposition 3.3 *Let K be a finite group. Then the relative oriented equivariant bordism groups are*

$$\Omega_2^{\mathrm{SO},K}\{\mathcal{A}K,\mathcal{P}K\} = \Bigl(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}} \mathbb{Z}\Bigr) \oplus \Bigl(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{R}}} \mathbb{Z}/2\Bigr), \tag{23}$$

$$\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K,\mathcal{P}K\} = \bigoplus_{\mathrm{Irr}^3_{\mathbb{R},\mathrm{SO}}(K)} \mathbb{Z}/2, \tag{24}$$

and the relative equivariant unitary bordism groups are

$$\Omega_2^{U,K}\{\mathcal{A}K,\mathcal{P}K\} = \Omega_2^U \oplus \bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)} \mathbb{Z}, \tag{25}$$

$$\Omega_3^{U,K}\{\mathcal{A}K,\mathcal{P}K\} = 0. \tag{26}$$

Proof Let us begin with the relative oriented equivariant bordism groups. Any manifold M in $\Omega_*^{\mathrm{SO}.K}\{\mathcal{A}K,\mathcal{P}K\}$ is equivalent in the bordism group to the normal bundle N around the fixed-point set M^K [5, Lemma 5.2]. Whenever M is connected, of dimension 2 and $M \neq M^K$, this normal bundle is classified by a map

$$M^K \to \coprod_{\mathrm{Irr}^1_{\mathbb{C}}(K)} BU(1), \tag{27}$$

where the K action on the bundle around the point is encoded by the irreducible representation (here we are using that $\mathrm{SO}(2) \cong U(1)$). Note that whenever V is a nontrivial 1–dimensional complex representation, the unit ball $B(\mathbb{R}\oplus V)$ bounds the union of $B(V)$ and $B(\overline{V})$, where $\overline{V}$ denotes the representation V with reverse orientation. This implies that in the relative oriented bordism group $\Omega_2^{K,\mathrm{SO}}\{\mathcal{A}K,\mathcal{P}K\}$ we have the equation $B(V)+B(\overline{V})=0$. Hence whenever V is of complex type, and therefore V is not isomorphic to $\overline{V}$, the relative oriented bordism group $\Omega_2^{K,\mathrm{SO}}\{\mathcal{A}K,\mathcal{P}K\}$ counts the difference between the number of K–fixed points with normal bundle isomorphic to $\overline{V}$ and the number of K–fixed points with normal bundle isomorphic to V; these are the integral invariants. If V is of real type, and hence V is isomorphic to $\overline{V}$, the ball $B(\mathbb{R}\oplus V)$ bounds $B(V)$ twice, and the relative oriented bordism group $\Omega_2^{K,\mathrm{SO}}\{\mathcal{A}K,\mathcal{P}K\}$ counts the parity of the number of points with normal bundle isomorphic to V; these are the $\mathbb{Z}/2$ invariants. This argument proves (23).

For the 3–dimensional case, the codimension-2 fixed points become circles, and since $\Omega_1^{\mathrm{SO}}(BU(1)) = 0$, we conclude that we only need to focus our attention on the isolated points of the K action. Around each isolated fixed point of the action we obtain a 3–dimensional real and oriented representation V

of K. This representation is irreducible in the category of oriented representations even though it may be not irreducible as a real representation. Note that the splitting of the representation as the product of two nonoriented representations implies that one must be a sign representation and the other must factor through a dihedral representation in $O(2)$. Hence the product of these two representations will be equivalent to a representation that factors through an oriented dihedral representation in SO(3) which is irreducible in the category of oriented representations. Now, the unit ball $B(\mathbb{R} \oplus V)$ bounds $B(V)$ twice because V and $\overline{V}$ are isomorphic. Therefore we can conclude that the isomorphism (24) counts the parity of the number of fixed points of K with the prescribed representation on its normal bundle.

The relative unitary bordism groups are much simpler. The 3–dimensional case (26) is trivial because both Ω_3^U and $\Omega_1^U(BU(1))$ are trivial. The 2–dimensional case (25) detects half of the first Chern number of the surface whenever the action is trivial, and it counts the number of fixed points with prescribed representation on their normal bundle. Here we are using that the isomorphism $\Omega_2^U \xrightarrow{\cong} \mathbb{Z}$ is given by the assignment $[\Sigma] \mapsto \frac{1}{2}c_1(\Sigma)$ where $c_1(\Sigma)$ is the first Chern number of the surface. □

As a consequence of the previous result, the 2–dimensional bordism classes of interest have no isolated fixed points for any subgroup K of G.

Corollary 3.4 *The torsion subgroups of both unitary and oriented equivariant bordism of surfaces are respectively isomorphic to the kernels of the associated fixed-point homomorphism*,

$$\mathrm{Tor}_{\mathbb{Z}}(\Omega_2^{U,G}) = \mathrm{Ker}(\phi_2^U) \quad \textit{and} \quad \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^{\mathrm{SO},G}) = \mathrm{Ker}(\phi_2^{\mathrm{SO}}). \tag{28}$$

Therefore the equivariant bordism groups $\mathrm{Ker}(\phi_2^{\mathrm{SO}})$ *and* $\mathrm{Ker}(\phi_2^U)$ *are generated by G–surfaces without isolated K–fixed points for any subgroup K of G; in the unitary case it is moreover required that the surfaces have trivial first Chern number.*

Proof Proposition 3.3 shows that the relative oriented and unitary bordism groups $\Omega_2^K\{\mathcal{A}K, \mathcal{P}K\}$ are torsion-free for all subgroups K of G, except in the oriented case whenever K has 1–dimensional complex representations of real type; such representations come from nontrivial elements in $\mathrm{Hom}(K, \mathbb{Z}/2)$. Whenever a closed oriented surface Σ has one K–fixed point whose normal bundle has the structure of a nontrivial element in $\mathrm{Hom}(K, \mathbb{Z}/2)$, the connected component of such a K–fixed point has an induced action of $\mathbb{Z}/2$. Since the Euler characteristic of the connected component is even, the number of fixed points of this $\mathbb{Z}/2$–action must also be even. Hence the original action of K on this connected component must have an even number of fixed points, and all of them will have isomorphic complex representation of real type on the normal bundles.

The previous argument shows that the image of the fixed-point homomorphism is torsion-free in both oriented and unitary cases. Therefore by Theorem 3.1, we can conclude that the torsion classes are generated by G–equivariant manifolds without isolated K–fixed points for any subgroup K of G, and in the unitary case it is furthermore required that the underlying surface has trivial first Chern number. □

The presence of the platonic groups A_4, $\mathfrak{S}_4$, A_5 or the dihedral groups D_{2k} as subgroups of a general group G makes the understanding of the bordism group $\Omega_3^{\mathrm{SO},G}$ more interesting. We need first a definition:

Definition 3.5 Let M be a G–manifold (oriented or unitary). Define the *ramification locus* of the G–action as the space

$$\overline{M} := \bigcup_{\substack{K\subset G\\ K\neq\{1\}}} M^K, \tag{29}$$

where M^K denotes the space of fixed points of the subgroup K.

Let us start with the dihedral groups:

Proposition 3.6 *The equivariant bordism groups $\Omega_3^{\mathrm{SO},D_{2k}}$ are generated by equivariant manifolds whose fixed points are all of codimension 0 or empty. In particular, the fixed-point homomorphism ϕ_3^{SO} is trivial.*

Proof Take M a closed oriented D_{2k}–equivariant manifold such that M/D_{2k} is connected. Let us first assume that no element in D_{2k} besides the identity acts trivially (we could always take the induced action on M of the group D_{2k}/L, where L is the subgroup that acts trivially and consider M as a D_{2k}/L–equivariant manifold). Hence the ramification locus $\overline{M}$ is the union of 1–dimensional and 0–dimensional manifolds.

Whenever the fixed-point set $M^{D_{2k}}$ is nonempty, it will consist of a finite number of isolated points. We will argue that the number of fixed points with isomorphic normal representations is even, thus implying that the image of the localization map (16) at D_{2k}–fixed points

$$f_{D_{2k}} : \Omega_3^{\mathrm{SO},D_{2k}} \to \Omega_3^{\mathrm{SO},D_{2k}}\{\mathcal{A}D_{2k}, \mathcal{P}D_{2k}\} \tag{30}$$

is trivial, and moreover that the fixed-point set $M^{D_{2k}}$ could be removed with an equivariant cobordism by attaching handles around pairs of fixed points with isomorphic normal representation.

If x belongs to $M^{D_{2k}}$, we claim that there is another fixed point $x' \in M^{D_{2k}}$, such that both have isomorphic representations of D_{2k} on their normal neighborhoods. The reason for this is the following. Consider the class $[x] \in \overline{M}/D_{2k}$ on the quotient of the ramification locus $\overline{M}$. The connected component of the fixed-point set of the cyclic subgroup $\mathbb{Z}/k$ around x defines a path on the quotient $\overline{M}/D_{2k}$ with the class $[x]$ at one end. Since $\overline{M}/D_{2k}$ is compact, the other end of this path ends at the class of the point $[x']$, where we have chosen x' to be on the same connected component as x on the fixed-point set $M^{\mathbb{Z}/k}$. The D_{2k} representations around x and x' are isomorphic because their restrictions to the group $\mathbb{Z}/k$ give representations with opposite orientations.

Note that whenever $k > 2$, the points x and x' are different. When $k = 2$ it could be the case that $x = x'$, and if this were the case, around $[x]$ in $\overline{M}/D_{2k}$ we would have a loop (the path we defined above from x to $x' = x$) and an extra path leaving from it. Following this third path from x, we will reach another point x'', which will be different from x.

We just have shown that the fixed points in $M^{D_{2k}}$ come in pairs with isomorphic representations. If the isomorphic representation is V and $B(V)$ denotes the unit ball in V, this pair of points could removed by the bordism that adds the handle $[0,1]\times B(V)$ on the normal neighborhoods of the pair of points.

The previous construction could be carried out on all the fixed points of the conjugacy classes of subgroups which are of dihedral type, and therefore we see that M is equivariantly cobordant to a manifold M' whose fixed points of its dihedral subgroups are empty. Hence the ramification locus $\overline{M'}$ is a 1–dimensional manifold, and therefore $\phi_3^{\mathrm{SO}}([M])=\phi_3^{\mathrm{SO}}([M'])=0$.

We could then choose as generators of $\Omega_3^{\mathrm{SO},D_{2k}}$ manifolds without 0– and 1–dimensional fixed points. □

Propositions 3.3 and 3.6 imply that the fixed-point homomorphism ϕ_3^{SO} is trivial on subgroups isomorphic to cyclic or dihedral groups. Nevertheless, the fixed-point homomorphism may be nontrivial when evaluated on subgroups isomorphic to the platonic groups A_4, $\mathfrak{S}_4$ and A_5. To understand the image of ϕ_3^{SO} for the platonic groups, we first need to define the *blowup* of a representation.

Definition 3.7 Let V be a finite-dimensional real G–representation. The *blowup* $\gamma(V)$ of V is the total space of the bundle of real lines $\mathbb{P}(V)$ of V,

$$\gamma(V):=\{(v,L)\in V\times\mathbb{P}(V)\mid v\in L\},\tag{31}$$

endowed with the natural G action: $g\cdot(v,L):=(gv,gL)$. Denote by $B(\gamma(V))$ and $S(\gamma(V))$ the unit ball and sphere bundles of $\gamma(V)$, respectively.

Note that the sphere bundle of $\gamma(V)$ and the sphere of the representation $S(V)$ are canonically isomorphic:

$$\rho\colon S(V)\xrightarrow{\cong}S(\gamma(V)),\quad v\mapsto(v,\langle v\rangle).\tag{32}$$

So one may glue $B(\overline{V})$, where $\overline{V}$ is V with the opposite orientation, to $B(\gamma(V))$ along their boundary,

$$Y(V):=B(\overline{V})\cup_\rho B(\gamma(V)),\tag{33}$$

thus constructing a closed oriented G–manifold.

What is interesting about the blowup is that, for faithful 3–dimensional oriented real representations V, the blowup $\gamma(V)$ only contains points with cyclic or dihedral isotropy groups. This is a key fact that will be used in what follows.

Proposition 3.8 *Let G be a finite subgroup of* $\mathrm{SO}(3)$. *Then the fixed-point homomorphism ϕ_3^{SO} is only nontrivial on subgroups isomorphic to the platonic groups A_4, $\mathfrak{S}_4$ and A_5. Moreover, its restriction*

$$\phi_3^{\mathrm{SO}}\colon\Omega_3^{\mathrm{SO},G}\to\bigoplus_{\substack{(K)\\ K\ \text{platonic}}}\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K,\mathcal{P}K\}^{W_K}\tag{34}$$

is surjective.

Proof Let (K) be a conjugacy class of subgroups of G with K isomorphic to any of the platonic groups A_4, $\mathfrak{S}_4$ or A_5. Denote by V_K the 3–dimensional real representation induced by the symmetries of the respective platonic solid. Note that V_K is isomorphic to the representation with the reverse orientation $\overline{V}_K$, and therefore the closed oriented K–manifold $Y(V_K)$ defined in (33) is diffeomorphic to $B(V_K) \cup_\rho B(\gamma(V_K))$. Note furthermore that $\Omega_3^K\{\mathcal{A}K, \mathcal{P}K\} \cong \mathbb{Z}/2$ since V_K is the only irreducible representation of dimension 3.

The localization map at K–fixed points of (16)

$$f_K \colon \Omega_3^{\mathrm{SO},K} \to \Omega_3^K\{\mathcal{A}K, \mathcal{P}K\} \cong \mathbb{Z}/2 \tag{35}$$

maps $Y(V_K)$ to the normal bundle of its K–fixed points $Y(V_K)^K$. Since the blowup $\gamma(V_K)$ has no K–fixed points, $Y(V_K)^K = B(V_K)^K$ and the fixed-point set consists of only one point. Hence $f_K(Y(V_K)) = B(V_K)$ with $[B(V_K)]$ the generator of the group $\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}$.

The commutativity of the diagram

$$\begin{tikzcd}
\Omega_3^{\mathrm{SO},K} \arrow[r, "i_K^G"] \arrow[d, "f_K"'] & \Omega_3^{\mathrm{SO},G} \arrow[d, "f_K \circ r_K^G"] \\
\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\} \arrow[r, "i_K^{N_G K}"'] & \Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}^{W_K}
\end{tikzcd} \tag{36}$$

where $i_H^L \colon \Omega_*^H \to \Omega_*^L$ given by $[M] \mapsto [L \times_H M]$ is the induction map for the inclusion of groups $H \subset L$, implies that the manifold $f_K \circ r_K^G(G \times_K Y(V_K))$ generates the group $\Omega_3^{\mathrm{SO},K}\{\mathcal{A}K, \mathcal{P}K\}^{W_K}$.

Note that whenever $K \subsetneq K'$, we have $(G \times_K Y(V_K))^{K'} = \varnothing$. Therefore we conclude that the images under ϕ_3^{SO} of the G–manifolds $G \times_K Y(V_K)$, where (K) runs over the conjugacy classes of platonic subgroups of G, provide the desired surjectivity. □

Let us see the previous result in an example. Let $G = A_5$ and take the A_5–manifolds $Y(V_{A_5})$ and $A_5 \times_{A_4} Y(V_{A_4})$ in $\Omega_3^{\mathrm{SO},A_5}$. The images under ϕ_3^{SO} of these two manifolds in

$$\Omega_3^{\mathrm{SO},A_5}\{\mathcal{A}A_5, \mathcal{P}A_5\} \oplus \Omega_3^{\mathrm{SO},A_4}\{\mathcal{A}A_4, \mathcal{P}A_4\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \tag{37}$$

are $(1,1)$ and $(0,1)$, respectively. The surjectivity of (34) follows.

3.2 Surfaces without isolated fixed points for any subgroup

Let $\mathcal{F}$ be a family of subgroups in G. Then denote by $\overline{\Omega}_2^G\{\mathcal{F}\}$ the subgroup of $\Omega_2^G\{\mathcal{F}\}$ generated by manifolds without isolated K–fixed points for all $K \in \mathcal{F}$, and whose underlying first Chern number is zero in the unitary case. Since Corollary 3.4 also implies that

$$\overline{\Omega}_2^G\{\mathcal{F}\} = \mathrm{Ker}(\phi_2|_{\Omega_2^G\{\mathcal{F}\}}) = \mathrm{Tor}_{\mathbb{Z}}(\Omega_2^G\{\mathcal{F}\}), \tag{38}$$

we may study the properties of $\overline{\Omega}_2^G$ restricted to families.

Lemma 3.9 *Let $\{\mathcal{F},\mathcal{F}'\}$ be an adjacent pair of families differing by the conjugacy class (K) of the subgroup $K\subset G$. Then the canonical map of bordism groups for families $\overline{\Omega}{}_2^G\{\mathcal{F}'\}\to\overline{\Omega}{}_2^G\{\mathcal{F}\}$ fits into the split exact sequence*

$$\overline{\Omega}{}_2^G\{\mathcal{F}'\}\to\overline{\Omega}{}_2^G\{\mathcal{F}\}\to\widetilde{\Omega}_2(BW_K)\to 0, \tag{39}$$

with $\widetilde{\Omega}_2$ the reduced bordism groups.

Proof A generator in $\overline{\Omega}{}_2^G\{\mathcal{F}\}$ not in the image of $\overline{\Omega}{}_2^G\{\mathcal{F}'\}$ is represented by a G–connected manifold M such that the fixed-point set M^K is a closed nonempty surface without boundary, and such that there is a G–equivariant homomorphism $G\times_{N_GK}M^K\xrightarrow{\cong}M$ given by $[(g,m)]\mapsto gm$. The closed surface M^K is endowed with a free action of the group W_K, thus producing a unique map up to homotopy $M^K/W_K\to BW_K$. The induction map

$$\widetilde{\Omega}_2(BW_K)\to\overline{\Omega}{}_2^G\{\mathcal{F}\}\quad\text{given by }\ L\mapsto G\times_{N_GK}L \tag{40}$$

produces the desired section.

For the unitary case we need only to see that the first Chern number of M is zero, if and only if the first Chern number of M^K is zero, if and only if the first Chern number of M^K/W_K is zero.

Here we have used the isomorphism

$$\widetilde{\Omega}{}_2^U(BW_K)\cong\operatorname{Ker}(\Omega_2^U(BW_K)\to\Omega_2^U), \tag{41}$$

where the forgetful map $\Omega_2^U(BW_K)\to\Omega_2^U$ simply takes a framed bordism $[\Sigma\to BW_K]$ and maps it to $[\Sigma]$. The kernel consists of framed surfaces whose underlying first Chern number is zero. In the oriented case $\widetilde{\Omega}{}_2^{\mathrm{SO}}(BW_K)=\Omega_2^{\mathrm{SO}}(BW_K)$. □

4 Bounding equivariant surfaces

In this section we present our main result, which is the calculation of the groups $\overline{\Omega}{}_2^G$. To do this we use the Conner–Floyd spectral sequence of Section 2.4 associated to the families of subgroups

$$\{1\}=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_l=\mathcal{A}G, \tag{42}$$

where all the pairs are adjacent, ie $\mathcal{F}_j-\mathcal{F}_{j-1}=(K_j)$ for some conjugacy class of subgroups (K_j), and such that the conjugacy classes (K_j) span all conjugacy classes of subgroups of G (and hence $l+1$ is the number of conjugacy classes of subgroups of G).

We may filter the group $\overline{\Omega}{}_2^G$ by the subgroups

$$F_p\overline{\Omega}{}_2^G:=\operatorname{Im}(\overline{\Omega}{}_2^G\{\mathcal{F}_p\}\to\overline{\Omega}{}_2^G) \tag{43}$$

whose associated graded groups are the quotients

$$\operatorname{Gr}_p\overline{\Omega}{}_2^G=F_p\overline{\Omega}{}_2^G/F_{p-1}\overline{\Omega}{}_2^G. \tag{44}$$

The commutative diagram with exact rows

$$
\begin{tikzcd}
\Omega_3^G\{\mathcal{A}G, \mathcal{F}_{p-1}\} \arrow[r] \arrow[d] & \overline{\Omega}_2^G\{\mathcal{F}_{p-1}\} \arrow[r] \arrow[d] & \overline{\Omega}_2^G \arrow[d, equal] \\
\Omega_3^G\{\mathcal{A}G, \mathcal{F}_p\} \arrow[r] & \overline{\Omega}_2^G\{\mathcal{F}_p\} \arrow[r] & \overline{\Omega}_2^G
\end{tikzcd} \tag{45}
$$

together with the result of Lemma 3.9 implies that the following sequence is exact:

$$\Omega_3^G\{\mathcal{A}G, \mathcal{F}_p\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p}) \to \mathrm{Gr}_p\, \overline{\Omega}_2^G \to 0. \tag{46}$$

We therefore need to understand the image of the boundary map

$$\Omega_3^G\{\mathcal{A}G, \mathcal{F}_p\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p}) \tag{47}$$

in order to determine the groups $\mathrm{Gr}_p\, \overline{\Omega}_2^G$.

Note that the image of the boundary map (47) is equivalent to the image of the boundary map

$$\Omega_3^{W_{K_p}}\{\mathcal{A}W_{K_p}, \{1\}\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p}). \tag{48}$$

This follows from the fact that the manifolds of interest will have trivial actions of the groups in the conjugacy class (K_p), and then one follows the same argument as presented in Lemma 3.9. Therefore we obtain the following result:

Lemma 4.1 *Consider the associated graded groups* $\mathrm{Gr}_*\, \overline{\Omega}_2^G$ *of* $\overline{\Omega}_2^G$ *induced by the families of subgroups presented in* (42). *Then*

$$\mathrm{Gr}_p\, \overline{\Omega}_2^G \cong \mathrm{Coker}(\Omega_3^{W_{K_p}}\{\mathcal{A}W_{K_p}, \{1\}\} \xrightarrow{\partial} \widetilde{\Omega}_2(BW_{K_p})). \tag{49}$$

Hence we need to understand which surfaces with free actions equivariantly bound.

4.1 Bounding free actions on surfaces

It turns out that the only free actions on surfaces that equivariantly bound are those on which the quotient surface is a torus. This result is originally due to the second author [29; 30] whenever the group G does not contain any subgroup isomorphic to the platonic groups $A_4, \mathfrak{S}_4, A_5$ or to the dihedral groups D_{2k}, and it motivated our investigation. Here we will produce an alternative proof, generalizing it for all finite groups. Let us first recall the definition of the Bogomolov multiplier of a finite group.

The cohomology group $H^2(G, \mathbb{C}^*)$ determines the isomorphism classes of central $\mathbb{C}^*$ group extensions of G, and therefore complex irreducible projective representations of the group G define elements in $H^2(G, \mathbb{C}^*)$. Schur [31] extensively studied this cohomology group, and therefore it was called the *Schur multiplier* of G [16].

Bogomolov [3] defined the subgroup $B_0(G)$ of the Schur multiplier consisting of all elements which vanish when restricted to all its abelian subgroups:

$$B_0(G) = \bigcap_{A \subset G \text{ abelian}} \mathrm{Ker}(\mathrm{res}_A^G : H^2(G, \mathbb{C}^*) \to H^2(A, \mathbb{C}^*)). \tag{50}$$

The interest in this group comes from, among other things, a result Bogomolov [3, Theorem 3.1] which states that whenever the field of G–invariants $\mathbb{C}[G]^G$ of the rational field $\mathbb{C}[G]$ is rational over $\mathbb{C}$, the Bogomolov multiplier of the group G vanishes.

Using the fact that for finite groups $H^2(G, \mathbb{C}^*) \cong H_2(G, \mathbb{Z})$, Moravec [22] showed that the Bogomolov multiplier group $B_0(G)$ is isomorphic to the group

$$\widetilde{B}_0(G) := H_2(G, \mathbb{Z})/M_0(G), \tag{51}$$

where $M_0(G)$ is the subgroup of $H_2(G, \mathbb{Z})$ generated by the images

$$\mathrm{Im}(H_2(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}) \to H_2(G, \mathbb{Z})) \tag{52}$$

of all homomorphisms $\mathbb{Z} \times \mathbb{Z} \to G$. This homology version of the Bogomolov multiplier was then used to calculate $B_0(G)$ for several types of finite groups [22].

In this homological form, the Bogomolov multiplier appeared much earlier in [24] in connection with SK–groups (cutting and pasting of manifolds) and in [25] as SK_1 in algebraic K-theory.

Using now the fact that there are canonical isomorphisms

$$\widetilde{\Omega}_2^U(BG) \xrightarrow{\cong} \Omega_2^{\mathrm{SO}}(BG) \xrightarrow{\cong} H_2(BG, \mathbb{Z}), \tag{53}$$

we present a generalization of a result which was established by the second author in [30]. First we need a lemma:

Lemma 4.2 *Let Σ be an oriented surface with free G–action that bounds equivariantly. Then Σ can be extended to an oriented G–manifold whose ramification locus is a* 1*–dimensional manifold* (*all the isotropy groups are all cyclic*).

Proof Let M be an oriented G–manifold whose boundary is the surface with free G–action Σ. Take a point x in the ramification locus $\overline{M}$ and denote by G_x its isotropy group. Since the G–action is free on the boundary, the action of G_x on the normal neighborhood of x must induce an injective homomorphism $G_x \to \mathrm{SO}(3)$. Hence G_x must be isomorphic to a cyclic group, a dihedral group or any of the platonic groups A_4, $\mathfrak{S}_4$ or A_5. Whenever G_x is cyclic, x is a smooth point in the ramification locus $\overline{M}$, because locally G_x acts by rotations. Whenever G_x is neither trivial nor cyclic, x is a singular point on the ramification locus. Simply note that the irreducible and oriented 3–dimensional representations of the dihedral and the platonic groups have the origin as a singular point. Therefore the obstruction for the ramification locus $\overline{M}$ to be a 1–dimensional manifold is the presence of points whose isotropy groups are isomorphic to the dihedral or the platonic groups (A_4, $\mathfrak{S}_4$ and A_5). Our goal is to modify M to build a new manifold without any such isotropies.

We briefly outline the overall strategy of our desingularizing process. There are three steps:

(i) Perform the blowup construction on the normal neighborhoods of the points whose isotropies are isomorphic to either A_5, $\mathfrak{S}_4$ or A_4; this produces a new manifold M' with the same boundary as M and no points with A_5, $\mathfrak{S}_4$ or A_4 isotropy.

(ii) In M', "cancel" as many pairs of distinct orbits of a given dihedral isotropy type possible; our cancellation method results in a manifold M'' that is equivariantly cobordant to M', relative to the boundary Σ. By canceling as many pairs as possible, we guarantee that for the action of G on M'', a given conjugacy class of a dihedral subgroup of G occurs on at most one orbit in M''.

(iii) The final step is the hardest. If x is a point in M'' with dihedral isotropy $G_x \leq G$ that is maximal, we show that the action of G on $G\cdot x$ possesses an "involutive" element g such that $y = gx \neq x$, $g^2x = x$, $G_y = G_x$ and g commutes with a preferred rotation $\sigma \in G_x$. We then classify the possibilities for $\langle G_x, g\rangle$, and build an appropriate equivariant handle that desingularizes the orbit $G\cdot x$. Inductively applying this construction to all dihedrally stabilized points, we arrive at our desired manifold M'''. In fact, this is oversimplifying; we must return to (ii) once at some point in this process, but the basic idea is as described.

Let us expand on (i). Take a point $x \in M$ whose isotropy G_x is isomorphic to A_5 (we will start with the larger isotropy first). Let N_x be a normal G_x–neighborhood of x such that $N_x \cap g\cdot N_x = \varnothing$ for all $g \in G - G_x$, and let

$$\sigma\colon B(V_{G_x}) \xrightarrow{\cong} N_x \tag{54}$$

be a G_x–equivariant diffeomorphism with V_{G_x} the faithful representation of G_x around x. Take $G\cdot N_x$ as a G–equivariant neighborhood around the orbit $G\cdot x$ and note that σ induces a G–equivariant diffeomorphism $G\times_{G_x} B(V_{G_x}) \xrightarrow{\cong} G\cdot N_x$. Construct the blowup $B(\gamma(V_{G_x}))$ presented in Definition 3.7 and note that the sphere bundles are G_x–diffeomorphic to the boundary of N_x:

$$S(\gamma(V_{G_x})) \cong S(V_{G_x}) \xrightarrow{\cong} \partial N_x. \tag{55}$$

Cut $G\cdot N_x$ from M and glue $G\times_{G_x} B(\gamma(V_{G_x}))$ along the boundary using the diffeomorphism σ. Define the new G–manifold

$$M' := (M - G\cdot N_x) \cup_{\partial(G\cdot N_x)} G\times_{G_x} B(\gamma(V_{G_x})) \tag{56}$$

and note that M' has the same boundary as M, but with the property that inside ∂N_x there are no more points with isotropy isomorphic to A_5. Cutting and pasting the blowups for every point with isotropy isomorphic to A_5 produces a manifold without points whose isotropy is isomorphic to A_5. Then a similar blowup procedure is carried out for points with isotropy isomorphic to $\mathfrak{S}_4$, and then to points with isotropy isomorphic to A_4. The resulting manifold M' has the same boundary as M, but it does not contain points with isotropy isomorphic to A_5, $\mathfrak{S}_4$ or A_4. The only isotropies that appear on M' are cyclic or dihedral groups. This concludes (i). (We note for the interested reader that M and M' are not necessarily relatively cobordant, even though they do have the same boundary.)

For (ii), suppose x and y are two points in M' with equal dihedral stabilizers $G_x = G_y$ such that x and y are not in the same G orbit, but the representation of G_x on a regular neighborhood of x is equivalent to the representation of $G_y = G_x$ on a regular neighborhood of y. Call this representation V. Choose local charts around x and y such that the angle of rotations of the elements in G_x agree in both charts to the angles of rotations in V. Now simply attach an equivariant 4–dimensional handle of the form $[0,1]\times(G\times_{G_x} B(V))$ to the equivariant neighborhood of $\{x,y\}$. Note that the only points in this handle with noncyclic isotropy are those in $G\cdot\{x,y\}$. Thus, the cobordant 3–manifold (where the open G–equivariant regular neighborhood of $G\cdot\{x,y\}$ is replaced with the vertical boundary of our handle) has fewer points with isotropy isomorphic to dihedral groups. Iterate this procedure, attaching such an equivariant handle anytime we see a pair x and y with the same isotropy group $G_x=G_y$, isomorphic local representations, and $y\notin G\cdot x$, until there are no more such pairs. Call the resulting manifold M''. Of course, since these handle attachments occur away from $\partial M'=\Sigma$, M'' still has boundary Σ. More precisely, M' and M'' are equivariantly bordant relative to Σ.

Step (iii) is the most involved. Let x be a point in M'' with dihedral isotropy $G_x\cong D_{2k}$. Define the set Λ_x of points y in M'' such that $G_y=G_x$ and whose local representations on regular neighborhoods around y and x, respectively, are isomorphic. By construction, if y is any other point in this set, then $y\in G\cdot x$. In fact, $\Lambda_x=N_G(G_x)\cdot x$ and Λ_x is bijective with the group $N_G(G_x)/G_x$. Our goal now is to build a G–handle that allows us to cancel the singularities in the *single* orbit $G\cdot x$ with one another in pairs in a G–equivariant fashion. In particular, we will need to show that $|G\cdot x|$ is even and admits a G–invariant matching.

Let us first show that there is an element $g\in N_G(G_x)$ such that its projection on $N_G(G_x)/G_x$ has order two. This g will allow us to define $y:=gx$ such that $x=gy$. We proceed by induction down the subgroup lattice of G and through the different isomorphism classes of faithful local representations of these point stabilizers. Let x be any point in M'' whose stabilizer $G_x\cong D_{2k}$ is maximal among all dihedral point stabilizers in M'' (with respect to subgroup inclusion) and consider the restricted action of just G_x on M''.

If $k>2$, let $\sigma\in G_x$ be an element of order k and take its fixed-point set $(M'')^\sigma$. Note that this fixed-point set is a disjoint union of embedded circles. The group $\mathbb{Z}_2\cong G_x/\langle\sigma\rangle$ acts on $(M'')^\sigma$ and the set of fixed points is precisely Λ_x. The Euler characteristic of $(M'')^\sigma$ being zero implies that Λ_x has an even number of points. Since $|\Lambda_x|=|N_{G_x}(G_x)/G_x|$, the group $N_{G_x}(G_x)/G_x$ has an element of order 2, and therefore we may choose $g\in N_{G_x}(G_x)$ as a lift of this element of order 2. If $y:=gx$, then by construction $x=gy$.

If $k=2$, then there is only one isomorphism class of local representations. Let

$$\Gamma_{G_x}=\{p\in M''\mid \mathrm{Stab}_{G_x}(p)\neq\{1\}\}$$

be the ramification locus of this action. Then Γ_{G_x} is a properly embedded topological graph. Because the action of G_x on $\partial M''=\Sigma$ is free, $\Gamma_{G_x}\cap\Sigma$ is empty, and so in particular every vertex in this graph has valence 6. The quotient graph Γ_{G_x}/G_x resides in the quotient manifold M''/G_x, and its vertices are in bijection with the vertices of Γ_{G_x}, and hence in bijection with the points in Λ_x. The vertices of Γ_{G_x}/G_x all have valence 3. Since twice the number of edges equals three times the number of vertices, we see that each connected component of Γ_{G_x}/G_x has an even number of vertices, and hence the same is true for each connected component of Γ_{G_x}. Note that this implies $|\Lambda_x|=|N_{G_x}(G_x)/G_x|$ is even. Now, as in the previous paragraph, we again choose $g\in N_{G_x}(G_x)$ lifting an element of order 2 in $N_{G_x}(G_x)/G_x$ and take $y:=gx$ with $x=gy$.

In both cases $k > 2$ and $k = 2$, note that $g^2 \in G_x$, and therefore the conjugation action of g on G_x squares to an inner automorphism of G_x. This is especially helpful when $k = 2$, ie when $G_x \cong D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, since in this case $\mathrm{Inn}(G_x) = 0$, and we can conclude that g acts on G_x by an automorphism of order either 1 or 2 (never 3).

We now specify a preferred "rotation subgroup generator" of G_x. When $k = 2$ and g conjugates G_x by a nontrivial automorphism (necessarily of order 2, as just discussed above), then we take σ in G_x to be the unique nontrivial element of G_x fixed by conjugation with g. If g conjugates G_x trivially and there is not a loop in Γ_{G_x}/G_x at x, then we take σ to be an arbitrarily chosen element of G_x; if there is a loop at x, then we take σ to correspond to the unique element in G_x that does not stabilize points in the preimage of the interior of the loop (here the preimage can be taken with respect to $\Gamma_{G_x} \to \Gamma_{G_x}/G_x$). When $k > 2$, our preferred σ is given essentially for free: pick either one of the two nontrivial elements of G_x with minimal (unsigned) rotation angle (in its action on N_x) and call it σ.

With these choices for σ, in either the $k = 2$ or $k > 2$ case we may parametrize G_x as $G_x = \langle \sigma, \alpha \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1} \rangle \cong D_{2k}$ for some arbitrarily chosen "reflection" α in G_x. We also know that g commutes with σ whenever $k = 2$ (because of how we picked σ), but when $k > 2$ it may be that $g\sigma g^{-1} = \sigma^{-1}$ since the local representations around x and y are isomorphic. If this were the case, replace g by αg and note that αg commutes with σ. Therefore we have found $g \in N_G(G_x)$ with $gx = y \neq x,\ gy = x$ and $g\sigma g^{-1} = \sigma$.

This in turn implies the following essential facts:

- When $k > 2$, no matter how σ conjugates G_x, we must have that $g^2 = \sigma^l$ for some $0 \le l \le k-1$.
- When $k = 2$, σ must conjugate $G_x = \{1, \sigma, \alpha, \sigma\alpha\} \cong D_4$ by an automorphism that leaves σ invariant. Thus σ either commutes with all of G_x, or else swaps α and $\sigma\alpha$.
  - If g swaps α and $\sigma\alpha$, notice that g does not commute with α or $\alpha\sigma$, and hence g^2 (which *does* commute with g) cannot equal α or $\alpha\sigma$. In other words, when g acts nontrivially on G_x, then we must have $g^2 = 1$ or $g^2 = \sigma$.
  - If g commutes with all of G_x, then in principle g^2 might equal any element of D_4. We will see below that in fact the only possibility is $g^2 = 1 \in D_4$.

Consider the group

$$K := \mathrm{Stab}_G\{x, y\} = \mathrm{Stab}_{N_G(G_x)}\{x, y\} = \langle \sigma, \alpha, g \rangle. \tag{57}$$

Notice that by construction, if h is any element of G such that $hx = y$, then in fact $h \in K$. (This is because $hx = gx$ implies $g^{-1}h \in G_x = G_y$, and hence $g^{-1}h \in G_x$ and so $h \in gG_x \subset K$.) In other words, we can build a G–equivariant matching on the orbit $G \cdot x$ by taking $\{x, y\}$ to be one pair in the matching, and inducing up to the entire orbit; the stabilizer of any edge in this matching is then conjugate to G_x. Therefore, if we can build a K–equivariant handle that allows us to desingularize the action of K on $N_x \sqcup N_y$, then we may induce this to a well-defined G–equivariant handle that desingularizes the action of G on the entire orbit $G \cdot N_x$.

We will now classify the possibilities for how K acts on $N_x \sqcup N_y$, and build nonsingular handles for each possibility. This will involve some casework, some of which depends on the integer $k \geq 2$ such that $G_x \cong D_{2k}$, and our success depends critically on the established fact that g commutes with σ.

Notice that K sits in an exact sequence

$$1 \to G_x \to K \to K/G_x \to 1, \tag{58}$$

where $K/G_x = \langle g \bmod G_x \rangle = \mathbb{Z}_2$. Recall that equivalence classes of such extensions can be placed in (noncanonical) bijection with the following pairs of data: homomorphisms $f: \mathbb{Z}_2 \to \mathrm{Out}(G_x)$ such that a certain canonically associated class in $H^3(\mathbb{Z}_2; Z(G_x))$ vanishes, together with a class $\omega \in H^2_f(\mathbb{Z}_2; Z(G_x))$, where $\mathbb{Z}_2$ acts on the coefficients $Z(G_x)$ in a manner induced by f.

However, not all homomorphisms $f: \mathbb{Z}_2 \to \mathrm{Out}(G_x)$ will be pertinent to our situation, because (except in the case $k = 2$ and g commutes with G_x), we already know that $g^2 = \sigma^l$ for some $0 \leq l \leq k-1$. Let us use some group cohomology to constrain the possibilities for K when $k > 2$.

If k is odd, then $Z(G_x) = \{0\}$, and so $H^2(K/G_x; Z(G_x)) = \{0\}$ and there is only one thing K could possibly be given that $g^2 = \sigma^l$, namely

$$K = \langle \sigma, \alpha, g \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1}, g\sigma g^{-1} = \sigma, g\alpha g^{-1} = \alpha\sigma^l, g^2 = \sigma^l \rangle \tag{59}$$

where $0 \leq l \leq k-1$.

If $k > 2$ is even, then $Z(G_x) = \langle \sigma^{k/2} \rangle \cong \mathbb{Z}_2$ and the action of K/G_x on the coefficients $\mathbb{Z}_2$ must be trivial no matter what l is; therefore $H^2(K/G_x; Z(G_x)) \cong \mathbb{Z}_2$ and we should expect two nonequivalent extensions for a given l. These are precisely:

$$K_* = \langle \sigma, \alpha, g \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1}, g\sigma g^{-1} = \sigma, g\alpha g^{-1} = \alpha\sigma^l, g^2 = \sigma^l \rangle,$$
$$K_\dagger = \langle \sigma, \alpha, g \mid \sigma^k = \alpha^2 = 1, \alpha\sigma\alpha = \sigma^{-1}, g\sigma g^{-1} = \sigma, g\alpha g^{-1} = \alpha\sigma^{l+(k/2)}, g^2 = \sigma^l \rangle,$$

where $0 \leq l \leq k-1$.

If $k = 2$, rather than use group cohomology to give an upper bound on the possibilities for K, we simply list the six known possibilities so far:

$$\begin{aligned}
K_1 &= \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha\sigma,\ g^2 = 1 \rangle \\
&= \langle \alpha, g\alpha \mid (g\alpha)^4 = \alpha^2 = 1,\ \alpha(g\alpha)\alpha = (g\alpha)^{-1} \rangle \cong D_8, \\
K_2 &= \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = \sigma \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4, \\
K_3 &= \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = \alpha \rangle \cong K_2, \\
K_4 &= \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = \sigma\alpha \rangle \cong K_2, \\
K_5 &= \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha\sigma,\ g^2 = \sigma \rangle = K_* \text{ (for } l = 1) \cong D_8, \\
K_6 &= \langle \sigma, \alpha, g \mid \sigma^2 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma,\ g\sigma g^{-1} = \sigma,\ g\alpha g^{-1} = \alpha,\ g^2 = 1 \rangle = K_* \text{ (for } l = 0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.
\end{aligned}$$

Each group has order 8 as needed, so no further constraints are required to specify K, beyond which of the two possible automorphisms fixing σ we have g act on G_x by, and which element $\sigma^2 \in G_x$ is.

All of the above listed possibilities for K are based on naive algebra. An algebraic classification of the possibilities for K is not immediately equivalent to a geometric classification of the different possible *faithful representations* of these K with $K \to \mathrm{Iso}_+(B^3 \sqcup B^3)$, which are, after all, what we need to desingularize. In particular, we will see that K_1 and K_2 (and therefore K_3 and K_4) in the case $k=2$ have *no* faithful representations into $\mathrm{Iso}_+(B^3 \sqcup B^3)$.

To understand how these possible abstract structures of the extension K relate to the geometry of the action of K on $N_x \sqcup N_y \cong B^3 \sqcup B^3$, we parametrize so that G_x acts on each copy of $B^3 \subset \mathbb{R}^3$ in the same standard way:

$$\sigma(x,y,z) = \Big(\cos\Big(\frac{2\pi l}{k}\Big)x - \sin\Big(\frac{2\pi l}{k}\Big)y, \sin\Big(\frac{2\pi l}{k}\Big)x + \cos\Big(\frac{2\pi l}{k}\Big)y, z\Big), \tag{60}$$

$$\alpha(x,y,z) = (x,-y,-z). \tag{61}$$

Here $0 < l < k$ and l is coprime with k.

Note that this standard action of D_{2k} on B^3 is unique up to a sign, meaning any two faithful representations $D_{2k} \to \mathrm{SO}(3)$ that have σ acting by rotation angle $\pm 2\pi l/k$ are related by a conjugacy in $\mathrm{SO}(3)$. With this, we see that the equivalence class of an isometric action of K on $N_x \sqcup N_y$ — *when it exists* — is entirely determined (in the relevant sense, namely, up to conjugation by $\mathrm{Iso}_+(N_x \sqcup N_y)$) by the representation of G_x on either component, and the diffeomorphism affected when g swaps N_x and N_y.

We will now show that none of K_1–K_4 in the $k=2$ case above are geometrically realizable. It suffices to show this for K_1 and K_2. In both cases, we assume without loss of generality that σ and α act on $B^3 \sqcup B^3$ in the standard way shown above.

For K_1 the only possibilities for g are $g(x,y,z) = (-y,x,z)$, $g(x,y,z) = (y,-x,z)$, $g(x,y,z) = (-y,-x,-z)$ or $g(x,y,z) = (y,x,-z)$, since g must leave the z–axis fixed and swap the x– and y–axes. The first two contradict $g^2 = 1$. The second two contradict that g commutes with α.

For K_2, since g commutes with all three generators, it leaves each axis fixed, and the possible actions are exactly $g(x,y,z) = (x,-y,-z)$, $g(x,y,z) = (-x,y,-z)$ or $g(x,y,z) = (-x,-y,z)$. None of these squares to σ.

Finally, we will show that all remaining K are geometrically realizable while simultaneously achieving our most important goal: a description of the desingularizing handle we are after for each possibility.

For each of them, we may define a faithful 4–dimensional real representation of $K = \langle \sigma, \alpha, g\rangle$ as follows:

$$\sigma(x,y,z,t) = \Big(\cos\Big(\frac{2\pi l}{k}\Big)x - \sin\Big(\frac{2\pi l}{k}\Big)y, \sin\Big(\frac{2\pi l}{k}\Big)x + \cos\Big(\frac{2\pi l}{k}\Big)y, z, t\Big), \tag{62}$$

$$\alpha(x,y,z,t) = (x,-y,-z,t), \tag{63}$$

$$g(x,y,z,t) = \Big(\cos\Big(\frac{\pi j}{k}\Big)x - \sin\Big(\frac{\pi j}{k}\Big)y, \sin\Big(\frac{\pi j}{k}\Big)x + \cos\Big(\frac{\pi j}{k}\Big)y, -z, -t\Big). \tag{64}$$

Here $j = l$ if k is odd, and $j = l$ or $l + \big(\frac{1}{2}k\big)$ if $k > 2$ is even and $K = K_*$ or $K = K_\dagger$, respectively. For the groups K_5 and K_6 we take $j = 1$ and $j = 2$, respectively. Note that we are continuing to assume (without loss of generality) that σ is the element in G_x that acts on N_x as rotation by the angle $2\pi l/k$.

Clearly (x,y,z,t) is a fixed point of g only if $z=t=0$. In the two cases with $k=2$, for K_5 we have

$$g(x,y,z,t)=(-y,x,-z,-t) \tag{65}$$

and for the group K_6 we have

$$g(x,y,z,t)=(-x,-y,-z,-t), \tag{66}$$

so other than the zero point $(0,0,0,0)$, g has no fixed points at all in $\mathbb{R}^4$. Thus $(x,y,0,0)\neq(0,0,0,0)$ is a fixed point of g only if $k>2$ and $j=l=0$. Nontrivial powers of σ never share a nonzero fixed point with g, as σ acts freely on the plane $z=t=0$. We conclude that any nonzero point in the $z=t=0$ plane has either a trivial stabilizer, a cyclic stabilizer generated by an element of G_x, or a stabilizer of the form $\langle\sigma^p\alpha,g\rangle\cong D_4$ for some $0\le p\le k-1$, and moreover, this third case can only occur when $k>2$ and $j=l=0$. This last fact — that noncycle stabilizers occur in this plane only when $k>2$ — is essential to the remainder of our argument.

Now consider the action of K on the unit sphere $S^3\subset\mathbb{R}^4$, ie on

$$S^3=\{(x,y,z,t)\in\mathbb{R}^4\mid x^2+y^2+z^2+t^2=1\}. \tag{67}$$

The isotropy group of both $(0,0,0,-1)$, and $(0,0,0,1)$ is the dihedral group $G_x=\langle\sigma,\alpha\rangle\cong D_{2k}$ and g swaps these two points. Moreover, every other point in $S^3\setminus\{(0,0,0,-1),(0,0,0,1)\}$ has either trivial or cyclic isotropy, with some minor exceptions: when $k>2$ and $j=l=0$ there are points in $S^3\setminus\{(0,0,0,-1),(0,0,0,1)\}$ with isotropy isomorphic to D_4. Denote by W a small K–equivariant ball around the union of $(0,0,0,-1)$ and $(0,0,0,1)$ and remove it from S^3. Attach the G–equivariant handle $G\times_K(S^3\backslash W)$ to the boundary of the G–equivariant normal neighborhood of $G\cdot x$ on M. For $k>2$, the resulting 3–manifold has fewer points with dihedral isotropy D_{2k}, although it may create new points with D_4 isotropy. Attach the handles inductively for all dihedral isotropies isomorphic to D_{2k} with $k>2$ (picking maximal such isotropies at every step) and all isomorphism classes of faithful irreducible representations, and arrive at a manifold whose isotropies are only cyclic or dihedral of order 4. Now repeat step (ii) of the proof to arrive at a manifold with only cyclic isotropies and dihedral isotropies of order 4 with, moreover, the property that for any x and y with $G_x=G_y\cong D_4$, we know x and y are in the same G orbit. Finally, return to step (iii) and desingularize any remaining D_4 isotropies as we did in the case with $k>2$. Since the handles we have constructed for the D_4 singularities have no noncyclic isotropies on their interior, attaching them to the remaining D_4 singularities gives our final manifold M''' with only cyclic isotropies. □

We are now ready to show which surfaces with free G–actions bound equivariantly.

Theorem 4.3 *Let G be a finite group. Then the oriented and unitary equivariant bordism of surfaces with free G–actions fits into the exact sequence*

$$\Omega_3^G\{\mathcal{A}G,\{1\}\}\xrightarrow{\partial}\widetilde{\Omega}_2(BG)\to\widetilde{B}_0(G)\to 0. \tag{68}$$

Proof Let us first show that the image of the boundary map consists of toral classes in $H_2(BG, \mathbb{Z})$, that is, homology classes coming from the image of maps of tori $S^1 \times S^1 \to BG$.

Let M be a 3–dimensional G–manifold (oriented or unitary) whose boundary ∂M has a free G–action; in the oriented case take M as shown in Lemma 4.2. Note that if the ramification locus $\overline{M}$ is not empty, then it is a smooth oriented 1–dimensional manifold; in the unitary case this follows from the fact that fixed points of all nontrivial subgroups can only have complex codimension 0 or 1.

If $\overline{M}$ is empty then M has a free G action and therefore the boundary surface $(\partial M)/G$ bounds. If $\overline{M}$ is not empty we may consider the G–equivariant tubular neighborhood N of $\overline{M}$ in M. The manifolds N and M define the same bordism class, since on $M - N$ the action of G is free, and therefore ∂M and ∂N are cobordant. The tubular neighborhood N is homeomorphic to the unit ball bundle $B\nu$ of the normal bundle ν of $\overline{M}$ in M. The sphere bundle $S\nu$ defines the S^1–principal bundle $S^1 \to S\nu \to \overline{M}$, and since every circle bundle over the circle is topologically a torus, the sphere bundle $S\nu$ is homeomorphic to a disjoint union of 2–dimensional tori. Hence ∂N is a disjoint union of 2–dimensional tori, and its quotient $\partial N/G$ is a torus (since M/G is connected and $\chi(\partial N/G) = \chi(\partial N)/|G|$). Hence we have now proved that the image of the boundary map ∂ of (68) consists only of toral classes in $H_2(BG, \mathbb{Z})$.

Now let us show the converse, namely that any toral class in $H_2(BG, \mathbb{Z})$ lies in the image of the boundary map of (68). Take any toral class defined by a homomorphism $\varphi\colon \mathbb{Z}\times\mathbb{Z} \to G$ and denote by $A := \varphi(\mathbb{Z}\times\{0\})$ and $C := \varphi(\{0\}\times\mathbb{Z})$ the cyclic subgroups of G that define the toral class. Denote by $a := \varphi(1,0)$ and $c := \varphi(0,1)$ the generators of A and C, respectively.

Let $N_G A$ be the normalizer of A in G and note that C is a subgroup of the normalizer. Denote by ι and $\bar\iota$ the homomorphism $\iota\colon \mathbb{Z} \to N_G A$, $\iota(n) := c^n$ and the homomorphism to the quotient $\bar\iota\colon \mathbb{Z} \to W_A$. Consider the irreducible representation $\rho\colon A \to U(1)$, $\rho(a) := e^{2\pi i/|A|}$, and define the $U(1)$ extension of W_A by the exact sequence of groups

$$U(1) \to U(1) \times_A N_G A \to W_A, \tag{69}$$

where $U(1) \times_A N_G A$ is defined by the equivalence relation $(\lambda\rho(\alpha), g) \sim (\lambda, \alpha g)$ for all $\alpha \in A,\ \lambda \in U(1)$ and $g \in N_G A$.

Consider the homomorphism $\tilde\iota\colon \mathbb{Z} \to U(1) \times_A N_G A$, $\tilde\iota(n) := [(1, \iota(c^n))]$, and note that its classifying map

$$B\tilde\iota\colon S^1 \to B(U(1) \times_A N_G A) \tag{70}$$

factors through the classifying map $B\bar\iota\colon S^1 \to BW_A$.

Let $E := (B\tilde\iota)^* E(U(1) \times_A N_G A)$ be the pullback of the universal bundle and note two things. First, E is a principal $U(1)\times_A N_G A$–bundle over the circle S^1, and therefore it is a surface. Second, the canonical homomorphism

$$N_G A \to U(1) \times_A N_G A \quad \text{given by } g \mapsto [(1, g)] \tag{71}$$

induces a free action of $N_G A$ on E. Now it is straightforward to notice that the homology class of the surface $E/N_G A \to BN_G A \to BG$ in $\widetilde{\Omega}_2(BG)$ agrees with the homology class defined by $B\varphi_*[S^1 \times S^1]$.

We still need to show that the surface E equivariantly bounds. Take the quotient $F = E/U(1)$ and note that F is homeomorphic to $(B\bar{\iota})^* EW_A$; hence F is the principal W_A–bundle over the circle that $\bar{\iota}$ defines (see the following commutative diagram):

$$
\begin{tikzcd}
U(1) \arrow[dd, dash] \arrow[dr] & & & & \\
& U(1) \times_A N_G A \arrow[rr, dash] \arrow[dr] \arrow[dd] & & U(1) \times_A N_G A \arrow[dr] \arrow[dd] & \\
U(1) \arrow[dr] & & W_A \arrow[rr] \arrow[dd] & & W_A \arrow[dd] \\
& E \arrow[rr] \arrow[dr] \arrow[dd] & & E(U(1) \times_A N_G A) \arrow[dr] \arrow[dd] & \\
& & F \arrow[rr] \arrow[dd] & & EW_A \arrow[dd] \\
& S^1 \arrow[rr, "B\tilde{\iota}"] \arrow[dr, equal] & & B(U(1) \times_A N_G A) \arrow[dr] & \\
& & S^1 \arrow[rr, "B\bar{\iota}"] & & BW_A
\end{tikzcd}
\tag{72}
$$

Then E is a principal $U(1)$–bundle over F, and therefore we may take the associated complex vector bundle

$$\mathbb{C} \to \mathbb{C} \times_{U(1)} E \to F. \tag{73}$$

The unit bundle $D(\mathbb{C} \times_{U(1)} E)$ is a unitary manifold endowed with the action of $N_G A$, whose boundary, the sphere bundle $S(\mathbb{C} \times_{U(1)} E)$, is homeomorphic to E:

$$\partial(D(\mathbb{C} \times_{U(1)} E)) = S(\mathbb{C} \times_{U(1)} E) \cong E. \tag{74}$$

Therefore we have just proved that

$$[G \times_{N_G A} D(\mathbb{C} \times_{U(1)} E)] \xrightarrow{\partial} [E/N_G A] = B\varphi_*[S^1 \times S^1], \tag{75}$$

thus showing that any toral class in $\widetilde{\Omega}_2(BG)$ equivariantly bounds. □

Now we can put the pieces together to understand the torsion of the equivariant bordism group of surfaces.

4.2 Torsion of the equivariant bordism group of surfaces

By Corollary 3.4,

$$\overline{\Omega}_2^G = \operatorname{Ker}(\phi_2) = \operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G). \tag{76}$$

Let us now determine explicitly these torsion subgroups.

Theorem 4.4 *Let G be a finite group. Then there is a canonical isomorphism*

$$\bigoplus_{(K)} \widetilde{B}_0(W_K) \cong \operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G), \tag{77}$$

where (K) runs over all conjugacy classes of subgroups of G, $W_K = N_G K/K$ and $\widetilde{B}_0(W_K)$ is the homology version of the Bogomolov multiplier of the group W_K.

Proof Denote by $\mathrm{Gr}_* \operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G)$ the associated graded groups of the G–equivariant, unitary or oriented, bordism groups of surfaces that are induced by the Conner–Floyd spectral sequence of the families of subgroups of (42). Lemma 4.1 and Theorem 4.3 imply that

$$\mathrm{Gr}_p \operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G) \cong \widetilde{B}_0(W_{K_p}), \tag{78}$$

and since all consecutive pairs of families are adjacent, we obtain the graded isomorphism

$$\mathrm{Gr}_* \operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G) \cong \bigoplus_{(K)} \widetilde{B}_0(W_K). \tag{79}$$

Now, for a fixed conjugacy class of subgroups (K), the canonical map

$$\widetilde{\Omega}_2(BW_K) \to \Omega_2^G \quad \text{given by } \Sigma/W_K \mapsto G \times_{N_G K} \Sigma, \tag{80}$$

which sends the quotient space of a surface Σ by the free W_K–action to the surface with G–action whose isotropy groups lie in (K), factors through $\widetilde{B}_0(W_K)$, thus producing a canonical homomorphism

$$\widetilde{B}_0(W_K) \to \Omega_2^G. \tag{81}$$

Bundling up all these homomorphisms we obtain a canonical map

$$\bigoplus_{(K)} \widetilde{B}_0(W_K) \to \operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G) \tag{82}$$

which becomes an isomorphism since it is compatible with the graded isomorphism (79). □

In particular, if G is a group whose Bogomolov multipliers vanish for all groups W_K with K a nontrivial subgroup, then $\operatorname{Tor}_{\mathbb{Z}}(\Omega_2^G) \cong \widetilde{B}_0(G)$. This is the case whenever G is one of the smallest p–groups with nontrivial Bogomolov multiplier. In the last section we present two p–groups of this kind.

We are now ready to provide an explicit calculation of the unitary and oriented equivariant bordism group of surfaces. Assembling Theorems 3.1 and 4.4, and Proposition 3.3, we obtain the following result:

Theorem 4.5 *Let G be a finite group. Then the unitary and oriented equivariant bordism of surfaces canonically decompose as follows*:

$$\Omega_2^{U,G} \cong \bigoplus_{(K)} \biggl(\widetilde{B}_0(W_K) \oplus \Omega_2^U \oplus \Bigl(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)} \mathbb{Z} \Bigr)^{W_K} \biggr), \tag{83}$$

$$\Omega_2^{\mathrm{SO},G} \cong \bigoplus_{(K)} \biggl(\widetilde{B}_0(W_K) \oplus \Bigl(\bigoplus_{\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}} \mathbb{Z} \Bigr)^{W_K} \biggr). \tag{84}$$

Here (K) runs over the conjugacy classes of subgroups of G, W_K is the Weyl group $N_G K/K$, $\mathrm{Irr}^1_{\mathbb{C}}(K)$ is the set of 1–dimensional nontrivial irreducible complex representations of K endowed with the natural W_K action, and $\mathrm{Irr}^1_{\mathbb{C}}(K)_{\mathbb{C}}/\mathrm{conj}$ denotes the representations of complex type modulo complex conjugation.

5 2–Dimensional SK–groups of classifying spaces

Jänich in [15; 14] started the study of the characterization of invariants with the additivity property of the Euler characteristic and the signature under cutting and pasting of manifolds.

Karras and Kreck in their diploma thesis extended the ideas of Jänich to cutting and pasting in the bundle situation. The book [17] presented and simplified these results with the definition of the SK–groups of a space (cutting and pasting groups from the German *Schneiden und Kleben*). Later Neumann [24] completely calculated the 2–dimensional SK–groups of a space in terms of what is now known as the Bogomolov multiplier of its fundamental group. We recall in this section the main results of [17; 24] that allow us to relate the SK–relation with the equivariant bordism relation on surfaces with free actions.

The Schneiden und Kleben groups $\mathrm{SK}_*(X)$ of a space are defined as the Grothendieck group of the semigroups obtained by defining the SK–equivalence on the class of continuous maps from oriented n–dimensional manifolds to X [17].

The SK–relation is defined as follows: given (M_i, f_i) with $f_i\colon M_i \to X$, we say that (M_1, f_1) and (M_2, f_2) are related by cutting and pasting along ∂N if $M_1 = N \cup_\phi -N'$, $M_2 = N \cup_\psi -N'$ and there are homotopies $f_1 |_N \simeq f_2 |_N$ and $f_1 |_{N'} \simeq f_2 |_{N'}$.

The Schneiden und Kleben bordism groups $\overline{\mathrm{SK}}_n(X)$ of a space are defined as the quotient of the oriented bordism groups by the equivalence relation generated by the SK–relation:

$$\overline{\mathrm{SK}}_*(X) = \Omega_*^{\mathrm{SO}}(X)/\sim . \tag{85}$$

The group $\overline{\mathrm{SK}}_2(BG)$ can be interpreted as the bordism group of surfaces with free G–actions modulo the SK–relations.

The following results summarize the main properties of the SK–relation [17, Lemmas 1.5 and 1.6].

(i) Any $f\colon S^1 \to X$ is zero in $\mathrm{SK}_1(X)$.

(ii) If M fibers over S^n with fiber F, then for any $f\colon M \to X$, in $\mathrm{SK}_*(X)$,

$$[M, f] = [S^n, *][F, f|_F]. \tag{86}$$

(iii) If $[M_2, f_2]$ is obtained from $[M_1, f_1]$ by surgery of type $(k+1, n-k)$, then in $\mathrm{SK}_*(X)$

$$[M_1, f_1] + [S^n, *] = [M_2, f_2] + [S^k \times S^{n-1}, *]. \tag{87}$$

Now, if I_* denotes the subgroup of $\mathrm{SK}_*(X)$ generated by the spheres with constant maps to X, which is isomorphic to the integers, we have:

Theorem 5.1 [17, Theorem 1.1] *For a connected space X, there is the exact sequence*

$$0 \to I_* \to \mathrm{SK}_*(X) \to \overline{\mathrm{SK}}_*(X) \to 0, \tag{88}$$

which is moreover split. The map $\frac{1}{2}(\chi - \tau)\colon \mathrm{SK}_n(X) \to \mathbb{Z} \cong I_n$ gives the splitting.

The groups $\overline{\mathrm{SK}}_*(X)$ fit into short exact sequences whose middle terms are the oriented bordism groups.

Theorem 5.2 [17, Theorem 1.2] *Let $F_n(X)$ be the submodule of $\Omega_n^{\mathrm{SO}}(X)$ generated by all elements which have a representative that fibers over S^1. Then $F_n(X)$ fits into the short exact sequence*

$$0 \to F_*(X) \to \Omega_*^{\mathrm{SO}}(X) \to \overline{\mathrm{SK}}_*(X) \to 0. \tag{89}$$

This theorem follows from the observations that any manifold that fibers over S^1 gives a class that is zero in $\overline{\mathrm{SK}}_*(X)$, and that the kernel of the homomorphism $\Omega_*^{\mathrm{SO}}(X) \to \overline{\mathrm{SK}}_*(X)$ consists of mapping tori. The key lemma for the opposite inclusion asserts that if (M_1, f_1) is obtained from (M_2, f_2) by cutting and pasting along N, then in $\Omega_*^{\mathrm{SO}}(X)$ the class of $(N \cup_\phi -N', f_1) - (N \cup_\psi -N', f_2)$ is equal to the mapping torus of the diffeomorphism of ∂N, $\phi^{-1} \circ \psi$. Any mapping torus fibers over S^1 and any fibration over S^1 is a mapping torus.

In dimensions 0 and 1 the groups $\overline{\mathrm{SK}}_n(X)$ are trivial. In dimension 2 the oriented manifolds that fiber over the circle are tori. Therefore by Theorem 5.2 we obtain the following result:

Theorem 5.3 [24, Theorem 2] *Let G be a discrete group. Then the 2–dimensional $\overline{\mathrm{SK}}$–group of BG is isomorphic to the Bogomolov multiplier of G:*

$$\overline{\mathrm{SK}}_2(BG) \cong \widetilde{B}_0(G). \tag{90}$$

Reinterpreting the SK–groups of BG in view of our previous results, we know by Theorem 4.3 that an element of $\overline{\mathrm{SK}}_2(BG)$ is zero whenever the associated G–cover of the surface is the boundary of a 3–dimensional manifold with a G–action. By Theorem 5.2, $\mathrm{SK}_2(BG) \cong \mathbb{Z} \oplus \widetilde{B}_0(G)$, and therefore a surface $\Sigma \to BG$ is zero in the group $\mathrm{SK}_2(BG)$ whenever the Euler characteristic of Σ is 0 and the G–cover $\widetilde{\Sigma}$ of Σ is the boundary of a 3–dimensional manifold with a G–action.

It would be interesting to explore the relation of this work with the higher-dimensional SK–groups of classifying spaces.

6 Small groups with nontrivial Bogomolov multiplier

We conclude this work by presenting some explicit examples of groups with nontrivial Bogomolov multiplier which induce nontrivial torsion subgroups in the equivariant bordism groups of surfaces. Some of the calculations were done with the help of the *Homological Algebra Programming* package for GAP [10].

6.1 2–Group of size 64

The smallest groups with nontrivial Bogomolov multiplier are 2–groups of order 64. There are nine of them, and all are in the same isoclinism class. By [23, Theorem 1.2] they all have isomorphic Bogomolov multipliers, and in this case it is the group $\mathbb{Z}/2$. Among the nine isoclinic groups we chose to study the group

$$C_8 \rtimes Q_8, \tag{91}$$

which is the semidirect product of the group of quaternions Q_8 with the cyclic group C_8 of order 8; this group is denoted by

```
SmallGroup(64,182)
```

in the GAP small groups library. Consider the presentations $Q_8 = \langle a, b \mid a^2 = b^2, aba^{-1} = b^{-1} \rangle$ and $C_8 = \langle c \mid c^8 = 1 \rangle$, and the action of Q_8 on C_8 given by the equations

$$ac = c^3, \quad bc = c^5 \quad \text{and} \quad (ab)c = c^7. \tag{92}$$

Since $H^2(C_8, \mathbb{C}^*) = 0 = H^2(Q_8, \mathbb{C}^*)$, we know by the Lyndon–Hochschild spectral sequence that

$$H^2(C_8 \rtimes Q_8, \mathbb{C}^*) \cong H^1(Q_8, H^1(C_8, \mathbb{C}^*)). \tag{93}$$

Define $\widehat{C}_8 := \mathrm{Hom}(C_8, \mathbb{C}^*) = H^1(C_8, \mathbb{C}^*)$ and let $\widehat{C}_8 = \langle \rho \mid \rho^8 = 1 \rangle$ with $\rho(c) = e^{2\pi i/8}$. Take the first two terms of the complex $C^*(Q_8, \widehat{C}_8)$,

$$\widehat{C}_8 \xrightarrow{\delta} \mathrm{Map}(Q_8, \widehat{C}_8), \tag{94}$$

and note that

$$\delta(\rho^k)(a^\pm) = \rho^{-2k}, \quad \delta(\rho^k)(b^\pm) = \rho^{4k}, \quad \delta(\rho^k)((ab)^\pm) = \rho^{2k} \quad \text{and} \quad \delta(\rho^k)(a^2) = \rho^0. \tag{95}$$

On the other hand, take the 1–cocycle $F : Q_8 \to \widehat{C}_8$ defined by the equations

$$F(a^\pm) = \rho^2, \quad F(b^\pm) = \rho^0, \quad F((ab)^\pm) = \rho^2 \quad \text{and} \quad F(a^2) = \rho^0, \tag{96}$$

and note that F does not bound but $F^2 = \delta(\rho^2)$. We have therefore that

$$H^1(Q_8, \widehat{C}_8) \cong \langle [F] \mid [F^2] = 0 \rangle \cong \mathbb{Z}/2. \tag{97}$$

Now any abelian subgroup of $C_8 \rtimes Q_8$ splits as a semidirect product of abelian groups $C \rtimes A$ with $C \subset C_8$ and $A \subset Q_8$. Since A can only be $\mathbb{Z}/4$ or $\mathbb{Z}/2$, it is now straightforward to check that $[F]|_{C \rtimes A} = 0$. Hence $[F]$ is the generator of the Bogomolov multiplier of $C_8 \rtimes Q_8$ and

$$\overline{\Omega}_2^{U, C_8 \rtimes Q_8} \cong \overline{\Omega}_2^{\mathrm{SO}, C_8 \rtimes Q_8} \cong \mathbb{Z}/2. \tag{98}$$

Finally, with the explicit description of F we can define a surface Σ_2 of genus 2 which defines the generator of $\widetilde{\Omega}_2^U(B(C_8 \rtimes Q_8))$. Consider the presentation of the fundamental group of the surface

$$\pi_1(\Sigma_2) = \langle x, y, z, w \mid [x, y][z, w] = 1 \rangle \tag{99}$$

and define the assignment

$$\Phi : \pi_1(\Sigma_2) \to C_8 \rtimes Q_8, \qquad x \mapsto a, \quad y \mapsto c, \quad z \mapsto ab, \quad w \mapsto c, \tag{100}$$

which induces a surjective homomorphism since

$$\Phi([x, y][z, w]) = aca^{-1}c^{-1}(ab)c(ab)^{-1}c^{-1} = c^3c^{-1}c^7c^{-1} = c^0. \tag{101}$$

The homomorphism Φ induces a map $B\Phi : \Sigma_2 \to B(C_8 \rtimes Q_8)$, and from the construction above of F, we deduce that $B\Phi_*[\Sigma_2]$ generates the group $H_2(B(C_8 \rtimes Q_8), \mathbb{Z})$.

Hence the surface

$$\tilde{\tilde{\Sigma}} := (B\Phi)^* E(C_8 \rtimes Q_8) \tag{102}$$

is a unitary surface with a free action of $C_8 \rtimes Q_8$ which does not equivariantly bound.

By Theorem 4.4, the class of $\tilde{\tilde{\Sigma}}$ is the generator of the torsion subgroup of $\Omega_2^{\mathrm{SO}, C_8 \rtimes Q_8}$:

$$\mathrm{Tor}_{\mathbb{Z}}\, \Omega_2^{\mathrm{SO}, C_8 \rtimes Q_8} = \langle [\tilde{\tilde{\Sigma}}] \rangle \cong \mathbb{Z}/2. \tag{103}$$

To make sure that the first Chern number vanishes, we take the bordism class

$$[\tilde{\tilde{\Sigma}}] - [(C_8 \rtimes Q_8) \times \Sigma_2] \in \overline{\Omega}_2^{U, C_8 \rtimes Q_8} \cong \mathbb{Z}/2, \tag{104}$$

and by Theorem 4.4 we conclude that this is indeed the generator of the torsion subgroup of $\Omega_2^{U, C_8 \rtimes Q_8}$:

$$\mathrm{Tor}_{\mathbb{Z}}\, \Omega_2^{U, C_8 \rtimes Q_8} = \langle [\tilde{\tilde{\Sigma}}] - [(C_8 \rtimes Q_8) \times \Sigma_2] \rangle \cong \mathbb{Z}/2. \tag{105}$$

6.2 3–Group of size 243

The smallest 3–groups with nontrivial Bogomolov multiplier are of order 243, and the three of them are isoclinic with Bogomolov multiplier the group $\mathbb{Z}/3$. We chose to study the group

$$G := (C_9 \rtimes C_9) \rtimes C_3, \tag{106}$$

which is defined by the presentation

$$G = \langle a, b, c \mid a^3 = c^3,\, a^9 = b^9 = 1,\, [a, b] = c^8 b^6,\, [b, c] = a^3,\, [a, c] = b^3 c^6 \rangle. \tag{107}$$

The left C_9 is generated by c, the right C_9 by b, and the C_3 by ab; their corresponding actions are

$$bcb^{-1} = c^4, \quad (ab)b(ab)^{-1} = c^8 b^7 \quad \text{and} \quad (ab)c(ab)^{-1} = cb^3. \tag{108}$$

This group corresponds to the small group

`SmallGroup(243,30)`

in the small groups library of GAP [10].

The second page of the Lyndon–Hochschild spectral sequence has for terms

$$H^2(C_9 \rtimes C_9, \mathbb{C}^*)^{C_3} = 0, \quad H^1(C_3, H^1(C_9 \rtimes C_9, \mathbb{C}^*)) = \mathbb{Z}/3 \quad \text{and} \quad H^2(C_3, \mathbb{C}^*) = 0, \tag{109}$$

where the middle term encodes the information of the Bogomolov multiplier.

Consider the surface Σ_2 of genus 2 as in (99), and define the assignment

$$\Phi\colon \pi_1(\Sigma_2) \to (C_9 \rtimes C_9) \rtimes C_3, \qquad x \mapsto a, \quad y \mapsto b^6, \quad z \mapsto c, \quad w \mapsto b, \tag{110}$$

which induces a surjective homomorphism since $[a, b^6] = a^3$, $[c, b] = a^6$ and

$$\Phi([x, y][z, w]) = [a, b^6][c, b] = 1. \tag{111}$$

The map $B\Phi\colon \Sigma_2 \to B((C_9 \rtimes C_9) \rtimes C_3)$ generates the Bogomolov multiplier, and therefore the surface $\widetilde{\Sigma} := (B\Phi)^* E((C_9 \rtimes C_9) \rtimes C_3)$ generates the torsion subgroup of the equivariant oriented bordism group of surfaces

$$\operatorname{Tor}_{\mathbb{Z}} \Omega_2^{\mathrm{SO},(C_9 \rtimes C_9) \rtimes C_3} = \langle [\widetilde{\Sigma}] \rangle \cong \mathbb{Z}/3. \tag{112}$$

In the unitary case,

$$\operatorname{Tor}_{\mathbb{Z}} \Omega_2^{U,(C_9 \rtimes C_9) \rtimes C_3} = \langle [\widetilde{\Sigma}] - [(C_9 \rtimes C_9) \rtimes C_3 \times \Sigma_2] \rangle \cong \mathbb{Z}/3. \tag{113}$$

Then $\widetilde{\Sigma}$ is a surface of genus 486 with a free action of $(C_9 \rtimes C_9) \rtimes C_3$ which does not equivariantly bound.

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References

[1] **A Ángel**, *A spectral sequence for orbifold cobordism*, from "Algebraic topology: old and new", Banach Center Publ. 85, Polish Acad. Sci. Inst. Math., Warsaw (2009) 141–154 MR Zbl

[2] **A Ángel**, **J M Gómez**, **B Uribe**, *Equivariant complex bundles, fixed points and equivariant unitary bordism*, Algebr. Geom. Topol. 18 (2018) 4001–4035 MR Zbl

[3] **F A Bogomolov**, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987) 485–516 MR Zbl In Russian; translated in Math. USSR-Izv. 30 (1988) 455–485

[4] **P E Conner**, **E E Floyd**, *Differentiable periodic maps*, Ergebnisse der Math. 33, Academic, New York (1964) MR Zbl

[5] **P E Conner**, **E E Floyd**, *Maps of odd period*, Ann. of Math. 84 (1966) 132–156 MR Zbl

[6] **T tom Dieck**, *Bordism of G–manifolds and integrality theorems*, Topology 9 (1970) 345–358 MR Zbl

[7] **T tom Dieck**, *Orbittypen und äquivariante Homologie, I*, Arch. Math. (Basel) 23 (1972) 307–317 MR Zbl

[8] **T tom Dieck**, *Transformation groups*, de Gruyter Stud. Math. 8, de Gruyter, Berlin (1987) MR Zbl

[9] **J E Domínguez**, **C Segovia**, *Extending free actions of finite groups on surfaces*, Topology Appl. 305 (2022) art. id. 107898 MR Zbl

[10] **GAP Group**, *GAP*: *groups*, *algorithms*, *and programming* (2021) Version 4.11.1 Available at `https://www.gap-system.org`

[11] **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 543, Amer. Math. Soc., Providence, RI (1995) MR Zbl

[12] **J P C Greenlees**, **J P May**, *Localization and completion theorems for MU–module spectra*, Ann. of Math. 146 (1997) 509–544 MR Zbl

[13] **B Hanke**, *Geometric versus homotopy theoretic equivariant bordism*, Math. Ann. 332 (2005) 677–696 MR Zbl

[14] **K Jänich**, *Charakterisierung der Signatur von Mannigfaltigkeiten durch eine Additivitätseigenschaft*, Invent. Math. 6 (1968) 35–40 MR Zbl

[15] **K Jänich**, *On invariants with the Novikov additive property*, Math. Ann. 184 (1969) 65–77 MR Zbl

[16] **G Karpilovsky**, *The Schur multiplier*, Lond. Math. Soc. Monogr. (N.S.) 2, Oxford Univ. Press (1987) MR Zbl

[17] **U Karras**, **M Kreck**, **W D Neumann**, **E Ossa**, *Cutting and pasting of manifolds*: SK*–groups*, Math. Lect. Ser. 1, Publish or Perish, Boston, MA (1973) MR Zbl

[18] **B Kunyavskiĭ**, *The Bogomolov multiplier of finite simple groups*, from "Cohomological and geometric approaches to rationality problems" (F Bogomolov, Y Tschinkel, editors), Progr. Math. 282, Birkhäuser, Boston, MA (2010) 209–217 MR Zbl

[19] **P S Landweber**, *Equivariant bordism and cyclic groups*, Proc. Amer. Math. Soc. 31 (1972) 564–570 MR Zbl

[20] **P Löffler**, *Bordismengruppen unitärer Torusmannigfaltigkeiten*, Manuscripta Math. 12 (1974) 307–327 MR Zbl

[21] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) MR Zbl

[22] **P Moravec**, *Unramified Brauer groups of finite and infinite groups*, Amer. J. Math. 134 (2012) 1679–1704 MR Zbl

[23] **P Moravec**, *Unramified Brauer groups and isoclinism*, Ars Math. Contemp. 7 (2014) 337–340 MR Zbl

[24] **W D Neumann**, *Manifold cutting and pasting groups*, Topology 14 (1975) 237–244 MR Zbl

[25] **R Oliver**, *SK_1 for finite group rings, II*, Math. Scand. 47 (1980) 195–231 MR Zbl

[26] **E Ossa**, *Unitary bordism of abelian groups*, Proc. Amer. Math. Soc. 33 (1972) 568–571 MR Zbl

[27] **R J Rowlett**, *The fixed-point construction in equivariant bordism*, Trans. Amer. Math. Soc. 246 (1978) 473–481 MR Zbl

[28] **R J Rowlett**, *Bordism of metacyclic group actions*, Michigan Math. J. 27 (1980) 223–233 MR Zbl

[29] **E Samperton**, *Schur-type invariants of branched G–covers of surfaces*, from "Topological phases of matter and quantum computation" (P Bruillard, C Ortiz Marrero, J Plavnik, editors), Contemp. Math. 747, Amer. Math. Soc., Providence, RI (2020) 173–197 MR Zbl

[30] **E G Samperton**, *Free actions on surfaces that do not extend to arbitrary actions on* 3*–manifolds*, C. R. Math. Acad. Sci. Paris 360 (2022) 161–167 MR Zbl

[31] **I Schur**, *Über die Darstellung der endlichen Gruppen durch gebrochen lineare Substitutionen*, J. Reine Angew. Math. 127 (1904) 20–50 MR Zbl

[32] **S Schwede**, *Global homotopy theory*, New Math. Monogr. 34, Cambridge Univ. Press (2018) MR Zbl

[33] **R E Stong**, *Notes on cobordism theory*: *mathematical notes*, Princeton Univ. Press (1968) MR Zbl

[34] **R E Stong**, *Complex and oriented equivariant bordism*, from "Topology of manifolds" (J C Cantrell, C H Edwards, Jr, editors), Markham, Chicago, IL (1970) 291–316 MR Zbl

[35] **B Uribe**, *The evenness conjecture in equivariant unitary bordism*, from "Proceedings of the International Congress of Mathematicians, II" (B Sirakov, P N de Souza, M Viana, editors), World Sci., Hackensack, NJ (2018) 1217–1239 MR Zbl

[36] **C T C Wall**, *Determination of the cobordism ring*, Ann. of Math. 72 (1960) 292–311 MR Zbl

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A spectral sequence for spaces of maps between operads

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Under mild conditions on topologically enriched operads P and Q, the derived mapping space $\mathrm{RHom}(P, Q)$ is the limit (sequential homotopy inverse limit) of a tower whose n^{th} layer admits a description in terms of certain (small) diagrams $J_n(P)$ and $J_n(Q)$. More precisely $J_n(P)$ is a 3–term diagram of spaces with action of Σ_n of the form $\mathrm{bound}_n(P) \to P(n) \to \mathrm{cobound}_n(P)$, where $P(n)$ is the space of n–ary operations in P. The statement takes some inspiration from manifold calculus, but the proof relies on the homotopical theory of dendroidal spaces and the concept of dendroidal nerve of an operad.

18M75, 55P48; 18N60

1 Introduction

Operads are tools well suited to describe and classify additional algebraic structures on objects in symmetric monoidal categories. On the other hand, they are a natural generalization of (enriched) categories allowing morphisms to have any finite number of sources. Operads (in a more restrictive one-object setting) were first defined by Peter May [1972]. Closely related notions can be seen in the earlier book by Boardman and Vogt [1968]. A specific operad emerged earlier still in [Stasheff 1963a; 1963b; Sugawara 1957]. For a very readable survey and exposition, see [Adams 1978, Section 2]. In his book, May proved the famous recognition principle, which gives an "operadic" characterization of based spaces which are homotopy equivalent to some n–fold loop space. Operads have since appeared in various branches of mathematics and mathematical physics. The principal aim of this investigation was to find a way to understand spaces of maps between operads. The "little disk" operads are important examples and test cases.

We will do this by translating the problem into the language of dendroidal spaces. These are contravariant functors from a certain category Ω of trees to the category sSet of simplicial sets. The theory of dendroidal sets and dendroidal spaces was introduced by Ieke Moerdijk and Ittay Weiss [2007] (see also [Weiss 2007]) and further investigated by Cisinski and Moerdijk [2011; 2013b; 2013a]. A simplicially enriched operad P determines a dendroidal space $N_d P$, known as the dendroidal nerve of P. There is a map of derived mapping spaces from $\mathrm{RHom}(P, Q)$ to $\mathrm{RHom}(N_d P, N_d Q)$, which is a weak equivalence in the cases we are interested in.

Although derived mapping spaces have a standard description in the context of model categories, we will mostly avoid this description and rely on the description due to Dwyer and Kan [1980] instead. They construct derived mapping spaces for objects in any category C equipped with a wide subcategory W (whose morphisms play the role of weak equivalences). If W happens to be the subcategory of weak equivalences in a model category, then these constructions yield weakly equivalent results. For our purposes a morphism of dendroidal spaces is a weak equivalence if and only if it is a levelwise equivalence of simplicial sets. The goal, then, is to understand the homotopy type of derived mapping spaces between dendroidal nerves of (some) operads.

We approach this problem by mapping the space $\mathrm{RHom}(N_d P, N_d Q)$ to a tower of derived mapping spaces obtained by restricting $N_d P$ and $N_d Q$ to certain subcategories $\Omega\langle k\rangle$, where $0 \leqslant k < \infty$. The subcategory $\Omega\langle k\rangle$ of Ω is the full subcategory on trees with vertices of valence $\leqslant k+1$ only (to put it differently, trees in which no vertex has more than k *incoming* edges). We note that these categories $\Omega\langle k\rangle$ are closed under grafting of trees. Contravariant functors from $\Omega\langle k\rangle$ to the category of simplicial sets will be called k–truncated dendroidal spaces. A morphism of truncated dendroidal spaces is a weak equivalence if it is a levelwise weak equivalence of simplicial sets. With these definitions it is clear that the restriction functor U_k from dendroidal spaces to k–truncated ones preserves all weak equivalences and thus induces maps $\mathrm{RHom}(X, Y) \to \mathrm{RHom}(U_k X, U_k Y)$ for all dendroidal spaces X and Y. We arrange these maps in a tower

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& \mathrm{RHom}(U_3 X, U_3 Y) \arrow[d] \\
& \mathrm{RHom}(U_2 X, U_2 Y) \arrow[d] \\
\mathrm{RHom}(X, Y) \arrow[r] \arrow[ur, bend left] \arrow[uur, bend left] \arrow[uuur, bend left] & \mathrm{RHom}(U_1 X, U_1 Y)
\end{tikzcd}$$

of derived mapping spaces.

In Section 3.1, we set up a "dévissage" mechanism for proving homotopical statements in categories of contravariant functors (with values in sSet) with levelwise weak equivalences. We show that every functor admits a weak equivalence from a free CW–functor, a more restrictive instance of the concept of CW–functor in [Dror Farjoun 1987]. These are functors admitting a CW-type decomposition into cells of the shape $\mathrm{Hom}(-, c) \times \Delta[k]$. We make use of this approximation to prove certain homotopical properties for contravariant functors. More precisely, we show that we can verify such a property by showing that it holds for representable functors and that it persists under formation of homotopy pushouts and disjoint unions. Our first application of this principle is the following statement (admittedly this is unsurprising, and it might have shorter proofs and might be regarded as obvious by some readers):

Lemma (Lemma 3.1.1 and Corollary 3.1.7) *The above tower of derived mapping spaces converges*, *ie for all dendroidal spaces X and Y the induced map*

$$\mathrm{RHom}(X,Y) \to \operatorname*{holim}_k \mathrm{RHom}(U_k X, U_k Y)$$

is a weak homotopy equivalence.

Under additional assumptions on our objects, the homotopy fibers of this tower can be simplified. A dendroidal Segal space is called 1–reduced if its values on the trivial tree and the 0–corolla are points and its value on the 1–corolla is contractible. The most important example is the dendroidal nerve of a 1–reduced simplicially enriched operad P. These are operads P that only have one object and satisfy $P(0) = *$ and $P(1) \simeq *$. This notion still captures our most important examples since all E_n–operads and many more are 1–reduced. In this setting we can, instead of working in the category sdSet of dendroidal spaces, restrict our attention to an easier category scdSet. This is based on a category of *closed* trees $\Omega_{\mathrm{cl}} \subset \Omega$.

Using this model, we define operadic boundary and coboundary objects reminiscent of the latching and matching objects, respectively, from the theory of Reedy categories. Let cc_k be the closed k–corolla, an important object of Ω_{cl} with a preferred action of Σ_k. Evaluating objects in scdSet at cc_k gives a functor

$$X \mapsto X_{\mathrm{cc}_k}$$

from scdSet to the category of simplicial sets with Σ_k–action. (If $X = N_d P$, where P is a 1–reduced operad, then $X_{\mathrm{cc}_k} \simeq P(k)$.) We define two more functors, bound_k and $\mathrm{cobound}_k$, from scdSet to simplicial sets with Σ_k–action, and natural Σ_k–maps

$$\mathrm{bound}_k X \to X_{\mathrm{cc}_k} \to \mathrm{cobound}_k X.$$

Let $J_k(X)$ be this diagram, and let $\partial J_k(X)$ be the shorter diagram

$$\mathrm{bound}_k X \to \mathrm{cobound}_k X$$

obtained by composing the two arrows in $J_k(X)$. Both of these are understood to be diagrams in the category of simplicial sets with an action of the symmetric group Σ_k. (For the present purposes a morphism of simplicial sets with Σ_k–action will be regarded as a weak equivalence if the underlying morphism in sSet is a weak equivalence.) Here is our main theorem (see also Remark 3.2.15 for a slightly different formulation):

Theorem 3.2.7 *Let X and Y be 1–reduced dendroidal Segal spaces*, *to be viewed as objects of* scdSet. *Then the following square is a homotopy pullback square*:

$$\begin{array}{ccc}
\mathrm{RHom}(U_k X, U_k Y) & \longrightarrow & \mathrm{RNat}(J_k X, J_k Y) \\
\downarrow & & \downarrow \\
\mathrm{RHom}(U_{k-1} X, U_{k-1} Y) & \longrightarrow & \mathrm{RNat}(\partial J_k X, \partial J_k Y)
\end{array}$$

This allows us to make some homotopical computations with $\mathrm{RHom}(E_n, E_{n+d})$.

Theorem *The homotopy fiber of*

$$\mathrm{RNat}(U_k E_n, U_k E_{n+d}) \to \mathrm{RNat}(U_{k-1} E_n, U_{k-1} E_{n+d})$$

is $((k-1)(d-2)+1)$*–connected.*

The above theorem and the corollary are reminiscent of fundamental results in the manifold calculus and can also be used in this context. A first application can be found in [Weiss 2021].

The operadic boundary and coboundary objects have also been investigated and used in [Fresse et al. 2017; Heuts 2021] in slightly different settings. Similar constructions can be seen in [Thumann 2017].

Authorship Apart from minor revisions, this is Göppl's PhD thesis (2019, Universität Münster), advised by Weiss. Göppl is no longer active in topology research, but his thesis was well received and, as time went by, the case for publishing it became only stronger. It fell to Weiss to revise and submit the work and act as corresponding author, although he is hardly an author or coauthor of the article.

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2 Operads and dendroidal objects

The purpose of this first section is to explain and motivate the notion of an operad. (The section is not a self-contained introduction to the homotopy theory of operads and dendroidal objects.) In the first part of the section we will give a short exposition of the basic definitions and theorems. The second part is devoted to some closely related notions more approachable by homotopical methods. The theory of dendroidal sets was introduced in [Moerdijk and Weiss 2007]. A dendroidal object is a contravariant functor on an indexing category of trees. An important subclass of trees are the linear ones and contravariant functors on this subcategory are simplicial objects. Most of the homotopical constructions for simplicial spaces generalize to the dendroidal setting. Our focus will be on the dendroidal analogue of (complete) Segal spaces [Rezk 2001].

Throughout this article we will make some use of the theory of model categories. A model structure on a bicomplete category C is defined by a triple $(\mathscr{C}o, \mathscr{W}, \mathscr{F}i)$ of wide subcategories of C. (A subcategory is called wide if it contains all identity morphisms.) These classes have to satisfy certain lifting properties analogous to the cofibrations, weak equivalences and fibrations of topological spaces. Although we are mostly interested in derived mapping spaces and these only depend on a class of weak equivalences, the additional structure given by fibrations and cofibrations provides useful tools for computations of mapping spaces and derived functors.

2.1 Operads

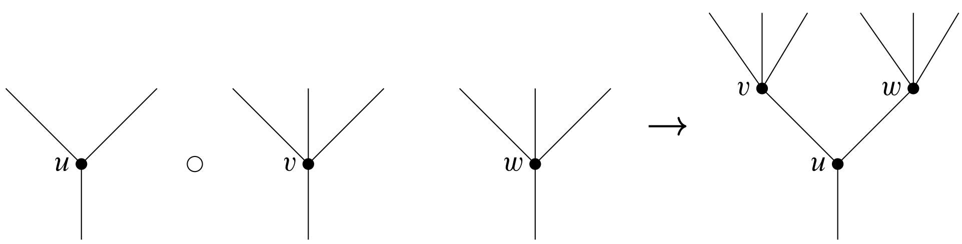
Definition 2.1.1 An *operad* P consists of a set of objects $\{x_i\}$ and, for every $(n+1)$–tuple $(x_1,\dots,x_n;x)$ of objects, a set of morphisms $P(x_1,\dots,x_n;x)$, subject to the following axioms:

- For every x, there is a morphism $\mathrm{Id}_x \in P(x;x)$, called the *identity of* x.
- There are associative composition morphisms
$$P(y_1,\dots,y_n;z)\times P(x_{1,1},\dots,x_{1,k_1};y_1)\times\cdots\times P(x_{n,1},\dots,x_{n,k_n};y_n)\to P(x_{1,1},\dots,x_{n,k_n};z).$$
- For every $(n+1)$–tuple $(x_1,\dots,x_n;y)$ and every $\sigma\in\Sigma_n$, there is a bijection
$$\sigma^*\colon P(x_1,\dots,x_n;y)\to P(x_{\sigma(1)},\dots,x_{\sigma(n)};y)$$
respecting the other structure.

A more complete definition is given in [Berger and Moerdijk 2007, 1.1].

A morphism of operads $f\colon P\to Q$ consists of a map between objects
$$\mathrm{ob}(P)\to\mathrm{ob}(Q)$$
and structure-preserving maps
$$P(x_1,\dots,x_n;x)\to Q(f(x_1),\dots,f(x_n);f(x)).$$
An operad is called *monochromatic* if it has only one object.

The operadic (multi)composition can be understood more easily by picturing the multimorphisms as special (planar) trees (so-called *corollas*) with several "input" edges and a unique "output" edge. Names of objects (sources and target) should be attached to edges and the name of the multimorphism can be attached to the unique vertex as a label. Grafting leads to a more complicated tree with several labels. The following depicts the grafting of a 2–morphism u with two 3–morphisms v and w in the monochromatic case, where the labeling of edges with objects is unnecessary:



It is up to the operadic structure to "simplify" the complicated tree in the right-hand side of the picture to a 6–corolla with a single label at the unique vertex. The simplification can be thought of as something induced (contravariantly) by a morphism from the 6–corolla to the complicated tree with three vertices and nine edges. (In this context, it is convenient to think of trees as partially ordered sets of edges. A morphism of trees is given by an order-preserving map of edge sets, subject to additional conditions, which will be spelled out below.)

For every symmetric monoidal category C it makes sense to speak of operads enriched over C. These still have a (discrete) set of objects. A *topological operad* is an operad enriched over the category of compactly generated weak Hausdorff spaces. We will more ambiguously speak of operads enriched over spaces to mean either topological operads or operads enriched over simplicial sets. The category of simplicially enriched operads will be denoted by sSetOp. For later use we say that a morphism between monochromatic topological operads is a *weak equivalence* if it is a levelwise weak homotopy equivalence.

To compare the theories of topological operads and simplicially enriched operads, we use a fact similar to [Berger and Moerdijk 2013, Corollary 1.14]: that a Quillen equivalence $\mathsf{V} \to \mathsf{V}'$ between suitably nice symmetric monoidal model categories induces a Quillen equivalence $\mathsf{V}\text{–}\mathsf{Op} \to \mathsf{V}'\text{–}\mathsf{Op}$ between the model structures on enriched operads. A (symmetric) monoidal category C equipped with a model structure is called a (*symmetric*) *monoidal model category* if it satisfies the following two axioms:

- For every pair of cofibrations $f : X \to Y$ and $f' : X' \to Y'$, the map
$$(X \otimes Y') \amalg_{(X \otimes X')} (Y \otimes X') \to Y \otimes Y'$$
is a cofibration. It is a weak equivalence if f or f' is.
- For every cofibrant X, the morphism
$$QI \otimes X \to I \otimes X \to X$$
is a weak equivalence. Here $QI \to I$ denotes a cofibrant replacement of the tensor unit I.

These axioms are called the *pushout–product axiom* and the *unit axiom*, respectively. Examples include the usual model categories of simplicial sets, compactly generated weak Hausdorff spaces, and chain complexes.

Example 2.1.2 Let $(\mathsf{C}, \otimes)$ be a closed symmetric monoidal category (a monoidal category is *closed* if the tensor product has a right adjoint, the *internal Hom*) and X an object of C. The *endomorphism operad* $\mathrm{End}(X)$ is the operad enriched in C on one object with morphism objects
$$\mathrm{End}(X)(n) = \underline{\mathrm{Hom}}_{\mathsf{C}}(X^{\otimes n}, X)$$
and the obvious multicomposition by insertion. The functor $\underline{\mathrm{Hom}}$ denotes the internal Hom functor of C.

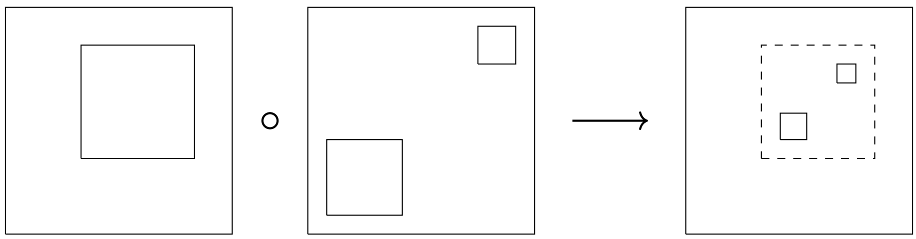
Endomorphism operads give a way for other operads to act on objects of C. In this way, operads classify additional algebraic structures.

Definition 2.1.3 An *algebra* A over a monochromatic C–operad P is an object A of C together with a map of operads $P \to \mathrm{End}(A)$.

Example 2.1.4 Let Com be the terminal topological operad. It has a single object and every mapping space is a point. Let X be a topological space. Then any map f from Com to $\mathrm{End}(X)$ turns X into an abelian topological monoid with operation $f(*_2) \in \mathrm{Map}(X \times X, X)$.

The little disk operads have been studied in great detail. May [1972] proved his famous recognition principle that a connected space is an algebra over the little n–disk operad if and only if it is weakly equivalent to an n–fold loop space. A more precise statement will be given after we define these operads.

Example 2.1.5 (the little disk operad) Let $D_n(k)$ denote the topological space of disjoint, rectilinear (ie respecting parallel lines) embeddings $\coprod_k I^n \to I^n$. Composition of (multi)morphisms is given by identifying the image of one morphism with an I^n in the domain of the next one. The following image shows a composition map $D_2(1) \times D_2(2) \to D_2(2)$:



Any topological operad weakly equivalent to the operad of little n–disks is called an E_n–operad.

We describe another model of topological E_n–operads, called Fulton–MacPherson operads. This one is less intuitive but has properties more closely related to objects we will investigate later on. It is built from a sequence of compactified euclidean configuration spaces. This construction is due to Fulton and MacPherson [1994] as an algebraic compactification of complex varieties and was later built in a topological way in [Axelrod and Singer 1994; Sinha 2004].

Example 2.1.6 (the Fulton–MacPherson operad [Sinha 2004; Getzler and Jones 1994]) For every n and k, the subgroup G_n of $\mathrm{Aff}(n)$, the group of affine automorphisms, generated by translations and scalar multiplication, of $\mathbb{R}^n$ acts freely on the ordered configuration space $\mathrm{Conf}(k, \mathbb{R}^n)$. The quotient $C[k, n]$ is a manifold of dimension $n(k-1)-1$ with an induced Σ_k–action. Consider the collection (or symmetric sequence) $\mathsf{F}_n(k)$ given by these manifolds for $k \geqslant 2$ and set $\mathsf{F}_n(0) = \mathsf{F}_n(1) = \varnothing$. The *Fulton–MacPherson E_n–operad* FM_n has the same underlying set as the free operad $\mathrm{Free}(\mathsf{F}_n)$ together with a point in degree zero. (The definition of symmetric sequences and the free operad construction will be given in Definition 2.1.8.) Its topology is constructed in such a way that every level $\mathrm{FM}_n(k)$ is a compact, connected manifold with corners. The interior of this manifold is $F_n(k)$ provided $k \geq 2$. The spaces $\mathrm{FM}_n(0)$ and $\mathrm{FM}_n(1)$ are one-point spaces.

We will now give an explicit construction. For all $(i, j) \in \binom{k}{2}$, define the maps

$$a_{(i,j)} \colon \mathrm{Conf}(k, \mathbb{R}^n) \to S^{n-1}, \quad x \mapsto \frac{x_i - x_j}{\|x_i - x_j\|}.$$

Furthermore, for all $(i, j, k) \in \binom{k}{3}$, define the maps

$$b_{(i,j,k)} \colon \mathrm{Conf}(k, \mathbb{R}^n) \to [0, \infty], \quad x \mapsto \frac{\|x_i - x_j\|}{\|x_i - x_k\|}.$$

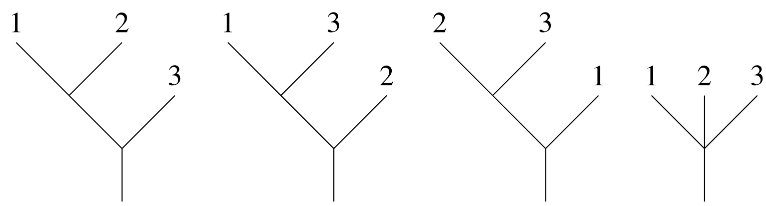
The configuration space $\mathrm{Conf}(k, \mathbb{R}^n)$ embeds into $\mathbb{R}^{nk} \times (S^{n-1})^{\binom{k}{2}} \times [0, \infty]^{\binom{k}{3}}$ via

$$x \mapsto \left(x, \prod_{i,j} a_{i,j}(x), \prod_{i,j,k} b_{i,j,k}(x) \right).$$

The closure of the image of this map shall be denoted by $C_k[\mathbb{R}^n]$. The action of G_n on the configuration space extends to an action on $C_k[\mathbb{R}^n]$. The quotients of this action assemble to the operad FM_n, ie $\mathrm{FM}_n(k) := C_k[\mathbb{R}^n]/G_n$. These quotient spaces are compact manifolds with corners. They have a natural stratification we can use to understand the operad structure on the collection FM_n.

The stratification is indexed over the category Ψ_k of rooted, labeled trees with k leaves (nonroot outer edges) and no vertices of valence one or two. The set of leaves shall be labeled by the set $\{1, 2, \dots, k\}$. The morphisms in Ψ_k are given by contraction of inner edges. So there is a map $S \to T$ if S can be turned into T by a sequence of contractions of inner edges. The stratum corresponding to a tree T with vertices $v_1, \dots, v_l$ of valence $k_1, \dots, k_l$ is homeomorphic to $\prod C[k_i - 1, n]$. Its closure is the union of all the strata indexed by trees mapping to T. In particular the interior of $\mathrm{FM}_n(k)$ is diffeomorphic to $C[k, n]$.

We try to illustrate this with some examples. The stratifications of the first three spaces ($\mathrm{FM}_n(0)$, $\mathrm{FM}_n(1)$ and $\mathrm{FM}_n(2)$) are trivial. There is no noncorolla tree with fewer than three inputs and no vertex of valence one or two. The first nontrivial stratification arises at level 3. There are four different trees in Ψ_3:

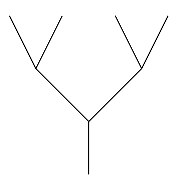


We see that there are three strata homeomorphic to $S^{n-1} \times S^{n-1}$ and the corolla stratum corresponding to the interior of $\mathrm{FM}_n(3)$. (In the case $n = 1$ the configurations in $\mathbb{R}^n$ have a canonical ordering and, by using this, we obtain $\mathrm{FM}_1(k) = \Sigma_k \times SP(k)$, where $SP(k)$ is a polytope found by Stasheff long before the work of Fulton and MacPherson.) The number of strata grows quickly with the level. There are already 26 trees in Ψ_4.

This stratification is compatible with the operadic structure. So, for example, the composition

$$\mathrm{FM}_n(2) \times (\mathrm{FM}_n(2) \times \mathrm{FM}_n(2)) \to \mathrm{FM}_n(4)$$

is an embedding whose image is the union of the strata corresponding to trees of the shape



The levelwise weak equivalences of simplicially enriched operads are the weak equivalences of a model structure on the category of monochromatic operads.

Theorem 2.1.7 [Cisinski and Moerdijk 2013b, Theorem 1.7] *The category* sSetOp_* *of monochromatic simplicially enriched operads carries a proper cofibrantly generated model structure such that the fibrations and weak equivalences are the levelwise fibrations and weak equivalences.*

Definition 2.1.8 Let C be a symmetric monoidal category. A *collection* (also known as a *symmetric sequence*) in C is a sequence of objects X_n (where $n \geq 0$) with actions of the symmetric group Σ_n. More formally, the category of collections in C is the product of functor categories

$$\mathsf{Coll}(\mathsf{C}) := \prod_{n\in\mathbb{N}} \mathsf{C}^{\Sigma_n},$$

where the groups are regarded as groupoids with one object. The forgetful functor taking a monochromatic C–operad to its underlying collection has a left adjoint, called the *free operad* functor. It is described at length in [Berger and Moerdijk 2003, Section 5.8]. In each level k a free operad is indexed by rooted trees with k leaves. Let $\mathbb{T}$ be the groupoid of finite rooted trees and isomorphisms. More precisely, $\mathbb{T}$ is the maximal subgroupoid of the dendrex category Ω defined in Section 2.2. Similarly, let $\mathbb{T}_\Lambda$ be the groupoid of finite, rooted trees together with a total order λ on their set of leaves. For every collection X, we can define a functor

$$\underline{X}\colon \mathbb{T}^{\mathrm{op}} \to \mathsf{C}$$

by setting $\underline{X}(\eta) = I$, the tensor unit of C. Every tree $T \in \mathbb{T}$ can inductively be written as a grafting $\mathrm{c}_n \circ (T_1, \dots, T_n)$. (The tree c_n is the n–corolla, the tree with a single vertex of valence $n+1$. These corollas will be introduced in Example 2.2.2.) We set

$$\underline{X}(T) := X(n) \otimes \underline{X}(T_1) \otimes \cdots \otimes \underline{X}(T_n).$$

The *free operad* on a collection X has the n^{th} space

$$\mathrm{free}(X)(n) \cong \coprod_{\substack{[(T,\lambda)]\in\pi_0\mathbb{T}_\Lambda \\ T \text{ has } n \text{ inputs}}} \underline{X}(T)/\mathrm{Aut}(T,\lambda).$$

Note that objects of $\mathbb{T}_\Lambda$ can have nontrivial automorphisms. There is an understanding that we choose a representative (T,λ) in each element of $\pi_0\mathbb{T}_\Lambda$. The action of Σ_n on $\mathrm{free}(X)(n)$ comes from the action of Σ_n on the total orderings of the leaves. The permutation $\sigma \in \Sigma_n$ sends (T,λ) to the chosen representative (T',λ') in the class of $(T,\sigma(\lambda))$. We need to choose an isomorphism from $(T,\sigma(\lambda))$ to the representative (T',λ') in order to get an isomorphism from $\underline{X}(T)$ to $\underline{X}(T')$. Consequently, that isomorphism is well defined only modulo the action of $\mathrm{Aut}(T,\lambda)$ on $\underline{X}(T)$. The operadic composition in a free operad is induced by grafting of trees in the obvious way.

Remark 2.1.9 [Goerss and Jardine 1999, Theorem 5.1] Let

$$F\colon \mathsf{C} \leftrightarrows \mathsf{D} : G$$

be an adjunction of categories. Let $(\mathscr{C}o, \mathscr{W}, \mathscr{F}i)$ be a cofibrantly generated model structure on C. A morphism $f : a \to b$ in D shall be called a fibration or weak equivalence if its image under G is. If

- G preserves filtered colimits, and
- every morphism of D with the left lifting property with respect to all fibrations is a weak equivalence,

then there exists a cofibrantly generated model structure on D with the above fibrations and weak equivalences. Furthermore, if I is the set of generating cofibrations of C and J the set of generating trivial cofibrations, then $F(I)$ and $F(J)$ are the sets of generating cofibrations and trivial cofibrations, respectively, of D. This model structure is called the (left) *transferred model structure* along the adjunction $(F \dashv G)$.

Since the model structure of Theorem 2.1.7 is transferred from the category of collections, we immediately see that any free operad on a cofibrant collection is cofibrant.

A functorial cofibrant replacement of monochromatic topological operads has been constructed in [Boardman and Vogt 1973] and generalized to the case of operads enriched in suitable model categories in [Berger and Moerdijk 2006] and to the multiobject case in [Berger and Moerdijk 2007]. To avoid unnecessary complexity, only the version for monochromatic topological operads will be presented here.

Definition 2.1.10 (the Boardman–Vogt construction) Let P be a topological operad. The *Boardman–Vogt W–construction* is a factorization

$$\mathrm{free}(P) \hookrightarrow WP \xrightarrow{\sim} P$$

of the counit $\mathrm{free}(P) \to P$ into a cofibration followed by a weak equivalence. The operad WP itself is also often called the Boardman–Vogt W–construction. Under a small hypothesis on P, the BV construction WP is a functorial cofibrant replacement of P.

To build this factorization, we start with the free operad $\mathrm{free}(P)$. Recall that its n–ary operations are the labellings of certain trees with n leaves. Each vertex in these trees (of valence $k+1$) is colored by an element of $P(k)$. To get WP, we furthermore equip the internal edges with a length $l_e \in [0,1]$. If some edge e has length 0, then this point in WP is identified with the one given by contracting the edge and composing the two adjacent operations.

Lemma 2.1.11 [Boardman and Vogt 1973; Berger and Moerdijk 2006] *If the underlying collection of P is Σ–cofibrant* (every space of the collection is cofibrant and the action of Σ_k on the k^{th} space of the collection is free for all k), *then the operad WP is cofibrant.*

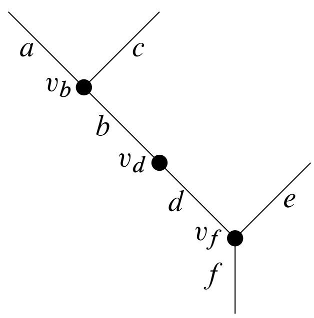
2.2 Dendroidal sets and spaces

The concept of dendroidal set was introduced in [Moerdijk and Weiss 2007] as a generalization of simplicial sets suited to describe and investigate the homotopy theory of operads. The homotopy theory of dendroidal sets and spaces was developed in [Cisinski and Moerdijk 2011; 2013b; 2013a]. Dendroidal sets correspond to (higher) operads in exactly the same way simplicial sets do to (higher) categories. Many constructions for simplicial objects have dendroidal analogues. A major tool in this article will be the notion of dendroidal complete Segal spaces.

In this section we will give a short introduction to these notions and quote the most important results for our further work.

Definition 2.2.1 A *tree* (or *dendrex*) T consists of a tuple $(T, \leq, L)$ such that $(T, \leq)$ is a partially ordered finite set (the set of edges) with a unique minimal element (called the *root*) and the property that, for each element $x \in T$, the set of elements smaller than x is linearly ordered. The set L is a subset of the set of maximal elements of T. The elements of T are called *edges* and the elements of L are called *leaves*. An edge is *inner* if it is neither a leaf nor the root. For any edge $x \in T \smallsetminus L$, the set $\mathrm{in}(x)$ of elements $y > x$ such that there is no z with $y > z > x$ is called the set of *incoming edges* (or *inputs*) of x. For any $x \in T \smallsetminus L$, the set $v_x := \{x\} \cup \mathrm{in}(x)$ is a *vertex* of T. (The set of vertices is in obvious bijection to $T \smallsetminus L$.)

These trees can be arranged into a category Ω. To define the morphisms of Ω, we note that every tree T determines an operad $\Omega(T)$ whose set of objects is the set of edges of T. Every vertex v_x contributes a generating operation whose input set is the set of incoming edges $\mathrm{in}(x)$ and whose output is x. For example, the operad generated by

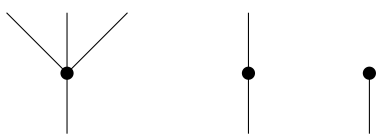


has six objects, morphisms $v_d \in \Omega(T)(b; d)$, $v_b \in \Omega(T)(a, c; b)$ and $v_f \in \Omega(T)(d, e; f)$ and their compositions $v_d \circ v_b \in \Omega(T)(a, c; d)$, $v_f \circ v_d \in \Omega(T)(b, e; f)$ and $v_f \circ v_d \circ v_b \in \Omega(T)(a, c, e; f)$. The set of morphisms in Ω between two trees is defined to be the set of morphisms between their corresponding operads,

$$\mathrm{Hom}_\Omega(T, T') := \mathrm{Hom}_{\mathsf{Op}}(\Omega(T), \Omega(T')).$$

Note that the morphisms do not have to preserve the root.

Example 2.2.2 The trees with exactly one vertex are of particular importance and are called *corollas*. The notation is c_n for a corolla with n leaves. The following figure shows the 3–corolla, the 1–corolla and the 0–corolla:



A presheaf on Ω is called a *dendroidal set*. More generally, for any symmetric monoidal category C, the objects of $\mathrm{Fun}(\Omega^{\mathrm{op}}, \mathsf{C})$ are called *dendroidal objects* in C. The category of dendroidal objects in C will be denoted by $d\mathsf{C}$. For every object T in Ω there is the dendroidal set represented by T; it is denoted by $\Omega[T]$.

The simplex category Δ embeds into Ω as a full subcategory by sending $[n]$ to the linear tree with n vertices and $n+1$ edges. The operads $\Omega(T)$ for T in the image of this embedding have no morphisms of higher degree and are thus equivalent to categories. They are easily seen to be the linear categories $[n]$. There is a tree η in Ω with exactly one edge; it is also the image of $[0]$ in Δ. Every operad which admits a morphism to $\Omega(\eta)$ cannot have higher morphisms and thus $\mathsf{Op}/\Omega(\eta) = \mathsf{Cat}$ and $\Omega/\eta \cong \Delta$ and $\mathsf{dSet}/\Omega[\eta] = \mathsf{sSet}$.

Several constructions on the category of simplicial sets can be generalized to the dendroidal setting and recovered by the description of sSet as the overcategory $\mathsf{dSet}/\Omega[\eta]$. One of the most important is the nerve construction. For an operad P, the *dendroidal nerve* $N_d P$ is the dendroidal set given by

$$N_d P(T) = \mathrm{Hom}_{\mathsf{Op}}(\Omega(T), P).$$

The nerve functor has a left adjoint τ_d. It can be described as the unique colimit-preserving functor that sends the represented presheaf $\Omega[T]$ to the operad $\Omega(T)$.

For any category regarded as an operad the dendroidal nerve reduces to the ordinary nerve of a category. Hence, the square of functors

$$\begin{tikzcd} \mathsf{Cat} \arrow[r] \arrow[d] & \mathsf{sSet} \arrow[d] \\ \mathsf{Op} \arrow[r] & \mathsf{dSet} \end{tikzcd}$$

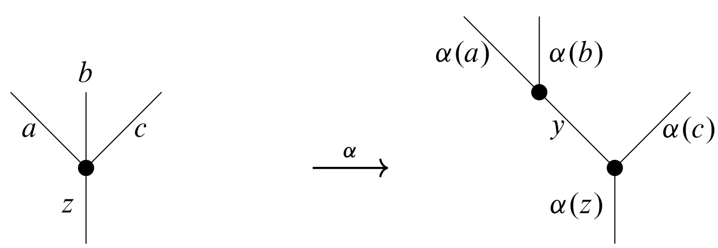
commutes.

Remark 2.2.3 For every T in Ω, there is an isomorphism of dendroidal sets

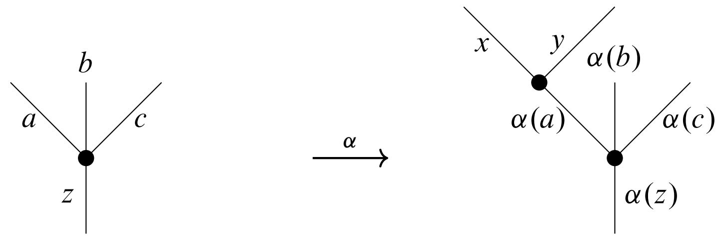
$$\Omega[T] \cong N_d \Omega(T).$$

We will now examine the category Ω more closely and describe the homotopy theory of dendroidal objects.

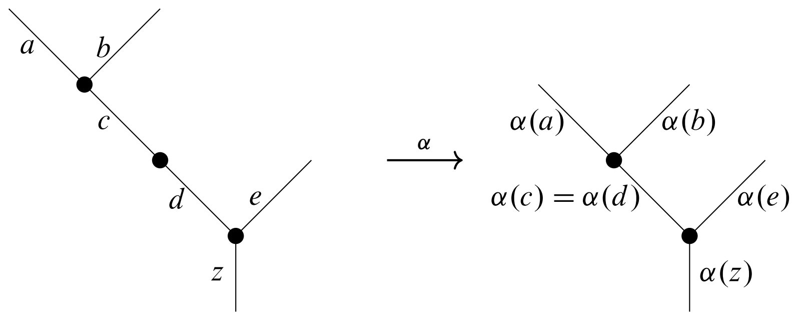
Definition 2.2.4 Morphisms (in Ω) of the following kind are called *inner face maps*:



They (contravariantly) correspond to operadic composition. Morphisms of the kind



are called *outer face maps*. The *degeneracies* are morphisms given by deleting an inner vertex of valence 2:



On the operads associated to these trees, this induces the map identifying the two adjacent objects and sending the unary morphism between them to the identity on the new object. By [Moerdijk and Weiss 2007, Lemma 3.1], every morphism in Ω factors up to isomorphism as a composition of degeneracies followed by a sequence of face maps.

The theory of *Segal spaces* was developed by Rezk [2001]. These Segal spaces are simplicial spaces behaving like an up-to-homotopy version of the nerve of a topological category.

Its dendroidal generalization was constructed in [Cisinski and Moerdijk 2013a]. This model has the merit of being less rigid than enriched operads in the sense that their composition law is only defined up to a contractible choice.

More exactly, our model will be based on simplicial dendroidal sets. As a category of simplicial presheaves, it is canonically tensored, cotensored and enriched over simplicial sets. The tensoring is given by taking a dendrexwise product of simplicial sets.

Definition 2.2.5 Let X and Y be dendroidal spaces. A morphism $f : X \to Y$ is called a *weak equivalence* if, for every tree T in Ω, the map $X_T \to Y_T$ is a weak equivalences of simplicial sets.

There are three standard choices for the classes of fibrations and cofibrations on the category of simplicial dendroidal sets if we fix the class of dendrexwise weak equivalences as our choice for the weak equivalences. The *projective* model structure is uniquely determined by defining a morphism to be a fibration if and only if it is a dendrexwise Kan fibration. Dually, the *injective* model structure is uniquely determined by the choice of dendrexwise cofibrations as its class of cofibrations.

There is an intermediate model structure taking into account the Reedy structure of Ω. Theorem 2.2.6 describes this in more detail. This model structure is a central starting point in [Cisinski and Moerdijk 2013a]. Since we are flexible in our choice of model structure, we will not need to use this result.

Theorem 2.2.6 [Cisinski and Moerdijk 2013b, Proposition 5.2] *The category* sdSet *of simplicial dendroidal sets can be equipped with a generalized Reedy model structure using the Reedy structure of* Ω. *It is cofibrantly generated and proper. The weak equivalences are the dendrexwise simplicial weak equivalences. A map of simplicial dendroidal sets* $X \to Y$ *is a fibration* (*resp. trivial fibration*) *if the relative matching maps*

$$X^{\Omega[T]} \to X^{\partial\Omega[T]} \times_{Y^{\partial\Omega[T]}} Y^{\Omega[T]}$$

are fibrations (*resp. trivial fibrations*) *for all* T. (See [Cisinski and Moerdijk 2013b, Section 2.1] for the meaning of $\partial\Omega[T]$.)

Definition 2.2.7 Let $T \in \Omega$ be a tree. If T has at least one vertex, the *spine* or *Segal core* $\mathrm{Sc}[T]$ of T is defined as a dendroidal subset of $\Omega[T]$ given by the union of all $\Omega[S]$ for subcorollas S of T. (There is one subcorolla for each vertex of T.) For the trivial tree η without vertices, we set $\mathrm{Sc}[\eta] = \Omega[\eta]$. Note that we recover the definition of a spine of a simplex by applying this definition to linear trees.

These Segal cores have a close connection to operads. Remember that a simplicial set X is the nerve of a category if and only if all maps

$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

induced by the spine inclusions are bijections. The following lemma is the generalization of this fact to the dendroidal setting:

Lemma 2.2.8 *A dendroidal set* X *is the nerve of an operad if and only if the map*

$$\mathrm{Hom}_{\mathsf{dSet}}(\Omega[T], X) \to \mathrm{Hom}_{\mathsf{dSet}}(\mathrm{Sc}[T], X)$$

induced by the Segal core inclusion is a bijection for all trees T.

Similarly, a dendroidal space X *is the nerve of a simplicially enriched operad if and only if the map of simplicial sets*

$$X^{\Omega[T]} \to X^{\mathrm{Sc}[T]}$$

is an isomorphism for all T.

For dendroidal spaces to model topological operads, we still want this equivalence to hold up to homotopy. The resulting notion will extend the classical definition of a complete Segal space as a model for $(\infty, 1)$–categories.

Definition 2.2.9 A dendroidal space X is called a *dendroidal Segal space* if, for all trees T, the map

$$X_T = \mathrm{Hom}(\Omega[T], X) = X^{\Omega[T]} \to \mathrm{RHom}(\mathrm{Sc}[T], X)$$

is a weak equivalence of simplicial sets.

Remark 2.2.10 Cisinski and Moerdijk [2013a] define the *model structure for dendroidal Segal spaces* as the left Bousfield localization of the generalized Reedy structure on sdSet at the set of Segal core inclusions.

(Definition 2.2.9 is not in full agreement with [Cisinski and Moerdijk 2013a], because they write $\mathrm{Hom}(\mathrm{Sc}[T], X)$ instead of $\mathrm{RHom}(\mathrm{Sc}[T], X)$ and insist that dendroidal Segal spaces be Reedy-fibrant to make up for that. Namely, the Segal core $\mathrm{Sc}[T]$ is Reedy-cofibrant. Therefore, $\mathrm{Hom}(\mathrm{Sc}[T], X)$ is weakly equivalent to $\mathrm{RHom}(\mathrm{Sc}[T], X)$ if X is Reedy fibrant.)

Lemma 2.2.11 [Boavida de Brito and Weiss 2018, Theorem 7.8; Boavida de Brito et al. 2019, Theorem 4.3] *Let P be a monochromatic simplicial operad. The dendroidal space $N_d P$ given by*

$$(N_d P)_T := \underline{P}(T),$$

using the notation of Definition 2.1.8, satisfies the Segal property. The assignment $P \mapsto N_d P$ is functorial and preserves all weak equivalences. Moreover for any two operads P and Q the morphism

$$\mathrm{RHom}(P, Q) \to \mathrm{RHom}(N_d P, N_d Q)$$

is a weak equivalence.

Remark 2.2.12 In [Boavida de Brito and Weiss 2018; Boavida de Brito et al. 2019], this result is attributed to Cisinski and Moerdijk, but it is not stated exactly in this form by Cisinski and Moerdijk. A small adjustment is required and Boavida de Brito et al. [2019] explain this in detail. Throughout this article we will only use the statement for 1–reduced operads. This implies that $N_d P$ is complete (and Segal). In general, $N_d P$ is not complete.

3 A tower of derived mapping spaces

3.1 Construction of the tower

In Section 1 we have introduced the notion of dendroidal Segal spaces as a model for the homotopy theory of topological operads. We want to use this model to describe the derived mapping spaces between two topological operads. For any two objects X and Y in any model category C, this derived mapping space can be defined as the space of maps $\mathrm{Hom}_{\mathsf{C}}(X^c, Y^f)$ from a cofibrant replacement of X to a fibrant replacement of Y. Although it is a slick definition, actual computations can be cumbersome because these objects tend to be unwieldy. Moreover, although this definition inherently depends on the choice of a model structure, the homotopy type of the derived mapping space only depends on the class of weak equivalences. There is another more general definition of a derived mapping space due to Dwyer and Kan [1980]. For every category C together with a subcategory W of weak equivalences, they define derived mapping spaces in terms of zigzags of morphisms. If W happens to be the class of weak equivalences of a model structure on C, then both definitions yield weakly equivalent derived mapping spaces.

We start this section with a general investigation of derived mapping spaces in categories of space-valued functors with levelwise weak equivalences under the assumption that the indexing category C can be written as a sequential colimit of full subcategories C_i. We prove a lemma that the derived space of natural transformations in $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{sSet})$ can be recovered up to homotopy from the mapping spaces between the restrictions of these functors to the subcategories C_i.

Lemma 3.1.1 *Let F and G be contravariant functors from C to sSet. We call a natural transformation $F \to G$ a weak equivalence if it is an objectwise weak equivalence of simplicial sets in the sense of Kan and Quillen. Let U_i denote the restriction functor from $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{sSet})$ to $\mathrm{Fun}(\mathsf{C}_i^{\mathrm{op}}, \mathsf{sSet})$. Then the natural morphism*

$$\mathrm{RHom}(F,G) \to \operatorname*{holim}_i \mathrm{RHom}(U_i F, U_i G)$$

is a weak equivalence.

We will prove this lemma in two steps. First we show that every contravariant functor admits a weak equivalence from a functor satisfying a cellularity property. These *free CW–functors* are a subclass of the CW–functors of [Dror Farjoun 1987, 1.16]. We then prove Lemma 3.1.1 for all free CW–functors F.

Definition 3.1.2 Let C be a category. A functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}$ is called a *free CW–functor* if there is a sequence

$$\varnothing = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_{i-1} \subset F_i \subset \cdots$$

of subfunctors of F such that the following properties are satisfied:

(1) $F(x) = \mathrm{colim}_i\, F_i(x)$ for all objects x of C.

(2) For all $i \geq 0$, there exists a pushout diagram

$$\begin{array}{ccc} K_i \times \partial\Delta[i] & \longrightarrow & F_{i-1} \\ \big\downarrow & & \big\downarrow \\ K_i \times \Delta[i] & \longrightarrow & F_i \end{array}$$

where K_i is a disjoint union of representable functors.

Example 3.1.3 Let $\underline{G}$ be a group regarded as a category with one object. Then the free CW–functor F is nothing but a simplicial set $F(*)$ with a free G–action. The subfunctors F_i of F can be the skeletons of $F(*)$.

Lemma 3.1.4 *For every functor $G\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{sSet}$, there is a free CW–functor F together with a natural equivalence $F \to G$.*

Proof Fix some $n \geq 0$ and suppose, for an induction argument, that we have already constructed a free CW–functor D together with a natural transformation $u\colon D \to G$ such that, for each c in C, the morphism $u_c\colon D(c) \to G(c)$ is $(n-1)$–connected. We want to get rid of the relative homotopy groups

$\pi_n(G(c), D(c))$ for all $c \in \mathsf{C}$. (Strictly speaking, we should write $\pi_n(Z(c), D(c))$, where $Z(c)$ is the mapping cylinder of u_c.) Let $x \in \pi_n(G(c), D(c))$ be a nontrivial element of this homotopy group and let

$$v_{c,x} \colon (K_{c,x}, L_{c,x}) \to G(c)$$

denote a representative of x, where $(K_{c,x}, L_{c,x})$ is a (possibly iterated) barycentric subdivision of the pair $(\Delta[n], \partial\Delta[n])$. Let $D_{c,x}$ be the pushout of

$$\mathrm{Hom}(-, c) \times K_{c,x} \hookleftarrow \mathrm{Hom}(-, c) \times L_{c,x} \to F$$

(where the right-hand arrow extends, and is determined by, $v_{c,x}$ restricted to $L_{c,x}$). Let E be the union along the common subfunctor D of the $D_{c,x}$, where c ranges over all objects of C and x ranges over all nontrivial elements of the homotopy groups $\pi_n(G(c), E(c))$. The choices $v_{c,x}$ together with u uniquely define a new natural transformation $v \colon E \to G$. By construction this specializes to an n–connected map $E(c) \to G(c)$ for every c in C. It remains to be shown that the functor E is again a free CW–functor. To do so, we show that the pairs $(K_{c,x}, L_{c,x})$ are pairs of cell complexes. Each nondegenerate simplex in $K_{c,x} \smallsetminus L_{c,x}$ contributes a free cell to $D_{c,x}$ which is not in D. It follows that $D_{c,x}$ is free CW. Since different choices of (c, x) lead to disjointly attached cells, the union E is free CW as well (and, what is more important, we have shown that it is free CW *relative* to D).

The functor F can now be constructed as the union (sequential colimit) of an increasing sequence

$$F^{-1} \subset F^0 \subset F^1 \subset F^2 \subset \cdots$$

of functors $\mathsf{C} \to \mathsf{sSet}$, each equipped with a morphism $w^n \colon F^n \to G$ such that w^n extends w^{n-1}. Define $F^{-1} = \varnothing$ and define F^n and w^n inductively so that F^n is to F^{n-1} as E is to D above, and w^n is to w^{n-1} as v is to u. The union of the w^n is a morphism $w \colon F \to G$ and it is a weak equivalence by construction. □

Proposition 3.1.5 *Suppose we have some property $\mathfrak{P}$ for contravariant functors from C to sSet. Assume the property $\mathfrak{P}$ is preserved under levelwise weak equivalence, disjoint unions over arbitrary indexing sets and homotopy pushouts and holds for all representable functors. Then $\mathfrak{P}$ holds for every contravariant functor F from C to sSet.*

Proof Without loss of generality, we assume F to be a free CW–functor. We will prove this in two steps. First we show by induction that all skeleta F_i have the desired property and then deduce the statement for the homotopy colimit. The 0–skeleton is just a disjoint union of representable functors and as such has the property $\mathfrak{P}$. To prove the induction step we have to show that F_{i+1} is a homotopy pushout of functors satisfying $\mathfrak{P}$. Note that, because of the homotopy invariance of $\mathfrak{P}$, every cell $\mathrm{Hom}(-, c) \times \Delta[i]$ has property $\mathfrak{P}$. Since we assumed the property to be preserved under disjoint union, the coproduct $\coprod \mathrm{Hom}(-, c) \times \Delta[i]$ still has the property. To conclude the induction step, we need to show our desired property for every functor of the form $\mathrm{Hom}(-, c) \times \partial\Delta[i]$. But this is already covered by the induction assumption because $\mathrm{Hom}(-, c) \times \partial\Delta[i]$ is a free CW–functor built from cells of dimension $i - 1$ and less.

Next, we have to show that the pushout diagram

$$\begin{tikzcd} K_i \times \partial\Delta[i] \arrow[r] \arrow[d] & F_{i-1} \arrow[d] \\ K_i \times \Delta[i] \arrow[r] & F_i \end{tikzcd}$$

is actually a homotopy pushout square. But the left-hand vertical morphism is levelwise injective and so the pushout square is levelwise a homotopy pushout.

So far we have shown the property $\mathfrak{P}$ for all k–skeleta F_k. We now show that the homotopy colimit F can be written as a homotopy pushout. The argument has already been presented by Milnor [1962]. Let

$$tF := F_0 \times [0,1] \cup F_1 \times [1,2] \cup F_2 \times [2,3] \cup \cdots,$$

understood as a subfunctor of $F \times [0,\infty)$. (The intervals can be taken as copies of $\Delta[1]$.) This construction is also known as the *telescope* associated to the skeletal filtration of F. Note that the inclusion of tF into $F \times [0,\infty)$ is a weak equivalence and thus tF has property $\mathfrak{P}$ if and only if F does. We want to show that tF decomposes as a homotopy pushout of functors with property $\mathfrak{P}$. To do so we define the subfunctors L_1 and L_2 of tF by setting

$$L_1 := F_0 \times [0,1] \cup F_2 \times [2,3] \cup \cdots \quad \text{and} \quad L_2 := F_1 \times [1,2] \cup F_3 \times [3,4] \cup \cdots$$

as the even and odd parts of tF, respectively. Their intersection $L_1 \cap L_2$ is the functor

$$L_1 \cap L_2 \cong F_0 \times \{1\} \cup F_1 \times \{2\} \cup \cdots.$$

We can write tF as the pushout of $L_1 \leftarrow L_1 \cap L_2 \rightarrow L_2$. The functors L_i and $L_1 \cap L_2$ are all weakly equivalent to disjoint unions of skeleta F_j and thus have property $\mathfrak{P}$ by the previous discussion. The pushout is also a homotopy pushout because both maps $L_1 \cap L_2 \rightarrow L_i$ are cofibrations. □

Using this principle we can prove Lemma 3.1.1.

Proof of Lemma 3.1.1 We need to verify the three assumptions of the previous lemma. We fix a levelwise fibrant functor $G \in \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{sSet})$ throughout this investigation.

First assume F to be representable by some object c and let C_k be the first subcategory of the sequence $\mathsf{C}_\bullet$ to contain c. Because we assumed all subcategories C_i to be full subcategories the restriction of F to C_k is isomorphic to the functor on C_k represented by c. The same holds for all C_n with $n \geqslant k$. It follows that

$$\mathrm{RHom}(U_n F, U_n G) \simeq U_n G(c) = G(c)$$

for all $n \geqslant k$. It is immediate that the homotopy limit of the tower of derived mapping spaces is weakly equivalent to $\mathrm{RHom}(F, G)$.

Now assume F is a disjoint union of functors F_i for which the tower converges. Since disjoint unions are certainly preserved under restrictions, we have $U_n F = \coprod U_n F_i$. It follows that

$$\mathrm{RHom}(U_n F, U_n G) = \prod \mathrm{RHom}(U_n F_i, U_n G)$$

for all n. We get a commutative square

$$\begin{tikzcd}
\mathrm{RHom}(F,G) \arrow[r] \arrow[d] & \prod \mathrm{RHom}(F_i,G) \arrow[d] \\
\operatorname{holim} \mathrm{RHom}(U_nF,U_nG) \arrow[r] & \operatorname{holim} \prod \mathrm{RHom}(U_nF_i,U_nG)
\end{tikzcd}$$

The two horizontal morphism are weak equivalences by assumption; the right-hand vertical morphism is a weak equivalence because we can commute the homotopy limit with the product. It follows that the morphism

$$\mathrm{RHom}(F,G) \to \operatorname{holim} \mathrm{RHom}(U_nF,U_nG)$$

is a weak equivalence.

For the last step, assume F is the homotopy pushout of $F_1 \leftarrow F_0 \rightarrow F_2$ and the tower converges for all F_i. We can arrange the derived mapping spaces in a commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathrm{RHom}(F,G) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{RHom}(F_2,G) \arrow[dl] \arrow[dd] \\
\mathrm{RHom}(F_1,G) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{RHom}(F_0,G) & \\
& \operatorname{holim}_n \mathrm{RHom}(U_nF,U_nG) \arrow[rr] \arrow[dl] & & \operatorname{holim}_n \mathrm{RHom}(U_nF_2,U_nG) \arrow[dl] \\
\operatorname{holim}_n \mathrm{RHom}(U_nF_1,U_nG) \arrow[rr] & & \operatorname{holim}_n \mathrm{RHom}(U_nF_0,U_nG) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Since the contravariant RHom functor turns homotopy pushouts into homotopy pullbacks, the upper horizontal square is a homotopy pullback. The lower horizontal square is a homotopy pullback because the truncation U_n preserves homotopy pushouts (and hence U_nF is the homotopy pushout of $U_nF_1 \leftarrow U_nF_0 \rightarrow U_nF_2$) and homotopy limits preserve homotopy pullbacks. We can thus regard this cube as a morphism between homotopy pullback squares. This morphism induces a weak equivalence in three columns

$$\mathrm{RHom}(F_i,G) \to \operatorname*{holim}_n \mathrm{RHom}(U_nF_i,U_nG)$$

and thus in the fourth column as well. □

We want to apply this machinery to the setting of dendroidal spaces. To do so we need to write the indexing category Ω of trees as an increasing union of full subcategories

$$\Omega\langle 0\rangle \subset \Omega\langle 1\rangle \subset \Omega\langle 2\rangle \subset \Omega\langle 3\rangle \subset \cdots \subset \Omega.$$

A natural choice for a filtration of Ω comes from the observation that every finite tree has a unique maximal valence among all its vertices. We will thus filter Ω by the maximal valence of the vertices.

Definition 3.1.6 For $n \geq 0$, let $\Omega\langle n\rangle$ denote the full subcategory of Ω on trees without vertices of valence $n+2$ or higher. An *n–truncated dendroidal space* is a contravariant functor from $\Omega\langle n\rangle$ to the category sSet of simplicial sets. The restriction functor along the inclusion $\Omega\langle n\rangle \hookrightarrow \Omega$ will be denoted by U_n.

The categories $\Omega\langle n\rangle$ have the property that their direct limit is the entire category Ω. We can map the derived mapping space $\mathrm{RHom}(X, Y)$ between two dendroidal spaces to a tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathrm{RHom}(U_3X, U_3Y) \arrow[d] \\
 & \mathrm{RHom}(U_2X, U_2Y) \arrow[d] \\
\mathrm{RHom}(X, Y) \arrow[r] \arrow[ur, bend left] \arrow[uur, bend left] \arrow[uuur, bend left] & \mathrm{RHom}(U_1X, U_1Y)
\end{tikzcd}$$

of derived mapping spaces between their truncations. The previous discussion implies the convergence of this tower.

Corollary 3.1.7 *For every pair X and Y of dendroidal spaces, this tower converges, ie the map $\mathrm{RHom}(X, Y) \to \operatorname{holim}_n \mathrm{RHom}(U_nX, U_nY)$ is a weak homotopy equivalence.*

Of special interest will be the mapping space between the little disk operads introduced in Example 2.1.5. Here X and Y are dendroidal spaces weakly equivalent to nerves of operads of type E_n and E_m, respectively.

3.2 The layers of the tower

We look for a description of the layers in the tower, ie the homotopy fibers of the forgetful map(s) $\mathrm{RHom}(U_nX, U_nY) \to \mathrm{RHom}(U_{n-1}X, U_{n-1}Y)$. There is such a description in the setting of 1–*reduced* dendroidal spaces. A dendroidal space X is 1–reduced if $X(\eta)$, $X(\mathrm{c}_0)$ and $X(\mathrm{c}_1)$ are contractible spaces. These correspond to monochromatic operads having contractible spaces in degrees 0 and 1.

Definition 3.2.1 Let $\Omega_{\mathrm{cl}} \subset \Omega$ be the full subcategory whose objects are the trees without any leaves, and let $\iota\colon \Omega_{\mathrm{cl}} \hookrightarrow \Omega$ be the inclusion functor. Objects of Ω_{cl} will be called *closed trees*. We abbreviate $\mathsf{cdSet} := \mathrm{Fun}(\Omega_{\mathrm{cl}}, \mathsf{Set})$ and $\mathsf{scdSet} := \mathrm{Fun}(\Omega_{\mathrm{cl}}, \mathsf{sSet})$. Objects of these categories will be called *closed dendroidal sets* and *closed dendroidal spaces*, respectively.

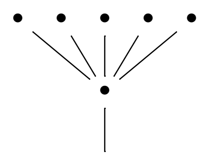
The full subcategory $\Omega_{\mathrm{cl}} \cap \Omega\langle n\rangle$ of Ω_{cl} will be denoted by $\Omega\langle n\rangle_{\mathrm{cl}}$. Its simplicial presheaves will be called *n–truncated closed dendroidal spaces* and their category denoted by $\mathsf{scdSet}\langle n\rangle$.

Remark 3.2.2 Morphisms in Ω_{cl} are much easier to understand than morphisms in Ω. Recall that an object of Ω_{cl} is a finite partially ordered set T (whose elements can be called edges) subject to some conditions. There is no need to specify a set of leaves, a subset L of T, because we are assuming that it is empty. A morphism from T_0 to T_1 in Ω_{cl} was defined to be a morphism of operads $\Omega(T_0) \to \Omega(T_1)$. But this boils down to a map $f : T_0 \to T_1$ which preserves the partial order relation $\le$ and which preserves *independence*. That is to say, if $x, y \in T_0$ are distinct and neither $x \le y$ nor $y \le x$ takes place, then neither $f(x) \le f(y)$ nor $f(y) \le f(x)$ takes place.

It does not follow that such an f is injective. But it does follow that, for every $z \in T_1$, the preimage $f^{-1}(z)$ is a linearly ordered subset of T_0. Moreover, if $x \in T_0 \smallsetminus f^{-1}(z)$ satisfies $x \ge y_0$ for some $y_0 \in f^{-1}(z)$, then it satisfies $x \ge y$ for all $y \in f^{-1}(z)$. (Suppose not; then there is $y_1 \in f^{-1}(z)$ such that x and y_1 are independent, but $f(x) > f(y_0) = f(y_1)$, a contradiction since f preserves independence.)

Definition 3.2.3 The *closed n–corolla* cc_n is the unique tree in Ω_{cl} with one vertex of valence $n+1$ and n vertices of valence 1.

Here is an artist's impression of cc_5:



Lemma 3.2.4 [Boavida de Brito and Weiss 2018, Lemma 7.12] *For all 1–reduced monochromatic topological operads P and Q, the restriction map $\mathrm{RHom}(N_d P, N_d Q) \to \mathrm{RHom}(\iota^* N_d P, \iota^* N_d Q)$ is a weak equivalence.*

In this restricted setting we can define levelwise boundaries and coboundaries, generalizing the levelwise boundaries in the description of the Fulton–MacPherson operad of Example 2.1.6.

Definition 3.2.5 Let $X \in \mathsf{sdSet}$ be a 1–reduced dendroidal Segal space. In this definition we only use the restriction of X to scdSet. The n^{th} *operadic boundary object* is the homotopy colimit

$$\mathrm{bound}_n X := \operatorname*{hocolim}_{(S,f)\in \mathrm{cc}_n/\Omega\langle n-1\rangle_{\mathrm{cl}}} X_S.$$

(This is a homotopy colimit of a contravariant functor, the functor which takes $(f : \mathrm{cc}_n \to S)$ to X_S.) The n^{th} *operadic coboundary object* is the homotopy limit

$$\mathrm{cobound}_n X := \operatorname*{holim}_{(S,f)\in \Omega\langle n-1\rangle_{\mathrm{cl}}/\mathrm{cc}_n} X_S.$$

Both spaces come with an obvious Σ_n–action. There is a natural Σ_n–map from $\mathrm{bound}_n X$ to X_{cc_n} induced by the various f in pairs (S, f), and similarly there is a natural Σ_n–map from X_{cc_n} to $\mathrm{cobound}_n X$ induced by the various f in pairs (S, f). The functor

$$J_n : \mathsf{sdSet} \to \mathrm{Fun}(\Sigma_n \times [2], \mathsf{sSet}), \quad X \mapsto (\mathrm{bound}_n X \to X_{\mathrm{cc}_n} \to \mathrm{cobound}_n X),$$

sends a reduced dendroidal Segal space to the diagram consisting of these two maps. By composing the two maps, we obtain

$$\partial J_n \colon \mathsf{sdSet} \to \mathrm{Fun}(\Sigma_n \times [1], \mathsf{sSet}), \qquad X \mapsto (\mathrm{bound}_n X \to \mathrm{cobound}_n X).$$

In other words, $\partial J_n = J_n \circ \rho$, where $\rho \colon \Sigma_n \times [1] \to \Sigma_n \times [2]$ is induced by the order-preserving injection $[1] \to [2]$ which does not take the value 1.

Example 3.2.6 Recall the Fulton–MacPherson operad FM_k, which we briefly introduced in Example 2.1.6. Then the space $\mathrm{bound}_n N_d \,\mathrm{FM}_k$ has the homotopy type of the boundary of the compact manifold-with-boundary $\mathrm{FM}_k(n)$. Informally, this can be seen because all the embeddings of substrata in the stratification of $\mathrm{FM}_k(n)$ are cofibrations and thus the homotopy colimit in the definition of the operadic boundary object is weakly equivalent to the colimit, which is exactly the boundary $\partial\mathrm{FM}_k(n)$. It follows that we can model the map $\mathrm{bound}_n N_d\, E_k \to E_k(n)$ (for the little k–disk operad E_k) by the inclusion $\partial\mathrm{FM}_k(n) \hookrightarrow \mathrm{FM}_k(n)$.

We next give a more detailed argument:

Proof of Example 3.2.6 In this proof we will use the notation of Definition 3.2.10 and Remark 3.2.11. To show this claim we first want to reduce the indexing category $\mathrm{cc}_n/\Omega\langle n-1\rangle_{\mathrm{cl}}$ to a smaller one. Every morphism $f\colon \mathrm{cc}_n \to T$ for $T \in \Omega\langle n-1\rangle_{\mathrm{cl}}$ factors uniquely through a maximal subtree T_0 of T with exactly n outermost edges and all of them (as well as the root of T_0) in the image of f. (By *subtree* we mean something connected, so that, if $x, z \in T_0$ and $y \in T$ satisfies $x \le y \le z$, then also $y \in T_0$.) Let I denote the subcategory of $\mathrm{cc}_n/\Omega\langle n-1\rangle_{\mathrm{cl}}$ on all pairs (S, g) such that S has exactly n outermost edges and all of them as well as the root are in the image of g. We have just seen that the inclusion functor for this subcategory has a right adjoint. Therefore, the inclusion

$$\operatorname*{hocolim}_{(S,f)\in I} \mathrm{FM}_k(S) \to \mathrm{bound}_n N_d\, \mathrm{FM}_k$$

(where $\mathrm{FM}_k(S)$ is short for $(N_d\,\mathrm{FM}_k)_S$) is a weak equivalence by [Dugger 2008, Theorem 6.7]. Note in passing that the information provided by the $f\colon \mathrm{cc}_n \to S$ in a pair (S, f) amounts to nothing more than a labeling of the outermost edges of S with labels $1, 2, \dots, n$. We take this as an excuse for writing S instead of (S, f), but the labeling of the outermost edges remains important and must be remembered.

Let I^s denote the full subcategory of I on those objects S which have no vertices of valence 2. The inclusion $I^s \to I$ has a left adjoint $\mathrm{sh}\colon I \to I^s$ and the unit morphisms for this adjunction induce isomorphisms $\mathrm{FM}_k(\mathrm{sh}(S)) \to \mathrm{FM}_k(S)$ for (S, f) in I. By [Dugger 2008, Theorem 6.16], the inclusion

$$\operatorname*{hocolim}_{S\in I^s} \mathrm{FM}_k(S) \to \operatorname*{hocolim}_{S\in I} \mathrm{FM}_k(S)$$

is a weak equivalence. Recall the category Ψ_n from Example 2.1.6 of the Fulton–MacPherson operad and let Ψ_n^- denote the full subcategory of Ψ_n on all trees not equal to the n–corolla. This category Ψ_n^- is equivalent to I^S. We can therefore view $S \mapsto \mathrm{FM}_k(S)$ as a functor on Ψ_n^-. It remains only to show that the map

$$\operatorname*{hocolim}_{S \in \Psi_n^-} \mathrm{FM}_k(S) \to \partial \mathrm{FM}_k(n)$$

from the homotopy colimit to the actual colimit of this functor is a homotopy equivalence. The plan is to show that this functor $\mathrm{FM}_k \colon \Psi_n^- \to \mathsf{sSet}/\mathrm{FM}_k(n)$ is projectively cofibrant. The category Ψ_n^- is directed in the sense that there is a faithful functor $\Psi_n^- \to \mathbb{N}$. This is trivial since Ψ_n^- is a finite poset, but here we have a preferred choice: the map which to every tree in Ψ_n associates the number of its vertices. Hence, Ψ_n^- becomes a Reedy category by defining the degree of a tree to be the negative of its number of vertices. Then every nonidentity morphism in Ψ_n^- raises this degree. Let M be some model category. A diagram $D \colon \Psi_n^- \to M$ is Reedy cofibrant if all its latching maps are cofibrations. But, by [Dugger 2008, Theorem 13.12], the Reedy model structure and the projective model structure agree on upwards-directed Reedy categories and thus D is Reedy cofibrant if and only if it is projectively cofibrant. For every projectively cofibrant diagram, its homotopy colimit is weakly equivalent to the actual colimit. We thus want to show that FM_k is Reedy cofibrant as a functor on Ψ_n^-. Let $T \in \Psi_n^-$ be a labeled tree. The latching object $\mathrm{Lat}_T(\mathrm{FM}_k)$ is the colimit over all maps $\mathrm{FM}_k(S) \to \mathrm{FM}_k(T)$ with $S \neq T$. But, using the description of the stratification of $\mathrm{FM}_k(n)$ given in Example 2.1.6, we see that this map is just the inclusion of a union of substrata of the (closure of the) stratum corresponding to T and thus a cofibration. □

Theorem 3.2.7 *Let X and Y be 1–reduced dendroidal Segal spaces. Then the following square of specialization or restriction maps is a homotopy pullback square*:

$$\begin{tikzcd}
\mathrm{RHom}(X|_{\Omega\langle n\rangle_{\mathrm{cl}}}, Y|_{\Omega\langle n\rangle_{\mathrm{cl}}}) \arrow[r] \arrow[d] & \mathrm{RNat}(J_n X, J_n Y) \arrow[d] \\
\mathrm{RHom}(X|_{\Omega\langle n-1\rangle_{\mathrm{cl}}}, Y|_{\Omega\langle n-1\rangle_{\mathrm{cl}}}) \arrow[r] & \mathrm{RNat}(\partial J_n X, \partial J_n Y)
\end{tikzcd}$$

In the remainder of this section we will make a reduction (by means of Proposition 3.2.9) of this theorem to an easier statement. The idea is to factor the inclusion $\Omega\langle n-1\rangle_{\mathrm{cl}} \to \Omega\langle n\rangle_{\mathrm{cl}}$ through certain intermediate subcategories V_n and W_n.

Definition 3.2.8 Let V_n be the full subcategory of Ω_{cl} on $\Omega\langle n-1\rangle_{\mathrm{cl}}$ and the closed n–corolla cc_n. Let W_n be the (slightly larger) full subcategory of $\Omega\langle n\rangle_{\mathrm{cl}}$ on all objects of $\Omega\langle n-1\rangle_{\mathrm{cl}}$ and all *extended* (closed) n–corollas. These are the objects of Ω_{cl} which are connected to cc_n by a sequence of degeneracies. For $n \neq 1$ they have a unique vertex of valence $n+1$ and only vertices of valence 2 and 1 otherwise.

Proposition 3.2.9 *Let $X, Y \in \mathsf{scdSet}$ be restrictions of 1–reduced dendroidal Segal spaces. Then the restriction map*

$$\mathrm{RHom}(X|_{\Omega\langle n\rangle_{\mathrm{cl}}}, Y|_{\Omega\langle n\rangle_{\mathrm{cl}}}) \to \mathrm{RHom}(X|_{\mathsf{V}_n}, Y|_{\mathsf{V}_n})$$

is a weak equivalence.

In the following proofs we will also need the notion of subtree of a given tree T. This has already been used in Example 3.2.6.

Definition 3.2.10 Let T be a tree in Ω_{cl}. A *subtree* S of T consists of a subset of edges of T such that the resulting graph is connected. In this case S should be understood as an object of Ω_{cl} such that the inclusion $S \subset T$ is a morphism in Ω_{cl}.

For example, the closed k–corolla cc_k can be realized as a subtree of the closed n–corolla cc_n, in $\binom{n}{k}$ different ways, assuming $k \neq 0$.

Remark 3.2.11 [Dugger 2008, Chapter 6] Let $\alpha\colon I \to J$ be a functor between small categories. For any $j \in J$, let $(j \downarrow \alpha)$ denote the category whose objects are pairs $(i, f : j \to \alpha(i))$ and morphisms $(i, f) \to (i', f')$ are given by commutative triangles

$$\begin{tikzcd} j \arrow[r, "f"] \arrow[dr, "f'"'] & \alpha(i) \arrow[d] \\ & \alpha(i') \end{tikzcd}$$

The functor α is called *homotopy terminal* if, for every j, the category $(j \downarrow \alpha)$ has a contractible classifying space. Homotopy terminal functors can be used to simplify homotopy colimits. More precisely, for any homotopy terminal α and any diagram $X : J \to \mathsf{sSet}$, there is a natural weak equivalence

$$\operatorname*{hocolim}_I \alpha^* X \to \operatorname*{hocolim}_J X.$$

There is a dual notion of a *homotopy initial* functor $\beta\colon I \to J$. It has the property that all overcategories $(\beta \downarrow j)$, defined dually to the undercategories above, are nonempty and contractible. In this case there is a natural weak equivalence

$$\operatorname*{holim}_I \beta^* Y \leftarrow \operatorname*{holim}_J Y$$

for all diagrams $Y : J \to \mathsf{sSet}$.

In the following we write RRan_F and RLan_F, respectively, for derived right and left Kan extensions along a functor F.

Lemma 3.2.12 *Let X in scdSet be a 1–reduced dendroidal Segal space. Let ϕ denote the inclusion of W_n in $\Omega\langle n\rangle_{\mathrm{cl}}$. Then the derived unit morphism*

$$X|_{\Omega\langle n\rangle_{\mathrm{cl}}} \to \mathrm{RRan}_\phi\, \phi^*(X|_{\Omega\langle n\rangle_{\mathrm{cl}}})$$

is a weak equivalence.

Proof We allow ourselves to write $\mathrm{RRan}_\phi\, \phi^* X$ instead of the more complicated expression in the statement. By definition, we have

$$(\mathrm{RRan}_\phi\, \phi^* X)_T \simeq \operatorname*{holim}_{(f : S \to T) \in \mathsf{W}_n/T} X_S.$$

(We may also write (S, f) instead of $(f : S \to T)$.) It suffices to show that $\mathrm{RRan}_\phi\, \phi^* X$ has the Segal property. Indeed, the unit maps $X_T \to (\mathrm{RRan}_\phi\, \phi^* X)_T$ are weak equivalences for all (closed) corollas T in $\Omega\langle n\rangle_{\mathrm{cl}}$. This follows from the fact that all these corollas are already in W_n. We can furthermore assume $\mathrm{RRan}_\phi\, \phi_* X$ to be projectively fibrant by choosing a fibrant model for a homotopy limit of Kan complexes.

We want to replace the indexing category W_n/T by an easier subcategory C such that its inclusion functor is homotopy terminal. The set of objects of C is the set of subtrees, as defined in Definition 3.2.10, of T, understood as pairs (A, a) of subtrees A (which are objects of W_n in their own right) with fixed inclusions $a\colon A \to T$. Maps $\tau\colon (A, a) \to (B, b)$ are given by commutative triangles

$$\begin{tikzcd} T & & A \arrow[ll, "a"'] \arrow[dl, "\tau"] \\ & B \arrow[ul, "b"] & \end{tikzcd}$$

The inclusion of C in W_n/T has a left adjoint. (This works only because we are using W_n instead of the smaller V_n.) Consequently, the inclusion of C into W_n/T is indeed homotopy terminal. It follows from the contravariant version of [Dugger 2008, Theorem 6.12] that the forgetful projection

$$\operatorname*{holim}_{(S,f)\in\mathsf{W}_n/T} X_S \to \operatorname*{holim}_{(A,a)\in\mathsf{C}} X_A$$

is a weak equivalence.

Let X' denote the induced functor $\mathsf{C} \to \mathsf{sSet}$. We want to prove that the decompositions of the X'_S given by the Segal property of X are natural in this reduced setting, ie every morphism $S \to S'$ in C induces a map $X'_{S'} \to X'_S$ which respects the product decomposition and can be defined factorwise. Let $g\colon S \to S'$ be any morphism in C. It is an inclusion of a subtree. Let $\{v_1, \dots, v_k\}$ be the set of vertices of T. Let $|v_j|_S$ denote the number of inputs of S at v_j. Since we assumed X to be 1–reduced and to satisfy the Segal property, we know that the morphism

$$X_S \to X^{\mathrm{Sc}[S]} \cong X_{\mathrm{cc}_{|v_1|_S}} \times \cdots \times X_{\mathrm{cc}_{|v_k|_S}}$$

induced by the Segal core inclusion $\mathrm{Sc}[S] \to \Omega_{\mathrm{cl}}[S]$ is a trivial Kan fibration. By functoriality of X, the square

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d] & X_S \arrow[d] \\ X^{\mathrm{Sc}[S']} \arrow[r] & X^{\mathrm{Sc}[S]} \end{tikzcd}$$

commutes. We only have to show that the maps $X^{\mathrm{Sc}[S']} \to X^{\mathrm{Sc}[S]}$ can be defined factorwise. But this is immediate because the map $\mathrm{Sc}[S] \to \mathrm{Sc}[S']$ is induced by a morphism $S \to S'$ over T. Let X'' denote the functor $\mathsf{C} \to \mathsf{sSet}$ defined by the composition $X' \circ \mathrm{Sc}$. We have a natural transformation $X' \to X''$ which is a levelwise trivial fibration. Hence,

$$\operatorname*{holim}_{S\in\mathsf{C}} X_S \simeq \operatorname*{holim}_{S\in\mathsf{C}} X''_S. \qquad \square$$

Lemma 3.2.13 *Let $X \in \mathsf{scdSet}$ be a 1–reduced dendroidal Segal space. Write ψ for the inclusion $\psi\colon \mathsf{V}_n \to \mathsf{W}_n$. The derived counit map*

$$\mathrm{RLan}_\psi\ \psi^*(X|_{\mathsf{W}_n}) \to X|_{\mathsf{W}_n}$$

is a weak equivalence.

Proof We have to show that this morphism induces a weak equivalence of simplicial sets at any tree T. For trees in V_n there is nothing to show. So let T be an extended (closed) n–corolla. The space $(\mathrm{RLan}_\psi\ \psi^*(X|_{\mathsf{W}_n}))_T$ is the homotopy colimit

$$\operatorname*{hocolim}_{(S,f)\in(T/\mathsf{V}_n)} X_S.$$

Here (S,f) is short for $f\colon T\to S$. We want to find an easier category C and a homotopy initial functor $\mathsf{C}\to T/\mathsf{V}_n$. Let the set of objects of C be the set of all pairs (S,f) such that S is a tree in V_n with exactly n vertices of valence 1 and $f\colon T\to S$ is a morphism such that every outermost edge (including the root) of S is in the image of f. A morphism $\tau\colon (S_0,f)\to(S_1,g)$ is given by a commutative triangle

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "g"'] & & S_0 \arrow[dl, "\tau"] \\ & S_1 & \end{tikzcd}$$

in W_n. The category C is a full subcategory of T/V_n. The inclusion functor $\mathsf{C}\to T/\mathsf{V}_n$ has a right adjoint. This implies that the inclusion is homotopy initial. By the contravariant version of [Dugger 2008, Theorem 6.7], we get a weak equivalence

$$\operatorname*{hocolim}_{(S,f)\text{ in }\mathsf{C}} \phi^* X_S \xrightarrow{\simeq} \mathrm{RLan}_\psi \psi^*\phi^* X_T.$$

In a second step we replace the indexing category C by another even easier one. Let C^{sh} be the full subcategory of C on the pairs (S,f) such that S has no vertices of valence 2. The inclusion $\alpha\colon \mathsf{C}^{\mathrm{sh}}\to\mathsf{C}$ has a left adjoint β. Clearly $\beta\alpha=\mathrm{Id}$ and the counit transformation for this adjunction is the identity. Let $\tau\colon \mathrm{Id}_{\mathsf{C}}\to\alpha\beta$ denote the unit transformation for this adjunction. Because X has the Segal property and is 1–reduced, the induced map $\tau_S^*\colon X_{\beta(S)}\to X_S$ is a weak equivalence for every (S,f) in C. The contravariant version of [Dugger 2008, Proposition 6.16] thus implies that the canonical morphism

$$\operatorname*{hocolim}_{(S,f)\text{ in }\mathsf{C}^{\mathrm{sh}}} X_S \to \operatorname*{hocolim}_{(S,f)\text{ in }\mathsf{C}} X_S$$

is a weak equivalence. The category C^{sh} has an initial object given by the closed n–corolla cc_n together with the unit map $T\to\mathrm{cc}_n$. Thus,

$$X_{\mathrm{cc}_n} \simeq \operatorname*{hocolim}_{(S,f)\text{ in }\mathsf{C}^{\mathrm{sh}}} X_S.$$

By our assumptions on X, this concludes the proof. □

Thus, Theorem 3.2.7 reduces to:

Theorem 3.2.14 *Let X and Y be 1–reduced dendroidal Segal spaces. The following square of specialization maps is a homotopy pullback*:

$$\begin{tikzcd}
\mathrm{RHom}(X|_{\vee_n}, Y|_{\vee_n}) \arrow[r] \arrow[d] & \mathrm{RNat}(J_n X, J_n Y) \arrow[d] \\
\mathrm{RHom}(X|_{\Omega\langle n-1\rangle_{\mathrm{cl}}}, Y|_{\Omega\langle n-1\rangle_{\mathrm{cl}}}) \arrow[r] & \mathrm{RNat}(\partial J_n X, \partial J_n Y)
\end{tikzcd}$$

This now follows from Theorem 4.1.2. We note that the maps in the square need careful definitions. They will be given in Section 4. The right-hand column of this homotopy pullback square can be modified as explained in the following remark:

Remark 3.2.15 The following square is a homotopy pullback square (and we switch from n to k):

$$\begin{tikzcd}
\mathrm{RNat}(J_k X, J_k Y) \arrow[r] \arrow[d] & \mathrm{RMap}_{\Sigma_k}(X_{\mathrm{cc}_k}, Y_{\mathrm{cc}_k}) \arrow[d] \\
\mathrm{RNat}(\partial J_k X, \partial J_k Y) \arrow[r] & \mathrm{RMap}_{\Sigma_k}(\mathrm{bound}_k X \to X_{\mathrm{cc}_k}, Y_{\mathrm{cc}_k} \to \mathrm{cobound}_k Y)
\end{tikzcd}$$

The horizontal maps are the obvious forgetful maps. The right-hand vertical arrow is explained by the diagram

$$\begin{tikzcd}
\mathrm{bound}_k X \arrow[d] \arrow[dr, dashed] & \\
X_{\mathrm{cc}_k} \arrow[r] \arrow[dr, dashed] & Y_{\mathrm{cc}_k} \arrow[d] \\
& \mathrm{cobound}_k Y
\end{tikzcd}$$

(A short argument for the homotopy pullback property: compare the horizontal homotopy fibers.) Therefore, the main theorem as stated in the introduction is equivalent to the statement that the square

$$\begin{tikzcd}
\mathrm{RHom}(U_k X, U_k Y) \arrow[r] \arrow[d] & \mathrm{RMap}_{\Sigma_k}(X_{\mathrm{cc}_k}, Y_{\mathrm{cc}_k}) \arrow[d] \\
\mathrm{RHom}(U_{k-1} X, U_{k-1} Y) \arrow[r] & \mathrm{RMap}_{\Sigma_k}(\mathrm{bound}_k X \to X_{\mathrm{cc}_k}, Y_{\mathrm{cc}_k} \to \mathrm{cobound}_k Y)
\end{tikzcd}$$

is a homotopy pullback square. Note that $U_k X$ can be read as $X|_{\Omega\langle k\rangle_{\mathrm{cl}}}$, etc. The interesting "news" here is that, for a homotopical description of k^{th} layer in the tower, we need to know only the diagrams $\mathrm{bound}_k X \to X_{\mathrm{cc}_k}$ (for the source) and $Y_{\mathrm{cc}_k} \to \mathrm{cobound}_k Y$ (for the target). As before, they are diagrams of spaces with an action of Σ_k.

3.3 The derived mapping spaces of little cube operads

In this section we will apply the machinery we developed to a concrete case. We will compute the connectivity of the layers of the tower for $\mathrm{RHom}(N_d E_n, N_d E_{n+d})$. (Beware that the d in N_d means dendroidal and everywhere else the d is used to denote the codimension.) Rational versions of the results of this section were obtained by Fresse, Turchin and Willwacher [Fresse et al. 2017, Chapter 10].

Remark 3.3.1 The derived mapping space $\mathrm{RHom}(N_d E_n|_{\Omega\langle 1\rangle_{\mathrm{cl}}}, N_d E_{n+d}|_{\Omega\langle 1\rangle_{\mathrm{cl}}})$ is contractible.

Lemma 3.3.2 *The pair $(E_n(k), \mathrm{bound}_k E_n)$ is homotopy equivalent to a CW pair (X, Y) with no relative cells of dimension above $n(k-1)-1$.*

Proof This follows from the construction of the Fulton–MacPherson operad. Notably, the inclusion of the boundary $\partial\mathrm{FM}_n(k) \to \mathrm{FM}_n(k)$ is a model for the operadic boundary inclusion map and the smooth manifold $\mathrm{FM}_n(k)$ has dimension $n(k-1)-1$. We have shown this in Example 3.2.6. □

Lemma 3.3.3 *The map $E_n(k) \to \mathrm{cobound}_k(E_n)$ is $((k-1)(n-2)+1)$–connected.*

Proof Munson and Volić [2015, Example 6.2.9] show that the k–cube of ordered configuration spaces defined by $S \mapsto \mathrm{Conf}(S, M)$ for $S \subset \{1, 2, \dots, k\}$ and a fixed n–dimensional manifold M is $((k-1)(n-2)+1)$–cartesian. □

Theorem 3.3.4 *Each homotopy fiber of*

$$\mathrm{RNat}(J_k E_n, J_k E_{n+d}) \to \mathrm{RNat}(\partial J_k E_n, \partial J_k E_{n+d})$$

is $((k-1)(d-2)+1)$–connected.

Proof By Remark 3.2.15, the homotopy fiber is equivalent to a total homotopy fiber of the square

$$\begin{tikzcd}
\mathrm{Map}(E_n(k), E_{n+d}(k)) \arrow[r] \arrow[d] & \mathrm{Map}(E_n(k), \mathrm{cobound}_k E_{n+d}) \arrow[d] \\
\mathrm{Map}(\mathrm{bound}_k E_n, E_{n+d}(k)) \arrow[r] & \mathrm{Map}(\mathrm{bound}_k E_n, \mathrm{cobound}_k E_{n+d})
\end{tikzcd}$$

By general principles, the connectivity of such a total homotopy fiber is not less than the connectivity of $\mathrm{hofib}(E_{n+d}(k) \to \mathrm{cobound}_k E_{n+d})$ minus the relative homotopical dimension of the inclusion $\mathrm{bound}_k E_n \hookrightarrow E_n(k)$. By Lemma 3.3.2, the first number is at least $(k-1)(n+d-2)$, and the second number is $n(k-1)-1$ by Lemma 3.3.3. The difference of these numbers turns out to be $(k-1)(d-2)+1$. □

Corollary 3.3.5 *Assume $d \geq 2$. The derived mapping space $\mathrm{RHom}(E_n, E_{n+d})$ is $(d-1)$–connected. Furthermore, all spaces of derived maps between their truncations are $(d-1)$–connected as well.*

4 Excision in categories

Let I be a small category. In this section we will investigate how spaces of natural transformations between two I–shaped diagrams of simplicial sets change if we remove or add a new object (with morphisms) from or to I. Throughout this section all model categories (of I–diagrams) are equipped with the projective model structure. We choose functorial fibrant and cofibrant replacements. (Their existence follows from the small object argument.) As a model for derived mapping spaces $\mathrm{RNat}(F, G)$, we choose the simplicial mapping space between the cofibrant and fibrant replacements $\mathrm{Map}(F^c, G^f)$.

4.1 Boundaries and coboundaries revisited

From now on we assume the indexing category I to be skeletal (distinct objects are not isomorphic). Let x be an object of I such that no object i (distinct from x) which admits a morphism to x receives a morphism from x. We also require that every endomorphism of x be an automorphism. This is, for example, the case if I is a direct category. The most important example for us is the case where $I = (\mathsf{V}_n)^{\mathrm{op}}$ and $x = \mathrm{cc}_n$.

The functor J_n of Definition 3.2.5 generalizes in an obvious way to a functor J_x for arbitrary I–shaped diagrams of spaces.

Definition 4.1.1 Let I and x be as above. For any diagram $F \in \mathrm{Fun}(I, \mathsf{sSet})$ we define

$$\mathrm{bound}_x(F) := \operatorname*{hocolim}_{(f : y \to x) \text{ in } I/x} F(y) \quad \text{and} \quad \mathrm{cobound}_x(F) := \operatorname*{holim}_{(g : x \to y) \text{ in } x/I} F(y).$$

These spaces serve as a replacement for the operadic boundary and coboundary space, respectively. They come with a natural action of the automorphism group of x. The functor

$$J_x : \mathrm{Fun}(I, \mathsf{sSet}) \to \mathsf{sSet}^{\mathrm{Aut}(x) \times [2]}$$

is now defined by sending a diagram F to the sequence

$$\mathrm{bound}_x(F) \to F(x) \to \mathrm{cobound}_x(F).$$

We note this is to be viewed as a functor from $\mathrm{Aut}(x) \times [2]$ to simplicial sets. Then ∂J_x is defined by sending F to the subsequence

$$\mathrm{bound}_x(F) \to \mathrm{cobound}_x(F)$$

with the same equivariance properties. Both functors respect levelwise weak equivalences. There is a functor ρ from $\mathsf{sSet}^{\mathrm{Aut}(x)\times[2]}$ to $\mathsf{sSet}^{\mathrm{Aut}(x)\times[1]}$ given by omitting the middle object and composing the two morphisms; we can write $\rho(J_x)$ instead of ∂J_x.

We obtain a commutative diagram of mapping spaces

$$\begin{tikzcd}
\mathrm{Map}(F^c, G^f) \arrow[r] \arrow[d] & \mathrm{Map}(J_x F^c, J_x G^f) \arrow[r] \arrow[d] & \mathrm{Map}((J_x F^c)^c, (J_x G^f)^f) \arrow[d] \\
\mathrm{Map}((F^c)^r, (G^f)^r) \arrow[r] \arrow[d] & \mathrm{Map}(\partial J_x F^c, \partial J_x G^f) \arrow[r] \arrow[d] & \mathrm{Map}((\partial J_x F^c)^c, (\partial J_x G^f)^f) \arrow[d] \\
\mathrm{Map}\big(((F^c)^r)^c, ((G^f)^r)^f\big) \arrow[r] & \mathrm{Map}\big(\partial J_x\big(((F^c)^r)^c\big), \partial J_x\big(((G^f)^r)^f\big)\big) \arrow[r] & \mathrm{Map}\big((\partial J_x (F^c)^c)^c, (\partial J_x (G^f)^f)^f\big)
\end{tikzcd}$$

where the superscripts c, f and r have the following meaning: c is for cofibrant replacement, f is for fibrant replacement and r is for restriction from I to $I \setminus x$, a full subcategory of I.

We will informally abbreviate the outer square to

$$\begin{tikzcd}
\mathrm{RNat}(F, G) \arrow[r] \arrow[d] & \mathrm{RNat}(J_x F, J_x G) \arrow[d] \\
\mathrm{RNat}(F^r, G^r) \arrow[r] & \mathrm{RNat}(\partial J_x F, \partial J_x G)
\end{tikzcd}$$

Justification: there are natural weak equivalences

$$\begin{gathered}
\mathrm{RNat}(F^r, G^r) \to \mathrm{Map}\big(((F^c)^r)^c, ((G^f)^r)^f\big), \\
\mathrm{RNat}(J_x F, J_x G) \to \mathrm{Map}((J_x F^c)^c, (J_x G^f)^f), \\
\mathrm{RNat}(\partial J_x F, \partial J_x G) \to \mathrm{Map}\big((\partial J_x (F^c)^c)^c, (\partial J_x (G^f)^f)^f\big)
\end{gathered}$$

given by suitable pre- and postcompositions.

Theorem 4.1.2 *Let I and $x \in I$ be as above. Let F and G be functors from I to* sSet *and let F^r and G^r be their restrictions to $I \smallsetminus x$. Then the following square is a homotopy pullback*:

$$\begin{tikzcd}
\mathrm{RNat}(F, G) \arrow[r] \arrow[d] & \mathrm{RNat}(J_x F, J_x G) \arrow[d] \\
\mathrm{RNat}(F^r, G^r) \arrow[r] & \mathrm{RNat}(\partial J_x F, \partial J_x G)
\end{tikzcd}$$

4.2 Dévissage

We will prove Theorem 4.1.2 in three steps using the principle we developed in Proposition 3.1.5. Throughout these steps, we will keep the notation of Theorem 4.1.2.

Lemma 4.2.1 *Let F the homotopy pushout of $F_1 \leftarrow F_0 \rightarrow F_2$. If Theorem 4.1.2 holds for the F_i, then it holds for F.*

Proof We observe that the two terms on the left-hand side turn homotopy pushouts into homotopy pullbacks. It follows that each vertical homotopy fiber in the left-hand column for F becomes a homotopy pullback of the respective fibers for the F_i.

To understand the right-hand homotopy fibers we note that we can arrange the resulting spaces into a cube

$$\begin{tikzcd}
& \mathrm{RHom}(J_x F, J_x G) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{RHom}(J_x F_1, J_x G) \arrow[dl] \arrow[dd] \\
\mathrm{RHom}(J_x F_2, J_x G) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{RHom}(J_x F_0, J_x G) & \\
& \mathrm{RHom}(\partial J_x F, \partial J_x G) \arrow[rr] \arrow[dl] & & \mathrm{RHom}(\partial J_x F_1, \partial J_x G) \arrow[dl] \\
\mathrm{RHom}(\partial J_x F_2, \partial J_x G) \arrow[rr] & & \mathrm{RHom}(\partial J_x F_0, \partial J_x G) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

We want to prove that this cube is homotopy cartesian. To do so we pick a point in the initial term $\mathrm{RHom}(\partial J_x F, \partial J_x G)$ of the lower square and thus in each term of the lower square. Then we obtain a square of vertical homotopy fibers and we want to show this square is homotopy cartesian.

One of these vertical homotopy fibers, the homotopy fiber of

$$\mathrm{RHom}(J_x F, J_x G) \to \mathrm{RHom}(\partial J_x F, \partial J_x G),$$

consists of $\mathrm{Aut}(x)$–equivariant lifts

$$\begin{tikzcd}
\mathrm{bound}_x F \arrow[r] \arrow[d] & F(x) \arrow[r] \arrow[d, dotted] & \mathrm{cobound}_x F \arrow[d] \\
\mathrm{bound}_x G \arrow[r] & G(x) \arrow[r] & \mathrm{cobound}_x G
\end{tikzcd}$$

Thus, a lift consists of an equivariant morphism $F(x) \to G(x)$, compatible homotopies $\mathrm{bound}_x F \times \Delta[1] \to G(x)$ and $F(x) \times \Delta[1] \to \mathrm{cobound}_x G$, and a homotopy of homotopies $\mathrm{bound}_x F \times (\Delta[1] \times \Delta[1]) \to \mathrm{cobound}_x G$. More formally, the space of lifts is the total homotopy fiber of the square

$$\begin{tikzcd}
\mathrm{RHom}(F(x), G(x)) \arrow[r] \arrow[d] & \mathrm{RHom}(F(x), \mathrm{cobound}_x G) \arrow[d] \\
\mathrm{RHom}(\mathrm{bound}_x F, G(x)) \arrow[r] & \mathrm{RHom}(\mathrm{bound}_x F, \mathrm{cobound}_x G)
\end{tikzcd}$$

over the (three) basepoints determined by the basepoint in $\mathrm{RHom}(\partial J_x F, \partial J_x G)$ which we selected. We obtain similar results for the other three vertical homotopy fibers by replacing F with F_i. From these descriptions, it is clear that the square formed by the vertical homotopy fibers is homotopy cartesian. Therefore, the cube is homotopy cartesian.

To conclude the statement, we note that there is a morphism of homotopy cartesian cubes from

$$\begin{tikzcd}
& \mathrm{RHom}(F, G) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{RHom}(F_1, G) \arrow[dl] \arrow[dd] \\
\mathrm{RHom}(F_2, G) \arrow[rr] \arrow[dd] & & \mathrm{RHom}(F_0, G) \arrow[dd] & \\
& \mathrm{RHom}(F^r, G^r) \arrow[rr] \arrow[dl] & & \mathrm{RHom}(F_1^r, G^r) \arrow[dl] \\
\mathrm{RHom}(F_2^r, G^r) \arrow[rr] & & \mathrm{RHom}(F_0^r, G^r) &
\end{tikzcd}$$

to

$$\begin{tikzcd}
& \mathrm{RHom}(J_x F, J_x G) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{RHom}(J_x F_1, J_x G) \arrow[dl] \arrow[dd] \\
\mathrm{RHom}(J_x F_2, J_x G) \arrow[rr] \arrow[dd] & & \mathrm{RHom}(J_x F_0, J_x G) \arrow[dd] & \\
& \mathrm{RHom}(\partial J_x F, \partial J_x G) \arrow[rr] \arrow[dl] & & \mathrm{RHom}(\partial J_x F_1, \partial J_x G) \arrow[dl] \\
\mathrm{RHom}(\partial J_x F_2, \partial J_x G) \arrow[rr] & & \mathrm{RHom}(\partial J_x F_0, \partial J_x G) &
\end{tikzcd}$$

which we can view as a (homotopy cartesian) 4–cube or, better, as a square of squares. By assumption, the three squares

$$\begin{array}{ccc} \mathrm{RNat}(F_i, G) & \longrightarrow & \mathrm{RHom}(J_x F_i, J_x G) \\ \big\downarrow & & \big\downarrow \\ \mathrm{RNat}(F_i^r, G^r) & \longrightarrow & \mathrm{RHom}(\partial J_x F_i, \partial J_x G) \end{array}$$

are homotopy cartesian. It follows that the square

$$\begin{array}{ccc} \mathrm{RNat}(F, G) & \longrightarrow & \mathrm{RHom}(J_x F, J_x G) \\ \big\downarrow & & \big\downarrow \\ \mathrm{RNat}(F^r, G^r) & \longrightarrow & \mathrm{RHom}(\partial J_x F, \partial J_x G) \end{array}$$

is homotopy cartesian as well. □

Lemma 4.2.2 *Let F be the levelwise disjoint union of functors F_α. If Theorem 4.1.2 holds for the F_α, then it holds for F.*

Proof The left-hand side of the square is easy to understand. If $F = \coprod_\alpha F_\alpha$, then $\mathrm{RNat}(F, G) = \prod_\alpha \mathrm{RNat}(F_\alpha, G)$; similarly, since restriction preserves disjoint unions, $F^r = \coprod_\alpha F_\alpha^r$ and $\mathrm{RNat}(F^r, G^r) = \prod_\alpha \mathrm{RNat}(F_\alpha^r, G^r)$.

On the right-hand side, we have $\mathrm{bound}_x F = \coprod_\alpha \mathrm{bound}_x F_\alpha$. The proof follows like the previous one by comparison of the vertical homotopy fibers. Each left-hand vertical homotopy fiber for F decomposes as a product of the corresponding left-hand vertical homotopy fibers for the F_α.

As we have seen in the proof of the previous lemma, each right-hand vertical homotopy fiber is the total homotopy fiber of

$$\begin{array}{ccc} \mathrm{RHom}(F(x), G(x)) & \longrightarrow & \mathrm{RHom}(\mathrm{bound}_x F, G(x)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{RHom}(F(x), \mathrm{cobound}_x G) & \longrightarrow & \mathrm{RHom}(\mathrm{bound}_x F, \mathrm{cobound}_x G) \end{array}$$

over the three basepoints obtained by choosing a point in $\mathrm{RNat}(\partial J_x F, \partial J_x G)$. Both terms $F(x)$ and $\mathrm{bound}_x F$ preserve disjoint unions and thus each right-hand vertical homotopy fiber splits as a product. By assumption, the squares

$$\begin{array}{ccc} \mathrm{RNat}(F_\alpha, G) & \longrightarrow & \mathrm{RHom}(J_x F_\alpha, J_x G) \\ \big\downarrow & & \big\downarrow \\ \mathrm{RNat}(F_\alpha^r, G^r) & \longrightarrow & \mathrm{RHom}(\partial J_x F_\alpha, \partial J_x G) \end{array}$$

are homotopy cartesian. □

Lemma 4.2.3 *Theorem 4.1.2 holds for F representable.*

Proof Let $F=\mathrm{Hom}(y,-)$; in other words, F is (co)represented by y. We will distinguish three different cases. First assume $y=x$. Then $\mathrm{RHom}(F,G)\simeq G(x)$ and the left-hand vertical arrow becomes

$$\begin{tikzcd} G(x) \arrow[d] \\ \mathrm{holim}_{x\to z} G(z) = \mathrm{cobound}_x G \end{tikzcd}$$

On the right-hand side, we have $J_x F=(\varnothing\to\mathrm{Hom}(x,x)\to\mathrm{cobound}_x F)$ and consequently

$$\partial J_x F=(\varnothing\to\mathrm{cobound}_x F).$$

Thus, a point in the right-hand vertical homotopy fiber consists of a choice of an $\mathrm{Aut}(x)$–equivariant lift

$$\begin{tikzcd} \mathrm{Hom}(x,x) \arrow[r, dotted] \arrow[d] & G(x) \arrow[d] \\ \mathrm{cobound}_x F \arrow[r] & \mathrm{cobound}_x G \end{tikzcd}$$

But, since $\mathrm{Hom}(x,x)$ is the free $\mathrm{Aut}(x)$–space on a point (by assumption), these lifts are in one-to-one correspondence with nonequivariant lifts of points in $\mathrm{cobound}_x G$ to $G(x)$. This space is homotopy equivalent to the left-hand homotopy fiber and the induced map between the two is a weak equivalence. Thus, the square of Theorem 4.1.2 is a homotopy pullback.

As a second case we assume that there is a morphism $y\to x$ but $y\neq x$. In this case, the left-hand vertical morphism is homotopic to the identity

$$\begin{tikzcd} G(y) \arrow[d] \\ G^r(y)=G(y) \end{tikzcd}$$

On the right-hand side, we see that

$$\mathrm{bound}_x F=\operatorname*{hocolim}_{z\to x}\mathrm{Hom}(y,z)\simeq\mathrm{Hom}(y,x).$$

This is because we can write $\mathrm{hocolim}_{z\to x}\mathrm{Hom}(y,z)$ as the classifying space of the category of diagrams of the form $y\to z\to x$ with fixed y and x. That category of diagrams has a subcategory consisting of the diagrams

$$y\xrightarrow{\mathrm{Id}}y\to x.$$

The inclusion of this subcategory has a right adjoint given by

$$(y\xrightarrow{f}z\xrightarrow{g}x)\mapsto(y\xrightarrow{\mathrm{Id}}y\xrightarrow{gf}x).$$

Thus, the morphism $\mathrm{bound}_x F \to F(x) = \mathrm{Hom}(y,x)$ is a weak equivalence. We are thus looking for (equivariant) solutions of

$$\begin{tikzcd} \mathrm{Hom}(y,x) \arrow[r] \arrow[d, equal] & \mathrm{bound}_x G \arrow[d] \\ \mathrm{Hom}(y,x) \arrow[r, dotted] \arrow[d] & G(x) \arrow[d] \\ \mathrm{cobound}_x F \arrow[r] & \mathrm{cobound}_x G \end{tikzcd}$$

(the broken arrow, two primary homotopies and a secondary homotopy). But the middle morphism was already determined to be the composition

$$\mathrm{RHom}(y,x) \to \mathrm{bound}_x G \to G(x)$$

and hence the right-hand homotopy fiber is contractible as well.

Now assume there is no morphism $y \to x$. Then $\mathrm{bound}_x F$ as well as $F(x)$ are empty sets. In this case, both vertical morphisms are isomorphisms and the square is a homotopy pullback square. Thus, the statement holds for all representable functors. $\square$

Remark 4.2.4 In the case $I^{\mathrm{op}} = \mathsf{V}_n$ and $x = \mathrm{cc}_n$, the object $\mathrm{bound}_x F$ is naturally weakly equivalent to $\mathrm{hocolim}_{\mathrm{cc}_n \to S} F(S)$, where we think of F as contravariant and we only allow morphisms $\mathrm{cc}_n \to S$ in V_n satisfying two conditions: the outer edges of S (including the root) are in the image and S has no vertices of valence 2. This was essentially proved in Example 3.2.6. Similarly, we can restrict the homotopy limit $\mathrm{holim}_{S \to \mathrm{cc}_n} F(S)$ to the category of subtrees (as defined in Definition 3.2.10) of cc_n. Therefore, the coboundary object is equivalent to a homotopy limit over a punctured n–cube.

References

[Adams 1978] **J F Adams**, *Infinite loop spaces*, Ann. of Math. Stud. 90, Princeton Univ. Press (1978) MR Zbl

[Axelrod and Singer 1994] **S Axelrod**, **I M Singer**, *Chern–Simons perturbation theory, II*, J. Differential Geom. 39 (1994) 173–213 MR Zbl

[Berger and Moerdijk 2003] **C Berger**, **I Moerdijk**, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. 78 (2003) 805–831 MR Zbl

[Berger and Moerdijk 2006] **C Berger**, **I Moerdijk**, *The Boardman–Vogt resolution of operads in monoidal model categories*, Topology 45 (2006) 807–849 MR Zbl

[Berger and Moerdijk 2007] **C Berger**, **I Moerdijk**, *Resolution of coloured operads and rectification of homotopy algebras*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007) 31–58 MR Zbl

[Berger and Moerdijk 2013] **C Berger**, **I Moerdijk**, *On the homotopy theory of enriched categories*, Q. J. Math. 64 (2013) 805–846 MR Zbl

[Boardman and Vogt 1968] **J M Boardman**, **R M Vogt**, *Homotopy-everything H–spaces*, Bull. Amer. Math. Soc. 74 (1968) 1117–1122 MR Zbl

[Boardman and Vogt 1973] **J M Boardman**, **R M Vogt**, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347, Springer (1973) MR Zbl

[Boavida de Brito and Weiss 2018] **P Boavida de Brito**, **M Weiss**, *Spaces of smooth embeddings and configuration categories*, J. Topol. 11 (2018) 65–143 MR Zbl

[Boavida de Brito et al. 2019] **P Boavida de Brito**, **G Horel**, **M Robertson**, *Operads of genus zero curves and the Grothendieck–Teichmüller group*, Geom. Topol. 23 (2019) 299–346 MR Zbl

[Cisinski and Moerdijk 2011] **D-C Cisinski**, **I Moerdijk**, *Dendroidal sets as models for homotopy operads*, J. Topol. 4 (2011) 257–299 MR Zbl

[Cisinski and Moerdijk 2013a] **D-C Cisinski**, **I Moerdijk**, *Dendroidal Segal spaces and ∞–operads*, J. Topol. 6 (2013) 675–704 MR Zbl

[Cisinski and Moerdijk 2013b] **D-C Cisinski**, **I Moerdijk**, *Dendroidal sets and simplicial operads*, J. Topol. 6 (2013) 705–756 MR Zbl

[Dror Farjoun 1987] **E Dror Farjoun**, *Homotopy and homology of diagrams of spaces*, from “Algebraic topology” (H R Miller, D C Ravenel, editors), Lecture Notes in Math. 1286, Springer (1987) 93–134 MR Zbl

[Dugger 2008] **D Dugger**, *A primer on homotopy colimits*, preprint (2008) `http://pages.uoregon.edu/~ddugger/hocolim.pdf`

[Dwyer and Kan 1980] **W G Dwyer**, **D M Kan**, *Simplicial localizations of categories*, J. Pure Appl. Algebra 17 (1980) 267–284 MR Zbl

[Fresse et al. 2017] **B Fresse**, **V Turchin**, **T Willwacher**, *The rational homotopy of mapping spaces of E_n operads*, preprint (2017) arXiv 1703.06123

[Fulton and MacPherson 1994] **W Fulton**, **R MacPherson**, *A compactification of configuration spaces*, Ann. of Math. 139 (1994) 183–225 MR Zbl

[Getzler and Jones 1994] **E Getzler**, **J D S Jones**, *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint (1994) arXiv hep-th/9403055

[Goerss and Jardine 1999] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progr. Math. 174, Birkhäuser, Basel (1999) MR Zbl

[Heuts 2021] **G Heuts**, *Goodwillie approximations to higher categories*, Mem. Amer. Math. Soc. 1333, Amer. Math. Soc., Providence, RI (2021) MR Zbl

[May 1972] **J P May**, *The geometry of iterated loop spaces*, Lecture Notes in Math. 271, Springer (1972) MR Zbl

[Milnor 1962] **J Milnor**, *On axiomatic homology theory*, Pacific J. Math. 12 (1962) 337–341 MR Zbl

[Moerdijk and Weiss 2007] **I Moerdijk**, **I Weiss**, *Dendroidal sets*, Algebr. Geom. Topol. 7 (2007) 1441–1470 MR Zbl

[Munson and Volić 2015] **B A Munson**, **I Volić**, *Cubical homotopy theory*, New Math. Monogr. 25, Cambridge Univ. Press (2015) MR Zbl

[Rezk 2001] **C Rezk**, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. 353 (2001) 973–1007 MR Zbl

[Sinha 2004] **D P Sinha**, *Manifold-theoretic compactifications of configuration spaces*, Selecta Math. 10 (2004) 391–428 MR Zbl

[Stasheff 1963a] **J D Stasheff**, *Homotopy associativity of H–spaces, I*, Trans. Amer. Math. Soc. 108 (1963) 275–292 MR Zbl

[Stasheff 1963b] **J D Stasheff**, *Homotopy associativity of H–spaces, II*, Trans. Amer. Math. Soc. 108 (1963) 293–312 MR Zbl

[Sugawara 1957] **M Sugawara**, *A condition that a space is group-like*, Math. J. Okayama Univ. 7 (1957) 123–149 MR Zbl

[Thumann 2017] **W Thumann**, *Operad groups and their finiteness properties*, Adv. Math. 307 (2017) 417–487 MR Zbl

[Weiss 2007] **I Weiss**, *Dendroidal sets*, PhD thesis, Utrecht University (2007) Available at `https://dspace.library.uu.nl/bitstream/handle/1874/22859/?sequence=7`

[Weiss 2021] **M S Weiss**, *Rational Pontryagin classes of Euclidean fiber bundles*, Geom. Topol. 25 (2021) 3351–3424 MR Zbl

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Classical homological stability from the point of view of cells

OSCAR RANDAL-WILLIAMS

We explain how to interpret the complexes arising in the "classical" homology stability argument (eg in the framework of Randal-Williams and Wahl) in terms of higher algebra, which leads to a new proof of homological stability in this setting. The key ingredient is a theorem of Damiolini on the contractibility of certain arc complexes. We also explain how to directly compare the connectivities of these complexes with that of the "splitting complexes" of Galatius, Kupers and Randal-Williams.

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1 Introduction

The goal of this note is to compare the classical approach to homological stability, specifically the formalisation of Quillen's approach given by Wahl and myself [17], with the more recent approach via cellular E_k–algebras developed by Galatius, Kupers, and myself [7]. It is an insight of Krannich [14] that the proper generality for the classical approach is to work in the category of $\mathbb{N}$–graded topological spaces, and start with a right E_1–module $\boldsymbol{M}$ over an E_2–algebra $\boldsymbol{R}$ equipped with compatible $\mathbb{N}$–gradings, a stabilising element $\sigma \in \boldsymbol{R}(1)$, and then ask about homological stability of the sequence of maps

$$\boldsymbol{M}(0) \xrightarrow{-\cdot\sigma} \boldsymbol{M}(1) \xrightarrow{-\cdot\sigma} \boldsymbol{M}(2) \xrightarrow{-\cdot\sigma} \boldsymbol{M}(3) \xrightarrow{-\cdot\sigma} \cdots .$$

In practice one may often take $\boldsymbol{M} = \boldsymbol{R}$ with its right $\boldsymbol{R}$–action, but it is clarifying to separate the two notions: it is then clear [14, Remark 2.19] that one may as well replace $\boldsymbol{R}$ by $\boldsymbol{E}_2^+(1_*(*))$, the free unital E_2–algebra on a single point in grading 1, and just consider the induced $\boldsymbol{E}_2^+(1_*(*))$–module structure on $\boldsymbol{M}$.

Viewed in this way, the constructions and results of [14; 17] beg to be explained from the point of view of an $\boldsymbol{E}_2^+(1_*(*))$–module cell-structure on $\boldsymbol{M}$. Our first main result does this: in Theorem 3.1 we will show that the cofibre of Krannich's [14, Section 2.2] "canonical resolution" $|R_\bullet(\boldsymbol{M})| \to \boldsymbol{M}$ may be identified with the derived $\boldsymbol{E}_2^+(1_*(*))$–module indecomposables of $\boldsymbol{M}$, so that the high-connectivity of the "spaces of destabilisations" $|W_\bullet(A)|$ implies a vanishing line for the $\boldsymbol{E}_2^+(1_*(*))$–module cells of $\boldsymbol{M}$ (at least after linearising). This leads to a new proof that the high-connectivity of the $|W_\bullet(A)|$ implies homological stability, which we explain in Section 4. It also has consequences for homology with twisted coefficients, and for representation stability.

Our second main result is particular to the set up of [17], where a braided monoidal groupoid G (satisfying certain axioms) yields an E_2–algebra $\boldsymbol{R} \simeq B\mathsf{G}$. In this setting, for a fixed stabilising object σ of G and each object A of G there is the space $|W_\bullet(A)|$ of destabilisations of A, as well as spaces $|Z_\bullet^{E_1}(A)|$ and $|Z_{\bullet,\bullet}^{E_2}(A)|$ of "E_1– and E_2–splittings of A". Proposition 7.1 will show that under appropriate conditions the homological connectivities of these three spaces are essentially equivalent.

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2 Recollections

There is some tension in comparing [14] and [7], because although they both deal with E_2–algebras and modules over them, these notions are implemented in technically different ways. Namely, in [14] Krannich considers a 2–coloured operad $\mathbb{O}$ equivalent to a certain suboperad the Swiss cheese operad $\mathcal{SC}_2$, whose algebras $(\boldsymbol{M}, \boldsymbol{R})$ are then considered as an E_1–module $\boldsymbol{M}$ over an E_2–algebra $\boldsymbol{R}$. On the other hand, [7] considers unital algebras $\boldsymbol{R}^+$ over the little 2–cubes operad, constructs a strictification $\overline{\boldsymbol{R}}$ of the underlying E_1–algebra to an associative monoid, and then considers modules $\boldsymbol{M}$ over this monoid. While Krannich's formulation is more elegant, to take advantage of the large amount of machinery already developed in [7] we find it necessary to work in that setting, and in Section 2.3 we will redevelop Krannich's ideas in that setting.

As the results we explain are principally of interest in the context of [7] we will freely use the basic notation and concepts of that paper without introducing them again, and only remind the reader of the most pressing or elaborate notions.

2.1 $\mathbb{N}$–graded spaces

We shall often work in the category $\mathsf{Top}^{\mathbb{N}} = \mathsf{Fun}(\mathbb{N}, \mathsf{Top})$ of $\mathbb{N}$–graded (compactly generated weak Hausdorff) topological spaces, where $\mathbb{N}$ is considered as a category with only identity morphisms. An object X of this category simply consists of a collection $\{X(n)\}_{n\in\mathbb{N}}$ of spaces. If V is a space, we write $n_*(V)$ for the $\mathbb{N}$–graded spaces which is V in grading n and empty otherwise.

We endow this category with the symmetric monoidal structure $\otimes$ given by Day convolution, with $\mathbb{N}$ considered as a symmetric monoidal category under addition. More prosaically, it is given by

$$(X \otimes Y)(n) = \coprod_{a+b=n} X(a) \times Y(n).$$

Using $\otimes$ we can therefore talk of associative algebra objects in $\mathsf{Top}^{\mathbb{N}}$, and of modules over them. We may also talk of E_k–algebras in this category, as explained in [7].

2.2 The associative algebra $\boldsymbol{S}$

Following [7, Section 12.2.1] we use the following model for $\overline{\boldsymbol{E}}_2(1_*(*))$, an associative unital monoid equivalent as an E_1–algebra to the free unital E_2–algebra on one generator in grading 1. Let $\mathscr{C}_2(n)$ denote the n^{th} space in the little 2–cubes operad, ie the space of tuples $e_1, e_2, \dots, e_n \colon I^2 \to I^2$ of rectilinear embeddings having disjoint interiors.

Definition 2.1 Let $\boldsymbol{S}$ be the $\mathbb{N}$–graded space with

$$\boldsymbol{S}(n) = (0,\infty) \times \mathscr{C}_2(n)/\Sigma_n$$

for $n > 0$, and $\boldsymbol{S}(0)$ given by a single point, considered as $\{(0,\varnothing)\}$. We think of $\boldsymbol{S}(n)$ as the space of pairs of a $t > 0$ and a set of n unordered rectilinear embeddings $I^2 \to [0,t] \times [0,1]$ with disjoint interiors, by the evident rescaling. In this interpretation, translation and disjoint union provide maps

$$\boldsymbol{S}(n) \times \boldsymbol{S}(m) \to \boldsymbol{S}(n+m)$$

making $\boldsymbol{S}$ an associative unital monoid in $\mathbb{N}$–graded spaces, with unit $(0,\varnothing)$. We write $\sigma := (1, \mathrm{id}_{I^2}) \in \boldsymbol{S}(1)$, or equivalently $\sigma \colon 1_*(*) \to \boldsymbol{S}$.

There is a homotopy equivalence between $\boldsymbol{S}(n)$ and the space $C_n(\mathbb{R}^2)$ of configurations of n unordered points in the plane (by passing first to the subspace with $t = 1$, then considering the map which sends a collection of embeddings $\{e_i \colon I^2 \to [0,1]^2\}$ to the collection of their centres $\left\{e_i\left(\frac{1}{2},\frac{1}{2}\right) \in (0,1)^2 \cong \mathbb{R}^2\right\}$, which is a fibration with contractible fibres). As such, $\boldsymbol{S}(n)$ is a model for the classifying space of Artin's braid group β_n on n strands. The map $-\cdot\sigma \colon \boldsymbol{S}(n-1) \to \boldsymbol{S}(n)$ corresponds to the homomorphism $\beta_{n-1} \to \beta_n$ which adds one strand (to the right). We record two well-known facts about these maps:

(i) The homomorphism $\beta_{n-1} \to \beta_n$ is injective for all n.

(ii) The homomorphism $\beta_{n-1} \to \beta_n$ induces an isomorphism on homology in degrees $* \le \frac{1}{2}(n-3)$, and an epimorphism in degrees $* \le \frac{1}{2}(n-1)$. Equivalently, the relative homology groups satisfy $H_*(\beta_n, \beta_{n-1}; \mathbb{Z}) = 0$ for $* < \frac{1}{2}n$.

The latter was first proved by Arnold [1], and there are many more recent proofs. The former is easy: the homomorphism lands in the subgroup $\beta_{n-1,1} \le \beta_n$ of those braids where the strand that starts at the rightmost point also ends at the rightmost point, and on this subgroup there is a splitting $\beta_{n-1,1} \to \beta_{n-1}$ given by forgetting this rightmost strand.

2.3 Strictifying Krannich's framework

Let $\boldsymbol{M}$ be a right $\boldsymbol{S}$–module, and let us define the analogue of Krannich's "canonical resolution" [14, Section 2.2]. This is almost a semisimplicial ($\mathbb{N}$–graded) space augmented over $\boldsymbol{M}$, but is indexed on a topological category $\widetilde{\Delta}_{\text{inj}}$ homotopy equivalent to, but not equal to, Δ_{inj}. We must first describe this category. We will work with Moore paths, write $\omega(\gamma)$ for the end of a Moore path γ, and write $*$ for concatenation of Moore paths.

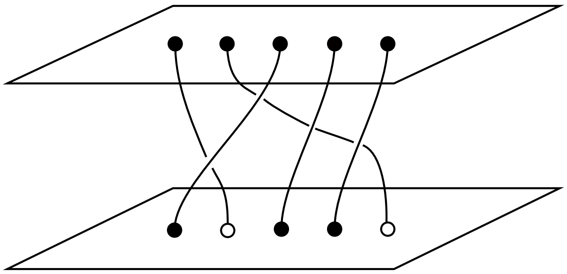

Figure 1: The braid μ, where the points $i([q])$ are shown open.

Definition 2.2 For $[q],[p] \in \Delta_{\mathrm{inj}}$ let $U([q],[p])$ denote the space of pairs of a $d \in \boldsymbol{S}(p-q)$ and a Moore path μ in $\boldsymbol{S}(p+1)$ from σ^{p+1} to $d \cdot \sigma^{q+1}$. There is a composition-law

$$U([l],[q]) \times U([q],[p]) \to U([l],[p])$$

given by $((e,\gamma),(d,\mu)) \mapsto (d \cdot e, \mu * (d \cdot \gamma))$, giving the structure of a topologically enriched category U with the same objects as Δ_{inj}.

For a morphism $i:[q] \to [p] \in \Delta_{\mathrm{inj}}$ there is a path component $U([q],[p])$ containing the point given by $d = \sigma^{p-q}$ and μ a Moore loop corresponding to the braid on $p+1$ strands where the first $q+1$ strands go behind the rest to end at $i([q]) \subset [p]$, as in Figure 1. We let $\widetilde{\Delta}_{\mathrm{inj}}([q],[p]) \subset U([q],[p])$ consist of such path components; one checks it defines a subcategory of U with the same objects as Δ_{inj}.

As in [14, Lemma 2.11] the space $\widetilde{\Delta}_{\mathrm{inj}}([q],[p])$ is homotopy discrete, and the map

$$\Delta_{\mathrm{inj}}([q],[p]) \to \pi_0 \widetilde{\Delta}_{\mathrm{inj}}([q],[p])$$

described above is a bijection. This yields a functor $\widetilde{\Delta}_{\mathrm{inj}} \to \Delta_{\mathrm{inj}}$ which is the identity on objects, and an equivalence on morphism spaces.

Definition 2.3 For a right $\boldsymbol{S}$–module $\boldsymbol{M}$, let $R_p(\boldsymbol{M})$ denote the $\mathbb{N}$–graded space which in grading n consists of pairs of a point $a \in \boldsymbol{M}(n-p-1)$ and a Moore path γ in $\boldsymbol{M}(n)$ ending at $a \cdot \sigma^{p+1}$. Consider this as an enriched functor $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathsf{Top}$ via the maps

$$\widetilde{\Delta}_{\mathrm{inj}}([q],[p]) \times R_p(\boldsymbol{M}) \to R_q(\boldsymbol{M})$$

given by $((d,\mu),(a,\gamma)) \mapsto (a \cdot d, \gamma * (a \cdot \mu))$. Evaluating the Moore path at 0 gives an augmentation $R_\bullet(\boldsymbol{M}) \to \boldsymbol{M}$.

One then sets

$$|R_\bullet(\boldsymbol{M})| := \operatorname*{hocolim}_{[p] \in \widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}} R_p(\boldsymbol{M})$$

by analogy with the geometric realisation of a semisimplicial space. Krannich's development of homological stability in this setting takes as its axiom the high-connectivity of the map $\epsilon_{\boldsymbol{M}} : |R_\bullet(\boldsymbol{M})| \to \boldsymbol{M}$, specifically that there is a $k \geq 2$ such that this map is $\lfloor (n-2+k)/k \rfloor$–connected in grading n for all n.

3 The canonical resolution and module indecomposables

Our main result is the following, showing that the homotopy cofibre of the "canonical resolution" has a conceptual meaning: it is the derived $\boldsymbol{S}$–module indecomposables.

Theorem 3.1 *If $\boldsymbol{M}$ is a right $\boldsymbol{S}$–module, there is an equivalence of $\mathbb{N}$–graded spaces between the homotopy cofibre of $\epsilon_{\boldsymbol{M}}\colon |R_\bullet(\boldsymbol{M})| \to \boldsymbol{M}$ and $Q^{\boldsymbol{S}}_{\mathbb{L}}(\boldsymbol{M})$.*

Let us write $\boldsymbol{S}_{>0}$ for the sub-$\mathbb{N}$–graded space of $\boldsymbol{S}$ which is empty in grading 0 and agrees with $\boldsymbol{S}$ otherwise. The following expands upon [14, Example 2.18].

Lemma 3.2 *The augmentation $\epsilon_{\boldsymbol{S}}\colon |R_\bullet(\boldsymbol{S})| \to \boldsymbol{S}$ is an equivalence onto $\boldsymbol{S}_{>0}$.*

Proof The space $R_0(\boldsymbol{S})(0)$ consists of a point $a \in \boldsymbol{S}(-1)$ and a Moore path to $a \cdot \sigma$, so is empty, and so the fibre of $\epsilon_{\boldsymbol{S}}$ over the point of grading 0 is indeed empty.

The homotopy fibre of $\epsilon_{\boldsymbol{S}}$ over $b \in \boldsymbol{S}(n)$ with $n > 0$ is, using [5, page 180], equivalent to the realisation of the $\widetilde{\Delta}_{\mathrm{inj}}$–space given by the homotopy fibres over b of the maps $R_p(\boldsymbol{S}) \to \boldsymbol{S}$, but as these maps are fibrations this is in turn the same as the realisation of the $\widetilde{\Delta}_{\mathrm{inj}}$–space $F_\bullet(b)$ with p–simplices given by the literal fibres of these maps, ie an $a \in \boldsymbol{S}(n-p-1)$ and a Moore path from b to $a \cdot \sigma^{p+1}$. There are maps

$$\operatorname*{hocolim}_{[p]\in\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}} F_p(b) \to \operatorname*{hocolim}_{[p]\in\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}} \pi_0 F_p(b) \to \operatorname*{hocolim}_{[p]\in\Delta^{\mathrm{op}}_{\mathrm{inj}}} \pi_0 F_p(b)$$

induced by $F_\bullet(b) \to \pi_0 F_\bullet(b)$, and by the fact that the functor $\pi_0 F_\bullet(b)$ on $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$ canonically factors through $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \Delta^{\mathrm{op}}_{\mathrm{inj}}$. The second map is an equivalence as $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \Delta^{\mathrm{op}}_{\mathrm{inj}}$ is an equivalence of enriched categories. The first map is an equivalence as each $F_p(b)$ is homotopy-discrete: this is because it is a homotopy fibre of the map $\boldsymbol{S}(n-p-1) \to \boldsymbol{S}(n)$, and by item (i) of Section 2.2 this is a map of $K(\pi,1)$'s which is injective on fundamental groups.

It remains to show that the semisimplicial set $[p] \mapsto \pi_0 F_p(b)$ has contractible geometric realisation. For any b this is the "space of destabilisations" of [17, Definition 2.1] in the case of the braid groups, which is described in [17, Section 5.6.2] as an arc complex. By a remarkable theorem of Damiolini [4, Theorem 2.48] (see [9, Proposition 3.2] for a published reference) this arc complex is contractible. □

Proof of Theorem 3.1 The augmented $\widetilde{\Delta}_{\mathrm{inj}}$–space $R_\bullet(\boldsymbol{S}) \to \boldsymbol{S}$ is constructed using the right $\boldsymbol{S}$–module structure on $\boldsymbol{S}$, so it admits a compatible left $\boldsymbol{S}$–module structure via

$$(b,(a,\gamma)) \mapsto (b\cdot a, b\cdot\gamma)\colon \boldsymbol{S} \otimes R_p(\boldsymbol{S}) \to R_p(\boldsymbol{S}).$$

Furthermore, contracting the Moore path gives a deformation retraction from $R_p(\boldsymbol{S})$ to the subspace where the Moore path is trivial, and this subspace is isomorphic to $\boldsymbol{S} \otimes (p+1)_*(*)$ as a $\mathbb{N}$–graded space, and as a left $\boldsymbol{S}$–module.

There is a map

$$\phi'_p\colon \boldsymbol{M} \otimes R_p(\boldsymbol{S}) \to R_p(\boldsymbol{M})$$

given by $(a,(b,\gamma)) \mapsto (a\cdot b, a\cdot\gamma)$. If $c \in \boldsymbol{S}$ then the map above satisfies $\phi'_p(a\cdot c, b) = \phi'_p(a, c\cdot b)$, and hence descends to a map $\phi_p\colon \boldsymbol{M} \otimes_{\boldsymbol{S}} R_p(\boldsymbol{S}) \to R_p(\boldsymbol{M})$ from the coequaliser. The composition

$$B(\boldsymbol{M},\boldsymbol{S},R_p(\boldsymbol{S})) \to \boldsymbol{M} \otimes_{\boldsymbol{S}} R_p(\boldsymbol{S}) \xrightarrow{\phi_p} R_p(\boldsymbol{M})$$

of the augmentation map and ϕ_p is an equivalence, using $R_p(\boldsymbol{M}) \simeq \boldsymbol{M} \otimes (p+1)_*(*)$ as well as $R_p(\boldsymbol{S}) \simeq \boldsymbol{S} \otimes (p+1)_*(*)$ and $B(\boldsymbol{M},\boldsymbol{S},\boldsymbol{S}) \simeq \boldsymbol{M}$. By commuting the bar construction with the homotopy colimit defining $|R_\bullet(\boldsymbol{S})|$, and using Lemma 3.2, we obtain equivalences

$$B(\boldsymbol{M},\boldsymbol{S},\boldsymbol{S}_{>0}) \xleftarrow{\sim} B\big(\boldsymbol{M},\boldsymbol{S},|R_\bullet(\boldsymbol{S})|\big) \xrightarrow{\sim} |R_\bullet(\boldsymbol{M})|$$

over $\boldsymbol{M}$. This identifies the homotopy cofibre of $\epsilon_{\boldsymbol{M}}$ with the homotopy cofibre of the composition

$$B(\boldsymbol{M},\boldsymbol{S},\boldsymbol{S}_{>0}) \to B(\boldsymbol{M},\boldsymbol{S},\boldsymbol{S}) \xrightarrow{\sim} \boldsymbol{M},$$

which is equivalent to the homotopy cofibre of the first map, ie $B(\boldsymbol{M},\boldsymbol{S},\boldsymbol{S}/\boldsymbol{S}_{>0}) \simeq B(\boldsymbol{M},\boldsymbol{S},\mathbb{1})$. The latter bar construction should be interpreted as being formed in $\mathbb{N}$–graded pointed spaces, with $\boldsymbol{M}$ and $\boldsymbol{S}$ included in this category by implicitly adding disjoint basepoints, and with $\mathbb{1} \simeq \boldsymbol{S}/\boldsymbol{S}_{>0}$ given by

$$\mathbb{1}(n) = \begin{cases} S^0 & \text{if } n = 0, \\ * & \text{if } n > 0. \end{cases}$$

By [7, Corollary 9.17], $B(\boldsymbol{M},\boldsymbol{S},\mathbb{1})$ agrees with $Q_{\mathbb{L}}^{\boldsymbol{S}}(\boldsymbol{M})$, as long as $\boldsymbol{S}$ and $\boldsymbol{M}$ are cofibrant in $\mathsf{Top}^{\mathbb{N}}$.

Now each $\boldsymbol{S}(n)$ has the structure of a smooth manifold with corners, so is cofibrant in Top; on the other hand the cofibrancy hypotheses in $\boldsymbol{M}$ may be neglected, for the following reason. If $\boldsymbol{M}^c \xrightarrow{\sim} \boldsymbol{M}$ is a cofibrant replacement of $\boldsymbol{M}$ as a $\boldsymbol{S}$–module (and so in particular $\boldsymbol{M}^c$ is cofibrant in Top) then the above applies to give $B(\boldsymbol{M}^c,\boldsymbol{S},\mathbb{1}) \simeq Q_{\mathbb{L}}^{\boldsymbol{S}}(\boldsymbol{M}^c)$. Now certainly $Q_{\mathbb{L}}^{\boldsymbol{S}}(\boldsymbol{M}^c) \to Q_{\mathbb{L}}^{\boldsymbol{S}}(\boldsymbol{M})$ is an equivalence, but also $B_\bullet(\boldsymbol{M}^c,\boldsymbol{S},\mathbb{1}) \to B_\bullet(\boldsymbol{M},\boldsymbol{S},\mathbb{1})$ is a levelwise equivalence (cartesian product preserves equivalences between *all* objects in Top) and so $B(\boldsymbol{M}^c,\boldsymbol{S},\mathbb{1}) \to B(\boldsymbol{M},\boldsymbol{S},\mathbb{1})$ is an equivalence too — geometric realisation of semisimplicial spaces preserves equivalences between *all* semisimplicial (compactly generated) spaces [6, Theorem 2.2]. □

4 Classical homological stability revisited

Theorem 3.1 leads to a new proof of homological stability in the setting of [14] or [17] (adapted as in Section 2.3), quite different from the standard proof but very similar in spirit to [7, Section 18]. It takes as given homological stability (of slope $\frac{1}{2}$) for the free E_2–algebra on one generator, ie configuration spaces of little cubes (or points) in the plane, or equivalently the braid groups.

In the terms we have been using, homological stability may be formulated as follows. First, if $\boldsymbol{X}$ is an $\mathbb{N}$–graded space then we define bigraded homology groups by $H_{n,d}(\boldsymbol{X}) := H_d(\boldsymbol{X}(n))$, and similarly reduced homology groups of $\mathbb{N}$–graded pointed spaces. Second, if $\boldsymbol{M}$ is a right $\boldsymbol{S}$–module then we may form the composition

$$-\cdot\sigma\colon \boldsymbol{M} \otimes 1_*(*) \xrightarrow{\boldsymbol{M}\otimes\sigma} \boldsymbol{M} \otimes \boldsymbol{S} \xrightarrow{\cdot} \boldsymbol{M}$$

in the category of $\mathbb{N}$–graded spaces, and write $\boldsymbol{M}/\sigma$ for its homotopy cofibre (considered as a $\mathbb{N}$–graded pointed space). Using that $(\boldsymbol{M} \otimes 1_*(*))(n) = \boldsymbol{M}(n-1)$, the associated long exact sequence on homology takes the form

$$\cdots \to \widetilde{H}_{n,d+1}(\boldsymbol{M}/\sigma) \to H_d(\boldsymbol{M}(n-1)) \xrightarrow{(-\cdot\sigma)_*} H_d(\boldsymbol{M}(n)) \to \widetilde{H}_{n,d}(\boldsymbol{M}/\sigma) \to \cdots .$$

Thus homological stability of the sequence of maps

$$\boldsymbol{M}(0) \xrightarrow{-\cdot\sigma} \boldsymbol{M}(1) \xrightarrow{-\cdot\sigma} \boldsymbol{M}(2) \xrightarrow{-\cdot\sigma} \boldsymbol{M}(3) \xrightarrow{-\cdot\sigma} \cdots \tag{4-1}$$

corresponds to the vanishing of the groups $\widetilde{H}_{n,d}(\boldsymbol{M}/\sigma)$ for $d \ll n$.

Henceforth $\Bbbk$ will always denote a commutative ring.

Proposition 4.1 *Let $\boldsymbol{M}$ be a right $\boldsymbol{S}$–module and $f : \mathbb{N} \to \mathbb{N}$ be such that*

$$H_{n,d}\big(\boldsymbol{M}, |R_\bullet(\boldsymbol{M})|; \Bbbk\big) = 0 \quad \text{for } d < f(n).$$

Then, setting $\bar{f}(n) := \min\{\lfloor f(p) + \frac{1}{2}(n-p)\rfloor \mid 0 \le p \le n\}$,

$$\widetilde{H}_{n,d}(\boldsymbol{M}/\sigma; \Bbbk) = 0 \quad \text{for } d < \bar{f}(n).$$

In particular, if f diverges then so does $\bar{f}$, ie (4-1) satisfies homological stability.

Proof By Theorem 3.1, the hypothesis of the proposition is equivalent to $H^{\boldsymbol{S}}_{n,d}(\boldsymbol{M}; \Bbbk) = 0$ for $d < f(n)$. Applying the symmetric monoidal functor $(-)_\Bbbk := \Bbbk[\mathrm{Sing}_\bullet(-)] \colon \mathsf{Top} \to \mathsf{sMod}_\Bbbk$ we obtain an associative monoid $\boldsymbol{S}_\Bbbk$ and a module $\boldsymbol{M}_\Bbbk$ over it in the category $\mathsf{sMod}_\Bbbk^{\mathbb{N}}$, satisfying $H^{\boldsymbol{S}_\Bbbk}_{n,d}(\boldsymbol{M}_\Bbbk) = 0$ for $d < f(n)$. By [7, Theorem 11.21] we may find an $\boldsymbol{S}_\Bbbk$–module cellular approximation $\boldsymbol{C} \xrightarrow{\sim} \boldsymbol{M}_\Bbbk$, such that $\boldsymbol{C}$ only has (n,d)–cells with $d \ge f(n)$. We write $\mathrm{sk}(\boldsymbol{C})$ for the filtered $\boldsymbol{S}_\Bbbk$–module given by the skeletal filtration of $\boldsymbol{C}$.

By considering the functor $(-)/\sigma = (-) \otimes_{\boldsymbol{S}_\Bbbk} \boldsymbol{S}_\Bbbk/\sigma$, which preserves homotopy cofibre sequences of right $\boldsymbol{S}_\Bbbk$–modules, we obtain a filtration of $\boldsymbol{C}/\sigma$ with associated graded

$$\mathrm{gr}(\boldsymbol{C}/\sigma) \simeq \mathrm{gr}(\boldsymbol{C})/\sigma \simeq \bigoplus_{d \ge 0} \bigoplus_{\alpha \in I_d} S^{n_\alpha, d} \otimes \boldsymbol{S}_\Bbbk/\sigma,$$

where $d \ge f(n_\alpha)$ for $\alpha \in I_d$.

By the discussion in Section 2.2 we have $H_{n,d}(\boldsymbol{S}_\Bbbk/\sigma) \cong H_d(\beta_n, \beta_{n-1}; \Bbbk)$ for β_n the n^{th} braid group, and by item (ii) of Section 2.2 and the universal coefficient theorem this vanishes for $d < \frac{1}{2}n$. It follows that the homology of $\mathrm{gr}(\boldsymbol{C}/\sigma)$ vanishes in bidegrees (n,d) such that $d < f(n_\alpha) + \frac{1}{2}(n - n_\alpha)$ for all cells α, so in particular for $d < \bar{f}(n)$. The same then holds for $\boldsymbol{C}/\sigma \simeq \boldsymbol{M}_\Bbbk/\sigma$ by the spectral sequence for the skeletal filtration of $\boldsymbol{C}/\sigma$. □

This is a simple application of Theorem 3.1, using only that $\boldsymbol{S}$ enjoys homological stability of slope $\frac{1}{2}$ with integral coefficients. But the principle behind the argument above shows that $\boldsymbol{M}$ will enjoy any homological stability pattern that $\boldsymbol{S}$ does, in a range of degrees controlled by the vanishing of $H_{*,*}\big(\boldsymbol{M}, |R_\bullet(\boldsymbol{M})|\big)$. (Of course this is only useful when the latter has a vanishing line of slope $> \frac{1}{2}$; Coxeter groups [11, Section 8] and Artin monoids [2, Theorem 8.1] give good families of examples.) As the homology of $\boldsymbol{S}$ is completely known, such patterns (meaning improved homological stability ranges with $\mathbb{Q}$– or $\mathbb{F}_p$–coefficients, or secondary and higher homological stability) can be easily analysed. A detailed analysis is given in [13, Corollary 2.12]; we will not spell out the (rather involved) formulation here.

A converse to Proposition 4.1 holds too:

Proposition 4.2 *Let $\boldsymbol{M}$ be a right $\boldsymbol{S}$–module, and $g\colon \mathbb{N} \to \mathbb{N}$ be such that*

$$\widetilde{H}_{n,d}(\boldsymbol{M}/\sigma; \Bbbk) = 0 \quad \textit{for } d < g(n).$$

Then, setting $\bar{g}(n) := \min\{g(p) + (n-p) \mid 0 \le p \le n\}$,

$$H_{n,d}\big(\boldsymbol{M}, |R_\bullet(\boldsymbol{M})|; \Bbbk\big) = 0 \quad \textit{for } d < \bar{g}(n). \qquad \square$$

In practice this is not usually sharp, in that $H_{*,*}\big(\boldsymbol{M}, |R_\bullet(\boldsymbol{M})|; \Bbbk\big)$ often vanishes with larger slope than $\widetilde{H}_{*,*}(\boldsymbol{M}/\sigma; \Bbbk)$ does. As mentioned above, this usually indicates the presence of secondary and higher order homological stability for $\boldsymbol{S}$.

In view of Theorem 3.1, a highbrow proof of Proposition 4.2 is the discussion in [7, Remark 19.3], allowing oneself to be more flexible with the form of the stability ranges. A middlebrow proof is to consider the Bousfield–Kan spectral sequence for the augmented $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$–space $\epsilon_{\boldsymbol{M}}\colon R_\bullet(\boldsymbol{M}) \to \boldsymbol{M}$, which — as the morphism spaces in $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$ are homotopy discrete — takes the form

$$E^1_{n,p,q} = H_{n,q}(R_p(\boldsymbol{M}); \Bbbk) \Rightarrow H_{n,p+q+1}\big(\boldsymbol{M}, |R_\bullet(\boldsymbol{M})|; \Bbbk\big)$$

for $p \ge -1$ with $R_{-1}(\boldsymbol{M}) := \boldsymbol{M}$. As $R_p(\boldsymbol{M}) \simeq \boldsymbol{M} \otimes (p+1)_*(*)$ we can write the E^1–page as $E^1_{n,p,q} \cong H_q(\boldsymbol{M}(n-p-1); \Bbbk)$, and recognise the d^1–differential $d^1\colon E^1_{n,p,q} \to E^1_{n,p-1,q}$ as the alternating sum of $p+1$ copies of the stabilisation map $(-\cdot\sigma)_*$. Thus this differential is zero if p is odd, and is $(-\cdot\sigma)_*$ if p is even. From the assumption it is then easy to see that $E^2_{n,p,q} = 0$ for $p+q+1 < \bar{g}(n)$. This is simply the usual spectral sequence argument for homological stability, with the logic reversed. (As with many middlebrow arguments, it even offers a slight improvement: in the definition of $\bar{g}$ one can take the minimum over those $0 \le p \le n$ having the same parity as n.)

5 An extension of Theorem 3.1

The discussion of Section 3 shows that the cofibre of the canonical resolution $\epsilon_{\boldsymbol{M}}\colon |R_\bullet(\boldsymbol{M})| \to \boldsymbol{M}$ is equivalent to the derived $\boldsymbol{S}$–module indecomposables $Q^{\boldsymbol{S}}_{\mathbb{L}}(\boldsymbol{M})$, so the high-connectivity of this cofibre means that $\boldsymbol{M}$ can be constructed as a cellular $\boldsymbol{S}$–module without using small-dimensional $\boldsymbol{S}$–module cells in large $\mathbb{N}$–grading. Usually, such high-connectivity is proved by establishing the high-connectivity of the fibres of $\epsilon_{\boldsymbol{M}}$; the fibre $W_\bullet(m)$ of $\epsilon_{\boldsymbol{M}}\colon R_\bullet(\boldsymbol{M}) \to \boldsymbol{M}$ over a point $m \in \boldsymbol{M}$ is called the "space of destabilisations" in [14, Definition 2.14(ii)]. The high-connectivity of a fibre is, of course, stronger than the high-connectivity of the corresponding cofibre. As such it might be expected that the high-connectivity of the fibres $W_\bullet(m)$ has more consequences than the high-connectivity of $Q^{\boldsymbol{S}}_{\mathbb{L}}(\boldsymbol{M})$. The goal of this section is to explain how this is so.

5.1 Formulation

The theory in [7] is developed not only in the category $\mathsf{Top}^{\mathbb{N}}$ of $\mathbb{N}$–graded spaces, but more generally in G–graded spaces for a (symmetric or braided) monoidal groupoid G. This allows for the treatment of homological stability with twisted coefficients, and is also the natural context for representation stability.

Let $(\mathsf{G}, \oplus, b, 0)$ be a braided monoidal groupoid. Let $r\colon \mathsf{G} \to \mathbb{N}$ be a strong monoidal functor, called the *rank*, and choose an $X \in \mathsf{G}$ with $r(X) = 1$. Assume furthermore that

(I) $0 \in \mathsf{G}$ is the only object of rank 0, and

(II) $\mathrm{Aut}_{\mathsf{G}}(0)$ is trivial.

Endow $\mathsf{Top}^{\mathsf{G}} = \mathsf{Fun}(\mathsf{G}, \mathsf{Top})$ with the braided monoidal structure given by Day convolution, and similarly $\mathsf{sMod}^{\mathsf{G}}_{\Bbbk}$.

In order to discuss E_2–algebras in a category which is only braided monoidal, in [7, Section 4.1] there is introduced the category FB_2 of "braided finite sets", and the category $\mathsf{Top}^{\mathsf{FB}_2}$ replaces the category of symmetric sequences. It is endowed with a composition product [7, Definition 4.3], monoids for which serve as a braided version of operads. In particular there is a braided version $\mathscr{C}_2^{\mathsf{FB}_2}$ of the nonunitary little 2–cubes operad [7, Definition 12.6]. This has $\mathscr{C}_2^{\mathsf{FB}_2}(n)$ contractible for each $n > 0$ (and empty for $n = 0$).

Using this we can make sense of E_2–algebras in $\mathsf{Top}^{\mathsf{G}}$ or $\mathsf{sMod}^{\mathsf{G}}_{\Bbbk}$, and in particular we can form the free E_2–algebra on the object $X_*(*) \in \mathsf{Top}^{\mathsf{G}}$,

$$\boldsymbol{E}_2(X_*(*)) \in \mathsf{Alg}_{E_2}(\mathsf{Top}^{\mathsf{G}}).$$

We can strictify $\boldsymbol{E}_2(X_*(*))$ to a unital associative algebra

$$\widetilde{\boldsymbol{S}} := \overline{\boldsymbol{E}}_2(X_*(*)),$$

which plays the role of $\boldsymbol{S}$ in this setting, and consider a right $\widetilde{\boldsymbol{S}}$–module $\widetilde{\boldsymbol{M}}$. The object $\widetilde{\boldsymbol{S}}$ is cofibrant in $\mathsf{Top}^{\mathsf{G}}$, and we will always assume that $\widetilde{\boldsymbol{M}}$ is too.

Taking left Kan extension along $r\colon \mathsf{G} \to \mathbb{N}$ gives

$$r_*\widetilde{\boldsymbol{S}} = r_*\overline{\boldsymbol{E}}_2(X_*(*)) = \overline{\boldsymbol{E}}_2(1_*(*)) = \boldsymbol{S} \quad \text{and} \quad r_*\widetilde{\boldsymbol{M}} =: \boldsymbol{M}$$

(as these objects were cofibrant in $\mathsf{Top}^{\mathsf{G}}$, this agrees with the homotopy left Kan extension), and $\boldsymbol{M}$ is a right $\boldsymbol{S}$–module. For each object $Y \in \mathsf{G}$ there is a quotient map $q\colon \widetilde{\boldsymbol{M}}(Y) \to \boldsymbol{M}(r(Y))$. This puts us in the setting of Section 2.3: there is the canonical resolution $\epsilon_{\boldsymbol{M}}\colon R_\bullet(\boldsymbol{M}) \to \boldsymbol{M}$, with fibre $W_\bullet(m)$ over $m \in \boldsymbol{M}$.

The following relates the spaces $|W_\bullet(m)|$, in particular their connectivities, with the derived $\widetilde{\boldsymbol{S}}$–module indecomposables.

Theorem 5.1 *Let $\widetilde{\boldsymbol{M}}$ be a right $\widetilde{\boldsymbol{S}}$–module which is cofibrant in $\mathsf{Top}^{\mathsf{G}}$. Then there is a morphism*

$$B(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{S}}_{>0}) \to \widetilde{\boldsymbol{M}} \tag{5-1}$$

with homotopy cofibre $Q_{\mathbb{L}}^{\widetilde{\boldsymbol{S}}}(\widetilde{\boldsymbol{M}})$ and homotopy fibre over $\tilde{m} \in \widetilde{\boldsymbol{M}}$ given by $|W_\bullet(q(\tilde{m}))|$.

In particular, if $|W_\bullet(q(\tilde{m}))|$ is k–connected for all $\tilde{m} \in \widetilde{\boldsymbol{M}}(Y)$, then $Q_{\mathbb{L}}^{\widetilde{\boldsymbol{S}}}(\widetilde{\boldsymbol{M}})(Y)$ is $(k+1)$–connected.

5.2 Proof of Theorem 5.1

We prove this theorem by analogy with Theorem 3.1, and so first construct an augmented $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$–object $R_\bullet(\widetilde{\boldsymbol{M}}) \to \widetilde{\boldsymbol{M}}$. For each object $Y \in \mathsf{G}$ and each $[p] \in \widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$ we define $R_p(\widetilde{\boldsymbol{M}})(Y)$ by the cartesian square

$$\begin{tikzcd}
R_p(\widetilde{\boldsymbol{M}})(Y) \arrow[r] \arrow[d] & \widetilde{\boldsymbol{M}}(Y) \arrow[d, "q"] \\
R_p(\boldsymbol{M})(r(Y)) \arrow[r] & \boldsymbol{M}(r(Y))
\end{tikzcd} \tag{5-2}$$

Repeatedly using the universal property of pullbacks, we see that these assemble to $R_p(\widetilde{\boldsymbol{M}}) \in \mathsf{Top}^{\mathsf{G}}$, and that in turn these assemble to an augmented $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$–object $R_\bullet(\widetilde{\boldsymbol{M}}) \to \widetilde{\boldsymbol{M}}$ in $\mathsf{Top}^{\mathsf{G}}$. Furthermore, when $\widetilde{\boldsymbol{M}} = \widetilde{\boldsymbol{S}}$ we see that this object consists of left $\widetilde{\boldsymbol{S}}$–modules.

As $\widetilde{\boldsymbol{M}}$ is assumed to be cofibrant, the quotient map $\widetilde{\boldsymbol{M}}(Y) \to \widetilde{\boldsymbol{M}}(Y)/\mathrm{Aut}_{\mathsf{G}}(Y)$ is a covering space and the latter is a union of path-components of $\boldsymbol{M}(r(Y))$, so the right-hand vertical map in (5-2) is a fibration; thus this square is also homotopy cartesian. It then follows that the square

$$\begin{tikzcd}
{|R_\bullet(\widetilde{\boldsymbol{M}})|(Y)} \arrow[r] \arrow[d] & \widetilde{\boldsymbol{M}}(Y) \arrow[d, "q"] \\
{|R_\bullet(\boldsymbol{M})|(r(Y))} \arrow[r] & \boldsymbol{M}(r(Y))
\end{tikzcd} \tag{5-3}$$

is also homotopy cartesian (Lemma 2.13 of [6] gives this for $\Delta_{\mathrm{inj}}^{\mathrm{op}}$–objects; it follows for $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$–objects by first homotopy Kan extending along the equivalence of enriched categories $\widetilde{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \Delta_{\mathrm{inj}}^{\mathrm{op}}$).

If we let $\widetilde{\boldsymbol{S}}_{>0} \in \mathsf{Top}^{\mathsf{G}}$ be the object that agrees with $\widetilde{\boldsymbol{S}}$ on objects Y with $r(Y) > 0$, and is the empty space on objects Y with $r(Y) = 0$ (recall that we have assumed that $0 \in \mathsf{G}$ is the only such object), then this obtains the structure of a left $\widetilde{\boldsymbol{S}}$–module. It follows from Lemma 3.2 and the homotopy cartesian square (5-3) that the augmentation gives an equivalence $|R_\bullet(\widetilde{\boldsymbol{S}})| \to \widetilde{\boldsymbol{S}}_{>0}$ of left $\widetilde{\boldsymbol{S}}$–modules.

Proof of Theorem 5.1 We proceed as in the proof of Theorem 3.1. Applying $B(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{S}}, -)$ to the homotopy cofibre sequence

$$\widetilde{\boldsymbol{S}}_{>0} \to \widetilde{\boldsymbol{S}} \to \mathbb{1},$$

and using $B(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{S}}) \xrightarrow{\simeq} \widetilde{\boldsymbol{M}}$, constructs the map (5-1) and identifies its homotopy cofibre with $B(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{S}}, \mathbb{1})$, which is equivalent to $Q_{\mathbb{L}}^{\widetilde{\boldsymbol{S}}}(\widetilde{\boldsymbol{M}})$ by [7, Corollary 9.17].

On the other hand there are equivalences

$$B(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{S}}_{>0}) \xleftarrow{\simeq} B\big(\widetilde{\boldsymbol{M}}, \widetilde{\boldsymbol{S}}, |R_\bullet(\widetilde{\boldsymbol{S}})|\big) \xrightarrow{\simeq} |R_\bullet(\widetilde{\boldsymbol{M}})|$$

over $\widetilde{\boldsymbol{M}}$, using as in the proof of Theorem 3.1 that $R_p(\widetilde{\boldsymbol{S}}) \simeq \widetilde{\boldsymbol{S}} \otimes (X^{\oplus p+1})_*(*)$ as a left $\widetilde{\boldsymbol{S}}$–module, and similarly for $\widetilde{\boldsymbol{M}}$. Finally, the homotopy fibre of $|R_\bullet(\widetilde{\boldsymbol{M}})|(Y) \to \widetilde{\boldsymbol{M}}(Y)$ over $\tilde{m} \in \widetilde{\boldsymbol{M}}(Y)$ is $|W_\bullet(q(\tilde{m}))|$ as (5-3) is homotopy cartesian. □

5.3 E_2–algebras coming from groupoids

A useful application of this result is as follows. As in [7, Section 17.1] (but replacing sSet by Top) there is a $\boldsymbol{T} \in \mathsf{Alg}_{E_2}(\mathsf{Top}^{\mathsf{G}})$ with $\boldsymbol{T}(A) \simeq *$ if $r(A) > 0$ and $\boldsymbol{T}(0) = \varnothing$, which is also cofibrant in $\mathsf{Alg}_{E_2}(\mathsf{Top}^{\mathsf{G}})$. Choosing an equivalence $* \to \boldsymbol{T}(X)$ we obtain by adjunction a map $X_*(*) \to \boldsymbol{T}$, which extends to an E_2–map $f\colon \boldsymbol{E}_2(X_*(*)) \to \boldsymbol{T}$. This can be strictified to a map $\bar{f}\colon \widetilde{\boldsymbol{S}} = \overline{\boldsymbol{E}}_2(X_*(*)) \to \overline{\boldsymbol{T}}$ of unital associative monoids in $\mathsf{Top}^{\mathsf{G}}$; furthermore these are cofibrant in this category by [7, Lemma 12.7(i)]. This gives $\overline{\boldsymbol{T}}$ the structure of a right $\widetilde{\boldsymbol{S}}$–module, cofibrant in $\mathsf{Top}^{\mathsf{G}}$, to which Theorem 5.1 can be applied.

The object $\boldsymbol{M} := r_*\overline{\boldsymbol{T}}$ satisfies

$$\boldsymbol{M}(n) \simeq \bigsqcup_{r(Y)=n} B\mathrm{Aut}_{\mathsf{G}}(Y)$$

because each $\overline{\boldsymbol{T}}(Y)$ is contractible. If in addition

(III) the map $-\oplus X\colon \mathrm{Aut}_{\mathsf{G}}(A \oplus X^{\oplus n}) \to \mathrm{Aut}_{\mathsf{G}}(A \oplus X^{\oplus n+1})$ is injective for all $n \geq 0$, and

(IV) $Y \oplus X^{\oplus m} \cong A \oplus X^{\oplus n}$ with $1 \leq m \leq n$ implies $Y \cong A \oplus X^{\oplus n-m}$,

then, as explained in [14, Section 7.3], for a point $m \in B\mathrm{Aut}_{\mathsf{G}}(A \oplus X^{\oplus n}) \subset \boldsymbol{M}$ the space $|W_\bullet(m)|$ is equivalent to the space $|W_n(A, X)_\bullet|$ of [17, Definition 2.1], also called "spaces of destabilisations". The following gives a conceptual meaning to these spaces of destabilisations, analogous to that given by Theorem 3.1.

Corollary 5.2 *Under the assumptions above there is an $\mathrm{Aut}_{\mathsf{G}}(A \oplus X^{\oplus n})$–equivariant equivalence between the unreduced suspension of $|W_n(A, X)_\bullet|$ and $Q_{\mathbb{L}}^{\widetilde{\boldsymbol{S}}}(\overline{\boldsymbol{T}})(A \oplus X^{\oplus n})$.*

Proof Apply Theorem 5.1 to $\widetilde{\boldsymbol{M}} = \overline{\boldsymbol{T}}$, and use that $\overline{\boldsymbol{T}}(Y) \simeq *$ so that the cofibre of (5-1) is the unreduced suspension of its fibre. □

Remark 5.3 In [17, Definition 2.8] there is formulated a simplicial complex $S_n(A, X)$ which is an "unordered version" of the semisimplicial sets $W_n(A, X)_\bullet$, and is mainly useful when the braided monoidal groupoid $(\mathsf{G}, \oplus, b, 0)$ is in fact symmetric monoidal. In this case $\boldsymbol{T}$ has the structure of an E_∞–algebra, and a similar analysis to that which we have carried out so far will show that $S_n(A,X)$ is $\mathrm{Aut}_{\mathsf{G}}(A \oplus X^{\oplus n})$–equivariantly equivalent to the indecomposables $Q_{\mathbb{L}}^{\overline{\boldsymbol{E}}_\infty(X_*(*))}(\overline{\boldsymbol{T}})(A \oplus X^{\oplus n})$ of $\overline{\boldsymbol{T}}$ as a module over the free E_∞–algebra on one generator. We leave the details of this argument to the appropriately motivated reader.

6 Coefficient systems, representation stability, and central stability

In [7, Section 19] it is discussed how to treat coefficient systems in the setting of Section 5.3. As a brief reminder, one fixes a commutative ring $\Bbbk$ and works in the category $\mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$ of functors from G to simplicial $\Bbbk$–modules. The constant functor $\underline{\Bbbk}$ with value $\Bbbk$ has the structure of a commutative algebra object in this category, and a *coefficient system* $\boldsymbol{A}$ is defined to be a right[1] $\underline{\Bbbk}$–module. It is called *discrete* if it takes values in $\Bbbk$–modules (considered as discrete simplicial $\Bbbk$–modules.)

Using $(-)_{\Bbbk} := \Bbbk[\mathrm{Sing}_\bullet(-)] \colon \mathsf{Top} \to \mathsf{sMod}_{\Bbbk}$ we can transport much of the previous discussion into the category of simplicial $\Bbbk$–modules. In particular there are unital associative monoids $\widetilde{\boldsymbol{S}}_{\Bbbk} \to \overline{\boldsymbol{T}}_{\Bbbk}$ which are cofibrant in $\mathsf{sMod}_{\Bbbk}^{\mathsf{G}}$, and as $\overline{\boldsymbol{T}}$ takes contractible values there is an equivalence of unital associative monoids $\overline{\boldsymbol{T}}_{\Bbbk} \xrightarrow{\sim} \underline{\Bbbk}$, which is a cofibrant replacement of $\underline{\Bbbk}$. Any coefficient system $\boldsymbol{A}$ can therefore be considered as a right $\overline{\boldsymbol{T}}_{\Bbbk}$–module, and if $\boldsymbol{A}^c \xrightarrow{\sim} \boldsymbol{A}$ is a cofibrant replacement as such then taking Kan extensions along $r \colon \mathsf{G} \to \mathbb{N}$ gives $\boldsymbol{R}_{\boldsymbol{A}} := r_*(\boldsymbol{A}^c) \simeq \mathbb{L}r_*(\boldsymbol{A})$ the structure of a right module over $\overline{\boldsymbol{R}}_{\Bbbk} := r_*(\overline{\boldsymbol{T}}_{\Bbbk})$. By definition of homotopy Kan extension,

$$H_{n,d}(\boldsymbol{R}_{\boldsymbol{A}}) = \bigoplus_{r(Y)=n} H_d(\mathrm{Aut}_{\mathsf{G}}(Y); \boldsymbol{A}(Y)).$$

Using the right $\overline{\boldsymbol{R}}_{\Bbbk}$–module structure and $\sigma \in H_{1,0}(\boldsymbol{R}_{\Bbbk})$ we can form the map $- \cdot \sigma \colon \boldsymbol{R}_{\boldsymbol{A}} \otimes S^{1,0} \to \boldsymbol{R}_{\boldsymbol{A}}$, and homological stability for the groups $\mathrm{Aut}_{\mathsf{G}}(Y)$ with coefficients in $\boldsymbol{A}(Y)$ can be phrased as a vanishing line for the homology of the cofibre $\boldsymbol{R}_{\boldsymbol{A}}/\sigma$.

Assuming that $\boldsymbol{A}$ is a discrete coefficient system we define

$$\mathrm{Tor}_p^{\underline{\Bbbk}}(\boldsymbol{A}, \Bbbk)(Y) := H_{Y,d}(B(\boldsymbol{A}, \underline{\Bbbk}, \Bbbk)),$$

and combining [7, Lemma 19.4] and [7, Theorem 19.2] shows that an appropriate vanishing line for these Tor–groups and homological stability for $\boldsymbol{R}_{\Bbbk}$ implies homological stability for $\boldsymbol{R}_{\boldsymbol{A}}$. A vanishing line for these Tor–groups sometimes goes under the name of *derived representation stability* for $\boldsymbol{A}$. These Tor–groups have a clear conceptual meaning: they measure how to construct $\boldsymbol{A}$ as a cellular $\underline{\Bbbk}$–module. (When G is the category of finite sets and bijections, then a $\underline{\Bbbk}$–module recovers the notion of an FI–module, and $\mathrm{Tor}_*^{\underline{\Bbbk}}(\Bbbk, \boldsymbol{A})$ recovers FI–homology in the sense of [3].)

There is another measure of the complexity of a coefficient system $\boldsymbol{A}$, namely the *central stability homology* $\widetilde{H}_*(\boldsymbol{A})$ of Putman and Sam [16] and Patzt [15]. Our main goal here is to give a similar conceptual interpretation of these homology groups, and hence to revisit Patzt's theorem [15, Theorem 5.7] relating $\widetilde{H}_*(\boldsymbol{A})$ and $\mathrm{Tor}_*^{\underline{\Bbbk}}(\boldsymbol{A}, \Bbbk)$.

[1] In [7, Section 19] left modules are considered, but there is no important difference.

Proposition 6.1 *A discrete coefficient system A may be considered as a right $\widetilde{S}_{\Bbbk}$–module via*

$$\widetilde{S}_{\Bbbk} \to \overline{T}_{\Bbbk} \xrightarrow{\sim} \underline{\Bbbk},$$

and then there are isomorphisms

$$H^{\widetilde{S}_{\Bbbk}}_{Y,d}(A) = H_{Y,d}(B(A, \widetilde{S}_{\Bbbk}, \Bbbk)) \cong \widetilde{H}_{d-1}(A)_Y.$$

Proof Following Section 5.2, the equivalence $|R_\bullet(\widetilde{S})| \to \widetilde{S}_{>0}$ and the cofibre sequence $\widetilde{S}_{>0} \to \widetilde{S} \to \mathbb{1}$ may be $\Bbbk$–linearised, and applying $B(A, \widetilde{S}_{\Bbbk}, -)$ and using that $B(A, \widetilde{S}_{\Bbbk}, \widetilde{S}_{\Bbbk}) \xrightarrow{\sim} A$ then gives a homotopy cofibre sequence

$$B\big(A, \widetilde{S}_{\Bbbk}, |R_\bullet(\widetilde{S}_{\Bbbk})|\big) \to A \to B(A, \widetilde{S}_{\Bbbk}, \Bbbk).$$

We may commute homotopy colimits and write the left-hand term as $|B(A, \widetilde{S}_{\Bbbk}, R_\bullet(\widetilde{S}_{\Bbbk}))|$. As

$$R_p(\widetilde{S}_{\Bbbk}) \simeq \widetilde{S}_{\Bbbk} \otimes (X^{\oplus p+1})_*(\Bbbk)$$

as a left $\widetilde{S}_{\Bbbk}$–module, we have $B(A, \widetilde{S}_{\Bbbk}, R_p(\widetilde{S}_{\Bbbk})) \simeq A \otimes (X^{\oplus p+1})_*(\Bbbk)$. The Bousfield–Kan spectral sequence for the augmented $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$–object $B(A, \widetilde{S}_{\Bbbk}, R_\bullet(\widetilde{S}_{\Bbbk})) \to A$ therefore takes the form

$$E^1_{Y,p,q} = H_{Y,q}(A \otimes (X^{\oplus p+1})_*(\Bbbk)) \Rightarrow H_{Y,p+q+1}(B(A, \widetilde{S}_{\Bbbk}, \Bbbk))$$

for $p \geq -1$. As $A \otimes (X^{\oplus p+1})_*(\Bbbk)$ is discrete this spectral sequence is supported along the line $q = 0$ and so collapses at E^2. By definition of Day convolution it has

$$E^1_{Y,p,0} = \operatorname*{colim}_{\substack{(Z,f) \text{ s.t.} \\ f\colon Z \oplus X^{\oplus p+1} \xrightarrow{\sim} Y}} A(Z)$$

and by definition of the $\widetilde{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$–object $R_\bullet(\widetilde{S}_{\Bbbk})$ the d^1–differential is given by the alternating sum of the maps

$$\delta_0, \delta_1, \dots, \delta_p\colon \operatorname*{colim}_{\substack{(Z,f) \text{ s.t.} \\ f\colon Z \oplus X^{\oplus p+1} \xrightarrow{\sim} Y}} A(Z) \to \operatorname*{colim}_{\substack{(Z',f') \text{ s.t.} \\ f'\colon Z' \oplus X^{\oplus p} \xrightarrow{\sim} Y}} A(Z'),$$

where δ_i braids the i^{th} copy of X in $X^{\oplus p+1}$ in front of the others to put it first, then adds it to Z to form $Z' := Z \oplus X$; it then applies $A(Z) \to A(Z \oplus X)$ given by the right $\widetilde{S}_{\Bbbk}$–module structure. Using [15, Proposition 4.3] one finds the same description of the complex that calculates central stability homology, so $\widetilde{H}_p(A)_Y \cong E^2_{p,0} \cong H_{Y,p+1}(B(A, \widetilde{S}_{\Bbbk}, \Bbbk))$, as claimed. □

For the following we strengthen assumption (III) of Section 5.3 to

(III′) the map $- \oplus -\colon \mathrm{Aut}_{\mathsf{G}}(U) \times \mathrm{Aut}_{\mathsf{G}}(V) \to \mathrm{Aut}_{\mathsf{G}}(U \oplus V)$ is injective for all $U, V \in \mathsf{G}$.

With the interpretations of $H^{\widetilde{S}_{\Bbbk}}_{*,*}(\overline{T}_{\Bbbk})$ given by Corollary 5.2 and of $H^{\widetilde{S}_{\Bbbk}}_{*,*}(A)$ given by Proposition 6.1, and the interpretation of a vanishing line for $\mathrm{Tor}^{\underline{\Bbbk}}_*(\Bbbk, A)$ in terms of a minimal $\underline{\Bbbk}$–module resolution of A, the following is then a version of Patzt's [15, Theorem 5.7].

Theorem 6.2 *Let $f:\mathbb{N}\to\mathbb{N}$ and assume that $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{Y,d}(\overline{\boldsymbol{T}}_\Bbbk)=0$ for $d<f(r(Y))$.*

(i) *If $g:\mathbb{N}\to\mathbb{N}$ is such that $\mathrm{Tor}^{\underline{\Bbbk}}_d(\underline{\Bbbk},\boldsymbol{A})(V)=0$ for $d<g(r(V))$, then $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{Y,d}(\boldsymbol{A})=0$ for $d<\bar{g}(r(Y))$, where $\bar{g}(n):=\min\{f(p)+g(n-p)\mid 0\le p\le n\}$.*

(ii) *If $h:\mathbb{N}\to\mathbb{N}$ is such that $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{U,d}(\boldsymbol{A})=0$ for $d<h(r(U))$, then $\mathrm{Tor}^{\underline{\Bbbk}}_d(\underline{\Bbbk},\boldsymbol{A})(Y)=0$ for $d<\bar{h}(r(Y))$, where $\bar{h}(n)$ is defined inductively by $\bar{h}(0)=h(0)$ and*

$$\bar{h}(n):=\min\{h(n),\ f(p)+\bar{h}(n-p)+1\mid 1\le p\le n\}.$$

Proof Consider $B(B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk),\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})$. By interchanging geometric realisations and using

$$B(\overline{\boldsymbol{T}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})\xrightarrow{\simeq}\boldsymbol{A}$$

this is equivalent to $B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\boldsymbol{A})$. On the other hand we may descendingly filter $B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk)$ by rank, as in [7, Remark 19.5]. The associated graded is equivalent to $B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk)$ but its $\widetilde{\boldsymbol{S}}_\Bbbk$–module structure is now trivial (ie induced via the augmentation $\widetilde{\boldsymbol{S}}_\Bbbk\to\underline{\Bbbk}$). Thus the induced filtration of $B(B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk),\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})$ has associated graded $B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk)\otimes B(\underline{\Bbbk},\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})$. Using (III′) we may apply [7, Lemma 10.6] to see that the associated spectral sequence takes the form

$$E^1_{Y,p,q}=\operatorname*{colim}_{\substack{U\oplus V\xrightarrow{\sim}Y\\ r(U)=p}}H_{p+q}\big(B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk)(U)\otimes B(\underline{\Bbbk},\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})(V)\big)\Rightarrow H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{Y,p+q}(\boldsymbol{A}).$$

For such U and V there is also a Künneth spectral sequence [7, Lemma 10.5]

$$\bigoplus_{t'+t''=q}\mathrm{Tor}^{\Bbbk}_s(H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{U,t'}(\overline{\boldsymbol{T}}_\Bbbk),H^{\overline{\boldsymbol{T}}_\Bbbk}_{V,t''}(\boldsymbol{A}))\Rightarrow H_{s+t}\big(B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk)(U)\otimes B(\underline{\Bbbk},\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})(V)\big).$$

By assumption $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{U,t'}(\overline{\boldsymbol{T}}_\Bbbk)=0$ for $t'<f(p)$ as $r(U)=p$. As $\boldsymbol{A}$ is assumed to be discrete, the discussion before [7, Lemma 19.4] gives $H^{\overline{\boldsymbol{T}}_\Bbbk}_{V,t''}(\boldsymbol{A})\cong\mathrm{Tor}^{\underline{\Bbbk}}_{t''}(\underline{\Bbbk},\boldsymbol{A})(V)$.

Supposing first that $\mathrm{Tor}^{\underline{\Bbbk}}_d(\underline{\Bbbk},\boldsymbol{A})(V)=0$ for all $d<g(r(V))$, the Künneth spectral sequence implies that

$$H_{p+q}\big(B(\underline{\Bbbk},\widetilde{\boldsymbol{S}}_\Bbbk,\overline{\boldsymbol{T}}_\Bbbk)(U)\otimes B(\underline{\Bbbk},\overline{\boldsymbol{T}}_\Bbbk,\boldsymbol{A})(V)\big)=0\quad\text{for } p+q<f(p)+g(r(Y)-p),$$

and so the first spectral sequence implies that $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{Y,d}(\boldsymbol{A})=0$ for $d<\bar{g}(r(Y))$, by definition of $\bar{g}$.

Suppose now that $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{Y,d}(\boldsymbol{A})=0$ for all $d<h(r(Y))$. Suppose for an induction that $\mathrm{Tor}^{\underline{\Bbbk}}_d(\underline{\Bbbk},\boldsymbol{A})(Y')=0$ for all $d<\bar{h}(r(Y'))$ and all $r(Y')<r(Y)$. The only object $U\in\mathsf{G}$ with $r(U)=0$ is $U=0$ by (I), and $H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{0,*}(\overline{\boldsymbol{T}}_\Bbbk)=\underline{\Bbbk}[0]$ consists of free $\Bbbk$–modules. Thus the Künneth spectral sequence collapses to give $E^1_{Y,0,q}=H^{\overline{\boldsymbol{T}}_\Bbbk}_{Y,q}(\boldsymbol{A})$. On the other hand if $r(U)>0$ then $r(V)<r(Y)$ and so by the inductive hypothesis $H^{\overline{\boldsymbol{T}}_\Bbbk}_{V,t''}(\boldsymbol{A})=0$ for $t''<\bar{h}(r(V))$; it then follows by the same argument as above that for $p>0$ we have $E^1_{Y,p,q}=0$ when $p+q<f(p)+\bar{h}(r(Y)-p)$. As the differentials have the form $d^r:E^r_{Y,0,d}\to E^r_{Y,r,d-r-1}$, the cokernel of the edge homomorphism

$$H^{\widetilde{\boldsymbol{S}}_\Bbbk}_{Y,d}(\boldsymbol{A})\to E^1_{Y,0,d}=H^{\overline{\boldsymbol{T}}_\Bbbk}_{Y,d}(\boldsymbol{A})$$

is trivial for $d-1 < \min\{f(p)+\bar{h}(r(Y)-p) \mid 1 \le p \le r(Y)\}$. As the domain of this edge homomorphism vanishes for $d < h(r(Y))$, it follows that $H^{\overline{T}_{\Bbbk}}_{Y,d}(\boldsymbol{A}) \cong \mathrm{Tor}^{\Bbbk}_{d}(\Bbbk, \boldsymbol{A})(Y)$ vanishes for $d < \bar{h}(r(Y))$. □

7 The space of destabilisations and the splitting complexes

In this section we continue to work in combinatorial setting of Section 5.3, and will explain the relationship between the connectivities of the spaces of destabilisations $|W_n(0,X)_\bullet|$ defined in [17, Definition 2.1], and the connectivities of the "E_1– and E_2–splitting complexes" $|Z^{E_1}_\bullet(X^{\oplus n})|$ and $|Z^{E_2}_{\bullet,\bullet}(X^{\oplus n})|$ defined in [7, Sections 17.2 and 17.3].

Proposition 7.1 *Let* $(\mathsf{G}, \oplus, b, 0)$ *be a braided monoidal groupoid satisfying (I), (II), (III) and (IV), and suppose* $r\colon \mathsf{G} \to \mathbb{N}$ *is a bijection on isomorphism classes of objects, with* $X \in \mathsf{G}$ *corresponding to* $1 \in \mathbb{N}$*. Let* $f\colon \mathbb{N} \to \mathbb{N}$ *be a function satisfying* $f(n) \le n$ *and* $f(n+m) \le f(n) + f(m)$*. Then the following are equivalent*:

(i) *The homology of* $|W_n(0,X)_\bullet|$ *vanishes in degrees* $* < f(n)-1$ *for all* $n > 1$.

(ii) *The homology of* $|Z^{E_1}_\bullet(X^{\oplus n})|$ *vanishes in degrees* $* < f(n)+1$ *for all* $n > 1$.

(iii) *The homology of* $|Z^{E_2}_{\bullet,\bullet}(X^{\oplus n})|$ *vanishes in degrees* $* < f(n)+2$ *for all* $n > 1$.

Hepworth [12, Theorem 13.2] has shown that (ii) implies (i) under slightly different connectivity hypotheses; see Example 7.4.

Proof of Proposition 7.1 We will first show that (iii) is equivalent to (i), and later the simpler statement that (iii) is equivalent to (ii). To do so we will use the abstract connectivity (see [7, Definition 11.1]) $\sigma\colon \mathsf{G} \to [-\infty,\infty]_{\ge}$ defined by $\sigma(A) := f(r(A))$, which by assumption satisfies $\sigma \le r$ and $\sigma * \sigma \ge \sigma$.

As before we may linearise (via $(-)_{\mathbb{Z}} := \mathbb{Z}[\mathrm{Sing}_\bullet(-)]\colon \mathsf{Top} \to \mathsf{sMod}_{\mathbb{Z}}$) the E_2–map $f\colon \boldsymbol{E}_2(X_*(*)) \to \boldsymbol{T}$ to obtain a map $f_{\mathbb{Z}}\colon \boldsymbol{E}_2(X_*(\mathbb{Z})) \to \boldsymbol{T}_{\mathbb{Z}}$ of E_2–algebras in $\mathsf{sMod}^{\mathsf{G}}_{\mathbb{Z}}$, with $\boldsymbol{T}_{\mathbb{Z}}(A) \simeq \mathbb{Z}$ if $r(A) > 0$ and $\boldsymbol{T}_{\mathbb{Z}}(0) = 0$. By [7, Proposition 17.14],

$$S^{0,2} \otimes Q^{E_2}_{\mathbb{L}}(\boldsymbol{T}_{\mathbb{Z}})(A) \simeq |\mathbb{Z}[Z^{E_2}_{\bullet,\bullet}(A)]|.$$

As $Q^{E_2}_{\mathbb{L}}\big(\boldsymbol{E}_2(X_*(\mathbb{Z}))\big) \simeq X_*(\mathbb{Z})$ is supported on the object X,

$$H^{E_2}_{X^{\oplus n},d}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) \cong H^{E_2}_{X^{\oplus n},d}(\boldsymbol{T}_{\mathbb{Z}}) \cong \widetilde{H}_{d+2}\big(|Z^{E_2}_{\bullet,\bullet}(X^{\oplus n})|\big) \tag{7-1}$$

as long as $n > 1$. On the other hand, as $Q^{\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))}_{\mathbb{L}}\big(\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))\big) \simeq 0_*(\mathbb{Z})$ is supported on the object 0, linearising the conclusion of Corollary 5.2 gives

$$H^{\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))}_{X^{\oplus n},d}\big(\overline{\boldsymbol{T}}_{\mathbb{Z}}, \overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))\big) \cong \widetilde{H}_{d-1}\big(|W_n(0,X)_\bullet|\big) \tag{7-2}$$

as long as $n > 0$.

The homology groups to which (iii) and (i) refer are the right-hand sides of (7-1) and (7-2) respectively; we will compare them using the interpretations given by the left-hand sides, as relative E_2–algebra indecomposables and $\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))$–module indecomposables respectively.

Suppose first that $(\mathsf{G}, \oplus, b, 0)$ is in fact *symmetric* monoidal. Then we may apply [7, Theorem 15.9] in the category $\mathsf{C} := \mathsf{sMod}_{\mathbb{Z}}^{\mathsf{G}}$ with $k = 2$, because G is 3–monoidal (= symmetric monoidal) and so C, with the Day convolution monoidal structure, is too. This theorem, applied to the morphism $f_{\mathbb{Z}}\colon \boldsymbol{E}_2(X_*(\mathbb{Z})) \to \boldsymbol{T}_{\mathbb{Z}}$ with $\rho = r$ and with σ an abstract connectivity such that $\sigma * \sigma \geq \sigma$ and $r \geq \sigma$, says the following: if $H^{E_2}_{X^{\oplus n},d}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) = 0$ whenever $d < \sigma(X^{\oplus n})$ then there is a morphism

$$H^{\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))}_{X^{\oplus n},d}\big(\overline{\boldsymbol{T}}_{\mathbb{Z}}, \overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))\big) \to H^{E_2}_{X^{\oplus n},d}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) \tag{7-3}$$

which is an isomorphism for $d < (\sigma * \sigma)(X^{\oplus n})$ and an epimorphism for $d < (\sigma * \sigma)(X^{\oplus n}) + 1$.

If σ is such that (iii) holds then by (7-1) the assumption for the above is satisfied, and so as $\sigma * \sigma \geq \sigma$ it follows that $H^{\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))}_{X^{\oplus n},d}\big(\overline{\boldsymbol{T}}_{\mathbb{Z}}, \overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))\big) = 0$ for $d < \sigma(X^{\oplus n})$, so by (7-2) it follows that the homology of $|W_n(0, X)_\bullet|$ vanishes in degrees $* < \sigma(X^{\oplus n}) - 1$ for $n > 1$.

In the other direction, if σ is such that (i) holds then $H^{\overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))}_{X^{\oplus n},d}\big(\overline{\boldsymbol{T}}_{\mathbb{Z}}, \overline{\boldsymbol{E}}_2(X_*(\mathbb{Z}))\big) = 0$ for $d < \sigma(X^{\oplus n})$ by (7-2). Define abstract connectivities σ_k by

$$\sigma_k(X^{\oplus n}) := \begin{cases} \sigma(X^{\oplus n}) & \text{if } n \leq k, \\ \sigma(X^{\oplus k}) & \text{if } n \geq k, \end{cases}$$

which satisfy $\sigma_k * \sigma_k \geq \sigma_k$ and $\sigma_k \leq r$. As $H^{E_2}_{X^{\oplus n},0}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) = 0$ for all n,

$$H^{E_2}_{X^{\oplus n},d}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) = 0 \quad \text{for } d < \sigma_1(X^{\oplus n}),$$

because $\sigma(X) \leq r(X) = 1$ by assumption. Suppose for an induction that $H^{E_2}_{X^{\oplus n},d}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) = 0$ for $d < \sigma_k(X^{\oplus n})$. Then by [7, Theorem 15.9] the map (7-3) is an epimorphism for $d < (\sigma_k * \sigma_k)(X^{\oplus n}) + 1$, and by assumption its source vanishes for $d < \sigma(X^{\oplus n})$, so we conclude that its target vanishes for

$$d < \inf(\sigma, \sigma_k * \sigma_k + 1)(X^{\oplus n}).$$

In particular it vanishes for $d < \inf(\sigma, \sigma_k + 1)(X^{\oplus n})$ and hence also for $d < \sigma_{k+1}(X^{\oplus n})$, as $\sigma_{k+1} \leq \sigma$ and $\sigma_{k+1} \leq \sigma_k + 1$. It follows by induction that $H^{E_2}_{X^{\oplus n},d}\big(\boldsymbol{T}_{\mathbb{Z}}, \boldsymbol{E}_2(X_*(\mathbb{Z}))\big) = 0$ for $d < \sigma_\infty(X^{\oplus n}) = \sigma(X^{\oplus n})$. Using (7-1) this translates to the homology of $|Z^{E_2}_{\bullet,\bullet}(X^{\oplus n})|$ vanishing in degrees $* < \sigma(X^{\oplus n}) + 2$ for $n > 1$. This finishes the proof that (iii) is equivalent to (i) if $(\mathsf{G}, \oplus, b, 0)$ is symmetric monoidal.

If $(\mathsf{G}, \oplus, b, 0)$ is only *braided* monoidal then we cannot appeal directly to [7, Theorem 15.9]; its proof uses [7, Theorem 15.3], which is false if $k = 2$ and G is only braided monoidal (see Example A.2). However, in the appendix we show that the conclusion of [7, Theorem 15.9] is nonetheless true when $k = 2$ and G is only braided monoidal. Given this, the above argument goes through to show that (iii) is equivalent to (i).

To see that (iii) and (ii) are equivalent we use the results of [7, Section 14] for transferring vanishing lines, along with

$$H^{E_1}_{X^{\oplus n},d}(\boldsymbol{T}_{\mathbb{Z}}) \cong \widetilde{H}_{d+1}\big(|Z^{E_1}_{\bullet}(X^{\oplus n})|\big) \tag{7-4}$$

from [7, Proposition 17.14]. If σ is such that (ii) holds then (7-4) shows that $H^{E_1}_{X^{\oplus n},d}(\boldsymbol{T}_{\mathbb{Z}}) = 0$ for $d < \sigma(X^{\oplus n})$ for $n > 1$, so letting

$$\rho(X^{\oplus n}) := \begin{cases} \sigma(X^{\oplus n}) + 1 & \text{if } n > 1, \\ n & \text{if } n \leq 1, \end{cases}$$

we have $\rho * \rho \geq \rho$ and $H^{E_1}_{X^{\oplus n},d}(\boldsymbol{T}_{\mathbb{Z}}) = 0$ for $d < \rho(X^{\oplus n}) - 1$, so by [7, Theorem 14.4] it follows that $H^{E_2}_{X^{\oplus n},d}(\boldsymbol{T}_{\mathbb{Z}}) = 0$ for $d < \rho(X^{\oplus n}) - 1$ (so for $d < \sigma(X^{\oplus n})$ and $n > 1$), which via (7-1) implies that (iii) holds. Using the same ρ, [7, Theorem 14.4] shows that (iii) implies (ii). □

Example 7.2 Let $(\mathsf{G}, \oplus, b, 0)$ be the free braided monoidal groupoid on one object X, so $\mathrm{Aut}_{\mathsf{G}}(X^{\oplus n}) \cong \beta_n$ is the braid group on n strands. In this case $\boldsymbol{T} \in \mathsf{Alg}_{E_2}(\mathsf{Top}^{\mathsf{G}})$ is the free E_2–algebra on $X_*(*)$, and so $|Z^{E_2}_{\bullet,\bullet}(A)|$ is the value at $A \in \mathsf{G}$ of the object $S^{0,2} \wedge X_*(S^0) \in \mathsf{Top}^{\mathsf{G}}_*$. This is S^2 when evaluated at X and contractible otherwise, so in general when evaluated at $X^{\oplus n}$ its homology vanishes in degrees $* < n + 2$ for all $n > 1$. By Proposition 7.1 it then follows that $|W_n(0, X)_\bullet|$ is homologically $(n-2)$–connected. The latter may be described as an arc complex [17, Section 5.6.2]. Note however that we used this connectivity (and in fact that it is contractible) in the proof of Lemma 3.2, so this is not new information.

Example 7.3 Similarly, if $(\mathsf{G}, \oplus, b, 0)$ is the free symmetric monoidal groupoid on one object X, so $\mathrm{Aut}_{\mathsf{G}}(X^{\oplus n}) \cong \Sigma_n$ is the n^{th} symmetric group, then $\boldsymbol{T}$ is the free E_∞–algebra on $X_*(*)$. Thus $|Z^{E_2}_{\bullet,\bullet}(A)| \simeq E_\infty(X_*(S^2))(A)$ by combining [7, Theorems 13.7, 13.8 and 17.4]. At $A = X^{\oplus n}$ this evaluates to $(E\Sigma_n)_+ \wedge_{\Sigma_n} (S^2)^{\wedge n}$ and so has trivial homology in degrees $* < 2n$, so in particular in degrees $* < n + 2$ for all $n > 1$. By Proposition 7.1 it then follows that $|W_n(0, X)_\bullet|$ is homologically $(n-2)$–connected. The latter may be identified with the "complex of injective words", which gives a (very complicated) new proof for the homological high-connectivity of this semisimplicial set.

Example 7.4 That $|Z^{E_1}_{\bullet}(X^{\oplus n})|$ be $(n-1)$–connected is called the "standard connectivity estimate" in [7, Definition 17.6], and several examples of braided monoidal groupoids are known to satisfy this: general linear groups over Dedekind domains [7, Section 18.2], mapping class groups of oriented surfaces [8, Theorem 3.4], and automorphism groups of free groups [12, Corollary 4.5]. In this case Proposition 7.1 applies with $f(n) = \frac{1}{2}n$ to show that $|W_n(0, X)_\bullet|$ has trivial homology in degrees $* < \frac{1}{2}(n-2)$, ie is homologically $\frac{1}{2}(n-3)$–connected. This recovers [12, Theorem 13.2] at the level of homology.

Example 7.5 In [17, Section 5] many examples are given of braided monoidal groupoids $(\mathsf{G}, \oplus, b, 0)$ such that $|W_n(0, X)_\bullet|$ is $\frac{1}{2}(n-3)$–connected. For example, the groupoids corresponding to: automorphism groups of free groups [17, Proposition 5.3 and Theorem 2.10], general linear groups of rings having stable rank ≤ 1 [17, Lemma 5.10], mapping class groups of orientable surfaces [17, Lemma 5.25] and certain 3–manifolds [17, Section 5.7]. Setting

$$f(n) := \begin{cases} \frac{1}{2}(n+1) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

we have $f(n) \leq n$ and $f(n+m) \leq f(n) + f(m)$, and $|W_n(0,X)_\bullet|$ has trivial homology in degrees $* < f(n)-1$. By Proposition 7.1 it then follows that for $n > 1$ the space $|Z_\bullet^{E_1}(X^{\oplus n})|$ has trivial homology in degrees $* < \frac{1}{2}(n+3)$, in all of these cases.

Appendix Comparing algebra and module cells, extended

The goal of this technical appendix is to relax very slightly the hypotheses of [7, Theorem 15.9] in the case $k = 2$, as follows. (In the following $\boldsymbol{S}$ no longer denotes the free E_2–algebra on one generator! The notation is parallel to [7, Theorem 15.9].)

Theorem A.1 *Suppose that* S *satisfies* [7, Axiom 11.19], *and that* G *is braided monoidal and Artinian. Let* $\rho, \sigma \colon \mathsf{G} \to [-\infty, \infty]_\geq$ *be abstract connectivities such that* $\rho * \rho \geq \rho$, $\sigma * \sigma \geq \sigma$ *and* $\rho * \sigma \geq \sigma * \sigma$. *If*

(i) $\boldsymbol{R} \in \mathsf{Alg}_{E_2}(\mathsf{C})$ *is such that* $H^{E_2}_{g,d}(\boldsymbol{R}) = 0$ *for* $d < \rho(g) - 1$,

(ii) $f \colon \boldsymbol{R} \to \boldsymbol{S}$ *is an* E_2*–algebra map such that* $H^{E_2}_{g,d}(\boldsymbol{S}, \boldsymbol{R}) = 0$ *for* $d < \sigma(g)$, *and*

(iii) $\boldsymbol{R}$ *and* $\boldsymbol{S}$ *are cofibrant in* C, 0*–connective, and reduced*,

then there is a map $H^{\overline{\boldsymbol{R}}}_{g,d}(\overline{\boldsymbol{S}}, \overline{\boldsymbol{R}}) \to H^{E_2}_{g,d}(\boldsymbol{S}, \boldsymbol{R})$ *which is an isomorphism for* $d < (\sigma * \sigma)(g)$, *and an epimorphism for* $d < (\sigma * \sigma)(g) + 1$.

The only change from the $k = 2$ case of [7, Theorem 15.9] is that G is only required to be braided monoidal, rather than symmetric monoidal.

Let us first explain the issue. The proof of [7, Theorem 15.9] uses [7, Theorem 15.3], which when G is symmetric monoidal provides an equivalence

$$\overline{\boldsymbol{E}}_2(A \vee B) \simeq \overline{\boldsymbol{E}}_2(A) \otimes E_2^+(E_1^+(S^1 \wedge A) \otimes B) \tag{A-1}$$

of left $\overline{\boldsymbol{E}}_2(A)$–modules. However, if G is only braided monoidal then there is no such equivalence.

To explain why, recall as in Section 5.1 that to discuss E_2–algebras in a category which is only braided monoidal we use the braided version $\mathscr{C}_2^{\mathsf{FB}_2}$ of the nonunitary little 2–cubes operad [7, Definition 12.6], which has $\mathscr{C}_2^{\mathsf{FB}_2}(n)$ contractible for each $n > 0$.

Example A.2 Let $\mathsf{G} = \mathsf{FB}_2$, the free braided monoidal groupoid on one generator, ie $\mathsf{G} = \bigsqcup_{n \geq 0} \{n\} /\!\!/ \beta_n$, and take $\mathsf{S} = \mathsf{sMod}_{\mathbb{Z}}$. Let $A = B = \{1\}_*(\mathbb{Z})$, with $\mathbb{Z}$ considered to be in degree 0. Then on the left-hand side of (A-1),

$$\overline{\boldsymbol{E}}_2(A \vee B)(\{n\}) \simeq (\mathbb{Z} \oplus \mathbb{Z})^{\otimes n}.$$

This is because, by definition of Day convolution, in $\mathsf{sMod}_{\mathbb{Z}}^{\mathsf{G}}$ the object $(A \vee B)^{\otimes n}$ is supported at $\{n\}$ and is here given by $\mathrm{Ind}_{\beta_1 \times \cdots \times \beta_1}^{\beta_n}((\mathbb{Z} \oplus \mathbb{Z})^{\otimes n})$, so when we apply $\mathscr{C}_2^{\mathsf{FB}_2}(n) \times_{\beta_n} -$ (see [7, Definition 12.6]) we obtain $\mathbb{Z}[\mathscr{C}_2^{\mathsf{FB}_2}(n)] \otimes (\mathbb{Z} \oplus \mathbb{Z})^{\otimes n} \simeq (\mathbb{Z} \oplus \mathbb{Z})^{\otimes n}$, using that $\mathscr{C}_2^{\mathsf{FB}_2}(n)$ is contractible. In particular, in each grading the homology of $\overline{\boldsymbol{E}}_2(A \vee B)$ is supported in degree zero.

On the other hand, the right-hand side of (A-1) contains as a retract $A \otimes (S^1 \wedge A) \otimes B$. This is supported on the object $\{3\}$ where it is given by

$$\mathrm{Ind}_{\beta_1 \times \beta_1 \times \beta_1}^{\beta_3}(\mathbb{Z} \otimes (S^1 \wedge \mathbb{Z}) \otimes \mathbb{Z}),$$

which has nontrivial first homology. Thus (A-1) cannot hold.

Our solution to this issue will be that although (A-1) need not hold when G is braided monoidal, a certain connectivity estimate for the natural morphism $B \to B(\mathbb{1}, \overline{\boldsymbol{E}}_2(A), \overline{\boldsymbol{E}}_2(A \vee B))$ that one would deduce from (A-1) does in any case hold, and it is only this connectivity estimate that is used in the proof of [7, Theorem 15.9]. The required connectivity estimate is as follows.

Proposition A.3 *Suppose that* S *satisfies* [7, Axiom 11.19], *and that* G *is braided monoidal and Artinian. Let* $\sigma, \rho\colon \mathsf{G} \to [-\infty, \infty]_{\geq}$ *be abstract connectivities with* $\sigma * \sigma \geq \sigma$, $\rho * \rho \geq \rho$ *and* $\rho * \sigma \geq \sigma * \sigma$. *If* $A \in \mathsf{C} := \mathsf{S}^{\mathsf{G}}$ *is homologically* $(\rho-1)$*–connective and* $B \in \mathsf{C}$ *is homologically* σ*–connective then the natural map*

$$B \to B(\mathbb{1}, \overline{\boldsymbol{E}}_2(A), \overline{\boldsymbol{E}}_2(A \vee B))$$

is homologically $\sigma * \sigma$*–connective.*

Proof Let $\beta_{a_1, a_2, \ldots, a_r}$ denote the subgroup of the braid group $\beta_{a_1 + a_2 + \cdots + a_r}$ consisting of those braids which induce a permutation which preserves the decomposition

$$\{1, 2, \ldots, a_1\} \sqcup \{a_1 + 1, \ldots, a_1 + a_2\} \sqcup \cdots \sqcup \{a_1 + a_2 + \cdots + a_{r-1} + 1, \ldots, a_1 + a_2 + \cdots + a_r\}.$$

In the braided monoidal category C,

$$(A \vee B)^{\otimes n} \cong \bigvee_{a+b=n} \mathrm{Ind}_{\beta_{a,b}}^{\beta_n}(A^{\otimes a} \otimes B^{\otimes b}),$$

and so

$$\overline{\boldsymbol{E}}_2(A \vee B) \cong \mathbb{1} \vee \bigvee_{a+b \geq 1} \mathrm{Res}_{\beta_{a,b}}^{\beta_{a+b}}((0,\infty) \times \mathscr{C}_2^{\mathsf{FB}_2}(a+b))_+ \wedge_{\beta_{a,b}} (A^{\otimes a} \otimes B^{\otimes b}).$$

Similarly, $\overline{\boldsymbol{E}}_2(A) \cong \mathbb{1} \vee \bigvee_{n \geq 1}((0,\infty) \times \mathscr{C}_2^{\mathsf{FB}_2}(n))_+ \wedge_{\beta_n} A^{\otimes n}$. Using these identities we may express $B(\mathbb{1}, \overline{\boldsymbol{E}}_2(A), \overline{\boldsymbol{E}}_2(A \vee B))$ as an analytic functor of the variables A and B in the form

$$\bigvee_{a,b \geq 0} |C(a,b)_\bullet| \wedge_{\beta_{a,b}} (A^{\otimes a} \otimes B^{\otimes b}) \tag{A-2}$$

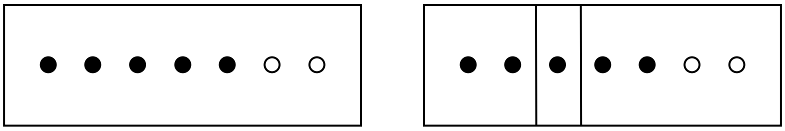

Figure 2: Left: the standard configuration with $a = 5$ and $b = 2$. Right: the standard 1–simplex $\sigma(2, 1, 2)$.

where $C(a,b)_\bullet$ is the semisimplicial pointed space (with free $\beta_{a,b}$–action) given as follows. The space $C(a,b)_p$ is

$$\bigvee_{a_1+\cdots+a_p+a_{p+1}=a} \mathrm{Ind}^{\beta_{a,b}}_{\beta_{a_1}\times\beta_{a_2}\times\cdots\times\beta_{a_p}\times\beta_{a_{p+1},b}} \left(\left(\prod_{i=1}^{p}(0,\infty)\times\mathscr{C}_2^{\mathsf{FB}_2}(a_i)\right)\times(0,\infty)\times\mathscr{C}_2^{\mathsf{FB}_2}(a_{p+1}+b)\right)_+,$$

with face maps given as in the two-sided bar construction.

Under the given connectivity assumptions $A^{\otimes a}\otimes B^{\otimes b}$ is $(\rho-1)^{*a}*\sigma^{*b}$–connective, ie $(\rho^{*a}*\sigma^{*b}-a)$–connective. As $|C(a,b)_\bullet|$ is a free $\beta_{a,b}$–space, the claim will follow from the decomposition (A-2) as long as $|C(a,b)_\bullet|$ is a–connective, and contractible for $b=0$. In this case $|C(a,b)_\bullet|\wedge_{\beta_{a,b}}(A^{\otimes a}\otimes B^{\otimes b})$ is contractible for $b=0$, and is $(\rho^{*a}*\sigma^{*b})$–connective otherwise, so is at least $\sigma*\sigma$–connective except when $(a,b)=(0,1)$.

To prove this connectivity statement we observe that the $C(a,b)_p$ are homotopy-discrete, because $(0,\infty)$ and $\mathscr{C}_2^{\mathsf{FB}_2}(n)$ are all contractible, so the semisimplicial pointed space $C(a,b)_\bullet$ is levelwise homotopy equivalent to the semisimplicial pointed set $\pi_0 C(a,b)_\bullet$ having

$$\pi_0 C(a,b)_p = \bigvee_{a_1+\cdots+a_p+a_{p+1}=a}\left(\frac{\beta_{a,b}}{\beta_{a_1}\times\beta_{a_2}\times\cdots\times\beta_{a_p}\times\beta_{a_{p+1},b}}\right)_+.$$

This semisimplicial pointed set admits a system of degeneracies, by setting $a_i=0$, making it a simplicial pointed set. The connectivity of this simplicial set can be analysed by the same argument as [8, Section 4], as we now explain.

Fix a configuration of a black points and b white points in $[0,a+b]\times[0,1]$, as shown in Figure 2, left, and let $\Sigma^{a,b}$ denote the surface given by this square with these marked points. Let the poset $\mathsf{S}(a,b)$ consist of the set of isotopy classes of smoothly embedded arcs $\alpha\colon[0,1]\to\Sigma^{a,b}$ disjoint from the marked points and with $\alpha(0)\in[0,a+b]\times\{0\}$ and $\alpha(1)\in[0,a+b]\times\{1\}$, such that the left-hand side of the arc contains a nonzero number of points, and the right-hand side contains all white points. Say that $[\alpha]\le[\alpha']$ if α and α' can be represented by disjoint embedded arcs with $\alpha(0)\le\alpha'(0)\in[0,a+b]$. Let $S(a,b)_\bullet$ denote the simplicial nerve of the poset $\mathsf{S}(a,b)$. The mapping class group of $\Sigma^{a,b}$, where diffeomorphisms must fix the boundary but are allowed to permute the black or the white marked points (but not interchange them), is the group $\beta_{a,b}$, and it acts on $S(a,b)_\bullet$. To a set of natural numbers $a_0>0, a_1, a_2, \ldots, a_{p+1}\ge 0$ there is an associated p–simplex $\sigma(a_0,\ldots,a_{p+1})\in S(a,b)_p$ as shown in Figure 2, right, given by vertical arcs partitioning the black points into groups of the indicated sizes. Every p–simplex is in the orbit of a

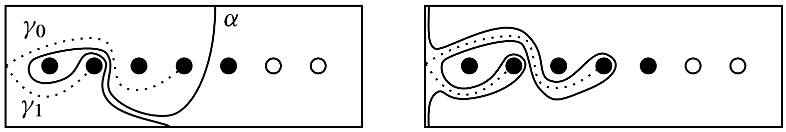

Figure 3: Left: dotted arcs γ_0 and γ_1, and a solid arc α in $F(\{[\gamma_0],[\gamma_1]\})$. Right: the maximal element of $F(\{[\gamma_0],[\gamma_1]\})$.

unique $\sigma(a_0,\dots,a_{p+1})$ (by counting the number of black points between the arcs). Furthermore, the stabiliser of this simplex under the $\beta_{a,b}$–action is the subgroup

$$\beta_{a_0}\times\cdots\times\beta_{a_p}\times\beta_{a_{p+1},b}.$$

We therefore recognise the pointed simplicial set $\pi_0 C(a,b)_\bullet$ as the suspension of the simplicial set $S(a,b)_\bullet$.

When $b=0$ the poset $\mathsf{S}(a,0)$ has a maximal element, given by the arc having no points to its right, and so $\pi_0 C(a,0)_\bullet$ is indeed contractible. When $b>0$ we must show that $|\pi_0 C(a,b)_\bullet|$ is a–connective, ie that $|S(a,b)_\bullet|$ is $(a-2)$–connected. We will do this by induction on a, using the nerve theorem as formulated in [8, Corollary 4.2]. It clearly holds for $a\le 1$.

Let $A(\Sigma^{a,b})$ denote the simplicial complex with vertices the isotopy classes of smoothly embedded arcs $\gamma\colon[0,1]\to\Sigma^{a,b}$ with $\gamma(0)=\bigl(0,\frac{1}{2}\bigr)$ and $\gamma(1)$ a black marked point. A collection $[\gamma_0],\dots,[\gamma_p]$ spans a simplex if the γ_i can be realised disjointly except at $\gamma_i(0)=\bigl(0,\frac{1}{2}\bigr)$. By a theorem of Hatcher and Wahl [10, Proposition 7.2] the simplicial complex $A(\Sigma^{a,b})$ is $(a-2)$–connected. We consider the functor

$$F\colon \mathsf{Simp}(A(\Sigma^{a,b}))^{\mathrm{op}}\to\{\text{downwards-closed subposets of } \mathsf{S}(a,b)^{\mathrm{op}}\}$$

which assigns to a simplex $\{[\gamma_0],\dots,[\gamma_p]\}$ of $A(\Sigma^{a,b})$ the subposet of $\mathsf{S}(a,b)^{\mathrm{op}}$ given by those $[\alpha]$'s such that the arcs $\alpha,\gamma_0,\dots,\gamma_p$ can be realised disjointly; see Figure 3, left. This is clearly a downwards-closed subposet, and defines a functor.

We apply the nerve theorem [8, Corollary 4.2] to this functor, with $t_{\mathsf{Simp}(A(\Sigma^{a,b}))}([\gamma_0],\dots,[\gamma_p]):=p$, $t_{\mathsf{S}(a,b)^{\mathrm{op}}}([\alpha]):=\#\{\text{black points to the right of }\alpha\}$, and $n:=a-1$. We verify the hypotheses of this theorem:

(i) $\mathsf{Simp}(A(\Sigma^{a,b}))$ is $(a-2)$–connected, by Hatcher and Wahl's theorem.

(ii) $\mathsf{Simp}(A(\Sigma^{a,b}))_{<\{[\gamma_0],\dots,[\gamma_p]\}}$ is the poset of simplices of the boundary of Δ^p, so is $(p-2)$–connected. The subposet $F([\gamma_0],\dots,[\gamma_p])\subset\mathsf{S}(a,b)^{\mathrm{op}}$ has a maximal element, given by an arc which has *precisely* the points $\gamma_0(1),\dots,\gamma_p(1)$ to its left and runs parallel to the γ_i as in Figure 3, right, so is contractible.

(iii) $(\mathsf{S}(a,b)^{\mathrm{op}})_{<[\alpha]}=(\mathsf{S}(a,b)_{>[\alpha]})^{\mathrm{op}}$, and if α has k black points to its right then $\mathsf{S}(a,b)_{>[\alpha]}\cong\mathsf{S}(k,b)$, which by induction may be supposed to be $(k-2)$–connected, ie $\bigl(t_{\mathsf{S}(a,b)^{\mathrm{op}}}([\alpha])-2\bigr)$–connected.

The subposet $\mathsf{Simp}(A(\Sigma^{a,b}))_{[\alpha]}$ may be identified with $\mathsf{Simp}(A(\Sigma^{a-k,0}))$, which, by Hatcher and Wahl's theorem, is $(a-k-2)$–connected, ie $\big((a-1)-t_{\mathsf{S}(a,b)^{\mathrm{op}}}([\alpha])-1\big)$–connected.

It follows from the nerve theorem that $\mathsf{S}(a,b)^{\mathrm{op}}$ is $(a-2)$–connected, as required. $\square$

References

[1] **V I Arnold**, *Certain topological invariants of algebraic functions*, Trudy Moskov. Mat. Obšč. 21 (1970) 27–46 MR Zbl In Russian; translated in Trans. Moscow Math. Soc. 21 (1970) 30–52

[2] **R Boyd**, *Homological stability for Artin monoids*, Proc. Lond. Math. Soc. 121 (2020) 537–583 MR Zbl

[3] **T Church**, **J S Ellenberg**, *Homology of FI-modules*, Geom. Topol. 21 (2017) 2373–2418 MR Zbl

[4] **C Damiolini**, *The braid group and the arc complex*, master's thesis, Universiteit Leiden (2013) Available at `https://hdl.handle.net/1887/3597297`

[5] **E Dror Farjoun**, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Math. 1622, Springer (1996) MR Zbl

[6] **J Ebert**, **O Randal-Williams**, *Semisimplicial spaces*, Algebr. Geom. Topol. 19 (2019) 2099–2150 MR Zbl

[7] **S Galatius**, **A Kupers**, **O Randal-Williams**, *Cellular E_k–algebras*, preprint (2018) arXiv 1805.07184 To appear under Astérisque

[8] **S Galatius**, **A Kupers**, **O Randal-Williams**, *E_2–cells and mapping class groups*, Publ. Math. Inst. Hautes Études Sci. 130 (2019) 1–61 MR Zbl

[9] **A Hatcher**, **K Vogtmann**, *Tethers and homology stability for surfaces*, Algebr. Geom. Topol. 17 (2017) 1871–1916 MR Zbl

[10] **A Hatcher**, **N Wahl**, *Stabilization for mapping class groups of 3–manifolds*, Duke Math. J. 155 (2010) 205–269 MR Zbl

[11] **R Hepworth**, *Homological stability for families of Coxeter groups*, Algebr. Geom. Topol. 16 (2016) 2779–2811 MR Zbl

[12] **R Hepworth**, *On the edge of the stable range*, Math. Ann. 377 (2020) 123–181 MR Zbl

[13] **Z Himes**, *Secondary homological stability for unordered configuration spaces*, Trans. Amer. Math. Soc. (2024)

[14] **M Krannich**, *Homological stability of topological moduli spaces*, Geom. Topol. 23 (2019) 2397–2474 MR Zbl

[15] **P Patzt**, *Central stability homology*, Math. Z. 295 (2020) 877–916 MR Zbl

[16] **A Putman**, **S V Sam**, *Representation stability and finite linear groups*, Duke Math. J. 166 (2017) 2521–2598 MR Zbl

[17] **O Randal-Williams**, **N Wahl**, *Homological stability for automorphism groups*, Adv. Math. 318 (2017) 534–626 MR Zbl

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Manifolds with small topological complexity

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We study closed orientable manifolds whose topological complexity is at most 3 and determine their cohomology rings. For some of the admissible cohomology rings we are also able to identify corresponding manifolds up to a homeomorphism.

55M30; 57N65

1 Introduction

Topological complexity of a (path-connected) space X, denoted by $\mathrm{TC}(X)$, is a numerical homotopy invariant introduced by M Farber [8] as a quantitative measure for the complexity of motion planning in a configuration space X of some robot device. Although configuration spaces of robots can be quite general topological spaces (see Kapovich and Millson [16] and Pavešić [20]), of particular importance are those that have the structure of a manifold (eg ordered configuration spaces of manifolds, see Cohen [3]; configuration spaces of spidery linkages, see O'Hara [19]; and of general parallel mechanisms, see Shvalb, Shoham and Blanc [24]). It is thus of interest to determine which closed manifolds M have a given value of $\mathrm{TC}(M)$. The case $\mathrm{TC}(M)=1$ is void, because a nontrivial closed manifold cannot be contractible. Grant, Lupton and Oprea [10, Corollary 1.2] showed that the only closed manifolds with topological complexity equal to 2 are the odd-dimensional spheres. In this paper we study closed oriented manifolds M with $\mathrm{TC}(M)=3$. Some examples immediately spring to mind: even-dimensional spheres S^{2n} by [8, Theorem 8] and products of two odd-dimensional spheres, by [8, Theorems 8 and 11]. Are there any other examples? Our main result is Theorem 3.2 in which we give an exact description of admissible cohomology rings of manifolds whose topological complexity is at most 3.

Theorem 3.2 *If M is a closed, orientable manifold with $\mathrm{TC}(M)\le 3$, then $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}$, and one of the following alternatives holds:*

(1) $H^*(M;\mathbb{Z})\cong\bigwedge(x_m)$, *or*

(2) $H^*(M;\mathbb{Z})\cong\bigwedge(x_k,x_l)$ *with k and l odd, $k\ge 1$, $l\ge 3$ and $k+l=m$, or*

(3) $H_i(M;\mathbb{Z})=0$ *for $i\ne 0,k,m$ with $k\ge 2$ and $m=2k+1$, and* $H^*(M;\mathbb{F}_2)\cong\bigwedge(x_k,x_{k+1})\otimes\mathbb{F}_2$.

The conditions in the theorem are necessary but not sufficient to guarantee that $\mathrm{TC}(M)=3$, as illustrated by the case of the symplectic group $\mathrm{Sp}(2)$ whose cohomology is of type (2), but $\mathrm{TC}(\mathrm{Sp}(2))=4$ (see Section 4).

In the next section we recall the definition and the main properties of the topological complexity. In Section 3 we state and prove our main result. Finally, in Section 4 we discuss specific manifolds whose cohomology ring is described in the mentioned theorem. We also obtain some specific results for closed orientable manifolds M that admit cellular decompositions with at most four cells: if $\mathrm{TC}(M) \le 3$, then certain Hopf invariants must vanish (Proposition 4.2); if in addition M is smooth and its dimension is even and smaller than 12, then M is the total space of an orthogonal sphere bundle over a sphere (Proposition 4.3).

2 Preliminaries on topological complexity

For a topological space X let X^I denote the space of continuous paths $\alpha\colon I \to X$, and let $\pi\colon X^I \to X \times X$ be the evaluation map $\pi(\alpha) := (\alpha(0), \alpha(1))$. *Topological complexity* of a path-connected topological space X is the least integer $\mathrm{TC}(X) = n$ for which there exists a covering $U_1, \dots, U_n$ of $X \times X$, where each U_i is open and admits a continuous section to the map $\pi\colon X^I \to X \times X$ [8, Definition 2]. Note that the topological complexity of X is not defined if X is not path-connected, because in that case the map π is not onto. We will thus assume throughout the paper that X (or M) is a path-connected space. Moreover, if X is a compact ANR space (which includes closed manifolds) then the requirement that the sets in the covering are open is superfluous, since by [21, Theorem 4.6], one can consider coverings of $X \times X$ by arbitrary subsets.

The main properties of topological complexity are listed in the following proposition, where the value of $\mathrm{TC}(X)$ is related to the Lusternik–Schnirelmann category $\mathrm{cat}(X)$ (for which we refer to the classical monograph [5]), and to the nilpotency of certain ideal in the cohomology ring of $X \times X$.

Note that in this work we use the nonnormalized versions of category and topological complexity for which $\mathrm{cat}(X) = \mathrm{TC}(X) = 1$ if X is a contractible space. Many authors use a normalized or reduced category and topological complexity, which is one less than in our definition, so that the category and the topological complexity of a contractible space are equal to 0. This holds in particular for the above mentioned monograph [5] and the article [10], so the reader should be careful when comparing results stated under different conventions.

Proposition 2.1 (1) $\mathrm{TC}(X) = 1$ *if and only if* X *is contractible.*

(2) *Homotopy invariance*:
$$X \simeq Y \implies \mathrm{TC}(X) = \mathrm{TC}(Y).$$

(3) *Category estimate*:
$$\mathrm{cat}(X) \le \mathrm{TC}(X) \le \mathrm{cat}(X \times X).$$

(4) *If* X *is a topological group*, *then* $\mathrm{TC}(X) = \mathrm{cat}(X)$.

(5) *Cohomological estimate*:
$$\mathrm{TC}(X) \ge \mathrm{nil}(\mathrm{Ker}\,\Delta^*),$$

where $\Delta^*\colon H^*(X\times X;R)\to H^*(X;R)$ *is the homomorphism induced by the diagonal map* $\Delta\colon X\to X\times X$ *on the cohomology with coefficients in a ring* R, *and* $\mathrm{nil}(\mathrm{Ker}\,\Delta^*)$ *is the minimal integer* k *for which all* k*–fold products in* $\mathrm{Ker}\,\Delta^*$ *are zero.*

(6) *Product formula*: *if* X *and* Y *are ANR spaces*, *then*

$$\mathrm{TC}(X\times Y)\le \mathrm{TC}(X)+\mathrm{TC}(Y)-1.$$

Recall that the value of Δ^* on the cross product $u\times v\in H^*(X\times X;R)$ of elements $u,v\in H^*(X;R)$ can be given in terms of their cup product as

$$\Delta^*(u\times v)=u\cdot v,$$

and the cup product of elements $u\times v$ and $u'\times v'$ is given as

$$(u\times v)\cdot(u'\times v')=(-1)^{|v|\cdot|u'|}(u\cdot u')\times(v\cdot v'),$$

where $|v|$ and $|u'|$ are the dimensions of cohomology classes v and u'; see [11, pages 215–216]. This explains why Farber [8, Definition 6] called $\mathrm{Ker}\,\Delta^*$ the *ideal of zero-divisors* of $H^*(X;R)$. For every $u\in H^*(X;R)$ we have

$$\Delta^*(u\times 1-1\times u)=u\cdot 1-1\cdot u=0,$$

therefore $(u\times 1-1\times u)\in \mathrm{Ker}\,\Delta^*$. Indeed, if $H^*(X;R)$ is a finitely generated free R–module (which implies that $H^*(X\times X;R)\cong H^*(X;R)\otimes H^*(X;R)$ by the Künneth theorem), then $\mathrm{Ker}\,\Delta^*$ is generated as an ideal by elements of the form $(u\times 1-1\times u)$ because $\Delta^*\bigl(\sum u_i\times v_i\bigr)=\sum u_i\cdot v_i=0$ implies

$$\sum u_i\times v_i=\sum(u_i\times v_i-1\times u_iv_i)=\sum(u_i\times 1-1\times u_i)\cdot(1\times v_i)\,.$$

3 Admissible cohomology rings

Computation of topological complexity of closed surfaces was completed in the orientable case by Farber [8, Theorem 9] and in the nonorientable case by Dranishnikov [6] and Cohen and Vandembroucq [4]. Thus we know that the only closed surfaces whose topological complexity is 3 are the sphere S^2 and the torus $S^1\times S^1$. To avoid making unnecessary exceptions, for the rest of this section let M denote a closed, orientable m–dimensional manifold with $m\ge 3$.

In this section we show that the condition $\mathrm{TC}(M)\le 3$ poses strong restrictions on the fundamental group and the cohomology ring of M. As a starting point we take the following consequence of a deep theorem proved by Dranishnikov, Katz and Rudyak [7].

Theorem 3.1 *If* $\mathrm{TC}(M)\le 3$, *then* $\pi_1(M)$ *is either trivial or isomorphic to* $\mathbb{Z}$.

Proof If $\mathrm{TC}(M)\le 3$, then $\mathrm{cat}(M)\le 3$ by Proposition 2.1(3), which by [7, Theorem 1.1] implies that $\pi_1(M)$ is a free group. Let us assume that the rank of $\pi_1(M)$ is at least 2 and consider the cup product pairing

$$H^1(M;\mathbb{Z})\times H^{m-1}(M;\mathbb{Z})\xrightarrow{\ \cdot\ } H^m(M;\mathbb{Z})$$

which is nonsingular by [11, Proposition 3.38]. Indeed, by the Hurewicz theorem $H_1(M;\mathbb{Z})$ is free abelian; therefore $H^1(M;\mathbb{Z})$ and $H^{m-1}(M;\mathbb{Z})$ are also free by the universal coefficients theorem and by Poincaré duality, respectively. Since the rank of $H^1(M;\mathbb{Z})$ is at least 2, nonsingularity of the pairing implies that there exist linearly independent elements $u, v \in H^1(M;\mathbb{Z})$ and $u', v' \in H^{m-1}(M;\mathbb{Z})$ such that $u \cdot u' = v \cdot v' = g$, where g is a generator of $H^m(M;\mathbb{Z})$, and furthermore $u \cdot v' = v \cdot u' = 0$. Then we obtain by direct computation a nontrivial four-fold product of zero-divisors,

$$(u \times 1 - 1 \times u) \cdot (u' \times 1 - 1 \times u') \cdot (v \times 1 - 1 \times v) \cdot (v' \times 1 - 1 \times v') = 2(g \times g) \neq 0.$$

Therefore by Proposition 2.1(6), if $\operatorname{rank}(\pi_1(M)) \geq 2$, then $\mathrm{TC}(M) \geq 5$. Thus, if $\mathrm{TC}(M) \leq 3$, then $\pi_1(M)$ is a free group or rank 0 or 1, as claimed. □

In the following theorem we determine all admissible cohomology rings for a manifold whose topological complexity is at most 3.

Theorem 3.2 *Assume that M is a closed, orientable manifold with $\mathrm{TC}(M) \leq 3 \leq \dim(M)$. Then $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}$ and one of the following alternatives holds*:

(1) $H^*(M;\mathbb{Z}) \cong \bigwedge(x_m)$, *or*

(2) $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ *with k and l odd, $k \geq 1$, $l \geq 3$ and $k + l = m$, or*

(3) $H_i(M;\mathbb{Z}) = 0$ *for $i \neq 0, k, m$ with $k \geq 2$ and $m = 2k + 1$, and $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$.*

Proof In order to prove the theorem we need to consider several cases and subcases. Let g denote the generator of the top-dimensional cohomology $H^m(M; R)$ and for every $u \in H^*(M; R)$ let

$$\hat{u} := u \times 1 - 1 \times u \in H^*(M \times M; R)$$

be the shorthand for the corresponding zero-divisor. By Theorem 3.1 we must consider two possibilities, $\pi_1(M) \cong \mathbb{Z}$ or $\pi_1(M) = 0$.

(1) If $\pi_1(M) \cong \mathbb{Z}$, let u be a generator of $H^1(M;\mathbb{Z}) \cong \mathbb{Z}$ and let, as in the proof of Theorem 3.1, $v \in H^{m-1}(M;\mathbb{Z})$ be such that $u \cdot v = g$. If $m - 1$ is even, then

$$\hat{v}^2 \cdot \hat{u} = -2(v \times v) \cdot \hat{u} = -2(g \times u - u \times g) \neq 0$$

(note that $v^2 = 0$ for dimensional reasons), and thus $\mathrm{TC}(M) \geq 4$ by Proposition 2.1(6). On the other hand, if $m - 1$ is odd, and if there exists a nonzero element $w \in H^i(M;\mathbb{Z})$ for some $2 \leq i \leq m - 2$, then

$$\hat{u} \cdot \hat{v} \cdot \hat{w} = w \times g - g \times w \pm uw \times v - v \times uw \neq 0,$$

so again $\mathrm{TC}(M) \geq 4$.

We conclude that if $\pi_1(M) \cong \mathbb{Z}$ and $\mathrm{TC}(M) = 3$, then $H^*(M;\mathbb{Z})$ is multiplicatively generated by two cohomology classes in dimensions 1 and $m - 1$, which are Poincaré duals to each other, and furthermore $m - 1$ must be odd. In other words, $H^*(M;\mathbb{Z}) \cong \bigwedge(x_1, x_k)$ for some odd integer $k > 1$.

(2) If M is simply connected, then we consider four subcases depending on the structure of the group

$$\widehat{H}(M;R) := \bigoplus_{i=2}^{m-2} H_i(M;R).$$

(2a) If $\widehat{H}(M;\mathbb{Q}) \neq 0$ we argue similarly as in case (1). First of all we note that $\widehat{H}(M;\mathbb{Z})$ is not all torsion, so by [11, Corollary 3.39] we may find homogeneous elements $u, v \in \widehat{H}(M;\mathbb{Z})$ of infinite order, such that $u \cdot v = g$. As in case (1), if either u or v is of even degree, then we can find a nontrivial product of three zero-divisors, and then $\mathrm{TC}(M) \geq 4$. Therefore, if $\mathrm{TC}(M) \leq 3$, then both u and v must be of odd degree, which as before implies that $H^*(M;\mathbb{Z})$ contains a subring of the form $\bigwedge(x_k, x_l)$ where k and l are odd integers and $1 < k \leq l < m-1$. Furthermore, if there exists an element $w \in H^*(M;\mathbb{Z})$ which is not contained in the mentioned subring, then $\hat{u} \cdot \hat{v} \cdot \hat{w} \neq 0$ similarly as in the second part of case (1). Thus, $\widehat{H}(M;\mathbb{Q}) \neq 0$ and $\mathrm{TC}(M) = 3$ imply $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$.

(2b) Let us now assume that $\widehat{H}(M;\mathbb{Q}) = 0$ but $\widehat{H}(M;\mathbb{F}_p) \neq 0$ for some odd prime p, and let k be the minimal $k \geq 2$ for which $H_k(M;\mathbb{Z})$ has p–torsion. By the universal coefficient theorem for cohomology (see [11, Theorem 3.2]) $H^i(M;\mathbb{F}_p) \neq 0$ for $i = k, k+1$. It then follows by Poincaré duality that $H^i(M;\mathbb{F}_p) \neq 0$ for $i = m-k-1, m-k$. Therefore, $H^i(M;\mathbb{F}_p) \neq 0$ in three different dimensions, unless $m = 2k+1$. In the first case, we may find (as in case (1)) three nontrivial cohomology classes u, v and w of different dimension (with $u \cdot v = g$ by [11, Corollary 3.39]), for which $\hat{u} \cdot \hat{v} \cdot \hat{w} \neq 0$ and thus $\mathrm{TC}(M) \geq 4$.

On the other hand, if $m = 2k+1$, then let $u \in H^k(M;\mathbb{F}_p)$ and $v \in H^{k+1}(M;\mathbb{F}_p)$ be such that $u \cdot v = g$. If k is even, then

$$\hat{u}^2 \cdot \hat{v} = 2(u \times g - g \times u) + v \times u^2 - u^2 \times v \neq 0.$$

Similarly, if k is odd, then $\hat{u} \cdot \hat{v}^2 \neq 0$, so in both cases $\mathrm{TC}(M) \geq 4$.

(2c) The next subcase arises if $\widehat{H}(M;\mathbb{Q}) = 0$ and $\widehat{H}(M;\mathbb{F}_p) = 0$ for p odd but $\widehat{H}(M;\mathbb{F}_2) \neq 0$. The argument is similar as in (2b), except if $m = 2k+1$, since in that case the proof that $\hat{u}^2 \cdot \hat{v} \neq 0$ for k even (or that $\hat{u} \cdot \hat{v}^2 \neq 0$ for k odd) breaks down because of 2–torsion. On the other hand, if $u \in H^k(M;\mathbb{F}_2)$ and $v \in H^{k+1}(M;\mathbb{F}_2)$ such that $u \cdot v = g$, and if additionally $u^2 \neq 0$, then

$$\hat{u}^2 \cdot \hat{v} = u^2 \times v + v \times u^2 \neq 0,$$

so $\mathrm{TC}(M) \geq 4$. Thus, under the assumptions of (2c), if $\mathrm{TC}(M) \leq 3$ then $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$.

(2d) The final possibility is that $\widehat{H}(M;R) = 0$ for all coefficient rings R, which clearly implies that $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k)$. □

4 Some manifolds with small TC

Theorem 3.2 shows that the condition $\mathrm{TC}(M) \leq 3$ is much more restrictive than the analogous condition $\mathrm{cat}(M) \leq 3$. Indeed the class of manifolds whose Lusternik–Schnirelmann category is at most 3 includes all surfaces, two-fold products of spheres, all $(n-1)$–connected $2n$–manifolds and a variety of other examples. In this section we study the actual manifolds M satisfying $\mathrm{TC}(M) \le 3$ (without the restriction that $\dim(M) \ge 3$). For some admissible cohomology rings we describe exactly the corresponding manifolds, while in other cases we are only able to present suitable candidates and compute their Lusternik–Schnirelmann category.

(1) The simplest case to consider are manifolds whose cohomology ring is given by Theorem 3.2(1). In fact, since the fundamental group of M is free, M must be simply connected (except in the trivial case $M = S^1$). This fact, together with $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k)$ immediately yields that M is homotopy equivalent to S^k. Finally, the positive solution to the Poincaré conjecture implies that M is actually homeomorphic to S^k.

(2) If $H^*(M;\mathbb{Z}) \cong \bigwedge(x_1, x_k)$ as in Theorem 3.2(2), then we can use the fact that $S^1 \simeq K(\mathbb{Z}, 1)$ to find a map $f_1\colon M \to S^1$ which represents the cohomology class

$$x_1 \in H^1(M;\mathbb{Z}) \cong [M, S^1].$$

Similarly, there is a map $f_k\colon M \to K(\mathbb{Z}, k)$ representing the cohomology class

$$x_k \in H^k(M;\mathbb{Z}) \cong [M, K(\mathbb{Z}, k)].$$

It is well known that $K(\mathbb{Z}, k)$ can be constructed by attaching cells of dimension bigger or equal to $k+2$ to the sphere S^k. Since the dimension of M is $m = k+1$, we may assume by cellular approximation theorem that the image of f_k is contained in S^k. Thus we obtain a map

$$(f_1, f_k)\colon M \to S^1 \times S^k,$$

which is clearly an isomorphism on the integral cohomology and is thus a homotopy equivalence, because $\pi_1(M) \cong \mathbb{Z}$. By a rigidity theorem of Kreck and Lück [17, Theorem 0.13(a)] we conclude that M is actually homeomorphic to $S^1 \times S^k$.

(3) If $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_k)$ with k odd, then M is a $(k-1)$–connected $2k$–dimensional manifold. Thus we may invoke C T C Wall's classification [27] by which $M \approx S^k \times S^k$ provided $k \equiv 3, 5, 7 \pmod 8$; see also [2, Theorem 3.1].

(4) The instances of Theorem 3.2(2) when $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ for $1 < k < l$ with k and l odd are more complicated. First of all, they include products of odd spheres of the form $S^k \times S^l$ and we know that $\mathrm{TC}(S^k \times S^l) = 3$. Moreover, by the above-mentioned theorem of Kreck and Lück [17, Theorem 0.13(a)], a manifold that is homotopy equivalent to a product of odd spheres is actually homeomorphic to that product.

The first example that is not a product of spheres is the special unitary group SU(3) whose cohomology is $H^*(\mathrm{SU}(3);\mathbb{Z}) \cong \bigwedge(x_3, x_5)$. Singhof [25, Theorem 1(a)] proved that $\mathrm{cat}(\mathrm{SU}(3)) = 3$; therefore by Proposition 2.1(4), we conclude that $\mathrm{TC}(\mathrm{SU}(3)) = 3$, as well.

The cohomology ring of the symplectic group Sp(2) is $H^*(\mathrm{Sp}(2);\mathbb{Z}) \cong \bigwedge(x_3, x_7)$. However, Schweitzer [23] used secondary cohomology operations to prove that $\mathrm{cat}(\mathrm{Sp}(2)) = 4$, which in turn implies that $\mathrm{TC}(\mathrm{Sp}(2)) = 4$. Hilton and Roitberg [13] discovered three more examples of H–spaces whose cohomology is isomorphic to $\bigwedge(x_3, x_7)$, which are usually denoted by $E_{3\omega}$, $E_{4\omega}$ and $E_{5\omega}$ (and Sp(2) corresponds to E_ω). Their Lusternik–Schnirelmann category (and thus topological complexity) is equal to 4; see [5, Chapter 4].

In fact, we have a complete description of manifolds that admit H–space structure and whose topological complexity is equal to 3. First observe, that by the classification of H–spaces of low rank (Hilton and Roitberg [14]; see also [12, Section III.2]), the following list exhausts all (homotopy types of) H–spaces whose cohomology ring is isomorphic to one of the rings listed in Theorem 3.2: spheres S^k for $k \in \{1,3,7\}$, products $S^k \times S^l$ for $k,l \in \{1,3,7\}$, SU(3), $E_{k\omega}$ for $k = 1,3,4,5$ and $\mathbb{R}P^3$. By a cup-length argument, $\mathrm{TC}(\mathbb{R}P^3) = \mathrm{cat}(\mathbb{R}P^3) = 4$, which together with the above discussion yields:

Proposition 4.1 *Let M be a closed orientable manifold with $\mathrm{TC}(M) = 3$. If M admits an H–space structure, then M is either $\mathrm{SU}(3)$ or $S^k \times S^l$ for $k,l \in \{1,3,7\}$.*

More generally, let us consider fibre bundles $p\colon M \to S^l$ with fibre S^k for some odd integers $1 < k < l$. The cohomology of M is easily computed using Gysin sequence, so we obtain $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ and the manifold itself admits a CW–decomposition of the form

$$M = S^k \cup_\alpha e^l \cup_\beta e^{k+l},$$

with attaching maps $\alpha\colon S^{l-1} \to S^k$ and $\beta\colon S^{k+l-1} \to S^k \cup_\alpha e^l$. If α is a suspension or more generally a coH–map (eg if $l < 2k-1$ so that $\pi_{l-1}(S^k)$ is in the stable range), then $S^k \cup_\alpha e^l$ is a coH–space and $\mathrm{cat}(S^k \cup_\alpha e^l) = 2$ (see [5]). Therefore, $\mathrm{cat}(M) \le 3$ but, since the cup length of M equals 2, we have that $\mathrm{cat}(M) = 3$. This yields many important examples like the complex and quaternionic Stiefel manifolds, $V_2(\mathbb{C}^n) = U(n)/U(n-2)$ whose cohomology ring is given as $H^*(V_2(\mathbb{C}^n);\mathbb{Z}) \cong \bigwedge(x_{2n-1}, x_{2n-3})$, and $V_2(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-2)$ with $H^*(V_2(\mathbb{H}^n);\mathbb{Z}) \cong \bigwedge(x_{4n-1}, x_{2n-5})$. It is known (see [15]) that except for the case $V_2(\mathbb{C}^4) = S^5 \times S^7$, the spaces $V_2(\mathbb{C}^n)$ and $V_2(\mathbb{H}^n)$ do not split as products of spheres.

If the attaching map α is not a coH–map, then $\mathrm{cat}(S^k \cup_\alpha e^l) = 3$. In that case $\mathrm{cat}(M) = 3$ if and only if certain set of Hopf invariants $\mathcal{H}(\beta)$ contains the zero class (see [5, Chapter 6], in particular Theorem 6.19 therein).

As we have seen, there are many sphere bundles over spheres whose category is 3. Unfortunately, we are currently lacking a general method to determine their topological complexity, so this remains an interesting open problem. Some cases can be settled by applying a method that was recently developed by Gonzalez, Grant and Vandembroucq [9] and which uses higher Hopf invariants. They computed topological complexity of many two-cell complexes, but the technical details are quite formidable, and the full analysis of three-cell complexes seems to be beyond reach at this point. Nevertheless, we were able to combine some of their computations with results from Pavešić [22] that relate topological complexity of a space with topological complexity of its skeleta, to show that some sphere bundles over spheres have topological complexity at least 4. We will work in the so-called *metastable range* and assume that $2k < l < 3k-1$. Under this assumption one can associate to every map $\alpha\colon S^{l-1} \to S^k$ a *generalized Hopf invariant* $H_0(\alpha)\colon S^{l-1} \to S^{2k-1}$ (see [9, Section 5] for relevant definitions and results), which allows us to determine $\mathrm{TC}(S^k \cup_\alpha e^l) \geq 4$.

Proposition 4.2 *Let k be an odd integer and let $2k < l < 3k-1$. Assume that M has a CW–decomposition of the form $M = S^k \cup_\alpha e^l \cup_\beta e^{k+l}$ with attaching maps $\alpha\colon S^{l-1} \to S^k$ and $\beta\colon S^{k+l-1} \to S^k \cup_\alpha e^l$ (this in particular applies if M is an S^l–bundle over S^k). If $H_0(\alpha) \neq 0$, then $\mathrm{TC}(M) \geq 4$.*

Proof Note that the inclusion $S^k \cup_\alpha e^l \hookrightarrow M$ is a $(k+l-1)$–equivalence because $S^k \cup_\alpha e^l$ is the $(k+l-1)$–skeleton of M. The topological complexity of $S^k \cup_\alpha e^l$ was bounded from below in [9, Theorem 5.6]: $\mathrm{TC}(S^k \cup_\alpha e^l) \geq 4$. On the other hand, [22, Theorem 3.6] implies that

$$\mathrm{cat}(M) \geq \mathrm{cat}(S^k \cup_\alpha e^l) = 3.$$

Therefore $\mathrm{TC}(M) \geq 3$. Then we may apply [22, Theorem 3.1], which states that if

$$2\dim(S^k \cup_\alpha e^l) < k(\mathrm{TC}(M) - 1) + (k + l - 1)$$

(which is clearly satisfied if $l < 3k-1$), then $\mathrm{TC}(M) \geq \mathrm{TC}(S^k \cup_\alpha e^l) \geq 4$. □

It turns out that up to dimension 10 the case of sphere bundles over spheres is generic for smooth, even-dimensional manifolds (that is quite relevant if one is mainly interested in configuration spaces of specific mechanical systems). In fact, we have the following result.

Proposition 4.3 *Let M be a smooth, orientable, closed manifold with $\mathrm{TC}(M) \leq 3$. If M is even-dimensional and $\dim(M) \leq 10$, then M is homotopy equivalent to the total space of an orthogonal sphere bundle over a sphere.*

Proof By the assumptions, the cohomology of M is given by cases (1) or (2) of Theorem 3.2. If M is not simply connected, then we already proved that M is homeomorphic to a product of spheres. If M is simply connected, then $\dim(M) \leq 10$ implies that its cohomology is isomorphic to either $\bigwedge(x_3, x_5)$ or to $\bigwedge(x_3, x_7)$. Thus we may apply [14, Theorem 6.1] to conclude that M is homotopy equivalent to the total space of an orthogonal S^3–bundle with base S^5 or S^7. □

For manifolds of dimension higher than 10 we may describe a convenient Morse decomposition of M. Smale [26, Theorem G] showed that if the dimension of M is at least 6, then it has a Morse decomposition with the minimal number of handles compatible with its homology. Therefore, if $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$, then M admits a decomposition with four handles whose indices are 0, k, l and $k+l$, respectively. The union of the 0– and k–handles depends on the framing which is given by an element of $\pi_{k-1}(O(l))$. This group is known to be trivial for $k \not\equiv 1 \pmod 8$, therefore the union of the first two handles is homeomorphic to $S^k \times D^l$. By the same argument, the union of the l– and $(k+l)$–handles is also homeomorphic to $S^k \times D^l$.

Proposition 4.4 *Let M be a smooth, orientable, closed manifold with $\dim(M) > 10$ and $\mathrm{TC}(M) = 3$. If $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ with $k \not\equiv 1 \pmod 8$, then M can be obtained by glueing together two copies of $S^k \times D^l$ along the common boundary $S^k \times S^{l-1}$.*

(5) Let us finally consider manifolds that satisfy condition (3) of Theorem 3.2. The lowest-dimensional case is a simply connected 5–dimensional manifold whose $\mathbb{F}_2$ cohomology is

$$H^*(M;\mathbb{F}_2) \cong \bigwedge(x_2, x_3) \otimes \mathbb{F}_2.$$

Barden [1] showed that every simply connected 5–dimensional manifolds can be decomposed as a connected sum of certain basic 5–manifolds. We are not dwelling into details but one can easily check that the only 5–manifold that satisfies the above condition is the famous Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. It admits a CW–decomposition $\mathrm{SU}(3)/\mathrm{SO}(3) = S^2 \cup e^3 \cup e^5$, where the 3–cell is attached by a degree 2 map; therefore the 3–skeleton of $\mathrm{SU}(3)/\mathrm{SO}(3)$ is the Moore space $M(\mathbb{Z}/2, 2)$. The category of a Moore space is 2; therefore the category of the Wu manifold is 3. However, we were not able to determine whether its topological complexity is also 3. One can construct higher analogues of the Wu manifold using handle decompositions, for example by gluing together two copies of a (twisted or untwisted, depending on the dimension) D^{k+1}–bundle over S^k along a suitable homeomorphism of the boundary. All of these spaces have a CW–decomposition with the top-cell attached to a suspension, so their category is equal to 3.

We should also mention an interesting result that was recently proved by S Mescher [18, Proposition 6.2]. He used weighted cohomology classes to show that a closed oriented manifold M with $\mathrm{TC}(M) \le 3$ is either a rational homology sphere or it admits a degree 1 map from a closed oriented manifold of the form $S^1 \times P$ (in other words, it is 1–*dominated* by a product of a $(\dim(M)-1)$–dimensional manifold with a circle).

Let us conclude with a brief discussion on two possible extensions of the presented results. Theorem 3.2 gives a precise description of cohomology rings of closed orientable manifolds whose topological complexity is at most 3, so it is natural to ask what can be said about nonorientable closed manifolds M with $\mathrm{TC}(M) \le 3$. As in the orientable case, the fundamental group $\pi_1(M)$ must be free. That rank of $\pi_1(M)$ cannot exceed 1 can be seen similarly as in Section 2. On the other hand, $\pi_1(M)$ cannot be trivial, because M is nonorientable. We thus conclude that $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_1, x_{m-1}) \otimes \mathbb{F}_2$, and the corresponding manifolds are the generalized Klein-bottles (nonorientable S^{m-1}–bundles over S^1). Their category is 3 but we do not know whether their topological complexity can be, at least in some cases, also equal to 3.

Another extension that could be pursued is determination of manifolds whose topological complexity is at most 4. Although the general case seems to be beyond reach because we have very little information on manifolds whose category is 4, we believe that some reasonable progress could be achieved on closed manifolds M satisfying $\mathrm{TC}(M) \le 4$ and $\mathrm{cat}(M) \le 3$.

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References

[1] **D Barden**, *Simply connected five-manifolds*, Ann. of Math. 82 (1965) 365–385 MR Zbl

[2] **I Bokor**, **D Crowley**, **S Friedl**, **F Hebestreit**, **D Kasproswki**, **M Land**, **J Nicholson**, *Connected sum decompositions of high-dimensional manifolds*, from "2019–20 MATRIX annals" (D R Wood, J de Gier, C E Praeger, T Tao, editors), MATRIX Book Ser. 4, Springer (2021) 5–30 MR Zbl

[3] **D C Cohen**, *Topological complexity of classical configuration spaces and related objects*, from "Topological complexity and related topics" (M Grant, G Lupton, L Vandembroucq, editors), Contemp. Math. 702, Amer. Math. Soc., Providence, RI (2018) 41–60 MR Zbl

[4] **D C Cohen**, **L Vandembroucq**, *Topological complexity of the Klein bottle*, J. Appl. Comput. Topol. 1 (2017) 199–213 MR Zbl

[5] **O Cornea**, **G Lupton**, **J Oprea**, **D Tanré**, *Lusternik–Schnirelmann category*, Math. Surv. Monogr. 103, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[6] **A Dranishnikov**, *On topological complexity of non-orientable surfaces*, Topology Appl. 232 (2017) 61–69 MR Zbl

[7] **A N Dranishnikov**, **M G Katz**, **Y B Rudyak**, *Small values of the Lusternik–Schnirelmann category for manifolds*, Geom. Topol. 12 (2008) 1711–1727 MR Zbl

[8] **M Farber**, *Topological complexity of motion planning*, Discrete Comput. Geom. 29 (2003) 211–221 MR Zbl

[9] **J González**, **M Grant**, **L Vandembroucq**, *Hopf invariants for sectional category with applications to topological robotics*, Q. J. Math. 70 (2019) 1209–1252 MR Zbl

[10] **M Grant**, **G Lupton**, **J Oprea**, *Spaces of topological complexity one*, Homology Homotopy Appl. 15 (2013) 73–81 MR Zbl

[11] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[12] **P Hilton**, **G Mislin**, **J Roitberg**, *Localization of nilpotent groups and spaces*, North-Holland Math. Stud. 15, North-Holland, Amsterdam (1975) MR Zbl

[13] **P Hilton**, **J Roitberg**, *On principal S^3–bundles over spheres*, Ann. of Math. 90 (1969) 91–107 MR Zbl

[14] **P J Hilton**, **J Roitberg**, *On the classification problem for H–spaces of rank two*, Comment. Math. Helv. 45 (1970) 506–516 MR Zbl

[15] **I M James**, **J H C Whitehead**, *The homotopy theory of sphere bundles over spheres, I*, Proc. Lond. Math. Soc. 4 (1954) 196–218 MR Zbl

[16] **M Kapovich**, **J J Millson**, *Universality theorems for configuration spaces of planar linkages*, Topology 41 (2002) 1051–1107 MR Zbl

[17] **M Kreck**, **W Lück**, *Topological rigidity for non-aspherical manifolds*, Pure Appl. Math. Q. 5 (2009) 873–914 MR Zbl

[18] **S Mescher**, *Spherical complexities with applications to closed geodesics*, Algebr. Geom. Topol. 21 (2021) 1021–1074 MR Zbl

[19] **J O'Hara**, *The configuration space of planar spidery linkages*, Topology Appl. 154 (2007) 502–526 MR Zbl

[20] **P Pavešić**, *A topologist's view of kinematic maps and manipulation complexity*, from "Topological complexity and related topics" (M Grant, G Lupton, L Vandembroucq, editors), Contemp. Math. 702, Amer. Math. Soc., Providence, RI (2018) 61–83 MR Zbl

[21] **P Pavešić**, *Topological complexity of a map*, Homology Homotopy Appl. 21 (2019) 107–130 MR Zbl

[22] **P Pavešić**, *Monotonicity of the Schwarz genus*, Proc. Amer. Math. Soc. 148 (2020) 1339–1349 MR Zbl

[23] **P A Schweitzer**, *Secondary cohomology operations induced by the diagonal mapping*, Topology 3 (1965) 337–355 MR Zbl

[24] **N Shvalb**, **M Shoham**, **D Blanc**, *The configuration space of arachnoid mechanisms*, Forum Math. 17 (2005) 1033–1042 MR Zbl

[25] **W Singhof**, *On the Lusternik–Schnirelmann category of Lie groups*, Math. Z. 145 (1975) 111–116 MR Zbl

[26] **S Smale**, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. 74 (1961) 391–406 MR Zbl

[27] **C T C Wall**, *Classification of* $(n-1)$*–connected* $2n$*–manifolds*, Ann. of Math. 75 (1962) 163–189 MR Zbl

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Steenrod problem and some graded Stanley–Reisner rings

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"What kind of ring can be represented as the singular cohomology ring of a space?" is a classic problem in algebraic topology, posed by Steenrod. We consider this problem when rings are the graded Stanley–Reisner rings, in other words the polynomial rings divided by an ideal generated by square-free monomials. We give a necessary and sufficient condition that a graded Stanley–Reisner ring is realizable when there is no pair of generators x, y such that $|x| = |y| = 2^n$ and $xy \neq 0$.

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1 Introduction

A classical problem in algebraic topology posed by Steenrod in [14] asks which graded rings occur as the cohomology ring of a space. Especially when the graded ring is polynomial ring, this problem was studied by many researchers, for example Adams and Wilkerson [1], Aguadé [2], Andersen and Grodal [4], Clark and Ewing [6], Dwyer, Miller and Wilkerson [8], Dwyer and Wilkerson [9; 10], Hubbuck [11], Sugawara and Toda [15] and Thomas [16]. This polynomial ring case was finally solved by Andersen and Grodal [3].

On the other hand, when the graded ring is a monomial ideal ring, in other words a polynomial ring divided by an ideal generated by monomials, some researchers studied this problem. The realizability of Stanley–Reisner rings, square-free monomial ideal rings, generated by degree 2 elements is proved by Davis and Januszkiewicz in [7]. Trevisan [17] generalize their construction and prove the realizability of monomial ideal rings generated by degree 2 elements. By using polyhedral products, the realizability of Stanley–Reisner rings of a certain class is proved by Bahri, Bendersky, Cohen and Gitler in [5]. So and Stanley [13] prove the realizability of graded monomial ideal ring modulo torsion. Thus there are results about the realizability of monomial ideal rings, but there are few results about necessary conditions for monomial ideal rings to be realizable.

In this paper we obtain a necessary and sufficiently condition for a graded Stanley–Reisner ring to be realizable when there is no pair of generators x, y such that $|x| = |y| = 2^n$ and $xy \neq 0$. At first, we define the graded Stanley–Reisner ring. A simplicial complex with the vertex set V is a subset of the power set of V which closed under taking subsets. In this paper we allow for there to exist $x \in V$ such that $\{x\} \notin K$, and we assume that the empty set is always a face of the simplicial complex. Let K be a simplicial complex with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$. Then the graded Stanley–Reisner ring $\mathrm{SR}(K,\phi)$ is defined by

$$\mathrm{SR}(K,\phi) \cong \mathbb{Z}[V]/I,$$

where $\mathbb{Z}[V]$ is the polynomial ring generated by $x \in V$ with $|x| = \phi(x)$ and I is the ideal generated by monomials $x_1x_2\cdots x_k$ with $\{x_1, x_2, \dots, x_k\} \notin K$ as a simplex. When $K = \{\varnothing\}$, there is an isomorphism $\mathrm{SR}(K,\phi) \cong \mathbb{Z}$.

To state the main theorem in this paper we set notation. A simplex of a simplicial complex is maximal when the simplex is not a face of a larger simplex in the simplicial complex. For a simplicial complex K with the vertex set V, we define a poset (not subcomplex) $P_{\max}(K) \subset K$, where we regard K as a subset of the power set of V. For $\sigma \in K$, $\sigma \in P_{\max}(K)$ if and only if there exist maximal simplices $\sigma_1, \dots, \sigma_n \in K$ such that $\bigcap \sigma_i = \sigma$. And for $\sigma, \tau \in P_{\max}$, we have $\sigma < \tau$ when σ is a face of τ in K.

Theorem 1.1 *Let $\mathrm{SR}(K,\phi)$ be the finitely generated graded Stanley–Reisner ring for a simplicial complex K with the vertex set V and $\phi\colon V \to 2\mathbb{Z}_{>0}$. Suppose that the graded Stanley–Reisner ring $\mathrm{SR}(K,\phi)$ satisfies the following*:

- *If generators $x, y \in V$ satisfy $\phi(x) = \phi(y) = 2^i$ for some $i \geq 2$, then $xy = 0$ in $\mathrm{SR}(K,\phi)$.*

Then there is a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$ if and only if $\mathrm{SR}(K,\phi)$ satisfies the following condition:

- *For $\sigma \in P_{\max}(K)$ the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2, 2, \dots, 2\}$, $\{4, 6, \dots, 2n+2\} \cup \{2, 2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, 2, \dots, 2\}$ as a multiset for some n.*

This is the main theorem in this paper.

Remark 1.2 In the main theorem there is an artificial assumption:

- If generators $x, y \in V$ satisfy $\phi(x) = \phi(y) = 2^i$ for some $i \geq 2$, then $xy = 0$ in $\mathrm{SR}(K,\phi)$.

We believe that this assumption in the main theorem can be replaced by the following condition:

- If generators $x, y \in V$ satisfy $\phi(x) = \phi(y) = 4$, then $xy = 0$ in $\mathrm{SR}(K,\phi)$.

This condition is the case that $i = 2$ in the upper assumption. Andersen and Grodal proved that the degree of the generators of realizable polynomial is a union of copies of $\{2\}$, $\{4, 6, \dots, 2n+2\}$ or $\{4, 8, \dots, 4n\}$. Since in polynomial case there is one generator with degree 4 except in the case $\{2\}$, this condition implies that the tensor products of two of polynomial rings with the case $\{4, 6, \dots, 2n+2\}$ and $\{4, 8, \dots, 4n\}$ is not included. Therefore this condition seems natural.

But now we are not able to prove the theorem that replaces the artificial assumption with this condition. The reason why the artificial assumption is required is in the latter part of this paper.

We can generalize the construction of a space X with $H^*(X;\mathbb{Z})$ being isomorphic to the graded Stanley–Reisner ring to a wider classes. The following theorem is proved in Section 3.

Theorem 1.3 *Let* $\mathrm{SR}(K,\phi)$ *be the finitely generated graded Stanley–Reisner ring for a simplicial complex* K *with vertex set* V *and* $\phi\colon V\to 2\mathbb{Z}_{>0}$. *If* $\mathrm{SR}(K,\phi)$ *satisfies the following condition, we can construct a space* X *as a homotopy colimit such that* $H^*(X;\mathbb{Z})\cong \mathrm{SR}(K,\phi)$:

- *There is a decomposition* $\coprod_i A_i = V$ *such that for all* i *and* $\sigma\in P_{\max}(K)$, *the set* $\{\phi(x)\mid x\in\sigma\cap A_i\}$ *is equal to* $\{2,2,\dots,2\}$, $\{4,6,\dots,2n\}\cup\{2,2,\dots,2\}$ *or* $\{4,8,\dots,4n\}\cup\{2,2,\dots,2\}$ *as a multiset for some* n.

In the first half of this paper, Sections 2, 3 and 4, we construct a space X with $H^*(X;\mathbb{Z})$ isomorphic to a graded Stanley–Reisner ring, and prove Theorem 1.3. In the latter half, Sections 5 and 6, we obtain the necessary condition that graded Stanley–Reisner rings occur as the cohomology ring of a space. At last, by combining these results, we prove the main theorem in Section 7.

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2 Homotopy colimit

In this section we recall a homotopy colimit and prove some lemmas we will use.

Let P be a finite poset. The order complex of P, $\Delta(P)$, is a simplicial complex whose faces are totally ordered subsets in P. We regard P as a category. For a functor $F\colon P\to \mathrm{Top}$, the homotopy colimit is defined as

$$\operatorname{hocolim}_P F = \coprod_{\sigma=(x_1<x_2<\cdots<x_k)\in\Delta(P)} |\sigma|\times F(x_k)/{\sim},$$

where the equivalence is $(\iota(x),y)\sim(x,F(\iota)(y))$ for $\iota\colon\tau\hookrightarrow\sigma$ and $x\in|\tau|$, $y\in F(\max(\sigma))$.

We write $P_{<a}=\{p\in P\mid p<a\}$ and $P_{\le a}=\{p\in P\mid p\le a\}$ for some $a\in P$.

Lemma 2.1 *Let* $(P,<)$ *be a finite poset and* $F\colon P\to\mathrm{Top}$ *be a functor. Let* $a\in P$ *be a maximal element. Then there is a pushout diagram*

$$\begin{tikzcd}
\operatorname{hocolim}_{P_{<a}} F \arrow[r] \arrow[d] & \operatorname{hocolim}_{P\setminus\{a\}} F \arrow[d] \\
\operatorname{hocolim}_{P_{\le a}} F \arrow[r] & \operatorname{hocolim}_P F
\end{tikzcd}$$

where for a subset $P'\subset P$ $\operatorname{hocolim}_{P'} F$ *means the homotopy colimit of the functor* $F|_{P'}\colon P'\to \mathit{Top}$.

Proof By the definition of homotopy colimit, we obtain that

$$\operatorname{hocolim}_{P\setminus\{a\}} F\cup \operatorname{hocolim}_{P_{\le a}} F = \operatorname{hocolim}_P F,$$
$$\operatorname{hocolim}_{P\setminus\{a\}} F\cap \operatorname{hocolim}_{P_{\le a}} F = \operatorname{hocolim}_{P_{<a}} F.$$

The inclusions
$$\operatorname{hocolim}_{P_{<a}} F \hookrightarrow \operatorname{hocolim}_{P\setminus\{a\}} F,$$
$$\operatorname{hocolim}_{P_{<a}} F \hookrightarrow \operatorname{hocolim}_{P_{\le\{a\}}} F$$
are cofibrations. By combining these we obtain this lemma. □

Next, we see the relation between the homotopy pushout and the graded Stanley–Reisner ring. For a subcomplex $K' \subset K$, let $V(K')$ be the vertex set of K'.

Lemma 2.2 *Let K be a simplicial complex with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$. Let K_1 and K_2 be subcomplexes of K. We assume the following*:

- *There is a space X with $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K_1 \cap K_2, \phi)$.*
- *For $i = 1, 2$ there are spaces X_i with $H^*(X_i;\mathbb{Z}) \cong \mathrm{SR}(K_i, \phi)$.*
- *For $i = 1, 2$ there are maps $\pi_i\colon X \to X_i$ such that π_i^* is identified with the natural projection $\mathrm{SR}(K_i, \phi) \to \mathrm{SR}(K_1 \cap K_2, \phi)$ in cohomology.*

Then the cohomology ring of the homotopy pushout of the diagram
$$\begin{array}{ccc} X & \xrightarrow{\pi_1} & X_1 \\ \big\downarrow{\scriptstyle \pi_2} & & \\ X_2 & & \end{array}$$
is isomorphic to $\mathrm{SR}(K_1 \cup K_2, \phi)$.

Proof Let $p_i\colon \mathrm{SR}(K_1 \cup K_2, \phi) \to \mathrm{SR}(K_i, \phi)$ be the natural projection for $i = 1, 2$. Then it is easy to see that the following sequence is a short exact sequence as a graded module
$$1 \to \mathrm{SR}(K_1 \cup K_2, \phi) \xrightarrow{p_1 \oplus p_2} \mathrm{SR}(K_1, \phi) \oplus \mathrm{SR}(K_2, \phi) \xrightarrow{\pi_1^* - \pi_2^*} \mathrm{SR}(K_1 \cap K_2, \phi) \to 1.$$
By the Mayer–Vietoris sequence for X_1 and X_2 of the pushout, we obtain that the cohomology of the homotopy pushout is isomorphic to $\mathrm{SR}(K_1 \cup K_2, \phi)$ as a graded module. Since the cohomology ring of the homotopy pushout is a graded subring of $H^*(X_1 \amalg X_2) \cong \mathrm{SR}(K_1, \phi) \oplus \mathrm{SR}(K_2, \phi)$, this isomorphism becomes an isomorphism as a graded ring. □

3 Construction of homotopy colimit

In this section we construct a homotopy colimit representation of a space X with $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K, \phi)$ for some graded Stanley–Reisner rings $\mathrm{SR}(K, \phi)$. This construction is an analogy to the construction in [7]. The Davis–Januszkiewicz space that first appeared in [7] is constructed by the union of the products of complex projective spaces. As far as looking at cohomology, our construction is like a graded version of their construction.

3.1 Maps between the classifying spaces of Lie groups

We define maps between the classifying spaces of Lie groups and see some properties. Consider the inclusions

$$\iota_1\colon \mathrm{SU}(n) \to \mathrm{SU}(n+1), \qquad \iota_1(A) = A \oplus 1 \quad \text{for } A \in \mathrm{SU}(n),$$
$$\iota_2\colon \mathrm{Sp}(n) \to \mathrm{Sp}(n+1), \qquad \iota_2(A) = A \oplus 1 \quad \text{for } A \in \mathrm{Sp}(n).$$

For the quaternion $\mathbb{H}$ and the set of complex 2×2–matrices $M(2,\mathbb{C})$, let $c\colon \mathbb{H} \to M(2,\mathbb{C})$ be the map

$$c(z + jw) = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \quad \text{for } z, w \in \mathbb{C}.$$

Let $\iota_3\colon \mathrm{Sp}(n) \to \mathrm{SU}(2n)$ be the map such that for $A = (a_{i,j})_{ij} \in \mathrm{Sp}(n)$,

$$\iota_3(A) = \begin{pmatrix} c(a_{1,1}) & c(a_{1,2}) & \dots \\ c(a_{2,1}) & c(a_{2,2}) & \\ \vdots & & \ddots \end{pmatrix} \in \mathrm{SU}(2n).$$

Since ι_i is a homomorphism, ι_i induces the map between classifying map. We denote these maps as same symbol ι_i. Since the diagram

$$\begin{tikzcd} \mathrm{Sp}(n) \arrow[r, "\iota_2"] \arrow[d, "\iota_3"] & \mathrm{Sp}(n+1) \arrow[d, "\iota_3"] \\ \mathrm{SU}(2n) \arrow[r, "\iota_1"] & \mathrm{SU}(2n+2) \end{tikzcd}$$

is commutative, there is a commutative diagram

$$\begin{tikzcd} B\mathrm{Sp}(n) \arrow[r, "\iota_2"] \arrow[d, "\iota_3"] & B\mathrm{Sp}(n+1) \arrow[d, "\iota_3"] \\ B\mathrm{SU}(2n) \arrow[r, "\iota_1"] & B\mathrm{SU}(2n+2) \end{tikzcd}$$

We recall the cohomology of these classifying spaces. There is an isomorphism

$$H^*(B\mathrm{SU}(n);\mathbb{Z}) \cong \mathbb{Z}[c_2, c_3, \dots, c_n],$$

where c_i is the i^{th} Chern class. For degree reasons, we obtain $\iota_3^*(c_{2n+1}) = 0$, and the next lemma holds.

Lemma 3.1 (cf [12, Chapter III, Theorem 5.8]) *There is an isomorphism*

$$H^*(B\mathrm{Sp}(n);\mathbb{Z}) \cong \mathbb{Z}[\iota_3^*(c_2), \iota_3^*(c_4), \dots, \iota_3^*(c_{2n})].$$

In this paper we take the generators of $H^*(B\mathrm{Sp}(n);\mathbb{Z})$ as in this lemma. Then there are equations for $\iota_1\colon B\mathrm{SU}(n) \to B\mathrm{SU}(n+1)$ and $\iota_2\colon B\mathrm{Sp}(n) \to B\mathrm{Sp}(n+1)$ (cf [12, Chapter III]):

$$\iota_1^*(c_i) = \begin{cases} c_i & \text{if } i \le n, \\ 0 & \text{if } i = n+1, \end{cases} \qquad \iota_2^*(\iota_3^*(c_{2i})) = \begin{cases} \iota_3^*(c_{2i}) & \text{if } i \le n, \\ 0 & \text{if } i = n+1. \end{cases}$$

In summary, ι_1, ι_2 and ι_3 are the maps that send each generator to its corresponding generator or 0 in cohomology.

3.2 Construction

We define a functor by using the maps ι_1, ι_2 and ι_3. Let K be a simplicial complex with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$ satisfying the following condition:

- There is a decomposition $\coprod_i A_i = V$ such that for all i and $\sigma \in P_{\max}(K)$, the set $\{\phi(x) \mid x \in \sigma \cap A_i\}$ is equal to $\{2, 2, \dots, 2\}$, $\{4, 6, \dots 2n+2\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ as a multiset for some n.

The simplicial complex K can be regarded as a poset by inclusions. We define a subposet $P \subset K$ satisfying

- $P_{\max}(K) \subset P$,
- for any $\sigma \in P$ and i, the set $\{\phi(x) \mid x \in \sigma \cap A_i\}$ is equal to $\{2, 2, \dots, 2\}$, $\{4, 6, \dots 2n+2\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ as a multiset for some n.

Then we regard the poset P as a category and we define a functor $F\colon P \to \mathrm{Top}$. For $\sigma \in K$,

$$X_\sigma = \begin{cases} B\mathrm{Sp}(n) \times \prod_{\{x \in \sigma \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty & \text{when } \{\phi(x) \mid x \in \sigma\} = \{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}, \\ B\mathrm{SU}(n+1) \times \prod_{\{x \in \sigma \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty & \text{when } \{\phi(x) \mid x \in \sigma\} = \{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}, \\ \text{point} & \text{when } \sigma \text{ is the empty set.} \end{cases}$$

For $\sigma \subset \tau \in K$, let

$$\iota\colon \prod_{\{x \in \sigma \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty \to \prod_{\{x \in \tau \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty$$

be the inclusion such that each vertex corresponds to the same vertex. Then let $f_{\sigma,\tau}\colon X_\sigma \to X_\tau$ be the map constructed by the product of the composition of ι_1, ι_2 and ι_3 between $B\mathrm{SU}(n)$ and $B\mathrm{Sp}(n)$, and ι between the products of $\mathbb{C}\mathrm{P}^\infty$. We define a functor $F\colon P \to \mathrm{Top}$ as follows:

- For $\sigma \in P$, put $F(\sigma) = \prod_i X_{\sigma \cap A_i}$.
- For $\sigma, \tau \in P$ with $\sigma \subset \tau$, the map between $F(\sigma) \to F(\tau)$ is defined by the product

$$\prod_i f_{\sigma \cap A_i, \tau \cap A_i}\colon \prod_i X_{\sigma \cap A_i} \to \prod_i X_{\tau \cap A_i}.$$

We define $X = \operatorname{hocolim}_P F$; then the following lemma holds.

Lemma 3.2 *Under the above notation, the cohomology ring of X is isomorphic to* $\mathrm{SR}(K, \phi)$.

Proof We prove this lemma by induction on $|P|$. Let σ be a maximal simplex in K. Let K' be the simplicial complex consisting of the faces of simplices in $P \setminus \{\sigma\}$. Then by the assumption of the induction,

$$\begin{aligned} H^*(\operatorname{hocolim}_{P \setminus \{\sigma\}} F; \mathbb{Z}) &\cong \mathrm{SR}(K', \phi), \\ H^*(\operatorname{hocolim}_{P_{\le \sigma}} F; \mathbb{Z}) &\cong \mathbb{Z}[\sigma], \\ H^*(\operatorname{hocolim}_{P_{<\sigma}} F; \mathbb{Z}) &\cong \mathrm{SR}(K', \phi)/(V \setminus \sigma) \cong \mathrm{SR}(K'', \phi), \end{aligned}$$

where K'' is the simplicial complex consisting of the simplices that a simplex in K' and a face of σ. By Lemma 2.1, X is represented by the following homotopy pushout diagrams

$$\begin{tikzcd} \operatorname{hocolim}_{P_{<\sigma}} F \arrow[r] \arrow[d] & \operatorname{hocolim}_{P\setminus\{\sigma\}} F \arrow[d] \\ \operatorname{hocolim}_{P_{\leq\sigma}} F \arrow[r] & X \end{tikzcd}$$

Since ι_1, ι_2 and ι_3 are the maps that send each generator to its corresponding generator or 0 in cohomology, the maps in the upper diagram satisfy the condition in Lemma 2.2. Therefore by Lemma 2.2, we obtain that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$. □

Proof of Theorem 1.3 By this discussion, we apply Lemma 3.2 to the case $P = P_{\max}(K)$, completing the proof. □

When the degree of generators of $\mathrm{SR}(K,\phi)$ are only 2 and 4, Theorem 1.3 becomes a well-known result. This corollary is directly proved by the result of Davis and Januszkiewicz [7], and a special case of [5, Theorem 2.34].

Corollary 3.3 *Let $\mathrm{SR}(K,\phi)$ be the finitely generated graded Stanley–Reisner ring for a simplicial complex K with the vertex set V and $\phi\colon V \to 2\mathbb{Z}_{>0}$. When the image of ϕ is in $\{2,4\}$, we can construct a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$.*

When, in $\mathrm{SR}(K,\phi)$, there is no pair of generators $x, y \in V$ such that $|x| = |y| = 4$ and $xy \neq 0$, we don't have to take the decomposition of the vertex set. In this case, we can restate Theorem 1.3 as follows.

Corollary 3.4 *Let $\mathrm{SR}(K,\phi)$ be the finitely generated graded Stanley–Reisner ring for a simplicial complex K with the vertex set V and $\phi\colon V \to 2\mathbb{Z}_{>0}$. We assume that there is no pair of generators $x, y \in V$ such that $|x| = |y| = 4$ and $xy \neq 0$ in $\mathrm{SR}(K,\phi)$. Then if $\mathrm{SR}(K,\phi)$ satisfies the following condition, we can construct a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$:*

- *For $\sigma \in P_{\max}(K)$, the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2,2,\ldots,2\}$, $\{4,6,\ldots,2n+2\} \cup \{2,2,\ldots,2\}$ or $\{4,8,\ldots,4n\} \cup \{2,2,\ldots,2\}$ as a multiset.*

4 Examples

In this section we look at some examples about Corollary 3.4.

Let $\mathrm{SR}[K,\phi] \cong \mathbb{Z}[x_4,x_6,x_8]/(x_6x_8)$. Then the corresponding diagram is

$$B\mathrm{SU}(3) \leftarrow B\mathrm{Sp}(1) \to B\mathrm{Sp}(2).$$

Let $\mathrm{SR}[K,\phi] \cong \mathbb{Z}[x_4, x_{6,1}, x_{6,2}, \dots, x_{6,n}, x_8]/(x_{6,j}x_{6,k} \text{ for } j \neq k)$, where $|x_{i,j}| = i$. Then the corresponding diagram is

$$\begin{tikzcd}
 & BSp(2) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
BSU(4) & BSU(4) & \cdots & BSU(4)
\end{tikzcd}$$

Let $\mathrm{SR}[K,\phi] \cong \mathbb{Z}[x_4, x_{6,1}, x_{6,2}, x_{8,1}, x_{8,2}]/(x_{6,1}x_{6,2}, x_{8,1}x_{8,2})$, where $|x_{i,j}| = i$. Then the corresponding diagram is

$$\begin{tikzcd}
 & & & BSU(4) \\
 & BSU(3) \arrow[urr] \arrow[rr] & & BSU(4) \\
 & & BSp(2) \arrow[uur] \arrow[dddr] & \\
BSp(1) \arrow[uur] \arrow[ur, to path={-- (\tikztotarget)}] \arrow[drr] \arrow[ddr] \arrow[urr] & & & \\
 & & BSp(2) \arrow[uuur] \arrow[ddr] & \\
 & BSU(3) \arrow[rr] \arrow[drr] & & BSU(4) \\
 & & & BSU(4)
\end{tikzcd}$$

5 Approach from algebra over the Steenrod algebra

This section discusses when a graded polynomial ring has an unstable algebra structure over mod p Steenrod algebra by using previous results. All of the properties in this section are similar to the properties used by Aguadé in [2]. There, Aguadé obtains which polynomial algebras over $\mathbb{Z}$ are realizable as the integral cohomology ring of a space when the orders of the generators are all different. To prove this, Aguadé observes which polynomial rings have an unstable algebra structure over the mod p Steenrod algebra by using the result of Adams and Wilkerson [1]. In this section, we consider which polynomial rings have an unstable algebra structure over the mod p Steenrod algebra under the condition that there is at most 1 generator with degree 4.

When a commutative graded algebra A^* over $\mathbb{Z}/p$ has an action of mod p Steenrod algebra with Cartan formula, we say A^* an algebra over the mod p Steenrod algebra. An algebra over the mod p Steenrod algebra A^* with $A^{2i+1} = 0$ for all i is unstable if and only if for all homogeneous elements $x \in A^{2d}$, there are equations

$$\mathcal{P}^k(x) = \begin{cases} x^p & \text{if } k = d, \\ 0 & \text{if } k > d \end{cases} \quad \text{when } p \geq 3, \qquad \text{or} \qquad \mathrm{Sq}^{2k}(x) = \begin{cases} x^2 & \text{if } k = d, \\ 0 & \text{if } k > d \end{cases} \quad \text{when } p = 2.$$

When the odd-degree part of A^* is equal to 0, the unstable condition can be defined by only these equations. Conversely, if the odd-degree part of A^* is not equal to 0, there are more equations needed to define the unstable condition.

The following theorem can be obtained by combining Theorems 1.1 and 1.2 in Adams and Wilkerson [1].

Theorem 5.1 (cf Adams and Wilkerson [1, Theorems 1.1 and 1.2]) *Let A^* be a graded polynomial algebra over $\mathbb{Z}/p$ for prime p. We assume that the following conditions hold*:

- *A^* is an unstable algebra over the mod p Steenrod algebra.*
- *A^* is evenly generated.*
- *A^* is finitely generated as ring.*
- *The degrees of generators of A^* are prime to p.*

Then there is an isomorphism

$$A^* \cong H^*(BT^n; \mathbb{Z}/p)^W$$

for some n and a group W generated by pseudoreflections.

By using this theorem, we can prove the next theorem.

Proposition 5.2 (cf Aguadé [2, Proposition 2]) *Let A^* be a graded polynomial algebra over $\mathbb{Z}$ satisfying the following condition*:

- *There is a number N such that for all prime numbers $p > N$, $A \otimes \mathbb{Z}/p$ has unstable algebra structure over the mod p Steenrod algebra.*

Then the degree of the generator of A^* is the union of the following list:

- $\{2\}$
- $\{4, 8, \dots, 4(n-1), 2n\}$ *for* $n \geq 4$
- $\{4, 10, 12, 16, 18, 24\}$
- $\{4, 16\}$
- $\{4, 6, \dots, 2n\}$
- $\{4, 12\}$
- $\{4, 12, 16, 20, 24, 28, 36\}$
- $\{4, 24\}$
- $\{4, 8, \dots, 4n\}$
- $\{4, 12, 16, 24\}$
- $\{4, 16, 24, 28, 36, 40, 48, 60\}$
- $\{4, 48\}$

We prove this proposition by the same method in the proof of [2, Proposition 2].

Proof Let $p_1, \dots, p_i$ be the primes larger than 7 which divide the degree of generators of A^*. Then by a theorem of Dirichlet we can take a prime number $p > N$ such that

$$p \equiv 7 \bmod 16, \quad p \equiv 2 \bmod 3, \quad p \equiv 3 \bmod 5, \quad p \equiv 3 \bmod 7, \quad p \equiv 2 \bmod p_i.$$

By Theorem 5.1, $A^* \otimes \mathbb{Z}/p$ is isomorphic to an invariant ring $H^*(BT^n; \mathbb{Z}/p)^W$ for some n and a group W generated by pseudoreflections. By the classification theorem of p–adic pseudoreflection groups (cf Clark and Ewing [6]), we obtain this proposition. □

For a graded algebra A^*, write $QA^* = A^*/(A^*_+)^2$. The following lemma is proved by Thomas.

Theorem 5.3 (Thomas [16, Theorem 1.4]) *Let A^* be a finitely generated polynomial algebra over $\mathbb{Z}/2$ and an unstable algebra over the mod 2 Steenrod algebra. Then for any number i and odd number $n \geq 3$, the map*

$$\mathrm{Sq}^{2^i} : QA^{2^i(n-1)} \to QA^{2^i n}$$

is a surjection.

Lemma 5.4 *Let A^* be a polynomial algebra over $\mathbb{Z}$ such that the degrees of generators are equal to one of the following list as a multiset:*

- $\{4, 8, \dots, 4(n-1), 2n\} \cup \{2, 2, \dots, 2\}$ *with $n \geq 4$ and n is not a power of 2,*
- $\{4, 12\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 12, 16, 24\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 10, 12, 16, 18, 24\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 12, 16, 20, 24, 28, 36\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 16, 24, 28, 36, 40, 48, 60\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 24\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 48\} \cup \{2, 2, \dots, 2\}$.

Then $A^* \otimes \mathbb{Z}/2$ doesn't have an unstable algebra structure over the mod 2 Steenrod algebra.

Proof We assume that $A^* \otimes \mathbb{Z}/2$ has an unstable algebra over the mod 2 Steenrod algebra. By Theorem 5.3, if there is a generator x with $|x| = 12$, then there must be a generator y with $|y| = 8$. Therefore the second, third, fourth and fifth cases don't have an unstable algebra structure over the mod 2 Steenrod algebra.

Similarly, if there is a generator x such that $|x| = 60, 24, 48$, then there must be a generator y with $|y| = 56, 16, 32$, respectively. Therefore the sixth, seventh and eighth cases don't have an unstable algebra structure over the mod 2 Steenrod algebra.

It remains to show the first case. In this case we can denote $n = 2^i m$ for an integer i and an odd number $m \geq 3$. When $i = 0$, by [2, Proposition 3], A^* doesn't have an unstable algebra structure over the mod 2 Steenrod algebra. When $i \geq 1$, by Theorem 5.3 $\mathrm{Sq}^{2^{i+1}} : QA^{2^{i+1}(m-1)} \to QA^{2^{i+1}m}$ must be a surjection. But $\dim(QA^{2^{i+1}(m-1)}) = 1$ and $\dim(QA^{2^{i+1}m}) = 2$; a contradiction. Therefore the first case doesn't have an unstable algebra structure over the mod 2 Steenrod algebra.

Combining these discussions, the proof is complete. □

Lemma 5.5 *Let A^* be a polynomial algebra over $\mathbb{Z}$ such that the degrees of generators are equal to $\{4, 16\} \cup \{2, 2, \dots, 2\}$ as a multiset. Then $A^* \otimes \mathbb{Z}/3$ doesn't have an unstable algebra over the mod 3 Steenrod algebra.*

Proof Let A^* be the polynomial ring with the degrees of generators are equal to $\{4,16\}\cup\{2,\dots,2\}$, and let x be the generator with degree 16 in A^*. We assume that $A^*\otimes\mathbb{Z}/3$ has an unstable algebra structure over the mod 3 Steenrod algebra. By the Adem relation, there is an equation $\mathcal{P}^8=-\mathcal{P}^1\mathcal{P}^7$. Since $\mathcal{P}^8(x)=x^3$, it follows that x^3 is in $\mathrm{Im}(\mathcal{P}^1)$. On the other hand since there is no generator y with $|y|\equiv 12 \bmod 16$, the term x^i is not included in the image $\mathcal{P}^1$. This is a contradiction. □

Proposition 5.6 *Let A^* be a nontrivial graded polynomial algebra over $\mathbb{Z}$ such that*

- *there is at most one generator with degree* 4, *and*
- *for all prime numbers p, $A\otimes\mathbb{Z}/p$ has an unstable algebra structure over the mod p Steenrod algebra.*

Then the degree of the generators of A^* is equal to the one of the following list as a multiset for some n:

- $\{2,2,\dots,2\}$
- $\{4,6,\dots,2n\}\cup\{2,2,\dots,2\}$
- $\{4,8,\dots,4n\}\cup\{2,2,\dots,2\}$
- $\{4,8,\dots,2^{n+1}-4,2^n\}\cup\{2,2,\dots,2\}$

Proof By Proposition 5.2 and the first condition, the degree of the generator of A^* is equal to the union of the one of the table in Proposition 5.2 and the copies of $\{2\}$. By Lemmas 5.4 and 5.5, the cases except for the cases $\{4,6,\dots,2n\}$, $\{4,8,\dots,4n\}$ or $\{4,8,\dots,2^{n+1}-4,2^n\}$ don't satisfies the second condition. Thus we obtain the proposition. □

Example 5.7 Let $\mathbb{Z}/2[t_1,\dots,t_{2^n}]\cong H^*(BT^{2^n};\mathbb{Z}/2)$ for $n\geq 2$. Take a subring of $H^*(BT^{2^n};\mathbb{Z}/2)$ as

$$\mathbb{Z}/2[t_1,t_2,\dots,t_{2^n-1}]^{W(\mathrm{Sp}(2^n-1))}\otimes\mathbb{Z}/2[{t_{2^n}}^{2^{n-1}}],$$

where $W(\mathrm{Sp}(2^n-1))$ is the Weyl group of $\mathrm{Sp}(2^n-1)$ and $\mathbb{Z}/2[t_1,t_2,\dots,t_{2^n}]^{W(\mathrm{Sp}(2^n-1))}$ is the invariant ring of the canonical $W(\mathrm{Sp}(2^n-1))$–action. Since $\mathbb{Z}/2[t_1,t_2,\dots,t_{2^n}]^{W(\mathrm{Sp}(2^n-1))}$ is isomorphic to $H^*(B\mathrm{Sp}(2^n-1);\mathbb{Z}/2)$, this subring preserve the action of mod 2 Steenrod operations, and the degree of generators of this subring is $\{4,8,\dots,2^{n+1}-4,2^n\}$. This subring has the unstable algebra structure over the mod 2 Steenrod algebra induced by $H^*(BT^{2^n};\mathbb{Z}/2)$.

When p is an odd prime number, the cohomology ring $H^*(B\mathrm{Spin}(2^n);\mathbb{Z}/p)$ is isomorphic to the polynomial ring with generator's degree $\{4,8,\dots,2^{n+1}-4,2^n\}$ (cf [12, Chapter III, Theorem 3.19]), and has the unstable algebra structure over the mod p Steenrod algebra. And the ring in this example has the unstable algebra structure over the mod 2 Steenrod algebra. Therefore by only using the method in this section, we cannot remove the case $\{4,8,\dots,2^{n+1}-4,2^n\}$ in Proposition 5.6.

6 Stanley–Reisner ring and Steenrod algebra

Let K be a simplicial set with the vertex set V, and $\phi\colon V\to 2\mathbb{Z}_{>0}$. For a polynomial $f\in\mathrm{SR}(K,\phi)$ and a monomial g, we write $g<f$ when $g\neq 0$ in $\mathrm{SR}(K,\phi)$ and the coefficient of g in f is not equal to 0. This notation is well-defined because the ideal I is generated by monomials.

Lemma 6.1 *Let X be a space such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$ for some graded Stanley–Reisner ring, and $\sigma \in K$ be a maximal simplex. Then for any prime number p the ideal $(V \setminus \sigma)$ in $H^*(X;\mathbb{Z}/p) \cong \mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p$ preserves the action of the mod p Steenrod algebra.*

Proof We assume that the ideal $(V \setminus \sigma)$ doesn't preserve the action of mod p Steenrod algebra. Then by the Cartan formula, there is $x \in V \setminus \sigma$ and a monomial $f \in \mathrm{SR}(K,\phi)$ such that $f \notin (V \setminus \sigma)$ and $f < \mathcal{P}^i(x)$ for some i. Now we can take f with i being minimal, ie $\mathcal{P}^j(x) \in (V \setminus \sigma)$ for $j < i$. We write $g = \prod_{y \in \sigma} y$. Since f is a monomial generated by σ, we get $fg \neq 0$ in $\mathrm{SR}(K,\phi)$. Then

$$fg < \mathcal{P}^i(x)g,$$

and since i is minimal,

$$fg \nless \sum_{j>0} \mathcal{P}^{i-j}(x)\mathcal{P}^j(g).$$

Therefore, by the Cartan formula,

$$fg < \mathcal{P}^i(x)g + \sum_{j>0} \mathcal{P}^{i-j}(x)\mathcal{P}^j(g) = \mathcal{P}^i(xg),$$

and we obtain

$$\mathcal{P}^i(xg) \neq 0.$$

Since $xg = 0$ in $\mathrm{SR}(K,\phi)$, this is a contradiction, so the assumption is false. This completes the proof. □

Proposition 6.2 *Let X be a space such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$ for some graded Stanley–Reisner ring. Let $\sigma_1,\dots,\sigma_m \in K$ be maximal simplexes. Then for any prime number p, the ring*

$$\mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p\mathbb{Z}/(V \setminus \sigma_1 \cap \cdots \cap \sigma_m)$$

has an unstable algebra structure over the mod p Steenrod algebra induced by the quotient map

$$H^*(X;\mathbb{Z}/p) \cong \mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p\mathbb{Z} \to \mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p\mathbb{Z}/(V \setminus \sigma_1 \cap \cdots \cap \sigma_m).$$

Proof By Lemma 6.1, for all $x \in V \setminus \sigma_k$ and i, we obtain

$$\mathcal{P}^i(x) \in (V \setminus \sigma_k) \subset (V \setminus \sigma_1 \cap \cdots \cap \sigma_n).$$

Therefore the ideal $(V \setminus \sigma_1 \cap \cdots \cap \sigma_m)$ preserves the action of the mod p Steenrod algebra. □

Theorem 6.3 *For a graded Stanley–Reisner ring $\mathrm{SR}(K,\phi)$, let X be a space such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$. We assume that there is no pair of generators $x, y \in V$ such that $\phi(x) = \phi(y) = 4$ and $xy \neq 0$ in $\mathrm{SR}(K,\phi)$. Then for $\sigma \in P_{\max}(K)$, the set $\{\phi(x) \mid x \in \sigma\}$ is equal to*

- $\{2,\dots,2\}$,
- $\{4,6,\dots,2n+2\} \cup \{2,\dots,2\}$,
- $\{4,8,\dots,4n\} \cup \{2,\dots,2\}$, *or*
- $\{4,8,\dots,2^{n+2}-8, 2^{n+2}-4, 2^{n+1}\} \cup \{2,\dots,2\}$

as a multiset for some $n \geq 1$.

Proof Since $\sigma \in K$, there is no relation between the generators in σ. Therefore there is an isomorphism

$$\mathrm{SR}(K,\phi)/(V \setminus \sigma) \cong \mathbb{Z}[\sigma].$$

By the definition of $P_{\max}(K)$ there are maximal simplexes $\sigma_1, \dots, \sigma_m \in K$ such that $\sigma = \bigcap_i \sigma_i$. Thus $\mathbb{Z}[\sigma]$ satisfies the condition of Proposition 6.2. By the assumption in the statement, for any $\sigma \in P_{\max}(K)$ there is at most one generator with degree 4 in σ. By Proposition 6.2 and this condition, the polynomial ring $\mathbb{Z}[\sigma]$ satisfies the condition in Proposition 5.6. Therefore the set $\{\phi(x) \mid x \in \bigcap_i \sigma_i\}$ is equal, as a multiset, to

- $\{2, \dots, 2\}$,
- $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$,
- $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$, or
- $\{4, 8, \dots, 2^{n+2}-8, 2^{n+2}-4, 2^{n+1}\} \cup \{2, \dots, 2\}$. □

Example 6.4 Let $\mathrm{SR}(K,\phi) \cong \mathbb{Z}[x_4, x_6]/(x_4x_6)$, where $|x_i| = i$. Then $P_{\max}(K) = \{\{4\}, \{6\}\}$. By Theorem 6.3, there is no space X with $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[x_4, x_6]/(x_4x_6)$.

7 Proof of the main theorem

By combining Corollary 3.4 and Theorem 6.3, we can prove Theorem 1.1.

Proof of Theorem 1.1 In Corollary 3.4, we prove that if $\mathrm{SR}(K,\phi)$ satisfies these conditions then there is a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$.

On the other hand, we assume that there is a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$. By assumption, in the statement for $i = 2$, $\mathrm{SR}(K,\phi)$ satisfies the condition of Theorem 6.3. By Theorem 6.3, for any $\sigma \in P_{\max}(K)$ the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2, \dots, 2\}$, $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$, $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 2^{n+2}-8, 2^{n+2}-4, 2^{n+1}\} \cup \{2, \dots, 2\}$ as a multiset. By the assumption in the statement, there is no pair of generators x, y such that $|x| = |y| = 2^n$ for some $n \geq 3$ and $xy \neq 0$ in $\mathrm{SR}(K,\phi)$. Since the case $\{4, 8, \dots, 2^{n+2}-8, 2^{n+1}-4, 2^n\} \cup \{2, \dots, 2\}$ for $n \geq 3$ includes such a pair of generators, this case doesn't appear. Therefore for any $\sigma \in P_{\max}(K)$ the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2, \dots, 2\}$, $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ as a multiset.

By combining these, the proof is complete. □

References

[1] **J F Adams**, **C W Wilkerson**, *Finite H–spaces and algebras over the Steenrod algebra*, Ann. of Math. 111 (1980) 95–143 MR Zbl

[2] **J Aguadé**, *A note on realizing polynomial algebras*, Israel J. Math. 38 (1981) 95–99 MR Zbl

[3] **K K S Andersen**, **J Grodal**, *The Steenrod problem of realizing polynomial cohomology rings*, J. Topol. 1 (2008) 747–760 MR Zbl

[4] **K K S Andersen**, **J Grodal**, *The classification of* 2*–compact groups*, J. Amer. Math. Soc. 22 (2009) 387–436 MR Zbl

[5] **A Bahri**, **M Bendersky**, **F R Cohen**, **S Gitler**, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. Math. 225 (2010) 1634–1668 MR Zbl

[6] **A Clark**, **J Ewing**, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. 50 (1974) 425–434 MR Zbl

[7] **M W Davis**, **T Januszkiewicz**, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991) 417–451 MR Zbl

[8] **W G Dwyer**, **H R Miller**, **C W Wilkerson**, *Homotopical uniqueness of classifying spaces*, Topology 31 (1992) 29–45 MR Zbl

[9] **W G Dwyer**, **C W Wilkerson**, *A cohomology decomposition theorem*, Topology 31 (1992) 433–443 MR Zbl

[10] **W G Dwyer**, **C W Wilkerson**, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. 139 (1994) 395–442 MR Zbl

[11] **J R Hubbuck**, *Generalized cohomology operations and H–spaces of low rank*, Trans. Amer. Math. Soc. 141 (1969) 335–360 MR Zbl

[12] **M Mimura**, **H Toda**, *Topology of Lie groups, I*, Transl. Math. Monogr. 91, Amer. Math. Soc., Providence, RI (1991) MR Zbl

[13] **T So**, **D Stanley**, *Realization of graded monomial ideal rings modulo torsion*, Algebr. Geom. Topol. 23 (2023) 733–764 MR Zbl

[14] **N E Steenrod**, *The cohomology algebra of a space*, Enseign. Math. 7 (1961) 153–178 MR Zbl

[15] **T Sugawara**, **H Toda**, *Squaring operations on truncated polynomial algebras*, Jpn. J. Math. 38 (1969) 39–50 MR Zbl

[16] **E Thomas**, *Steenrod squares and H–spaces, II*, Ann. of Math. 81 (1965) 473–495 MR Zbl

[17] **A J Trevisan**, *Generalized Davis–Januszkiewicz spaces, multicomplexes and monomial rings*, Homology Homotopy Appl. 13 (2011) 205–221 MR Zbl

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Dehn twists and the Nielsen realization problem for spin 4–manifolds

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We prove that for a closed oriented smooth spin 4–manifold X with nonzero signature, the Dehn twist about a $(+2)$– or (-2)–sphere in X is not homotopic to any finite-order diffeomorphism. In particular, we negatively answer the Nielsen realization problem for each group generated by the mapping class of a Dehn twist. We also show that there is a discrepancy between the Nielsen realization problems in the topological category and smooth category for connected sums of copies of $K3$ and $S^2 \times S^2$. The main ingredients of the proofs are Y Kato's $10/8$–type inequality for involutions and a refinement of it.

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1 Main results

Given a smooth manifold X, let $\mathrm{Diff}(X)$ denote the group of diffeomorphisms. The *Nielsen realization problem* asks whether a given finite subgroup G of the mapping class group $\pi_0(\mathrm{Diff}(X))$ can be realized as a subgroup of $\mathrm{Diff}(X)$, ie whether there exists a (group-theoretic) section $s\colon G \to \mathrm{Diff}(X)$ of the natural map $\mathrm{Diff}(X) \to \pi_0(\mathrm{Diff}(X))$ over G. If there is a section, we say that G is *realizable in* $\mathrm{Diff}(X)$. When X is of $\dim = 2$ and oriented closed, which is the classical case of the Nielsen realization problem, Kerckhoff [18] proved that every G is realizable.

In contrast, Raymond and Scott [30] showed that, in every dimension ≥ 3, there are nilmanifolds for which the Nielsen realization fails essentially using their nontrivial fundamental groups. Focusing on dimension 4 and simply connected manifolds, it was recently proven by Baraglia and the author [4] and Farb and Looijenga [9] that the Nielsen realization fails for $K3$, the underlying smooth 4–manifold of a $K3$ surface. However, to the best of the author's knowledge, the nilmanifolds in [30] and $K3$ are the only examples of 4–manifolds X that are shown to admit finite subgroups of $\pi_0(\mathrm{Diff}(X))$ that are not realizable in $\mathrm{Diff}(X)$. The purpose of this paper is to expand the list of such 4–manifolds considerably. In particular, we give infinitely many examples of simply connected 4–manifolds with distinct intersection forms for which the Nielsen realization fails.

For a general 4–manifold, it is not obvious to find a potential example of nonrealizable finite subgroups of the mapping class group. Following Farb and Looijenga [9], we consider *Dehn twists*, which are sources of interesting examples. Given a $(+2)$– or (-2)–sphere S embedded in a 4–manifold X, one has a diffeomorphism $T_S\colon X \to X$ called the Dehn twist, whose mapping class $[T_S]$ generates an order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$ (see Section 5.1). Our first main result is:

Theorem 1.1 *Let X be a closed oriented smooth spin 4–manifold with nonzero signature and S be a smoothly embedded 2–sphere in X with $[S]^2 = 2$ or $[S]^2 = -2$. Then the Dehn twist $T_S\colon X \to X$ about S is not homotopic to any finite-order diffeomorphism of X. In particular, the order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$ generated by the mapping class $[T_S]$ is not realizable in* $\mathrm{Diff}(X)$.

Theorem 1.1 generalizes the case when $X = K3$ due to Farb and Looijenga [9, Corollary 1.10] (see Remark 5.2 for the comparison), and Theorem 1.1 immediately implies that the Nielsen realization fails for quite many 4–manifolds, such as $\#_m K3 \#_n S^2 \times S^2$ with $m > 0$ and also infinitely many examples of irreducible 4–manifolds. See Example 5.3 for details.

Theorem 1.1 makes a striking contrast to a recent result by Lee [21, Corollary 1.5, Remark 1.7], which implies that the Dehn twist about every (± 2)–sphere in $\mathbb{CP}^2 \# n(-\mathbb{CP}^2)$ with $n \leq 8$ is topologically isotopic (hence homotopic) to a smooth involution. This means that an analogous statement to Theorem 1.1 does not hold for *nonspin* 4–manifolds.

Another result of this paper concerns a comparison between the Nielsen realization problems in the topological category and the smooth category. Let $\mathrm{Homeo}(X)$ denote the group of homeomorphisms of a manifold X. As well as the smooth Nielsen realization, we say that a subgroup G of $\pi_0(\mathrm{Homeo}(X))$ is *realizable in* $\mathrm{Homeo}(X)$ if there is a section $s\colon G \to \mathrm{Homeo}(X)$ of the natural map

$$\mathrm{Homeo}(X) \to \pi_0(\mathrm{Homeo}(X))$$

over G. In [4, Theorem 1.2], Baraglia and the author showed that some order-2 subgroup of $\pi_0(\mathrm{Diff}(K3))$ is not realizable in $\mathrm{Diff}(K3)$, even when the corresponding subgroup in $\pi_0(\mathrm{Homeo}(K3))$ is realizable in $\mathrm{Homeo}(K3)$. We generalize this result to connected sums of copies of $K3$ and $S^2 \times S^2$:

Theorem 1.2 *For $m > 0$ and $n \geq 0$, set $X = mK3 \# nS^2 \times S^2$. Then there exists an order-2 subgroup G of $\pi_0(\mathrm{Diff}(X))$ with the following properties*:

- *The group G is not realizable in* $\mathrm{Diff}(X)$. *Moreover, a representative of the generator of G is not homotopic to any finite-order diffeomorphism of X.*
- *The subgroup $G' \subset \pi_0(\mathrm{Homeo}(X))$ defined as the image o G under the natural map*

$$\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$$

  is a nontrivial group, and G' is realizable in $\mathrm{Homeo}(X)$.

In other words, a representative $g \in \mathrm{Diff}(X)$ of the generator of G in Theorem 1.2 is not homotopic to any finite-order diffeomorphism, although g^2 is smoothly isotopic to the identity and g is topologically isotopic to some topological involution with nontrivial mapping class. Theorem 1.2 gives also an alternative proof of a result by Baraglia [2, Proposition 1.2] about the realization problem along $\mathrm{Diff}(X) \to \mathrm{Aut}(H_2(X;\mathbb{Z}))$ (see Section 7).

Theorems 1.1 and 1.2 shall be derived from the following constraint on the induced actions of finite-order diffeomorphisms on homology. Let $\sigma(X)$ denote the signature of an oriented closed 4–manifold

X and $b_+(X)$ denote the maximal-dimension of positive-definite subspaces of $H_2(X;\mathbb{R})$. For an involution φ on the intersection lattice, we denote by $b^\varphi_+(X)$ (resp. $b^\varphi_-(X)$) the maximal-dimension of positive-definite (resp. negative-definite) subspaces of the φ–invariant part $H_2(X;\mathbb{R})^\varphi$, and we set $\sigma^\varphi(X) = b^\varphi_+(X) - b^\varphi_-(X)$.

Theorem 1.3 *Let X be a closed oriented smooth 4–manifold with $\sigma(X) < 0$, and let $\mathfrak{s}$ be a spin structure on X. Let $g\colon X \to X$ be a finite-order diffeomorphism that preserves orientation of X and $\mathfrak{s}$, and let $\varphi\colon H_2(X;\mathbb{Z})/\mathrm{Tor} \to H_2(X;\mathbb{Z})/\mathrm{Tor}$ denote the action on homology induced from g. Suppose that φ is of order 2 and that $\sigma^\varphi(X) \neq \sigma(X)/2$. Then*

$$-\tfrac{1}{16}\sigma(X) \leq b_+(X) - b^\varphi_+(X). \tag{1}$$

Moreover, if $b_+(X) - b^\varphi_+(X) > 0$, then

$$-\tfrac{1}{16}\sigma(X) + 1 \leq b_+(X) - b^\varphi_+(X).$$

The main ingredients of the proof of Theorem 1.3 are Y Kato's $10/8$–type inequality for involutions [17] (Theorem 2.2) coming from Seiberg–Witten theory and a refinement of it (Theorem 3.1). This refinement is necessary to show the "moreover" part of Theorem 1.3, which shall be used to obtain the results on Dehn twists (Theorem 1.1) for both $(+2)$– and (-2)–spheres.

Here is an outline of the contents of this paper. In Section 2, we recall Kato's $10/8$–type inequality for a smooth involution on a spin 4–manifold. In Section 3, we give a refinement of Kato's inequality by proving a new Borsuk–Ulam-type theorem using equivariant K–theory. In Section 4, we prove Theorem 1.3 based on Kato's inequality and the refinement of it in Section 3. Sections 5 and 6 are devoted to prove Theorems 1.1 and 1.2 respectively. We conclude by giving remarks on another kind of Dehn twist and other variants of the Nielsen realization problem in Section 7.

2 Kato's $10/8$–type inequality for involutions

Henceforth, for an oriented closed 4–manifold X, we identify $H_2(X)$ with $H^2(X)$ via Poincaré duality. For an involution ι on X, we set $b^\iota_+(X) = b^{\iota_*}_+(X)$, and similarly define $b^\iota_-(X)$ and $\sigma^\iota(X)$. Note that, if X has nonvanishing signature, all diffeomorphisms of X are orientation-preserving, namely, we have $\mathrm{Diff}(X) = \mathrm{Diff}^+(X)$, the group of orientation-preserving diffeomorphisms.

First, we recall the notion of even and odd involutions following [1; 6]. Let X be an oriented closed smooth 4–manifold and $\mathfrak{s}$ be a spin structure on X. Let $\iota\colon X \to X$ be an orientation-preserving diffeomorphism of order 2, and suppose that ι preserves (the isomorphism class of) $\mathfrak{s}$. Then there are exactly two lifts of ι to $\mathfrak{s}$ as automorphisms of the spin structure. We have either both lifts are of order 2 or both are of order 4. We say that the involution ι is *of even type* if the lifts are of order 2, and say that ι is *of odd type* if the lifts are of order 4. When the fixed-point set X^ι is nonempty, the codimension of all components of X^ι are the same, which is either 4 or 2, and the parity of ι determines which of them arises: X^ι is of codimension-4 if ι is of even type, and X^ι is of codimension-2 if ι is of odd type [1, Proposition 8.46]; see also [31].

Lemma 2.1 *Let X be an oriented closed smooth 4–manifold and $\mathfrak{s}$ be a spin structure on X. Let $\iota\colon X \to X$ be an orientation-preserving diffeomorphism of order 2, and suppose that ι preserves (the isomorphism class of) $\mathfrak{s}$ and is of even type. Then $\sigma^\iota(X) = \sigma(X)/2$.*

Proof By Hirzebruch's signature theorem (see for example [16, equation (12), page 177]), $\sigma^\iota(X)$ can be obtained by adding $\sigma(X)/2$ to contributions from fixed surfaces of ι. (Note that, for a general involution, the contribution from isolated fixed points is zero.) However, X^ι does not contain surfaces since ι is even. □

An important ingredient of this paper is the following $10/8$–type constraint on odd smooth involutions, proven by Y Kato [17] using Seiberg–Witten theory and $\mathbb{Z}/4$–equivariant K–theory:

Theorem 2.2 (Kato [17, Theorem 2.3]) *Let $(X, \mathfrak{s})$ be a smooth closed oriented spin 4–manifold. Let $\iota\colon X \to X$ be a smooth orientation-preserving involution, and suppose that ι preserves $\mathfrak{s}$ and is of odd type. Then*

$$-\tfrac{1}{16}\sigma(X) \le b_+(X) - b_+^\iota(X). \tag{2}$$

Remark 2.3 In [17], the result corresponding to Theorem 2.2 is stated using a quantity $b_+^I(X)$, where I acts on $H^2(X;\mathbb{R})$ as $I = -\iota^*$. By Poincaré duality, it immediately follows that $b_+^I(X) = b_+(X) - b_+^\iota(X)$.

3 A refinement of Kato's inequality

To deal with Dehn twists about both $(+2)$– and (-2)–spheres in Theorem 1.1, we shall need the following refinement of Kato's inequality (Theorem 2.2), which we call the *refined Kato's inequality*:

Theorem 3.1 *Let $(X, \mathfrak{s})$ be a smooth closed oriented spin 4–manifold. Let $\iota\colon X \to X$ be a smooth orientation-preserving involution, and suppose that ι preserves $\mathfrak{s}$ and is of odd type. Suppose that $b_+(X) - b_+^\iota(X) > 0$. Then*

$$-\tfrac{1}{16}\sigma(X) + 1 \le b_+(X) - b_+^\iota(X).$$

This shall be proven in Section 3.2 using a Borsuk–Ulam-type theorem (Theorem 3.3), which we first give in Section 3.1.

3.1 $\mathbb{Z}/4$–equivariant K–theory

To show Theorem 3.1, we prove a new Borsuk–Ulam-type theorem using $\mathbb{Z}/4$–equivariant K–theory. As in Kato's argument [17], the following approach is modeled on Bryan's argument [6] for Pin(2)–equivariant K–theory. A difference from Kato's argument is that we shall use the structure of the whole representation ring $R(\mathbb{Z}/4)$.

Set $G = \mathbb{Z}/4$ and let j denote a generator; $G = \{1, j, -1, -j\}$. (The symbol j stands for a unit quaternion $j \in \mathrm{Pin}(2) \subset \mathbb{H}$, which is a symmetry that the Seiberg–Witten equations admit.) Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_-$ be complex 1–dimensional representations of G determined by

$$\mathrm{tr}_j\,\mathbb{C} = 1, \quad \mathrm{tr}_j\,\mathbb{C}_+ = i, \quad \mathrm{tr}_j\,\mathbb{C}_- = -i,$$

where tr_j denotes the trace of the action of j and $i = \sqrt{-1}$. Namely, $\mathbb{C}$ is the trivial 1–dimensional representation, and $\mathbb{C}_\pm$ are representations given as $\pm i$–multiplication of the fixed generator of G. Let $\widetilde{\mathbb{R}}$ denote a real 1–dimensional representation of G defined through the surjective homomorphism $G \to \mathbb{Z}/2$ and multiplication of $\mathbb{Z}/2 = \{\pm 1\}$. Set $\widetilde{\mathbb{C}} = \widetilde{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Recall that the complex representation ring $R(G)$ is given by

$$R(G) = \mathbb{Z}[t]/(t^4 - 1), \tag{3}$$

where $t = \mathbb{C}_+$.

Here we recall a general fact, which holds for any compact Lie group G, called tom Dieck's formula by Bryan [6]. Let V and W be finite-dimensional unitary representations of G. Let V^+ denote the one-point compactification of V, naturally acted by G. We regard the point at infinity as the base point of V^+. Let $f\colon V^+ \to W^+$ be a pointed G–continuous map. By the equivariant K–theoretic Thom isomorphism, we have that $\widetilde{K}_G(V^+)$ and $\widetilde{K}_G(W^+)$ are free $\widetilde{K}_G(S^0) = R(G)$–modules generated by the equivariant K–theoretic Thom classes $\tau_G^K(V)$ and $\tau_G^K(W)$ respectively, and thus one may define the equivariant K–theoretic mapping degree $\alpha_f \in R(G)$ of f characterized by

$$f^*\tau_G^K(W) = \alpha_f \tau_G^K(V).$$

For an element $g \in G$, let V^g and W^g denote the fixed-point set for g, and let $(V^g)^\perp$ and $(W^g)^\perp$ denote the orthogonal complement of V^g and W^g in V and W respectively. Let $d(f^g) \in \mathbb{Z}$ denote the mapping degree, defined using just the ordinary cohomology, of the fixed-point set map $f^g\colon (V^g)^+ \to (W^g)^+$. For $\beta \in R(G)$, define $\lambda_{-1}\beta \in R(G)$ to be $\sum_{i \ge 0} (-1)^i \Lambda^i \beta$. Then tom Dieck's formula is:

Proposition 3.2 ([7, Proposition 9.7.2], see also [6, Theorem 3.3]) *In the above setup, we have*

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\,\mathrm{tr}_g(\lambda_{-1}((W^g)^\perp - (V^g)^\perp)).$$

Now we are ready to prove the Borsuk–Ulam-type theorem we need:

Theorem 3.3 *Let $G = \mathbb{Z}/4$. For natural numbers $m_0, m_1, n_0, n_1 \ge 0$ with $m_0 < m_1$, suppose that there exists a G–equivariant pointed continuous map*

$$f\colon (\widetilde{\mathbb{C}}^{m_0} \oplus (\mathbb{C}_+ \oplus \mathbb{C}_-)^{n_0})^+ \to (\widetilde{\mathbb{C}}^{m_1} \oplus (\mathbb{C}_+ \oplus \mathbb{C}_-)^{n_1})^+ \tag{4}$$

with $f(0) = 0$. Then

$$n_0 - n_1 + 1 \le m_1 - m_0. \tag{5}$$

Remark 3.4 This Borsuk–Ulam-type theorem, Theorem 3.3, may be of independent interest. Especially, it is worth noting that Theorem 3.3 allows us to give a proof of Furuta's celebrated 10/8–inequality [11] using only the $\mathbb{Z}/4$–symmetry of the Seiberg–Witten equations, while the original proof used a bigger symmetry, $\mathrm{Pin}(2)$. See Remark 3.5 for further comments on this.

Also, Theorem 3.3 generalizes a result by Pfister and Stolz [28, Theorem, page 286], where they proved Theorem 3.3 for the case that $m_0 = 0$ and $n_1 = 0$. The argument of Pfister and Stolz is also based on K–theory, but in a slightly different way than ours.

Proof of Theorem 3.3 Let $\alpha = \alpha_f \in R(G)$ denote the equivariant K–theoretic mapping degree of f. We shall obtain constraints on α from the actions of -1 and j. First, note that the (-1)–fixed point set map for f is given by $f^{-1}\colon (\widetilde{\mathbb{C}}^{m_0})^+ \to (\widetilde{\mathbb{C}}^{m_1})^+$, and thus the assumption $m_0 < m_1$ implies $d(f^{-1}) = 0$. Hence it follows from Proposition 3.2 that $\mathrm{tr}_{-1}(\alpha) = 0$. Thanks to the ring structure (3) of $R(G)$, α can be expressed in the form

$$\alpha = \sum_{k=0}^{3} a_k t^k, \tag{6}$$

where $a_k \in \mathbb{Z}$. Since $\mathrm{tr}_{-1}(t) = -1$, it follows that $\mathrm{tr}_{-1}(\alpha) = a_0 - a_1 + a_2 - a_3$. Thus,

$$a_0 - a_1 + a_2 - a_3 = 0. \tag{7}$$

Next, let us consider the j–action on α. First, note that f^j is just the identity map on $S^0 = \{0\} \cup \{\infty\}$, and hence $d(f^j) = 1$. In general, for complex rank 1 (virtual) representations $L_1, \dots, L_N \in R(G)$, one has $\lambda_{-1}\big(\sum_{i=1}^N L_i\big) = \prod_{i=1}^N \lambda_{-1} L_i$. Thus, again using Proposition 3.2,

$$\begin{aligned}
\mathrm{tr}_j(\alpha) &= \mathrm{tr}_j\big(\lambda_{-1}(\widetilde{\mathbb{C}}^{m_1-m_0} \oplus (\mathbb{C}_+ \oplus \mathbb{C}_-)^{n_1-n_0})\big) \\
&= \mathrm{tr}_j\big(\lambda_{-1}((m_1-m_0)t^2 + (n_1-n_0)t + (n_1-n_0)t^3)\big) \\
&= \mathrm{tr}_j\big((1-t^2)^{m_1-m_0}(1-t)^{n_1-n_0}(1-t^3)^{n_1-n_0}\big) \\
&= (1+1)^{m_1-m_0}(1-i)^{n_1-n_0}(1+i)^{n_1-n_0} \\
&= 2^{m_1-m_0+n_1-n_0}.
\end{aligned} \tag{8}$$

On the other hand, from the expression (6) of α, we have $\mathrm{tr}_j(\alpha) = a_0 - a_2 + (a_1 - a_3)i$. Since $\mathrm{tr}_j(\alpha) \in \mathbb{R}$ by (8), we have $a_1 - a_3 = 0$, and this combined with (7) implies that

$$\mathrm{tr}_j(\alpha) = a_0 - a_2 = 2(a_1 - a_2). \tag{9}$$

Since $a_1 - a_2 \in \mathbb{Z}$, the desired inequality (5) follows from (8) and (9). □

Note that the divisibility by 2 over $\mathbb{Z}$ of the right-hand side of (9) contributes to the "+1" term in the inequality (5), which is the source of the refined Kato's inequality.

3.2 Proof of Theorem 3.1

Now we are ready to prove the refined Kato's inequality:

Proof of Theorem 3.1 Set $G = \mathbb{Z}/4$. Kato proved in [17] that the odd involution ι gives rise to an involutive symmetry I on the Seiberg–Witten equations on $(X, \mathfrak{s})$, and the complexification of a finite-dimensional approximation of the I–invariant part of the Seiberg–Witten equations is a G–equivariant pointed continuous map f of the form (4) with $f(0) = 0$, where the natural numbers m_0, m_1, n_0 and n_1 in (4) satisfy

$$m_1 - m_0 = b_+(X) - b_+^{\iota}(X), \quad n_0 - n_1 = -\tfrac{1}{16}\sigma(X).$$

By the assumption $b_+(X) - b_+^{\iota}(X) > 0$, we may apply Theorem 3.3 to this f. □

Remark 3.5 Furuta's $10/8$–inequality [11] was proved using the $\mathrm{Pin}(2)$–symmetry of the Seiberg–Witten equations for a closed spin 4–manifold X. Using our Borsuk–Ulam-type theorem, Theorem 3.3, we may recover Furuta's $10/8$–inequality using only the $\mathbb{Z}/4$–symmetry of the Seiberg–Witten equations as follows. Note that $G = \mathbb{Z}/4 = \langle j \rangle$ is a subgroup of $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{H}$. Restricting the $\mathrm{Pin}(2)$–symmetry to the $\mathbb{Z}/4$–symmetry in Furuta's construction [11], we have that the complexification of a finite-dimensional approximation of the Seiberg–Witten equations is a G–equivariant pointed continuous map f of the form (4) with $f(0) = 0$ for natural numbers m_0, m_1, n_0 and n_1 with

$$m_1 - m_0 = b_+(X), \quad n_0 - n_1 = -\tfrac{1}{8}\sigma(X).$$

Applying Theorem 3.3 to f, we obtain

$$-\tfrac{1}{8}\sigma(X) + 1 \le b_+(X)$$

provided that $b_+(X) > 0$. This inequality is equivalent to the $10/8$–inequality [11, Theorem 1].

4 Proof of Theorem 1.3

Proof of Theorem 1.3 First, we reduce the problem to involutions following [9, Proof of Corollary 1.10]. Since the subgroup of $\mathrm{Diff}(X)$ generated by g has a surjective homomorphism onto $\langle \varphi \rangle \cong \mathbb{Z}/2$, the order of g is even. Let $2m$ be the order of g; then g^m is a smooth involution. Set $\iota = g^m$. Since $g_* = \varphi$ is of order 2, either $\iota_* = \varphi$ or $\iota_* = \mathrm{id}$. By the condition that $g^*\mathfrak{s} \cong \mathfrak{s}$, ι also preserves $\mathfrak{s}$.

If $\iota_* = \varphi$, we have $\sigma^{\iota}(X) \neq \sigma(X)/2$ from the assumption that $\sigma^{\varphi}(X) \neq \sigma(X)/2$. If $\iota_* = \mathrm{id}$, we have $\sigma^{\iota}(X) \neq \sigma(X)/2$ since we supposed $\sigma(X) \neq 0$. Thus, in any of these cases, $\sigma^{\iota}(X) \neq \sigma(X)/2$, and hence it follows from Lemma 2.1 that ι is of odd type. It then follows from Kato's inequality, Theorem 2.2, that

$$-\tfrac{1}{16}\sigma(X) \le b_+(X) - b_+^{\iota}(X) \le b_+(X) - b_+^{\varphi}(X). \tag{10}$$

To see the "moreover" part of the theorem, suppose that $b_+(X) - b_+^{\iota}(X) > 0$. Then we can replace the left-hand side of (10) with $-\sigma(X)/16 + 1$ by the refined Kato's inequality, Theorem 3.1. □

5 Proof of Theorem 1.1

5.1 Dehn twists about (± 2)–spheres

First, we recall 4–dimensional Dehn twists associated with (± 2)–spheres. We refer readers to a lecture note by Seidel [32, Section 2] for details. While the construction of the Dehn twist in [32] is described for a Lagrangian sphere in a symplectic 4–manifold, which is always a (-2)–sphere, the construction works for any (-2)–sphere in a general 4–manifold without any change, and it is easy to obtain an analogous diffeomorphism for a $(+2)$–sphere, described below.

Given a (-2)–sphere S in an oriented 4–manifold X, namely a smoothly embedded 2–dimensional sphere S with $[S]^2 = -2$, one may construct a diffeomorphism $T_S\colon X \to X$ called the *Dehn twist* about S, which is supported in a tubular neighborhood of S in X, as follows. First, note that a tubular neighborhood of S is diffeomorphic to T^*S^2 since S is a (-2)–sphere, and fix an embedding $T^*S^2 \hookrightarrow X$. The Dehn twist T_S is the extension by the identity of some compactly supported diffeomorphism τ of T^*S^2 called the *model Dehn twist*, which is given as the monodromy around the nodal singular fiber of the family $\mathbb{C}^3 \to \mathbb{C}$, $(z_1, z_2, z_3) \mapsto z_1^2 + z_2^2 + z_3^2$ over the origin of $\mathbb{C}$. The model Dehn twist τ acts on the zero-section S^2 as the antipodal map and τ^2 is smoothly isotopic to the identity through compactly supported diffeomorphisms of T^*S^2 [32, Proposition 2.1]. Hence the induced action of T_S on homology is nontrivial, more precisely, $(T_S)_*\colon H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})$ is given as

$$(T_S)_*(x) = x + (x \cdot [S])[S],$$

and T_S^2 is smoothly isotopic to the identity. Thus the mapping class $[T_S]$ is nontrivial and it generates an order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$.

Next, consider the situation that a $(+2)$–sphere S in an oriented 4–manifold X is given. Then a tubular neighborhood of S is diffeomorphic to TS^2. Via an isomorphism between TS^2 and T^*S^2 obtained by fixing a metric on S^2, we may implant the model Dehn twist into X as well as the (-2)–sphere case above. We denote by $T_S\colon X \to X$ also this diffeomorphism, and call T_S the Dehn twist as well. This Dehn twist also generates an order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$, since the corresponding statement for a (-2)–sphere follows just from a property of the model Dehn twist, and the action on $H_2(X)$ is given by

$$(T_S)_*(x) = x - (x \cdot [S])[S].$$

We note that every Dehn twist preserves every spin structure:

Lemma 5.1 *Let X be a closed oriented smooth 4–manifold, and suppose that X admits a spin structure $\mathfrak{s}$. Let S be a $(+2)$– or (-2)–sphere in X. Then the Dehn twist T_S preserves $\mathfrak{s}$.*

Proof Recall that T_S is just the identity map on the complement of a tubular neighborhood of S in X, which is diffeomorphic to the disk cotangent bundle $D(T^*S^2)$. Thus it suffices to show that, given a spin structure $\mathfrak{t}$ on $\partial D(T^*S^2) = S(T^*S^2)$, an extension of $\mathfrak{t}$ to $D(T^*S^2)$ is unique. By the relative obstruction theory for a natural fibration $B(\mathbb{Z}/2) \to B\,\mathrm{Spin}(4) \to B\mathrm{SO}(4)$, it follows that the extensions of $\mathfrak{t}$ are classified by $H^1(D(T^*S^2), S(T^*S^2); \mathbb{Z}/2)$, which is the trivial group by the mod 2 Thom isomorphism for $T^*S^2 \to S^2$. □

5.2 Proof of Theorem 1.1

Now we are ready to prove our main result on Dehn twists:

Proof of Theorem 1.1 By reversing the orientation, we may suppose that $\sigma(X) < 0$. Note that a (± 2)–sphere turns into a (∓ 2)–sphere if we reverse the orientation of X. First we consider the case that a (-2)–sphere is given in X with $\sigma(X) < 0$. Let S be a (-2)–sphere, and let φ denote the induced automorphism of $H_2(X; \mathbb{Z})$ from the Dehn twist T_S. Let us calculate b_+^φ, b_-^φ and σ^φ. As described above, φ is given by $\varphi(x) = x + (x \cdot [S])[S]$, namely, φ acts on $H_2(X)$ as the reflection with respect to the orthogonal complement of the subspace generated by $[S]$. Here the orthogonal complement is with respect to the intersection form, and hence the complement contains a maximal-dimensional positive-definite subspace. Thus,

$$b_+^\varphi(X) = b_+(X), \quad b_-^\varphi(X) = b_-(X) - 1, \quad \sigma^\varphi(X) = \sigma(X) + 1.$$

From this we have that $\sigma^\varphi(X) \neq \sigma(X)/2$, since we supposed that $\sigma(X) < 0$ and hence $|\sigma(X)| \geq 8$ since $H_2(X; \mathbb{Z})$ is an even lattice. Moreover, we also have $-\sigma(X)/16 > b_+(X) - b_+^\varphi(X)$, again by $\sigma(X) < 0$. Now the claim of Theorem 1.1 for (-2)–spheres in X with $\sigma(X) < 0$ follows from Theorem 1.3 combined with Lemma 5.1.

Next, we consider the case that a $(+2)$–sphere S in X with $\sigma(X) < 0$ is given. Note that, as in the (-2)–sphere case above, $\varphi = (T_S)_*$ is the reflection with respect to the orthogonal complement of the subspace generated by $[S]$, but now $[S]$ has positive self-intersection. Thus,

$$b_+^\varphi(X) = b_+(X) - 1, \quad b_-^\varphi(X) = b_-(X), \quad \sigma^\varphi(X) = \sigma(X) - 1.$$

Again because $|\sigma(X)| \geq 8$, it follows that $\sigma^\varphi(X) \neq \sigma(X)/2$. Moreover,

$$b_+(X) - b_+^\varphi(X) = 1 < -\tfrac{1}{16}\sigma(X) + 1.$$

Now the desired claim follows from the "moreover" part of Theorem 1.3 combined with Lemma 5.1. □

Note that the "moreover" part of Theorem 1.3, which was derived from the refined Kato's inequality (Theorem 3.1), was effectively used to deal with $(+2)$–spheres in X with $\sigma(X) < 0$ in the above proof of Theorem 1.1.

Remark 5.2 For $X = K3$, the above proof of Theorem 1.1 gives an alternative proof of [9, Corollary 1.10] by Farb and Looijenga. They gave two different proofs of [9, Corollary 1.10], and one of them is based on Seiberg–Witten theory. We also used Seiberg–Witten theory, but in a slightly different manner: our proof uses Kato's result [17], rather than a result due to Bryan [6] used by Farb and Looijenga.

Kato's inequality (2) is useful to obtain a result for general spin 4–manifolds as in Theorem 1.1, not only $K3$. This is essentially because b_+ is replaced with $b_+ - b_+^\iota$ in Kato's inequality (2).

Example 5.3 Theorem 1.1 tells us that quite many spin 4–manifolds X have (many) nonrealizable order-2 subgroups of $\pi_0(\mathrm{Diff}(X))$. Indeed, there are many spin 4–manifolds that admit (± 2)–spheres. For example, $S^2 \times S^2$ admits both $(+2)$– and (-2)–spheres. A $K3$ surface, more generally, a spin complete intersection surface M admits a (-2)–sphere. Except for $M = S^2 \times S^2$ we have $\sigma(M) < 0$ for such M, and thus we may apply Theorem 1.1 to M and obtain a nonrealizable subgroup, and, of course, we may apply Theorem 1.1 also to the connected sum of M with any spin 4–manifold with $\sigma \leq 0$. (For the fact that M contains a (-2)–sphere, see the proof of Theorem 1.5 in [32, page 255]. In fact, one may find a Lagrangian sphere in M, whose self-intersection is always -2. See also [15, pages 23–24] for the topology of M, including when a complete intersection is spin.)

6 Proof of Theorem 1.2

Given an oriented closed simply connected smooth 4–manifold X, let $\mathrm{Aut}(H_2(X;\mathbb{Z}))$ denote the automorphism group of $H_2(X;\mathbb{Z})$ equipped with the intersection form. Since the space of maximal-dimensional positive-definite subspaces of $H^2(X;\mathbb{R})$ is known to be contractible, it makes sense whether a given $\varphi \in \mathrm{Aut}(H_2(X;\mathbb{Z}))$ preserves a given orientation of the positive part of $H^2(X;\mathbb{R})$. Let us recall the following classical fact:

Theorem 6.1 [5; 8; 23] *Let $\Gamma(K3) \subset \mathrm{Aut}(H_2(K3;\mathbb{Z}))$ denote the image of the natural map*

$$\pi_0(\mathrm{Diff}(K3)) \to \mathrm{Aut}(H_2(K3;\mathbb{Z})).$$

Then $\Gamma(K3)$ is the index-2 subgroup of $\mathrm{Aut}(H_2(K3;\mathbb{Z}))$ which consists of automorphisms that preserve a given orientation of $H^+(K3)$.

We shall also use:

Theorem 6.2 [4, Theorem 1.1] *There exists a (group-theoretic) section $s\colon \Gamma(K3) \to \pi_0(\mathrm{Diff}(K3))$ of the natural map $\pi_0(\mathrm{Diff}(K3)) \to \mathrm{Aut}(H_2(K3;\mathbb{Z}))$.*

Proof of Theorem 1.2 First, we recall a construction of a topological involution f_K on $K3$ (ie order-2 element of $\mathrm{Homeo}(K3)$) in [4, Section 3]. Let $-E_8$ denote the negative-definite E_8–manifold, namely, simply connected closed oriented topological 4–manifold whose intersection form is the negative-definite E_8–lattice. Let $f_S\colon S^2 \times S^2 \to S^2 \times S^2$ be the diffeomorphism defined by $(x,y) \mapsto (y,x)$. Since f_S has nonempty fixed-point set, which is of codimension-2, we can form an equivariant connected sum of three copies of $(S^2 \times S^2, f_S)$. Take a point x_0 of $3S^2 \times S^2$ outside the fixed-point set of $\#_3 f_S$, and attach two copies of $-E_8$ with $3S^2 \times S^2$ at x_0 and $(\#_3 f_S)(x_0)$. Now we have got a topological involution $\tilde{f}\colon 3S^2 \times S^2 \# 2(-E_8) \to 3S^2 \times S^2 \# 2(-E_8)$. Let $h\colon K3 \to 3S^2 \times S^2 \# 2(-E_8)$ be a homeomorphism obtained from Freedman theory [10], and define $f_K\colon K3\to K3$ by $f_K=h^{-1}\circ\tilde f\circ h$, which is a topological involution on $K3$.

Define a topological involution $f\colon X\to X$ by an equivariant connected sum $f=\#_m f_K\,\#_n\,f_S$ on $X=mK3\#nS^2\times S^2$ along fixed points, which acts on homology as follows. Recall that $H^+(S^2\times S^2)$ is generated by $[S^2\times\mathrm{pt}]+[\mathrm{pt}\times S^2]$ and $H^-(S^2\times S^2)$ is generated by $[S^2\times\mathrm{pt}]-[\mathrm{pt}\times S^2]$. Hence f_0 acts trivially on $H^+(S^2\times S^2)$, and acts on $H^-(S^2\times S^2)$ by (-1)–multiplication. Thus, $b_+^{f_S}(S^2\times S^2)=1$ and $b_-^{f_S}(S^2\times S^2)=0$, and hence

$$b_+^{f_K}(K3)=3,\qquad b_-^{f_K}(K3)=8,\qquad \sigma^{f_K}(K3)=-5,\tag{11}$$

$$b_+^{f}(X)=3m+n,\qquad b_-^{f}(X)=8m,\qquad \sigma^{f}(X)=-5m+n.\tag{12}$$

It follows from (11) that $(f_K)_*$ preserves an orientation of $H^+(K3)$, and hence $(f_K)_*$ lies in $\Gamma(K3)$ by Theorem 6.1. Using the section $s\colon\Gamma(K3)\to\pi_0(\mathrm{Diff}(K3))$ given in Theorem 6.2, set $\Phi=s((f_K)_*)$. Then Φ is a nontrivial element of $\pi_0(\mathrm{Diff}(K3))$ of order 2, and hence a representative $g_K\colon K3\to K3$ of Φ is a diffeomorphism whose square g_K^2 is smoothly isotopic to the identity. By smooth isotopy, we may take g_K such that g_K pointwise fixes a 4–disk in $K3$. Similarly, we may obtain a diffeomorphism g_S of $S^2\times S^2$ which is smoothly isotopic to f_S and which fixes a 4–disk pointwise. Fixing disjoint disks $D^4_1,\dots,D^4_{m+n}$ in S^4, form a diffeomorphism

$$g=\#_m g_K\,\#_n\,g_S\colon X\to X$$

by attaching g_K's and g_S's with (S^4,id_{S^4}) along the fixed disks of the g_K's and g_S's and $D^4_1,\dots,D^4_{m+n}$. It is clear that g is supported outside $S^4_0:=S^4\setminus\bigsqcup_{i=1}^{m+n}D^4_i$.

We claim that g^2 is smoothly isotopic to the identity. First, for a simply connected closed oriented 4–manifold M, let $\mathrm{Diff}(M,D^4)$ denote the group of diffeomorphisms fixing pointwise an embedded 4–disk D^4 in M. It follows from [12, Proposition 3.1] that we have an exact sequence

$$1\to\ker p\to\pi_0(\mathrm{Diff}(M,D^4))\xrightarrow{p}\pi_0(\mathrm{Diff}(M))\to 1,$$

where p is an obvious homomorphism and $\ker p$ is isomorphic to either $\mathbb{Z}/2$ or 0, which is generated by the mapping class of the Dehn twist τ_M along the 3–sphere parallel to the boundary. Set $\tau_K=\tau_{K3}$ and $\tau_S=\tau_{S^2\times S^2}$. Note that the relative mapping class $[\tau_K]_\partial$ is nontrivial in $\pi_0(\mathrm{Diff}(K3,D^4))$ by [19, Proposition 1.2], while $[\tau_S]_\partial$ is trivial since τ_S can be absorbed into the S^1–action on $S^2\times S^2$ given by the rotation of one S^2–component. Thus we obtain from $[g_K]^2=1$ and $[g_S]^2=1$ that $[g_K]^2_\partial=[\tau_K]_\partial\neq 1$ and $[g_S]^2_\partial=1$. Hence $[g]^2$ is the product of the Dehn twists along necks between m–copies of $K3$ and S^4_0.

On the other hand, let $\tau_{S^4_0}\colon S^4_0\to S^4_0$ be the diffeomorphism defined as the simultaneous Dehn twists near all ∂D^4_i. It follows from Lemma 6.3 below that $\tau_{S^4_0}$ is smoothly isotopic to the identity relative to ∂S^4_0. Thus, $[g]^2=[(\tau_{S^4_0}\#\mathrm{id}_{X\setminus S^4_0})\circ g^2]$. Note that $\tau_{S^4_0}$ restricted to the neck between each $K3$ and S^4_0 cancels the Dehn twist τ_K, but $\tau_{S^4_0}$ yields the Dehn twist on each of the necks between the $S^2\times S^2$'s and S^4_0. As a result, $[g]^2$ is the product of the Dehn twists along the necks between all of the $S^2 \times S^2$ and S^4_0. But each of these Dehn twists can be absorbed into the rotation of $S^2 \times S^2$ as above. Thus we get $[g]^2 = 1$.

Let G be the subgroup of $\pi_0(\mathrm{Diff}(X))$ generated by the mapping class $[g]$. We claim that this group G is the desired one. First, by construction, $g_* = f_*$ on $H_2(X;\mathbb{Z})$. By a theorem of Quinn [29] and Perron [27], this implies that g and f are topologically isotopic to each other. Thus the image G' of G under the map $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$ lifts to the order-2 subgroup of $\mathrm{Homeo}(X)$ generated by f. Since G' is a nontrivial group as g acts homology nontrivially, this proves the statement on G' in the theorem.

What remains to prove is that g is not homotopic to any finite-order diffeomorphism of X. However, using $g_* = f_*$, (12), and $m > 0$, it is straightforward to see that $\varphi = g_*$ violates the inequality (1) and that $\sigma^{\varphi}(X) \neq \sigma(X)/2$. Thus the desired assertion follows from Theorem 1.3. □

The following lemma and how to use it in the proof of Theorem 1.2 were suggested to the author by David Baraglia:

Lemma 6.3 *Let $N > 0$ and S^4_0 be an N–punctured 4–sphere, $S^4_0 = S^4 \setminus \bigsqcup_{i=1}^N D^4_i$. Let $\tau_{S^4_0}\colon S^4_0 \to S^4_0$ be the diffeomorphism defined as the simultaneous Dehn twists near all ∂D^4_i. Then $\tau_{S^4_0}$ is smoothly isotopic to the identity relative to ∂S^4_0.*

Proof Regard S^4 as the unit sphere of $\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^3$, and let S^1 act on S^4 by the standard rotation of the $\mathbb{R}^2$–component. The fixed-point set Σ of the S^1–action is given by $S(0 \oplus \mathbb{R}^3) \cong S^2$. We may assume that D^4_i are embedded disks in S^4 whose centers p_i are on Σ. Then the normal tangent space N_{p_i} of Σ at p_i in S^4 is acted on by S^1 as the standard rotation.

Pick a disk $\widehat{D}^4_i$ in S^4 that contains D^4_i such that $\widehat{D}^4_i \setminus D^4_i$ is diffeomorphic to the annulus $S^3 \times [0,1]$. Set $\widehat{S}^4_0 = S^4 \setminus \bigsqcup_{i=1}^N \widehat{D}^4_i$. The S^1–action on S^4 gives rise to an isotopy $\{\varphi_t\}_{t\in[0,1]} \subset \mathrm{Diff}(\widehat{S}^4_0)$ from $\mathrm{id}_{\widehat{S}^4_0}$ to itself such that $\{\varphi_t|_{\partial \widehat{D}^4_i}\}_t$ gives the homotopically nontrivial loop in $\mathrm{SO}(4) \subset \mathrm{Diff}(S^3) \cong \mathrm{Diff}(\partial \widehat{D}^4_i)$.

On the other hand, recall that the Dehn twist τ on $S^3 \times [0,1]$ is defined by $\tau(y,t) = (g(t)\cdot y, t)$, where $g\colon [0,1] \to \mathrm{SO}(4)$ is the homotopically nontrivial loop in $\mathrm{SO}(4)$. By definition, τ is isotopic to $\mathrm{id}_{S^3\times[0,1]}$ by an isotopy
$$\psi_t \in \mathrm{Diff}(S^3 \times [0,1], S^3 \times \{1\}),$$
through the diffeomorphism group fixing $S^3 \times \{1\}$ pointwise, such that $\{\psi_t|_{S^3\times\{0\}}\}_t$ gives the homotopically nontrivial loop in $\mathrm{Diff}^+(S^3)$.

Let ψ^i_t be copies of ψ_t, regarded as isotopies on $\widehat{D}^4_i \setminus D^4_i$. By gluing φ_t with ψ^i_t along $\bigsqcup_{i=1}^N \partial \widehat{D}^4_i$, we obtain an isotopy from $\tau_{S^4_0}$ to $\mathrm{id}_{S^4_0}$ relative to ∂S^4_0. □

Remark 6.4 For $X = K3$, the above proof of Theorem 1.2 gives a slight alternative proof of [4, Theorem 1.2], which used the adjunction inequality rather than Kato's result [17].

7 Additional remarks

7.1 Another kind of Dehn twist

A kind of Dehn twist different from that in Theorem 1.1 is the Dehn twist along an embedded annulus $S^3 \times [0,1]$ in a 4–manifold, defined using the generator of $\pi_1(\mathrm{SO}(4)) \cong \mathbb{Z}/2$, as described in the previous section. The square of the Dehn twist of this kind is smoothly isotopic to the identity. Recently, Kronheimer and Mrowka [19] proved that the Dehn twist τ along the neck of $K3 \# K3$ is not smoothly isotopic to the identity, and J Lin [22] showed that the extension of τ to $K3 \# K3 \# S^2 \times S^2$ by the identity of $S^2 \times S^2$ is also not smoothly isotopic to the identity. Hence it turns out that these Dehn twists generate order-2 subgroups of the mapping class groups. We remark that these subgroups also give counterexamples to the Nielsen realization problem:

Proposition 7.1 (i) *Let τ be the Dehn twist along the neck of $K3 \# K3$. Then the order-2 subgroup of $\pi_0(\mathrm{Diff}(K3 \# K3))$ generated by the mapping class of τ is not realized in* $\mathrm{Diff}(K3 \# K3)$.

(ii) *Let τ' be the extension of τ by the identity to $K3 \# K3 \# S^2 \times S^2$. Then the order-2 subgroup of $\pi_0(\mathrm{Diff}(K3 \# K3 \# S^2 \times S^2))$ generated by the mapping class of τ' is not realized in* $\mathrm{Diff}(K3 \# K3 \# S^2 \times S^2)$.

Proof By a result of Matumoto [24] and Ruberman [31], a simply connected closed spin 4–manifold with nonzero signature does not admit a homologically trivial locally linear involution. Since the Dehn twist τ is homologically trivial, the claim of the proposition immediately follows. □

7.2 Other variants of the realization problem

Given a manifold X of any dimension, one may also consider the realization problem for *infinite* subgroups of $\pi_0(\mathrm{Diff}(X))$ along $\mathrm{Diff}(X) \to \pi_0(\mathrm{Diff}(X))$ (or along $\mathrm{Diff}^+(X) \to \pi_0(\mathrm{Diff}^+(X))$ when $\mathrm{Diff}(X) \neq \mathrm{Diff}^+(X)$). To answer this problem negatively, several authors developed cohomological obstructions, which can be thought of as descendants of an argument started by Morita [25] for surfaces. In dimension 4, concrete results on the nonrealization were obtained in [14; 33] in this direction (see also [13]). Concretely, Giansiracusa, Kupers and Tshishiku [14] studied $X = K3$, and Tshishiku [33] considered manifolds of any dimension, but especially the result [33, Theorem 9.1] treated 4–manifolds whose fundamental groups are isomorphic to nontrivial lattices, which does not have overlap with 4–manifolds that we considered in this paper.

Another variant of the realization problem is about the realization along the natural map

$$\mathrm{Diff}^+(X) \to \pi_0(\mathrm{Homeo}^+(X))$$

for a subgroup of the image of this map. If X is a simply connected 4–manifold, the natural map $\pi_0(\mathrm{Homeo}^+(X)) \to \mathrm{Aut}(H_2(X;\mathbb{Z}))$ is isomorphic [27; 29], and hence this version of realization problem is equivalent to the realization along the map $\mathrm{Diff}^+(X) \to \mathrm{Aut}(H_2(X;\mathbb{Z}))$, which has been extensively studied by Nakamura [26], Baraglia [2; 3], and Lee [20; 21]. As noted in Section 1, Theorem 1.2 gives an alternative proof of [2, Proposition 1.2] about the realization of an involution of $H_2(X;\mathbb{Z})$.

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References

[1] **M F Atiyah**, **R Bott**, *A Lefschetz fixed point formula for elliptic complexes, II: Applications*, Ann. of Math. 88 (1968) 451–491 MR Zbl

[2] **D Baraglia**, *Obstructions to smooth group actions on 4–manifolds from families Seiberg–Witten theory*, Adv. Math. 354 (2019) art. id. 106730 MR Zbl

[3] **D Baraglia**, *Constraints on families of smooth 4–manifolds from Bauer–Furuta invariants*, Algebr. Geom. Topol. 21 (2021) 317–349 MR Zbl

[4] **D Baraglia**, **H Konno**, *A note on the Nielsen realization problem for $K3$ surfaces*, Proc. Amer. Math. Soc. 151 (2023) 4079–4087 MR Zbl

[5] **C Borcea**, *Diffeomorphisms of a $K3$ surface*, Math. Ann. 275 (1986) 1–4 MR Zbl

[6] **J Bryan**, *Seiberg–Witten theory and $\mathbb{Z}/2^p$ actions on spin 4–manifolds*, Math. Res. Lett. 5 (1998) 165–183 MR Zbl

[7] **T tom Dieck**, *Transformation groups and representation theory*, Lecture Notes in Math. 766, Springer (1979) MR Zbl

[8] **S K Donaldson**, *Polynomial invariants for smooth four-manifolds*, Topology 29 (1990) 257–315 MR Zbl

[9] **B Farb**, **E Looijenga**, *The Nielsen realization problem for K3 surfaces*, J. Differential Geom. 127 (2024) 505–549 MR

[10] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR Zbl

[11] **M Furuta**, *Monopole equation and the $\frac{11}{8}$–conjecture*, Math. Res. Lett. 8 (2001) 279–291 MR Zbl

[12] **J Giansiracusa**, *The stable mapping class group of simply connected 4–manifolds*, J. Reine Angew. Math. 617 (2008) 215–235 MR Zbl

[13] **J Giansiracusa**, *The diffeomorphism group of a $K3$ surface and Nielsen realization*, J. Lond. Math. Soc. 79 (2009) 701–718 MR Zbl

[14] **J Giansiracusa**, **A Kupers**, **B Tshishiku**, *Characteristic classes of bundles of $K3$ manifolds and the Nielsen realization problem*, Tunis. J. Math. 3 (2021) 75–92 MR Zbl

[15] **R E Gompf**, **A I Stipsicz**, *4–manifolds and Kirby calculus*, Graduate Studies in Math. 20, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[16] **F Hirzebruch**, **D Zagier**, *The Atiyah–Singer theorem and elementary number theory*, Math. Lect. Ser. 3, Publish or Perish, Boston, MA (1974) MR Zbl

[17] **Y Kato**, *Nonsmoothable actions of* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *on spin four-manifolds*, Topology Appl. 307 (2022) art. id. 107868 MR Zbl

[18] **S P Kerckhoff**, *The Nielsen realization problem*, Ann. of Math. 117 (1983) 235–265 MR Zbl

[19] **P B Kronheimer**, **T S Mrowka**, *The Dehn twist on a sum of two* $K3$ *surfaces*, Math. Res. Lett. 27 (2020) 1767–1783 MR Zbl

[20] **S E B Lee**, *The Nielsen realization problem for high degree del Pezzo surfaces*, preprint (2021) arXiv 2112.13500

[21] **S E B Lee**, *Isotopy classes of involutions of del Pezzo surfaces*, Adv. Math. 426 (2023) art. id. 109086 MR Zbl

[22] **J Lin**, *Isotopy of the Dehn twist on* $K3\#K3$ *after a single stabilization*, Geom. Topol. 27 (2023) 1987–2012 MR Zbl

[23] **T Matumoto**, *On diffeomorphisms of a* $K3$ *surface*, from "Algebraic and topological theories" (M Nagata, S Araki, A Hattori, editors), Kinokuniya, Tokyo (1986) 616–621 MR Zbl

[24] **T Matumoto**, *Homologically trivial smooth involutions on* $K3$ *surfaces*, from "Aspects of low-dimensional manifolds" (Y Matsumoto, S Morita, editors), Adv. Stud. Pure Math. 20, Kinokuniya, Tokyo (1992) 365–376 MR Zbl

[25] **S Morita**, *Characteristic classes of surface bundles*, Invent. Math. 90 (1987) 551–577 MR Zbl

[26] **N Nakamura**, *Smoothability of* $\mathbb{Z}\times\mathbb{Z}$*–actions on* 4*–manifolds*, Proc. Amer. Math. Soc. 138 (2010) 2973–2978 MR Zbl

[27] **B Perron**, *Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique*, Topology 25 (1986) 381–397 MR Zbl

[28] **A Pfister**, **S Stolz**, *On the level of projective spaces*, Comment. Math. Helv. 62 (1987) 286–291 MR Zbl

[29] **F Quinn**, *Isotopy of* 4*–manifolds*, J. Differential Geom. 24 (1986) 343–372 MR Zbl

[30] **F Raymond**, **L L Scott**, *Failure of Nielsen's theorem in higher dimensions*, Arch. Math. (Basel) 29 (1977) 643–654 MR Zbl

[31] **D Ruberman**, *Involutions on spin* 4*–manifolds*, Proc. Amer. Math. Soc. 123 (1995) 593–596 MR Zbl

[32] **P Seidel**, *Lectures on four-dimensional Dehn twists*, from "Symplectic 4–manifolds and algebraic surfaces" (F Catanese, G Tian, editors), Lecture Notes in Math. 1938, Springer (2008) 231–267 MR Zbl

[33] **B Tshishiku**, *Cohomological obstructions to Nielsen realization*, J. Topol. 8 (2015) 352–376 MR Zbl

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Sequential parametrized topological complexity and related invariants

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Parametrized motion planning algorithms have a high degree of universality and flexibility; they generate the motion of a robotic system under a variety of external conditions. The latter are viewed as parameters and constitute part of the input of the algorithm. The concept of sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ is a measure of the complexity of such algorithms. It was studied by Cohen, Farber and Weinberger (2021, 2022) for $r = 2$ and by Farber and Paul (2022) for $r \geq 2$. We analyze the dependence of the complexity $\mathsf{TC}_r[p\colon E \to B]$ on an initial bundle with structure group G and on its fibre X viewed as a G–space. Our main results estimate $\mathsf{TC}_r[p\colon E \to B]$ in terms of certain invariants of the bundle and the action on the fibre. Moreover, we also obtain estimates depending on the base and the fibre. Finally, we develop a calculus of sectional categories featuring a new invariant $\mathsf{secat}_f[p\colon E \to B]$ which plays an important role in the study of sectional category of towers of fibrations.

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1 Introduction

Motion planning algorithms of robotics control autonomous robots in engineering; see [LaValle 2006]. A motion planning algorithm takes as input the initial and the desired states of the system and produces as output a motion of the system starting at the initial states and ending at the desired states. A robot is "told" where it needs to go and the execution of this task, including selection of a specific route of motion, is made by the robot itself, ie by the robot's motion planning algorithm. Typically it is understood that the external conditions (such as the positions of the obstacles and the geometry of the enclosing domain) are known and are constant during the motion.

In [Cohen et al. 2021; 2022], motion planning algorithms of a new type were analyzed. These are *parametrized motion planning algorithms*, which, besides the initial and desired states, take as input the parameters characterizing the external conditions. The output of a parametrized motion planning algorithm is a continuous motion of the system from the initial to the desired state, respecting the given external conditions. The papers [Cohen et al. 2021; 2022] laid out the new formalism and analyzed in full detail the problem of moving an arbitrary number n of robots in the domain with m a priori unknown obstacles.

The recent paper [Farber and Paul 2022] developed a generalization where the robot must perform *a sequence* of tasks. The topological complexity of such an algorithm is called *sequential parametrized topological complexity* $\mathsf{TC}_r[p\colon E \to B]$, where $r = 2, 3, \dots$, and the case $r = 2$ corresponds to the situations analyzed in [Cohen et al. 2021; 2022]. Formally, $\mathsf{TC}_r[p\colon E \to B]$ is an integer associated with a fibration $p\colon E \to B$ where the points of the base $b \in B$ parametrize the external conditions (for example, positions of the obstacles) and for each $b \in B$ the fibre $X_b = p^{-1}(b) \subset E$ is the space of all admissible configurations of the system under the external conditions b. To make the present work independent, we include the definition of the concept $\mathsf{TC}_r[p\colon E \to B]$ and its major properties in Section 2.

In this paper we further analyze the invariant $\mathsf{TC}_r[p\colon E \to B]$ trying to understand its dependence on classical invariants of the initial bundle $p\colon E \to B$; in particular, on its base B and on its fibre X. As with all such invariants, exact calculation is generally hard and the development of lower and upper bounds is an essential part of the subject. This is the focus of our work.

We first show that, if the bundle $p\colon E \to B$ has structure group G with fibre X a G–space, then the equivariant sequential topological complexity of X — developed in [Bayeh and Sarkar 2020; Colman and Grant 2012] — serves as an upper bound for $\mathsf{TC}_r[p\colon E \to B]$; see (9). The case when G acts freely on X is especially interesting and leads to several somewhat surprising estimates. But using equivariant topological complexity as an upper bound is fraught with danger since it can be infinite in what appear to be innocuous situations.

As an alternative we develop the notion of *weak sequential equivariant complexity*, denoted by $\mathsf{TC}^w_{r,G}(X)$, and its variant $\mathsf{TC}^w_{r,G}(X; P)$ which we are tempted (but loathe) to call *weak sequential equivariant complexity with coefficients P* (see Section 7). We will give several examples showing that these invariants are finite even when equivariant topological complexity is infinite, so they offer the opportunity for effective estimation in many situations. Indeed, our main result Theorem 8.1 gives lower and upper bounds for $\mathsf{TC}_r[p\colon E \to B]$ in terms of these invariants. To state it one needs to recall the invariant

$$G\text{–}\mathsf{cat}[p\colon E \to B]$$

introduced by I M James [1978, page 342]. It is defined as the smallest integer $k \geq 0$ such that the base B admits an open cover $B = U_0 \cup U_1 \cup \cdots \cup U_k$ with the property that over each set U_i the bundle E is trivial *as a G–bundle*. Clearly, G–$\mathsf{cat}[p\colon E \to B]$ equals the sectional category $\mathsf{secat}[\tau\colon P \to B]$ of the associated principal bundle that constructs $p\colon E \to B$. In general,

$$G\text{–}\mathsf{cat}[p\colon E \to B] \leq \mathsf{cat}(B) \leq \dim B \tag{1}$$

and if the group G is 2–connected (as is the case for simply connected compact Lie groups for instance) then we can say

$$G\text{–}\mathsf{cat}[p\colon E \to B] \leq \left\lceil \tfrac{1}{4}(\dim B - 3) \right\rceil \tag{2}$$

as follows by applying [Schwartz 1962, Theorem 5] to $\mathsf{secat}[\tau\colon P \to B]$. If the structure group G is discrete then instead of (1) one has a stronger inequality

$$G\text{–}\mathsf{cat}[p\colon E \to B] \leq \mathsf{cat}_1(B), \tag{3}$$

where $\mathsf{cat}_1(B)$ is the sectional category of the universal cover $\widetilde{B} \to B$. Our main result (Theorem 8.1) then is the following.

Theorem *For a locally trivial bundle $p\colon E = X \times_G P \to B = P/G$ one has the inequalities*

$$\mathsf{TC}^w_{r,G}(X; P) \leq \mathsf{TC}_r[p\colon E \to B] \leq G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{r,G}(X). \tag{4}$$

Note that the first summand in the right-hand side of (4) is independent of r and is bounded above by the Lusternik–Schnirelmann category of the base B; the second term is the weak equivariant sequential topological complexity of the fibre X. In our view, this estimate gets at the heart of the matter. After all, what *is* a bundle? It is just a principal bundle together with an action of the structure group on the fibre and our upper bound is expressed exactly in numerical quantities derived from these objects. In Example 8.4 we shall see that the right-hand side can be an equality, so at least in some cases the upper bound can be sharp. Such a posteriori knowledge then warrants a deeper study of the invariants $\mathsf{TC}^w_{r,G}(X)$ and $\mathsf{TC}^w_{r,G}(X; P)$ and we hope the present work elicits this.

Beyond defining and applying the new invariants $\mathsf{TC}^w_{r,G}(X)$ and $\mathsf{TC}^w_{r,G}(X; P)$, in Sections 4, 5 and 6 we develop a calculus of sectional categories, including a new notion denoted by $\mathsf{secat}_f[p\colon E \to B]$ where $f\colon B \to C$ is a continuous map. It is this notion that allows us to estimate the sectional category of towers of fibrations which serves as the crucial technical tool in the proof of our main results. We believe that $\mathsf{secat}_f[p\colon E \to B]$ holds independent interest and should find application in many situations orbiting the twin galaxies of Lusternik–Schnirelmann category and topological complexity.

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2 The concept of sequential parametrized topological complexity

In this section we recall the notion of sequential parametrized topological complexity introduced in [Farber and Paul 2022]. It is a generalization of the concept of topological complexity [Farber 2003] and its parametrized version [Cohen et al. 2021].

Let $p\colon E \to B$ be a Hurewicz fibration with fibre X. Fix an integer $r \geq 2$ and set

$$E^r_B = \{(e_1, \dots, e_r) \in E^r \mid p(e_1) = \cdots = p(e_r)\}.$$

The symbol $I = [0, 1]$ denotes the unit interval. Let $E^I_B \subset E^I$ be the space of all paths $\gamma\colon I \to E$ such that the path $p \circ \gamma\colon I \to B$ is constant. Fix r points

$$0 \leq t_1 < t_2 < \cdots < t_r \leq 1$$

and consider the evaluation map

$$\Pi_r\colon E^I_B \to E^r_B, \quad \Pi_r(\gamma) = (\gamma(t_1), \gamma(t_2), \dots, \gamma(t_r)). \tag{5}$$

Π_r is a Hurewicz fibration; see [Cohen et al. 2022, Appendix]. The fibre of Π_r is $(\Omega X)^{r-1}$, the Cartesian $(r-1)^{\text{st}}$ power of the based loop space ΩX. A section $s\colon E^r_B \to E^I_B$ of the fibration Π_r can be interpreted as *a parametrized motion planning algorithm*, ie a function which assigns to every sequence of points $(e_1, e_2, \dots, e_r) \in E^r_B$ a continuous path $\gamma\colon I \to E$ (representing motion of the system) satisfying $\gamma(t_i) = e_i$ for every $i = 1, 2, \dots, r$ and such that the path $p \circ \gamma\colon I \to B$ is constant. The latter condition means that the system moves under *constant external conditions* (such as positions of the obstacles etc).

Typically, the fibration Π_r does not admit continuous sections; see [Farber and Weinberger 2023a, Corollary 1 and Lemma 1], which deal with the case $r = 2$; when $r > 2$ the arguments are similar. Therefore the motion planning algorithms are necessarily discontinuous in most situations.

The following definition [Farber and Paul 2022] gives a measure of complexity of sequential parametrized motion planning algorithms.

Definition 2.1 The r^{th} *sequential parametrized topological complexity* of the fibration $p\colon E \to B$, denoted by $\mathsf{TC}_r[p\colon E \to B]$, is defined as the sectional category of the fibration Π_r, ie

$$\mathsf{TC}_r[p\colon E \to B] := \mathsf{secat}(\Pi_r). \tag{6}$$

In more detail, $\mathsf{TC}_r[p\colon E \to B]$ is the minimal integer k such that there is an open cover $\{U_0, U_1, \dots, U_k\}$ of E^r_B with the property that each open set U_i admits a continuous section $s_i : U_i \to E^I_B$ of Π_r.

Under some mild assumptions, instead of open covers one can consider totally general partitions:

Proposition 2.2 [Farber and Paul 2022, Proposition 3.6] *Let E and B be metrizable separable ANRs and let $p\colon E \to B$ be a locally trivial fibration. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ equals the smallest integer n such that E^r_B admits a partition*

$$E^r_B = F_0 \sqcup F_1 \sqcup \cdots \sqcup F_n, \qquad F_i \cap F_j = \varnothing \quad \text{for } i \neq j,$$

with the property that on each set F_i there exists a continuous section $s_i\colon F_i \to E^I_B$ of Π_r.

If two fibrations $p\colon E \to B$ and $p'\colon E' \to B$ are fibrewise homotopy equivalent then

$$\mathsf{TC}_r[p\colon E \to B] = \mathsf{TC}_r[p'\colon E' \to B];$$

see [Farber and Paul 2022, Corollary 4.2].

The following upper bound is a reformulation of [Farber and Paul 2022, Proposition 6.1]:

Proposition 2.3 *Let $p\colon E \to B$ be a locally trivial fibration with fibre X, where E, B and X are CW–complexes. Assume that the fibre X is k–connected, where $k \geq 0$. Then*

$$\mathsf{TC}_r[p\colon E \to B] \leq \left\lceil \frac{r \dim X + \dim B - k}{1+k} \right\rceil. \tag{7}$$

We refer the reader to [Farber and Paul 2022] for proofs and further detail.

3 Relation with the equivariant sequential topological complexity

In this section we show that $\mathsf{TC}_r[p\colon E \to B]$ admits as an upper bound the sequential equivariant topological complexity [Bayeh and Sarkar 2020; Colman and Grant 2012] of the fibre X. This leads to simple estimates in terms of the dimension of the fibre in the case when the structure group G of the fibration acts freely on X; see Lemma 3.5 and Corollary 3.6.

3.1 Equivariant topological complexity

We shall recall a sequential analogue of the notion of equivariant topological complexity introduced by M Bayeh and S Sarkar [2020]; it generalizes the concept of equivariant topological complexity originally introduced and studied by H Colman and M Grant [2012].

Let G be a topological group acting on a topological space X from the left. The papers [Bayeh and Sarkar 2020; Colman and Grant 2012] require G to be compact but we do not impose this assumption at this stage.

The symbol X^I denotes the space of all continuous paths $\gamma\colon I \to X$ where $I = [0,1]$ is equipped with the compact–open topology. The group G acts naturally on X^I, where $(g\gamma)(t) = g\gamma(t)$ for $t \in I$.

Fix an integer $r \geq 2$ and consider the Cartesian power $X^r = X \times X \times \cdots \times X$ (r times). We shall consider the diagonal action of G on X^r.

Fix r points $0 = t_1 < t_2 < \cdots < t_r = 1$ in the unit interval $I = [0, 1]$ and consider the evaluation map

$$\rho_r \colon X^I \to X^r, \tag{8}$$

where $\rho_r(\gamma) = (\gamma(t_1), \ldots, \gamma(t_r))$. Clearly, ρ_r a G–equivariant map.

Definition 3.1 For a path-connected G–space X, we denote by $\mathsf{TC}_{r,G}(X)$ the smallest integer $k \geq 0$ such that the Cartesian power $X^r = X \times X \times \cdots \times X$ (r times) admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ with the following properties: each set U_i is G–invariant and admits a continuous G–equivariant section $s_i \colon U_i \to X^I$ of the fibration ρ_r. If no such cover exists we set $\mathsf{TC}_{r,G}(X) = \infty$.

The invariant $\mathsf{TC}_{2,G}(X)$ coincides with the equivariant topological complexity $\mathsf{TC}_G(X)$ of Colman and Grant [2012].

It is obvious from Definition 3.1 that

$$\mathsf{TC}_r(X) \leq \mathsf{TC}_{r,G}(X),$$

where $\mathsf{TC}_r(X)$ is the sequential topological complexity of X introduced by Rudyak [2010].

An alternative definition of $\mathsf{TC}_{r,G}(X)$ is obtained as follows (compare [Farber and Paul 2022, Lemma 3.5]). Let K be a path-connected locally compact metrizable space and let $k_1, k_2, \ldots, k_r \in K$ be a set of r pairwise distinct points. Consider the set X^K of continuous maps $\alpha \colon K \to X$ equipped with compact–open topology. The evaluation map

$$\rho_r^K \colon X^K \to X^r,$$

where $\Pi_r^K(\alpha) = (\alpha(k_1), \ldots, \alpha(k_r)) \in X^r$, is continuous and G–equivariant, where we view X^r with the diagonal action of G.

Lemma 3.2 *For any path-connected locally compact metrizable space K, the number $\mathsf{TC}_{r,G}(X)$ equals the smallest integer $k \geq 0$ such that the Cartesian power X^r admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ with the following properties: each set U_i is G–invariant and admits a continuous G–equivariant section $s_i \colon U_i \to X^K$ of ρ_r^K.*

Proof Consider the commutative diagram

$$\begin{tikzcd} X^I \arrow[rr, shift left, "F'"] \arrow[dr, "\rho_r"'] & & X^K \arrow[ll, shift left, "F"] \arrow[dl, "\rho_r^K"] \\ & X^r & \end{tikzcd}$$

where the maps $F \colon X^K \to X^I$ and $F' \colon X^I \to X^K$ are defined as follows. Fix a path $\gamma \colon I \to K$ satisfying $\gamma(t_i) = k_i$ for all $i = 1, \ldots, r$. Then $F(\alpha) = \alpha \circ \gamma \colon I \to X$, where $\alpha \in X^K$.

To define the map $F'\colon X^I \to X^K$ we first construct a continuous function $f: K \to I$ satisfying $f(k_i) = t_i$ for all $i = 1, \dots, r$. Applying the Tietze extension theorem we find continuous functions $\psi_j : K \to [0,1]$ with $\psi_j(t_i) = \delta_{ij}$ where $j = 1, \dots, r$. Then the function $f = \min\{1, \sum_{i=1}^r t_i \psi_r\}$, $f: K \to I$, has the required properties. The map $F'\colon X^I \to X^K$ is defined by $F'(\alpha) = \alpha \circ f$ where $\alpha \in X^I$.

Clearly the maps F and F' are G–equivariant. For an open G–invariant subset $U \subset X^r$ any G–equivariant section $s\colon U \to X^I$ of ρ_r defines the G–equivariant section $s' = F' \circ s\colon U \to X^K$ of ρ_r^K. And vice versa, any G–equivariant section $s'\colon U \to X^K$ defines $s = F \circ s'\colon U \to X^I$, an equivariant section of ρ_r. □

Yet another equivalent characterization of $\mathsf{TC}_{r,G}(X)$ is given by the following (see [Bayeh and Sarkar 2020]):

Lemma 3.3 *For a G–space X and $r \geq 2$ the integer $\mathsf{TC}_{r,G}(X)$ equals the smallest $k \geq 0$ such that X^r admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ by G–invariant open sets U_i with the property that each inclusion $U_j \to X^r$ is G–homotopic to a map with values in the diagonal $X \subset X^r$.*

Now we can state our result relating the sequential parametrized topological complexity of a fibration with the equivariant sequential topological complexity of the fibre:

Theorem 3.4 *Consider a locally trivial bundle $p\colon E \to B$ with path-connected fibre X and structure group G. Let $\tau\colon P \to B$ be a G–principal bundle such that $p\colon E \to B$ coincides with the associated bundle $p\colon E = X \times_G P = (X \times P)/G \to P/G = B$. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ is bounded above by $\mathsf{TC}_{r,G}(X)$, ie*

$$\mathsf{TC}_r[p\colon E \to B] \leq \mathsf{TC}_{r,G}(X). \tag{9}$$

Note that the right-hand side of inequality (9) depends only on the fibre X viewed as a G–space, where G is the structure group of the bundle.

Proof First we note that there exists the commutative diagram

$$\begin{tikzcd}
X^I \times_G P \arrow[r, "\alpha"] \arrow[d, "\rho_r \times_G 1"'] & E_B^I \arrow[d, "\Pi_r"] \\
X^r \times_G P \arrow[r, "\beta"'] & E_B^r
\end{tikzcd}$$

where α and β are homeomorphisms. Therefore,

$$\mathsf{TC}_r[p\colon E \to B] = \mathsf{secat}[\Pi_r : E_B^I \to E_B^r] = \mathsf{secat}[\rho_r \times_G 1\colon X^I \times_G P \to X^r \times_G P].$$

For $k = \mathsf{TC}_{r,G}(X)$ let $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ be an open cover as in Definition 3.1. Consider the sets

$$W_i = (U_i \times P)/G \subset (X^r \times P)/G.$$

They are open and cover $(X^r \times P)/G$. Any G–equivariant section $s_i\colon U_i \to X^I$ of the fibration ρ_r obviously defines the section $\sigma_i\colon W_i \to (X^I \times P)/G$ of the orbit spaces; here σ_i is the map induced by $s_i \times 1_P$ on the spaces of orbits. This shows that

$$\mathsf{TC}_r[p\colon E \to B] = \mathsf{secat}[\rho_r \times_G 1\colon X^I \times_G P \to X^r \times_G P] \le k. \qquad \square$$

As mentioned in [Bayeh and Sarkar 2020; Colman and Grant 2012], in some cases the number $\mathsf{TC}_{r,G}(X)$ is infinite. In particular, one has $\mathsf{TC}_{r,G}(X) = \infty$ if for a subgroup $H \subset G$ the fixed-point set X^H is not path-connected. In such situations the upper bound (9) becomes meaningless. We discuss below situations when the number $\mathsf{TC}_{r,G}(X)$ is finite and admits useful upper bounds.

The following lemma uses the notion of G–equivariant homotopy lifting property (G–HLP) applied to a map $q\colon X \to X/G$. This property means that the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] & X \arrow[d, "q"] \\ Y \times I \arrow[r, "F"'] & X/G \end{tikzcd}$$

where X and Y are separable metric spaces and the map $f\colon Y \to X$ is G–equivariant, can be completed by a G–equivariant map $H\colon Y \times I \to X$ extending f and such that $q \circ H = F$. A theorem of R Palais (see [Bredon 1972, Theorem II.7.3]) states that this property is automatically satisfied for free actions of compact Lie groups.

Lemma 3.5 *Consider a locally trivial bundle $p\colon E \to B$ with fibre X (a path-connected separable metric space) and structure group G. Assume that the group G acts freely on X and, moreover, that the quotient map $q_r\colon X^r \to X^r/G$ possesses the G–HLP. Then*

$$\mathsf{TC}_r[p\colon E \to B] \le \mathsf{TC}_{r,G}(X) \le \mathsf{cat}(X^r/G) \le \dim(X^r/G).$$

Proof In view of Theorem 3.4 we only need to prove the inequality $\mathsf{TC}_{r,G}(X) \le \mathsf{cat}(X^r/G)$. Consider an open covering $X^r/G = V_0 \cup V_1 \cup \cdots \cup V_k$, where $k = \mathsf{cat}(X^r/G)$ and each inclusion $U_i \subset X^r/G$ is homotopic to the constant map into a point $x_0 \in X/G \subset X^r/G$; here $X \subset X^r$ is the diagonal. By our assumption, the projection $q_r\colon X^r \to X^r/G$ has the G–homotopy lifting property. The sets $U_i = q_r^{-1}(V_i) \subset X^r$ are G–invariant, where $i = 0, 1, \dots, k$, and applying the G–homotopy lifting property to the homotopy of V_i to x_0 we find a homotopy $h_t^i\colon U_i \to X^r$ (where $t \in [0,1]$ and $i \in \{0, 1, \dots, r\}$) such that h_0^i is the inclusion $U_i \to X^r$, each map h_t^i is G–equivariant and $h_1^i(U_i) \subset X \subset X^r$. Applying Lemma 3.3 we obtain $\mathsf{TC}_{r,G}(X) \le k$. $\square$

Corollary 3.6 *Consider a locally trivial bundle $p\colon E \to B$ with fibre X (which is a path-connected separable metric space) and a compact Lie group G acting freely on X, as the structure group. Then for any $r \ge 2$,*

$$\mathsf{TC}_r[p\colon E \to B] \le \mathsf{cat}(X^r/G) \le r \dim X - \dim G. \tag{10}$$

Proof First we note that due to the theorem of Palais [Bredon 1972, Theorem II.7.3] the assumptions of Lemma 3.5 are satisfied. We are only left to note that $\dim X^r/G = \dim X^r - \dim G \le r \dim X - \dim G$; see [Palais 1960, Corollary 1.7.32]. □

One can use Lemma 3.5 to give an alternative proof of [Farber and Paul 2022, Proposition 3.3] — see also [Cohen et al. 2021, Proposition 4.3] — with some minor additional assumptions:

Corollary 3.7 *Let $G \to P \xrightarrow{\tau} B$ be a principal bundle, where G is a path-connected topological group which has the topology of a separable metric space. Then*

$$\mathsf{TC}_r[\tau\colon P \to B] = \mathsf{cat}(G^{r-1}) \quad \textit{for any } r \ge 2.$$

Proof By [Farber and Paul 2022, Section 3] we know that $\mathsf{TC}_r[\tau\colon P \to B] \ge \mathsf{TC}_r(G) = \mathsf{cat}(G^{r-1})$. We view the fibre G as acting on itself by left translations and acting diagonally on G^r. The quotient map $q_r\colon G^r \to G^r/G$ admits a section $s\colon G^r/G \to G^r$, given by

$$s(g_1, g_2, \dots, g_r) = (e, g_1^{-1}g_2, g_1^{-1}g_3, \dots, g_1^{-1}g_r).$$

Therefore, we explicitly obtain a G–homeomorphism $G^r \cong G^r/G \times G$, so q_r is a trivial bundle $G^r/G \times G \to G^r/G$ and, as such, has the G–HLP. Lemma 3.5 then applies and gives the upper bound $\mathsf{TC}_r[\tau\colon P \to B] \le \mathsf{cat}(G^r/G) = \mathsf{cat}(G^{r-1})$. Comparing, we see that both bounds are in fact equalities. □

4 Calculus of sectional categories

In this section we introduce a new invariant $\mathsf{secat}_f[p\colon E \to B]$ which generalizes the concept of sectional category of a fibration. This invariant plays a role in estimating sectional category of towers of fibrations, see Theorem 5.1.

Let $p\colon E \to B$ be a fibration and let $f\colon B \to C$ be a continuous map.

Definition 4.1 We define the invariant

$$\mathsf{secat}_f[p\colon E \to B]$$

to be the smallest integer $k \ge 0$ such that C admits a family of open subsets $U_0, U_1, \dots, U_k$ with the properties

(a) $U_0 \cup U_1 \cup \dots \cup U_k \supset f(B)$ or, equivalently, $B = \bigcup_{i=1}^k f^{-1}(U_i)$;

(b) the fibration $p\colon E \to B$ admits a continuous section over each open set $f^{-1}(U_i)$ for $i = 0, 1, \dots, k$.

We set $\mathsf{secat}_f[p\colon E \to B] = \infty$ if no such family exists.

Open sets of the form $f^{-1}(U) \subset B$, where $U \subset C$, can be called *f–saturated*. Definition 4.1 can be rephrased as dealing with covers of the base B by f–saturated open sets admitting continuous sections of the fibration $p\colon E \to B$.

4.1 Finiteness

The following lemma summarizes information about finiteness of the invariant $\mathsf{secat}_f[p\colon E \to B]$.

Lemma 4.2 *Let $p\colon E \to B$ be a fibration and let $f : B \to C$ be a continuous map.*

(A) *If $\mathsf{secat}[p\colon p^{-1}f^{-1}(x) \to f^{-1}(x)] > 0$ for some $x \in f(B) \subset C$ then $\mathsf{secat}_f[p\colon E \to B] = \infty$.*

(B) *If B is compact and every point $x \in f(B) \subset C$ has an open neighbourhood $U \subset C$ such that $\mathsf{secat}[p\colon p^{-1}f^{-1}(U) \to f^{-1}(U)] = 0$ then $\mathsf{secat}_f[p\colon E \to B] < \infty$.*

Proof Under assumption (A) there is no open set $U \subset C$ containing x with $f^{-1}(U)$ having a continuous section. Statement (B) is obvious. □

In our applications we shall typically have the map $f : B \to C$ be surjective, and more specifically, it will often be the quotient map with respect to a group action. However, it is convenient to make no additional assumptions at this stage.

4.2 Dependence on f

In the special case when the map $f : B \to C = B$ is the identity map, the number $\mathsf{secat}_f[p\colon E \to B]$ turns into the usual sectional category $\mathsf{secat}[p\colon E \to B]$. In general, obviously,

$$\mathsf{secat}[p\colon E \to B] \le \mathsf{secat}_f[p\colon E \to B] \tag{11}$$

and

$$\mathsf{secat}[p\colon E \to B] = \mathsf{secat}_f[p\colon E \to B] \tag{12}$$

assuming that $\mathsf{secat}[p\colon E \to B] = 0$.

Moreover, for $B \xrightarrow{f} C \xrightarrow{g} C'$ one clearly has

$$\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_{gf}[p\colon E \to B]. \tag{13}$$

Lemma 4.3 *Let $p\colon E \to B$ be a fibration and let $f : B \to C$ and $f' : B \to C'$ be two continuous maps.*

(a) *If there is a continuous map $h\colon C \to C'$ such that $f' = h \circ f$, then*

$$\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_{f'}[p\colon E \to B].$$

(b) *Moreover, if the restriction of $h\colon C \to C'$ induces a homeomorphism $f(B) \to f'(B)$, then*

$$\mathsf{secat}_f[p\colon E \to B] = \mathsf{secat}_{f'}[p\colon E \to B].$$

Proof Statement (a) follows from inequality (13). To prove (b) assume that $U \subset C$ is an open subset with the property that $f^{-1}(U)$ admits a section of p. Then

$$h(U \cap f(B)) = U' \subset f'(B)$$

is an open subset of $f'(B)$ and hence there exists an open subset $V \subset C'$ with $V \cap f'(B) = U'$. Then $f'^{-1}(V) = f^{-1}(U)$ admits a section of p. Thus any family of open sets $U_0 \cup U_1 \cup \dots \cup U_k \supset f(B)$

such that $f^{-1}(U_i)$ admits a section of p determines a family of open subsets of the same cardinality, $V_0 \cup V_1 \cup \cdots \cup V_k \supset f'(B)$ with the preimages $f'^{-1}(V_j)$ admitting sections of p. This shows the inverse inequality $\mathsf{secat}_f[p\colon E \to B] \geq \mathsf{secat}_{f'}[p\colon E \to B]$. □

4.3 Induced fibrations

Lemma 4.4 *Assume that a fibration $p\colon E' \to B'$ is induced from the fibration $p\colon E \to B$ via the map $\alpha\colon B' \to B$ as shown on the diagram*

$$\begin{tikzcd} E' \arrow[r, "\beta"] \arrow[d, "p'"'] & E \arrow[d, "p"] & \\ B' \arrow[r, "\alpha"'] & B \arrow[r, "f"'] & C \end{tikzcd}$$

For $f\colon B \to C$ set $f' = f \circ \alpha$. Then

$$\mathsf{secat}_{f'}[p'\colon E' \to B'] \leq \mathsf{secat}_f[p\colon E \to B].$$

Proof Assuming that there is a continuous section $s\colon f^{-1}(U) \to E$ of $p\colon E \to B$, for $U \subset C$ open, define $\phi\colon f'^{-1}(U) \to E$ by $\phi = s \circ \alpha$. Then we have $p \circ \phi = \alpha$ and by the pullback property there is a continuous map $s'\colon f'^{-1}(U) \to E'$ with $p' \circ s' =$ inclusion, ie s' is a section of p'. Since $f'(B) \subset f(B) \subset C$, we see that the statement of the lemma follows. □

Lemma 4.5 (maps of fibrations) *If for two fibrations $p\colon E \to B$ and $p'\colon E' \to B$ over the same base B there exists a map $\phi\colon E \to E'$ such that the diagram*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

commutes up to homotopy, then $\mathsf{secat}_f[p'\colon E' \to B] \leq \mathsf{secat}_f[p\colon E \to B]$.

Proof If $U \subset C$ is such that p admits a continuous section s over $f^{-1}(U) \subset B$ then p' admits a homotopy section $\phi \circ s$ over the same subset. Since p' satisfies the homotopy lifting property, the homotopy section can be made a genuine section. The statement now follows from the definition. □

Lemma 4.6 *Suppose that for two fibrations $p\colon E \to B$ and $p'\colon E' \to B'$ there exist continuous maps G, α, β and $\hat{\alpha}$ shown on the diagram*

$$\begin{tikzcd} & E' \arrow[r, "G"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B \arrow[r, "\alpha"'] \arrow[d, "f"'] & B' \arrow[r, "\beta"'] \arrow[d, "f'"] & B \\ C \arrow[r, "\hat{\alpha}"'] & C' & \end{tikzcd}$$

such that the bottom left square is commutative, the upper right square is homotopy commutative and $\beta \circ \alpha\colon B \to B$ is homotopic to the identity $\mathrm{Id}_B\colon B \to B$. Then $\mathsf{secat}_f[p\colon E \to B] \leq \mathsf{secat}_{f'}[p'\colon E' \to B']$.

Proof Consider the fibration $q\colon \bar E \to B$ induced by the map $\alpha\colon B \to B'$ from $p'\colon E' \to B'$. It appears in the commutative diagram

$$\begin{tikzcd} \bar E \arrow[r, "\psi"] \arrow[d, "q"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\alpha"'] & B' \end{tikzcd}$$

Using Lemmas 4.3 and 4.4 one obtains

$$\mathsf{secat}_f[q\colon \bar E \to B] \le \mathsf{secat}_{\hat\alpha\circ f}[q\colon \bar E \to B] \le \mathsf{secat}_{f'}[p'\colon E' \to B']. \tag{14}$$

Next we note that the diagram

$$\begin{tikzcd} \bar E \arrow[rr, "G\circ\psi"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

homotopy commutes:

$$p\circ G\circ\psi \simeq \beta\circ p'\circ\psi = \beta\circ\alpha\circ q \simeq q.$$

Applying Lemma 4.5 we obtain the inequality $\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_f[q\colon \bar E \to B]$ which together with (14) implies $\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_{f'}[p'\colon E' \to B']$, as claimed. □

Corollary 4.7 *Assume that in the diagram*

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "p"'] & E' \arrow[r, "G"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ B \arrow[r, "\alpha"'] \arrow[d, "f"'] & B' \arrow[r, "\beta"'] \arrow[d, "f'"] & B \arrow[d, "f"] \\ C \arrow[r, "\hat\alpha"'] & C' \arrow[r, "\hat\beta"'] & C \end{tikzcd}$$

the maps p and p' are fibrations, the lower squares are commutative, the upper squares are homotopy commutative and the maps α and β are mutually inverse homotopy equivalences. Then

$$\mathsf{secat}_f[p\colon E \to B] = \mathsf{secat}_{f'}[p'\colon E' \to B'].$$

Proof This follows from applying Lemma 4.6 twice: to the diagram

$$\begin{tikzcd} & E \arrow[r, "F"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B' \arrow[r, "\beta"'] \arrow[d, "f'"'] & B \arrow[r, "\alpha"'] \arrow[d, "f"] & B' \\ C' \arrow[r, "\hat\beta"'] & C & \end{tikzcd}$$

and to the diagram of Lemma 4.6. □

Corollary 4.8 *Suppose that in the commutative diagram*

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "\alpha"] \arrow[d, "f"'] & B' \arrow[d, "f'"] \\ C \arrow[r, "\hat{\alpha}"] & C' \end{tikzcd}$$

the maps p', f and f' are fibrations and p is the induced fibration. If α and $\hat{\alpha}$ are homotopy equivalences then

$$\mathsf{secat}_f[p\colon E \to B] = \mathsf{secat}_{f'}[p'\colon E' \to B'].$$

Proof By Lemmas 4.3 and 4.4 we have $\mathsf{secat}_{f'}[p'\colon E' \to B'] \geq \mathsf{secat}_{\hat{\alpha} f}[p\colon E \to B] \geq \mathsf{secat}_f[p\colon E \to B]$ so we must only show the inverse inequality. Since f and f' are fibrations and α and $\hat{\alpha}$ are homotopy equivalences, applying the proposition on page 53 of [May 1999] we see that there exist homotopy inverses β and $\hat{\beta}$ for α and $\hat{\alpha}$, respectively, such that the diagram

$$\begin{tikzcd} B' \arrow[r, "\beta"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C' \arrow[r, "\hat{\beta}"] & C \end{tikzcd}$$

commutes. We obtain the commutative diagram

$$\begin{tikzcd} & E \arrow[r, "F"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B' \arrow[r, "\beta"'] \arrow[d, "f'"'] & B \arrow[r, "\alpha"'] \arrow[d, "f"] & B' \\ C' \arrow[r, "\hat{\beta}"'] & C & \end{tikzcd}$$

with the composition $\alpha \circ \beta\colon B' \to B'$ homotopic to the identity map. Lemma 4.6 now gives

$$\mathsf{secat}_{f'}[p'\colon E' \to B'] \leq \mathsf{secat}_f[p\colon E \to B]. \qquad \square$$

4.4 Homotopical dimension

For a topological space A having the homotopy type of a finite-dimensional CW–complex we shall denote by $\mathsf{hdim}(A)$ *the homotopical dimension of* A; it is defined as the minimal dimension of a CW–complex homotopy equivalent to A.

The following lemma will be used later.

Lemma 4.9 *Consider a locally trivial bundle $p\colon E \to B$ where E and B are separable metric spaces and the base B and the fibre F have the homotopy type of finite-dimensional CW–complexes. Assume also that the fibre F of $p\colon E \to B$ has finite covering dimension* $\dim F$*. Then the total space E has the homotopy type of a finite-dimensional CW–complex and, moreover,*

$$\mathsf{hdim}(E) \le \mathsf{hdim}(B) + \dim F. \tag{15}$$

Proof Let $g\colon B' \to B$ be a homotopy equivalence where B' is a CW–complex satisfying $\dim B' = \mathsf{hdim} B$. Consider the diagram

$$\begin{tikzcd} E' \arrow[r, "G"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "g"'] & B \end{tikzcd}$$

where $p'\colon E' \to B'$ is the fibration induced by g. Clearly G is a homotopy equivalence and

$$\dim E' \le \dim B' + \dim F.$$

By [Fritsch and Piccinini 1990, Theorem 5.4.2] the space E' has homotopy type of a CW–complex. Hence,

$$\mathsf{hdim}(E) = \mathsf{hdim}(E') \le \dim(E') \le \dim B' + \dim F = \mathsf{hdim}(B) + \dim F. \qquad \square$$

4.5 An upper bound

The following statement gives a useful upper bound for the invariant $\mathsf{secat}_f[p\colon E \to B]$.

Proposition 4.10 *Assume that E, B and C are separable metric spaces. Let $p\colon E \to B$ be a fibration and let $f\colon B \to C$ be a locally trivial bundle such that*

- (a) *the space C and the fibre F_0 of $f\colon B \to C$ have the homotopy type of CW–complexes*;
- (b) *the fibre F_1 of the fibration $p\colon E \to B$ is $(k-1)$–connected, where $k \ge 0$*;
- (c) *the fibre F_0 of the fibration $f\colon B \to C$ is d–dimensional, where $0 \le d \le k$.*

Then one has

$$\mathsf{secat}_f[p\colon E \to B] \le \left\lceil \frac{\dim B - k}{1+k-d} \right\rceil. \tag{16}$$

Proof First we shall prove the statement under an additional assumption that C is a simplicial complex. We shall remove this assumption afterwards.

Consider the skeleta $C^{(i)} \subset C$ of C, where $i = 0, 1, \dots$. We know that for any two integers $0 \le i < j$ the complement $C^{(i)} - C^{(j)}$ is homotopy equivalent to a simplicial complex of dimension at most $i - j - 1$; see for example [Farber et al. 2019, Corollary 5.3].

We may find a chain of open subsets $U_0 \subset U_1 \subset U_2 \subset \cdots$ of C such that each set U_i contains $C^{(i)}$ as a strong deformation retract.

Setting $r = k - d$, consider the skeleta

$$C^{(r)} \subset C^{(2r+1)} \subset C^{(3r+2)} \subset \cdots \subset C^{((c+1)r+c)},$$

where c is the smallest integer satisfying $\dim C \le (c+1)r + c$, ie

$$c = \left\lceil \frac{\dim C - r}{1+r} \right\rceil = \left\lceil \frac{\dim B - k}{1+k-d} \right\rceil.$$

Each complement,

$$X_i = C^{((i+1)r+i)} - C^{(ir+i-1)}, \quad i = 0, 1, \dots, c,$$

has the homotopy type of a simplicial complex of dimension $\le r$. The open set

$$Y_i = U_{(i+1)r+i} - C^{(ir+i-1)} \subset C$$

deformation retracts onto X_i and therefore $\mathsf{hdim}(Y_i) \le r$. Applying Lemma 4.9 we obtain

$$\mathsf{hdim}(V_i) \le r + d = k,$$

where

$$V_i = f^{-1}(Y_i) \subset B, \quad i = 0, 1, \dots, c.$$

The fibre F_1 of $p\colon E \to B$ is $(k-1)$–connected, and thus we may apply the well-known result of the obstruction theory stating that the fibration $p\colon E \to B$ admits a continuous section over each open set V_i, where $i = 0, 1, \dots, c$. Since $B = V_0 \cup V_1 \cup \cdots \cup V_c$, it shows that $\mathsf{secat}_f[p\colon E \to B] \le c$. This completes the proof in the case when C is a simplicial complex.

Consider now the general case, ie we shall only assume that C has the homotopy type of a CW–complex. We can find a simplicial complex C' and a homotopy equivalence $\hat{\alpha}\colon C' \to C$; see [Fritsch and Piccinini 1990, Theorem 5.2.1]. Consider the fibration $f'\colon B' \to C'$ induced by $\hat{\alpha}$ from $f\colon B \to C$. The map α shown on the diagram

$$\begin{tikzcd} E' \arrow[r, "F"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "\alpha"'] \arrow[d, "f'"'] & B \arrow[d, "f"] \\ C' \arrow[r, "\hat{\alpha}"'] & C \end{tikzcd}$$

is a homotopy equivalence. The map α induces the fibration $p'\colon E' \to B'$. Applying Corollary 4.8 we obtain that

$$\mathsf{secat}_{f'}[p'\colon E' \to B'] = \mathsf{secat}_f[p\colon E \to B].$$

Hence the upper bound (16) which we proved above for $\mathsf{secat}_{f'}[p'\colon E' \to B']$ applies to $\mathsf{secat}_f[p\colon E \to B]$ as well. □

Remark 4.11 In [Farber et al. 2019] an upper bound for topological complexity was derived that made use of an invariant which was called

$$\widetilde{\mathsf{TC}}(X) = \widetilde{\mathsf{secat}}(E \xrightarrow{p} \overline{X} \xrightarrow{q} X)$$

there, but which we recognize in fact to be $\mathsf{secat}_q[p\colon E \to \overline{X}]$ here. In [Farber et al. 2019] it was further shown that $\widetilde{\mathsf{TC}}(X)$ could be identified with the notion of strongly invariant topological complexity $\mathsf{TC}^*_\pi(\widetilde{X})$ introduced by A Dranishnikov [2015] earlier. A K Paul and D Sen [2020] extended both the invariant $\widetilde{\mathsf{TC}}(X)$ and the strongly invariant topological complexity to the realm of sequential topological complexity and proved the analogous identification. This identification, in some sense, was the genesis of our calculus of sectional categories and together with Theorem 3.4 begs the question of exactly how parametrized topological complexity and various forms of equivariant topological complexity are intertwined, especially in the case of locally trivial fibre bundles.

5 Sectional category of towers of fibrations

Consider a tower of fibrations

$$E_r \xrightarrow{p_r} E_{r-1} \xrightarrow{p_{r-1}} E_{r-2} \to \cdots \xrightarrow{p_1} E_0$$

and the total fibration

$$p = p_1 p_2 \cdots p_r \colon E_r \to E_0.$$

We shall assume that all spaces E_i are normal.

Theorem 5.1 *The sectional category* $\mathsf{secat}[p\colon E_r \to E_0]$ *of the total fibration admits the lower and upper bounds*

$$\begin{aligned}(17)\qquad \mathsf{secat}[p_1\colon E_1 \to E_0] &\le \mathsf{secat}[p\colon E_r \to E_0]\\ &\le \mathsf{secat}[p_1\colon E_1 \to E_0] + \sum_{i=1}^{r-1} \mathsf{secat}_{(p_1 p_2 \cdots p_i)}[p_{i+1}\colon E_{i+1} \to E_i].\end{aligned}$$

Here $p_1 p_2 \cdots p_i\colon E_i \to E_0$ *denotes the composition.*

Lemma 5.2 below will be used in the proof of Theorem 5.1.

Lemma 5.2 *Let* C *be a normal space. Consider properties* $A_1, A_2, \ldots, A_r$ *of open subsets of* C*, such that each property* A_i *is inherited by open subsets and disjoint unions. Assume that for each* $i = 1, 2, \ldots, r$ C *admits an open cover consisting of* $n_i + 1$ *open sets satisfying the property* A_i*. Then* C *admits an open cover consisting of* $N + 1$ *open sets, where* $N = \sum_{i=1}^r n_i$*, satisfying all the properties* $A_1, \ldots, A_r$*.*

Proof For $r = 2$ this statement was proven in [Oprea and Strom 2011, Lemma 4.3]. The case $r > 2$ follows from this by induction. □

Proof of Theorem 5.1 Since the left inequality in (17) is obvious we shall concentrate on the right one and use Lemma 5.2 to prove it. Consider the following properties $A_1, A_2, \dots, A_r$ of open subsets of E_0. We shall say that an open subset $U \subset E_0$ satisfies A_1 if U has a continuous section of the fibration p_1. For $2 \le i \le r$, we shall say that an open subset $U \subset E_0$ satisfies the property A_i if the open set $p_{i-1}^{-1} \cdots p_2^{-1} p_1^{-1}(U) \subset E_{i-1}$ admits a continuous section of p_i. By definition, for any $i = 1, 2, \dots, r$, the set E_0 admits an open cover of cardinality $\mathsf{secat}_{(p_1 p_2 \cdots p_{i-1})}[p_i : E_i \to E_{i-1}] + 1$ with each set satisfying A_i. Applying Lemma 5.2, we obtain that E_0 admits an open cover $\{U_j\}$ of cardinality $\sum_{i=1}^r n_i + 1$ such that each set U_j satisfies all the properties $A_1, \dots, A_r$. This means that there exists a continuous section $s_0 \colon U_j \to E_1$ of p_1 and for any $i = 1, 2, \dots, r-1$, there exists a continuous section

$$s_i : p_i^{-1} \cdots p_2^{-1} p_1^{-1}(U_j) \to E_{i+1}$$

of the fibration p_i. Hence, the composition

$$s = s_{r-1} s_{r-2} \cdots s_1 s_0 \colon U_j \to E_r$$

is a well-defined continuous section of the composition $p = p_1 p_2 \cdots p_r \colon E_r \to E_0$. This gives the inequality (17). □

For convenience of references, we state below the special case $r = 2$ of Theorem 5.1 which we combine with the dimension-connectivity upper bound of Proposition 4.10:

Corollary 5.3 *Consider a tower of fibrations $E_2 \xrightarrow{p_2} E_1 \xrightarrow{p_1} E_0$ of separable metric spaces. Assume that $p_1 \colon E_1 \to E_0$ is locally trivial. Then the sectional category $\mathsf{secat}[p \colon E_2 \to E_0]$ of the total bundle*

$$p = p_2 \circ p_1 \colon E_2 \to E_0$$

lies between $\mathsf{secat}[p_1 \colon E_1 \to E_0]$ and

$$\mathsf{secat}[p_1 \colon E_1 \to E_0] + \mathsf{secat}_{p_1}[p_2 \colon E_2 \to E_1]. \tag{18}$$

Moreover, under the additional assumptions that

(a) *the fibre of $p_2 \colon E_2 \to E_1$ is $(k-1)$–connected,*

(b) *the space E_0 and the fibre of $p_1 \colon E_1 \to E_0$ have the homotopy type of CW–complexes,*

(c) *the fibre of $p_1 \colon E_1 \to E_0$ has dimension $\le d$ where $0 \le d \le k$,*

one has

$$\mathsf{secat}_{p_1}[p_2 \colon E_2 \to E_1] \le \left\lceil \frac{\dim E_1 - k}{1 + k - d} \right\rceil. \tag{19}$$

6 Product inequalities

Lemma 5.2 distills the main results of [Dranishnikov 2009; 2010; Oprea and Strom 2011; Ostrand 1965], but for the product inequalities which we describe below we need more specific information about open covers.

An open cover $\mathcal{W} = \{W_0, \dots, W_{m+k}\}$ of a space C is an $(m+1)$*–cover* if every subcollection

$$\{W_{j_0}, W_{j_1}, \dots, W_{j_m}\}$$

of $m+1$ sets from $\mathcal{W}$ also covers C. The following simple observation (see [Farber et al. 2019] for instance) is the basis for many arguments in this approach.

Lemma 6.1 *A cover $\mathcal{W} = \{W_0, W_1, \dots, W_{k+m}\}$ is an $(m+1)$–cover of C if and only if each $x \in C$ is contained in at least $k+1$ sets of $\mathcal{W}$.*

An open cover can be lengthened to a $(k+1)$–cover, while retaining certain essential properties of the sets in the cover.

Theorem 6.2 [Dranishnikov 2009; Ostrand 1965] *Let $\mathcal{U} = \{U_0, \dots, U_k\}$ be an open cover of a normal space C. Then, for any $m = k, k+1, \dots, \infty$, there is an open $(k+1)$–cover of C, $\{U_0, \dots, U_m\}$, extending $\mathcal{U}$ such that for $n > k$, U_n is a disjoint union of open sets that are subsets of the U_j, $0 \le j \le k$.*

We use these facts to obtain inequalities for product fibrations.

Lemma 6.3 (product inequality, I) *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations and let $f : B \to C$ and $f'\colon B' \to C'$ be continuous maps. Assume that the spaces $f(B)$ and $f'(B')$ with topology induced from C and C', respectively, are normal. Then the sectional category of the product fibration*

$$\mathsf{secat}_{f\times f'}[p \times p'\colon E \times E' \to B \times B']$$

is bounded above by the sum

$$\mathsf{secat}_f[p\colon E \to B] + \mathsf{secat}_{f'}[p'\colon E' \to B']$$

and it is bounded below by

$$\max\{\mathsf{secat}_f[p\colon E \to B], \mathsf{secat}_{f'}[p'\colon E' \to B']\}.$$

Proof First we deal with the lower bounds. Fix a point $b'_0 \in B'$ and embed B into $B \times B'$ via $b \mapsto (b, b'_0)$; also, embed C into $C \times C'$ via $x \mapsto (x, x'_0)$ where $x'_0 = f'(b'_0)$. For an open subset $U \subset C \times C'$, a section of $p \times p'$ over $(f \times f')^{-1}(U) \subset B \times B'$ determines obviously a section of p over $f^{-1}(U \cap (C \times x'_0))$. This implies the inequality $\mathsf{secat}_{f\times f'}[p \times p'\colon E \times E' \to B \times B'] \ge \mathsf{secat}_f[p\colon E \to B]$. Similarly, one obtains $\mathsf{secat}_{f\times f'}[p \times p'\colon E \times E' \to B \times B'] \ge \mathsf{secat}_{f'}[p'\colon E' \to B']$.

Now we prove the upper bound. Let $\mathsf{secat}_f[p\colon E \to B] = k$ be realized by open sets $U_0, \dots, U_k \subset C$ covering $f(B) \subset C$, with continuous sections $s_j : f^{-1}(U_j) \to E$ of p, and let $\mathsf{secat}_{f'}[p'\colon E' \to B'] = m$ be realized by open sets $V_0, \dots, V_m \subset C'$ covering $f'(B')$, with sections $s'_j : f'^{-1}(V_j) \to E'$ of p'. By Theorem 6.2 we can extend the family $U_0, \dots, U_k$ to a family of open subsets $U_0, \dots, U_{k+m}$ of C such that any $k+1$ members of this family cover $f(B)$. Similarly, we can find a family $V_0, \dots, V_{k+m}$ of open subsets of C' extending the initial family $V_0, \dots, V_m$ such that any $m+1$ members of this extended family cover $f'(B')$. Theorem 6.2 guarantees that every set of the form $f^{-1}(U_j)$ or $f'^{-1}(V_j)$ admits a continuous section of p or p' respectively, where $j = 0, 1, \dots, k+m$.

Letting $W_j = U_j \times V_j$, where $j = 0, \dots, k+m$, we see that each set $(f \times f')^{-1}(W_j) = f^{-1}(U_j) \times f'^{-1}(V_j)$ admits a continuous section of $p \times p'$. We show below that the sets W_j cover $f(B) \times f'(B')$, which implies that $\mathsf{secat}_{f \times f'}[p \times p' \colon E \times E' \to B \times B'] \le k+m$.

Suppose that a point $(x, y) \in f(B) \times f'(B')$ is not in any of the sets W_j, where $j = 0, \dots, k+m$. Since any $k+1$ sets U_j cover $f(B)$, we know that x belongs to at least $m+1$ of the U_j, by Lemma 6.1. Without loss of generality, we may assume that $x \in U_0 \cap U_1 \cap \dots \cap U_m$. Then $y \notin V_0 \cup V_1 \cup \dots \cup V_m$, in view of our assumption. Therefore, y can only lie in the sets $V_{m+1}, \dots, V_{k+m}$ which is a contradiction since y belongs to at least $k+1$ of the sets V_j, by Lemma 6.1. □

Next we state another product inequality dealing with fibrations over the same base.

Lemma 6.4 (product inequality, II) *Let $p \colon E \to B$ and $p' \colon E' \to B$ be two fibrations, and let $f \colon B \to C$. We shall assume that $f(B)$ is normal in the topology induced from C. Then the sectional category*

$$\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B]$$

of the fibrewise product is bounded below by

$$\max\{\mathsf{secat}_f[p \colon E \to B], \mathsf{secat}_f[p' \colon E' \to B]\}$$

and is bounded above by the sum

$$\mathsf{secat}_f[p \colon E \to B] + \mathsf{secat}_f[p' \colon E' \to B].$$

Moreover,

$$\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B] = \mathsf{secat}_f[p \colon E \to B]$$

if $\mathsf{secat}[p' \colon E' \to B] = 0$, ie if p' admits a section.

Proof The projection $\mathrm{pr} \colon E \times_B E' \to E$ appears in the commutative diagram

$$\begin{tikzcd} E \times_B E' \arrow[rr, "\mathrm{pr}"] \arrow[dr, "p \times_B p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

and Lemma 4.5 gives $\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B] \ge \mathsf{secat}_f[p \colon E \to B]$. Similarly one gets the lower bound using $\mathsf{secat}_f[p' \colon E' \to B]$, which proves the statement concerning the lower bound. Next we note that

$$\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B] \le \mathsf{secat}_{f \times f}[p \times p' \colon E \times E' \to B \times B]. \tag{20}$$

Indeed, the fibration $p \times_B p' \colon E \times_B E' \to B$ is induced from the product fibration $p \times p' \colon E \times E' \to B \times B$ by the diagonal map $\Delta \colon C \to C \times C$. Lemma 4.4 gives the inequality

$$\mathsf{secat}_{(f\times f)\circ\Delta}[p \times_B p' \colon E \times_B E' \to B] \le \mathsf{secat}_{f\times f}[p \times p' \colon E \times E' \to B \times B].$$

Finally, we can apply Lemma 4.3 and replace $(f \times f) \circ \Delta$ by f. Combining (20) with Lemma 6.3 we obtain the upper bound.

The last statement obviously follows by combining the lower and upper bounds. □

7 Weak equivariant topological complexity $\mathsf{TC}^w_{r,G}(X)$

Let $p \colon E \to B$ be a bundle with fibre X and structure group G which is associated to a principal bundle $\tau \colon P \to B$. In other words, $E = X \times_G P$.

As in Section 2, we fix $r \ge 2$ points $0 = t_1 < t_2 < \cdots < t_r = 1$ and consider the evaluation map

$$\rho_r \colon X^I \to X^r, \qquad \rho_r(\gamma) = (\gamma(t_1), \gamma(t_2), \dots, \gamma(t_r)), \quad \text{where } \gamma \in X^I.$$

Consider also the quotient map

$$q_r \colon X^r \to X^r/G,$$

where we view G acting diagonally on X^r.

The following invariant plays an important role in our main Theorem 8.1:

$$\mathsf{TC}^w_{r,G}(X) = \mathsf{secat}_{q_r}[\rho_r \colon X^I \to X^r]. \tag{21}$$

Explicitly, we have:

Definition 7.1 The invariant $\mathsf{TC}^w_{r,G}(X)$ equals the smallest integer $k \ge 0$ such that X^r admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ by G–invariant open sets such that for each $i = 0, 1, \dots, k$ there is a continuous section $s_i \colon U_i \to X^I$ of π_r.

Note that the section s_i in Definition 7.1 is not required to be G–equivariant, unlike in the case of $\mathsf{TC}_{r,G}(X)$. This explain the adjective "weak" and the symbol "w" in the notation. We obviously have

$$\mathsf{TC}_r(X) \le \mathsf{TC}^w_{r,G}(X) \le \mathsf{TC}_{r,G}(X), \tag{22}$$

where the left inequality is a special case of (11). All these inequalities become equalities when the action of G is trivial.

Lemma 7.2 *For any G–space P,*

$$\mathsf{TC}^w_{r,G}(X) = \mathsf{secat}_{q_r\times\epsilon}[\rho_r \times 1 \colon X^I \times P \to X^r \times P],$$

where $\epsilon \colon P \to *$ is the map onto a singleton.

Proof This follows from Lemma 6.3 since clearly $\mathsf{secat}_\epsilon[1 \colon P \to P] = 0$. □

Next we state the dimension-connectivity upper bound:

Lemma 7.3 *Assume that X is a k–connected simplicial complex and G is a topological group homeomorphic to a CW–complex acting freely on X and such that the map $q_r\colon X^r \to X^r/G$ is a locally trivial bundle. If $\dim G \le k$ then*

$$\mathsf{TC}^w_{r,G}(X) \le \left\lceil \frac{r\dim X - k}{1+k-\dim G} \right\rceil. \tag{23}$$

Proof We apply Proposition 4.10 having in mind that the fibre $(\Omega X)^{r-1}$ of fibration ρ_r is $(k-1)$–connected. □

As a special case of Lemma 7.3 we mention:

Corollary 7.4 *If X is k–connected, where $k \ge 0$, and the group G is discrete and the quotient map $q_r\colon X^r \to X^r/G$ is a covering map then*

$$\mathsf{TC}^w_{r,G}(X) \le \left\lceil \frac{r\dim X - k}{1+k} \right\rceil. \tag{24}$$

We shall be discussing yet another invariant $\mathsf{TC}^w_{r,G}(X;P)$ given by

$$\mathsf{TC}^w_{r,G}(X;P) = \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P] \tag{25}$$

with $Q\colon X^r \times P \to X^r \times_G P$ being the natural projection; here X and P are G–spaces and ρ_r is the fibration (8). Comparing with Lemma 7.2 we see that it is similar to $\mathsf{TC}^w_{r,G}(X)$ with the only distinction that the map $q_r \times \epsilon$ is replaced by Q.

Lemma 7.5 *One has*

$$\mathsf{TC}_r(X) \le \mathsf{TC}^w_{r,G}(X;P) \le \mathsf{TC}^w_{r,G}(X). \tag{26}$$

Proof Consider the commutative diagram

$$\begin{tikzcd}
X^I \times P \arrow[r, "\rho_r \times 1"] \arrow[d, "p_1"'] & X^r \times P \arrow[r, "Q"] \arrow[d, "p_2"'] & X^r \times_G P \arrow[d, "p_3"] \\
X^I \arrow[r, "\rho_r"'] & X^r \arrow[r, "q_r"'] & X^r/G
\end{tikzcd}$$

where the maps p_1, p_2 and p_3 are projections on the first factor. Since the fibration $\rho_r \times 1$ is induced from ρ_r via p_2, we may apply Lemma 4.4 to conclude

$$\begin{aligned}
\mathsf{TC}^w_{r,G}(X) &= \mathsf{secat}_{q_r}[\rho_r\colon X^I \to X^r] \\
&\ge \mathsf{secat}_{q_r \circ p_2}[\rho_r \times 1\colon X^I \times P \to X^r \times P] \\
&= \mathsf{secat}_{p_3 \circ Q}[\rho_r \times 1\colon X^I \times P \to X^r \times P] \\
&\ge \mathsf{secat}_{Q}[\rho_r \times 1\colon X^I \times P \to X^r \times P] \\
&= \mathsf{TC}^w_{r,G}(X;P).
\end{aligned}$$

On the third line we used Lemma 4.3(a). This proves the right inequality in (26). The left inequality follows from

$$\begin{aligned}\mathsf{TC}^{w}_{r,G}(X;P) &= \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P]\\ &\geq \mathsf{secat}[\rho_r \times 1\colon X^I \times P \to X^r \times P]\\ &= \mathsf{secat}[\rho_r\colon X^I \to X^r]\\ &= \mathsf{TC}_r(X),\end{aligned}$$

where on the second line we used inequality (11) and on the third line Lemma 6.3. □

The next result gives a dimension-connectivity upper bound for $\mathsf{TC}^{w}_{r,G}(X;P)$ which holds for weaker assumptions on X compared to Lemma 7.3.

Lemma 7.6 *Assume that X is a k–connected simplicial complex and G is a topological group homeomorphic to a CW–complex. Suppose that $P \to P/G$ is a locally trivial bundle. If $\dim G \leq k$ then*

$$\mathsf{TC}^{w}_{r,G}(X;P) \leq \left\lceil \frac{r\dim X + \dim P - k}{1 + k - \dim G} \right\rceil. \tag{27}$$

Proof This follows by applying Proposition 4.10 to the definition (25). □

Example 7.7 Consider the unit circle $S^1 \subset \mathbb{C}$ with the action of the cyclic group of order two $G = \mathbb{Z}_2$ acting as the complex conjugation, $z \mapsto \bar{z}$. We know from [Colman and Grant 2012] that in this case $\mathsf{TC}_{2,G}(S^1)$ is infinite due to the fact that the set of fixed points is disconnected.

On the other hand one can consider the open cover $S^1 \times S^1 = U_0 \cup U_1$ where $U_0 = \{(z_1, z_2) \mid z_1 \neq -z_2\}$ and $U_1 = \{(z_1, z_2) \mid z_1 \neq z_2\}$. These sets are G–invariant and over each of these sets one has the well-known continuous sections. Thus, $\mathsf{TC}^{w}_{2,G}(S^1) = 1$.

Example 7.8 Consider the more general case of a sphere S^n, where $n \geq 1$, with an action of a discrete group G. First we apply the upper bound (24) with $k = n - 1$ to obtain

$$\mathsf{TC}^{w}_{r,G}(S^n) \leq r \quad \text{for any } r \geq 2.$$

Second, using (26) and the result of Y Rudyak [2010] (stating that $\mathsf{TC}_r(S^n)$ equals r for n even and $r-1$ for n odd), we obtain that for any even n

$$\mathsf{TC}^{w}_{r,G}(S^n) = r. \tag{28}$$

For n odd our inequalities imply that $\mathsf{TC}^{w}_{r,G}(S^n)$ equals either $r-1$ or r.

Example 7.9 Let S^1 act on S^2 by rotations about the z–axis. The fixed-point set of the action is the disconnected set $\{N, S\}$, where N and S are the north and south poles, respectively, so the equivariant topological complexity is infinite: $\mathsf{TC}_{r,S^1}(S^2) = \infty$ for all $r \geq 2$.

Let us now examine the weak equivariant topological complexity $\mathsf{TC}^w_{2,S^1}(S^2)$. Fix an orbit $O \subset S^2$ given by the equator and fix an orientation of O. Consider the open cover $S^2 \times S^2 = U_0 \cup U_1 \cup U_2$ where

$$\begin{aligned} U_0 &= \{(x,y) \mid x \neq -y\}, \\ U_1 &= \{(x,y) \mid x \neq y\} - \{(N,S),(S,N)\}, \\ U_2 &= \{(x,y) \mid x \notin O \text{ and } y \notin O\}. \end{aligned}$$

Clearly, the sets U_0, U_1 and U_2 are S^1–invariant. We may define the motion planning rules over each of the sets U_i as follows. For $(x,y) \in U_0$, go from x to y along the shortest geodesic arc. For $(x,y) \in U_1$ the point x moves along the shortest geodesic arc first to the closest point of O, then along O in the positive direction to the point closest to y, and finally to y. For $(x,y) \in U_2$ the point x moves along the shortest geodesic arc to the closest pole (N or S), then to the closest pole to y along a fixed path and then to y; the first and the third portions are along the shortest geodesic arc on the sphere S^2. Hence $\mathsf{TC}^w_{2,S^1}(S^2) \leq 2$. Since $2 = \mathsf{TC}(S^2) \leq \mathsf{TC}^w_{2,S^1}(S^2)$, we see that $\mathsf{TC}^w_{2,S^1}(S^2) = 2$.

8 Bounds for the sequential parametrized topological complexity

Finally we are in position to state and prove the main result of this paper:

Theorem 8.1 *Let $p\colon E \to B$ be a locally trivial fibre bundle with structure group G, the fibre X and the associated principal bundle $\tau\colon P \to B$. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ admits the upper and lower bounds*

$$\mathsf{TC}^w_{r,G}(X;P) \leq \mathsf{TC}_r[p\colon E \to B] \leq G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{r,G}(X;P). \tag{29}$$

Proof Since $E = X \times_G P$,

$$E^r_B = X^r \times_G P \quad \text{and} \quad E^I_B = X^I \times_G P \quad \text{for any } r \geq 2.$$

The map $\Pi_r\colon E^I_B \to E^r_B$ becomes $\rho_r \times 1\colon X^I \times_G P \to X^r \times_G P$, where $\rho_r(\gamma) = (\gamma(t_0), \dots, \gamma(t_r))$. Consider the commutative diagram

$$\begin{tikzcd} X^I \times P \arrow[r, "Q'"] \arrow[d, "\rho_r \times 1"'] & X^I \times_G P \arrow[d, "\rho_r \times_G 1"] \\ X^r \times P \arrow[r, "Q"'] & X^r \times_G P \end{tikzcd} \tag{30}$$

where $Q\colon X^r \times P \to X^r \times_G P$ and $Q'\colon X^I \times P \to X^I \times_G P$ are the natural projections. Using Lemma 4.5 and Theorem 5.1,

$$\begin{aligned} \mathsf{TC}_r[p\colon E \to B] &= \mathsf{secat}[\rho_r \times_G 1\colon X^I \times_G P \to X^r \times_G P] \\ &\leq \mathsf{secat}[(\rho_r \times_G 1) \circ Q'\colon X^I \times P \to X^r \times_G P] \\ &= \mathsf{secat}[(Q \circ (\rho_r \times 1)\colon X^I \times P \to X^r \times_G P] \\ &\leq \mathsf{secat}[Q\colon X^r \times P \to X^r \times_G P] + \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P]. \end{aligned}$$

Next we observe that

$$\mathsf{secat}[X^r \times P \to X^r \times_G P] \le \mathsf{secat}[\tau\colon P \to B] = G\text{–}\mathsf{cat}[p\colon E \to B]$$

and

$$\mathsf{secat}_Q[X^I \times P \to X^r \times P] = \mathsf{TC}^w_{r,G}(X;P).$$

Thus, we obtain the right inequality in (29).

For the left inequality in (29) we consider again diagram (30) and observe that the fibration

$$\pi_r \times 1\colon X^I \times P \to X^r \times P$$

is induced from $\pi_r \times_G 1\colon X^I \times_G P \to X^r \times_G P$ via Q. Therefore, using Lemma 4.4 we obtain

$$\begin{aligned}\mathsf{TC}_r[p\colon E \to B] &= \mathsf{secat}[\rho_r \times 1\colon X^I \times_G P \to X^r \times_G P]\\ &\ge \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P]\\ &= \mathsf{TC}^w_{r,G}(X;P).\end{aligned}$$

□

Remark 8.2 Due to the right inequality in (26), the upper bound in (29) gives

$$\mathsf{TC}_r[p\colon E \to B] \le G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{r,G}(X). \tag{31}$$

The right-hand side of this inequality has two terms, one depending only on the initial bundle $p\colon E \to B$ and the other depending only on the fibre, X viewed as a G–space.

Theorem 8.1 implies that for the trivial bundle $p\colon E \to B$ with fibre X one has $\mathsf{TC}_r[p\colon E \to B] = \mathsf{TC}_r(X)$; see [Farber and Paul 2022, Example 3.2]. Indeed, in this case

$$G\text{–}\mathsf{cat}[p\colon E \to B] = 0 \quad\text{and}\quad \mathsf{TC}^w_{r,G}(X,P) = \mathsf{TC}_r(X);$$

hence the statement follows from (29).

Example 8.3 The Klein bottle K is the total space of the bundle $p\colon K = S^1 \times_{\mathbb{Z}/2} S^1 \to S^1$ with the associated principal bundle the 2–fold covering $\tau\colon S^1 \to S^1$ and the action of $G = \mathbb{Z}/2$ on the fibre S^1 being given by reflection in the last coordinate. The inequality (31) with $r = 2$ and the result of Example 7.7 give

$$\mathsf{TC}[p\colon K \to S^1] = \mathsf{TC}_2[p\colon K \to S^1] \le 1 + 1 = 2. \tag{32}$$

Mark Grant observed that (32) is in fact an equality. The inequality $\mathsf{TC}[p\colon K \to S^1] \ge 2$ can be obtained by applying [Farber and Weinberger 2023a, Theorem 2]. The bundle $p\colon K \to S^1$ is the unit sphere bundle of a rank 2 vector bundle ξ over the circle S^1. One has $\mathrm{w}_2(\xi) = 0$ (for dimensional reasons) and $\mathrm{w}_1(\xi) \ne 0$ (since ξ is not orientable) and therefore the relative height $\mathfrak{h}(\mathrm{w}_1(\xi) \mid \mathrm{w}_2(\xi))$ equals one. Theorem 2 from [Farber and Weinberger 2023a] now applies and gives an equality $\mathsf{TC}[p\colon K \to S^1] = 2$.

Example 8.4 Consider the principal G–bundle $\tau\colon P \to B$ where $G = S^1$, $P = S^{2n+1}$ and $B = \mathbb{CP}^n$ (the Hopf bundle). Here the sphere S^{2n+1} is viewed as the unit sphere in $\mathbb{C}^{n+1}$ and the circle S^1 acts on it by complex multiplication. Let $X = S^2$ with S^1–action given by rotations about the z–axis, as in Example 7.9. Consider the fibre bundle $p\colon E \to B$ with fibre $X = S^2$ where $E = X \times_G P$. Applying (31) with $r = 2$ we obtain

$$\mathsf{TC}[p\colon E \to B] = \mathsf{TC}_2[p\colon E \to B] \le \mathsf{secat}[\tau\colon P \to B] + \mathsf{TC}^w_{2,G}(X) \tag{33}$$

and from Example 7.9 we know that $\mathsf{TC}^w_{2,G}(X) = 2$. On the other hand, since $\mathsf{cat}(\mathbb{CP}^n) = n$, we have $\mathsf{secat}[\tau\colon P \to B] \le \mathsf{cat}(B) = n$ (in fact, this is an equality by a cup-length argument). Thus, (33) gives $\mathsf{TC}[p\colon E \to B] \le n + 2$.

In [Farber and Weinberger 2023b] the authors studied parametrized topological complexity of sphere bundles. The sphere bundle $p\colon E \to B$ which was discussed in the previous paragraph is the unit sphere bundle associated with the rank 3 vector bundle over $B = \mathbb{CP}^n$ which is the Whitney sum $\eta \oplus \epsilon$ where η is the canonical complex line bundle over $\mathbb{CP}^n$ and ϵ is a trivial real line bundle. The result of [Farber and Weinberger 2023b, Example 20] states that $\mathsf{TC}[p\colon E \to B] \le n+2$ and moreover $\mathsf{TC}[p\colon E \to B] = n+2$ for any even n.

Here the point is that, in the example above, the upper bound (31) is in fact sharp; that is, we have an equality

$$\mathsf{TC}[p\colon E \to B] = G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{2,G}(S^2).$$

In fact, since in general $\mathsf{TC}^w_{r,G}(X; P) \le \mathsf{TC}^w_{r,G}(X)$, we see that (29) in this case is an equality as well. This emphasizes the fact that these upper bounds can sometimes detect parametrized topological complexity precisely.

References

[Bayeh and Sarkar 2020] **M Bayeh**, **S Sarkar**, *Higher equivariant and invariant topological complexities*, J. Homotopy Relat. Struct. 15 (2020) 397–416 MR Zbl

[Bredon 1972] **G E Bredon**, *Introduction to compact transformation groups*, Pure Appl. Math. 46, Academic, New York (1972) MR Zbl

[Cohen et al. 2021] **D C Cohen**, **M Farber**, **S Weinberger**, *Topology of parametrized motion planning algorithms*, SIAM J. Appl. Algebra Geom. 5 (2021) 229–249 MR Zbl

[Cohen et al. 2022] **D C Cohen**, **M Farber**, **S Weinberger**, *Parametrized topological complexity of collision-free motion planning in the plane*, Ann. Math. Artif. Intell. 90 (2022) 999–1015 MR Zbl

[Colman and Grant 2012] **H Colman**, **M Grant**, *Equivariant topological complexity*, Algebr. Geom. Topol. 12 (2012) 2299–2316 MR Zbl

[Dranishnikov 2009] **A N Dranishnikov**, *On the Lusternik–Schnirelmann category of spaces with* 2*–dimensional fundamental group*, Proc. Amer. Math. Soc. 137 (2009) 1489–1497 MR Zbl

[Dranishnikov 2010] **A Dranishnikov**, *The Lusternik–Schnirelmann category and the fundamental group*, Algebr. Geom. Topol. 10 (2010) 917–924 MR Zbl

[Dranishnikov 2015] **A Dranishnikov**, *On topological complexity of twisted products*, Topology Appl. 179 (2015) 74–80 MR Zbl

[Farber 2003] **M Farber**, *Topological complexity of motion planning*, Discrete Comput. Geom. 29 (2003) 211–221 MR Zbl

[Farber and Paul 2022] **M Farber**, **A K Paul**, *Sequential parametrized motion planning and its complexity*, Topology Appl. 321 (2022) art. id. 108256 MR Zbl

[Farber and Weinberger 2023a] **M Farber**, **S Weinberger**, *Parametrized motion planning and topological complexity*, from "Algorithmic foundations of robotics, XV" (S M LaValle, J M O'Kane, M Otte, D Sadigh, P Tokekar, editors), Springer Proc. Adv. Robot. 25, Springer (2023) 1–17 MR Zbl

[Farber and Weinberger 2023b] **M Farber**, **S Weinberger**, *Parametrized topological complexity of sphere bundles*, Topol. Methods Nonlinear Anal. 61 (2023) 161–177 MR Zbl

[Farber et al. 2019] **M Farber**, **M Grant**, **G Lupton**, **J Oprea**, *An upper bound for topological complexity*, Topology Appl. 255 (2019) 109–125 MR Zbl

[Fritsch and Piccinini 1990] **R Fritsch**, **R A Piccinini**, *Cellular structures in topology*, Cambridge Stud. Adv. Math. 19, Cambridge Univ. Press (1990) MR Zbl

[James 1978] **I M James**, *On category, in the sense of Lusternik–Schnirelmann*, Topology 17 (1978) 331–348 MR Zbl

[LaValle 2006] **S M LaValle**, *Planning algorithms*, Cambridge Univ. Press (2006) MR Zbl

[May 1999] **J P May**, *A concise course in algebraic topology*, Univ. Chicago Press (1999) MR Zbl

[Oprea and Strom 2011] **J Oprea**, **J Strom**, *Mixing categories*, Proc. Amer. Math. Soc. 139 (2011) 3383–3392 MR Zbl

[Ostrand 1965] **P A Ostrand**, *Dimension of metric spaces and Hilbert's problem* 13, Bull. Amer. Math. Soc. 71 (1965) 619–622 MR Zbl

[Palais 1960] **R S Palais**, *The classification of G–spaces*, Mem. Amer. Math. Soc. 36, Amer. Math. Soc., Providence, RI (1960) MR Zbl

[Paul and Sen 2020] **A K Paul**, **D Sen**, *An upper bound for higher topological complexity and higher strongly equivariant complexity*, Topology Appl. 277 (2020) art. id. 107172 MR Zbl

[Rudyak 2010] **Y B Rudyak**, *On higher analogs of topological complexity*, Topology Appl. 157 (2010) 916–920 MR Zbl

[Schwartz 1962] **A S Schwartz**, *The genus of a fibre space*, Tr. Mosk. Mat. Obs. 11 (1962) 99–126 MR Zbl In Russian; translated in "Eleven papers on topology and algebra", Amer. Math. Soc. Transl. (2) 55, Amer. Math. Soc., Providence, RI (1966) 49–140

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The multiplicative structures on motivic homotopy groups

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We reconcile the multiplications on the homotopy rings of motivic ring spectra used by Voevodsky and Dugger. While the connection is elementary and similar phenomena have been observed in situations like supersymmetry, neither we nor other researchers we consulted were aware of the conflicting definitions and the potential consequences. Hence this short note.

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The homotopy groups of a motivic spectrum E form a $\mathbb{Z}\times\mathbb{Z}$–graded abelian group $\pi_{*,\star}E$. If E is a motivic ring spectrum, then the multiplication induces a ring structure on $\pi_{*,\star}E$, which, if E is commutative, should be graded commutative, as explained in [Dugger 2014]. Voevodsky [2003] displays the dual Steenrod algebra $\mathscr{A}_{*,\star}$ as a ring with graded commutativity $x\cdot y = y\cdot x\cdot(-1)^{ac}$ for $x\in\mathscr{A}_{a,b}$ and $y\in\mathscr{A}_{c,d}$ — the same convention is used in [Hoyois et al. 2017; Spitzweck 2018] — while [Dugger 2014] yields $x\cdot y = y\cdot x\cdot(-1)^{(a-b)(c-d)}\cdot(-1)^{bd}$. These are different formulas: for instance, Voevodsky claims $\tau_0\tau = \tau\tau_0$ and, according to [Dugger 2014], we must have that $\tau_0\tau = -\tau\tau_0$.

The authors were distressed to discover this, and, worryingly enough, none of those we consulted had discovered the discrepancy (although [Dugger 2014] claims that the Betti realization is not a ring map). Was there a subtle mistake buried in the literature somewhere? Something was surely wrong. But what?

Don't panic

Fortunately, the results are not irreconcilable, and in fact the solution is already to be found in [Dugger 2014, Proposition 7.2]:

> "The" homotopy ring of a motivic ring spectrum A is not canonical.

Let us recall the outline of this story:

(1) Taking as given the usual bigraded family of spheres $S^{p,q}$, one obtains a bigraded abelian group $\pi_{*,\star}A = \bigoplus_{p,q}\pi_{p,q}A$. But equipping this with a product requires fixing a choice of isomorphisms $\phi_{a,b}\colon S^{a_1,a_2}\wedge S^{b_1,b_2}\cong S^{a_1+b_1,a_2+b_2}$ in the stable homotopy category. For the product to be associative, a set of familiar pentagonal diagrams has to commute; when this happens, let us say that the collection of ϕ–isomorphisms is *coherent*.

(2) Let $\mathbb{S}$ denote the motivic sphere spectrum. The set of coherent collections of ϕ–isomorphisms is a torsor for the group $Z^2(\mathbb{Z}\times\mathbb{Z}, (\pi_{0,0}\mathbb{S})^*)$ of reduced 2–cocycles on the group $\mathbb{Z}\times\mathbb{Z}$ with values in the group of units in the ring $\pi_{0,0}\mathbb{S}$. In other words, if we fix one collection of coherent ϕ–isomorphisms, then any other such collection differs from it by such a reduced 2–cocycle. Recall here that a function $\alpha\colon \mathbb{Z}\times\mathbb{Z} \to (\pi_{0,0}\mathbb{S})^*$ is a 2–cocycle when $\alpha(u+v,w)\cdot\alpha(u,v) = \alpha(v,w)\cdot\alpha(u,v+w)$ for $u,v,w \in \mathbb{Z}^2$, and is reduced when $\alpha(0,0) = 1$.

(3) Two different choices of coherent ϕ–isomorphisms typically lead to two different ring structures on $\pi_{*,\star}A$. The difference 2–cocycle is a coboundary precisely when there is a bigraded isomorphism between these rings that multiplies elements of each bidegree $a = (a_1, a_2)$ by a fixed unit $e_a \in \pi_{0,0}(\mathbb{S})^*$. Such isomorphisms are called *standard isomorphisms* in [Dugger 2014].

See [Dugger 2014, Section 7] for details on the above.

It turns out that the ϕ–isomorphisms chosen in [Dugger 2014] lead to a different ring structure on $\pi_{*,\star}A$ than the one used by Voevodsky, even up to standard isomorphism. Of course, we can still translate between the two rings, and it is not exactly that one choice is right and one is wrong — if a person keeps their wits about them as far as remembering the different conventions, there are no contradictions. But below we will analyze a collection of different choices and make some suggestions about which ones seem ideal. We stress that the underlying symmetric monoidal structure of motivic spectra and the definition of homotopy groups are the same in [Dugger 2014; Voevodsky 2003]; it is only the choice of coherent ϕ–isomorphisms (not explicitly spelled out in [Voevodsky 2003], but in some sense there implicitly) that differs.

That multigraded objects have flexibility in sign conventions has been observed in situations other than motivic homotopy theory, for instance in supersymmetry [Deligne and Morgan 1999]. We comment on this, as well as on the connection to equivariant theory, in Remarks 2 and 3 below.

The signs they are a-changin'

Regardless of the base scheme, $\pi_{0,0}\mathbb{S}$ always contains the following four (not necessarily distinct) square roots of 1: 1, -1, ϵ and $-\epsilon$, where -1 and ϵ are given by $g \mapsto g^{-1}$ on the topological and Tate circles, S^1 and $\mathbb{G}_m$, respectively. When choosing the coherent isomorphisms

$$S^{a_1,a_2} \wedge S^{b_1,b_2} \cong S^{a_1+b_1,a_2+b_2},$$

where $S^{a_1,a_2} = (S^1)^{\wedge(a_1-a_2)} \wedge \mathbb{G}_m^{\wedge a_2}$, the convention in [Dugger 2014] was as follows: every time two S^1's are moved past each other, the sign -1 appears, and every time two $\mathbb{G}_m$'s are moved past each other, we get an ϵ. But swapping S^1's and $\mathbb{G}_m$'s is *not* assigned any punishment in [Dugger 2014]. This convention makes sense if S^1 and $\mathbb{G}_m$ are regarded as generic objects without any special relation between them, which was the case in the more general settings treated in [Dugger 2014]. However, this particular choice raises a problem: when the ground field is the complex numbers, Betti realization sends $\mathbb{G}_m$ to $\mathbb{G}_m(\mathbb{C}) \simeq S^1$, so moving a $\mathbb{G}_m$ past an S^1 *is* detected in topology. Consequently, with these conventions, the Betti realization map $\pi_{*,\star}X \to \pi_* X(\mathbb{C})$ is not a ring homomorphism — there is an annoying sign that comes up (see [Dugger 2014, Proposition 1.19]).

A better approach is to recognize that the isomorphism $S^{a_1,a_2} \wedge S^{b_1,b_2} \cong S^{a_1+b_1,a_2+b_2}$ should involve $a_2(b_1-b_2)$ swaps of $\mathbb{G}_m$'s past S^1's and we can choose to include a "generalized sign" factor to track this. To this end, choose once and for all a unit $u \in \pi_{0,0}\mathbb{S}$. In our applications we will have $u^2 = 1$ and u will play the role of a "generalized sign", but the basic setup only needs u to be invertible. If A is any motivic ring spectrum with unit map $\eta\colon \mathbb{S} \to A$, we may consider the $\mathbb{Z}\times\mathbb{Z}$–graded ring $(\pi_{*,\star}A, \cdot)$ provided by [Dugger 2014] and we may consider the alternative $(\pi_{*,\star}A, \cdot_u)$ with

$$x \cdot_u y = x \cdot y \cdot \eta u^{a_2(b_1-b_2)}$$

when $x \in \pi_{a_1,a_2}A$ and $y \in \pi_{b_1,b_2}A$ ("punishing" each swap of $\mathbb{G}_m$'s past S^1's by multiplying with u). Here $\alpha_u((a_1,a_2),(b_1,b_2)) = \eta u^{a_2(b_1-b_2)}$ is the 2–cocycle from our story. The cocycle condition gives associativity of $\cdot_u$, and the other axioms for a ring follow readily. If A is commutative then the same proof as for [Dugger 2014, Proposition 1.18] shows that $x \cdot y = y \cdot x \cdot (-1)^{(a_1-a_2)(b_1-b_2)}\epsilon^{a_2b_2}$. So

$$\begin{aligned} x \cdot_u y &= y \cdot_u x \cdot (-1)^{(a_1-a_2)(b_1-b_2)}\eta(\epsilon^{a_2b_2}u^{a_2(b_1-b_2)}u^{-b_2(a_1-a_2)}) \\ &= y \cdot_u x \cdot (-1)^{(a_1b_1+a_1b_2+a_2b_1+a_2b_2)}\eta(\epsilon^{a_2b_2}u^{a_2b_1-a_1b_2}) \\ &= y \cdot_u x \cdot (-1)^{a_1b_1}\eta(-u)^{a_2b_1-a_1b_2}\eta(-\epsilon)^{a_2b_2}. \end{aligned}$$

In particular, if $\eta(\epsilon) = \eta(u) = -1$ then $\eta(-\epsilon) = \eta(-u) = 1$ and thus

$$x \cdot_u y = y \cdot_u x \cdot (-1)^{a_1b_1}.$$

This is exactly Voevodsky's convention for commutativity in the dual Steenrod algebra: graded commutativity with respect to the total grading (see [Voevodsky 2003, Theorem 2.2]).

Remark 1 We used a special 2–cocycle in the above computations, but this wasn't necessary. For any reduced 2–cocycle α, we can define $x \cdot_\alpha y = x \cdot y \cdot \alpha((a_1,a_2),(b_1,b_2))$, and then there is an associated commutativity formula of the form

$$x \cdot_\alpha y = y \cdot_\alpha x \cdot w((a_1,a_2),(b_1,b_2)),$$

where w is a 2–cocycle that is skew-symmetric in the sense of $w(a,b) = w(b,a)^{-1}$. In fact,

$$w(a,b) = (-1)^{(a_1-a_2)(b_1-b_2)}\epsilon^{a_2b_2}\alpha(a,b)^{-1}\alpha(b,a).$$

Proposition *An invertible element u in $\pi_{0,0}\mathbb{S}$ gives a functor $A \mapsto (\pi_{*,\star}A, \cdot_u)$ from motivic ring spectra to $\mathbb{Z}\times\mathbb{Z}$–graded rings.*

Choosing $u = -1$ or $u = \epsilon$ gives graded rings conforming with Voevodsky's commutativity formulas for ring spectra A having the property that $\eta_A(\epsilon) = -1$. Choosing $u = 1$ gives the multiplication in [Dugger 2014].

Also, over the complex numbers, when choosing $u = -1$ or $u = \epsilon$, Betti realization gives a map of (commutative) graded rings by forgetting weight.

For a given choice of u, we can ask whether the rings $(\pi_{*,\star}A, \cdot)$ and $(\pi_{*,\star}A, \cdot_u)$ happen to be isomorphic via a standard isomorphism. Deciding this is equivalent to checking whether α_u is a coboundary. But, if β is a 1–cochain, then $(\delta\beta)(a,b) = \beta(a) - \beta(a+b) + \beta(b)$ and is therefore symmetric in a and b. As α_u is not symmetric, it is not a coboundary.

If $u^2 = 1$ then the subgroup $\langle u \rangle$ of $(\pi_{0,0}\mathbb{S})^*$ is just $\mathbb{Z}/2$, and since $B(\mathbb{Z}\times\mathbb{Z})$ is the 2–torus we have $H^2(\mathbb{Z}\times\mathbb{Z};\mathbb{Z}/2) = \mathbb{Z}/2$. So, as far as twisting by u goes (once u is fixed), there are only two different standard isomorphism classes of homotopy rings that can arise: these are represented by the products $\cdot$ and $\cdot_u$ that we saw above. Allowing arbitrary twists from the subgroup $\{1, -1, \epsilon, -\epsilon\}$ increases the number of possibilities to four.

Remark 2 Of course, these considerations hold in situations other than motivic homotopy theory. An interesting example is that of C_2–equivariant spectra. When over the real numbers, evaluating at complex points gives a symmetric monoidal functor from motivic spectra to C_2–equivariant spectra, where $S^1(\mathbb{C})$ corresponds to the trivial representation and $\mathbb{G}_m(\mathbb{C})$ to the sign representation σ. Thus, choosing your u's in the same way in the motivic and in the C_2–equivariant setting gives that Betti realization induces a map of (commutative) bigraded rings.

In this C_2–equivariant context, in addition to the forgetful map to nonequivariant spectra there is also the fixed-point functor $A \mapsto \phi A$. This induces maps of groups $\pi_{p,q}(A) \to \pi_{p-q}(\phi A)$, and so we can ask whether $\pi_{*,\star}(A) \to \pi_*(\phi A)$ is a ring homomorphism. For $u = -1$ it is not, but for $u = \epsilon$ it is. For this reason we suggest that $u = \epsilon$ is the best choice for both motivic and C_2–equivariant homotopy. With this convention, the graded-commutativity formula for the homotopy ring of a ring spectrum is

$$xy = yx \cdot (-1)^{a_1b_1}(-\epsilon)^{a_2b_1+a_1b_2+a_2b_2} = yx \cdot (-\epsilon)^{(a_1-a_2)(b_1-b_2)} \cdot \epsilon^{a_1b_1}$$

for $x \in \pi_{a_1,a_2}A$ and $y \in \pi_{b_1,b_2}A$.

Remark 3 We mention a connection to supersymmetry. Choosing $u = 1$ corresponds to the "Deligne convention" (see [Deligne and Morgan 1999, 1.2.8]), where commuting something in degree $a + b\sigma$ with something in degree $c + d\sigma$ would introduce the penalty $(-1)^{ac+bd\sigma}$ (where $(-1)^\sigma$ is the twist on the sign representation) while choosing $u = \epsilon$ would result in the "Bernstein convention" with sign $(-1)^{(a+b\sigma)(c+d\sigma)}$.

Remark 4 Another approach to these issues is to grade the stable homotopy ring by invertible objects rather than by isomorphism classes of invertible objects. This is sometimes referred to as a Pic–grading, though that terminology can be confusing since Pic is often used for isomorphism classes of invertible objects. For example, in the context of G–equivariant stable homotopy theory, one has to remember that $\pi_{V-W}X$ depends on the pair of representations (V, W) and not just on the class $V - W$ in $RO(G)$. In the Pic–grading, $\pi_{V\oplus W}X$ and $\pi_{W\oplus V}X$ are different groups, albeit isomorphic ones.

The Pic–grading eliminates all questions of sign choices: everything works out canonically. However, the cost is that one does not have a ring graded by a manageable collection of objects, so this approach is not conducive for computation. The sign issues considered in this paper arise when one tries to reduce the Pic–grading to something practical for computation.

The effect on the motivic stable homotopy ring

These considerations led us to wonder whether any well-known relations in the motivic stable homotopy groups change under the different sign conventions. For example, do any of the relations in [Dugger and Isaksen 2013] depend on the sign convention? See [Isaksen and Østvær 2020] for a recent survey article on motivic stable homotopy groups.

First of all, the relation $(1-\epsilon)\eta^2 = 0$ witnesses that ϵ *must* play a role in graded commutativity. When we commute the element η in $\pi_{1,1}$ past itself, a factor of ϵ appears. Note that $2\eta^2$ is not zero in general; this is detected in the $\mathbb{R}$–motivic homotopy groups.

Consider the list

$$\rho, \quad \eta, \quad \nu, \quad \sigma, \quad \eta_{\mathrm{top}}, \quad \nu_{\mathrm{top}}, \quad \sigma_{\mathrm{top}}$$

of elements of degrees

$$(-1,-1), \quad (1,1), \quad (3,2), \quad (7,4), \quad (1,0), \quad (3,0), \quad (7,0),$$

respectively. These seven elements are defined in the motivic stable homotopy ring over any base. As far as we are aware, the only way to produce additional "universal" examples is to assemble these elements with Toda brackets.

By inspection, it turns out the commutativity relations amongst these elements are the same when $u = 1$ or $u = \epsilon$. The "error" factor $\epsilon^{a_2 b_1 + a_1 b_2}$ is not equal to one in some cases. However, in all such cases, we are saved by the relations $(1-\epsilon)\rho = 0$ and $(1-\epsilon)\eta = 0$.

This observation led us to search further for an explicit example where the cases $u = 1$ and $u = \epsilon$ give different commutativity relations in the motivic stable homotopy ring. We inspected the 2–complete $\mathbb{R}$–motivic stable homotopy ring in a large range [Belmont and Isaksen 2022], and we found no possible differences. Similarly, a brief, speculative investigation of 3–complete homotopy yielded no examples.

On the other hand, assume that τ detects a stable homotopy element of degree $(0,-1)$. This assumption holds, for example, in the p–complete context over the field $\mathbb{C}$. Then the cases $u = 1$ and $u = \epsilon$ give different commutativity relations. For example, if $u = 1$, then $\tau\nu = -\nu\tau$ in $\pi_{3,1}$; but if $u = \epsilon$, then $\tau\nu = -\epsilon\nu\tau$.

This investigation led us to notice a pattern in the 2–complete $\mathbb{R}$–motivic stable homotopy groups that had not been previously observed.

Conjecture *Let α have degree (s, w) in the 2–complete $\mathbb{R}$–motivic stable homotopy ring. If $(1-\epsilon)\alpha$ is nonzero, then w is even.*

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References

[Belmont and Isaksen 2022] **E Belmont**, **D C Isaksen**, *$\mathbb{R}$–motivic stable stems*, J. Topol. 15 (2022) 1755–1793 MR Zbl

[Deligne and Morgan 1999] **P Deligne**, **J W Morgan**, *Notes on supersymmetry* (*following Joseph Bernstein*), from "Quantum fields and strings: a course for mathematicians, I" (P Deligne, P Etingof, D S Freed, L C Jeffrey, D Kazhdan, J W Morgan, D R Morrison, E Witten, editors), Amer. Math. Soc., Providence, RI (1999) 41–97 MR Zbl

[Dugger 2014] **D Dugger**, *Coherence for invertible objects and multigraded homotopy rings*, Algebr. Geom. Topol. 14 (2014) 1055–1106 MR Zbl

[Dugger and Isaksen 2013] **D Dugger**, **D C Isaksen**, *Motivic Hopf elements and relations*, New York J. Math. 19 (2013) 823–871 MR Zbl

[Hoyois et al. 2017] **M Hoyois**, **S Kelly**, **P A Østvær**, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. 19 (2017) 3813–3849 MR Zbl

[Isaksen and Østvær 2020] **D C Isaksen**, **P A Østvær**, *Motivic stable homotopy groups*, from "Handbook of homotopy theory" (H Miller, editor), CRC, Boca Raton, FL (2020) 757–791 MR Zbl

[Spitzweck 2018] **M Spitzweck**, *A commutative $\mathbb{P}^1$–spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. France 157, Soc. Math. France, Paris (2018) MR Zbl

[Voevodsky 2003] **V Voevodsky**, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. 98 (2003) 1–57 MR Zbl

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Coxeter systems with 2–dimensional Davis complexes, growth rates and Perron numbers

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We study growth rates of Coxeter systems with Davis complexes of dimension at most 2. We show that if the Euler characteristic χ of the nerve of a Coxeter system is vanishing (resp. positive), then its growth rate is a Salem (resp. Pisot) number. In this way, we extend results due to Floyd (1992) and Parry (1993). In the case where χ is negative, we provide infinitely many nonhyperbolic Coxeter systems whose growth rates are Perron numbers.

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1 Introduction

Let Γ be a finitely generated group with generating set S. For an element $x \in \Gamma$, we write $|x|_S$ for the word length with respect to S. The growth rate of (Γ, S) is defined by

$$\tau(\Gamma, S) = \limsup_{\ell\to\infty} \sqrt[\ell]{a_\ell},$$

where a_ℓ is the number of elements of Γ of word length ℓ. Gromov's polynomial growth theorem [1981] states that Γ has a nilpotent subgroup of finite index if and only if there exist positive constants $C > 0$ and $d > 0$ such that $a_\ell \le C\ell^d$ for $\ell \ge 0$. If (Γ, S) satisfies the latter property, then we say that (Γ, S) has *polynomial growth*. In this case, one has $\tau(\Gamma, S) = 1$. The pair (Γ, S) is said to have *exponential growth* when $\tau(\Gamma, S) > 1$. Note that there exist pairs of groups and finite generating sets which have neither polynomial growth nor exponential growth (see [Grigorchuk 1984] for example).

Suppose that (Γ, S) is an abstract Coxeter system; that is, Γ is generated by S and has the presentation

$$\Gamma = \langle s_1, \dots, s_N \mid (s_i s_j)^{k_{ij}} \text{ for } 1 \le i, j \le N\rangle,$$

where $k_{ii} = 1$ and $k_{ij} \ge 2$ (see Section 2.1). There are three types of Coxeter systems: spherical, affine, and otherwise. If (Γ, S) is spherical or affine, then it has polynomial growth. Therefore, our interest lies in the growth rates of nonspherical, nonaffine Coxeter systems. For instance, cofinite hyperbolic Coxeter systems are such Coxeter systems (see Section 2.2).

In the study of the growth rates of hyperbolic Coxeter systems, three kinds of real algebraic integers appear: Salem numbers, Pisot numbers, and Perron numbers (see Section 2.3). By results of Parry [1993], the growth rates of 2– and 3–dimensional cocompact hyperbolic Coxeter systems are Salem numbers. Floyd [1992] showed that the growth rates of 2–dimensional cofinite hyperbolic Coxeter systems are Pisot numbers. Moreover, their growth rates are limits of growth rates of 2–dimensional cocompact hyperbolic Coxeter systems. Yukita [2017; 2018] proved that the growth rates of 3–dimensional cofinite hyperbolic Coxeter systems are Perron numbers. Kolpakov [2012] proved that the growth rates of particular 3–dimensional cofinite hyperbolic Coxeter systems are Pisot numbers. With all the above considerations, we are interested in the relation between the geometric properties of Coxeter systems and the arithmetic nature of their growth rates as follows.

Let (Γ, S) be an abstract Coxeter system. Its *nerve* $L(\Gamma, S)$ is the abstract simplicial complex defined as follows (see Section 2.2). The vertex set is S. For a nonempty subset $T = \{s_{i_1}, \dots, s_{i_n}\} \subset S$, the vertices $s_{i_1}, \dots, s_{i_n}$ span an $(n-1)$–simplex if and only if T generates a finite subgroup of Γ. By abuse of notation, we write $L(\Gamma, S)$ for its geometric realization (see [Munkres 1984, Chapter 1, Section 3] for details). The *dimension of* (Γ, S) is defined as the maximal rank of a spherical parabolic subgroup of Γ, that is a subgroup generated by a subset of S. It coincides with the dimension of the Davis complex of (Γ, S); see [Davis 2008; Felikson and Tumarkin 2010].

In this paper, we study the arithmetic nature of the growth rates of nonspherical, nonaffine Coxeter systems (Γ, S) of dimension at most 2. We will prove the following main theorems.

Theorem A *If $\chi(L(\Gamma, S)) = 0$, then the growth rate $\tau(\Gamma, S)$ is a Salem number.*

Theorem B *If $\chi(L(\Gamma, S)) \geq 1$, then the growth rate $\tau(\Gamma, S)$ is a Pisot number. Moreover, there exists a sequence of Coxeter systems (Γ_n, S_n) with vanishing Euler characteristic such that the growth rate $\tau(\Gamma_n, S_n)$ converges to $\tau(\Gamma, S)$ from below.*

This paper is organized as follows. In Section 2, we provide the necessary background about Coxeter systems, their nerves, and their growth rates. Theorem A is discussed in Section 3 where we consider Coxeter systems with vanishing Euler characteristic. This extends the result by Parry [1993]. Section 4 is devoted to the study of Coxeter systems with positive Euler characteristic where we prove Theorem B generalizing Floyd's result [1992]. In Section 5, we provide some examples of infinite sequences of Coxeter systems with negative Euler characteristic whose growth rates are Perron numbers; see Proposition 5.1.

2 Preliminaries

2.1 Coxeter systems

For a group Γ with generating set $S = \{s_1, \dots, s_N\}$, the pair (Γ, S) is called a *Coxeter system* if Γ has the presentation

$$\Gamma = \langle s_1, \dots, s_N \mid (s_i s_j)^{k_{ij}} \text{ for } 1 \leq i, j \leq N \rangle,$$

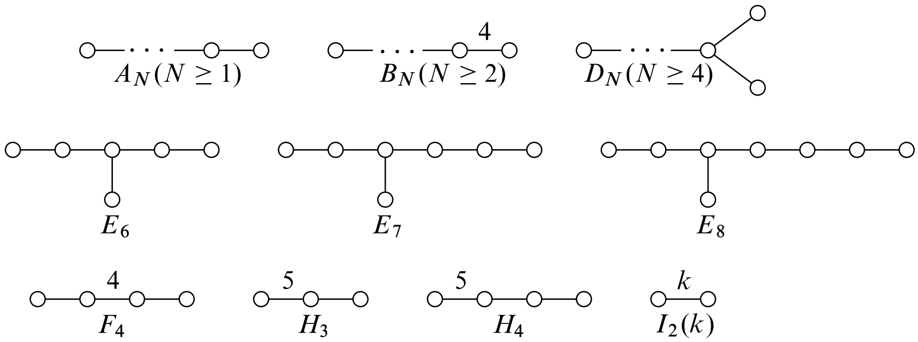

Figure 1: Irreducible spherical Coxeter systems of rank N.

where $k_{ii} = 1$ and $k_{ij} \geq 2$. In the case where $s_i s_j$ has infinite order, we put $k_{ij} = \infty$. The *rank* of a Coxeter system (Γ, S) is defined as the cardinality $\#S$ of S. For a subset $T \subset S$, the subgroup Γ_T of Γ generated by T is called a *parabolic subgroup* of Γ, with $\Gamma_\varnothing = \{1\}$ by convention.

Given a Coxeter system (Γ, S) of rank N, define the *cosine matrix* associated to (Γ, S) as the symmetric matrix $C_{(\Gamma,S)} = (c_{ij}) \in M_N(\mathbb{R})$ with entries

$$c_{ij} = \begin{cases} -\cos(\pi/k_{ij}) & \text{if } k_{ij} < \infty, \\ -1 & \text{if } k_{ij} = \infty. \end{cases}$$

The Coxeter system (Γ, S) is said to be *spherical* (resp. *affine*), if $C_{(\Gamma,S)}$ is positive definite (resp. positive semidefinite).

In this paper, a graph X is said to be *simple* if X has no loops or multiple edges. We associate to a Coxeter system (Γ, S) two kinds of edge-labeled simple graphs: the *Coxeter diagram* $\mathrm{Cox}(\Gamma, S)$ and the *presentation diagram* $X(\Gamma, S)$.

The *Coxeter diagram* $\mathrm{Cox}(\Gamma, S)$ is defined as follows. The vertex set is S. Two vertices s_i and s_j are connected by an edge if and only if $k_{ij} \geq 3$. The edge between s_i and s_j is labeled by k_{ij} if $k_{ij} \in \{4, 5, \dots\} \cup \{\infty\}$. A Coxeter system (Γ, S) is said to be *irreducible* if the underlying graph of $\mathrm{Cox}(\Gamma, S)$ is connected. It is known that a spherical (resp. affine) Coxeter system decomposes into a

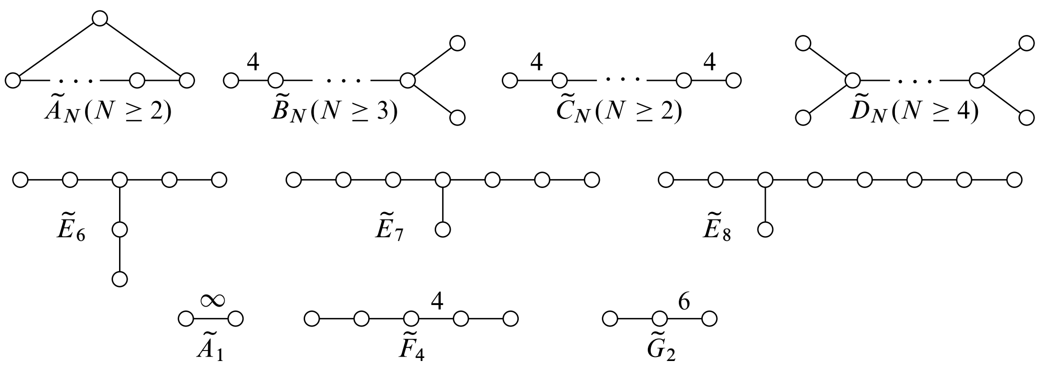

Figure 2: Irreducible affine Coxeter systems of rank $N + 1$.

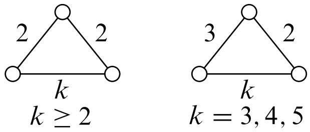

Figure 3: The presentation diagrams of the spherical Coxeter systems of rank 3.

direct product of irreducible spherical (resp. spherical and affine) Coxeter systems. The Coxeter diagrams of irreducible spherical and affine Coxeter systems are depicted in Figures 1 and 2, respectively (see [Humphreys 1990, pages 32 and 34]).

The *presentation diagram* $X(\Gamma, S)$ is defined as follows. The vertex set is S. Two vertices s_i and s_j are connected by an edge labeled by k_{ij} when $k_{ij} < \infty$. It follows that the underlying graphs of the presentation diagrams of spherical Coxeter systems of rank N are complete graphs with N vertices. For example, Figure 3 shows the presentation diagrams of the spherical Coxeter systems of rank 3.

Remark 2.1 If the Coxeter diagram $\mathrm{Cox}(\Gamma, S)$ (resp. the presentation diagram $X(\Gamma, S)$) of a Coxeter system (Γ, S) is disconnected, then (Γ, S) is a direct product (resp. a free product) of the Coxeter systems corresponding to the connected components.

2.2 Geometric Coxeter groups and nerves

For more details about geometric Coxeter groups and nerves of Coxeter systems we refer to [Davis 2008; Ratcliffe 1994].

Let us denote by $\mathbb{X}^n$ the n–dimensional spherical space $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$, or hyperbolic space $\mathbb{H}^n$. An *n–dimensional Coxeter polytope* $P \subset \mathbb{X}^n$ is the intersection of finitely many half-spaces whose interior is nonempty and dihedral angles are of the form π/k for $k \geq 2$ or equal to zero. Given an n–dimensional Coxeter polytope $P \subset \mathbb{X}^n$, the set S_P of the reflections in the bounding hyperplanes of P generates a discrete subgroup Γ_P of $\mathrm{Isom}\,(\mathbb{X}^n)$. The pair (Γ_P, S_P) is a Coxeter system, and is called an *n–dimensional geometric Coxeter system associated with P*. The group Γ_P is called the *n–dimensional geometric Coxeter group associated with P*. It is known that P is a fundamental polytope for Γ_P and the orbit $\{gP \mid g \in \Gamma_P\}$ of P gives rise to an exact tessellation of $\mathbb{X}^n$. Furthermore, Γ_P is said to be *cocompact* (resp. *cofinite*) when P is compact (resp. not compact but of finite volume). For a hyperbolic Coxeter polytope P, we say that Γ_P is *ideal* when every vertex of P lies on the boundary at infinity $\partial\mathbb{H}^n$. For each irreducible spherical (resp. affine) Coxeter system (Γ, S), there exists a spherical (resp. compact Euclidean) Coxeter polytope P such that $(\Gamma, S) = (\Gamma_P, S_P)$. Therefore, if (Γ, S) is a spherical (resp. affine) Coxeter system, then Γ is finite (resp. virtually nilpotent). In contrast to this, if (Γ, S) is nonspherical and nonaffine, then Γ contains a free group of rank at least 2; see [de la Harpe 1987].

Let (Γ, S) be an abstract Coxeter system. The *nerve* $L(\Gamma, S)$ is an abstract simplicial complex defined as follows. The vertex set is S, and for a nonempty subset $T = \{s_{i_1}, \ldots, s_{i_n}\} \subset S$, the vertices $s_{i_1}, \ldots, s_{i_n}$

span an $(n-1)$–simplex if and only if the parabolic subgroup Γ_T is finite. For simplicity of notation, we continue to write $L(\Gamma, S)$ for its geometric realization (see [Munkres 1984, Chapter 1, Section 3] for details). The *dimension of* (Γ, S), denoted by $\dim(\Gamma, S)$, is defined as the maximal rank of a spherical parabolic subgroup of Γ, that is a subgroup generated by a subset of S. It coincides with the dimension of the Davis complex of (Γ, S); see [Davis 2008; Felikson and Tumarkin 2010].

In this paper, we consider Coxeter systems of dimension at most 2. In particular, such a class of Coxeter systems contains hyperbolic Coxeter groups of dimension 2 and ideal hyperbolic Coxeter groups of dimension 3. Indeed, for such groups, maximal spherical subgroups are of rank at most 2. For a Coxeter system (Γ, S) of dimension at most 2, it is easy to see that the underlying graph of $X(\Gamma, S)$ is the geometric realization of the nerve $L(\Gamma, S)$. Therefore the Euler characteristic $\chi(L(\Gamma, S))$ equals the one of the underlying graph of $X(\Gamma, S)$. It is known that the Euler characteristic of a graph is the number of vertices minus the number of edges.

2.3 Growth rates of Coxeter systems

Let (Γ, S) be a Coxeter system. For $x \in \Gamma$, we define its *word length with respect to* S by

$$|x|_S = \min\{n \in \mathbb{N} \mid x = s_1 \cdots s_n \ (s_1, \dots, s_n \in S)\}.$$

By convention, $|1|_S = 0$. The *growth series* $f_{(\Gamma,S)}(z)$ of (Γ, S) is defined by

$$f_{(\Gamma,S)}(z) = \sum_{\ell \geq 0} a_\ell z^\ell,$$

where a_ℓ is the number of the elements of Γ of word length ℓ. If (Γ, S) is spherical, then $f_{(\Gamma,S)}(z)$ is a polynomial and called the *growth polynomial* of (Γ, S).

By a result of Solomon [1966], the growth polynomials of spherical Coxeter systems can be computed in terms of its exponents. For the list of exponents, see [Humphreys 1990]. For example, the exponents of A_N are given by $1, 2, \dots, N$, and those of $I_2(k)$ are $1, k-1$. For positive integers $m, m_1, \dots, m_r$, we put

$$[m] = 1 + z + \cdots + z^{m-1} \quad \text{and} \quad [m_1, \dots, m_r] = [m_1] \cdots [m_r].$$

Solomon's formula states that for a spherical Coxeter system (Γ, S) with the exponents $m_1, \dots, m_r$, one has $f_{(\Gamma,S)}(z) = [m_1 + 1, \dots, m_r + 1]$.

If (Γ, S) is nonspherical, then the inverse of the radius of convergence of $f_{(\Gamma,S)}(z)$ is called the *growth rate* of (Γ, S), denoted by $\tau(\Gamma, S)$. The Cauchy–Hadamard formula gives

$$\tau(\Gamma, S) = \limsup_{\ell \to \infty} \sqrt[\ell]{a_\ell}.$$

Since free abelian groups of finite rank have polynomial growth [Wolf 1968], and any affine Coxeter system contains a free abelian subgroup of finite rank and finite index, the growth rate of an affine Coxeter system is 1.

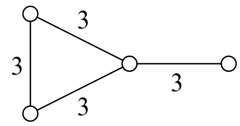

Figure 4: The presentation diagram of $(\Gamma_\star, S_\star)$.

Remark 2.2 If a Coxeter system (Γ, S) decomposes as $(\Gamma_1, S_1) \times (\Gamma_2, S_2) \times \cdots \times (\Gamma_l, S_l)$, then its growth series satisfies $f_{(\Gamma,S)} = \Pi_i f_{(\Gamma_i,S_i)}$. It follows for the growth rate that $\tau(\Gamma, S) = \max_i \tau(\Gamma_i, S_i)$. This does not hold when (Γ, S) decomposes as a free product.

The following formula, established by Steinberg, is an important tool to compute the growth series of Coxeter systems.

Theorem 2.3 (Steinberg's formula [1968]) *Let (Γ, S) be a Coxeter system. Then the identity*

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \sum_{\substack{T\subset S\\ \#\Gamma_T<\infty}} \frac{(-1)^{\#T}}{f_{(\Gamma_T,T)}(z)} \tag{2-1}$$

holds for the growth series $f_{(\Gamma,S)}(z)$.

Steinberg's formula implies that the growth series is a rational function and satisfies that

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are monic polynomials with integer coefficients. It follows that the growth rate $\tau(\Gamma, S)$ is the real root of $P(z)$ whose modulus is maximal among the roots of $P(z)$, and hence $\tau(\Gamma, S) \geq 1$ is a real algebraic integer.

Example 2.4 Consider the abstract Coxeter system $(\Gamma_\star, S_\star)$ whose presentation diagram is depicted in Figure 4. The spherical subgroups are A_1 and A_2, both with multiplicity four. By Steinberg's formula (2-1), we compute its growth series

$$\frac{1}{f_{(\Gamma_\star,S_\star)}(z^{-1})} = 1 - \frac{4}{[2]} + \frac{4}{[2,3]} = \frac{[2,3]-4[3]+4}{[2,3]}.$$

We write $P(z)$ for the numerator of $1/f_{(\Gamma_\star,S_\star)}(z^{-1})$; that is,

$$P(z) = 1 - 2z - 2z^2 + z^3.$$

One easily sees that $P(-1) = 0$ and that the greatest positive root of $P(z)$ is given by

$$\tau(\Gamma_\star, S_\star) = \frac{3+\sqrt{5}}{2} = \frac{1}{(\varphi-1)^2},$$

where φ is the golden ratio.

Example 2.5 If (Γ, S) is a Coxeter system of rank N whose presentation diagram $X(\Gamma, S)$ has no edges, then $\tau(\Gamma, S) = N - 1$. Indeed, we compute by Steinberg's formula (2-1)

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} = \frac{z-(N-1)}{[2]}.$$

Example 2.6 If (Γ, S) is a Coxeter system of rank N whose presentation diagram $X(\Gamma, S)$ is a tree with edges labeled by 2 only, then

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} + \frac{N-1}{[2,2]} = \frac{[2,2]-N[2]+N-1}{[2,2]} = \frac{z(z-(N-2))}{(1+z)^2}.$$

Observe that the growth series does not depend on the isomorphism type of the tree, only on the number of its vertices. Therefore, the growth rate is given by $\tau(\Gamma, S) = N - 2$.

From now on, we focus on the growth rates of nonspherical, nonaffine Coxeter systems. Three kinds of real algebraic integers appear in the study of the growth rates of hyperbolic Coxeter systems: Salem numbers, Pisot numbers, and Perron numbers (see [Bertin et al. 1992, page 84]).

An algebraic integer $\tau > 1$ is called a *Salem number* if it is a quadratic unit or is such that the inverse τ^{-1} is a Galois conjugate of τ and the other Galois conjugates lie on the unit circle. The minimal polynomial of a Salem number is called a *Salem polynomial*. Parry showed that the growth rates of 2– and 3–dimensional cocompact hyperbolic Coxeter systems are Salem numbers [Parry 1993].

An algebraic integer $\tau > 1$ is called a *Pisot number* if τ is an integer or if all of its other Galois conjugates are contained in the unit open disk. The minimal polynomial of a Pisot number is called a *Pisot polynomial*. Floyd showed that the growth rates of 2–dimensional cofinite hyperbolic Coxeter systems are Pisot numbers [Parry 1993]. Moreover, for a 2–dimensional cofinite hyperbolic Coxeter systems (Γ, S), there exists a sequence of 2–dimensional cocompact hyperbolic Coxeter systems (Γ_n, S_n) whose growth rates $\tau(\Gamma_n, S_n)$ converges to $\tau(\Gamma, S)$ from below.

An algebraic integer $\tau > 1$ is called a *Perron number* if τ is an integer or if all of its other Galois conjugates are strictly less than τ in absolute value. Note that Salem numbers and Pisot numbers are Perron numbers. Yukita [2017; 2018]] showed that the growth rates of 3–dimensional cofinite hyperbolic Coxeter systems are Perron numbers. Note that Komori and Yukita [2015] and Nonaka and Kellerhals [2017] showed that the growth rates of cofinite 3–dimensional hyperbolic ideal Coxeter systems are Perron numbers. For a 4–dimensional cocompact Coxeter system (Γ_P, S_P), Kellerhals and Perren [2011] proved that the growth rates are Perron numbers for $\#S_P = 5$ and 6. In particular, they conjectured that the growth rates of hyperbolic Coxeter systems are Perron numbers.

This is a motivation to relate geometric properties of Coxeter systems to the arithmetic nature of their growth rates. The aim of this paper is to extend the results of Floyd and Parry to *nonspherical*, *nonaffine*, *and nonhyperbolic* Coxeter systems of dimension at most 2. Note that Charney and Davis [1991] studied the relationship between the geometry of nerves and reciprocity of the growth series.

We use the partial order on the set of Coxeter systems defined by McMullen [2002]. Let (Γ, S) and (Γ', S') be Coxeter systems. Write $(\Gamma, S) \preceq (\Gamma', S')$ when there exists an injection $\iota : S \to S'$ such that $k(s,t) \le k'(\iota(s), \iota(t))$, where $k(s,t)$ and $k'(\iota(s), \iota(t))$ are the orders of st and $\iota(s)\iota(t)$, respectively.

Theorem 2.7 [Terragni 2016, Corollary 3.2] *If $(\Gamma, S) \preceq (\Gamma', S')$, then $\tau(\Gamma, S) \le \tau(\Gamma', S')$.*

For a finitely generated group Γ with ordered finite generating set S with $\#S = N$, we call the pair (Γ, S) an *N–marked group*. Given two N–marked groups (Γ, S) and (Γ', S') we say that they are isomorphic as marked groups when the map $\iota : S \to S'$ sending s_i to s_i' extends to a group isomorphism between Γ and Γ'. The *space of N–marked groups* is the set of isomorphism classes of N–marked groups equipped with a metric topology, given by the Chabauty–Grigorchuk topology; see [Grigorchuk 1984]. Let us denote by $\mathscr{C}_N$ the set of marked Coxeter systems of rank N. Yukita [2024] studied the space $\mathscr{C}_N$ and showed that $\mathscr{C}_N$ is compact.

Theorem 2.8 [Yukita 2024, Theorems 3.2 and 3.5] *Let $\{(\Gamma_n, S_n)\}$ and (Γ, S) be marked Coxeter systems of rank N. We write $k_{ij}(n)$ (resp. k_{ij}) for the order of $s_i(n)s_j(n)$ in Γ_n (resp. $s_i s_j$ in Γ).*

1. *The sequence $\{(\Gamma_n, S_n)\}$ converges to (Γ, S) if and only if $\lim_{n\to\infty} k_{ij}(n) = k_{ij}$ for $1 \le i, j \le N$.*
2. *If $\lim_{n\to\infty}(\Gamma_n, S_n) = (\Gamma, S)$, then $\lim_{n\to\infty} \tau(\Gamma_n, S_n) = \tau(\Gamma, S)$.*

3 Growth rates of Coxeter systems with vanishing Euler characteristic

Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 such that $\chi(L(\Gamma, S)) = 0$, where $L(\Gamma, S)$ denotes the geometric realization of its nerve. In this section, we prove that the growth rate $\tau(\Gamma, S)$ is a Salem number.

We write N (resp. E) for the number of vertices (resp. edges) of the presentation diagram $X(\Gamma, S)$. Recall that the Euler characteristic of a graph is the number of vertices minus the number of edges. Since the dimension of (Γ, S) is at most 2, the underlying graph of $X(\Gamma, S)$ coincides with $L(\Gamma, S)$, and hence $N = E$. Suppose that the set of labels of the edges of $X(\Gamma, S)$ is $\{k_1, \dots, k_r\}$. Let us denote by E_i the number of edges of $X(\Gamma, S)$ labeled by k_i.

We obtain the equality

$$\begin{aligned}\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2,k_i]} &= 1 - \frac{E_1 + \cdots + E_r}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2,k_i]} \\ &= 1 + \sum_{i=1}^{r} \frac{E_i}{[2]}\left(\frac{1}{[k_i]} - 1\right) \\ &= 1 + \sum_{i=1}^{r} \frac{E_i}{[2]}\left(\frac{z-1}{z^{k_i}-1} - 1\right) = 1 + \sum_{i=1}^{r} E_i \frac{z - z^{k_i}}{(z+1)(z^{k_i}-1)}\end{aligned}$$

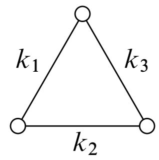

Figure 5: The presentation diagram in the case $N = 3$.

by Steinberg's formula (2-1); see also [Parry 1993, page 413]. Hence,

$$\frac{z+1}{(z-1)f_{(\Gamma,S)}(z^{-1})} = \frac{z+1}{z-1} + \sum_{i=1}^{r} E_i \frac{z-z^{k_i}}{(z-1)(z^{k_i}-1)}. \tag{3-1}$$

The following lemma is fundamental for the proof.

Lemma 3.1 [Parry 1993, Corollary 1.8] *Given integers $k_1, \dots, k_r \geq 2$ and $E_1, \dots, E_r \geq 1$, suppose that*

$$\sum_{i=1}^{r}\Big(1-\frac{1}{k_i}\Big)E_i > 2. \tag{3-2}$$

Let $R(z)$ be the rational function defined by

$$R(z) = \frac{z+1}{z-1} + \sum_{i=1}^{r} E_i \frac{z-z^{k_i}}{(z-1)(z^{k_i}-1)}.$$

Then $R(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are relatively prime monic polynomials with integer coefficients and equal degrees, and $P(z)$ is a product of distinct irreducible cyclotomic polynomials and exactly one Salem polynomial.

Theorem 3.2 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2. If $\chi(L(\Gamma, S)) = 0$, then the growth rate $\tau(\Gamma, S)$ is a Salem number.*

Proof We apply Lemma 3.1 to (3-1). The proof is divided into three cases: the cases $N = 3$, $N = 4$, and $N \geq 5$.

(i) Assume $N = 3$. By assumption, $N = E = 3$, and hence the presentation diagram of $X(\Gamma, S)$ is as in Figure 5.

Since (Γ, S) is nonspherical and nonaffine,

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1.$$

Therefore,

$$\Big(1-\frac{1}{k_1}\Big) + \Big(1-\frac{1}{k_2}\Big) + \Big(1-\frac{1}{k_3}\Big) > 2.$$

(ii) Assume $N = 4$. The presentation diagram $X(\Gamma, S)$ is one of the diagrams in Figure 6. We show that one of the labels of $X(\Gamma, S)$ is at least 3.

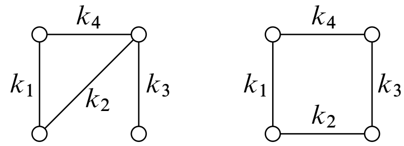

Figure 6: The presentation diagrams in the case $N = 4$.

Suppose that $X(\Gamma, S)$ is the diagram in Figure 6, left. If $k_1 = k_2 = k_4 = 2$, then the vertices of the triangle generates a spherical parabolic subgroup of Γ of rank 3. This contradicts the fact that the dimension of (Γ, S) is at most 2. Therefore, one of the labels is at least 3. Suppose that $X(\Gamma, S)$ is the diagram in Figure 6, right. If $k_1 = k_2 = k_3 = k_4 = 2$, then the Coxeter diagram $\mathrm{Cox}(\Gamma, S)$ is made of two connected components $\widetilde{A}_1$ (see Figure 2 for $\widetilde{A}_1$). This is a contradiction to the fact that (Γ, S) is nonspherical and nonaffine. Therefore, one of the labels is at least 3. Hence,

$$\left(1-\frac{1}{k_1}\right)+\left(1-\frac{1}{k_2}\right)+\left(1-\frac{1}{k_3}\right)+\left(1-\frac{1}{k_4}\right) \geq 3\left(1-\tfrac{1}{2}\right)+\left(1-\tfrac{1}{3}\right) > 2.$$

(iii) Assume $N \geq 5$. It follows that

$$\sum_{i=1}^{r}\left(1-\frac{1}{k_i}\right)E_i = \sum_{i=1}^{r} E_i - \sum_{i=1}^{r}\frac{E_i}{k_i} = N - \sum_{i=1}^{r}\frac{E_i}{k_i} \geq N - \sum_{i=1}^{r}\frac{E_i}{2} = \frac{N}{2} \geq \frac{5}{2} > 2.$$

Therefore, (3-2) holds, and the assertion follows from Lemma 3.1. □

For later use, we show the following.

Lemma 3.3 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2. Suppose that the growth series $f_{(\Gamma,S)}(z)$ satisfies the equality*

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \frac{P(z)}{[2, k_1, \dots, k_r]},$$

where $P(z)$ is a monic polynomial with integer coefficients. If $\chi(L(\Gamma, S)) = 0$, then $P(z)$ is a product of cyclotomic polynomials and exactly one Salem polynomial.

Proof As in the proof of Theorem 3.2, we apply Lemma 3.1 to (3-1):

$$\frac{z+1}{(z-1)\,f_{(\Gamma,S)}(z^{-1})} = \frac{P_0(z)}{Q_0(z)},$$

where $P_0(z)$ and $Q_0(z)$ are the relatively prime polynomials with integer coefficients. P_0 is a product of distinct irreducible cyclotomic polynomials and exactly one Salem polynomial. By assumption, we have

$$\frac{P(z)}{[2, k_1, \dots, k_r]} = \frac{(z-1)\,P_0(z)}{(z+1)\,Q_0(z)}. \tag{3-3}$$

Since every factor of the polynomial $[2, k_1, \dots, k_r]$ is a cyclotomic polynomial, the equality (3-3) implies that $P(z)$ is a product of cyclotomic polynomials and exactly one Salem polynomial. □

4 Growth rates of Coxeter systems with positive Euler characteristic

Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 such that $\chi(L(\Gamma, S)) \geq 1$, where $L(\Gamma, S)$ denotes the geometric realization of its nerve. Recall that $\chi(L(\Gamma, S))$ equals the Euler characteristic of the underlying graph of $X(\Gamma, S)$. In this section, we prove that the growth rate $\tau(\Gamma, S)$ is a Pisot number.

Lemma 4.1 *Let (Γ, S) be a nonspherical, nonaffine marked Coxeter system of dimension at most 2 and rank N. Suppose that either the presentation diagram $X(\Gamma, S)$ is disconnected, or has an edge labeled by $k \geq 3$. If $\chi(L(\Gamma, S)) \geq 1$, then there exists a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n \geq 7}$ of rank N such that for $n \geq 7$,*

(1) $(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S)$;

(2) $\dim(\Gamma_n, S_n) \leq 2$;

(3) $\chi(L(\Gamma_n, S_n)) = \chi(L(\Gamma, S)) - 1$;

(4) *the sequence $\{(\Gamma_n, S_n)\}_{n \geq 7}$ converges to (Γ, S) in the space $\mathscr{C}_N$ of marked Coxeter systems of rank N.*

Proof Set $S = \{s_1, \dots, s_N\}$. We denote by E and k_{ij} the number of edges of $X(\Gamma, S)$ and the order of the product $s_i s_j$, respectively.

Suppose first that the underlying graph of the presentation diagram $X(\Gamma, S)$ is disconnected. Let s_p and s_q be two vertices of different connected components of the underlying graph of $X(\Gamma, S)$. It follows that $k_{pq} = \infty$. For $n \geq 7$, we define a marked Coxeter system (Γ_n, S_n) of rank N by the presentation

$$\Gamma_n = \langle s_1(n), \dots, s_N(n) \mid (s_i(n) s_j(n))^{k_{ij}(n)} = 1 \text{ for } 1 \leq i, j \leq N \rangle,$$

where

$$k_{ij}(n) = \begin{cases} n & \text{if } \{i, j\} = \{p, q\}, \\ k_{ij} & \text{otherwise.} \end{cases}$$

We will show that (Γ_n, S_n) satisfies the desired properties. For $1 \leq i, j \leq N$ and $n \geq 7$, we have $k_{ij}(n) \leq k_{ij}(n+1) \leq k_{ij}$, so

$$(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S).$$

In order to show that $\dim(\Gamma_n, S_n) \leq 2$, it is sufficient to see that the presentation diagram $X(\Gamma_n, S_n)$ does not contain any of the diagrams depicted in Figure 3. Since $\dim(\Gamma, S) \leq 2$, no such diagram is contained in $X(\Gamma, S)$. The presentation diagram $X(\Gamma_n, S_n)$ is obtained from $X(\Gamma, S)$ by adding an edge between s_p and s_q labeled by n (see Figure 7). In Figure 7, we do not put labels of the edges other than the added edge for simplicity. Since the vertices s_p and s_q lie in different connected components of the underlying graph of $X(\Gamma, S)$, every cycle of the underlying graph of $X(\Gamma_n, S_n)$ comes from one of $X(\Gamma, S)$. Hence we see that $X(\Gamma_n, S_n)$ does not contain any of the diagrams depicted in Figure 3. The Euler characteristics of the underlying graphs of $X(\Gamma_n, S_n)$ and $X(\Gamma, S)$ are equal to $\chi(L(\Gamma_n, S_n))$

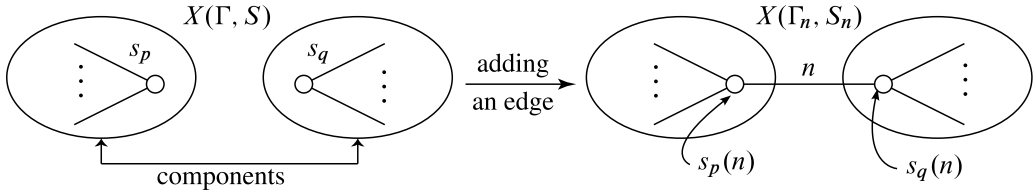

Figure 7: Adding an edge between s_p and s_q.

and $\chi(L(\Gamma, S))$, respectively. This observation implies that $\{(\Gamma_n, S_n)\}_{n\geq 7}$ satisfies the property (3). By definition of (Γ_n, S_n), we have $\lim_{n\to\infty} k_{ij}(n) = k_{ij}$ for $1 \leq i, j \leq N$. Property (1) of Theorem 2.8 implies that $\{(\Gamma_n, S_n)\}_{n\geq 7}$ converges to (Γ, S) in $\mathscr{C}_N$.

Suppose next that the underlying graph of $X(\Gamma, S)$ is connected, and let us show that the underlying graph is a tree. Since every connected graph with the Euler characteristic 1 is a tree, it is sufficient to show that $\chi(L(\Gamma, S)) = 1$. By the connectivity of the underlying graph of $X(\Gamma, S)$, there exists a spanning tree T of the graph. We denote by N_T and E_T the number of vertices and of edges of T, respectively. It follows that $N = N_T$, $E_T \leq E$, and $N_T - E_T = 1$. Since $\chi(L(\Gamma, S)) = N - E \geq 1$,

$$1 \leq N - E \leq N - E_T = N_T - E_T = 1,$$

and hence $\chi(L(\Gamma, S)) = 1$.

Since Coxeter systems of rank at most 2 are spherical or affine, our assumption implies that $N \geq 3$. Also by assumption, there exists an edge e between vertices s_p and s_q of $X(\Gamma, S)$, labeled by $k_{pq} \geq 3$. Since the underlying graph of $X(\Gamma, S)$ is a tree with at least 3 vertices, we can find an edge e' incident with e. Without loss of generality we can assume that e and e' share the vertex s_q. We write s_r for the endpoint of e' other than s_q. Since the underlying graph of $X(\Gamma, S)$ is a tree, the vertices s_p and s_r are not joined by an edge. It follows that $k_{pr} = \infty$. For $n \geq 7$, we define a marked Coxeter system (Γ_n, S_n) of rank N by the presentation

$$\Gamma_n = \langle s_1(n), \ldots, s_N(n) \mid (s_i(n)s_j(n))^{k_{ij}(n)} = 1 \text{ for } 1 \leq i, j \leq N \rangle,$$

where

$$k_{ij}(n) = \begin{cases} n & \text{if } \{i, j\} = \{p, r\}, \\ k_{ij} & \text{otherwise.} \end{cases}$$

We will show that (Γ_n, S_n) satisfies the desired properties. For $1 \leq i, j \leq N$ and $n \geq 7$, we have $k_{ij}(n) \leq k_{ij}(n+1) \leq k_{ij}$, so $(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S)$.

The presentation diagram $X(\Gamma_n, S_n)$ is obtained from $X(\Gamma, S)$ by adding an edge between s_p and s_r labeled by n (see Figure 8). In Figure 8, we do not put labels of the edges other than three edges joining two of s_p, s_q, and s_r for simplicity.

Since the underlying graph of $X(\Gamma, S)$ is a tree, the graph of $X(\Gamma_n, S_n)$ has only one cycle and the cycle consists of three edges joining two of s_p, s_q, and s_r. Therefore, the presentation diagram $X(\Gamma_n, S_n)$ does

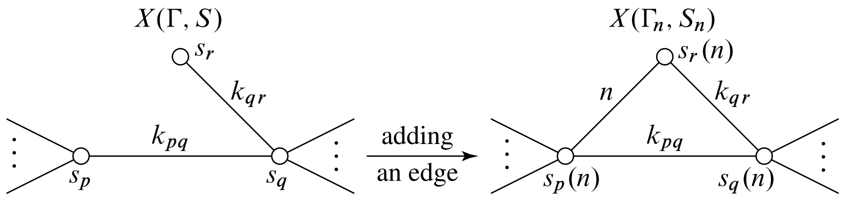

Figure 8: Adding an edge between s_p and s_r.

not contain any of the diagrams in Figure 3, which is due to the fact that $k_{pq} \geq 3$ and $n \geq 7$. It follows that $\dim(\Gamma_n, S_n) \leq 2$. The same reasoning as before allows one to conclude that

$$\chi(L(\Gamma_n, S_n)) = \chi(L(\Gamma, S)) - 1.$$

By definition of (Γ_n, S_n), we have $\lim_{n\to\infty} k_{ij}(n) = k_{ij}$ for $1 \leq i, j \leq n$, and Property (1) of Theorem 2.8 implies that $\{(\Gamma_n, S_n)\}_{n\geq 7}$ converges to (Γ, S) in $\mathscr{C}_N$. □

Remark 4.2 Suppose that (Γ, S) is a nonspherical, nonaffine Coxeter system of at most dimension 2 such that $\chi(L(\Gamma, S)) \geq 1$. If (Γ, S) does not satisfy the hypothesis in Lemma 4.1, the presentation diagram $X(\Gamma, S)$ is connected and its edges are labeled by 2 only. As shown in the proof, in this case, the positivity of the Euler characteristic forces $X(\Gamma, S)$ to be a tree.

Corollary 4.3 *Let (Γ, S) be a nonspherical, nonaffine marked Coxeter system of dimension at most 2 and rank N such that $\chi(L(\Gamma_n, S_n)) \geq 1$. Suppose that either the presentation diagram $X(\Gamma, S)$ is disconnected, or has an edge labeled by $k \geq 3$. Then there exists a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n\geq 7}$ of rank N such that for $n \geq 7$,*

(1) $(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S)$;

(2) $\dim(\Gamma_n, S_n) \leq 2$;

(3) $\chi(L(\Gamma_n, S_n)) = 0$;

(4) *the sequence $\{(\Gamma_n, S_n)\}$ converges to (Γ, S) in the space $\mathscr{C}_N$ of marked Coxeter systems of rank N.*

Proof We take a sequence of marked Coxeter systems $\{(\Gamma_{n_1}, S_{n_1})\}_{n_1\geq 7}$ of rank N as in Lemma 4.1. If $\chi(L(\Gamma, S)) = 1$, then for $n_1 \geq 7$,

$$\chi(L(\Gamma_{n_1}, S_{n_1})) = \chi(L(\Gamma, S)) - 1 = 0.$$

Hence the sequence $\{(\Gamma_{n_1}, S_{n_1})\}_{n_1\geq 7}$ satisfies the properties in Corollary 4.3.

Suppose that $\chi(L(\Gamma, S)) \geq 2$. The presentation diagram $X(\Gamma_{n_1}, S_{n_1})$ has an edge labeled by $n_1 \geq 7$ and $\chi(L(\Gamma_{n_1}, S_{n_1})) = \chi(L(\Gamma, S)) - 1 \geq 1$. For each $n_1 \geq 7$, by applying Lemma 4.1 to (Γ_{n_1}, S_{n_1}), there exists a sequence of marked Coxeter systems $\{(\Gamma_{n_1,n_2}, S_{n_1,n_2})\}_{n_2\geq 7}$ of rank N satisfying the properties in

Lemma 4.1. Moreover, we may assume that $(\Gamma_{n_1,n_2}, S_{n_1,n_2}) \preceq (\Gamma_{n'_1,n'_2}, S_{n'_1,n'_2})$ for $n_1 \le n'_1$ and $n_2 \le n'_2$. If $\chi(L(\Gamma, S)) = 2$, then for $n_1, n_2 \ge 7$,

$$\chi(L(\Gamma_{n_1,n_2}, S_{n_1,n_2})) = \chi(L(\Gamma_{n_1}, S_{n_1})) - 1 = \chi(L(\Gamma, S)) - 2 = 0.$$

Therefore, the diagonal subsequence $\{(\Gamma_{n,n}, S_{n,n})\}_{n\ge 7}$ satisfies the properties in Corollary 4.3. Repeating this procedure until the Euler characteristic vanishes completes the proof. □

Let (Γ, S) be a nonspherical, nonaffine marked Coxeter system of dimension at most 2 with $\chi(L(\Gamma, S)) \ge 1$. For simplicity, we write χ instead of $\chi(L(\Gamma, S))$.

We denote by N (resp. E) the number of vertices (resp. edges) of the presentation diagram $X(\Gamma, S)$. It follows that $N - E = \chi \ge 1$. Suppose that the set of labels of the edges of $X(\Gamma, S)$ is $\{k_1, \dots, k_r\}$. Let us write E_i for the number of edges of $X(\Gamma, S)$ labeled by k_i, so $E = E_1 + \cdots + E_r$.

We obtain the equality

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2, k_i]} = 1 - \frac{E+\chi}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2, k_i]}$$

by Steinberg's formula (2-1); see also [Floyd 1992, page 479]. Therefore,

$$\begin{aligned}
\frac{1}{f_{(\Gamma,S)}(z^{-1})} &= \frac{[2, k_1, \dots, k_r] - (E+\chi)[k_1, \dots, k_r] + \sum_{i=1}^{r} E_i [k_1, \dots, \hat{k}_i, \dots, k_r]}{[2, k_1, \dots, k_r]} \\
&= \frac{[2, k_1, \dots, k_r] + \sum_{i=1}^{r} E_i (1 - [k_i])[k_1, \dots, \hat{k}_i, \dots, k_r] - \chi[k_1, \dots, k_r]}{[2, k_1, \dots, k_r]} \\
&= \frac{[2, k_1, \dots, k_r] - \sum_{i=1}^{r} E_i z [k_i - 1][k_1, \dots, \hat{k}_i, \dots, k_r] - \chi[k_1, \dots, k_r]}{[2, k_1, \dots, k_r]} \\
&= \frac{[2, k_1, \dots, k_r] - \sum_{i=1}^{r} E_i z [k_1, \dots, k_i - 1, \dots, k_r] - \chi[k_1, \dots, k_r]}{[2, k_1, \dots, k_r]}.
\end{aligned}$$

If $\chi = 1$, then

$$\begin{aligned}
\frac{1}{f_{(\Gamma,S)}(z^{-1})} &= \frac{([2]-1)[k_1, \dots, k_r] - \sum_{i=1}^{r} E_i z [k_1, \dots, k_i - 1, \dots, k_r]}{[2, k_1, \dots, k_r]} \\
&= \frac{z\bigl([k_1, \dots, k_r] - \sum_{i=1}^{r} E_i [k_1, \dots, k_i - 1, \dots, k_r]\bigr)}{[2, k_1, \dots, k_r]}.
\end{aligned}$$

We define the polynomial $P(z)$ as

$$P(z) = \begin{cases} [k_1, \dots, k_r] - \sum_{i=1}^{r} E_i [k_1, \dots, k_i - 1, \dots, k_r] & \text{if } \chi = 1, \\ [2, k_1, \dots, k_r] - \sum_{i=1}^{r} E_i z [k_1, \dots, k_i - 1, \dots, k_r] - \chi[k_1, \dots, k_r] & \text{if } \chi \ge 2. \end{cases}$$

It follows that

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \begin{cases} zP(z)/[2, k_1, \dots, k_r] & \text{if } \chi = 1, \\ P(z)/[2, k_1, \dots, k_r] & \text{if } \chi \ge 2. \end{cases}$$

In order to show that $P(z)$ is a product of cyclotomic polynomials and exactly one Pisot polynomial, we use the following; see [Floyd 1992].

Lemma 4.4 [Floyd 1992, Lemma 1] *Let $P(z)$ be a monic polynomial with integer coefficients. We denote the reciprocal polynomial of $P(z)$ by $\widetilde{P}(z)$; that is, $\widetilde{P}(z) = z^{\deg P} P(z^{-1})$. Suppose that $P(z)$ satisfies*

(i) $P(0) \neq 0$ *and* $P(1) < 0$;

(ii) $P(z) \neq \widetilde{P}(z)$;

(iii) *for sufficiently large integer m, $(z^m P(z) - \widetilde{P}(z))/(z-1)$ is a product of cyclotomic polynomials and exactly one Salem polynomial.*

Then the polynomial $P(z)$ is a product of cyclotomic polynomials and exactly one Pisot polynomial.

Theorem 4.5 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 with $\chi(L(\Gamma, S)) \geq 1$. Then the growth rate $\tau(\Gamma, S)$ is a Pisot number.*

Proof Assume that (Γ, S) of rank N satisfies the hypothesis of the theorem. Since (Γ, S) is a nonspherical, nonaffine Coxeter system, we have that $N \geq 3$. If the presentation diagram $X(\Gamma, S)$ has no edges, the growth rate $\tau(\Gamma, S) = N - 1 \geq 2$ is a Pisot number; see Example 2.5. From now on, we assume that the presentation diagram $X(\Gamma, S)$ has at least one edge. Denote by $E \geq 1$ the number of edges of $X(\Gamma, S)$.

Considering Remark 4.2, we divide the proof into two cases: the presentation diagram $X(\Gamma, S)$ is a tree all of whose edges are labeled by 2, and otherwise.

In the first case, we have $E = N - 1$. Without loss of generality, we can assume that $N \geq 4$ since (Γ, S) is nonaffine. Therefore, by Example 2.6, the growth rate $\tau(\Gamma, S) = N - 2 \geq 2$ is a Pisot number.

In the other case, either the presentation diagram $X(\Gamma, S)$ is disconnected or it has an edge labeled by $k \geq 3$. We fix an ordering of the generating set S. Let us take a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n \geq 7}$ of rank N as in Corollary 4.3. It follows from property (3) that the number of edges of $X(\Gamma_n, S_n)$ equals $E + \chi(L(\Gamma, S))$. In particular, for every $n \geq 7$ different from $k_1, \dots, k_r$, the number of edges of $X(\Gamma_n, S_n)$ labeled by n is equal to $\chi(L(\Gamma, S))$. For simplicity, we write χ instead of $\chi(L(\Gamma, S))$. By Steinberg's formula (2-1),

$$\frac{1}{f_{(\Gamma_n, S_n)}(z^{-1})} = 1 - \frac{N}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2, k_i]} + \frac{\chi}{[2, n]} = \frac{P_n(z)}{[2, k_1, \dots, k_r, n]},$$

where

$$P_n(z) = [2, k_1, \dots, k_r, n] - N[k_1, \dots, k_r, n] + \sum_{i=1}^{r} E_i[k_1, \dots, \hat{k}_i, \dots, k_r, n] + \chi[k_1, \dots, k_r].$$

From the equality $N = E_1 + \dots + E_r + \chi$, we obtain that

$$P_n(z) = [2, k_1, \dots, k_r, n] - \sum_{i=1}^{r} E_i z[k_1, \dots, k_i - 1, \dots, k_r, n] - \chi z[k_1, \dots, k_r, n-1].$$

Define the polynomials $P(z)$ as

$$P(z)=\begin{cases}[k_1,\dots,k_r]-\sum_{i=1}^r E_i[k_1,\dots,k_i-1,\dots,k_r] & \text{if } \chi=1,\\ [2,k_1,\dots,k_r]-\sum_{i=1}^r E_i z[k_1,\dots,k_i-1,\dots,k_r]-\chi[k_1,\dots,k_r] & \text{if } \chi\geq 2,\end{cases}$$

and $\widetilde{P}(z)=z^{\deg P}P(z^{-1})$. Then

$$(z-1)P_n(z)=\begin{cases}z^{n+1}P(z)-\widetilde{P}(z) & \text{if } \chi=1,\\ z^nP(z)-\widetilde{P}(z) & \text{if } \chi\geq 2.\end{cases}$$

Since $\chi(L(\Gamma_n,S_n))=0$, by Lemma 3.3, the polynomial $P_n(z)$ is a product of cyclotomic polynomials and exactly one Salem polynomial. In order to apply Lemma 4.4 to $P(z)$, we need to show that $P(0)\neq 0$, $P(1)<0$, and that $P(z)$ is not reciprocal. First,

$$P(0)=\begin{cases}1-E & \text{if } \chi=1,\\ 1-\chi & \text{if } \chi\geq 2.\end{cases}$$

It follows that $P(0)\neq 0$. Since $P(z)$ is monic, we also conclude that $P(z)$ is not reciprocal. Finally, we see that $P(1)<0$ as follows.

In the case $\chi=1$,

$$P(1)=\prod_{i=1}^r k_i-\sum_{i=1}^r\biggl(E_i\cdot\prod_{j=1}^r k_j\cdot\frac{k_i-1}{k_i}\biggr)=\prod_{i=1}^r k_i\cdot\biggl\{1-\sum_{i=1}^r E_i\Bigl(1-\frac{1}{k_i}\Bigr)\biggr\}.$$

If $N\geq 4$, then

$$\sum_{i=1}^r E_i\Bigl(1-\frac{1}{k_i}\Bigr)\geq\sum_{i=1}^r E_i\Bigl(1-\frac{1}{2}\Bigr)=\frac{E}{2}=\frac{N-1}{2}\geq\frac{3}{2}>1.$$

It follows that $P(1)<0$ from

$$1-\sum_{i=1}^r E_i\Bigl(1-\frac{1}{k_i}\Bigr)<0.$$

For $N=3$, the presentation diagram is made of two edges with labels k_1 and k_2. We necessarily have $k_1\geq 3$ or $k_2\geq 3$, so

$$1-\Bigl(1-\frac{1}{k_1}\Bigr)-\Bigl(1-\frac{1}{k_2}\Bigr)\leq 1-\frac{1}{2}-\frac{2}{3}=-\frac{1}{6}<0.$$

Hence $P(1)<0$.

In the case $\chi\geq 2$,

$$P(1)=2\prod_{i=1}^r k_i-\sum_{i=1}^r\biggl(E_i\cdot\prod_{j=1}^r k_j\cdot\frac{k_i-1}{k_i}\biggr)-\chi\prod_{i=1}^r k_i=\prod_{i=1}^r k_i\cdot\biggl\{2-\chi-\sum_{i=1}^r E_i\Bigl(1-\frac{1}{k_i}\Bigr)\biggr\}.$$

Since $X(\Gamma,S)$ has at least one edge,

$$P(1)<\prod_{i=1}^r k_i\cdot(2-\chi)\leq 0.$$

By Lemma 4.4, the polynomial $P(z)$ is a product of cyclotomic polynomials and exactly one Pisot polynomial, and hence the growth rate $\tau(\Gamma,S)$ is a Pisot number. □

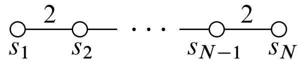

Figure 9: The presentation diagram $X(\widehat{\Gamma}, \widehat{S})$.

Theorem 4.6 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 with $\chi(L(\Gamma, S)) \geq 1$. Then, there exists a sequence of Coxeter systems (Γ_n, S_n) of dimension at most 2 with vanishing Euler characteristic such that the growth rate $\tau(\Gamma_n, S_n)$ converges to $\tau(\Gamma, S)$ from below.*

Proof We denote by N the rank of (Γ, S). As in the proof of Theorem 4.5, we divide the proof into two cases: either the presentation diagram $X(\Gamma, S)$ is disconnected or has an edge labeled by $k \geq 3$, and otherwise.

In the first case, we fix an ordering of S and we take a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n\geq 7}$ of rank N as in Corollary 4.3. By combining Theorems 2.7, 2.8, and 3.2, we conclude that the growth rate $\tau(\Gamma_n, S_n)$ is a Salem number and the sequence $\{\tau(\Gamma_n, S_n)\}_{n\geq 7}$ converges to $\tau(\Gamma, S)$ from below.

In the other case, by Remark 4.2, the presentation diagram $X(\Gamma, S)$ is a tree with all edges labeled by 2. Since (Γ, S) is nonspherical and nonaffine, it forces $N \geq 4$. It was shown in Example 2.6 that the growth rate of (Γ, S) does not depend on the isomorphism type of the tree, only on the number of its vertices, and that $\tau(\Gamma, S) = N - 2 \geq 2$.

Consider the marked Coxeter system $(\widehat{\Gamma}, \widehat{S})$ of rank N whose presentation diagram $X(\widehat{\Gamma}, \widehat{S})$ is depicted in Figure 9.

Let (Γ_n, S_n) be the marked Coxeter system of rank N whose presentation diagram $X(\Gamma_n, S_n)$ is obtained by adding an edge labeled by $n \geq 3$ between s_1 and s_N. As a direct consequence, (Γ_n, S_n) converges to $(\widehat{\Gamma}, \widehat{S})$ in the space of marked Coxeter systems $\mathscr{C}_N$ of rank N. Since $\tau(\widehat{\Gamma}, \widehat{S}) = \tau(\Gamma, S)$, by combining Theorems 2.7, 2.8, and 3.2, the assertion follows. □

Remark 4.7 We mention that for hyperbolic groups, Fujiwara and Sela [2023] have studied the convergence properties of growth rates with respect to all their finite generating sets; see also [Yukita 2024]. However, they did not characterize the arithmetic nature of growth rates.

5 Examples for the growth rates of Coxeter systems with negative Euler characteristic

In this section, we consider Coxeter systems of dimension at most 2 with negative Euler characteristic. We provide some infinite sequences of such Coxeter systems, and prove by a classical approach that their growth rates are Perron numbers; see also Remark 5.2.

Let $(\Gamma_\star, S_\star)$ be the Coxeter system with presentation diagram depicted in Figure 10. As discussed in Example 2.4, the radius of convergence of its growth series is given by $r_\star = 1/\tau(\Gamma_\star, S_\star) = (\varphi - 1)^2$, where φ is the golden ratio.

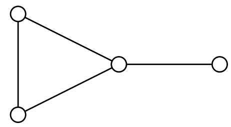

Figure 10: The presentation diagram of $(\Gamma_\star, S_\star)$.

For all the Coxeter systems (Γ, S) discussed below, we assume that $(\Gamma_\star, S_\star) \preceq (\Gamma, S)$; see Section 2.3.

We provide examples in terms of the underlying graphs of their presentation diagrams; see Figure 11. For terminology, we refer to [Gallian 1998]. Such Coxeter systems all satisfy $\chi \leq -1$. For instance, the family of *wheel graphs* W_N, for all $N \geq 6$, formed by a cycle of length $N-1$ and a universal vertex, that is, a central vertex linked to each other vertex. In that case the number of edges of the graph is given by $E = 2(N-1)$. The same goes for the *windmill graphs* of type $\mathscr{W}(4, l)$, with $l \geq 2$, made of l copies of complete graphs K_4 joined at common central vertex. The family of *friendship graphs* $F_l = \mathscr{W}(3, l)$ for $l \geq 3$ satisfies $E = \frac{3}{2}(N-1)$. Several variations of those graphs can be constructed. For example, we defined the *triangulated bouquet* $\mathscr{T}(c, l)$ as the graph formed by l copies of c–cycles glued in a common vertex v, such that any other vertex is linked to v. In this case, v is universal and one has

$$E = \frac{2c-1}{c-1}(N-1).$$

Proposition 5.1 *Let $(\Gamma_{k,N}, S)$ be a nonspherical, nonaffine Coxeter system of dimension at most 2 and rank N, such that all edges of the presentation diagram $X(\Gamma_{k,N}, S)$ are labeled by the same $k \geq 3$. Denote by E the number of edges of $X(\Gamma_{k,N}, S)$.*

If $(\Gamma_{k,N}, S)$ satisfies that

(i) $(\Gamma_\star, S_\star) \preceq (\Gamma_{k,N}, S)$,

(ii) $E = a(N-1)$ *for a rational number* $1 < a \leq \frac{1}{3}(1+\varphi)^2$,

then the growth rate $\tau(\Gamma_{k,N}, S)$ is a Perron number.

Proof We give an outline of the proof, which is classical, and omit details. Assume that $(\Gamma_{k,N}, S)$ satisfies the hypothesis of Proposition 5.1. In what follows, we denote by $f_{k,N}(z) = Q_{k,N}(z)/P_{k,N}(z)$ the growth series of $(\Gamma_{k,N}, S)$, by $r_{k,N}$ its radius of convergence, and by $\tau_{k,N}$ the growth rate of $(\Gamma_{k,N}, S)$. Recall that $r_{k,N}$ is the smallest positive real root of $P_{k,N}(z)$.

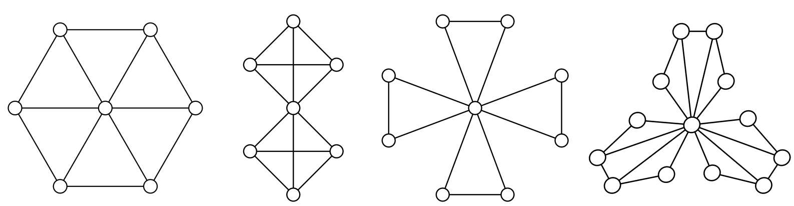

Figure 11: The graphs W_7, $\mathscr{W}(4,2)$, $F_4 = \mathscr{W}(3,4)$, and $\mathscr{T}(5,3)$.

By Steinberg's formula (2-1),

$$\frac{1}{f_{k,N}(z^{-1})} = 1 - \frac{N}{[2]} + \frac{a(N-1)}{[2,k]} = \frac{[2,k] - N[k] + a(N-1)}{[2,k]}. \tag{5-1}$$

Therefore, the denominator of $f_{k,N}(z)$ is given by

$$P_{k,N}(z) = 1 + (2-N)(z + z^2 + \cdots + z^{k-1}) + (a-1)(N-1)z^k = h_N(z) + R_{k,N}(z),$$

where $h_N(z)$ is the quadratic polynomial $h_N(z) = 1 + (2-N)(z + z^2)$ and $R_{k,N}(z)$ is the remaining part.

By hypothesis, $(\Gamma_\star, S_\star) \preceq (\Gamma_{k,N}, S)$; therefore by Theorem 2.8, we conclude that $\tau(\Gamma_\star, S_\star) \leq \tau_{k,N}$. It follows that the associated radii of convergence all satisfy $r_{k,N} \leq r_\star$. In order to prove that $r_{k,N}$ is the unique root with smallest modulus of $P_{k,N}(z)$, we use Rouché's theorem on the open disk $D(0, r_\star)$. We first observe that $h_N(z)$ has a unique root in $D(0, r_\star)$, and we prove $|h_N(z)| - |R_{k,N}| > 0$ on $|z| = r_\star$.

We assume that $N \geq 9$; the case where $N \leq 8$ can be done by applying similar reasoning. An easy analysis of the roots shows that for any N, $h_N(z)$ admits a unique root in the open disk $D(0, r_\star)$. Moreover, on the circle $|z| = r_\star$, one has

$$|h_N(z)| \geq |1 + (2-N)(r_\star^2 - r_\star)|. \tag{5-2}$$

Let z be such that $|z| = r_\star$, and put $\Delta_{k,N}(z) = |h_{k,N}(z)| - |R_{k,N}(z)|$. Since $a > 1$, by the triangle inequality,

$$|R_{k,N}(z)| \leq (N-2)(r_\star^3 + \cdots + r_\star^{k-1}) + (a-1)(N-1)r_\star^k.$$

Also, by (5-2), one has $|h_N(z)| \geq |1 + (2-N)(r_\star^2 - r_\star)| \geq 1 + (2-N)(r_\star^2 - r_\star)$. It follows that

$$\Delta_{k,N}(z) \geq N\left(1 + 2r - \frac{1 - r_\star^k}{1 - r_\star} - (a-1)r_\star^k\right) - 3 - 4r_\star + 2\frac{1 - r_\star^k}{1 - r_\star} + (a-1)r_\star^k.$$

By analysis of each term, one can prove that $\Lambda_{k,N}$ increases with respect to N for all $k \geq 3$, and that $\Lambda_{k,N}$ decreases with respect to k for all $N \geq 9$. Therefore,

$$\Delta_{k,N}(z) \geq \lim_{k\to\infty} \Lambda_{k,N} = N\left(1 + 2r_\star - \frac{1}{1-r_\star}\right) - 3 - 4r_\star + \frac{2}{1-r_\star}.$$

We obtain that $\Delta_{k,N}(z) > 0$ when

$$N > \frac{3 + 4r_\star - 2\frac{1}{1-r_\star}}{1 + 2r_\star - \frac{1}{1-r_\star}} = \frac{11 + \sqrt{45}}{2}.$$

This is true for any $N \geq 9$, which finishes the proof. □

Remark 5.2 A Coxeter system is said to be ∞–spanned if there exists a spanning tree of its Coxeter diagram with edges labeled ∞ only. Kolpakov and Talambutsa [2022] proved that the growth rate of ∞–spanned Coxeter systems are Perron numbers. By the existence of a universal vertex in the presentation diagram of the Coxeter systems discussed above, such a spanning tree cannot be found in the corresponding

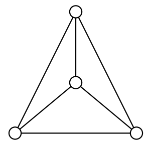

Figure 12: The presentation diagram of (Γ_0, S_0).

Coxeter diagrams. However, the growth series of such a Coxeter system (Γ, S) of dimension 2 at most coincides with the growth series of a ∞–spanned Coxeter system obtained as follows.[1] Construct a complete graph with N vertices, with E of its edges labeled by $k \geq 3$ and the remaining ones by ∞. If a vertex is chosen so that all its emanating edges are labeled by ∞, the resulting graph encodes a ∞–spanned Coxeter system whose growth series equals the original growth series.

In Theorems 3.2 and 4.5, we proved that growth rates of Coxeter systems of dimension at most 2 with positive and vanishing Euler characteristic are Salem and Pisot numbers respectively. By Proposition 5.1 and Remark 5.2, the growth rates of infinitely many Coxeter systems with negative Euler characteristic are Perron numbers.

Note that, there exist Coxeter systems of dimension at most 2 such that $\chi \leq -1$ whose growth rates are Perron numbers but *are neither* Pisot numbers nor Salem numbers. For instance, the 3–dimensional hyperbolic *ideal* Coxeter system (Γ_0, S_0) whose presentation diagram admits labels 3 only and is depicted in Figure 12.

Inspired by these observations, we make the following claim.

Conjecture *The growth rate of any Coxeter system of dimension at most 2 is a Perron number.*

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References

[Bertin et al. 1992] **M-J Bertin**, **A Decomps-Guilloux**, **M Grandet-Hugot**, **M Pathiaux-Delefosse**, **J-P Schreiber**, *Pisot and Salem numbers*, Birkhäuser, Basel (1992) MR Zbl

[Charney and Davis 1991] **R Charney**, **M Davis**, *Reciprocity of growth functions of Coxeter groups*, Geom. Dedicata 39 (1991) 373–378 MR Zbl

[Davis 2008] **M W Davis**, *The geometry and topology of Coxeter groups*, Lond. Math. Soc. Monogr. Ser. 32, Princeton Univ. Press (2008) MR Zbl

[1] We thank the referee for pointing out this fact to us.

[Felikson and Tumarkin 2010] **A Felikson**, **P Tumarkin**, *Reflection subgroups of Coxeter groups*, Trans. Amer. Math. Soc. 362 (2010) 847–858 MR Zbl

[Floyd 1992] **W J Floyd**, *Growth of planar Coxeter groups, PV numbers, and Salem numbers*, Math. Ann. 293 (1992) 475–483 MR Zbl

[Fujiwara and Sela 2023] **K Fujiwara**, **Z Sela**, *The rates of growth in a hyperbolic group*, Invent. Math. 233 (2023) 1427–1470 MR Zbl

[Gallian 1998] **J A Gallian**, *A dynamic survey of graph labeling*, Electron. J. Combin. 1998 (1998) DS6 MR Zbl

[Grigorchuk 1984] **R I Grigorchuk**, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984) 939–985 MR Zbl In Russian; translated in Math. USSR-Izv. 25 (1985) 259–300

[Gromov 1981] **M Gromov**, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. 53 (1981) 53–73 MR Zbl

[de la Harpe 1987] **P de la Harpe**, *Groupes de Coxeter infinis non affines*, Expo. Math. 5 (1987) 91–96 MR Zbl

[Humphreys 1990] **J E Humphreys**, *Reflection groups and Coxeter groups*, Cambridge Stud. Adv. Math. 29, Cambridge Univ. Press (1990) MR Zbl

[Kellerhals and Perren 2011] **R Kellerhals**, **G Perren**, *On the growth of cocompact hyperbolic Coxeter groups*, European J. Combin. 32 (2011) 1299–1316 MR Zbl

[Kolpakov 2012] **A Kolpakov**, *Deformation of finite-volume hyperbolic Coxeter polyhedra, limiting growth rates and Pisot numbers*, European J. Combin. 33 (2012) 1709–1724 MR Zbl

[Kolpakov and Talambutsa 2022] **A Kolpakov**, **A Talambutsa**, *Growth rates of Coxeter groups and Perron numbers*, Int. Math. Res. Not. 2022 (2022) 14675–14696 MR Zbl

[Komori and Yukita 2015] **Y Komori**, **T Yukita**, *On the growth rate of ideal Coxeter groups in hyperbolic* 3*–space*, Proc. Japan Acad. Ser. A Math. Sci. 91 (2015) 155–159 MR Zbl

[McMullen 2002] **C T McMullen**, *Coxeter groups, Salem numbers and the Hilbert metric*, Publ. Math. Inst. Hautes Études Sci. 95 (2002) 151–183 MR Zbl

[Munkres 1984] **J R Munkres**, *Elements of algebraic topology*, Addison-Wesley, Menlo Park, CA (1984) MR Zbl

[Nonaka and Kellerhals 2017] **J Nonaka**, **R Kellerhals**, *The growth rates of ideal Coxeter polyhedra in hyperbolic* 3*–space*, Tokyo J. Math. 40 (2017) 379–391 MR Zbl

[Parry 1993] **W Parry**, *Growth series of Coxeter groups and Salem numbers*, J. Algebra 154 (1993) 406–415 MR Zbl

[Ratcliffe 1994] **J G Ratcliffe**, *Foundations of hyperbolic manifolds*, Graduate Texts in Math. 149, Springer (1994) MR Zbl

[Solomon 1966] **L Solomon**, *The orders of the finite Chevalley groups*, J. Algebra 3 (1966) 376–393 MR Zbl

[Steinberg 1968] **R Steinberg**, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. 80, Amer. Math. Soc., Providence, RI (1968) MR Zbl

[Terragni 2016] **T Terragni**, *On the growth of a Coxeter group*, Groups Geom. Dyn. 10 (2016) 601–618 MR Zbl

[Wolf 1968] **J A Wolf**, *Growth of finitely generated solvable groups and curvature of Riemannian manifolds*, J. Differential Geom. 2 (1968) 421–446 MR Zbl

[Yukita 2017] **T Yukita**, *On the growth rates of cofinite* 3*–dimensional hyperbolic Coxeter groups whose dihedral angles are of the form* $\frac{\pi}{m}$ *for* $m = 2, 3, 4, 5, 6$, from "Geometry and analysis of discrete groups and hyperbolic spaces" (M Fujii, N Kawazumi, K Ohshika, editors), RIMS Kôkyûroku Bessatsu B66, RIMS, Kyoto (2017) 147–165 MR Zbl

[Yukita 2018] **T Yukita**, *Growth rates of* 3*–dimensional hyperbolic Coxeter groups are Perron numbers*, Canad. Math. Bull. 61 (2018) 405–422 MR Zbl

[Yukita 2024] **T Yukita**, *On the continuity of the growth rate on the space of Coxeter systems*, Groups Geom. Dyn. 18 (2024) 109–126

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