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Complex hypersurfaces in direct products of Riemann surfaces

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We study smooth complex hypersurfaces in direct products of closed hyperbolic Riemann surfaces and give a classification in terms of their fundamental groups. This answers a question of Delzant and Gromov on subvarieties of products of Riemann surfaces in the smooth codimension one case. We also answer Delzant and Gromov's question of which subgroups of a direct product of surface groups are Kähler for two classes: subgroups of direct products of three surface groups, and subgroups arising as the kernel of a homomorphism from the product of surface groups to $\mathbb{Z}^3$. These results will be a consequence of answering the more general question of which subgroups of a direct product of surface groups are the image of a homomorphism from a Kähler group, which is induced by a holomorphic map, for the same two classes. This provides new constraints on Kähler groups.

32J27; 20F65, 20J05, 32Q15

1 Introduction

A *Kähler group* is a group that can be realized as fundamental group of a compact Kähler manifold.

Convention Throughout this work, S_g will denote a closed orientable surface of genus $g \geq 2$ and $\Gamma_g = \pi_1(S_g)$ its fundamental group. Furthermore, a *surface group* will always be a group isomorphic to Γ_g for some $g \geq 2$.

Kähler groups have attracted much interest over the last decades and have been studied from many different points of view. An important motivation for studying them is that they are closely linked to the study of the topology of smooth complex projective varieties. Historically, a key technique for understanding Kähler groups is through their homomorphisms onto surface groups. For some examples of how surface groups are used in the study of Kähler groups, as well as for general background on Kähler groups, we refer the reader to [Amorós et al. 1996] (and also [Biswas and Mj 2017; Burger 2011] for more recent developments).

A central objective of this work will be to develop new constraints on homomorphisms from Kähler groups onto surface groups by studying complex hypersurfaces in direct products of Riemann surfaces. More precisely, we will address the following questions, raised by Delzant and Gromov [2005] in their fundamental work on cuts in Kähler groups:

Question 1 [Delzant and Gromov 2005] *Which subgroups of direct products of surface groups are Kähler?*

Question 2 [Delzant and Gromov 2005] *Given a subgroup $G \leq \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r})$, when does there exist an algebraic variety $V \subset S_{g_1} \times \cdots \times S_{g_r}$ of a given dimension n such that the image of the fundamental group of V is G?*

Question 2 can be seen as a more general version of Question 1. This is particularly apparent from the following group-theoretic reformulation:

Question 3 *When is a subgroup $G \leq \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r})$ the image of a homomorphism $\pi_1(X) \to \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r})$ which is induced by a holomorphic map $X \to S_{g_1} \times \cdots \times S_{g_r}$ from a compact Kähler manifold X?*

Answers to these questions in concrete situations provide new constraints on Kähler groups and can thus have interesting applications. Indeed, one such application of Theorem 1.1 has been provided recently by Llosa Isenrich and Py [2021]. They apply it to obtain constraints on Kodaira fibrations admitting more than two fiberings, thereby making progress on the question [Salter 2015; Catanese 2017] of whether such Kodaira fibrations can exist.

Delzant and Gromov [2005] give criteria for when a Kähler group admits a homomorphism to a direct product of surface groups. These results have been extended by [Py 2013; Delzant and Py 2019]. A key consequence of their works is that many actions of Kähler groups on CAT(0) cube complexes factor through homomorphisms to direct products of surface groups. Combined with the important role that CAT(0) cube complexes have played in recent advances in geometric group theory and low-dimensional topology (eg [Agol 2013]), this motivates Delzant and Gromov's questions.

The first nontrivial examples of Kähler subgroups of direct products of surface groups were constructed by Dimca, Papadima and Suciu [Dimca et al. 2009] with the purpose of showing that there is a Kähler group which does not have a classifying space which is a quasiprojective variety. They arise as fundamental groups of generic fibres of holomorphic maps from a direct product of Riemann surfaces onto an elliptic curve, which restrict to ramified coverings of degree two on the factors. These examples have been generalized by Llosa Isenrich [2019] and Biswas, Mj and Pancholi [Biswas et al. 2014]. All of these examples are fundamental groups of smooth complex hypersurfaces in direct products of closed Riemann surfaces. More general classes of Kähler subgroups of direct products of surface groups have been constructed from holomorphic maps onto higher-dimensional tori [Llosa Isenrich 2020]. They include examples coming from subvarieties of all possible codimensions. On the other hand, Kähler subgroups of direct products of surface groups must satisfy strong constraints and the same remains true for subgroups arising as images of homomorphisms which are induced by holomorphic maps [Llosa Isenrich 2020]. We will provide more details on these results in Section 2.

The combination of the diversity of examples and constraints reveals the subtle conditions that a complete answer to Delzant and Gromov's question needs to satisfy. However, as discussed above, solutions even in specific cases provide new tools for studying Kähler groups, enabling interesting applications. This work is thus concerned with finding natural situations in which complete answers can be obtained. For this we combine insights from previous works with Albanese maps and a careful analysis of complex hypersurfaces in direct products of closed Riemann surfaces.

Our first result is an answer to Question 3 for direct products of three surface groups.

Definition For a direct product $G_1 \times \cdots \times G_r$ of groups, denote by $p_i\colon G_1 \times \cdots \times G_r \to G_i$ the projection onto the i^{th} factor. A subgroup $H \leq G_1 \times \cdots \times G_r$ is called

- *subdirect* if $p_i(H) = G_i$ for $1 \leq i \leq r$, and
- *full* if $H \cap G_i := H \cap (1 \times \cdots \times 1 \times G_i \times 1 \times \cdots \times 1)$ is nontrivial for $1 \leq i \leq r$.

Theorem 1.1 *Let $G = \pi_1(X)$ be the fundamental group of a compact Kähler manifold X, and let $\phi\colon G \to \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ be a homomorphism with finitely presented full subdirect image $\overline{G} := \phi(G)$ of infinite index. Assume that $\ker(p_i \circ \phi)$ is finitely generated for $1 \leq i \leq 3$.*

Then there are finite-index subgroups $\Gamma_{\gamma_i} \leq \Gamma_{g_i}$, a complex elliptic curve E and a holomorphic map

$$f = \sum_{i=1}^{3} f_i\colon S_{\gamma_1} \times S_{\gamma_2} \times S_{\gamma_3} \to E,$$

induced by branched holomorphic coverings $f_i\colon S_{\gamma_i} \to E$, such that $\overline{G}_0 = \ker(f_*) \cong \pi_1(H) \leq \overline{G}$ is a finite-index subgroup, where H is the smooth generic fibre of f and $f_*\colon \Gamma_{\gamma_1} \times \Gamma_{\gamma_2} \times \Gamma_{\gamma_3} \to \pi_1(E)$ is the induced map on fundamental groups.

We emphasize that the condition that $\ker(p_i \circ \phi)$ is finitely generated in Theorem 1.1 implies that the homomorphism ϕ is induced by a holomorphic map, and, conversely, that every homomorphism to a surface group induced by a holomorphic map will have finitely generated kernel, after possibly passing to a finite ramified cover. Thus, our result does really provide an answer to Question 3 for direct products of three surface groups.

Remark 1.2 Theorem 1.1 also provides constraints on homomorphisms to products of more than three surface groups satisfying the remaining assumptions of the theorem. To see this, we use that, for subdirect products of surface groups, finite presentability is equivalent to satisfying the virtual surjection to pairs property (VSP) [Bridson et al. 2013, Theorem D]. Thus, finite presentability is preserved under projections to factors, allowing us to apply Theorem 1.1 to every composition of such a homomorphism with a projection to three of the surface group factors.

We also give a description of all possible images of homomorphisms with ϕ as in Theorem 1.1 when the image is not a full subdirect product (see Theorem 4.3). However, in this case the homomorphism will not always be induced by a holomorphic map.

As a consequence of Theorem 4.3, we obtain the following answer to Question 1 in the three factor case:

Corollary 1.3 *Let $G = \pi_1(X) \leq \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ for X a compact Kähler manifold. Then there is a finite-index subgroup $G_0 \leq G$ such that either*

(1) *$G_0 \cong \mathbb{Z}^{2k} \times \Gamma_{h_1} \times \cdots \times \Gamma_{h_s}$ for $h_1, \ldots h_s \geq 2$ and $0 \leq 2k + s \leq 3$; or*

(2) *G_0 is the kernel of an epimorphism $\psi\colon \Gamma_{\gamma_1} \times \Gamma_{\gamma_2} \times \Gamma_{\gamma_3} \to \mathbb{Z}^2$ which is induced by a surjective holomorphic map $f = \sum_{i=1}^{3} f_i\colon S_{\gamma_1} \times S_{\gamma_2} \times S_{\gamma_3} \to E$ with the same properties as the map f in Theorem 1.1.*

Conversely, every group which satisfies one of the conditions (1) *and* (2) *is Kähler.*

We remark that Theorem 1.1 and Corollary 1.3 will hold for any choice of compact Kähler manifold X with $G = \pi_1(X)$. However, the complex structures on E and S_{γ_i} obtained in the proof will depend on the complex structure of X, since we will make use of the fact that there is a holomorphic map $X \to S_{g_1} \times S_{g_2} \times S_{g_3}$ which realizes the homomorphism $G \to \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$. Both results will be consequences of the more general criterion provided by Theorem 3.1. Theorem 3.1 also allows us to classify connected smooth complex hypersurfaces in a direct product of r closed Riemann surfaces in terms of the image of their fundamental groups, thus providing a complete answer to Question 2 for this case.

Theorem 1.4 *Let $X \subset S_{g_1} \times \cdots \times S_{g_r}$ be a connected smooth complex hypersurface in a product of closed Riemann surfaces of genus $g_i \geq 2$. Then there are finite unramified covers $X_0 \to X$ and $S_{\gamma_i} \to S_{g_i}$, and a holomorphic embedding $\iota\colon X_0 \hookrightarrow S_{\gamma_1} \times \cdots \times S_{\gamma_r}$ such that one of the following holds*:

(1) *ι_* is surjective on fundamental groups.*

(2) *X_0 is a direct product of $r-1$ Riemann surfaces.*

(3) *There is $3 \leq s \leq r$, an elliptic curve E and surjective holomorphic maps $h_i\colon S_{\gamma_i} \to E$ for $1 \leq i \leq s$ such that $X_0 = H \times S_{g_{s+1}} \times \cdots \times S_{g_r}$ for H the smooth generic fibre of $h = \sum_{i=1}^{s} h_i\colon S_{\gamma_1} \times \cdots \times S_{\gamma_s} \to E$.*

Moreover, if (3) *holds, then h induces a short exact sequence*

$$1 \to \pi_1(H) \to \pi_1(S_{\gamma_1}) \times \cdots \times \pi_1(S_{\gamma_s}) \to \pi_1(E) \to 1.$$

Finally, the techniques used to prove Theorem 3.1 can be adapted to give a complete classification of Kähler subgroups of direct products of surface groups arising as kernels of homomorphisms to $\mathbb{Z}^3$, hence also answering Question 1 for this case. We refer to Section 6 for the precise statement and results.

Structure

In Section 2 we will give some additional background and motivation for this work. In Section 3 we will prove Theorem 3.1, which is the main technical result of this work. We apply this result in Section 4 to prove Theorem 1.1 and Corollary 1.3 and in Section 5 to prove Theorem 1.4. In Section 6 we explain how the techniques used in the proof of Theorem 3.1 can be applied to kernels of homomorphisms from direct products of surface groups to $\mathbb{Z}^3$.

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2 Background

When approaching Delzant and Gromov's questions, it is helpful to use our understanding of the nature of subgroups of direct products of surface groups from geometric group theory. The work of Bridson, Howie, Miller and Short [Bridson et al. 2009; 2013] and other authors (eg [Kochloukova 2010; Kuckuck 2014]) shows that finiteness properties play a key role in this context. We say that a group has finiteness type $\mathscr{F}_k$ if it has a classifying CW–complex with finitely many cells of dimension $\leq k$. Note that type $\mathscr{F}_1$ is equivalent to being finitely generated, while type $\mathscr{F}_2$ is equivalent to being finitely presented. A subgroup of type $\mathscr{F}_r$ of a direct product of r surface groups is virtually a direct product of finitely many free groups and surface groups [Bridson et al. 2009; 2013]. Thus, all "nontrivial" subgroups of such a product must have exotic finiteness properties. Moreover, for groups which are not of type $\mathscr{F}_r$, stronger finiteness properties mean stronger constraints on the type of group. For more details we refer to [Bridson et al. 2009; 2013; Kochloukova 2010; Kuckuck 2014].

As explained in the introduction, finding a complete answer to Delzant and Gromov's question is far from trivial. However, there are interesting subclasses of direct products of surface groups in which finding an answer seems more feasible. Indeed, a first class are the subgroups G of type $\mathscr{F}_\infty$: since any such G is virtually a direct product of surface groups and free groups, one deduces readily that G being Kähler is equivalent to G being virtually a product $\mathbb{Z}^{2k} \times \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_s})$ for $k \geq 0$, $s \geq 0$ and $g_1, \ldots, g_s \geq 2$.

In terms of finiteness properties, the first nontrivial class of subgroups of a direct product of r surface groups is given by the ones which are of type $\mathscr{F}_{r-1}$ but not $\mathscr{F}_r$. The examples constructed in [Dimca et al. 2009] show the existence of Kähler groups of this type for every $r \geq 3$. They are obtained as fundamental groups of complex hypersurfaces in direct products of Riemann surfaces. Their construction was subsequently generalized in [Biswas et al. 2014; Llosa Isenrich 2019]. All known examples of this type can be obtained from the following result:

Theorem 2.1 [Dimca et al. 2009; Llosa Isenrich 2019] *Let $r \geq 3$, let E be an elliptic curve and let $f_i\colon S_{g_i} \to E$ be branched covers for $1 \leq i \leq r$. Define the map $f := \sum_{i=1}^{r}\colon S_{g_1} \times \cdots \times S_{g_r} \to E$ using the additive structure in E. Assume that the induced map f_* on fundamental groups is surjective and let H be the smooth generic fibre of f. Then f induces a short exact sequence*

$$1 \to \pi_1(H) \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \pi_1(E) \to 1.$$

The group $\pi_1(H)$ is Kähler of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$. Moreover, $\pi_1(H) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is an irreducible full subgroup.

When passing to subgroups with more general finiteness properties, the situation turns out to be more subtle. Indeed, the class of Kähler subgroups of direct products of surface groups that one can then obtain is much larger: they can attain any possible finiteness properties and can arise from subvarieties of all codimensions [Llosa Isenrich 2020]. Moreover, there is no apparent correlation between the codimension of a smooth subvariety realizing a subgroup and its finiteness properties (see [Llosa Isenrich 2020, Theorems 1.2 and 4.1] for precise statements of these results).

On the other hand, it is not hard to see that Kähler subgroups of a direct product of surface groups have to satisfy many restrictions. It is well known that a Kähler subgroup of a direct product of surface groups must be isomorphic to a subdirect product of a free abelian group of even rank and finitely many surface groups. Even among subgroups of this form, strong constraints hold [Llosa Isenrich 2020, Sections 6–9]. For instance, every Kähler full subdirect product of r surface groups which is of type $\mathcal{F}_k$ with $k > \frac{1}{2}r$ must virtually be isomorphic to the kernel of an epimorphism $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^{2m}$ for some $m \geq 0$ and $g_1, \ldots, g_r \geq 2$; a similar result holds for finitely presented images of homomorphisms from Kähler groups to direct products of surface groups which are induced by holomorphic maps.

Given the explicit nature of Theorem 2.1, one may now wonder if these constraints can be strengthened to show that all Kähler subgroups of direct products of r surface groups are of the form of this theorem if they are of type $\mathcal{F}_{r-1}$ but not $\mathcal{F}_r$. Theorem 1.1, Corollary 1.3 and Theorem 6.4 show that this is indeed the case after imposing additional assumptions and that the same remains true even when we consider images of homomorphisms to direct products of surface groups. The common key to these results is that our assumptions will allow us to reduce to situations in which all interesting Kähler subgroups are fundamental groups of smooth complex hypersurfaces.

We now turn to explaining in more detail why the condition that $\ker(p_i \circ \phi)$ is finitely generated in Theorem 1.1 arises naturally. For this recall the following classical result about Kähler groups:

Theorem 2.2 *Let $G = \pi_1(X)$, for X a compact Kähler manifold. Fix $h \geq 2$. The following properties are equivalent*:

1. *There exists a surjective homomorphism $\phi\colon G \twoheadrightarrow \Gamma_h$.*
2. *There exists $g \geq h$ and a holomorphic map $f\colon X \to S_g$ with connected fibres.*

(3) *There exists* $g \geq h$ *and a holomorphic map* $\hat{f}\colon X \to S_{g,\underline{n}}$ *with connected and nonmultiple fibres such that the kernel of the induced homomorphism* $\hat{f}_*\colon G \to \pi_1^{\mathrm{orb}}(S_{g,\underline{n}})$ *is finitely generated, where* $S_{g,\underline{n}}$ *is a closed hyperbolic Riemann orbisurface with cone points of orders* $\underline{n} = (n_1, \dots, n_k)$.

Moreover, if (1) *is satisfied, then we can choose a map* f *satisfying* (2) *such that* ϕ *factors through* $f_*\colon \pi_1(X) \to \Gamma_g$. *Similarly, if* (2) *is satisfied, then we can choose a map* $\hat{f}$ *satisfying* (3) *such that* f *factors through* $\hat{f}$.

The equivalence of (1) and (2) is due to Siu [1987] and Beauville [1988], while the orbifold version was proved by Catanese [2003] (although it seems to have been known earlier; see [Kotschick 2012] for further details). Conversely, every homomorphism from a Kähler group onto a closed hyperbolic orbisurface group with finitely generated kernel is induced by a holomorphic map (see [Catanese 2008; Delzant 2016, Theorem 2]). For further background and definitions on maps from compact Kähler manifolds to hyperbolic orbisurfaces, we refer the reader to [Delzant 2016, Section 2].

Note that every hyperbolic orbisurface group has a finite-index subgroup which is a surface group. Considering that all of the main results in this paper require us to pass to finite-index subgroups, we will thus restrict ourselves to considering surface groups for the remainder of this work.

We conclude this section by fixing some notation and definitions which we will require later. For a direct product $G_1 \times \cdots \times G_r$ of groups and $1 \leq i_1 < \cdots < i_k \leq r$, we denote by $p_{i_1,\dots,i_k}\colon G_1 \times \cdots \times G_r \to G_{i_1} \times \cdots \times G_{i_k}$ the projection homomorphism. We say that a subgroup $K \leq G_1 \times \cdots \times G_r$ *surjects onto* k*–tuples* if $p_{i_1,\dots,i_k}(K) = G_{i_1} \times \cdots \times G_{i_k}$, *virtually surjects onto* k*–tuples* if $p_{i_1,\dots,i_k}(K) \leq G_{i_1} \times \cdots \times G_{i_k}$ is a finite-index subgroup, and *virtually surjects onto pairs* (VSP) if K virtually surjects onto 2–tuples for all $1 \leq i_1 < \cdots < i_k \leq r$.

We call a subgroup $K \leq G_1 \times \cdots \times G_r$ *coabelian* if it is the kernel of an epimorphism $\psi\colon G_1 \times \cdots \times G_r \to \mathbb{Z}^k$ for some $k \geq 0$, and *coabelian of even rank* if k is even.

Moreover, for a product of surfaces $S_{g_1} \times \cdots \times S_{g_r}$ and $1 \leq i_1 < \cdots < i_k \leq r$, we will denote by $q_{i_1,\dots,i_k}\colon S_{g_1} \times \cdots \times S_{g_r} \to S_{g_{i_1}} \times \cdots \times S_{g_{i_k}}$ the projection. We say that a subset $X \subset S_{g_1} \times \cdots \times S_{g_r}$ *geometrically surjects onto* k*–tuples* if $q_{i_1,\dots,i_k}(X) = S_{g_{i_1}} \times \cdots \times S_{g_{i_k}}$ for all $1 \leq i_1 < \cdots < i_k \leq r$. We say that X is *geometrically subdirect* if it geometrically surjects onto 1–tuples.

3 From homomorphisms to complex hypersurfaces

In this section we will prove the main result of this work. The results described in the introduction will be consequences of this result and the techniques developed in its proof.

Theorem 3.1 *Let* $r \geq 3$, *let* X *be a compact Kähler manifold and let* $G = \pi_1(X)$. *Let* $\phi\colon G \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ *be a homomorphism with full subdirect image which can be realized by a holomorphic* *map* $f: X \to S_{g_1} \times \cdots \times S_{g_r}$. *Assume that*

- $\phi(G)$ *is coabelian and a proper subgroup of* $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$; *and*
- *for* $1 \le i_1 < \cdots < i_{r-1} \le r$, *the composition* $q_{i_1,\dots,i_{r-1}} \circ f: X \to S_{g_{i_1}} \times \cdots \times S_{g_{i_{r-1}}}$ *is surjective.*

Then there is an elliptic curve B *and branched covers* $h_i: S_{g_i} \to B$ *such that* $\phi(G) = \pi_1(H)$, *where* H *is the connected smooth generic fibre of the holomorphic map* $h = \sum_{i=1}^r h_i: S_{g_1} \times \cdots \times S_{g_r} \to B$.

Moreover, $f(X)$ *is a (possibly singular) fibre of* h.

The proof of Theorem 3.1 uses the following simple and well-known result:

Lemma 3.2 *Let* X *and* Y *be complex tori and let* $f: X \to Y$ *be a surjective holomorphic homomorphism. Then* $f_*(\pi_1(X)) \le \pi_1(Y)$ *is a finite-index subgroup.*

Proof of Theorem 3.1 Let $A(X)$ be the Albanese torus of X, let $A_i = A(S_{g_i})$ be the Albanese torus of S_{g_i} for $1 \le i \le r$, and denote by $a_X: X \to A(X)$ and $a_i: S_{g_i} \to A_i$ the respective Albanese maps. By the universal property of the Albanese map, we obtain a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & S_{g_1} \times \cdots \times S_{g_r} & & \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle (a_1,\dots,a_r)} & \searrow{\scriptstyle h} & \\ A(X) & \xrightarrow{\bar f} & A_1 \times \cdots \times A_r & \longrightarrow & B \end{array} \tag{3-1}$$

where B is the complex torus $(A_1 \times \cdots \times A_r)/\bar f(A(X))$ (this quotient is well defined, since the induced map on complex tori is a holomorphic homomorphism with image a complex subtorus). Denote by $b: A_1 \times \cdots \times A_r \to B$ the quotient map. It is the sum $b = \sum_{i=1}^r b_i$ of the restrictions $b_i: A_i \to B$.

Surjectivity of the map $q_{1,\dots,r-1} \circ f: X \to S_{g_1} \times \cdots \times S_{g_{r-1}}$ implies that, for every $(s_1, \dots, s_{r-1}) \in S_{g_1} \times \cdots \times S_{g_{r-1}}$, there are $x \in X$ and $s_{x,r} \in S_{g_r}$ with $f(x) = (s_1, \dots, s_{r-1}, s_{x,r})$. By commutativity of (3-1), we obtain that

$$(t_1, \dots, t_{r-1}, t_{x,r}) := (a_1(s_1), \dots, a_{r-1}(s_{r-1}), a_r(s_{x,r})) = \bar f(a_X(x)).$$

Denote by $\Sigma_i := b_i(a_i(S_{g_i}))$ the image of S_{g_i} in B. Since $\bar f(A(X)) = \ker(b)$, we obtain that $b(t_1, \dots, t_{r-1}, t_{x,r}) = 0 \in B$ and hence $\sum_{i=1}^{r-1} b_i(t_i) = -b_r(t_{x,r}) \in -\Sigma_r$. Irreducibility of S_{g_i} implies that Σ_i is an irreducible subvariety of dimension at most one in B. Thus, the holomorphic map

$$\sum_{i=1}^{r-1} b_i: a_1(S_{g_1}) \times \cdots \times a_{r-1}(S_{g_{r-1}}) \to -\Sigma_r, \quad (t_1, \dots, t_{r-1}) \mapsto \sum_{i=1}^{r-1} b_i(t_i),$$

is either trivial or surjective. It follows that the image $b_i(a_i(S_{g_i}))$ is either a point or a translate of $-\Sigma_r$ for $1 \le i \le r-1$. If, moreover, at least one of the images $b_i(a_i(S_{g_i}))$ is nontrivial, then $-\Sigma_r \subset B$ is nontrivial and therefore an irreducible subvariety of dimension one. A repeated application of the same argument to all $j \in \{1, \dots, r\}$ shows that, if at least one of the images Σ_i of S_{g_i} in B is one-dimensional, then all of the Σ_i are one-dimensional and translates of each other.

It follows that either

(1) Σ_i is a point for all $i \in \{1, \dots, r\}$, or

(2) Σ_i is a one-dimensional irreducible projective variety and Σ_i is a translate of Σ_j for all $i, j \in \{1, \dots, r\}$.

Consider the case when the image of all of the S_{g_i} is one-dimensional in B. Then the restriction of the holomorphic map

$$\sum_{i=1}^{r-1} b_i \circ a_i \colon S_{g_1} \times \dots \times S_{g_{r-1}} \to -\Sigma_r$$

to $\{(s_1, \dots, s_{j-1})\} \times S_{g_j} \times \{(s_{j+1}, \dots, s_{r-1})\}$ is a surjective holomorphic map for every $j \in \{1, \dots, r-1\}$, $(s_1, \dots, s_{j-1}) \in S_{g_1} \times \dots \times S_{g_{j-1}}$ and $(s_{j+1}, \dots, s_r) \in S_{g_{j+1}} \times \dots \times S_{g_{r-1}}$. By symmetry, the same holds for $\sum_{i=1, i \neq j}^{r} b_i \circ a_i$ for $1 \leq j \leq r$.

By assumption, $r \geq 3$. It follows that, for any choice of points $s_{1,0} \in S_{g_1}$ and $s_{r,0} \in S_{g_r}$, we have

$$\begin{aligned} -\Sigma_r + b_r(a_r(s_{r,0})) &= h(S_{g_1} \times \dots \times S_{g_{r-1}} \times \{s_{r,0}\}) \\ &= h(\{s_{1,0}\} \times S_{g_2} \times \dots \times S_{g_{r-1}} \times \{s_{r,0}\}) \\ &= h(\{s_{1,0}\} \times S_{g_2} \times \dots \times S_{g_r}) = b_1(a_1(s_{1,0})) - \Sigma_1. \end{aligned}$$

Hence, $-\Sigma_r + b_r(a_r(s_{r,0})) = b_1(a_1(s_{1,0})) - \Sigma_1$ is independent of $s_{1,0}$ and $s_{r,0}$ and therefore the image $h(S_{g_1} \times \dots \times S_{g_r}) = b_r(a_r(s_{r,0})) - \Sigma_r$ is one-dimensional and a translate of $-\Sigma_r$. Furthermore, the restriction $h|_{\{(s_1,\dots,s_{j-1})\} \times S_{g_j} \times \{(s_{j+1},\dots,s_r)\}}$ maps onto $b_r(a_r(s_{r,0})) - \Sigma_r$ for every $j \in \{1, \dots, r\}$, $(s_1, \dots, s_{j-1}) \in S_{g_1} \times \dots \times S_{g_{j-1}}$ and $(s_{j+1}, \dots, s_r) \in S_{g_{j+1}} \times \dots \times S_{g_r}$.

Choose $s_{1,0} \in S_{g_1}$ such that there is an open neighbourhood $U \subset S_{g_1}$ of $s_{1,0}$ in which the restriction $b_1 \circ a_1 \colon U \to b_1(a_1(U)) \subset \Sigma_1$ is biholomorphic. In particular, $b_1(a_1(U))$ is a smooth one-dimensional complex manifold.

Surjectivity of the restriction $\beta|_{\{(s_{1,0})\} \times S_{g_2} \times \{(s_3,\dots,s_r)\}}$ for every $(s_3, \dots, s_r) \in S_{g_3} \times \dots \times S_{g_r}$ implies that, for every $z \in a_r(b_r(s_{r,0})) - \Sigma_r$, there is a point $s_{2,z} \in S_{g_2}$ such that $h(s_1, s_{2,z}, s_3, \dots, s_{r,0}) = z$. Then the map

$$U \to a_r(b_r(s_{r,0})) - \Sigma_r, \quad u \mapsto b_1(a_1(u)) + b_2(a_2(s_{2,z})) + \sum_{i=3}^{r} b_i(a_i(s_i))$$

is a biholomorphic map from U onto a neighbourhood of $z \in \Sigma_r$. Hence, z is a smooth point of Σ_r and it follows that Σ_r is a smooth connected projective variety of dimension one.

The S_{g_i} are finite-sheeted branched coverings of the closed Riemann surface $a_r(b_r(s_{r,0})) - \Sigma_r$ and thus the image of $\pi_1(S_{g_i})$ in $\pi_1\big(a_r(b_r(s_{r,0}))-\Sigma_r\big)$ is a finite-index subgroup for $1 \le i \le r$. Since $r \ge 2$, there is a $\mathbb{Z}^2$ subgroup in $\pi_1\big(a_r(b_r(s_{r,0})) - \Sigma_r\big)$ and the only closed Riemann surface with a $\mathbb{Z}^2$ subgroup in its fundamental group is an elliptic curve. Thus, $a_r(b_r(s_{r,0})) - \Sigma_r$ is an elliptic curve.

Surjectivity of the maps $a_{i*}\colon \pi_1(S_{g_i}) \to \pi_1(A_i)$ on fundamental groups and the fact that the fibres of the quotient map $A_1 \times \cdots \times A_r \to B$ are connected imply that the map h is surjective on fundamental groups. Hence, $a_r(b_r(s_{r,0})) - \Sigma_r = B$, h is surjective holomorphic, and the restrictions $h|_{S_{g_j}}$ for $1 \le j \le r$ are branched covers. Theorem 2.1 implies that h induces a short exact sequence

$$1 \to \pi_1(H) \to \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_r}) \xrightarrow{h_*} \pi_1(B) = \mathbb{Z}^2 \to 1$$

on fundamental groups, where H is the connected smooth generic fibre of h.

Since $\phi(G) \le \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is coabelian, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \phi(G) & \longrightarrow & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} & \longrightarrow & \mathbb{Z}^l & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & (\phi(G))_{\mathrm{ab}} & \longrightarrow & (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} & \longrightarrow & \mathbb{Z}^l & \longrightarrow & 1 \end{array} \tag{3-2}$$

where the lower sequence is exact by right-exactness of abelianization.

We now use the same line of argument as in the proof of [Llosa Isenrich 2020, Lemma 6.1] to show that $l = \mathrm{rk}_{\mathbb{Z}}(\pi_1(B))$. Since it is short, we include it here for the readers convenience:

By definition of the Albanese map, the commutative diagram (3-1) induces a commutative diagram

$$\begin{array}{ccccccc} \pi_1(X) & \xrightarrow{f_*} & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} & \longrightarrow & \mathbb{Z}^l & \longrightarrow & 1 \\ \downarrow & & \downarrow & \searrow^{h_*} & & & \\ \pi_1(A(X)) = (\pi_1(X))_{\mathrm{ab}} & \xrightarrow{\bar{f}_* = f_{*,\mathrm{ab}}} & \pi_1(A_1) \times \cdots \times \pi_1(A_r) = (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} & \twoheadrightarrow & \pi_1(B) & & \end{array} \tag{3-3}$$

The map $\phi\colon \pi_1(X) \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ factors through $\phi(G)$; thus, the map $(\pi_1(X))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}$ factors through $(\phi(G))_{\mathrm{ab}}$. It follows that

$$\mathrm{im}\big((\pi_1(X))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}\big) = \mathrm{im}\big((\phi(G))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}\big),$$

and exactness of the bottom horizontal sequence in (3-2) implies that

$$(\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}/\mathrm{im}\big((\pi_1(X))_{\mathrm{ab}} \to (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}}\big) \cong \mathbb{Z}^l.$$

The commutative diagram (3-3) can be extended to a commutative diagram

$$\begin{array}{ccccccc} \pi_1(X) & \xrightarrow{f_*} & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} & \twoheadrightarrow & \mathbb{Z}^l & \longrightarrow & 1 \\ \downarrow & & \downarrow & \times & \downarrow\!\!\downarrow & & \\ \pi_1(A(X)) = (\pi_1(X))_{\mathrm{ab}} & \xrightarrow{\bar{f}_* = f_{*,\mathrm{ab}}} & \pi_1(A_1) \times \cdots \times \pi_1(A_r) = (\Gamma_{g_1} \times \cdots \times \Gamma_{g_r})_{\mathrm{ab}} & \twoheadrightarrow & \pi_1(B) & & \end{array}$$

Hence, the fundamental group $\pi_1(B)$ is a quotient of $\mathbb{Z}^l$. By Lemma 3.2, we have $\mathrm{rk}_{\mathbb{Z}}\,\bar{f}_*\big(\pi_1(A(X))\big) = \mathrm{rk}_{\mathbb{Z}}\pi_1\big(\bar{f}(A(X))\big)$. Thus, we obtain

$$\begin{aligned}\mathrm{rk}_{\mathbb{Z}}(\pi_1(B)) &= 2\cdot\dim_{\mathbb{C}} B = 2\cdot\dim_{\mathbb{C}}(A_1\times\cdots\times A_r) - 2\cdot\dim_{\mathbb{C}}\bar{f}(A(X))\\ &= \mathrm{rk}_{\mathbb{Z}}(\Gamma_{g_1}\times\cdots\times\Gamma_{g_r})_{\mathrm{ab}} - \mathrm{rk}_{\mathbb{Z}}\,\bar{f}_*\big(\pi_1(A(X))\big) = l.\end{aligned}$$

It follows that the epimorphism $\mathbb{Z}^l \to \pi_1(B)$ is an isomorphism and therefore we obtain an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \phi(G) & \longrightarrow & \Gamma_{g_1}\times\cdots\times\Gamma_{g_r} & \longrightarrow & \mathbb{Z}^l & \longrightarrow & 1\\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 1 & \longrightarrow & \pi_1(H) & \longrightarrow & \Gamma_{g_1}\times\cdots\times\Gamma_{g_r} & \xrightarrow{h_*} & \pi_1(B) & \longrightarrow & 1\end{array}$$

If Σ_i is a point, then the same argument shows that B is a point and the isomorphism of short exact sequences implies that $\phi(G)\cong\pi_1(H)\cong\Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$ is not a proper subgroup.

Finally, observe that, since $h\circ f\colon X\to B$ factors through the Albanese torus $A(X)$ of X, the image of X in B is trivial. Hence, $f(X)$ is contained in a fibre of h. Since $f(X)$ is the image of a smooth complex manifold under a proper holomorphic map, it is an irreducible subvariety of a fibre of h. The map h has isolated singularities, since the restriction of h to every surface factor is a branched covering of B, and its fibres (singular or nonsingular) are connected.

If $f(X)$ is contained in a smooth generic fibre of h, then it is equal to this fibre, since smooth projective varieties are irreducible. So assume that $f(X)$ is contained in one of the finitely many singular fibres H_s of h and let $z\in H_s$ be a singular point. By Milnor's theory [1968] of isolated hypersurface singularities, a neighbourhood of z in H_s is homeomorphic to a cone over a smooth manifold K (called the link of the singularity). Furthermore, K is $(n-2)$–connected for n the complex dimension of H_s. In particular, K is connected if $n\geq 2$. Since the complex dimension of H_s is $r-1\geq 2$, it follows that K is connected. Thus, the complement of the cone point in the cone over K is connected. Connectedness of H_s then implies that the complement of the finite set of singular values in H_s is a connected smooth complex manifold. It follows that H_s is an irreducible variety and thus $H_s = f(X)$. □

4 The three factor case

By combining Theorem 3.1 with the following results, we can complete the classification of Kähler subgroups of direct products of three surface groups up to passing to finite-index subgroups.

Proposition 4.1 [Llosa Isenrich 2020, Proposition 9.5] *Let $r\geq 2$, let X be a compact Kähler manifold and let $G=\pi_1(X)$. Let $\phi\colon G\to\Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$ be a homomorphism with finitely presented full subdirect image such that the projections $p_i\circ\phi\colon G\to\Gamma_{g_i}$, $1\leq i\leq r$, have finitely generated kernel.*

Then ϕ is induced by a holomorphic map $f: X\to S_{g_1}\times\cdots\times S_{g_r}$ and the composition $q_{i,j}\circ f: X\to S_{g_i}\times S_{g_j}$ is surjective for $1\leq i<j\leq r$.

Theorem 4.2 [Llosa Isenrich 2020, Theorem 6.13] *Let X be a compact Kähler manifold and let $G=\pi_1(X)$. Let $\psi\colon G\to\Gamma_{g_1}\times\Gamma_{g_2}\times\Gamma_{g_3}$ be a homomorphism such that the projection $p_i\circ\psi$ has finitely generated kernel for $1\le i\le r$ and the image $\overline{G}:=\psi(G)$ is finitely presented. Then one of the following holds*:

1. $\overline{G}=\pi_1(R)$ *for R a closed Riemann surface of genus* ≥ 0.
2. $\overline{G}=\mathbb{Z}^k$ *for* $k\in\{1,2,3\}$.
3. $\overline{G}$ *is virtually a direct product* $\mathbb{Z}^k\times\Gamma_{h_1}\times\Gamma_{h_2}$ *for* $h_1,h_2\geq 2$ *and* $k\in\{0,1\}$.
4. $\overline{G}$ *is virtually* $\mathbb{Z}^k\times\Gamma_h$ *for* $h\geq 2$ *and* $k\in\{1,2\}$.
5. $\overline{G}$ *is virtually subdirect and coabelian of even rank.*

As a consequence one can obtain a constraint on Kähler subgroups of direct products of surface groups by imposing the evenness condition on the first Betti number for (1)–(5) in Theorem 4.2. Note that, while groups of the form $\pi_1(R)$, $\Gamma_{h_1}\times\Gamma_{h_2}$ and $\mathbb{Z}^2\times\Gamma_h$ are Kähler, the same turns out to not be true in general for coabelian subgroups of $\Gamma_{h_1}\times\Gamma_{h_2}\times\Gamma_{h_3}$ of even rank. In fact, many such subgroups are not even the image of a homomorphism from a Kähler group which is induced by a holomorphic map. As an application of Theorem 3.1, we can make this statement precise and thus prove Theorem 1.1 and Corollary 1.3.

Theorem 4.3 *Let $G=\pi_1(X)$ be Kähler and let $\psi\colon G\to\Gamma_{g_1}\times\Gamma_{g_2}\times\Gamma_{g_3}$ be a homomorphism such that the projections $p_i\circ\psi\colon G\to\Gamma_{g_i}$ have finitely generated kernel for $1\le i\le 3$ and the image is finitely presented. Then there is a finite-index subgroup $\overline{G}_0\le\overline{G}=\psi(G)$ such that either*

1. $\overline{G}_0\cong\mathbb{Z}^k\times\Gamma_{h_1}\times\cdots\times\Gamma_{h_s}$ *with* $0\le k+s\le 3$; *or*
2. *there are finite-index subgroups $\Gamma_{\gamma_i}\le\Gamma_{g_i}$, an elliptic curve E and branched holomorphic coverings $f_i\colon S_{\gamma_i}\to E$ for $1\le i\le 3$ such that $\overline{G}_0\cong\pi_1(H)\cong\ker(f_*)$, where H is the smooth generic fibre of the surjective holomorphic map $f=\sum_{i=1}^3 f_i$.*

Conversely, any group satisfying one of the conditions (1) *and* (2) *is the image of a homomorphism satisfying the above hypotheses.*

Proof By Theorem 4.2, it suffices to consider the case when $\overline{G}$ is virtually coabelian of even rank. Then there are finite-index subgroups $\Gamma_{\gamma_i}\le\Gamma_{g_i}$ for $l\geq 0$ and an epimorphism $\phi\colon\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}\to\mathbb{Z}^{2l}$ such that $\overline{G}_0:=\ker\phi\le G$ is a finite-index subgroup and $\overline{G}_0\le\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}$ is a finitely presented full subdirect product. We may further assume that $\overline{G}_0\le\Gamma_{\gamma_1}\times\Gamma_{\gamma_2}\times\Gamma_{\gamma_3}$ is a proper subgroup (if not, then (1) holds with $k=0$ and $s=3$).

Let $X_0\to X$ be the finite-sheeted holomorphic cover corresponding to the subgroup $\psi^{-1}(\overline{G}_0)\le G$. Then X_0 is a compact Kähler manifold with $\psi(\pi_1(X_0))=\overline{G}_0=\ker\phi$ and the projections

$$p_i\circ\psi|_{\pi_1(X_0)}\colon\pi_1(X_0)\to\Gamma_{\gamma_i}$$

have finitely generated kernel. Proposition 4.1 implies that $\psi|_{\pi_1(X_0)}$ is induced by a holomorphic map $f\colon X_0\to S_{\gamma_1}\times S_{\gamma_2}\times S_{\gamma_3}$ with the property that $q_{i,j}\circ f\colon X_0\to S_{\gamma_i}\times S_{\gamma_j}$ is a surjective holomorphic map for $1 \leq i < j \leq 3$. Hence, all assumptions of Theorem 3.1 are satisfied. It follows that $\overline{G}_0$ satisfies (2). The converse direction follows easily by taking quotients of Kähler groups of the form $\mathbb{Z}^{2s} \times \Gamma_{h_1} \times \cdots \times \Gamma_{h_s}$ and from Theorem 2.1. □

Proof of Theorem 1.1 If in Theorem 4.3 the group $\overline{G}$ is a full subdirect product, then (1) can only hold if $\overline{G}_0 \leq \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ has finite index. Hence, we must be in case (2). □

To reduce Corollary 1.3 to Theorem 4.3, we will apply the following result of Bridson and Miller:

Theorem 4.4 [Bridson and Miller 2009, Theorem 4.6] *Let $\Gamma_{g \geq 2}$ be a surface group, let A be any group and let $G \leq \Gamma_g \times A$. Assume that G is finitely presented and that the intersection $G \cap \Gamma_g$ is nontrivial. Then $G \cap A$ is finitely generated.*

Proof of Corollary 1.3 Let $G = \pi_1(X) \leq \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ be a nontrivial Kähler group; in particular, G is finitely presented. Let $\psi\colon \pi_1(X) \hookrightarrow \Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ be the canonical inclusion. To apply Theorem 4.3, we need to show that $\ker(p_i \circ \psi)$ is finitely generated for $1 \leq i \leq 3$.

Assume first that $G \cap \Gamma_{g_i}$ is nontrivial for $1 \leq i \leq 3$. Then Theorem 4.4 implies that $\ker(p_1 \circ \psi) = G \cap (\Gamma_{g_2} \times \Gamma_{g_3})$ is finitely generated and that, similarly, $\ker(p_2 \circ \psi)$ and $\ker(p_3 \circ \psi)$ are finitely generated. If some of the intersections $G \cap \Gamma_{g_i}$ are trivial, then, by reordering factors and projecting away from factors with trivial intersection, we may assume that G is a full subgroup of $\Gamma_{g_1} \times \cdots \times \Gamma_{g_s}$ with $1 \leq s \leq 2$. In particular, we may assume that the embedding of G in $\Gamma_{g_1} \times \Gamma_{g_2} \times \Gamma_{g_3}$ has trivial projection to the last $3-s$ factors. For $s = 1$, it is now trivially true that $\ker(p_i \circ \psi)$ is finitely generated for $1 \leq i \leq 3$, and for $s = 2$ the same follows from another application of Theorem 4.4.

Thus, we can apply Theorem 4.3 in all cases. The first part of the result is then a direct consequence of the fact that Kähler groups have even first Betti number.

Conversely, groups satisfying condition (1) and having even first Betti number are clearly Kähler and $\pi_1(H)$ in (2) is Kähler as the fundamental group of H. □

Remark 4.5 Corollary 1.3 provides a classification of Kähler subgroups of direct products of three surface groups up to passing to finite-index subgroups. This statement can be made more precise in the cases corresponding to (1): when $k = 0$, finite extensions of these groups are Kähler if they are subdirect products of surface groups; and when $k = 2$, the group G is either a finite-index subgroup of a direct product $\mathbb{Z}^2 \times \Gamma_{h'}$ with $h \geq h' \geq 2$ or $\cong \mathbb{Z}^2$.

The following example shows that it may be necessary to pass to finite-index subgroups:

Example 4.6 Let $\Gamma_{g_1} \times \Gamma_{g_2}$ be a direct product of surface groups. For $m \geq 2$, consider the canonical epimorphisms $\nu_i\colon H_1(\Gamma_{g_i}, \mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}$ obtained by mapping a basis of $H_1(\Gamma_{g_i}, \mathbb{Z})$ to $1 \in \mathbb{Z}/m\mathbb{Z}$. Denote by $\widehat{\nu}_i\colon \Gamma_{g_1} \to \mathbb{Z}/m\mathbb{Z}$ the composition of ν_i with the abelianization map and define $\widehat{\nu} := \nu_1 + \nu_2\colon \Gamma_{g_1} \times \Gamma_{g_2} \to \mathbb{Z}/m\mathbb{Z}$. The finite-index subgroup $\ker \widehat{\nu} \leq \Gamma_{g_1} \times \Gamma_{g_2}$ is Kähler and virtually a direct product $\ker \nu_1 \times \ker \nu_2$ of surface groups, but is not itself a direct product of surface groups.

5 Complex hypersurfaces

In this section we prove Theorem 1.4. We consider an embedded connected smooth complex hypersurface $\iota\colon X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ in a direct product of closed Riemann surfaces of genus $g_i \geq 2$. Observe that we may assume that all projections $q_i \circ \iota\colon X \to S_{g_i}$ are nonconstant. Indeed, if one of the projections $q_i \circ \iota\colon X \to S_{g_i}$ in Lemma 5.2 is constant, say $q_r \circ \iota$, then $X = S_{g_1} \times \cdots \times S_{g_{r-1}}$ is a direct product of $r-1$ surfaces. Hence, we do not lose much by excluding this case.

Lemma 5.1 *Let $r \geq 2$ and let $\iota_X : X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ be a geometrically subdirect embedding of a connected smooth complex hypersurface in a direct product of closed Riemann surfaces. Then there is $2 \leq s \leq r$ such that $X = Y \times S_{g_{s+1}} \times \cdots \times S_{g_r}$ with $\iota_Y : Y \hookrightarrow S_{g_1} \times \cdots \times S_{g_s}$ an embedded smooth complex hypersurface which geometrically surjects onto $(s-1)$–tuples.*

Proof The result follows by induction on the number of factors $r \geq 2$. For $r = 2$, the result holds due to the assumption that the embedding is geometrically subdirect. If X does not geometrically surject onto $(r-1)$–tuples, then there is an $(r-1)$–tuple $1 \leq i_1 < \cdots < i_{r-1} \leq r$ such that the irreducible variety $\overline{X} = q_{i_1,\dots,i_{r-1}}(X)$ is $(r-2)$–dimensional; we may assume $i_j = j$. Hence, the smooth generic fibre of $q_{1,\dots,r-1} : X \to S_{g_1} \times \cdots \times S_{g_{r-1}}$ is one-dimensional and therefore equal to S_{g_r}. Let $\overline{X}^* \subset \overline{X}$ be the locus of nonsingular values. Then $\overline{X}^* \times S_{g_r} \subset X$ is an open dense submanifold. It follows that $X = \overline{X} \times S_{g_r}$ with $\overline{X} \hookrightarrow S_{g_1} \times \cdots \times S_{g_{r-1}}$ a connected smooth embedded hypersurface. Clearly $\overline{X}$ is geometrically subdirect. The result follows by induction. □

Lemma 5.2 *Let $r \geq 1$ and let $\iota\colon X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ be a connected smooth complex hypersurface such that the projections $q_i \circ \iota\colon X \to S_{g_i}$ are nontrivial. Then there are finite regular covers $S_{h_i} \to S_{g_i}$ for $1 \leq i \leq r$ such that ι lifts to an embedding $j : X \hookrightarrow S_{h_1} \times \cdots \times S_{h_r}$ with $i_*(\pi_1(X)) \cong j_*(\pi_1(X)) \leq \Gamma_{h_1} \times \cdots \times \Gamma_{h_r}$ a subdirect product.*

Proof The projections $q_i \circ \iota\colon X \to S_{g_i}$ are proper holomorphic maps between compact Kähler manifolds. Thus, $\Gamma_{h_i} := (q_i \circ \iota)_*(\pi_1(X)) \leq \pi_1(S_{g_i})$ is a finite-index subgroup for $1 \leq i \leq r$. Let $f_i : S_{h_i} \to S_{g_i}$ be the associated unramified coverings. Then ι factors through a continuous map $j : X \to S_{h_1} \times \cdots \times S_{h_r}$ making the diagram

$$\begin{array}{ccc} & & S_{h_1} \times \cdots \times S_{h_r} \\ & \overset{j}{\nearrow} & \downarrow \\ X & \xrightarrow{\iota} & S_{g_1} \times \cdots \times S_{g_r} \end{array}$$

commutative. Since ι and the f_i are holomorphic, the map j defines a holomorphic embedding and, by choice of the f_i, the group $j_*(\pi_1(X)) \leq \Gamma_{h_1} \times \cdots \times \Gamma_{h_r}$ is subdirect. □

We may in fact assume that the image $\iota_*(\pi_1(X)) \leq \Gamma_{h_1} \times \cdots \times \Gamma_{h_r}$ is full subdirect.

Lemma 5.3 *Let $r \geq 2$ and let $\iota\colon X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ be an embedded connected smooth complex hypersurface such that $\Lambda := \iota_*(\pi_1(X)) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is a subdirect product. If Λ is not full in $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$, then (after possibly reordering factors) X is biholomorphic to $R_\gamma \times S_{g_3} \times \cdots \times S_{g_r}$ with $j\colon R_\gamma \hookrightarrow S_{g_1} \times S_{g_2}$ an embedded Riemann surface such that $j_*(\pi_1(R_\gamma)) \cong \Gamma_{g_2}$, the projection $R_\gamma \to S_{g_i}$ for $i = 1, 2$ is a branched covering, and $\Gamma_{g_1} \cap j_*(\pi_1(R_\gamma)) = \{1\}$.*

Proof After applying Lemma 5.1 and splitting off direct surface factors from X, we may assume that X geometrically surjects onto $(r-1)$–tuples for $r \geq 2$. If Λ is not full, then there is a factor Γ_{g_i} with $\Gamma_{g_i} \cap \Lambda = \{1\}$, say $i = 1$. Hence, the projection $q_{2,\dots,r}\colon S_{g_1} \times \cdots \times S_{g_r} \to S_{g_2} \times \cdots \times S_{g_r}$ induces an isomorphism $\Lambda \cong q_{2,\dots,r,*}(\Lambda) =: \overline{\Lambda} \leq \Gamma_{g_2} \times \cdots \times \Gamma_{g_r}$. Since X geometrically surjects onto $(r-1)$–tuples, the map $q_{2,\dots,r}\colon X \to S_{g_2} \times \cdots \times S_{g_r}$ is a surjective holomorphic map between closed complex manifolds. It follows that $\overline{\Lambda} \leq \Gamma_{g_2} \times \cdots \times \Gamma_{g_r}$ is a finite-index subgroup and thus a full subdirect product.

The epimorphism $p_1\colon \Lambda \to \Gamma_{g_1}$ induces an epimorphism $\bar{p}_1\colon \overline{\Lambda} \to \Gamma_{g_1}$. By the universal property of full subdirect products of limit groups (see [Bridson et al. 2013, Theorem C(3)]), $\bar{p}_1$ is induced by a homomorphism $\Gamma_{g_2} \times \cdots \times \Gamma_{g_r} \to \Gamma_{g_1}$ and thus factors through the projection $\Gamma_{g_2} \times \cdots \times \Gamma_{g_r} \to \Gamma_{g_i}$ for some $2 \leq i \leq r$ (else the image Γ_{g_1} would contain an element with noncyclic centralizer), say $i = 2$. It follows that the projection $\Lambda \to \Gamma_{g_1} \times \Gamma_{g_2}$ factors through the projection to Γ_{g_2} and thus has image isomorphic to Γ_{g_2}. However, this contradicts geometric surjection to $(r-1)$–tuples unless $r = 2$ (since, as above, $q_{1,\dots,r-1,*}(\Lambda) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_{r-1}}$ is a finite-index subgroup).

This leaves us with the situation when $X = R_\gamma$ is a closed Riemann surface of genus $\gamma \geq 2$ with the property that $\Lambda = \iota_*(\pi_1(X)) \cong \Gamma_{g_2}$. Since $\iota_*(\pi_1(X))$ is subdirect, the projections onto factors induce finite-sheeted branched coverings $R_\gamma \to S_{g_i}$ for $i = 1, 2$. □

Proof of Theorem 1.4 If X is not geometrically subdirect, then (2) holds. Hence, we can assume that X is geometrically subdirect. By Lemma 5.1, reduce to the case that $X = Y \times S_{g_{s+1}} \times \cdots \times S_{g_r}$ with $j\colon Y \hookrightarrow S_{g_1} \times \cdots \times S_{g_s}$ an embedded smooth complex hypersurface that geometrically surjects onto $(s-1)$–tuples. If $s = 1$, then Y is a point and we are in case (2). If $s = 2$ then Y is a smooth Riemann surface and we are again in case (2). Hence, we may assume that $s \geq 3$. By Lemmas 5.2 and 5.3, we may further assume that $\Lambda := j_*(\pi_1(Y)) \leq \pi_1(S_{g_1}) \times \cdots \times \pi_1(S_{g_s})$ is a full subdirect product.

Since Y geometrically surjects onto $(s-1)$–tuples, the projections

$$q_{1,\dots,i-1,i+1,\dots,s} \circ j\colon Y \to S_{g_1} \times \cdots \times S_{g_{i-1}} \times S_{g_{i+1}} \times \cdots \times S_{g_s}$$

are surjective holomorphic maps between closed complex manifolds of the same dimension. Hence, $(q_{1,\dots,i-1,i+1,\dots,s,*} \circ j)(\pi_1(Y)) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_{i-1}} \times \Gamma_{g_{i+1}} \times \cdots \times \Gamma_{g_s}$ is a finite-index subgroup for $1 \leq i \leq s$. Hence, Corollary 3.6 of [Kuckuck 2014] implies that there are finite-index subgroups $\Gamma_{\gamma_i} \leq \Gamma_{g_i}$ and an epimorphism $\phi\colon \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s} \to \mathbb{Z}^k$ such that $\Lambda_0 := \ker\phi = \Lambda \cap (\Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s}) \leq \Lambda$ is a finite-index subgroup and the restriction of ϕ to every factor is surjective. Note that, in particular, $\Lambda_0 \leq \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s}$ is a full subdirect product.

Denote by $Y_0 \to Y$ the finite-sheeted covering associated to the finite-index subgroup $j_*^{-1}(\Lambda_0) \leq \pi_1(Y)$. Then there is a holomorphic embedding $\iota\colon Y_0 \hookrightarrow S_{\gamma_1} \times \cdots \times S_{\gamma_s}$ making the diagram

$$\begin{array}{ccc} Y_0 & \xrightarrow{\iota} & S_{\gamma_1} \times \cdots \times S_{\gamma_s} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{j} & S_{g_1} \times \cdots \times S_{g_s} \end{array}$$

commutative. By construction, we have $\iota_*(\pi_1(Y_0)) = \Lambda_0$ and that Y_0 geometrically surjects onto $(s-1)$–tuples

If $\Lambda_0 \leq \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_s}$ is a finite-index subgroup, then we are in case (1). Hence, we may assume that Λ_0 has infinite index. In particular, $k \geq 1$ and all conditions of Theorem 3.1 are satisfied. Hence, there is an elliptic curve E and branched covers $h_i\colon S_{\gamma_i} \to E$ such that Y_0 is equal to a fibre of the holomorphic map $h = \sum_{i=1}^{s} h_i\colon S_{\gamma_1} \times \cdots \times S_{\gamma_s} \to E$.

The map h has isolated singularities and all fibres are irreducible varieties by the proof of Theorem 3.1. In particular, the map h is a submersion in all but finitely many points. It follows that h has reduced fibres and thus the fibres of h over singular values are singular varieties and, in particular, cannot be smooth manifolds (see eg [Milnor 1968, page 13]). Since Y_0 is a smooth subvariety of $S_{\gamma_1} \times \cdots \times S_{\gamma_s}$, it follows that Y_0 is a smooth generic fibre of h. □

Remark 5.4 We want to mention that case (2) in Theorem 1.4 splits into three cases (after reordering factors):

- (i) X_0 has trivial image in one factor, say S_{γ_r}, and thus $X_0 = S_{\gamma_1} \times \cdots \times S_{\gamma_{r-1}}$.
- (ii) $\iota_*(\pi_1(X_0)) \leq \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ is not full. In this case, the proof of Lemma 5.3 shows that $X_0 = R_h \times S_{\gamma_3} \times \cdots \times S_{\gamma_r}$ with $R_h \hookrightarrow S_{\gamma_1} \times S_{\gamma_2}$ an embedded curve and $\iota_*(\pi_1(X_0)) \cong \Gamma_{\gamma_2} \times \cdots \times \Gamma_{\gamma_r}$.
- (iii) $s = 2$, $X_0 = R_h \times S_{\gamma_3} \times \cdots \times S_{\gamma_r}$ with $R_h \hookrightarrow S_{\gamma_1} \times S_{\gamma_2}$ an embedded curve and $\iota_*(\pi_1(X_0)) = \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r}$ This happens for instance when R_h is a generic hyperplane section of $S_{g_1} \times S_{g_2}$. Note that in this case ι_* is not injective and furthermore this is precisely the case when (1) and (2) both hold in Theorem 1.4.

Remark 5.5 In case (1) of Theorem 1.4, the epimorphism $\iota\colon \pi_1(X_0) \to \Gamma_{\gamma_1} \times \cdots \times \Gamma_{\gamma_r}$ is not necessarily injective. For instance, X_0 can be as in Remark 5.4(iii). However, it can be an isomorphism: Take X to be a smooth generic hyperplane section of $S_{g_1} \times \cdots \times S_{g_r}$. If $r \geq 3$ the Lefschetz hyperplane theorem implies that $X \hookrightarrow S_{g_1} \times \cdots \times S_{g_r}$ induces an isomorphism on fundamental groups.

Remark 5.6 In the light of Theorem 1.4, it is natural to ask if one can also classify smooth subvarieties X of codimension $k \geq 2$ in a direct product of Riemann surfaces $S_{g_1} \times \cdots \times S_{g_r}$ in terms of their fundamental groups. The examples constructed in [Llosa Isenrich 2020] show that the class of fundamental groups of such subvarieties will be much larger. Furthermore, the Lefschetz hyperplane theorem will allow us to realize any fundamental group of a smooth subvariety of codimension $l < k$ as the fundamental group of a smooth subvariety of codimension k whenever $k \le r-2$. These two observations show that any such classification will have to allow a much wider variety of fundamental groups. One observation that seems worth mentioning is that, for $k < \frac{1}{2}r$, the image of $\pi_1(X)$ in $\Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ has to be isomorphic to a virtually coabelian subgroup of even rank in a direct product of $\le r$ surface groups (we might need to get rid of some factors and replace others by finite-index subgroups).

To see this, we first split off direct factors, using the same methods as above, to obtain a codimension k subvariety X_0 in a product of $s \le r$ surfaces which geometrically surjects onto $(s-k)$–tuples. Then we combine results of Kuckuck [2014] with the fact that the inclusion $X_0 \hookrightarrow S_{g_1} \times \cdots \times S_{g_s}$ is holomorphic and thus the images $q_{i_1,\dots,i_{s-k},*}(\pi_1(X)) \le \Gamma_{g_{i_1}} \times \cdots \times \Gamma_{g_{i_{s-k}}}$ are finite-index subgroups for $1 \le i_1 < \cdots < i_{s-k} \le s$ (see [Llosa Isenrich 2020, Sections 5 and 6] for details, in particular Proposition 6.3).

6 Maps to $\mathbb{Z}^3$

Another situation in which we can give a complete answer to Delzant and Gromov's question is the case of coabelian subgroups of rank two. Our proof will make use of [Bridson et al. 2013].

Theorem 6.1 [Bridson et al. 2013, Theorem D] *Let $G \le \Lambda_1 \times \cdots \times \Lambda_r$ be a finitely generated full subdirect product of nonabelian limit groups Λ_i for $1 \le i \le r$.*

Then G is finitely presented if and only if G virtually surjects onto pairs.

Theorem 6.2 *Let X be compact Kähler, let $G = \pi_1(X)$ and let $\phi\colon G \to \Gamma_{g_1} \times \cdots \times \Gamma_{g_r}$ be a homomorphism with finitely presented full subdirect image which is induced by a holomorphic map $f\colon X \to S_{g_1} \times \cdots \times S_{g_r}$. Assume that there is an epimorphism $\psi\colon \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^2$ such that $\ker\psi = \phi(G)$.*

Then (after possibly reordering factors) there is $s \ge 3$, an elliptic curve E and branched covering maps $f_i\colon S_{g_i} \to E$ for $1 \le i \le s$ such that $\phi(G) = \pi_1(H) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$, where H is the connected smooth generic fibre of the holomorphic map $f = \sum_{i=1}^{s} f_i\colon S_{g_1} \times \cdots \times S_{g_s} \to E$, $f_* = \psi|_{\Gamma_{g_1} \times \cdots \times \Gamma_{g_s}}$, and $\psi|_{\Gamma_{g_i}}$ trivial for $i \ge s+1$.

Proof With the same notation as in the proof of Theorem 3.1, consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & S_{g_1} \times \cdots \times S_{g_r} & & \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle (a_1,\dots,a_r)} & \overset{h}{\searrow} & \\ A(X) & \xrightarrow{\bar{f}} & A_1 \times \cdots \times A_r & \longrightarrow & B \end{array}$$

Arguing as in the proof of Theorem 3.1 (see diagram (3-3) and subsequent discussion) we obtain that $\mathrm{rk}_{\mathbb{Z}}\pi_1(B) = 2$ and that the map ψ is induced by the holomorphic map $h\colon S_{g_1} \times \cdots \times S_{g_r} \to B$. Since the restriction $h|_{S_{g_i}}\colon S_{g_i} \to B$ is a holomorphic map, either it is surjective or $h(S_{g_i})$ is a point.

A surjective holomorphic map between closed Riemann surfaces is a branched covering. Hence, there is $1 \le s \le r$ such that (after reordering factors):

- $h\colon S_{g_i} \to B$ is a branched holomorphic covering for $1 \le i \le s$.
- $h(S_{g_i})$ is a point for $s+1 \le i \le r$.

It follows that

$$\begin{aligned}\phi(G) = \ker h_* &= \ker((h|_{S_{g_1}\times\cdots\times S_{g_s}})_*) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}\\ &= \ker\psi = \ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}.\end{aligned}$$

Since $\Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$ is finitely generated and $\phi(G)$ is finitely presented, the full subdirect product $\ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) \cong \phi(G)/(\Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}) \le \Gamma_{g_1} \times \cdots \times \Gamma_{g_s}$ is finitely presented.

If $s=1$, then being a full subdirect product implies that $\ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) = \Gamma_{g_1}$, and, if $s=2$, then Theorem 6.1 implies that the group $\ker(\psi|_{\Gamma_{g_1}\times\Gamma_{g_2}}) \le \Gamma_{g_1}\times\Gamma_{g_2}$ is a finite-index subgroup. However, ψ is an epimorphism onto the infinite group $\mathbb{Z}^2$. It follows that $s \ge 3$.

Hence, the restriction $h|_{S_{g_1}\times\cdots\times S_{g_s}}$ satisfies all conditions of Theorem 2.1, so $\ker(\psi|_{\Gamma_{g_1}\times\cdots\times\Gamma_{g_s}}) = \pi_1(H)$ for H the smooth generic fibre of the restriction $h|_{S_{g_1}\times\cdots\times S_{g_s}}$. Thus, $\phi(G) = \pi_1(H) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$. □

As a consequence of Theorem 6.2, we can now classify all Kähler subgroups arising as kernels of homomorphisms from a direct product of surface groups to $\mathbb{Z}^3$. For this we will require the following result:

Theorem 6.3 [Llosa Isenrich 2020, Corollary 1.6] *Let $k \ge 0$ and $g_1,\dots,g_r \ge 2$. If $\phi\colon \Gamma_{g_1}\times\cdots\times\Gamma_{g_r} \to \mathbb{Z}^{2k+1}$ is a surjective homomorphism, then $\ker\phi$ is not Kähler.*

Theorem 6.4 *Let $r \ge 1$, let $\phi\colon \Gamma_{g_1}\times\cdots\times\Gamma_{g_r} \to \mathbb{Z}^3$ be a homomorphism, let $G = \ker\phi \le \Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$ and let $p_i(G) = \Gamma_{\gamma_i} \le \Gamma_{g_i}$ be the projection of G to the i^{th} factor. Then the following are equivalent:*

(1) *G is Kähler.*

(2) *Either $G = \Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$, or there is $r \ge s \ge 3$, an elliptic curve E and surjective holomorphic maps $f_i\colon S_{\gamma_i} \to E$ for $1 \le i \le s$ such that $G = \pi_1(H) \times \Gamma_{g_{s+1}} \times \cdots \times \Gamma_{g_r}$ (after possibly reordering factors), where H is the connected smooth generic fibre of the holomorphic map $f = \sum_{i=1}^{s} f_i\colon S_{\gamma_1}\times\cdots\times S_{\gamma_s} \to E$, $f_* = \phi|_{\Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_s}}\colon \Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_s} \to \pi_1(E) \cong \phi(\Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_s})$ and $\phi|_{\Gamma_{g_i}}$ is trivial for $i \ge s+1$.*

Theorem 6.4 shows in particular that the image of ϕ is either trivial or isomorphic to $\mathbb{Z}^2$.

Proof By Theorem 2.1, (2) implies (1). Assume that G is Kähler. If ϕ is trivial, then $G = \Gamma_{g_1}\times\cdots\times\Gamma_{g_r}$ is Kähler, and, if $\mathrm{im}(\phi) \le \mathbb{Z}^3$ has odd rank, then, by Theorem 6.3, G is not Kähler. Thus, we may assume that G is a finitely presented full subdirect product of $\Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_r}$ which is the kernel of an epimorphism $\phi\colon \Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_r} \to \mathbb{Z}^2 = \mathrm{im}(\phi)$, where, by slight abuse of notation, ϕ now denotes the restriction of ϕ to $\Gamma_{\gamma_1}\times\cdots\times\Gamma_{\gamma_r}$.

Since G is finitely presented, we apply Theorem 4.4 as in the proof of Corollary 1.3 to show that the kernels of the projections of $\ker(\phi)$ to factors are finitely generated. Let X be a compact Kähler manifold with $G = \pi_1(X)$. Then Proposition 4.1 implies that ϕ is induced by a holomorphic map $f: X \to S_{\gamma_1} \times \cdots \times S_{\gamma_s} \times S_{g_{s+1}} \times \cdots \times S_{g_r}$. Hence, all conditions of Theorem 6.2 are satisfied and we obtain (2). □

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